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There is Always a prime between n^2 and $(n+1)^2$

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ABSTRACT

I give an answer in the affirmative to the following unanswered question:

Is there always a prime between n^2 and $(n+1)^2$ Where n is natural number?

The question represents a famous unsolved problem in Mathematics.

I employ some familiar ideas in Number Theory.

Mathematics Subject Classification: Number Theory, General Mathematics.

PROOF

Seeking for a contradiction, I assume that there exists a natural number n, such that there exist no prime numbers between n^2 and $(n+1)^2$

I apply the well-known theorem:

There is a constant A such that:

$$(1)\quad \sum_{p\leq x} \frac{1}{p} = \log\log x + A + O\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2$$

O is the big oh notaion, The summation is taken over all primes

P less than or equal to x. The "log" is the natural logarithm.

According to formula (1)

$$(2)\quad \sum_{p\leq n^2} \frac{1}{p} = \log\log n^2 + A + O\left(\frac{1}{\log n^2}\right)$$

$$(3)\quad \sum_{p\leq (n+1)^2} \frac{1}{p} = \log\log\left((n+1)^2\right) + A + O\left(\frac{1}{\log (n+1)^2}\right)$$

Since I have assumed that there exist no prime numbers between

n^2 and $(n+1)^2$, we deduce that the left – hand sides of (2) and (3) are equal, hence.

(4) $\log\log n^2 + A + O\left(\frac{1}{\log n^2}\right) = \log\log\left((n+1)^2\right) + A + O\left(\frac{1}{\log(n+1)^2}\right)$

Hence we get

(5) $\log\log\left((n+1)^2\right) - \log\log n^2 = O\left(\frac{1}{\log n^2}\right) - O\left(\frac{1}{\log(n+1)^2}\right)$

We know that

(6) $\frac{\pi(X)}{x} = O\left(\frac{1}{\log x}\right)$ for all $x \geq 2$

$\pi(x)$ is the number of primes less than or equal to x

Now I assume that (n+1) is prime.

Thus

(7) $\pi(n+1) = \pi(n) + 1$

We have that

(8) $\frac{\pi(n^2)}{n^2} = O\left(\frac{1}{\log n^2}\right) = O\left(\frac{1}{2\log n}\right) = O\left(\frac{1}{\log n}\right)$

(9) $\frac{\pi((n+1)^2)}{(n+1)^2} = O\left(\frac{1}{\log(n+1)n^2}\right) = O\left(\frac{1}{2\log(n+1)}\right) = O\left(\frac{1}{\log(n+1)}\right)$

Hence equation (5) becomes

(10) $\log\log((n+1)^2) - \log\log n^2 = O\left(\frac{1}{\log n}\right) - O\left(\frac{1}{\log(n+1)}\right)$

We know that, since log is the natural logarithm, and since the function $\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right)$ is a decreasing function,

(11) $\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right) < \frac{1}{2(n+1)}$ for all $n > 10$

we know also that

(12) $O\left(\frac{1}{\log n}\right) = \frac{\pi(n)}{n}$

(13) $O\left(\frac{1}{\log(n+1)}\right) = \frac{\pi(n+1)}{(n+1)}$

Now using (11), (12), and (13), we can rewrite equation (10) in the form:

(14) $\log\log((n+1)^2) - \log\log n^2 = \frac{\pi(n)}{(n)} - \frac{\pi(n+1)}{(n+1)} + t$ (for n > 10)

where t is small number ($t < \frac{1}{2(n+1)}$)

if we substitute from equation (7) we get:

(15) $\log\log((n+1)^2 - \log\log n^2 = = \frac{\pi(n)}{(n)} - \frac{\pi(n)+1}{(n+1)} + t = = \frac{\pi(n)}{n(n+1)} - \frac{1}{(n+1)} + t$

that is

(16) $\log\log((n+1)^2) - \log\log n^2 = \frac{1}{(n+1)} \left(\frac{\pi(n)}{(n)} - 1 + t(n+1)\right)$

since

(17) $\frac{\pi(n)}{(n)} < 0.5$ for n > 10

and

(18) $t(n+1) < \frac{1}{2(n+1)} \times (n+1) = 0.5$

We conclude that the right – hand side of equation (16) is negative, and its left – hand side is positive. Hence we arrive at a contradiction.

Now I assume that (n+1) is a composite number. Let r be the greatest prime less than or equal to n.

According to a theorem in Number Theory, if k is a natural number, then there is a prime between k and 2k.

Hence if n is even we have:

(19) $\frac{n}{2} \leq r < n$

and if n is odd

(20) $\frac{n-1}{2} \leq r \leq n$

In both cases we have

(21) $O\left(\frac{1}{\log r}\right) = O\left(\frac{1}{\log(n+1)}\right)$

(22) $O\left(\frac{1}{\log(r-1)}\right) = O\left(\frac{1}{\log n}\right)$

substituting in equation (10) we get

(23) $\log\log((n+1)^2) - \log\log n^2 = O\left(\frac{1}{\log(r-1)}\right) - O\left(\frac{1}{\log r}\right)$

we know that:

(24) $\frac{\pi(r)}{r} = O\left(\frac{1}{\log r}\right)$

(25) $\frac{\pi(r-1)}{(r-1)} = O\left(\frac{1}{\log(r-1\_}\right)$

we know that, since log is the natural logarithm, and since the function:

$\left(\frac{1}{\log(r-1)}\right) - \left(\frac{1}{\log r}\right)$ is a decreasing function,

(26) $\left(\frac{1}{\log(r-1)}\right) - \left(\frac{1}{\log r}\right) < \frac{1}{2r}$ for all $r > 10$

Now using (24), (25), and (26), we can rewrite equation (23) in the form:

(27) $\log\log((n+1)^2) - \log\log n^2 = \frac{\pi(r-1)}{(r-1)} - \frac{\pi(r)}{(r)} + t$ (for $r > 10$)

where t is small number ($t < \frac{1}{2r}$)

But we know that

(28) $\pi(r) = \pi(r-1) + 1$

hence equation (27) takes the form:

(29) $\log\log((n+1)^2) - \log\log n^2 = \frac{\pi(r-1)}{(r-1)} - \frac{\pi(r-1)+1}{r} + t$

$= \frac{\pi(r-1)}{r(r-1)} - \frac{1}{r} + t$

$= \frac{1}{r}\ (\frac{\pi(r-1)}{(r-1)} - 1 + tr)$

that is

(30) $\log\log((n+1)^2 - \log\log n^2 = = \frac{1}{r} + (\frac{\pi(r-1)}{(r-1)} - 1 + tr)$

we know that

(31) $(\frac{\pi\,(r-1)}{(r-1)} - < 0.5$ for $r > 11$

and

(32) $t < \frac{1}{2r}$

that is

(33) $tr < 0.5$

hence we conclude that the right – hand side of equation (30) is negative, and its left – hand side is positive. This is a contradiction.

This ends my proof.

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