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Distributed control for cooperative Parabolic systems with conjugation conditions

H. M. Serag [1],S. A. El-Zahaby [2] and L. M. Abd-Elrahman [2]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt.

[2] Department of Mathematics, Faculty of Science, Al-Azhar University [For Girls], Nasr City, Cairo, Egypt.

Abstract

In this paper, we consider cooperative Parabolic systems defined on bounded, continuous and strictly Lipschitz domain of R^n with conjugation conditions. We study the optimal control for these systems with Dirichlet conditions. Also, we establish the problem with Neumann conditions .The control in our problems is of distributed type.

Keywords: Distributed control; existence of solution; Cooperative Parabolic Systems; Conjugation Conditions; Dirichlet and Neumann Conditions; Laplace Operator; Case without Constraints.

1. Introduction

The optimal control problems for systems governed by finite order partial differential equations (Elliptic, Parabolic and Hyperbolic) defined on finite dimensional spaces have been studied by Lions [6,7]. The control problems described by either infinite order operators or operators with an infinite number of variables have been discussed by Gali et al in [3-5]. These results have been extended to cooperative [1,2,5,8-11,13,17]or non-cooperative[18] systems. In [14-16], Sergienko and Deineka introduced some control problems of distributed systems with conjugation conditions and quadratic cost functions.Here,we consider cooperative parabolic systems with conjugation conditions.Our paper is organized as follows: In section two,some definitions and notations which will be used later,are introduced. In section three the existence and uniqueness of the state for cooperative Dirichlet Parabolic systems with conjugation conditions is proved ,then, the set of equations and inequalities that characterizes the optimal control of systems is found. The case without constraints is also considered. The problem with Neumann under conjugation conditions is studied for cooperative Parabolic systems, in section four.

2. Definitions and Notations[14]

Let Ω_1 and Ω_2, with boundary $\partial\Omega_1, \partial\Omega_2$ respectively, be bounded, continuous and strictly Lipschitz domains from n -dimensional Euclidean space R^n such that $\Omega = (\Omega_1 \cup \Omega_2)$, $(\Omega_1 \cap \Omega_2) = \phi$ and $\overline{\Omega} = (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, $\Gamma = (\partial\Omega_1 \cup \partial\Omega_2) \setminus \gamma$ ($\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi$) be the boundary of the domain $\overline{\Omega}$, $\gamma_T = \gamma_T^+ \cup \gamma_T^-$, $\gamma_T^+ = (\partial\Omega_2 \cap \gamma) \times (0,T)$, $\gamma_T^- = (\partial\Omega_1 \cap \gamma) \times (0,T)$, $Q = \Omega_T = \Omega \times (0,T)$ be a complicated cylinder and $\Sigma = \Gamma \times (0,T)$ be the lateral surface of a cylinder $\Omega_T \cup \gamma_T$.

Let us define

$$V \times V = \{Y(x,t) = (y_1, y_2)|_{\Omega_i} \in (H^1(\Omega_i))^2, i = 1, 2 \ \forall t \in (0,T), Y|_{\Sigma} = 0\},$$

and introduce the space $L^2(0;T,V \times V)$ of functions $t \to f(t)$ that map an interval $(0,T)$ into the space $V \times V$ such that

$$\| Y(t)\|^2_{L^2(0;T,V\times V)} = \int_{(0,T)} \| Y(t)\|^2 dt < \infty.$$

Finally ,we introduce the Hilbert space:

$$W(0,T) = \{Y : Y \in L^2(0,T;V\times V), \frac{\partial Y}{\partial t} \in L^2(0,T;V\times V)\},$$

With the norm :

$$\| Y(t)\|^2_{W(0,T)} = (\int_{(0,T)} \| Y(t)\|^2 dt + \int_{(0,T)} \| \frac{dY}{dt} \|^2 dt).$$

Definition 2.0.1 *System* $\frac{\partial y_i}{\partial t} - \nabla\cdot(\beta\nabla y_i) + \sum_{j=1}^{n} h_{ij} y_j = f_i(x,t) \quad (x,t)\in\Omega_T$ *, is called cooperative system if* $h_{ij} > 0$ *for* $i \neq j$ *otherwise is called non -cooperative system [2].*

3. Cooperative Parabolic systems with Dirichlet and Conjugation Conditions

In this section, we consider the following initial boundary value problem :

$$\begin{cases} \begin{bmatrix} \frac{\partial y_1}{\partial t} \\ \frac{\partial y_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\beta\nabla) + h_{11} & h_{12} \\ h_{21} & \nabla\cdot(\beta\nabla) + h_{22} \end{bmatrix} \begin{bmatrix} y_1(x,t) \\ y_2(x,t) \end{bmatrix} + \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix} & in \quad Q, \\ \begin{bmatrix} y_1(x,0) \\ y_2(x,0) \end{bmatrix} = \begin{bmatrix} y_{1,0}(x) \\ y_{2,0}(x) \end{bmatrix}, \quad y_{1,0}(x), y_{2,0}(x) \in L^2(\Omega) & in \quad \Omega, \\ \begin{bmatrix} y_1(x,t) \\ y_2(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & on \quad \Sigma, \end{cases} \tag{3.0.1}$$

under conjugation conditions :

$$\begin{cases} \begin{bmatrix} \left[\beta \frac{\partial y_1}{\partial \nu_A}\right] \\ \left[\beta \frac{\partial y_2}{\partial \nu_A}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad on \quad \gamma_T, \end{cases} \tag{3.0.2}$$

$$\left\{\begin{bmatrix}\left\{\beta\dfrac{\partial y_1}{\partial \nu_A}\right\}^{\pm}\\ \\ \left\{\beta\dfrac{\partial y_2}{\partial \nu_A}\right\}^{\pm}\end{bmatrix}=\begin{bmatrix}r[y_1]\\ \\ r[y_2]\end{bmatrix}\quad on \quad \gamma_T,\right. \tag{3.0.3}$$

where $f_k \in C(\Omega_{kT})$, $|f_k| < \infty$, $Q = \Omega_{kT} = \Omega_k \times (0,T)$, and $\Omega_T = \bigcup_k \Omega_{kT}$, $\beta = \beta(x)$ is a positive function having discontinuity along γ,

$$0 \le r = r(x) \le r_1 < \infty, \qquad r_1 = is \quad a \quad positive \quad constant, \qquad r \in C(\gamma), \tag{3.0.4}$$

