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Cubic H-ideals in BCK-Algebras

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Abstract: The notions of cubic H-ideals of BCK-algebras are introduced and several related properties are investigated.
Keywords: BCK-algebras; cubic sets; cubic H-ideals.

1. INTRODUCTION AND PRELIMINAREIS

The study of BCK–algebras was initiated by Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus, after, a large volume of literature has been produced on the theory of BCK-algebras. The concept of fuzzy sets were introduced by Zadeh in 1965, several researchers explored the generalization of the notion of fuzzy sets. The notion of interval-valued fuzzy sets were first introduced by Zadeh as an extension of fuzzy sets. Based on the interval valued fuzzy sets, Jun et al. [6] introduced the notion of cubic sets. Zhan and Tan [3] introduced the fuzzy H-ideals in BCK-algebras.

In this paper we introduce the concept of cubic H-ideals and investigate some of its properties.

A BCK-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the following axioms:

(1).$(x * y) * (x * z) \leq (z * y)$

(2). $x * (x * y) \leq y$,

(3). $x \leq x$,

(4). $x \leq y$ and $y \leq x$ imply $x = y$,

(5).$0 \leq x$ for all x in X.

A BCK-algebra can be partially ordered by $x \leq y \Leftrightarrow x * y = 0$ this ordering is called BCK-ordering. The following statements are true in a BCK-algebra:

$$(a)\ x * 0 = x, \quad (b)(x * y) * z \leq (x * z) * y, \ (c)\ x * y \leq x$$

$$(d)(x * y) * z \leq (x * z) * (y * z), \quad (e)\ x \leq y \Rightarrow x * z \leq y * z \text{ and } z * y \leq z * x.$$

Definition 1.1:[7] A Subset I of a BCK-algebra $(X, *, 0)$ is called an ideal of X, for any $x, y \in X$.

I_1. $0 \in I$,

I_2. $x * y$ and $y \in I \Rightarrow x \in I$.

Definition 1.2:[7] An ideal I of a BCK-algebra $(X, *, 0)$ is called closed if

I_3. $0 * x \in I$, for all $x \in I$.

Definition 1.3:[7] A non empty subset I of a BCK-algebra X is called a H-ideal of X, if it satisfies I_1 and I_4.

$x * (y * z) \in I$ *and* $y \in I \Rightarrow x * z \in I$.

Definition 1.4:[7] A fuzzy subset H in a BCK-algebra X is called fuzzy H-ideal of X, if

(FH-1). $\mu(0) \geq \mu(x)$

(FH-2). $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}\ \forall x, y, z \in X$.

The fuzzy set μ in X is called fuzzy sub algebra of X, if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for $x, y \in X$.

Definition 1.5:[7] For fuzzy sets μ and λ of X and $s, t \in [0,1]$. The sets $U(\mu; t) = \{x \in X : \mu(x) \geq t\}$ is called upper t-level cut of μ and $L(\lambda; s) = \{x \epsilon X : \lambda(x) \leq s\}$ is called lower s-level cut of λ. Let X be the collection of objects denoted generally by x. Then the fuzzy set A in X is defined as $A = \{(x, \mu_A(x)) : x \epsilon X\}$ where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$.

Now we recall the concept of interval-valued fuzzy sets:

By the interval number D we mean an interval $[a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+]$. We define

- Min $(D_1, D_2) = D_1 \cap D_2 = \min([a_1^-, b_1^+], [a_2^-, b_2^+])$
  $= [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$
- Max $(D_1, D_2) = D_1 \cup D_2 = \max([a_1^-, b_1^+], [a_2^-, b_2^+])$
  $= [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^-.a_2^-,\ b_1^+ + b_2^+ - b_1^+.b_2^+]$

And put

- $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^-$ *and* $b_1^+ \leq b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ *and* $b_1^+ = b_2^+$
- $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$ *and* $D_1 \neq D_2$
- $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, *where* $0 \leq m \leq 1$.

An interval valued fuzzy set A over X is an object having the form $A = \{(x, \tilde{\mu}_A) : x \in X\}$ where $\tilde{\mu}_A(x) : X \longrightarrow D[0,1]$ is the set of all sub-intervals of [0, 1]. The interval $\tilde{\mu}_A(x)$ denotes the intervals of the degree of membership of element x to the set A where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [5] described a method to find max/sup and min/inf between two intervals or set of intervals.

