



Blow-up for Semidiscretizations of some Semilinear Parabolic Equations with a Convection Term

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Abstract

This paper concerns the study of the numerical approximation for the following parabolic equations with a convection term

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), & 0 < x < 1, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, & 0 \leq x \leq 1, \end{cases}$$

where $p > 1$.

We obtain some conditions under which the solution of the semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We also prove that the semidiscrete blow-up time converges to the real one, when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate our analysis.

Keywords: Burgers' equation; semidiscretizations; discretizations; parabolic equations; convection term; blow-up; blow-up time; convergence.

1. Introduction

Consider the following boundary value problem

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \quad t > 0, \quad (1)$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \quad (2)$$

$$u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \quad (3)$$

where $p > 1$, $u_0 \in C^2([0,1])$, u_0 is nondecreasing on the interval $(0,1)$ and verifies

$$u_0'(0) = 0, \quad u_0'(1) = 0, \quad (4)$$

$$u_0''(x) - u_0(x)u_0'(x) + u_0^p(x) \geq 0, \quad 0 \leq x \leq 1, \quad (5)$$

$$u_0(x) > -p(p-1)u_0'(x), \quad 0 < x < 1, \quad (6)$$

1.1 Definition

We say that the solution u of (1)-(3) blows up in a finite time if there exists a finite time T_b such that $\|u(\cdot, t)\|_\infty < \infty$ for $t \in [0, T_b)$ but

$$\lim_{t \rightarrow T_b} \|u(\cdot, t)\|_\infty = \infty.$$

The time T_b is called the blow-up time of the solution u .

The above problem arises in fluid mechanics and is called viscous Burgers' equation in one dimension with a reaction term. The solution $u(x, t)$ represents the motion field of the fluid in space and time. Burgers' equation with a reaction term is a transport equation with a convection term. The term uu_x is called convection term. It's a nonlinear term that ensures the movement, generates instability and also responsible for the turbulent appearance (here we'll refer to it as intermittent since we are in one dimension) when it happens. In the general case the term u_{xx} is replaced by νu_{xx} with $\nu > 0$. The term νu_{xx} is the viscous term, which has the opposite effect of slicking and making it appear laminar that is ordered. The constant ν , coefficient of the viscous term, is called the kinematic viscosity (normalized by the density) of the fluid. The fluid's flow ability is inversely proportional to the size of the viscosity. The term u^p (the reaction term) is the external force which is generally a white and Gaussian noise within the time scale which forces the fluid to flow faster, slower or make it mill around. It's the quantitative relation between the convection term and viscous, called Reynolds number that will condition the appearance of the flow in the case when there is no external force. The Burgers' equation occurs in various areas of applied mathematics such as modelling of gas dynamics and traffic flow. It was in 1939 that the Dutch scientist Johannes Martinus Burgers simplified the Navier-Stokes equation by just dropping the pressure term (see [2], [23]).

The theoretical study of blow-up solutions for the parabolic equations with a convection term has been the subject of investigations of many authors (see [3], [6], [7], [8], [9],[19], [20], [21] and the references cited therein). Local in time existence and uniqueness of the solution have been proved(see [4], [5], [24], [26] and the references cited therein). Here, we are interesting in the numerical study using a semidiscrete form of (1)-(3). We give some assumptions under which the solution of a semidiscrete form of (1)-(3) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1] and [26].

The paper is organized as follows. In the next section, we present a semidiscrete scheme of (1)-(3) and give some lemmas which will be used throughout the paper. In section 3, under some conditions, we prove that the solution of the semidiscrete form of (1)-(3) blows up in a finite time. In section 4, we study the convergence of the semidiscrete blow-up time. Finally, in last section, taking some discrete forms of (1)-(3), we give some numerical experiments.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the semidiscrete scheme. Let I be a positive integer and let $h=1/I$. Define the grid $x_i=ih, 0 \leq i \leq I$ and approximate the solution u of (1)-(3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - U_i(t) \delta^0 U_i(t) + U_i^p(t), \quad 1 \leq i \leq I-1, \quad t \in (0, T_b^h), \tag{7}$$

$$\frac{dU_0(t)}{dt} = \delta^2 U_0(t) + U_0^p(t), \quad t \in (0, T_b^h), \tag{8}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + U_I^p(t), \quad t \in (0, T_b^h), \tag{9}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{10}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

$$\delta^0 U_i(t) = \frac{U_{i+1}(t) - U_{i-1}(t)}{2h}, \quad 1 \leq i \leq I-1,$$

$$\delta^0 U_0(t) = 0, \quad \delta^0 U_I(t) = 0,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \quad 0 \leq i \leq I-1,$$

$$\delta^+ \varphi_i \leq 0, \quad 0 \leq i \leq I-1,$$

$$\varphi_i^{p-1} > -p(p-1)h\varphi_{i-1}^{p-2}\delta^0 \varphi_i, \quad 1 \leq i \leq I-1, \quad p \geq 2.$$

Here, $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, where

$$\|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|.$$

When the time T_b^h is finite, we say that the solution $U_h(t)$ of (8)-(10) blows up in a finite time, and the time T_b^h is called the blow-up time of the solution $U_h(t)$.

