

Optimal Auctions and Pricing with Limited Information

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ABSTRACT

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Information availability plays a fundamental role in decision-making for business operations. The present dissertation aims to develop frameworks and algorithms in order to guide a decision-maker in environments with limited information. In particular, in the first part, we study the fundamental problem of designing optimal auctions while relaxing the widely used assumption of *common prior*. We are able to characterize (near-)optimal mechanisms and associated performance. In the second part of the dissertation, we focus on data-driven pricing in the low sample regime. More precisely, we study the fundamental problem of a seller pricing a product based on historical information consisting of one sample of the willingness-to-pay distribution. By drawing connection with the statistical theory of reliability, we propose a novel approach, using dynamic programming, to characterize near-optimal data-driven pricing algorithms and their performance. In the last part of the dissertation, we delve into the detailed practical operations of the online display advertising marketplace from an information structure perspective. In particular, we analyze the tactical role of intermediaries within this marketplace and their impact on the value chain. In turn, we make the case that under some market conditions, there is a potential for Pareto improvement by adjusting the role of these intermediaries.

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Introduction

Decision-making is embedded at all strategic and operational levels of organizations. With the widespread emergence of technology during the last decades, there is an unprecedented need for theory and frameworks to guide real-time decisions in a scalable fashion.

One key building block of any decision-making task is the information available to the decision-maker. Depending on the availability and the structure of the information at hand, the approach might differ significantly. For example, in the e-commerce industry, if a seller has tailored information about the preferences of her customers, then she might be able to segment these customers and make the decision to present personalized offers to each segment. More broadly, given the information gathered, a decision-maker needs to be armed with suitable algorithms, tools and models that would guide her toward the “best” decision. Such a need was at the inception of many streams of literature across different research communities. In particular, some main streams of literature across Operations Research, Economics, Computer Science have led different approaches to tackle this need. For instance, some streams develop different stochastic models and optimization frameworks in order to guide a decision maker. Some other key streams study models in order to analyze the market equilibrium and strategic behavior of agents that a decision maker might face. Also, some important streams analyze and design algorithms and study approximation solutions in order to help a decision maker approach the best decision. Moreover, within the machine learning community, there has been recently significant developments of algorithms

in order to train models from existing data to learn to guide decision-making.

Despite the difference in approaches and interests, at the heart of these communities lays some fundamental questions: How to model the information available to the decision-maker? Given this model of available information, how to find the best decision? As a result, what insights or recommendations the decision maker needs to follow? What is the impact of such decisions on the other agents?

There has been a massive body of research trying to address these questions across different communities with important questions still open. This present dissertation aims to contribute to this stream, focusing on the design of selling mechanisms under various information structure: a decision maker, aka a seller, with limited information, aims to design a selling mechanism to sell goods to buyers. In the first two chapters, we analyze the case when the seller has limited information about the process generating the buyers' values for the goods. Especially, in the first, chapter, we focus on characterizing the optimal selling mechanism in the presence of competition under limited information. In the second chapter, we study a complementary direction, where there is no competition and the seller has some available data to design the optimal selling mechanism. In the last chapter, we analyze a model where the information structure is impacted by the market structure rather than the lack of knowledge of buyers' characteristics. More precisely, we study the impact of the presence of intermediaries between the seller and buyers on the information structure and ultimately on the welfare of each agents involved.

Chapter 1. We will be analyzing the questions raised under the lens of auctions. In other words, the central question of the first chapter is what is the optimal auction or optimal decision of an auctioneer, when she has limited information about the preferences of potential buyers.

The motivation of this question is rooted around different practical and theoretical

reasons. Indeed, auctions have been a central tool in selling goods and services across history. Nowadays, auctions are used in many of industries. Governments for instance use auction to sell public goods such as spectrum licenses. Auctions are also used to sell unique and historical art pieces. Given the rise of technology and new industries, auctions have gained even more prevalence. In online advertisement, there are millions of auctions taking place daily.

This importance of auctions has led different communities such as Operations Research, Economics and Computer Science to elaborate and analyze models, algorithms and tools in order to support decision making in auctions. Although there has been extensive work on designing auctions under different settings, a large body of the literature is anchored around a crucial informational assumption: that of a *common prior*. In more details, this assumption means that the seller as well as the buyers share some common prior on the process generating the values for the object.

Since the seminal works of Vickrey (1961) and Myerson (1981), this assumption has allowed many subsequent works to derive tractable models and solutions for the optimal auctions. However, at the same time, this assumption might create a potential disconnect between practice and theory. Indeed, there are many settings in which this assumption can be clearly violated. For instance, in the case of a brand new product, it is not clear how a decision maker can determine the exact process generating the values of buyers. Another setting is motivated by emerging privacy policies. In particular, even though an auctioneer can have access to large data-sets, then due to regulation constraints, she might be constrained not to use the full data when fine tuning the auction. More generally, the buyers themselves might not know apriori their own value and might need to go through some complex process to determine the value of the auctioned good, and in turn it is not clear how an auctioneer might make assumptions on such a complex process. Hence, practice motivates the need to relax this assumption.

This need to relax this assumption to have more “realistic” models is typically referred as the “Wilson doctrine” (Wilson, 1987). Motivated by this doctrine, a recent stream of literature at the frontier of Operations Research, Economics and Computer Science has emerged to close this gap. This first chapter contributes to this growing stream.