Definition 1.6:[5] Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$ then $rmin(D_1, D_2)$ $=[min(a_1^-, a_1^+), min(a_2^-, a_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus if $D_i = [a_i^-, a_i^+] \in D[0,1]$ for $i = 1,2,3 \ldots$ then we define $r\, sup_i(D_i) = [sup_i(a_i^-), sup_i(a_i^+)]$ i.e,$\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Now we call $D_1 \geq D_2$ iff $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. similarly , the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Based on (interval valued) fuzzy sets, Jun et al.[9] introduced the notion of (internal, external) cubic sets and investigated several properties.

Definition 1.7:[6] Let X be a non-empty set. A cubic set A in X is a Structure $A = \{(x, \tilde{\mu}_A(x), \lambda_A(x)) ; x \in X\}$ which is briefly denoted by $A = (\tilde{\mu}_A, \lambda_A)$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval valued fuzzy set in X and λ_A is fuzzy set in X.

2. Cubic H-ideals of BCK-algebras

Let X denotes a BCK-algebra unless otherwise specified. Combined the definitions of fuzzy H-ideal over a crisp set and the idea of cubic set we define cubic fuzzy H-ideal. After that, we give some important consequences of this representation

Definition 2.1: Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set in X, where X is a BCK-algebra, then the set A is cubic H-ideal of X if,

(CH-1). $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$

(CH-2). $\tilde{\mu}_A(x * z) \geq rmin\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$

(CH-3). $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$ for all $x, y, z \in X$.

Example 2.2 Let $X = \{0, x, y, z\}$ be a BCK-algebra with the following Cayley table

*	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	x	0	0
z	z	y	x	0

We define a cubic set $A = (X, \tilde{\mu}_A, \lambda_A)$ by $\tilde{\mu}_A(0) = \tilde{\mu}_A(x) = [0.6, 0.7]$,
$\tilde{\mu}_A(y) = \tilde{\mu}_A(z) = [0.2, 0.3]$, $\lambda_A(0) = 0.1$, $\lambda_A(x) = 0.3$, $\lambda_A(y) = \lambda_A(z) = 0.4$

By routine calculations we know that $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic H-ideal of X.

Definition 2.3: A cubic set $A = (X, \tilde{\mu}_A, \lambda_A)$ in a BCK-algebra X is called a cubic closed H-ideal if it satisfies the following

(CCH-1). $\tilde{\mu}_A(0 * x) \geq \tilde{\mu}_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$

(CCH-2). $\tilde{\mu}_A(x * z) \geq rmin\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$

(CCH-3). $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$ for all $x, y, z \in X$.

Theorem 2.4: Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic H-ideal of X, if there is a sequence $\{x_n\}$ in X such that

(i) $\lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1,1]$ then $\tilde{\mu}_A(0) = [1, 1]$ and

(ii) $\lim_{n\to\infty} \lambda_A(x_n) = 0$ then $\lambda_A(0) = 0$.

Proof: Since $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ for all $x \in X$. Therefore $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x_n)$ for every positive integer n.

Consider $[1,1] \geq \tilde{\mu}_A(0) \geq \lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1,1]$. Hence $\tilde{\mu}_A(0) = [1,1]$.

Since $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in X$. Thus $\lambda_A(0) \leq \lambda_A(x_n)$ for every positive integer n, Now $0 \leq \lambda_A(0) \leq)$ $\lim_{n\to\infty} \lambda_A(x_n) = 0$. Hence $\lambda_A(0) = 0$.

Theorem 2.5: A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in X is a cubic H-ideal of X if and only if μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X.

Proof : Let μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X and $x, y, z \in X$. Then

$\mu_A^-(0) \geq \mu_A^-(x), \mu_A^+(0) \geq \mu_A^+(x),\ \mu_A^-(x*z) \geq min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\}$,

$\mu_A^+(x*z) \geq min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\}$ and $\lambda_A(x*z) \leq max\{\lambda_A(x*(y*z)), \lambda_A(y)\}$.