Lemma 2.1 Let $a_h(t), b_h(t) \in C^0([0, T], \mathfrak{R}^{I+1})$ and let $V_h(t) \in C^1([0, T], \mathfrak{R}^{I+1})$ where $b_h(t)\delta^0 V_h(t) \leq 0$, such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t)\delta^0 V_i(t) + a_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0, T), \quad (11)$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \quad (12)$$

Then we have

$$V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0, T). \quad (13)$$

Proof. Let T_0 be any quantity satisfying the inequality $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that

$$a_i(t) - \lambda > 0 \text{ for } 0 \leq i \leq I, \quad t \in [0, T_0].$$

Let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} Z_i(t)$. Since, $Z_i(t)$ is a continuous vector on the compact $[0, T_0]$, there exists $i_0 \in \{0, \dots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \rightarrow 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I, \quad (14)$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I-1, \quad (15)$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \text{ if } i_0 = 0, \quad (16)$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \text{ if } i_0 = I. \quad (17)$$

From (11), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + b_{i_0}(t_0)\delta^0 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \quad (18)$$

It follows from (14)-(18) that

$$(a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0, \quad (19)$$

which implies that $Z_{i_0}(t_0) \geq 0$ because $a_{i_0}(t_0) - \lambda > 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete.

Lemma 2.2 Let $V_h(t), W_h(t) \in C^1([0, T], \mathfrak{R}^{I+1})$ and $f \in C^1(\mathfrak{R} \times \mathfrak{R}, \mathfrak{R})$ such that

$$\begin{aligned} \frac{dV_i(t)}{dt} - \delta^2 V_i(t) + V_i(t)\delta^0 V_i(t) + f(V_i(t), t) &< \frac{dW_i(t)}{dt} - \\ \delta^2 W_i(t) + W_i(t)\delta^0 W_i(t) + f(W_i(t), t), \quad 0 \leq i \leq I, \quad t \in (0, T), \end{aligned} \quad (20)$$

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I. \quad (21)$$

Then we have

$$V_i(t) < W_i(t), \quad 0 \leq i \leq I, \quad t \in (0, T).$$

Proof. Define the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. We remark that

$$\begin{aligned} \frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k \rightarrow 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I, \\ \delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I-1, \\ \delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if } i_0 = 0, \\ \delta^2 Z_{i_0}(t_0) &= \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if } i_0 = I, \end{aligned}$$

Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + W_{i_0}(t_0)\delta^0 W_{i_0}(t_0) - V_{i_0}(t_0)\delta^0 V_{i_0}(t_0) + f(V_{i_0}(t_0), t_0) - f(W_{i_0}(t_0), t_0) \leq 0,$$

which contradicts the first strict inequality of the lemma and this end the proof.

Lemma 2.3 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_i(t) > 0 \quad \text{for } 0 \leq i \leq I, \quad t \in (0, T_b^h). \quad (22)$$

Proof. Assume that there exists a time $t_0 \in (0, T_b^h)$ such that $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. We observe that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \rightarrow 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 U_{i_0}(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} > 0 \quad \text{if } i_0 = 0,$$

$$\delta^2 U_{i_0}(t_0) = \frac{2U_{I-1}(t_0) - 2U_I(t_0)}{h^2} > 0 \quad \text{if } i_0 = I.$$

which implies that

$$\frac{dU_0(t_0)}{dt} - \delta^2 U_0(t_0) - U_0^p(t_0) < 0,$$

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) + U_{i_0}(t_0) \delta^0 U_{i_0}(t_0) - U_{i_0}^p(t_0) < 0, \quad 1 \leq i_0 \leq I-1,$$

$$\frac{dU_I(t_0)}{dt} - \delta^2 U_I(t_0) - U_I^p(t_0) < 0.$$

But these inequalities contradict (7)-(9) and we obtain the desired result.

Lemma 2.4 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_{i+1}(t) < U_i(t) \quad \text{for } 0 \leq i \leq I-1, \quad t \in (0, T_b^h). \quad (23)$$

Proof. Introduce the vector $Z_h(t)$ defined as follows $Z_i(t) = U_{i+1}(t) - U_i(t)$ for $0 \leq i \leq I-1$. Let t_0 be the first $t > 0$ such that $Z_i(t) < 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I-1\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \rightarrow 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \geq 0, \quad 0 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} < 0, \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} < 0 \quad \text{if } i_0 = 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + U_{i_0+1}(t_0) \delta^0 U_{i_0+1}(t_0) - U_{i_0}(t_0) \delta^0 U_{i_0}(t_0) + U_{i_0}^p(t_0) - U_{i_0+1}^p(t_0) > 0, \quad 1 \leq i_0 \leq I-1,$$

$$\frac{dZ_0(t_0)}{dt} - \delta^2 Z_0(t_0) + U_0^p(t_0) - U_1^p(t_0) > 0.$$

Therefore, we have a contradiction because of (7)-(8). This ends the proof.

