Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

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RESEARCH ORGANISATION|

Volume 5, Issue 2
Published online: September 01, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

On a Stability Theorem of the Optimal Control Problem For Quasi Optic Equation

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Abstract

In this paper, the finite difference method is applied to the optimal control problem of system stationary equation of Quasi-Optic. The optimal control problem has been covered to finite dimensional optimization problem and difference approximations are obtained. The estimation of stability of difference scheme is proved.

Keywords: Quasi optic; Schrödinger equation; optimal control.

Mathematics Subject classification: 49J20 , 65M06, 35Q93.

1. Introduction

Optimal control theory play an important role in many area in science and engineeering. The optimal control problems governed by schrödinger equation is one of those areas. Quasi optic equation is a form of the Schrödinger equation with complex potential. Such problems have arise various branches of non-linear optic, modern physics and quantum mechanic. These problems have been studied by many researchers in [1-7]. Difference methods for such control problems are investigated in studies [11-13].

In this paper, we prove the stability estimate of difference approximations of the optimal control problem governed by quasi optic equation with control in coefficient. The set of admissible controls is a set of square integrable functions. Considered the optimal control problem differs from previous studies because of its statement and cost functional.

2. Formulation of the problem and its difference scheme

Let us consider the problem of finding the minimum of the functional

$$J(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} \tag{1}$$

in the set

$$V \equiv \left\{ v = (v_0, v_1), v_m \in L_2(0,L), \|v_m\|_{L_2(0,L)} \le b_m, \ v_1(z) \ge 0 , \overset{0}{\forall} z \in (0,L), m = 0,1 \right\}$$

subject to

$$i\frac{\partial \psi_k}{\partial z} + a_0 \frac{\partial^2 \psi_k}{\partial x^2} - a(x)\psi_k + v_0(z)\psi_k + iv_1(z)\psi_k = f_k(x,z),$$

$$(x,z) \in \Omega , \ k = 1,2 , \tag{2}$$

$$\psi_k(x,0) = \varphi_k(x), x \in (0,l), \ \ k = 1,2 , \tag{3}$$

$$\psi_1(0,z) = \psi_1(l,z) = 0 , \ z \in (0,L) , \tag{4}$$

$$\frac{\partial \psi_2(0,z)}{\partial x} = \frac{\partial \psi_2(l,z)}{\partial x} = 0 , z \in (0,L). \tag{5}$$

where $\psi_k = \psi_k(x,z)$ is a wave function, $i = \sqrt{-1}$, $a_0 > 0$, $l > 0$, $L > 0$, $b_m > 0$ $(m = 0,1)$ are given numbers, $a(x)$ is a measurable bounded function that satisfies the following conditions:

$$0 < \mu_0 \le a(x) \le \mu_1 , \quad \left|\frac{da(x)}{dx}\right| \le \mu_2, \quad \left|\frac{d^2 a(x)}{dx^2}\right| \le \mu_3 , \forall x \in (0,l), \ \mu_m = constant > 0.$$

$\varphi_k(x)$ and $f_k(x,z)$ are given functions that satisfy the condition

$$\varphi_1 \in \overset{0}{W}{}_2^2(0,l) \ , \varphi_2 \in W_2^2(0,l), \ \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0 \tag{6}$$

$$f_1 \in \overset{0}{W}{}_2^{2,0}(\Omega) , \ f_2 \in W_2^{2,0}(\Omega), \ \frac{\partial f(0,z)}{\partial x} = \frac{\partial f(l,z)}{\partial x} = 0. \tag{7}$$

The spaces $W_l^{k,m}(\Omega)$ are Sobolev spaces defined as in Ladyzenskaja et al. (1968).
In study [14], it was shown that the problem (1) to (4) has unique solution for each $v \in V$ and the following estimation is valid for this solution

$$\|\psi_1\|_{\overset{0}{W}{}_2^{2,1}(\Omega)} \le c_1(\|\varphi_1\|_{\overset{0}{W}{}_2^{2}(0,l)} + \|f_1\|_{\overset{0}{W}{}_2^{2,0}(\Omega)})$$

$$\|\psi_2\|_{W_2^{2,1}(\Omega)} \le c_2(\|\varphi_2\|_{W_2^{2}(0,l)} + \|f_2\|_{W_2^{2,0}(\Omega)})$$

for each $z \in (0, L)$.

Now, we shall find approximation of the optimal control problem (1) to (5). For discretization, let us transform the region Ω into the following scheme:

$$\{(x_j, z_k)_n\}, n = 1,2, \ldots \quad, x_j = jh - h/_2, j = \overline{1, M_{n-1}}, z_k = k\tau, k = \overline{1, N_n}$$

$$h = h_n = l/(M_n - 1), \tau = \tau_n = \tau/N_n, M = M_n, N = N_n .$$

and let us write the following assignments:

$$\delta_{\bar{x}}\phi_{jk} = \frac{\phi_{jk} - \phi_{jk-1}}{h}, \delta_{\bar{z}}\phi_{jk} = \frac{\phi_{jk} - \phi_{jk-1}}{\tau}, \delta_x\phi_{jk} = \frac{\phi_{j+1k} - \phi_{jk}}{h},$$

$$\delta_{x\bar{x}}\phi_{jk} = \frac{\phi_{j+1k} - 2\phi_{jk} - \phi_{j-1k}}{h^2}.$$