ν is an ort of a normal to γ that is called simply a normal to γ and it is directed into the domain Ω_2, $\dfrac{\partial y}{\partial \nu_A}$ is directional derivative of y. In addition,

$$[y] = y^+ - y^-,$$

$$y^+ = \{y\}^+ = y(x,t) \quad for\,(x,t) \in \gamma_T^+,$$
$$y^- = \{y\}^- = y(x,t) \quad for\,(x,t) \in \gamma_T^-$$

The model of our system is given by : $A \in \pounds\big(W(0,T), L^2(0,T;V \times V)\big)$,

$$AY(x) = A(y_1,y_2) = (\frac{\partial y_1}{\partial t} - \nabla\cdot(\beta\nabla y_1) - h_{11}y_1 - h_{12}y_2, \frac{\partial y_2}{\partial t} - \nabla\cdot(\beta\nabla y_2) - h_{21}y_1 - h_{22}y_2).$$

For a control $u = (u_1,u_2) \in (L^2(Q))^2$, the state $Y(x,t;u) = (y_1(u), y_2(u)) \in$ W(0,T) is given as a generalized solution of

$$\begin{cases} \begin{bmatrix} \dfrac{\partial y_1(u)}{\partial t} \\ \\ \dfrac{\partial y_2(u)}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\beta\nabla)+h_{11} & h_{12} \\ \\ h_{21} & \nabla\cdot(\beta\nabla)+h_{22} \end{bmatrix} \begin{bmatrix} y_1(x,t;u) \\ \\ y_2(x,t;u) \end{bmatrix} + \begin{bmatrix} f_1+u_1 \\ \\ f_2+u_2 \end{bmatrix} & in \quad Q, \\ \begin{bmatrix} y_1(x;0,u) \\ \\ y_2(x;0,u) \end{bmatrix} = \begin{bmatrix} y_{1,0}(x) \\ \\ y_{2,0}(x) \end{bmatrix}, \quad y_{1,0}(x), y_{2,0}(x) \in L^2(\Omega), & in \quad \Omega \\ \begin{bmatrix} y_1(x;t,u) \\ \\ y_2(x;t,u) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} & on \quad \Sigma, \end{cases} \tag{3.0.5}$$

and by conditions (3.0.2), (3.0.3). Specify the observation by the following expression:

$$Z(u) = (z_1(u), z_2(u)) = Y(u) = (y_1(u), y_2(u)).$$

For a given $Z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$, the cost functional is given by

$$J(u) = \|y_1(u) - z_{1d}\|^2_{L^2(Q)} + \|y_2(u) - z_{2d}\|^2_{L^2(Q)} + (\bar{a}u_1, u_1)_{L^2(Q)} + (\bar{a}u_2, u_2)_{L^2(Q)}. \tag{3.0.6}$$

Where

$$\bar{a}(x) \in C(\Omega)\,,\, 0 < a_0 \le \bar{a}(x) \le a_1 < \infty, \qquad a_0, a_1 = constant. \tag{3.0.7}$$

The control problem then is to find :

$$\begin{cases} u = (u_1, u_2) \in U_{ad}(closed \quad convex \quad subset \quad of\,(L^2(Q))^2 such \quad that: \\ \\ J(u) = inf \;\; J(v) \qquad \forall v \in U_{ad}. \end{cases} \tag{3.0.8}$$

The generalized problem corresponds to initial boundary value problem (3.0.5), (3.0.2), (3.0.3) and mean to find $Y(x,t;u) = (y_1(u), y_2(u)) \in W(0,T)$ that satisfies the following equations

$$\begin{aligned} &\int_\Omega \frac{\partial y_1}{\partial t}\varphi_1 dx + \int_\Omega \frac{\partial y_2}{\partial t}\varphi_2 dx + \int_\Omega \beta(x)\nabla y_1 \nabla\varphi_1 dx + \int_\Omega \beta(x)\nabla y_2 \nabla\varphi_2 dx + \\ &\int_\Omega (-h_{11}y_1\phi_1 - h_{12}y_2\phi_1)dx + \int_\Omega (-h_{21}y_1\phi_2 - h_{22}y_2\phi_2)dx + \\ &\int_\gamma r[y_1][\phi_1]d\gamma + \int_\gamma r[y_2][\phi_2]d\gamma \\ &= (f_1,\varphi_1) + (f_2,\varphi_2) + (u_1,\varphi_1) + (u_2,\varphi_2). \end{aligned} \tag{3.0.9}$$

and

$$\int_\Omega y_i(x,0;u)\varphi_i dx = \int_\Omega y_{i,0}(x)\varphi_i dx. \tag{3.0.10}$$

$\forall \Phi = (\phi_1, \phi_2) \in V_0 \times V_0 = \{Y = (y_1, y_2)|_{\Omega_i} \in (H^1(\Omega_i))^2, Y|_\Gamma = 0, i = 1,2\}$. Let us define on $L^2(0;T,V\times V)$,for each t a bilinear form

$$a(Y,\Phi) : (H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2 \to R$$

by

$$\begin{aligned} a(Y,\Phi) &= \int_\Omega (\beta(x)\nabla y_1 \nabla \phi_1 + \beta(x)\nabla y_2 \nabla \phi_2)\, dx \\ &0.5cm - \int_\Omega (h_{11}y_1\phi_1 + h_{12}y_2\phi_1 + h_{21}y_1\phi_2 + h_{22}y_2\phi_2)\, dx + \int_\gamma r[y_1][\phi_1]\, d\gamma + \int_\gamma r[y_2][\phi_2]\, d\gamma. \end{aligned} \tag{3.0.11}$$

This bilinear form is continuous ,since

$$|a(Y,\Phi)| \le k_1 \|Y\| \|\Phi\|. \tag{3.0.12}$$

Lemma 3.0.1 *The bilinear form* (3.0.11) *is coercive on* $(H_0^1(\Omega))^2$*; that is, there exists a positive constants* k *and* α *such that:*

$$a(Y,Y) + k\|Y\|^2_{(L^2(\Omega))^2} \ge \alpha \|Y\|^2_{(H_0^1(\Omega))^2} \qquad \forall\, Y = (y_1, y_2) \in (H_0^1(\Omega))^2 \tag{3.0.13}$$

Proof.

$$\begin{aligned} a(Y,Y) = \; & \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_1|^2 + \beta(x)|\nabla y_1|^2)dx + \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_2|^2 + \beta(x)|\nabla y_2|^2)dx - \\ & \int_\Omega y_1 y_2 dx - \frac{h_{11}}{(h_{12}+h_{21})}\int_\Omega |y_1|^2 dx - \frac{h_{22}}{(h_{12}+h_{21})}\int_\Omega |y_2|^2 dx + \\ & \int_\gamma r[y_1]^2 d\gamma + \int_\gamma r[y_2]^2 d\gamma \end{aligned}$$