Lemma 2.5 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$\frac{dU_i(t)}{dt} > 0 \quad \text{for } 0 \leq i \leq I, \quad t \in (0, T_b^h).$$

Proof. Consider the vector $Z_h(t)$ with $Z_i(t) = \frac{dU_i(t)}{dt}$, $0 \leq i \leq I$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the above equality. We get

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \rightarrow 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0 \text{ if } i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_0(t_0)}{h^2} > 0 \text{ if } i_0 = I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + U_{i_0}(t_0) \delta^0 Z_{i_0}(t_0) + (\delta^0 U_{i_0}(t_0) - pU_{i_0}^{p-1}(t_0)) Z_{i_0}(t_0) < 0 \text{ if } 1 \leq i_0 \leq I-1,$$

$$\frac{dZ_0(t_0)}{dt} - \delta^2 Z_0(t_0) - pU_0^{p-1}(t_0) < 0,$$

$$\frac{dZ_I(t_0)}{dt} - \delta^2 Z_I(t_0) - pU_I^{p-1}(t_0) < 0.$$

But these inequalities contradict (7)-(9) and leads to the desired result.

Lemma 2.6 Let $U_h(t)$ be the solution of (7)-(10). Then, we have, for $p \geq 2$,

$$U_i^{p-1}(t) > -p(p-1)hU_{i-1}^{p-2}(t)\delta^0 U_i(t) \text{ for } 1 \leq i \leq I-1, \quad t \in (0, T_b^h).$$

Proof. Define the vectors $Z_h(t)$, $K_h(t)$ and $V_h(t)$ such that $Z_i(t) = K_i(t) - V_i(t)$ with $K_i(t) = U_i^{p-1}(t)$ and $V_i(t) = -p(p-1)hU_{i-1}^{p-2}(t)\delta^0 U_i(t)$ for $1 \leq i \leq I-1$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \dots, I-1\}$. We may suppose that i_0 is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \rightarrow 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + K_{i_0}(t_0) \delta^0 K_{i_0}(t_0) - V_{i_0}(t_0) \delta^0 V_{i_0}(t_0) + V_{i_0}^p(t_0) - K_{i_0}^p(t_0) < 0, \quad 1 \leq i_0 \leq I-1,$$

But this inequality contradicts (7) and we obtain the desired result.

Lemma 2.7 Let $U_h \in C^1([0, T], \mathfrak{R}^{I+1})$ such that $U_h > 0$. Then, we have,

$$\delta^2 U_i^p \geq p U_i^{p-1} \delta^2 U_i \text{ for } 0 \leq i \leq I, p \geq 2.$$

Proof. Using Taylor's expansion, we get

$$\delta^2 U_0^p = p U_0^{p-1} \delta^2 U_0 + (U_1 - U_0)^2 \frac{p(p-1)}{2h^2} \theta_0^{p-2},$$

$$\delta^2 U_i^p = p U_i^{p-1} \delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p-1)}{2h^2} \xi_i^{p-2} + (U_{i-1} - U_i)^2 \frac{p(p-1)}{2h^2} \theta_i^{p-2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_I^p = p U_I^{p-1} \delta^2 U_I + (U_{I-1} - U_I)^2 \frac{p(p-1)}{2h^2} \theta_I^{p-2},$$

Where $p \geq 2$, $\theta_0 \in (U_1, U_0)$, $\theta_i \in (U_i, U_{i-1})$, $\xi_i \in (U_{i+1}, U_i)$ and $\theta_I \in (U_I, U_{I-1})$.

The result follows taking into account the fact that $U_h > 0$.

Lemma 2.8 Let $U_h \in C^1([0, T], \mathfrak{R}^{I+1})$ such that $U_h > 0$. Then, we have,

$$-U_i \delta^0 U_i^p \geq -p U_i^p \delta^0 U_i - p(p-1)h U_{i-1}^{p-2} (\delta^0 U_i)^2, \quad 1 \leq i \leq I-1, p \geq 2.$$

Proof. Applying Taylor's expansion, we obtain

$$\delta^0 U_i^p = p U_{i-1}^{p-1} \delta^0 U_i + (U_{i+1} - U_{i-1})^2 \frac{p(p-1)}{4h} U_{i-1}^{p-2}, \quad 1 \leq i \leq I-1, p = 2,$$

$$\delta^0 U_i^p = p U_{i-1}^{p-1} \delta^0 U_i + (U_{i+1} - U_{i-1})^2 \frac{p(p-1)}{4h} U_{i-1}^{p-2} + (U_{i+1} - U_{i-1})^3 \frac{p(p-1)(p-2)}{12h} \zeta_i^{p-3}, \quad 1 \leq i \leq I-1, p \geq 3,$$

Where $\zeta_i \in (U_{i+1}, U_{i-1})$.

Using Lemma 2.4 and $U_h > 0$, we have the desired result.

3. Semidiscrete Blow-up solutions

In this section under some assumptions, we show that the solution U_h of (7)-(10) blows up in a finite time and estimate its semidiscrete blow-up time.

Theorem 3.1 Let U_h be the solution of (7)-(10), then the solution U_h blows up in a finite time T_b^h with following estimate

$$T_b^h \leq \frac{1}{(p-1)} \frac{1}{(\min_{0 \leq i \leq I} (\varphi_i))^{p-1}}. \tag{24}$$

Proof. Consider the following differential equation

$$\dot{\alpha}(t) = \alpha^p(t), t \in (0, T_\alpha), p \geq 2, \tag{25}$$

$$\alpha(0) = \min_{0 \leq i \leq I} (\varphi_i), \tag{26}$$

with $T_\alpha = \frac{1}{(p-1)} \frac{1}{(\min_{0 \leq i \leq I} (\varphi_i))^{p-1}}$.