Now $\tilde{\mu}_A(x*z) = [\mu_A^-(x*z), \mu_A^+(x*z)]$

$$\geq [min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\}, min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\}]$$

$$=rmin\{[\mu_A^-(x*(y*z)), \mu_A^+(x*(y*z))], [\mu_A^-(y),, \mu_A^+(y)]\}$$

$$=rmin\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}.$$

Therefore A is cubic H-ideal of X.

Conversely assume that A is a cubic H-ideal of X. For any $x, y, z \in X$,

$$[\mu_A^-(x*z), \mu_A^+(x*z)] = \tilde{\mu}_A(x*z) \geq rmin\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$$

$$= rmin\{[\mu_A^-(x*(y*z)), \mu_A^+(x*(y*z))], [\mu_A^-(y),, \mu_A^+(y)]\}$$

$$= [min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\}, min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\}]$$

Thus

$\mu_A^-(x*z) \geq min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\}$, $\mu_A^+(x*z) \geq min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\}$ and $\lambda_A(x*z) \leq max\{\lambda_A(x*(y*z)), \lambda_A(y)\}$.

Hence μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X.

Theorem2.6: If A = $(X, \tilde{\mu}_A, \lambda_A)$ is cubic H-ideal of a BCK algebra x, then we have the following:

$x \leq$ a $\Longrightarrow \tilde{\mu}_A(x) \geq \tilde{\mu}_A(a)$ and $\lambda_A(x) \leq \lambda_A(a)$ for all $x, a \in$ X.

Proof : Let $x, a \in$ X such that $x \leq$ a $\Longrightarrow x *$ a=0

Consider $\tilde{\mu}_A(x) = \tilde{\mu}_A(x*0) \geq rmin\{\tilde{\mu}_A(x*(a*0)), \tilde{\mu}_A(a)\}$

$$= rmin\{\tilde{\mu}_A(x*a), \tilde{\mu}_A(a)\}$$

$$= rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(a)\} = \tilde{\mu}_A(a)$$

and $\lambda_A(x) = \lambda_A(x*0) \leq \max\{\lambda_A(x*(a*0)), \lambda_A(a)\}$

$$= \max\{\lambda_A(x*a), \lambda_A(a)\}$$

$$= \max\{\lambda_A(0), \lambda_A(a)\}$$

$$= \lambda_A(a).$$

Therefore $\lambda_A(x) \leq \lambda_A(a)$.

Definition 2.7: Let A = $(\tilde{\mu}_A, \lambda_A)$ be a cubic set in X. For $[s_{1,} s_2\ ] \epsilon$ D (0,1) and t ϵ [0,1]

The set U $(\tilde{\mu}_A$; $[s_{1,} s_2$]) = { $x \epsilon$ X/ $\tilde{\mu}_A(x) \ \geq [\ s_{1,} s_2\ ]$} is called upper $[s_{1,} s_2\ ]$ level of A and L $(\lambda_A$, t) = { $x \epsilon$ X / $\lambda_A\ (x) \leq$ t } is called lower t-level cut of A.

Theorem 2.8: If A = $(\tilde{\mu}_A, \lambda_A)$ is a cubic H-ideal of X, then the upper [$s_{1,} s_2$] level and lower t-level of A are H-ideals of X.

Proof: Let $x \in$ U $(\tilde{\mu}_A, [\ s_{1,} s_2\ ])$

$\Longrightarrow \tilde{\mu}_A(x) \ \geq [\ s_{1,} s_2\ ]$

$\Longrightarrow [s_{1,} s_2\ ] \leq \tilde{\mu}_A(x) \leq\ \tilde{\mu}_A(0)$

$\Longrightarrow \tilde{\mu}_A(0) \geq [\ s_{1,} s_2\ ]$

$\Longrightarrow 0 \in$ U ($\tilde{\mu}_A$; [$s_{1,} s_2$])

$x *$ (y$*$z) $\in$ U ($\tilde{\mu}_A$; [$s_{1,} s_2$]) , y $\in$ U ($\tilde{\mu}_A$; [$s_{1,} s_2$])

$\tilde{\mu}_A\big(x * (y * z)\big) \geq [\ s_{1,} s_2\ ]$ and $\tilde{\mu}_A$ (y) $\geq$ [$s_{1,} s_2$]

Consider $\tilde{\mu}_A$ (x $*$z) $\geq rmin\{\tilde{\mu}_A\big(x * (y * z)\big), \tilde{\mu}_A(y)\}$

