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Matrix Power, Determinant and Polynomials

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Abstract: In this paper we provide work on an approach integrating the division algorithm over the polynomial ring, and determinants in computing the large matrix powers.

Keywords: Matrix Theory; Division Algorithm; Ring of Polynomials; Matrix Power; Determinants.

1. Introduction

Commonly used approaches to calculating large matrix powers require the eigenvalues of matrices, and often eigenvalue computation can become a demanding and tedious work [3]. Abu-Saris and Ahmad provided in their paper [1] an approach that makes use of polynomials to calculate large powers of matrices. Their approach does not require eigenvalue calculation. The approach begins with the characteristic polynomial of square matrices, and continues with the use of the division algorithm along with the application of Cayley-Hamilton Theorem [2, p.210].This approach produces a recursive algorithm for the computation of the coefficients of polynomials [1].

In their paper [1], Abu-Saris and Ahmad provided a detail discussion for the derivation of their recursive formula and the remainder polynomials. They further provided a theorem giving a relation between the recursive algorithm and the determinant of specific matrices whose entries formed by the coefficients of the characteristic polynomials of square matrices. This relation however was not proven in their article [1]. In our paper, we will use an elementary approach integrating mathematical induction to prove the theorem.

Before we give the proof, let us first give the basic entities from the recursive algorithm [1]. Let's consider, $p(x)=\sum_{i=0}^{k} a_i x^i$ as the characteristic polynomial of a kxk matrix A. Then, as stated in Abu-Saris and Ahmad [1], due to the division algorithm $(A^n=q_n(A)p(A)+r_n(A), n \geq k)$, and Cayley-Hamilton theorem, we have $A^n = \sum_{i=0}^{k-1} b_i(n)A^i$ where

$$b_i(n) = -b_{k-1}(n-1)a_i + b_{i-1}(n-1) \text{ for } i=0,\ldots k-1. \tag{1}$$

This algorithm needs only the computation of the powers of A up to k-1 to obtain the nth power of A, $n \geq k$. One can find further details of the recursive algorithm in Abui-Saris and Ahmad [1].Here, we should note that the remainder polynomials, $r_n(x)$, can also be obtained from a simple long division procedure of x^n by p(x).

2. Determinant Approach for A^n

Abu-Saris and Ahmad [1] provided the following theorem stating determinant as a tool to calculate coefficients, $b_i(n)$, of $\sum_{i=0}^{k-1} b_i(n)x^i$ rather than applying the recursive algorithm given in line (1) .

Theorem(see page 452 in [1]): *if A is a kxk matrix with characteristic polynomial* $p(x)=\sum_{i=0}^{k} a_i x^i$ *then for the remainder polynomial* $r_n(x)=\sum_{i=0}^{k-1} b_i(n)x^i$ *,its coefficients,* $b_i(n)$*, can be obtained from* $b_i(n)=(-1)^{n-k+1}\det(T_i(n))$ *where* $T_i(n)$ *is an(n-k+1)x(n-k+1) matrix,*

$$T_i(n)=\begin{bmatrix} a_i & a_{i-1} & \dots & a_{i-(n-k-1)} & a_{i-(n-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n-k-1)} & a_{k-(n-k)} \\ 0 & 1 & \dots & a_{k-(n-k-2)} & a_{k-(n-k-1)} \\ \dots & \dots\dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{bmatrix}$$ *with* $a_t=0$ *whenever* $t<0$.

Proof: Let p(x) $=\sum_{i=0}^{k} a_i x^i$ be the characteristic polynomial of a *kxk* matrix A, and $r_n(x)=\sum_{i=0}^{k-1} b_i(n)x^i$ be the remainder polynomial for A^n. For the case $n=k$, we have the remainder polynomial, $r_k(x)=\sum_{i=0}^{k-1} -a_i x^i$ (see Abu-Saris and Ahmad [1]). Thus, we get $r_k(x)=\sum_{i=0}^{k-1} b_i(k)x^i=\sum_{i=0}^{k-1} -a_i x^i$ giving us $b_i(k)=-a_i$ *for* $i=0,\dots,k-1$.

For the case $n=k+1$, we have $b_i(k+1)=-b_{k-1}(k)a_i+b_{i-1}(k)$ from the recursive algorithm online (1). Thus, integrating the case n=k yields:

$$b_i(k+1)=-b_{k-1}(k)a_i+b_{i-1}(k)=a_{k-1}a_i-a_{i-1},\ \textit{for } i=0,\dots,k-1. \quad \textbf{(2)}$$

Next, considering the determinant of the matrix, $T_i(k+1)$ gives us:

$$\det(T_i(k+1))=\begin{vmatrix} a_i & a_{i-1} \\ 1 & a_{k-1} \end{vmatrix}=a_i a_{k-1}-a_{i-1}\ , \quad \textit{for } i=0,\dots,k-1.$$

Thus,

$$b_i(k+1) = (-1)^{(k+1-k+1)}[\det(T_i(k+1)] = (-1)^2(a_i a_{k-1} - a_{i-1}) = a_{k-1}a_i - a_{i-1} \quad \textbf{(3)}$$

Comparing the expressions in lines (2) and (3), it is clear that they are identical.