Introduce the vector $V_h(t)$ such that $V_i(t) = \alpha(t)$, $0 \leq i \leq I$, $t \in (0, T_\alpha)$. Let the vector $Z_h(t)$ define as follow
 $Z_h(t) = U_h(t) - V_h(t)$. It not hard to see that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + U_i(t) \delta^0 Z_i(t) + (\delta^0 V_i(t) - p \beta_i^{p-1}(t)) Z_i(t) \geq 0 \text{ if } 0 \leq t \leq I, t \in (0, T_1),$$

$$Z_i(0) = 0,$$

where $\beta_i(t) \in (V_i(t), U_i(t))$ and $T_1 = \min \{T_\alpha, T_b^h\}$

Due to Lemma 2.2, we have $U_i(t) \geq V_i(t)$, $0 \leq t \leq I$, $t \in (0, T_1)$. We deduce that

$$T_b^h \leq T_\alpha \leq \frac{1}{(p-1)} \frac{1}{(\min_{0 \leq i \leq I} (\varphi_i))^{p-1}}.$$

The following theorem gives a best result than the previous.

Theorem 3.2 Let U_h be the solution of (7)-(10). Suppose that there exists a positive integer λ such that

$$\delta^2 U_i(0) - U_i(0) \delta^0 U_i(0) + U_i^p(0) \geq \lambda U_i^p(0), \quad 0 \leq i \leq I. \quad (27)$$

Then, the solution U_h blows up in a finite time T_b^h and we have the following estimate

$$T_b^h \leq \frac{1}{\lambda} \frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

Proof. Let $(0, T_p^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty < \infty$. Our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} - \lambda U_i^p(t), \quad 0 \leq i \leq I. \quad (28)$$

A straightforward calculation gives

$$\frac{dJ_i}{dt} - \delta^2 J_i + U_i \delta^0 J_i = \frac{d^2 U_i}{dt^2} - \lambda p U_i^{p-1} \frac{dU_i}{dt} - \delta^2 \frac{dU_i}{dt} + \lambda \delta^2 U_i^p + U_i \delta^0 \left(\frac{dU_i}{dt} \right) - \lambda U_i \delta^0 U_i^p, \quad 1 \leq i \leq I-1.$$

From Lemma 2.7, we have $\delta^2 U_i^p \geq p U_i^{p-1} \delta^2 U_i$ for $0 \leq i \leq I$, $p \geq 2$, which implies that

$$\frac{dJ_i}{dt} - \delta^2 J_i + U_i \delta^0 J_i \geq \frac{d}{dt} \left(\frac{d}{dt} U_i - \delta^2 U_i \right) - \lambda p U_i^{p-1} \left(\frac{d}{dt} U_i - \delta^2 U_i \right) + U_i \delta^0 \left(\frac{d}{dt} U_i \right) - \lambda U_i \delta^0 U_i^p, \quad 1 \leq i \leq I-1.$$

Using (7)-(9), we arrive at

$$\frac{dJ_i}{dt} - \delta^2 J_i + U_i \delta^0 J_i + (\delta^0 U_i - p U_i^{p-1}) J_i \geq -\lambda U_i \delta^0 U_i^p + \lambda p U_i^p \delta^0 U_i - \lambda U_i^p \delta^0 U_i, \quad 1 \leq i \leq I-1.$$

$$\frac{dJ_0}{dt} - \delta^2 J_0 - p U_0^{p-1} J_0 \geq 0,$$

$$\frac{dJ_I}{dt} - \delta^2 J_I - p U_I^{p-1} J_I \geq 0.$$

Using Lemma 2.8 we get

$$\frac{dJ_i}{dt} - \delta^2 J_i + U_i \delta^0 J_i + (\delta^0 U_i - pU_i^{p-1})J_i \geq -\lambda U_i \delta^0 U_i (U_i^{p-1} + p(p-1)hU_{i-1}^{p-2} \delta^0 U_i), 1 \leq i \leq I-1,$$

$$\frac{dJ_0}{dt} - \delta^2 J_0 - pU_0^{p-1} J_0 \geq 0,$$

$$\frac{dJ_I}{dt} - \delta^2 J_I - pU_I^{p-1} J_I \geq 0.$$

From Lemma 2.6, we have $U_i^{p-1} > -p(p-1)hU_{i-1}^{p-2} \delta^0 U_i$ for $1 \leq i \leq I-1$ and using the fact that $-\lambda U_i \delta^0 U_i \geq 0$, we get finally

$$\frac{dJ_i}{dt} - \delta^2 J_i + U_i \delta^0 J_i + (\delta^0 U_i - pU_i^{p-1})J_i \geq 0, 1 \leq i \leq I-1,$$

$$\frac{dJ_0}{dt} - \frac{(2J_1 - 2J_0)}{h^2} - pU_0^{p-1} J_0 \geq 0,$$

$$\frac{dJ_I}{dt} - \frac{(2J_{I-1} - 2J_I)}{h^2} - pU_I^{p-1} J_I \geq 0.$$

From (27), we observe that

$$J_i(0) = \delta^2 U_i(0) - U_i(0) \delta^0 U_i(0) + U_i^p(0) \geq \lambda U_i^p(0), \quad 0 \leq i \leq I.$$

