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Fractional integrated semi groups and nonlocal Cauchy problem for abstract nonlinear fractional differential equations

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Abstract

Some classes of fractional abstract differential equations with α-integrated semi groups are studied in Banach space. The existence of a unique solution of the nonlocal Cauchy problem is studied. Some properties are given.

Key words: α-Integrated semi groups-Nonlinear fractional abstract differential equations- Nonlocal Cauchy problem.

AMS classifications: 47D60- 47D62- 35A05- 34G20.

1 Introduction

Consider the following abstract fractional differential equation:

$$\frac{d^{\beta}u(t)}{dt^{\beta}} = Au(t) + f(t, B(t)u(t)) + s(t)\sum_{i=1}^{k} c_i u(t_i), \qquad (1.1)$$

with the initial condition

$$u(0) = u_0 \quad , \qquad (1.2)$$

where $0 \le t_1 < \cdots < t_k \le T$, $c_1, ..., c_k$ are real numbers, A is a linear closed operator defined on a dense set S in a Banach Space E,

$$B(t)u = (B_1(t)u, ..., B_r(t)u), B_i(t), i = 1, ..., r$$

is a family of linear closed operators defined on dense sets $S_1, ..., S_r \supset S$, respectively in E to E , f is a given abstract function defined on JXE^r to E, $0 < \beta \le 1$, u_0 is a given element in S and s is a real function , which has continuous derivative

$$r(t) = \frac{d^{\alpha\beta} s(t)}{dt^{\alpha\beta}}, \ on \ \ J = [0,T], \ s(0) = 0.$$

It is assumed that A generates α - times integrated semi groups Q(t), $t \geq 0$ with the following Properties:

$C_1 : Q(t) : t \geq 0$ is family of strongly continuous operator.

C_2 : There exist positive constants M and c such that $\| \| Q(t) \| \| \leq Me^{ct}$, where $\| \cdot \|$ is the norm in E.

C_3 : The interval (c, ∞) is contained in the resolvent set $\rho(A)$ of A and,

$C_4 : (I\lambda - A)^{-1} = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} Q(t) dt$, for all $\lambda > c$,

(I is the identity operator), $0 < \alpha \leq 1$,([1-9]).

Let $C_S(J)$ be the set of all continuous functions u on J with values in S. By a strong Solution of the Cauchy problem (1.1), (1.2), we mean a function u such that:

$$u \in C_S(J),$$

$$\frac{d^{\beta} u(t)}{dt^{\beta}} \in C_E(J),$$

u satisfies the following equation :

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha\beta)} \int_0^t (t-\theta)^{\beta-1} [Au(\theta) + f(\theta, B(\theta)u(\theta)] d\theta$$
$$+ \frac{1}{\Gamma(\alpha\beta)} \int_0^t (t-\theta)^{\beta-1} s(\theta) \sum_{i=1}^{k} c_i u(t_i) d\theta, \qquad (1.3)$$

Where $\Gamma(.)$ is the gamma function. In section 2, we shall consider the linear case. In other words when f depends only on t. In this case the solution can be obtained in a closed form. Also the stability of solutions can be established. In section 3, we shall solve equation (1.3) under suitable conditions on f and the operators $B_1, \cdots, B_r$.

It is assumed that:

$C_5 : \| B_i(t_2) Q(t_1) h \| \leq \frac{K}{t_1^c} \| h \|$, for all $t_1 > 0, h \in E$,

$C_6 : B_1(t)h, \cdots, B_r(t)h$ are uniformly Holder in $t \in J$ for all $h \in \bigcap_i S_i$.

It is assumed also that there exists a function g such that:

$$C_7 : f(t, B(t)u(t)) = \frac{1}{\Gamma(\alpha\beta)} \int_0^t (t-\theta)^{\alpha\beta-1} g(\theta, B(\theta)u(\theta)) d\theta$$

(This means that f(0,w)=0 , $\frac{d^{\alpha\beta} f}{dt^{\alpha\beta}} = g$ exists) , where g is continuous on JXE, with the following properties:

C_8 : g satisfies a uniform Holder condition in $t \in J$ and a Lipschitz condition with respect to $B_1(t)u, \cdots, B_r(t)u$. There are many important applications of the theory of integrated semi groups and the nonlocal Cauchy problem for fractional differential equation. The applications can be found in the theory of quantum mechanics and the theory of elasticity. [1-8].