From (3.0.4), we get

$$\begin{aligned} a(Y,Y) + \frac{h_{11}}{(h_{12}+h_{21})}\int_\Omega |y_1|^2 dx + \frac{h_{22}}{(h_{12}+h_{21})}\int_\Omega |y_2|^2 dx \ge \; & \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_1|^2 + \beta(x)|\nabla y_1|^2)dx + \\ & \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_2|^2 + \beta(x)|\nabla y_2|^2)dx - \int_\Omega y_1 y_2 dx. \end{aligned}$$

By Cauchy Schwartz inequality and from (Friedrichs inequality)

$$\int_\Omega |y|^2 dx \le \mu(\Omega)\int_\Omega |\nabla y|^2 dx \;\; , \quad \mu(\Omega) > 0. \tag{3.0.14}$$

We deduce

$$\begin{aligned} a(Y,Y) + max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})})[\|y_1\|^2_{L^2(\Omega)} + \|y_2\|^2_{L^2(\Omega)}] \ge & \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_1|^2 + (\beta(x))^{-1}(\mu(\Omega))^{-1}|y_1|^2)dx + \\ & \frac{1}{2(h_{12}+h_{21})}\int_\Omega (\beta(x)|\nabla y_2|^2 + (\beta(x))^{-1}(\mu(\Omega))^{-1}|y_2|^2)dx - \\ & (\int_\Omega |y_1|^2 dx)^{\frac{1}{2}} (\int_\Omega |y_2|^2 dx)^{\frac{1}{2}}. \end{aligned}$$

This inequality is equivalent to

$$a(Y,Y)+max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})},\frac{1}{2})(\|y_1\|^2_{L^2(\Omega)}+\|y_1\|^2_{L^2(\Omega)})\geq$$
$$\frac{1}{2(h_{12}+h_{21})}min(\beta(x),(\beta(x))^{-1}(\mu(\Omega))^{-1})(\|y_1\|^2_{H_0^1(\Omega)}+\|y_2\|^2_{H_0^1(\Omega)})+$$
$$(\frac{1}{\sqrt{2}}\|y_1\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|\ y_2\|_{L^2(\Omega)})^2.$$

Therefore,

$$\begin{aligned} a(Y,Y)+k\|Y\|^2_{(L^2(\Omega))^2} &\geq \alpha[\|y_1\|^2_{H_0^1(\Omega)}+\|y_2\|^2_{H_0^1(\Omega)}] \\ \geq \qquad &\alpha\|Y\|^2_{(H_0^1(\Omega))^2} \qquad \forall Y\in(H_0^1(\Omega))^2, \end{aligned}$$

where

$$k=max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})},\frac{1}{2}),\alpha=\frac{1}{2(h_{12}+h_{21})}min(\beta(x),(\beta(x))^{-1}(\mu(\Omega))^{-1}),\beta(x)\ is\ a\ positive\ constant\geq 0,$$

which proves the coerciveness condition .

Let $\Phi\to L(\Phi)$ be a linear defined on $L^2(0,T;V\times V)$ by

$$L(\Phi)=\int_\Omega f_1(x,t)\varphi_1(x)dx+\int_\Omega f_2(x,t)\varphi_2(x)dx,$$

this linear is continuous ,since :

$$|L(\Phi)|\leq c_3(\|\varphi_1\|_{(H_0^1(\Omega))}+\|\varphi_2\|_{(H_0^1(\Omega))}\leq c_3\|\Phi\|_{(H_0^1(\Omega))^2}\quad ,c_3\ is\ a\ constant. \tag{3.0.15}$$

Based on (3.0.12), (3.0.13), (3.0.15) , and Lax - Milgram lemma(se also [16]),we have

Theorem 3.0.2 *For a given* $f=(f_1,f_2)\in L^2(0,T;V\times V)$ *and* $y_{1,0}(x),y_{2,0}(x)\in L^2(\Omega)$ *there exists a unique solution* $Y=(y_1,y_2)\in W(0,T)$ *for system* (3.0.9), (3.0.10).

Now, rewrite the cost functional (3.0.6) as(see[6]):

$$J(u)=\pi(u,u)-2h(u)+\|y_1(0)-z_{1d}\|^2_{L^2(Q)}+\|y_2(0)-z_{2d}\|^2_{L^2(Q)},$$

where :

$$\begin{aligned}\pi(u,v)= \quad &(y_1(u)-y_1(0),y_1(v)-y_1(0))_{L^2(Q)}+\\ &(y_2(u)-y_2(0),y_2(v)-y_2(0))_{L^2(Q)}+\\ &(\bar{a}(x)u_1,u_1)_{L^2(Q)}+(\bar{a}(x)u_2,u_2)_{L^2(Q)}),\end{aligned} \tag{3.0.16}$$

is a continuous bilinear form and

$$\begin{aligned}h(v)= \quad &(z_{1d}-y_1(0),y_1(v)-y_1(0))_{L^2(Q)}+\\ &(z_{2d}-y_2(0),y_2(v)-y_2(0))_{L^2(Q)},\end{aligned} \tag{3.0.17}$$

is a continuous linear form .

Since

$$\pi(u,u)\geq(\bar{a}u,u)_{(L^2(Q))^2},$$

the general theory of Lions [6] gives :

Theorem 3.0.4 *If the state of our system is determined as a solution to problem* (3.0.9), (3.0.10), *and if the cost functional is given by* (3.0.6)*, there exists a unique distributed control* $u=(u_1,u_2)\in(L^2(Q))^2$ *of problem* (3.0.8)*; Moreover, it is characterized by the following equations and inequalities:*

$$\begin{cases}\begin{bmatrix}\dfrac{-\partial p_1(u)}{\partial t}\\ \\ \dfrac{-\partial p_2(u)}{\partial t}\end{bmatrix}=\begin{bmatrix}\nabla\cdot(\beta\nabla)+h_{11} & h_{21}\\ \\ h_{12} & \nabla\cdot(\beta\nabla)+h_{22}\end{bmatrix}\begin{bmatrix}p_1(u)\\ \\ p_2(u)\end{bmatrix}+\begin{bmatrix}y_1(u)-z_{1d}\\ \\ y_2(u)-z_{2d}\end{bmatrix} & in \quad Q\\ \begin{bmatrix}p_1(x;T,u)\\ \\ p_2(x;T,u)\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix} & in \quad \Omega\\ \begin{bmatrix}p_1(u)\\ \\ p_2(u)\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix} & on \quad \Sigma,\end{cases} \tag{3.0.18}$$