We deduce from Lemma 2.1 that $J_h(t) \geq 0$ for $t \in (0, T_b^h)$, which implies that

$$\frac{dU_i(t)}{dt} \geq \lambda U_i^p(t), \quad 0 \leq i \leq I, \quad t \in (0, T_b^h). \tag{29}$$

These estimates may be rewritten in the following form

$$U_i^{-p} dU_i \geq \lambda dt, \quad 0 \leq i \leq I.$$

Integrating the above inequalities over (t, T_b^h) , we arrive at

$$T_b^h - t \leq \frac{1}{\lambda} \frac{(U_i(t))^{1-p}}{(p-1)}. \tag{30}$$

which implies that

$$T_b^h \leq \frac{1}{\lambda} \frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

Remark 3.1 The inequalities (30) implies that

$$T_b^h - t_0 \leq \frac{1}{\lambda} \frac{\|U_h(t_0)\|_\infty^{1-p}}{(p-1)} \text{ if } 0 < t_0 < T_b^h.$$

4. Convergence of the semidiscrete blow-up time

In this section, under some assumptions, we show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. In order to obtain the convergence of semidiscrete blow-up time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1 Assume that (1)-(3) has a solution $u \in C^{4,1}([0,1] \times [0, T])$ and the initial condition at (10) satisfies

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \text{ as } h \rightarrow 0, \quad (31)$$

Where $u_h(t) = (u(x_0, t), \dots, u(x_I, t))^T$. Then, for h sufficiently small, the problem (7)-(10) has a unique solution $U_h \in C^1([0, T_b^h], \mathfrak{R}^{I+1})$ such that

$$\max_{0 \leq t \leq T} \|U_h(t) - u_h(t)\|_\infty = O(\|U_h^0 - u_h(0)\|_\infty + h^2) \text{ as } h \rightarrow 0. \quad (32)$$

Proof. Let $K > 0$ be such that

$$\|u\|_\infty \leq K. \quad (33)$$

The problem (7)-(10) has for each h , a unique solution $U_h \in C^1([0, T_b^h], \mathfrak{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \text{ for } t \in (0, t(h)). \quad (34)$$

The relation (31) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. By the triangular inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u(\cdot, t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \text{ for } t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq 1 + K \text{ for } t \in (0, t^*(h)). \quad (35)$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0, t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) + u(x_i, t) \delta^0 e_i(t) = p\beta_i^{p-1}(t)e_i(t) - e_i(t) \delta^0 u(x_i, t) - \frac{h^2}{6} u(x_i, t) u_{xxx}(\tilde{x}_i, t),$$

$$\frac{de_0(t)}{dt} - \frac{(2e_1(t) - 2e_0(t))}{h^2} = p\beta_0^{p-1}(t)e_0(t) + \frac{h^2}{12} u_{xxxx}(\tilde{x}_0, t),$$

$$\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} = p\beta_I^{p-1}(t)e_I(t) + \frac{h^2}{12} u_{xxxx}(\tilde{x}_I, t),$$

where $\beta_i \in (U_i(t), u(x_i, t))$ for $i \in \{0, \dots, I\}$.

Using (35), there exists a constant $M > 0$ such that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) + u(x_i, t) \delta^0 e_i(t) \leq M|e_i(t)| + Mh^2, \quad 1 \leq i \leq I-1, \quad (36)$$

$$\frac{de_0(t)}{dt} - \frac{(2e_1(t) - 2e_0(t))}{h^2} \leq M|e_0(t)| + Mh^2, \quad (37)$$

$$\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} \leq M|e_I(t)| + Mh^2. \quad (38)$$

Consider the vector W_h such that

$$W_i(t) = e^{(M+1)t} (\|U_h^0 - u_h(0)\|_\infty + Mh^2), \quad 0 \leq i \leq I.$$

A direct calculation yields

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + u(x_i, t) \delta^0 W_i(t) > M|W_i(t)| + Mh^2, \quad 1 \leq i \leq I-1, \quad (39)$$

$$\frac{dW_0(t)}{dt} - \frac{(2W_1(t) - 2W_0(t))}{h^2} > M|W_0(t)| + Mh^2, \quad (40)$$

$$\frac{dW_I(t)}{dt} - \frac{(2W_{I-1}(t) - 2W_I(t))}{h^2} > M|W_I(t)| + Mh^2, \quad (41)$$

$$W_i(0) > e_i(0), \quad 0 \leq i \leq I. \quad (42)$$

It follows from Lemma 2.2 that

$$W_i(t) > e_i(t) \text{ for } t \in (0, t^*(h)), \quad 0 \leq i \leq I.$$

By the same way, we also prove that

$$W_i(t) > -e_i(t) \text{ for } t \in (0, t^*(h)), \quad 0 \leq i \leq I,$$

which implies that

$$W_i(t) > |e_i(t)| \text{ for } t \in (0, t^*(h)), \quad 0 \leq i \leq I.$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(M+1)T} (\|U_h^0 - u_h(0)\|_\infty + Mh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t^*(h)$. From (34), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(M+1)T} (\|U_h^0 - u_h(0)\|_\infty + Mh^2). \quad (43)$$

Since $e^{(M+1)T} (\|U_h^0 - u_h(0)\|_\infty + Mh^2) \rightarrow 0$ when $h \rightarrow 0$, we deduce from (43) that $1 \leq 0$, which is impossible.