2 The linear case

Let us consider the case when f depends only on t. Denote by $\psi_1(t)$ and $\psi_2(t)$ the following operator - valued functions:

$$\psi_1(t) = d^{\alpha\beta} dt^{\alpha\beta} \int_0^\infty \zeta_\beta(\theta) Q(t^\beta \theta) d\theta,$$

$$\psi_2(t) = \beta \int_0^\infty \zeta_\beta(\theta) Q(t^\beta \theta) t^{\beta-1} d\theta,$$

Where ζ_β is a probability density function defined on $(0,\infty)$, see [9].

It is clear that $\|\psi_2(t)\| \le Kt^{\beta-1}$ on $(0,T]$, for some constant $K < 0$.

Let us suppose that $\sum_{i=1}^k |c_i| < \frac{1}{KMT^\beta}$, where $M = \sup_J |r(t)|$. Under this condition and the properties of the operators Q, ψ_2 , one gets

$$\sum_{i=1}^k |c_i| \int_0^{t_i} \| r(\eta)\psi_2(t_i - \eta) \| d\eta < 1. \tag{2.1}$$

Theorem 1. *If g is continuous on J and is an element of S for every t in J and if* u_0 *is an element in the domain of definition of the operator* A^2 *, then the strong solution of (1.3) is given by*

$$u(t) = u_0 + \frac{1}{\Gamma(1-\alpha\beta)} \int_0^t (t-\eta)^{-\alpha\beta} \psi_2(\eta) A u_0 d\eta$$

$$+\int_0^t r(\eta)\psi_2(t-\eta) \sum_{i=1}^k c_i u(t_i) d\eta$$

$$+\int_0^t \psi_2(t-\eta) g(\eta) d\eta, \tag{2.2}$$

Where

$$\sum_{i=1}^k c_i u(t_i) = \phi[\sum_{i=1}^k \{c_i u_0 + c_i \Gamma(1-\alpha\beta) \int_0^{t_i} (t_i - \eta)^{-\alpha\beta} \psi_2(\eta) A u_0 d\eta\}]$$

$$+\phi\sum_{i=1}^{k}c_i\int_0^{t_i}\psi_2(t_i-\eta)g(\eta)d, \tag{2.3}$$

ϕ *is the inverse bonded operator:*

$$\phi=[I-\sum_{i=1}^{k}c_i\int_0^{t_i}r(\eta)\psi_2(t_i-\eta)d\eta]^{-1}.$$

Proof. Using our previous results [9-16], and the conditions (C_1)-(C_4), we can write

$$u(t)=\psi_1(t)u_0+\int_0^t\psi_2(t-\eta)g(\eta)d\eta+\int_0^t r(\eta)\psi_2(t-\eta)\sum_{i=1}^{k}c_iu(t_i)d\eta$$

Using the facts:

$$Q(t)h=\frac{t^\alpha}{\Gamma(1+\alpha)}h+\int_0^t Q(\theta)Ahd\theta,\ \textit{for all}\ \ t>0,\ h\in S,$$

$$\frac{d^{\alpha\beta}f}{dt^{\alpha\beta}}=\frac{1}{\Gamma(1-\alpha\beta)}\frac{d}{dt}\int_0^t(t-\theta)^{-\alpha\beta}f(\theta)d\theta=\frac{1}{\Gamma(1-\alpha\beta)}\int_0^t(t-\theta)^{-\alpha\beta}\frac{df(\theta)}{d\theta}d\theta$$

for all continuous f such that f(0)=0 , $0\le\alpha\beta<1$, one gets:

$$u(t)=u_0+\frac{1}{\Gamma(1-\alpha\beta)}\frac{d}{dt}\int_0^t(t-\eta)^{-\alpha\beta}\psi_2(\eta)Au_0d\eta$$

$$+\int_0^t r(\eta)\psi_2(t-\eta)\sum_{i=1}^{k}c_iu(t_i)d\eta+\int_0^t\psi_2(t-\eta)g(\eta)d\eta.$$

Consequently

$$\sum_{j=1}^{k}c_ju(t_j)=\sum_{i=1}^{k}c_j\{u_0+\frac{c_i}{\Gamma(1-\alpha\beta)}\int_0^{t_j}(t_j-\eta)^{-\alpha\beta}\psi_2(\eta)Au_0d\eta\}$$

$$+\sum_{j=1}^{k}c_j\int_0^{t_j}\psi_2(t_j-\eta)g(\eta)d\eta+\sum_{j=1}^{k}c_j\int_0^t r(\eta)\psi_2(t_j-\eta)\sum_{i=1}^{k}c_iu(t_i)d\eta,$$

From (2.2) and (2.3), we deduce that u satisfies the conditions (I), (II) and (III). Now it is easy to see that the considered strong solution is unique and more over the Cauchy problem (1.3) is correctly formulated. Is other words: If $\|u_0\|+\|Au_0\|+\|g\|\le\epsilon$, for sufficiently small $\epsilon>0$, then $\|u\|\le K\epsilon$, for some constant positive constant K.