under conjugation conditions :

$$\begin{cases}\begin{bmatrix}\left[\beta\dfrac{\partial p_1(u)}{\partial \nu_A*}\right]\\ \\ \left[\beta\dfrac{\partial p_2(u)}{\partial \nu_A*}\right]\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix} \quad on \quad \gamma_T,\end{cases}$$

$$\begin{cases}\begin{bmatrix}\left\{\beta\dfrac{\partial p_1(u)}{\partial \nu_A*}\right\}^{\pm}\\ \\ \left\{\beta\dfrac{\partial p_2(u)}{\partial \nu_A*}\right\}^{\pm}\end{bmatrix}=\begin{bmatrix}r[p_1(u)]\\ \\ r[p_2(u)]\end{bmatrix} \quad on \quad \gamma_T,\end{cases}$$

and

$$\forall \qquad v=(v_1,v_2)\in U_{ad},$$

$$\int_Q (p_1(u)+\bar{a}_1u_1)(v_1-u_1)\,dxdt+\int_Q (p_2(u)+\bar{a}_2u_2)(v_2-u_2)\,dxdt \geq 0, \qquad (3.0.19)$$

together with (3.0.5), where $P(u)=(p_1(u),p_2(u))$ is the adjoint state .

Proof. The optimal control $u=(u_1,u_2)\in(L^2(Q))^2$ is characterized by (see[6,16]) :

$$\pi(u,v-u)\geq h(v-u) \qquad \forall\, v=(v_1,v_2)\in U_{ad}.$$

By (3.0.16), and (3.0.17):

$$\begin{aligned}\pi(u,v-u)-h(v-u) = \;& (y_1(u)-z_{1d},y_1(v)-y_1(u))_{L^2(Q)}+\\ & (y_2(u)-z_{2d},y_2(v)-y_2(u))_{L^2(Q)}+\\ & (\bar{a}_1u_1,v_1-u_1)_{L^2(Q)}+(\bar{a}_2u_2,v_2-u_2)_{L^2(Q)}\geq 0.\end{aligned} \qquad (4.0.20)$$

Now, since

$$((\frac{-\partial}{\partial t}+A^*)P(u),Y(u))=((P(u),(\frac{\partial}{\partial t}+A)Y(u))),$$

then:

$$\begin{aligned}(P(u),((\frac{\partial}{\partial t}+A)Y(u)))_{(L^2(Q))^2} = \;& (p_1(u),\frac{\partial y_1(u)}{\partial t}-\nabla\cdot(\beta\nabla y_1(u))-h_{11}y_1(u)-h_{12}y_2(u))_{L^2(Q)}+\\ & (p_2(u),\frac{\partial y_2(u)}{\partial t}-\nabla\cdot(\beta\nabla y_2(u))-h_{21}y_1(u)-h_{22}y_2(u))_{L^2(Q)}\;.\end{aligned}$$

Applying Green's formula, we obtain

$$\begin{aligned}(P(u),(\frac{\partial}{\partial t}+A)Y(u)) = \;& (\frac{-\partial p_1(u)}{\partial t}-\nabla\cdot(\beta\nabla p_1(u))-h_{11}p_1(u)-h_{21}p_2(u),y_1(u))\\ & +(\frac{-\partial p_2(u)}{\partial t}-\nabla\cdot(\beta\nabla p_2(u))-h_{12}p_1(u)-h_{22}p_2(u),y_2(u))=((\frac{-\partial}{\partial t}+A^*)P(u),Y(u)).\end{aligned}$$

Hence $A^*P(u)=A^*(p_1(u),p_2(u))$

$$=(\frac{-\partial p_1(u)}{\partial t}-\nabla\cdot(\beta\nabla p_1(u))-h_{11}p_1(u)-h_{21}p_2(u),\frac{-\partial p_2(u)}{\partial t}-\nabla\cdot(\beta\nabla p_2(u))-h_{12}p_1(u)-h_{22}p_2(u)).$$

$$=(y_1(u)-z_{1d},y_2(u)-z_{2d})in\ Q.$$

Therefore, (3.0.20) is equivalent to

$$\begin{aligned}&\int_Q(\frac{-\partial p_1(u)}{\partial t}-\nabla\cdot(\beta\nabla p_1(u))-h_{11}p_1(u)-h_{21}p_2(u),y_1(v)-y_1(u))\,dxdt+\\ &\int_Q(\frac{-\partial p_2(u)}{\partial t}-\nabla\cdot(\beta\nabla p_1(u))-h_{12}p_1(u)-h_{22}p_2(u),y_2(v)-y_2(u))\,dxdt+\\ &\int_Q\bar{a}_1u_1(v_1-u_1)\,dxdt+\int_Q\bar{a}_2u_2(v_2-u_2)\,dxdt\geq 0.\end{aligned}$$

So

$$(p_1(u),\frac{\partial}{\partial t}(y_1(v)-y_1(u)))_{L^2(Q)}+(p_2(u),\frac{\partial}{\partial t}(y_2(v)-y_2(u)))_{L^2(Q)}+$$
$$(p_1(u)-\Delta(y_1(v)-y_1(u)))_{L^2(Q)}+(p_1(u),\frac{\partial(y_1(v)-y_1(u))_{L^2(\Sigma)}}{\partial\nu_A})+$$
$$(p_2(u)-\Delta(y_2(v)-y_2(u)))_{L^2(Q)}+(p_2(u),\frac{\partial(y_2(v)-y_2(u))_{L^2(\Sigma)}}{\partial\nu_A})+$$
$$(p_1(u),-h_{11}(y_1(v)-y_1(u)))_{L^2(Q)}+(p_2(u),-h_{21}(y_1(v)-y_1(u)))_{L^2(Q)}+$$
$$(p_1(u),-h_{12}(y_2(v)-y_2(u)))_{L^2(Q)}+(p_2(u),-h_{22}(y_2(v)-y_2(u)))_{L^2(Q)}+$$
$$(\bar{a}_1u_1,v_1-u_1)_{L^2(Q)}+(\bar{a}_2u_2,v_2-u_2)_{L^2(Q)}\geq 0.$$

Using equation (3.0.5), we obtain

$$\int_Q(p_1(u)+\bar{a}_1u_1)(v_1-u_1)dxdt+\int_Q(p_2(u)+\bar{a}_2u_2)(v_2-u_2)dxdt\geq 0. \tag{3.0.21}$$

Remark 3.0.5 *: If the constraints are absent ,i.e. When* $U_{ad}=U$ *, then the equality*

$$p_1(u)+\bar{a}_1u_1=0 \ \ and \ \ p_2(u)+\bar{a}_2u_2=0$$

is obtained from inequality (3.0.21) .