Consequently $t^*(h) = T$, and we conclude the proof.

Theorem 4.2 Suppose that the solution u of (1)-(3) blows up in a finite time T_b such that $u \in C^{4,1}([0,1] \times [0, T_b])$ and the initial condition at (10) satisfies

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \text{ as } h \rightarrow 0. \quad (44)$$

Assume that there exists a constant $\lambda > 0$ such that

$$\delta^2 U_i(0) - U_i(0) \delta^0 U_i(0) + U_i^p(0) \geq \lambda U_i^p(0), \quad 0 \leq i \leq I. \quad (45)$$

Then the solution U_h of (7)-(10) blows up in a finite time T_b^h and

$$\lim_{h \rightarrow 0} T_b^h = T_b. \quad (46)$$

Proof. Let $\varepsilon > 0$. There exists N such that

$$\frac{1}{\lambda} \frac{y^{1-p}}{(p-1)} \leq \frac{\varepsilon}{2} < \infty \text{ for } y \in [N, +\infty[. \quad (47)$$

Since $\lim_{t \rightarrow T_b} \max_{x \in [0,1]} |u(x,t)| = +\infty$, then, there exists T_I such that

$|T_1 - T_b| \leq \frac{\varepsilon}{2}$ and $\|u(x,t)\|_\infty \geq 2N$ for $t \in [T_1, T_b]$. Let $T_2 = \frac{T_1 + T_b}{2}$, then $\sup_{t \in [0, T_2]} |u(x,t)| < +\infty$.

It follows from Theorem 4.1 that $\sup_{t \in [0, T_2]} |U_h(t) - u_h(t)| < N$. Applying the triangular inequality, we get

$$\|U_h(t)\|_\infty \geq \|u(\cdot, t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty, \text{ which leads to } \|U_h(t)\|_\infty \geq N \text{ for } t \in [0, T_2].$$

From Theorem 3.2, $U_h(t)$ blows up at the time T_b^h . We deduce from Remark 3.1 and (47) that

$$|T_b - T_b^h| \leq |T_b - T_2| + |T_b^h - T_2| \leq \frac{\varepsilon}{2} + \frac{1}{\lambda} \frac{\|U_h(T_2)\|_\infty^{1-p}}{(p-1)} \leq \varepsilon,$$

which leads us to the desired result.

5. Numerical results

In this section, we present some numerical approximations to the blow-up time of (1)-(3). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - U_i^{(n)} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} + (U_i^{(n)})^p, \quad 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (U_I^{(n)})^p,$$

where $n \geq 0$, $p \geq 2$, $\Delta t_n = \min \left\{ \frac{h^2}{2}, \tau \|U_h^{(n)}\|_\infty^{1-p} \right\}$ with $\tau = cont \in (0,1)$.

Also we use the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - U_i^{(n)} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} + (U_i^{(n)})^p, \quad 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (U_I^{(n)})^p,$$

where $n \geq 0$, $p \geq 2$, $\Delta t_n = \tau \|U_h^{(n)}\|_\infty^{1-p}$ with $\tau = cont \in (0,1)$.

In the tables 1-8, in rows, we present the numerical blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical blow-up

time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}$. The order(s) of the method is

$$\text{computed from } s = \frac{\log((T_{4h} - T_{2h}) / (T_{2h} - T_h))}{\log(2)}.$$

First case: $U_i^{(0)} = \frac{1}{2}$, $p = 2$ and $\tau = \frac{h^2}{2}$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	2.003306	15788	-	-
32	2.000827	60265	-	-
64	2.000207	229656	2	1.99
128	2.000052	873150	9	1.99
256	2.000013	3310849	63	2.00
512	2.000003	12516533	464	2.00
1024	2.000001	47158825	3458	2.00

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	2.003906	15631	-	-
32	2.000977	59637	-	-
64	2.000244	227142	2	2.00
128	2.000061	863093	12	2.00
256	2.000015	3270629	89	2.00
512	2.000004	12355655	672	2.00
1024	2.000001	46515309	5159	2.00

Second case: $U_i^{(0)} = 2$, $p = 2$ and $\tau = \frac{h^2}{2}$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.500977	15631	-	-
32	0.500244	59637	-	-
64	0.500061	227142	2	2.00
128	0.500015	863093	9	2.00
256	0.500004	3270629	62	2.00
512	0.500001	12355655	457	2.00
1024	0.500000	46515309	3407	2.00

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.500977	15631	-	-
32	0.500244	59637	-	-
64	0.500061	227142	2	2.00
128	0.500015	863093	12	2.00
256	0.500004	3270629	90	2.00
512	0.500001	12355655	674	2.00
1024	0.500000	46515309	5070	2.00

Third case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1 - (ih)^2)^2$, $p = 2$ and $\tau = \frac{h^2}{2}$.