For the arbitrary natural number, $n \geq 1$, let us consider the minimizing problem of the function

$$I_n([v]_n) = h\sum_{j=1}^{M-1}\left|\phi_{jN}^1 - \phi_{jN}^2\right|^2 \qquad (8)$$

in the set

$$V_n \equiv \{[v]_n : [v]_n = ([v_0]_n, [v_1]_n), v_{1k} \geq 0, k = \overline{1, N}, [v_p] = (v_{p1}, v_{p2}, \ldots, v_{pN}), \left(h\sum_{k=1}^{N}\left|v_{pk}\right|^2\right)^{1/2} \leq b_p, p = 0,1, k = \overline{1, N}\}$$

subject to

$$i\delta_{\bar{z}}\phi_{jk}^p + a_0\delta_{x\bar{x}}\phi_{jk}^p - a_j\phi_{jk}^p + v_{0k}\phi_{jk}^p + iv_{1k}\phi_{jk}^p = f_{jk}^p, j = \overline{1, M-1}, k = \overline{1, N}, (9)$$

$$\phi_{j0}^p = \varphi_j^p, j = \overline{0, M}, p = 1,2, \qquad (10)$$

$$\phi_{0k}^1 = \phi_{Mk}^1 = 0, k = \overline{1, N}, \qquad (11)$$

$$\delta_{\bar{x}}\phi_{1k}^2 = \delta_{\bar{x}}\phi_{Mk}^2 = 0, k = \overline{1, N}. \qquad (12)$$

where the scheme functions $a_j, \varphi_j^p, f_{jk}^p, p = 1,2$ are defined by

$$a_j = \frac{1}{h}\int_{x_j - h/_2}^{x_j + h/_2} a(x)dx \quad, j = \overline{1, M-1} \qquad (13)$$

$$\varphi_j^p = \frac{1}{h} \int_{x_j - h/2}^{x_j + h/2} \varphi_p(x)\,dx \quad , p = 1,2 \ , j = \overline{1, M-1} \qquad (14)$$

$$\varphi_0^1 = \varphi_M^1 = 0 \ , \varphi_0^2 = \varphi_1^2 \ , \ \varphi_M^2 = \varphi_{M-1}^2$$

$$f_{jk}^p = \frac{1}{\tau h} \int_{z_{k-1}}^{z_k} \int_{x_j - h/2}^{x_j + h/2} f_p(x,z)\,dx\,dz \quad , p = 1,2 \ , \quad j = \overline{1, M-1} \ , k = \overline{1, N}. \qquad (15)$$

As we have seen discrite problem (8)- (12) is the same as problem (1)- (5). Hence the problem (8)- (12) has at least solution.

3. The Stability Theorem

In this section, we shall show the stability estimate for the problem (8)- (12).

Theorem 3.1: For each $[v]_n \in V_n$, the stability of the difference scheme (8) -(12) satisfies the following estimation:

$$h \sum_{j=1}^{M-1} \left|\phi_{jk}^p\right|^2 \leq c_3 \left(h \sum_{j=1}^{M-1} \left|\varphi_j^p\right|^2 + \tau h \sum_{k=1}^{N} \sum_{j=1}^{M-1} \left|f_{jk}^p\right|^2 \right) , m = 1,2 \ldots , N \ , p = 1,2 \, . \quad (16)$$

where $c_3 > 0$ is a constant that does not depend on τ and h.

Proof : It is clear that the following identity is valid for $z = z_k$:

$$h \sum_{j=1}^{M-1} \left(i \delta_z \phi_{jk}^1 \bar{\eta}_{jk}^1 \right) - h \sum_{j=1}^{M-1} \left(a_0 i \delta_x \phi_{jk}^1 \delta_x \bar{\eta}_{jk}^1 \right) - h \sum_{j=1}^{M-1} \left(a_j \phi_{jk}^1 \bar{\eta}_{jk}^1 \right) +$$

$$+ h \sum_{j=1}^{M-1} \left(v_{0j} \phi_{jk}^1 \bar{\eta}_{jk}^1 \right) + h \sum_{j=1}^{M-1} \left(v_{1j} \phi_{jk}^1 \bar{\eta}_{jk}^1 \right) = h \sum_{j=1}^{M-1} \left(f_{jk}^1 \bar{\eta}_{jk}^1 \right) , k = \overline{1, N} \, , \qquad (17)$$

$$h \sum_{j=1}^{M-1} \left(i \delta_z \phi_{jk}^2 \bar{\eta}_{jk}^2 \right) - h \sum_{j=1}^{M-1} \left(a_0 i \delta_x \phi_{jk}^2 \delta_x \bar{\eta}_{jk}^2 \right) - h \sum_{j=1}^{M-1} \left(a_j \phi_{jk}^2 \bar{\eta}_{jk}^2 \right) +$$