So the control

$$u_1=-\frac{p_1}{\bar{a}_1},u_2=-\frac{p_2}{\bar{a}_2},(x,t)\in Q \tag{3.0.22}$$

is found from the latter equality. On the basis of equalities (3.0.5) and (3.0.18) the problem is obtained: Find a vector -function

$$(Y,P)^T\in(H)^2=\{\Phi=(\Phi_1,\Phi_2)=((y_1,p_1)^T,(y_2,p_2)^T):\Phi_i(x,t)|_{\Omega_i}\in(H^1(\Omega_i))^2, i=1,2\ \forall t\in(0,T),\Phi|_\Sigma=0\},$$

that satisfies the equality systems

$$a(Y,\Phi)=L(-\frac{P}{\bar{a}},\Phi),\ a(P,\Phi)=h(Y,\Phi)\ \ \forall\ \ \Phi\in V_0\times V_0, \tag{3.0.23}$$

and the vector solution $(Y,P)^T$ is found from this system along with the optimal control

$$u_1=-\frac{p_1}{\bar{a}_1},u_2=-\frac{p_2}{\bar{a}_2},(x,t)\in Q.$$

If the vector solution $(Y,P)^T$ to problem (3.0.23) is smooth enough on $\overline{Q}$,then the differential problem of finding the vector - function $(Y,P)^T$, that satisfies the relations

$$\begin{bmatrix}\frac{\partial y_1}{\partial t}\\ \frac{\partial y_2}{\partial t}\end{bmatrix}+\begin{bmatrix}-\nabla\cdot(\beta\nabla)-h_{11} & -h_{12}\\ -h_{21} & -\nabla\cdot(\beta\nabla)-h_{22}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}\frac{p_1}{\bar{a}_1}\\ \frac{p_2}{\bar{a}_2}\end{bmatrix}=\begin{bmatrix}f_1\\ f_2\end{bmatrix} inQ, \tag{3.0.24}$$

$$\begin{bmatrix} \dfrac{-\partial p_1}{\partial t} \\ \\ \dfrac{-\partial p_2}{\partial t} \end{bmatrix} + \begin{bmatrix} -\nabla\cdot(\beta\nabla) - h_{11} & -h_{21} \\ \\ -h_{12} & -\nabla\cdot(\beta\nabla) - h_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ \\ p_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} -z_{1d} \\ \\ -z_{2d} \end{bmatrix} in Q, \tag{3.0.25}$$

$$\begin{cases} \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \quad on \quad \Sigma \\ \\ \begin{bmatrix} p_1 \\ \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \quad on \quad \Sigma, \end{cases} \tag{3.0.26}$$

and the conjugation conditions :

$$\begin{cases} \begin{bmatrix} \left[\beta \dfrac{\partial y_1}{\partial \nu_A}\right] \\ \\ \left[\beta \dfrac{\partial y_2}{\partial \nu_A}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \quad on \quad \gamma_T, \\ \\ \begin{bmatrix} \left\{\beta \dfrac{\partial y_1}{\partial \nu_A}\right\}^{\pm} \\ \\ \left\{\beta \dfrac{\partial y_2}{\partial \nu_A}\right\}^{\pm} \end{bmatrix} = \begin{bmatrix} r[y_1] \\ \\ r[y_2] \end{bmatrix} \quad on \quad \gamma_T, \end{cases} \tag{3.0.27}$$

$$\begin{cases} \begin{bmatrix} \left[\beta \dfrac{\partial p_1}{\partial \nu_A *}\right] \\ \\ \left[\beta \dfrac{\partial p_2}{\partial \nu_A *}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \quad on \quad \gamma_T, \\ \\ \begin{bmatrix} \left\{\beta \dfrac{\partial p_1}{\partial \nu_A *}\right\}^{\pm} \\ \\ \left\{\beta \dfrac{\partial p_2}{\partial \nu_A *}\right\}^{\pm} \end{bmatrix} = \begin{bmatrix} r[p_1] \\ \\ r[p_2] \end{bmatrix} \quad on \quad \gamma_T, \end{cases} \tag{3.0.28}$$

$$\begin{cases} \begin{bmatrix} y_1(x,0) \\ \\ y_2(x,0) \end{bmatrix} = \begin{bmatrix} y_{1,0}(x) \\ \\ y_{2,0}(x) \end{bmatrix} & in \quad \Omega \\ \begin{bmatrix} p_1(x,T) \\ \\ p_2(x,T) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} & in \quad \Omega, \end{cases} \tag{3.0.29}$$

corresponds to problem (3.0.23).

Definition 3.0.6 A generalized (weak)solution to boundary -value problem (3.0.24)-(3.0.29) is called a vector -function $(Y,P)^T \in (H)^2$ that satisfies the equation

$$(\frac{\partial \Phi}{\partial t},\Psi)+a(t;\Phi,\Psi)=L(\Psi)\ \forall\ \Phi\in(H)^2 \quad , \tag{3.0.30}$$

where $L(\Psi)$ defined on $(H)^2$ by :

$$L(\Psi)=\int_\Omega((f_1-z_{1d})\psi_1)dx+\int_\Omega((f_2-z_{2d})\psi_2)dx, \tag{3.0.31}$$

and a bilinear form

$$a(t;\Phi,\Psi):(H)^2\times(H)^2\to R$$

defined by

$$\begin{aligned} a(\Phi,\Psi)= \quad & \int_\Omega\{\beta(x)\nabla y_1\nabla\psi_1-h_{11}y_1\psi_1-h_{12}y_2\psi_1+\frac{p_1}{\bar{a}_1}\psi_1\}dx+ \\ & \int_\Omega\{\beta(x)\nabla y_2\nabla\psi_2-h_{21}y_1\psi_2-h_{22}y_2\psi_2+\frac{p_2}{\bar{a}_2}\psi_2\}dx+ \\ & \int_\Omega\{\beta(x)\nabla p_1\nabla\psi_1-h_{11}p_1\psi_1-h_{21}p_2\psi_1-y_1\psi_1\}dx+ \\ & \int_\Omega\{\beta(x)\nabla p_2\nabla\psi_2-h_{12}p_1\psi_2-h_{22}p_2\psi_2-y_2\psi_2\}dx+ \\ & \int_\gamma r[y_1][\psi_1]d\gamma+\int_\gamma r[y_2][\psi_2]d\gamma+ \\ & \int_\gamma r[p_1][\psi_1]d\gamma+\int_\gamma r[p_2][\psi_2]d\gamma \quad . \end{aligned} \tag{3.0.32}$$

It is easy to check that

$$|\,a(t;\Phi,\Psi)\,|\le k_2\|\,\Phi\|_{(H)^2}\ \|\,\Psi\|_{(H)^2}, \tag{3.0.33}$$

is a continuous bilinear form .and

$$|\,L(\Psi)\,|\le k_3\|\,\Psi\|_{(H)^2}, \tag{3.0.34}$$

is a continuous linear form .

