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On Characterizations of n–Inner Product Spaces

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Abstract

In this paper, some necessary and sufficient conditions for an $n-$ normed spaces to be an $n-$inner product spaces are given.

Keywords: $n-$inner product spaces; characterizations.

1. Introduction

Chen and Song [1], showed that a necessary and sufficient conditions for an $n-$normed spaces to be an $n-$inner product spaces is the following extended parallelogram law:

$$\|x+y,a_2,\dots,a_n\|^2+\|x-y,a_2,\dots,a_n\|^2=2(\|x,a_2,\dots,a_n\|^2+\|y,a_2,\dots,a_n\|^2) \qquad (1.1)$$

holds, in such case the $n-$inner product spaces is given by

$$\langle x,y|a_2,\dots,a_n\rangle=\frac{1}{4}(\|x+y,a_2,\dots,a_n\|-\|x-y,a_2,\dots,a_n\|). \qquad (1.2)$$

for all $x,y,a_2,\dots,a_n\in X$.

Recently, Soenjaya [3] called that above law a characterization of $n-$inner product spaces.

For this work we need the following definitions:

Definition 2.1.[2] Let X be a real vector spaces of $\dim\geq n$. An $n-$norm on X is a mapping $\|\cdot,\dots,\cdot\|:X^n\to\mathbb{R}$, which satisfies the following four conditions:

nN 1: $\|x_1,\dots,x_n\|=0$, if and only if $x_1,\dots,x_n$ are linearly dependent,

nN 2: $\|x_1,\dots,x_n\|=\|x_{i_1},\dots,x_{i_n}\|$, for every permutation $(i_1,\dots i_n)$ of $(1,\dots,n)$,

nN 3: $\|\alpha x_1,\dots,x_n\|=|\alpha|\|x_1,\dots,x_n\|$ for $\alpha\in\mathbb{R}$,

nN 4: $\|x_1+\acute{x}_1,x_2,\dots,x_n\|\leq\|x_1,x_2,\dots,x_n\|+\|\acute{x}_1,x_2,\dots,x_n\|$,

for all $x_1,\acute{x}_1,x_2,\dots,x_n\in X$. The pair $(X,\|\cdot,\dots,\cdot\|)$ is called an $n-$normed spaces.

Definition 2.2. [2] A real-valued function $\langle\cdot,\cdot\,|\,\cdot,\dots,\cdot\rangle$ on X^{n+1} satisfied the following properties:

nI 1: $\langle x_1, x_1|x_2, \dots, x_n\rangle \geq 0$ and $\langle x_1, x_1|x_2, \dots, x_n\rangle = 0$,

if and only if $x_1, x_2 \dots, x_n$ are linearly dependent.

nI 2: $\langle x_1, x_1|x_2, \dots, x_n\rangle = \langle x_{i_1}, x_{i_1}|x_{i_2}, \dots, x_{i_n}\rangle$, for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$.

nI 3: $\langle x_1', x_1|x_2, \dots, x_n\rangle = \langle x_1, x_1'|x_2, \dots, x_n\rangle$,

nI 4: $\langle \alpha x_1, x_1|x_2, \dots, x_n\rangle = \alpha\langle x_1, x_1|x_2, \dots, x_n\rangle$, for every $\alpha \in \mathbb{R}$.

nI 5: $\langle x_0 + x_0', x_1|x_2, \dots, x_n\rangle = \langle x_0, x_1|x_2, \dots, x_n\rangle + \langle x_0', x_1|x_2, \dots, x_n\rangle$.

is called an $n-$inner product on a vector spaces X. The pair $(X, \langle\cdot,\cdot\,|\,\cdot, \dots, \cdot\rangle)$ is called an $n-$inner product spaces.

3. Main results

Theorem 3.1. A characterization of $n-$inner product spaces $(X, \langle\cdot,\cdot\,|\,\cdot, \dots, \cdot\rangle)$ where $(X, \|\cdot, \dots, \cdot\|)$ be an $n-$normed spaces on $\mathbb{C}$ are:

i. $2(\|x, x_2, \dots, x_n\|^2 - \|y, x_2, \dots, x_n\|^2) = \|x + iy, x_2, \dots, x_n\|^2 + \|x - iy, x_2, \dots, x_n\|^2$, (3.1)

ii. $\langle x, y|x_2, \dots, x_n\rangle = \frac{1}{8}\begin{pmatrix} \|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2 \\ + \\ i(\|x + iy, x_2, \dots, x_n\|^2 - \|x - iy, x_2, \dots, x_n\|^2) \end{pmatrix}$, (3.2)

for every $x, y, x_2, \dots, x_n \in X$.

Proof.

To prove equation (3.1) of (i),

R.H.S. $= \|x + iy, x_2, \dots, x_n\|^2 + \|x - iy, x_2, \dots, x_n\|^2$

$= \langle x + iy, x + iy|x_2, \dots, x_n\rangle + \langle x - iy, x - iy|x_2, \dots, x_n\rangle$

$= \langle x, x|x_2, \dots, x_n\rangle - 2i\langle x, y|x_2, \dots, x_n\rangle - \langle y, y|x_2, \dots, x_n\rangle$

$+$

$\langle x, x|x_2, \dots, x_n\rangle + 2i\langle x, y|x_2, \dots, x_n\rangle - \langle y, y|x_2, \dots, x_n\rangle$

$= 2\langle x, x|x_2, \dots, x_n\rangle - 2\langle y, y|x_2, \dots, x_n\rangle$

$= 2(\|x, x_2, \dots, x_n\|^2 - \|y, x_2, \dots, x_n\|^2)$.

Which is the L.H.S. of equation (3.1).

To prove equation (3.2) of (ii),

R.H.S. $= \frac{1}{8}\begin{pmatrix} \|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2 \\ + \\ i(\|x + iy, x_2, \dots, x_n\|^2 - \|x - iy, x_2, \dots, x_n\|^2) \end{pmatrix}$

$$= \frac{1}{8}\left(4\langle x, y|x_2, \dots, x_n\rangle + i(-4i\langle x, y|x_2, \dots, x_n\rangle)\right)$$

$$= \langle x, y|x_2, \dots, x_n\rangle,$$

which is the L.H.S. of equation (3.2).

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