$$+ h \sum_{j=1}^{M-1} \left(v_{0j} \phi_{jk}^2 \bar{\eta}_{jk}^1 \right) + h \sum_{j=1}^{M-1} \left(v_{1j} \phi_{jk}^2 \bar{\eta}_{jk}^2 \right) = h \sum_{j=1}^{M-1} \left(f_{jk}^2 \bar{\eta}_{jk}^2 \right) , k = \overline{1, N} \, . \qquad (18)$$

where the functions $\bar{\eta}_{jk}^p$, $p = 1,2$ are the complex conjugate of any functions η_{jk}^p , $p = 1,2$ defined in the scheme $\{(x_j, z_k)_n\}$ such that $\eta_{0k}^1 = \eta_{Mk}^1 = 0$, $\delta_x \eta_{1k}^2 = \delta_x \eta_{Mk}^2 = 0$. Let take functions $\tau \bar{\phi}_{jk}^p$, $p = 1,2$ instead of functions $\bar{\eta}_{jk}^p$, $p = 1,2$ in this identity and then, subtracting its complex conjugate from obtained equality, we get the following equation:

$$h\sum_{j=1}^{M-1}\tau\left(\delta_{\bar{t}}\phi_{jk}^{p}\bar{\phi}_{jk}^{p}+\delta_{\bar{t}}\bar{\phi}_{jk}^{p}\phi_{jk}^{p}\right)=2\tau h\sum_{j=1}^{M-1}v_{1k}\left|\phi_{jk}^{p}\right|^{2}-2\tau h\sum_{j=1}^{M-1}Im(f_{jk}^{p}\,\bar{\phi}_{jk}^{p})\qquad(19)$$

$$k=\overline{1,M},p=1,2\ .$$

If we use equation (20)

$$\tau\left(\delta_{\bar{t}}\phi_{jk}^{p}\bar{\phi}_{jk}^{p}+\delta_{\bar{t}}\bar{\phi}_{jk}^{p}\phi_{jk}^{p}\right)=\left|\phi_{jk}^{p}\right|^{2}-\left|\phi_{jk-1}^{p}\right|^{2}+\left|\phi_{jk}^{p}-\phi_{jk-1}^{p}\right|^{2}\qquad(20)$$

we obtain

$$h\sum_{j=1}^{M-1}\left(\left|\phi_{jk}^{p}\right|^{2}-\left|\phi_{jk-1}^{p}\right|^{2}+\left|\phi_{jk}^{p}-\phi_{jk-1}^{p}\right|^{2}\right)\leq 2\tau h\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\bar{\phi}_{jk}^{p}\right|\qquad(21).$$

Summing this equality in k from 1 to $m\leq N$ and applying $\varepsilon-cauchy$'s inequality, we obtain

$$h\sum_{j=1}^{M-1}\left|\phi_{jm}^{p}\right|^{2}+h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\phi_{jk}^{p}-\phi_{jk-1}^{p}\right|^{2}\leq h\sum_{j=1}^{M-1}\left|\varphi_{j}^{p}\right|^{2}+\varepsilon\tau h\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}+$$

$$+\frac{1}{\varepsilon}\tau h\sum_{j=1}^{M-1}\left|\phi_{jm}\right|^{2}+2\tau h\sum_{j=1}^{m-1}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\bar{\phi}_{jk}^{p}\right|.\qquad(22).$$

The last inequality is written for $\varepsilon=2\tau$ as follows:

$$h\sum_{j=1}^{M-1}\left|\phi_{jm}^{p}\right|^{2}+2h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|\phi_{jk}^{p}-\phi_{jk-1}^{p}\right|^{2}\leq 2h\sum_{j=1}^{M-1}\left|\varphi_{j}^{p}\right|^{2}+4\tau\tau h\sum_{j=1}^{M-1}\left|f_{jm}^{p}\right|^{2}+$$

$$+4\tau h\sum_{j=1}^{m-1}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|\left|\bar{\phi}_{jk}^{p}\right|.\qquad(23).$$

Considering that second term of the left side of the inequality (23) is not negative we get the following inequality:

$$h\sum_{j=1}^{M-1}\left|\phi_{jm}^{p}\right|^{2}\leq 2h\sum_{j=1}^{M-1}\left|\varphi_{j}^{p}\right|^{2}+2\tau h\sum_{k=0}^{m-1}\sum_{j=1}^{M-1}\left|\phi_{jk}^{p}\right|^{2}+(4\tau+2)\tau h\sum_{k=1}^{m}\sum_{j=1}^{M-1}\left|f_{jm}^{p}\right|^{2},$$

$$m=1,2\ldots\ldots,N\quad,p=1,2\qquad(24).$$

Using Gronwall's lemma in study [15] the inequaliy (24), we get the estimation

$$h\sum_{j=1}^{M-1}\left|\phi_{jm}^{p}\right|^{2}\le c_{3}\left(h\sum_{j=1}^{M-1}\left|\varphi_{j}^{p}\right|^{2}+\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^{p}\right|^{2}\right),\qquad m=1,\ldots\ldots,N\ ,p=1,2.$$

where $c_3 < 0$ does not depend on τ and h. Thus, the proof is completed.

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