$\geq rmin \{\ [\ s_{1,} s_2\ ], [\ s_{1,} s_2\ ]\}$

$=\{\min\ [\ s_{1,} s_1\ ], min\ [\ s_{1,} s_2\ ]\}$

$= [\ s_{1,} s_2\ ]$

Therefore x $*$z $\in$ U ($\tilde{\mu}_A$; [$s_{1,} s_2$])

Hence U $(\tilde{\mu}_A$; [$s_{1,} s_2$]) is an H-ideal of X

For $x \ \epsilon$ L $(\lambda_A$; t) $\Longrightarrow\ \lambda_A\ (x) \leq$ t

$\Longrightarrow\ \lambda_A\ (0) \leq \lambda_A\ (x) \leq$ t

$\Longrightarrow 0 \in$ L $(\lambda_A; t)$

Let $x * (y$ $*$z) $\in$ L $(\lambda_A; t)$ and y $\in$ L $(\lambda_A; t)$

$\Longrightarrow\ \lambda_A\ (x * (y$ $*$z)) $\leq t$ and $\lambda_A\ (y)\ \leq t$

Since for all x, y, z $\in\ X$,

$\lambda_A\ (x$ $*$z) $\leq max$ { $\lambda_A\ (x * (y$ $*$z)}, λ_A (y) }

$\leq max$ {t ,t} = t

$\Longrightarrow\ \ \lambda_A\ (x$ $*$z) $\leq$ t.

Therefore x $*$z ϵ L (λ_A ; t) for all x, y,z $\in$ X

Hence L (λ_A ; t) is an H-ideal of x.

3. Homomorphism of cubic H-ideal

Let F be a mapping from a set X into a set Y. Let B= ($\tilde{\mu}_{B,}\ \lambda_B$) be cubic set in y. Then the inverse image of B is defined as f^{-1}(B) = { (x, (f^{-1} $(\tilde{\mu}_{B,})$, f^{-1} (λ_B)) / $x \in X$ } with the membership function and non-membership function respectively are given by f^{-1} $(\tilde{\mu}_B)(x)$ = $\tilde{\mu}_B$ (f(x)) and f^{-1} $(\lambda_B)(x)$ = λ_B(f(x)). It can be shown that f^{-1} (B) is a cubic set.

Theorem 3.1: Let $f: X \to Y$ be a homomorphism of BCK-algebras. If B = ($\tilde{\mu}_{B,}$ λ_B) is a cubic H-ideal of Y, then the pre image f^{-1} (B) = $\{(x, f^{-1}(\tilde{\mu}_B), f^{-1}(\lambda_B)) \,/\, x \in X\}$ of B under f is a cubic H-ideal of X.

Proof: Let B = ($\tilde{\mu}_{B,}$ λ_B) is a cubic H-ideal of Y. Let x, y,z ∈ X ⟹ $f(x), f(y), f(z) \in Y$.

Consider $f^{-1}(\tilde{\mu}_B)(0) = \tilde{\mu}_B(f(0)) \geq \tilde{\mu}_B(f(x)) = f^{-1}(\tilde{\mu}_B)(x)$ and

$$f^{-1}(\lambda_B)(0) = \lambda_B(f(0)) \leq \lambda_B(f(x)) = f^{-1}(\lambda_B)(x)$$

Thus $f^{-1}(\tilde{\mu}_B)(x * z) = \tilde{\mu}_B(f(x * z)) \geq rmin\{\tilde{\mu}_B(f(x * (y * z))), \tilde{\mu}_B(f(y))\}$

$$= rmin\{f^{-1}(\tilde{\mu}_B)(x * (y * z)), f^{-1}(\tilde{\mu}_B)(y)\}$$

And $f^{-1}(\lambda_B)(x * z) = \lambda_B(f(x * z)) \leq max\{\lambda_B(f(x * (y * z))), \lambda_B(f(y))\}$

$$= max\{f^{-1}(\lambda_B)(x * (y * z)), f^{-1}(\lambda_B)(y)\}$$

Therefore f^{-1} (B) = $\{(x, f^{-1}(\tilde{\mu}_{B,}), f^{-1}(\lambda_B)) \,/\, x \in X\}$ is a cubic H-ideal of Y.

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