Lemma 3.0.7 *The bilinear form* (3.0.32) *is coercive on* $(H)^2$ *; that is, there exists a positive constants* k *,* α *such that:*

$$a(\Phi,\Phi)+k(\|\,Y\|^2_{(L^2(\Omega))^2}+\|\,P\|^2_{(L^2(\Omega))^2})\ge\alpha(\|Y\|^2_{(H^1_0(\Omega))^2}+\|P\|^2_{(H^1_0(\Omega))^2}) \qquad \forall Y,P\in(H^1_0(\Omega))^2. \tag{3.0.35}$$

Proof.

$$a(\Phi,\Phi)= \frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_1|^2+\beta(x)|\nabla y_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_2|^2+\beta(x)|\nabla y_2|^2)dx+$$
$$\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_1|^2+\beta(x)|\nabla p_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_2|^2+\beta(x)|\nabla p_2|^2)dx-$$
$$\int_\Omega y_1y_2dx-\int_\Omega p_1p_2dx-\frac{h_{11}}{(h_{12}+h_{21})}\int_\Omega(|y_1|^2+|p_1|^2)dx-\frac{h_{22}}{(h_{12}+h_{21})}\int_\Omega(|y_2|^2+|p_2|^2)dx+$$
$$(\frac{1}{\bar{a}_1}-1)(\frac{1}{h_{12}+h_{21}})\int_\Omega p_1y_1dx+(\frac{1}{\bar{a}_2}-1)(\frac{1}{h_{12}+h_{21}})\int_\Omega p_2y_2dx+$$
$$\int_\gamma r[y_1]^2d\gamma+\int_\gamma r[y_2]^2d\gamma\int_\gamma r[p_1]^2d\gamma+\int_\gamma r[p_2]^2d\gamma\,.$$

From (3.0.4), we get

$$a(\Phi,\Phi)+\frac{h_{11}}{(h_{12}+h_{21})}\int_\Omega(|y_1|^2+|p_1|^2)dx+\frac{h_{22}}{(h_{12}+h_{21})}\int_\Omega(|y_2|^2+|p_2|^2)dx\geq$$
$$\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_1|^2+\beta(x)|\nabla y_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_2|^2+\beta(x)|\nabla y_2|^2)dx+$$
$$\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_1|^2+\beta(x)|\nabla p_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_2|^2+\beta(x)|\nabla p_2|^2)dx-$$
$$\int_\Omega y_1y_2dx-\int_\Omega p_1p_2dx+(\frac{1}{\bar{a}_1}-1)(\frac{1}{h_{12}+h_{21}})\int_\Omega p_1y_1dx+$$
$$(\frac{1}{\bar{a}_2}-1)(\frac{1}{h_{12}+h_{21}})\int_\Omega p_2y_2dx\,.$$

By Cauchy Schwartz inequality and from (3.0.14), we deduce

$$a(\Phi,\Phi)+max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})})\{[\|y_1\|^2L^2(\Omega)+\|p_1\|^2_{L^2}(\Omega)]+[\|y_2\|^2_{L^2(\Omega)}+\|p_2\|^2_{L^2(\Omega)}]\}\geq$$
$$\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_1|^2+(\beta(x))^{-1}(\mu(\Omega))^{-1}|y_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla y_2|^2+(\beta(x))^{-1}(\mu(\Omega))^{-1}|y_2|^2)dx-$$
$$\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_1|^2+(\beta(x))^{-1}(\mu(\Omega))^{-1}|p_1|^2)dx+\frac{1}{2(h_{12}+h_{21})}\int_\Omega(\beta(x)|\nabla p_2|^2+(\beta(x))^{-1}(\mu(\Omega))^{-1}|p_2|^2)dx-$$
$$(\int_\Omega|y_1|^2dx)^{\frac{1}{2}}(\int_\Omega|y_2|^2dx)^{\frac{1}{2}}-(\int_\Omega|p_1|^2dx)^{\frac{1}{2}}(\int_\Omega|p_2|^2dx)^{\frac{1}{2}}+$$
$$(\frac{1}{\bar{a}_1}-1)(\frac{1}{h_{12}+h_{21}})(\int_\Omega|y_1|^2dx)^{\frac{1}{2}}(\int_\Omega|p_1|^2dx)^{\frac{1}{2}}+$$
$$(\frac{1}{\bar{a}_2}-1)(\frac{1}{h_{12}+h_{21}})(\int_\Omega|y_1|^2dx)^{\frac{1}{2}}(\int_\Omega|p_1|^2dx)^{\frac{1}{2}}.$$

Since

$$\bar{a}(x)\in C(\Omega),$$
$$0<a_0\leq\bar{a}(x)\leq a_1<\infty,\qquad a_0,a_1=constant\ \ ,$$

then we have

$$a(\Phi,\Phi)+max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})},\frac{1}{2})\{[\|y_1\|^2_{L^2(\Omega)}+\|p_1\|^2_{L^2(\Omega)}]+[\|y_2\|^2_{L^2(\Omega)}+\|p_2\|^2_{L^2(\Omega)}]\}\geq$$
$$\frac{1}{2(h_{12}+h_{21})}min(\beta(x),(\beta(x))^{-1}(\mu(\Omega))^{-1})\{(\|y_1\|^2_{H^1_0(\Omega)}+\|y_2\|^2_{H^1_0(\Omega)})+(\|p_1\|^2_{H^1_0(\Omega)}+\|p_2\|^2_{H^1_0(\Omega)})\}+$$
$$(\frac{1}{\sqrt{2}}\|y_1\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|y_2\|_{L^2(\Omega)})^2+(\frac{1}{\sqrt{2}}\|p_1\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|p_2\|_{L^2(\Omega)})^2.$$