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.861878	16031	-	-
32	0.860194	60831	-	-
64	0.859773	231919	1	2.00
128	0.859668	882201	9	2.00
256	0.859641	3347050	66	2.00
512	0.859635	12661342	488	2.00
1024	0.859633	47738061	3670	2.00

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.861680	15961	-	-
32	0.860144	60832	-	-
64	0.859761	231919	2	2.00
128	0.859665	882201	13	2.00
256	0.859641	3347052	93	2.00
512	0.859635	12661342	699	2.00
1024	0.859633	47738061	5294	2.00

Fourth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1 - (ih)^2)^2$, $p = 4$ and $\tau = \frac{h^2}{2}$.

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.012839	4693	-	-
32	0.012798	17825	-	-
64	0.012788	67544	-	1.98
128	0.012786	255190	3	1.99
256	0.012785	960804	18	1.99
512	0.012785	3603268	135	1.99
1024	0.012785	13452596	1002	1.99

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.012841	4694	-	-
32	0.012799	17825	-	-
64	0.012788	67545	1	1.98
128	0.012786	255190	4	1.99
256	0.012785	960804	27	1.99
512	0.012785	3603268	197	1.99
1024	0.012785	13452596	1490	1.99

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I=16$ and $p=2$. In Figures 1 and 2, we can appreciate that the discrete solution blows up in a finite time where the initial data is a constant. In Figures 3 and 4, we see that the blow-up is faster when the initial data is not a constant. The Figures 5, 6, 7 and 8 show the effect of the convection term on the evolution of the solution. In Figures 9, 10, 11 and 12, we observe that the solution of our problem blows up in a finite time $t \approx 2$ when the initial data is $\frac{1}{2}$ and $t \approx 0.86$ when the initial data is

$$\left(\frac{1}{2}\right)^{3-p} + (1 - (ih)^2)^2.$$

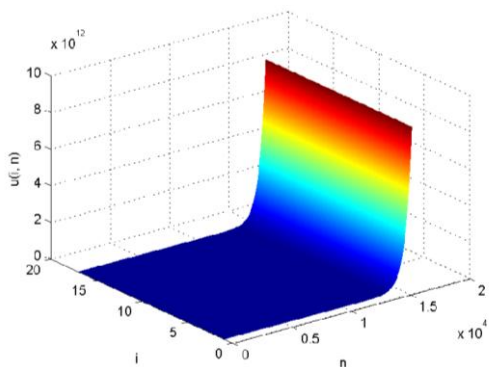


Figure 1: Evolution of the discrete solution(Explicit scheme)

$$U_i^{(0)} = 2, p = 2$$

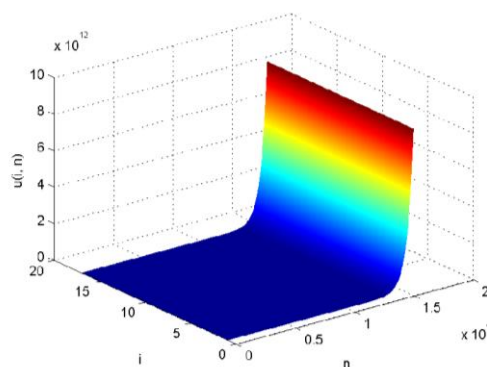


Figure 2: Evolution of the discrete solution(Implicit scheme)

$$U_i^{(0)} = 2, p = 2$$

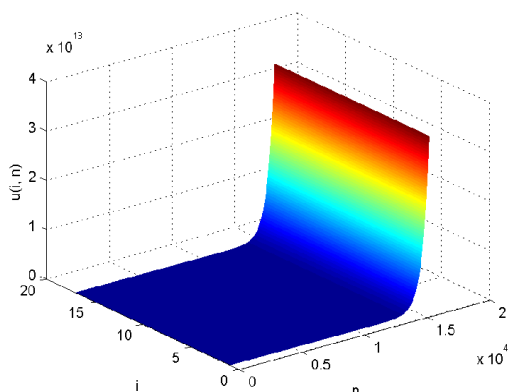


Figure 3: Evolution of the discrete solution(Explicit scheme)

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1 - (ih)^2)^2, p = 2$$

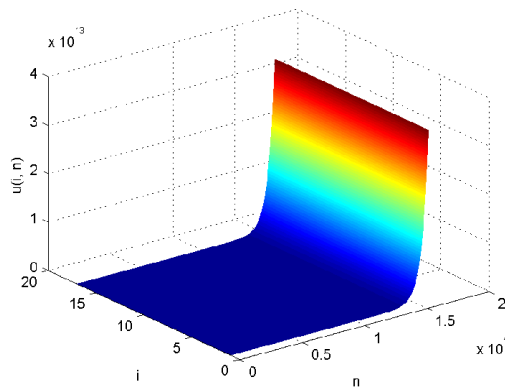


Figure 4: Evolution of the discrete solution(Implicit scheme)