At this point, we assume that the equation, $b_i(n) = (-1)^{n-k+1}\det(T_i(n))$, is held for n. Next, we will show $b_i(n+1) = (-1)^{(n+1)-k+1}\det(T_i(n+1))$ for $n+1$. Using the recursive algorithm in line (1), we write,

$$b_i(n+1) = -b_{k-1}(n)a_i + b_{i-1}(n). \quad \textbf{(4)}$$

If we apply the following two determinant forms for n to the recursive equation in (4),

1. $$b_{k-1}(n) = (-1)^{n-k+1}\det(T_{k-1}(n)) = (-1)^{n-k+1}\begin{vmatrix} a_{k-1} & a_{k-2} & \dots & a_{k-1-(n-k-1)} & a_{k-1-(n-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n-k-1)} & a_{k-(n-k)} \\ 0 & 1 & \dots & a_{k-(n-k-2)} & a_{k-(n-k-1)} \\ \dots & \dots. & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix}$$

(5)

2. $$b_{i-1}(n) = (-1)^{n-k+1}\det(T_{i-1}(n)) = (-1)^{n-k+1}\begin{vmatrix} a_{i-1} & a_{i-2} & \dots & a_{i-1-(n-k-1)} & a_{i-1-(n-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n-k-1)} & a_{k-(n-k)} \\ 0 & 1 & \dots & a_{k-(n-k-2)} & a_{k-(n-k-1)} \\ \dots & \dots. & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix},$$

Then, we obtain:

$$b_i(n+1) = -(-1)^{n-k+1}\begin{vmatrix} a_{k-1} & a_{k-2} & \dots a_{k-1-(n-k-1)} & a_{k-1-(n-k)} \\ 1 & a_{k-1} & \dots a_{k-(n-k-1)} & a_{k-(n-k)} \\ 0 & 1 & \dots a_{k-(n-k-2)} & a_{k-(n-k-1)} \\ \dots & \dots. & \dots \quad \dots & \dots \\ 0 & 0 & \dots \quad 1 & a_{k-1} \end{vmatrix} a_i + (-1)^{n-k+1}\begin{vmatrix} a_{i-1} & a_{i-2} & \dots & a_{i-1-(n-k-1)} & a_{i-1-(n-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n-k-1)} & a_{k-(n-k)} \\ 0 & 1 & \dots & a_{k-(n-k-2)} & a_{k-(n-k-1)} \\ \dots & \dots. & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix}$$

(6)

Now, let's briefly turn our attention to the minors of $det(T_i(n+1))$. From the cofactors expansion:

$$\det(T_i(n+1)) = \begin{vmatrix} a_i & a_{i-1} & \dots & a_{i-(n+1-k-1)} & a_{i-(n+1-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n+1-k-1)} & a_{k-(n+1-k)} \\ 0 & 1 & \dots & a_{k-(n+1-k-2)} & a_{k-(n+1-k-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix}$$

$$= a_i \begin{vmatrix} a_{k-1} & a_{k-2} & \dots & a_{k-(n+1-k-1)} & a_{k-(n+1-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n+1-k-2)} & a_{k-(n+1-k-1)} \\ 0 & 1 & \dots & a_{k-(n+1-k-3)} & a_{k-(n+1-k-2)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix} - \begin{vmatrix} a_{i-1} & a_{i-2} & \dots & a_{i-(n+1-k-1)} & a_{i-(n+1-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n+1-k-2)} & a_{k-(n+1-k-1)} \\ 0 & 1 & \dots & a_{k-(n+1-k-3)} & a_{k-(n+1-k-2)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix} \quad \textbf{(7)}$$

we get the following two minors that can easily be verified by comparing matrices on (5) and (7) to be the same as *det(T*$_{k-1}$*(n))* and *det(T*$_{i-1}$*(n))* (For instance, *k-(n+1-k)=k-1-(n-k)*):

$$\det(T_{k-1}(n)) = \begin{vmatrix} a_{k-1} & a_{k-2} & \dots & a_{k-(n+1-k-1)} & a_{k-(n+1-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n+1-k-2)} & a_{k-(n+1-k-1)} \\ 0 & 1 & \dots & a_{k-(n+1-k-3)} & a_{k-(n+1-k-2)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix} \text{ and } \det(T_{i-1}(n)) = \begin{vmatrix} a_{i-1} & a_{i-2} & \dots & a_{i-(n+1-k-1)} & a_{i-(n+1-k)} \\ 1 & a_{k-1} & \dots & a_{k-(n+1-k-2)} & a_{k-(n+1-k-1)} \\ 0 & 1 & \dots & a_{k-(n+1-k-3)} & a_{k-(n+1-k-2)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{vmatrix}$$

Thus,

$$\det(T_i(n+1)) = \det(T_{k-1}(n))a_i - \det(T_{i-1}(n)), \text{ for } i=0,\dots,k-1 \quad \textbf{(8)}$$

Then, applying the identities on lines (5) and (8) to the equation online (6) gives us the desired equality:

$$b_i(n+1) = -(-1)^{n-k+1}\det(T_{k-1}(n))a_i + (-1)^{n-k+1}\det(T_{i-1}(n)) =$$

$$-(-1)^{n-k+1}[\det(T_{k-1}(n))a_i - \det(T_{i-1}(n))] = (-1)^{(n+1)-k+1}\det(T_i(n+1)),\ i = 0,\dots,k-1.$$

This completes the proof.

Example:

The division of x^{3+1} by x^3-2x^2+4x-5 gives the remainder polynomial: $r_1(x) = \sum_{i=0}^{n-1} b_i(1)x^i = -3x+10$

where:

$$b_0(1) = (-1)^2\begin{vmatrix} a_0 & a_{-1} \\ 1 & a_2 \end{vmatrix} = a_0a_2 - a_{-1} = 10$$

$$b_1(1) = (-1)^2\begin{vmatrix} a_1 & a_0 \\ 1 & a_2 \end{vmatrix} = a_1a_2 - a_0 = -3$$

$$b_2(1) = (-1)^2 \begin{vmatrix} a_2 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 a_2 - a_1 = 0$$

Note that if we were to evaluate the coefficients of the remainder polynomial for the division of x^{3+10} by x^3-2x^2+4x-5 then we would need the determinant of three separate matrices of size 11 by 11. For further discussion on this approach see [1-3].

References

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