Therefore

$$a(\Phi,\Phi)+k(\|Y\|^2_{(L^2(\Omega))^2}+\|P\|^2_{(L^2(\Omega))^2})\geq\alpha(\|Y\|^2_{(H^1_0(\Omega))^2}+\|P\|^2_{(H^1_0(\Omega))^2})\qquad \forall Y,P\in(H^1_0(\Omega))^2,$$

this inequality is equivalent to

$$a(\Phi,\Phi)+k\|\Phi\|^2_{(H)^2}\geq\ \alpha\|\Phi\|^2_{(H)^2}\qquad \forall\Phi=((y_1,p_1)^T,(y_2,p_2)^T),$$

where

$$k=max(\frac{(h_{11},h_{22})}{(h_{12}+h_{21})},\frac{1}{2}),\ \alpha=\frac{1}{2(h_{12}+h_{21})}min(\beta(x),(\beta(x))^{-1}(\mu(\Omega))^{-1}),$$

which proves the coerciveness condition .

Since $\Psi=(\Psi_1,\Psi_2)^T$ be arbitrary elements of the Hilbert space $(H)^2$ with the norm

$$\|\Phi\|^2_{(H)^2}=\|\Phi_1\|^2_{(H^1_0(\Omega))^2}+\|\Phi_2\|^2_{(H^1_0(\Omega))^2}\quad .$$

Based on (3.0.33) - (3.0.35) and Lax-Milgram Lemma , there exists a unique vector solution $(Y,P)^T\in(H)^2$ to the boundary value problem (3.0.30).

4. Cooperative Neumann parabolic systems under conjugation conditions

In this section ,we discuss the following 2×2 cooperative Parabolic systems with non - homogenous Neumann conditions:

$$\begin{cases} \begin{bmatrix} \dfrac{\partial y_1}{\partial t} \\ \\ \dfrac{\partial y_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\beta\nabla)+h_{11} & h_{12} \\ \\ h_{21} & \nabla\cdot(\beta\nabla)+h_{22} \end{bmatrix} \begin{bmatrix} y_1(x,t) \\ \\ y_2(x,t) \end{bmatrix} + \begin{bmatrix} f_1(x,t) \\ \\ f_2(x,t) \end{bmatrix} & in \quad Q, \\ \begin{bmatrix} y_1(x,0) \\ \\ y_2(x,0) \end{bmatrix} = \begin{bmatrix} y_{1,0}(x) \\ \\ y_{2,0}(x) \end{bmatrix}, \quad y_{1,0}(x), y_{2,0}(x) \in L^2(\Omega) & in \quad \Omega, \\ \begin{bmatrix} \beta\dfrac{\partial y_1}{\partial \nu_A} \\ \\ \beta\dfrac{\partial y_2}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} g_1 \\ \\ g_2 \end{bmatrix} & on \quad \Sigma, \end{cases} \tag{4.0.36}$$

with conjugation conditions (3.0.2) ,(3.0.3). Where $(g_1, g_2) \in (L^2(\Sigma))^2$ are given functions. Let us define

$$V_c \times V_c = \{Y(x,t) = (y_1, y_2)|_{\Omega_i} \in (H^1(\Omega_i))^2, i = 1, 2\ \forall t \in (0,T)\}.$$

For a control $u = (u_1, u_2) \in (L^2(Q))^2$, the state $Y(x,t;u) = (y_1(u), y_2(u)) \in$ W(0,T) is given as a generalized solution of

$$\begin{cases} \begin{bmatrix} \dfrac{\partial y_1(u)}{\partial t} \\ \\ \dfrac{\partial y_2(u)}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\beta\nabla)+h_{11} & h_{12} \\ \\ h_{21} & \nabla\cdot(\beta\nabla)+h_{22} \end{bmatrix} \begin{bmatrix} y_1(x,t;u) \\ \\ y_2(x,t;u) \end{bmatrix} + \begin{bmatrix} f_1 + u_1 \\ \\ f_2 + u_2 \end{bmatrix} & in \quad Q, \\ \begin{bmatrix} y_1(x;0,u) \\ \\ y_2(x;0,u) \end{bmatrix} = \begin{bmatrix} y_{1,0}(x) \\ \\ y_{2,0}(x) \end{bmatrix}, \quad y_{1,0}(x), y_{2,0}(x) \in L^2(\Omega) & in \quad \Omega, \\ \begin{bmatrix} \beta\dfrac{\partial y_1(u)}{\partial \nu_A} \\ \\ \beta\dfrac{\partial y_2(u)}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} g_1 \\ \\ g_2 \end{bmatrix} & on \quad \Sigma \end{cases} \tag{4.0.37}$$

and by conjugation conditions (3.0.2), (3.0.3). For a given $Z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$, the cost functional is given a gain by (3.0.6). The control problem then is to find :

$$\begin{cases} u = (u_1, u_2) \in U_{ad} (closed \quad convex \quad subset \quad of\, (L^2(Q))^2 \, such \quad that: \\ \\ J(u) = inf \;\; J(v) \qquad \forall v \in U_{ad}. \end{cases} \tag{4.0.38}$$

The generalized problem corresponds to initial boundary value problem (4.0.37), (3.0.2), (3.0.3) and mean to find $Y(x,t;u) = (y_1(u), y_2(u)) \in$ W(0,T) that satisfies the following equations

$$\begin{aligned} &\int_\Omega \frac{\partial y_1}{\partial t}\varphi_1 dx + \int_\Omega \frac{\partial y_2}{\partial t}\varphi_2 dx + \int_\Omega \beta(x)\nabla y_1 \nabla \varphi_1 dx + \int_\Omega \beta(x)\nabla y_2 \nabla \varphi_2 dx + \\ &\int_\Omega (-h_{11} y_1 \phi_1 - h_{12} y_2 \phi_1) dx + \int_\Omega (-h_{21} y_1 \phi_2 - h_{22} y_2 \phi_2) dx + \\ &\int_\gamma r[y_1][\phi_1] d\gamma + \int_\gamma r[y_2][\phi_2] d\gamma \\ &= (f_1, \varphi_1) + (f_2, \varphi_2) + \int_\Gamma g_1(x)\varphi_1 d\Gamma + \int_\Gamma g_2(x)\varphi_2 d\Gamma + (u_1, \varphi_1) + (u_2, \varphi_2). \end{aligned} \tag{4.0.39}$$

and

$$\int_\Omega y_i(x,0;u)\varphi_i dx = \int_\Omega y_{i,0}(x)\varphi_i dx. \tag{4.0.40}$$