3 The Nonlinear case

Let V satisfy (formally), the following equation:

$$\partial^{\beta} u(t)\partial t^{\beta} - Au(t) = \frac{1}{\Gamma(\alpha\beta)}\int_0^t (t-\theta)^{-\alpha\beta} V(\theta)d\theta$$

$$= \frac{1}{\Gamma(\alpha\beta)}\int_0^t (t-\theta)^{-\alpha\beta} g(\theta, B(\theta)u(\theta))d\theta$$

$$+\frac{1}{\Gamma(\alpha\beta)}\int_0^t (t-\theta)^{-\alpha\beta} r(\theta)\sum_{i=1}^{k} c_i u(t_i)d\theta.$$

Thus we can write formally

$$u(t) = \psi_1(t)u_0 + \int_0^t \psi_2(t-\eta)V(\eta)d\eta. \tag{3.1}$$

We shall solve the following equation:

$$V(t) = g(t, B(t)u(t)) + r(t)\sum_{i=1}^{k} c_i u(t_i). \tag{3.2}$$

Theorem 2. *If equation (1.3) has a strong solution, then that solution is unique.*

Proof. Set

$$V_j(t) = g(t, B(t)u_j(t)) + r(t)\sum_{i=1}^{k} c_i u_j(t_i), \quad j = 1, 2,$$

where u_1 and u_2 are two solutions of equation (1.3) .

Using conditions (C_5),(C_6) and (C_8) ,one gets:

$$\| V_1(t) - V_2(t) \leq M\int_0^t (t-\theta)^{-1} \| V_1(\theta) - V_2(\theta) \| d\theta$$

$$+M\sum_{i=1}^{k} | c_i | \int_0^{t_i} (t_i - \theta)^{-1} \| V_1(\theta) - V_2(\theta)d\theta,$$

Where $\mu = \beta(1-c)$ and M is a positive constant Set $\gamma = \max_J \| e^{-bt}[V_1(t) - V_2(t)] \|$, where b is a sufficiently large positive number. It is easy to see that:

$$\int_0^t (t-\theta)^{\alpha-1} \| V_1(\theta) - V_2(\theta) \| d\theta$$

$$\leq b^{1-\alpha}\gamma\int_0^{t-1b} e^{b\theta}d\theta + \gamma\int_{t-1b}^{t} e^{b\theta}(t-\theta)^{\alpha-1}d\theta \leq (1b)^{\alpha}[1+1\alpha]\gamma$$

Thus for some positive constant M and for sufficiently large b, one gets $\gamma \leq v\gamma$, where

$v = M(1b)^{\alpha}[1+1\alpha] < 1$. This means that $V_1(t) = V_2(t)$ on J ,so $u_1(t) = u_2(t)$ on J .

Theorem 3. *Equation (1.3) has a strong unique solution.*

Proof. The uniqueness is already proved. Let us prove the existence. Using the method of successive approximations, we set

$$V_n(t) = g(t, Bu_n(t)) + r(t)\sum_{i=1}^{k} c_i u_n(t_i).$$

Thus

$$\max_J \| e^{-bt}[V_{n+1}(t) - V_n(t)] \| \leq v \max_J \| e^{-bt}[V_n(t) - V_{n-1}(t)] \|,$$

So

$$\max_J \| e^{-bt}[V_{n+1}(t) - V_n(t)] \| \leq v^n \| e^{-bt}[V_1(t) - V_0(t)] \|$$

Where $V_0(t)$ is the zero approximation, which can be taken the zero element in E.

Thus the sequence $V_n(t)$ uniformly converges in the space $C_E(J)$ to a continuous function V (t), which satisfies equation (3.2). Since $V \in C_E(J)$,it follows that $u \in C_E(J)$, where u is given by equation (3.1). To prove that $u(t) \in S$, for all $t \in J$, it suffices to prove that V satisfies a uniform Holder condition. Using similar arguments as in [10,15], we see that V satisfies a uniform Holder condition, [Comp 17 -29]. This completes the proof of the theorem.

Conclusion

Using the theory of α -integrated semigroups, we have proved existence and uniqueness theorems for general abstract fractional nonlinear differential equations in Banach space. We have got generalizations of some of our old results.

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