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1 - (ih)^2)^2, p = 2$$

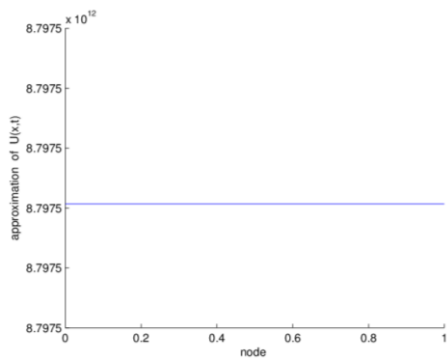


Figure 5: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = 2, p = 2$

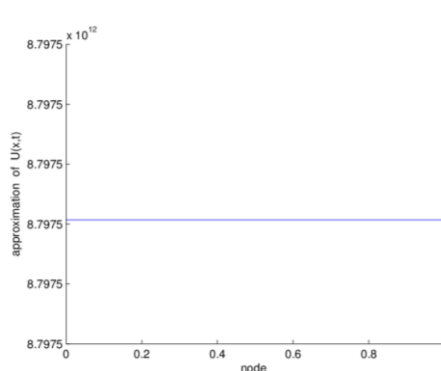


Figure 6: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = 2, p = 2$

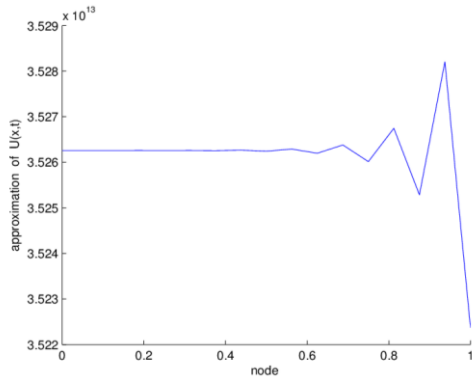


Figure 7: Evolution of $U(x,t)$ according to the node (explicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1 - (ih)^2\right)^2, p = 2$$

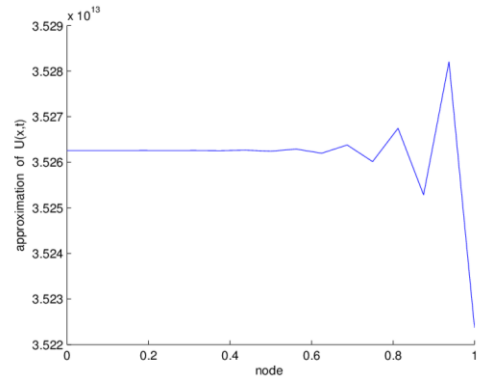


Figure 8: Evolution of $U(x,t)$ according to the node (implicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1 - (ih)^2\right)^2, p = 2$$

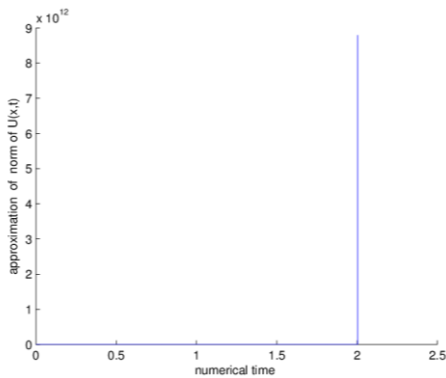


Figure 9: Evolution of norm of $U(x,t)$ according to the time (explicit scheme), $U_i^{(0)} = 2, p = 2$

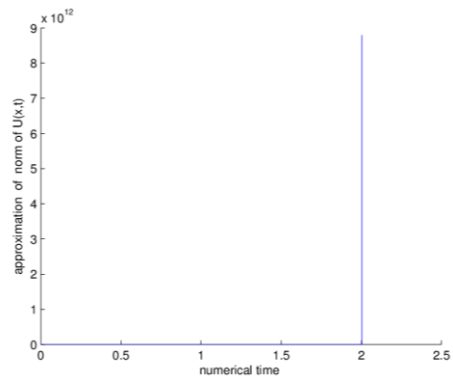


Figure 10: Evolution of norm of $U(x,t)$ according to the time (implicit scheme), $U_i^{(0)} = 2, p = 2$

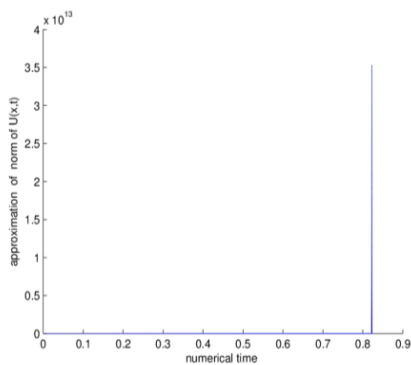


Figure 11: Evolution of norm of $U(x,t)$ according to the time (explicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1 - (ih)^2\right)^2, p = 2$$

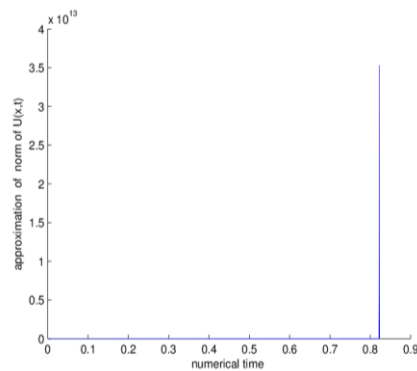


Figure 12: Evolution of norm of $U(x,t)$ according to the time (implicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1 - (ih)^2\right)^2, p = 2$$

Remark 5.1

We observe that the blow-up phenomenon occurs faster for the large values of the initial data and the exponent p . In the case where the initial data is a constant, the solution of our problem blows up in a finite time for all $p \geq 2$,

but slowly. This slowness is due to the absence of the turbulence effect, generated by the convection term. Therefore the blow-up only depends on the reaction term. When the initial data is not a constant, the convection term, head of turbulence, accelerates the blow-up created by the reaction term.

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