$\forall \Phi = (\phi_1, \phi_2) \in V_d \times V_d = \{Y = (y_1, y_2)|_{\Omega_i} \in (H^1(\Omega_i))^2, i = 1,2\}$. Since

$$(H_0^1(\Omega))^2 \subseteq (H^1(\Omega))^2.$$

We introduce again the bilinear form (3.0.11) which is coercive on $(H^1(\Omega))^2$, that is, there exists a positive constants k and α such that:

$$a(Y,Y) + k\|Y\|^2_{(L^2(\Omega))^2} \geq \alpha \|Y\|^2_{(H^1(\Omega))^2} \qquad \forall\, Y = (y_1, y_2) \in (H^1(\Omega))^2. \tag{4.0.41}$$

This bilinear form is continuous ,since

$$|a(Y,\Phi)| \leq k_1 \|Y\| \|\Phi\|. \tag{4.0.42}$$

Let $\Phi \to L_g(\Phi)$ be a linear form defined on $L^2(0,T;V_c \times V_c)$ by

$$L_g(\Phi) \;\; = \int_\Omega (f_1(x,t)\varphi_1(x) + f_2(x,t)\varphi_2(x)) dx + \int_\Gamma (g_1(x,t)\varphi_1(x) + g_2(x,t)\varphi_2(x)) d\Gamma$$

this linear form is continuous since :

$$\begin{aligned} |L_g(\Phi)| \leq &\|f_1\|_{L^2(\Omega)} \|\varphi_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} \|\varphi_2\|_{L^2(\Omega)} + \\ &\|g_1\|_{L^2(\Gamma)} \|\varphi_1\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)} \|\varphi_2\|_{L^2(\Gamma)}. \end{aligned}$$

The inequalities

$$\|\varphi\|_{L^2(\Omega)} \leq c_1 \|\varphi\|_{H^1(\Omega)} \quad ,$$

and

$$\|\varphi\|_{L^2(\Gamma)} \leq c_2 \|\varphi\|_{H^1(\Omega)},$$

imply

$$
\begin{aligned}
|L_g(\Phi)| \le & c_1\|f_1\|_{L^2(\Omega)}\|\varphi_1\|_{H^1(\Omega)} + c_1\|f_2\|_{L^2(\Omega)}\|\varphi_2\|_{H^1(\Omega)} + \\
& c_2\|g_1\|_{L^2(\Gamma)}\|\varphi_1\|_{H^1(\Omega)} + c_2\|g_2\|_{L^2(\Gamma)}\|\varphi_2\|_{H^1(\Omega)} \\
& \le [c_1\|f_1\|_{L^2(\Omega)} + c_2\|g_1\|_{L^2(\Gamma)}]\|\varphi_1\|_{H^1(\Omega)} + \\
& [c_1\|f_2\|_{L^2(\Omega)} + c_2\|g_2\|_{L^2(\Gamma)}]\|\varphi_2\|_{H^1(\Omega)}.
\end{aligned}
$$

Hence

$$
|L_g(\Phi)| \le c_3\|\Phi\|_{(H^1(\Omega))^2} \quad ,c_3 \text{ is a constant}, \tag{4.0.43}
$$

Based on (4.0.41)- (4.0.43) and Lax - Milgram lemma ,we have

Theorem 4.0.8 *For a given* $f=(f_1,f_2)\in L^2(0,T;V\times V)$ *and* $y_{1,0}(x),y_{2,0}(x)\in L^2(\Omega)$ *there exists a unique solution* $Y=(y_1,y_2)\in W(0,T)$ *for system* (4.0.39) , (4.0.40).

Now, rewrite The cost functional :

$$
J(u)=\pi(u,u)-2L(u)+\|y_1(0)-z_{1d}\|^2_{L^2(Q)}+\|y_2(0)-z_{2d}\|^2_{L^2(Q)}.
$$

Then the general theory of Lions [6]gives :

Theorem 4.0.9 *If the state of our system is determined as a solution to problem* (4.0.39), (4.0.40), *and if the cost functional is given by* (4.0.6), *there exists a unique distributed control* $u=(u_1,u_2)\in(L^2(Q))^2$ *of problem* (4.0.38) *;Moreover, it is characterized by the following equations and inequalities:*

$$
\begin{cases}
\begin{bmatrix} \dfrac{-\partial p_1(u)}{\partial t} \\ \\ \dfrac{-\partial p_2(u)}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\beta\nabla)+h_{11} & h_{21} \\ \\ h_{12} & \nabla\cdot(\beta\nabla)+h_{22} \end{bmatrix}\begin{bmatrix} p_1(u) \\ \\ p_2(u) \end{bmatrix} + \begin{bmatrix} y_1(u)-z_{1d} \\ \\ y_2(u)-z_{2d} \end{bmatrix} & in \quad Q, \\
\begin{bmatrix} p_1(x;T,u) \\ \\ p_2(x;T,u) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} & in \quad \Omega, \\
\begin{bmatrix} \beta\dfrac{\partial p_1(u)}{\partial \nu_A{}^*} \\ \\ \beta\dfrac{\partial p_2(u)}{\partial \nu_A{}^*} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} & on \quad \Sigma,,
\end{cases} \tag{4.0.44}
$$

under conjugation conditions :

$$\left\{\begin{bmatrix}\left[\beta\dfrac{\partial p_1(u)}{\partial \nu_A *}\right]\\ \\ \left[\beta\dfrac{\partial p_2(u)}{\partial \nu_A *}\right]\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix}\quad on\quad \gamma_T,\right.$$

$$\left\{\begin{bmatrix}\left\{\beta\dfrac{\partial p_1(u)}{\partial \nu_A *}\right\}^{\pm}\\ \\ \left\{\beta\dfrac{\partial p_2(u)}{\partial \nu_A *}\right\}^{\pm}\end{bmatrix}=\begin{bmatrix}r[p_1(u)]\\ \\ r[p_2(u)]\end{bmatrix}\quad on\quad \gamma_T,\right.$$

and

$$\forall \qquad v=(v_1,v_2)\in U_{ad},$$

$$\int_Q (p_1(u)+\bar{a}_1u_1)(v_1-u_1)\,dxdt+\int_Q (p_2(u)+\bar{a}_2u_2)(v_2-u_2)\,dxdt\geq 0, \tag{4.0.45}$$

together with (4.0.37), where

$$P(u)=(p_1(u),p_2(u))\text{is the adjoint state.}$$

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