In particular, we study the design of optimal *prior-independent* selling mechanisms: buyers do not have any information about their competitors and the seller does not know the distribution of values, but only knows a general class it belongs to. Anchored on the canonical model of two buyers with independent and identically distributed values, we analyze a competitive ratio objective, in which the seller attempts to optimize the worst-case fraction of revenues garnered compared to those of an oracle with knowledge of the distribution. We characterize properties of optimal mechanisms, and in turn establish fundamental impossibility results through upper bounds on the maximin ratio. By also deriving lower bounds on the maximin ratio, we are able to crisply characterize the *optimal* performance for a spectrum of families of distributions. In particular, our results imply that a second price auction is an *optimal* mechanism when the seller only knows that the distribution of buyers has a monotone non-decreasing hazard rate, and guarantees at least 71.53% of oracle revenues against any distribution within this class. Furthermore, a second price auction is near-optimal when the class of admissible distributions is that of those with non-decreasing virtual value function (aka regular). Under this class, it guarantees a fraction of 50% of oracle revenues and no mechanism can guarantee more than 55.6%. Finally, we extend our results to the case of an unknown and adversarially selected number of buyers and show that the same bounds above hold.

Chapter 2. In the first chapter, we have focused on the case in which the decision maker does not use any side or historical information. In the second chapter, we

study a complementary setting. Especially, under limited information, we assume that the decision maker has access to some side information or historical data and facing only one buyer, meaning there is no competition among buyers. In other words, we study the canonical model of one seller selling one indivisible good based on existing data. This falls under the data-driven pricing. The main problem we are interested by is how to design the optimal algorithm that maps available data to the pricing decision. This stream has also received attention from different angles. Some streams have studied the sample complexity in order to achieve a near-optimal revenue compared to the oracle monopoly revenue under full information. The online learning community has studied this problem in a dynamic fashion. The focus of that stream is to minimize the regret by optimally trading-off between exploration and exploitation. In both of the last streams, the mode of analysis is asymptotic, meaning either the seller has a large amount of data or has a long time horizon to learn. In the recent years, there has been an interest on developing models in finite sample regimes.

Chapter 2 focuses on the regime where the seller has only access to one observation. In more details, we analyze the following fundamental problem: how should a decision-maker optimally price based on a single sample of the willingness-to-pay (WTP) of customers. The decision-maker's objective is to select a general pricing policy with maximum competitive ratio when the WTP distribution is only known to belong to some broad set. We characterize optimal performance across a spectrum of non-parametric families of distributions, α -strongly regular distributions, two notable special cases being regular and monotone hazard rate distributions. We develop a general approach to obtain structural lower and upper bounds on the maximin ratio characterized by novel dynamic programming value functions. In turn, we develop a tractable procedure to obtain near-optimal mechanisms and near-worst-case distributions, allowing to characterize the maximin ratio for all values of α in $[0, 1]$.

Chapter 3. In the last two chapters, we have analyzed how the optimal auctions and pricing are impacted by the available information to the seller. Sometimes, the information availability is also impacted by the structure of the market. In particular, the presence of intermediaries between a seller and potential buyers might distort the information flow. In this third chapter, we study how does the structure of a marketplace, especially the presence of intermediaries, impact the information and the performance of different agents within this market? This is particularly motivated by the marketplace of Online Display Advertising.

Online Advertising is becoming one of the most prominent channels for companies and brands to reach their customers. In particular, with the advance of the technology, online advertising allows these companies to target in real time their customers. This selling mechanism is typically referred as *Real-time Bidding*. In more details, while a user is loading a page with an advertisement banner, advertisers can now participate in auctions and send in real time their bids based on the available information about this user. This high flexibility of targeting has attracted over the last decades lot of interest in industry and has created a value chain of different actors. Especially, there has been the emergence of intermediaries between the advertisers and publishers across the value chain. In other words, advertisers can go through some intermediaries that will manage the bidding process on their behalf.

In practice, these intermediaries allow to reduce the cost of participating in this real-time bidding process for advertisers because of the economy of scale and the need of technical expertise. But at the same time, these intermediaries impact the information structure within this marketplace given the fact that they may manage couple of advertisers simultaneously. In turn, intermediaries, when bidding on behalf of their customers, strategize to maximize some internal objective and may only submit a single bid to limit competition on a given item. In this chapter, we propose a framework to analyze the implications of such a campaign coordination role by

intermediaries, taking as a benchmark the case in which each intermediary would manage the bidding process of each advertiser it represents independently of other buyers, a case we refer to as multi-bidding. We show that the adoption of multi-bidding by all intermediaries would lead to an increase in both the social welfare and the seller's revenues. Furthermore, we analyze the impact on buyers in two regimes: i.) without competition among intermediaries and ii.) with competition, with a large number of intermediaries and buyers in an appropriate asymptotic regime. Quite remarkably, we establish that multi-bidding would also lead to an increase in the buyers' side surplus under a very broad set of market characteristics. In particular, as long as the average number of buyers interested in an item is moderate and the coefficient of variation of buyers' values is not too small, moving from coordinated campaigns to multi-bidding leads to a Pareto improvement in the value chain.

In each chapter, we discuss in more details, the contributions and the positioning within the existing literature, as well as potential extensions.

Prior-Independent Optimal Auctions

1.1 Introduction

Auctions have been run for many centuries and play today a prominent role in applications as diverse as e-commerce, spectrum allocation, antique sales, online advertising and procurement. In turn, auction design has been a central topic of research at the intersection of Operations Research, Computer Science and Economics. The monograph of Krishna (2009) provides an overview of auction theory and Talluri and Van Ryzin (2006) details many revenue management applications. While there is an elegant theory of auction design dating back to the seminal works of, e.g., Vickrey (1961) and Myerson (1981), the classical theory of auctions is anchored around a fundamental assumption: that of a common prior. This assumption stipulates that the seller as well as the buyers share the same common prior on the process generating the values for the object. In turn, this assumption leads naturally to the buyers using this common prior to play equilibrium bidding strategies, forming a Bayesian Nash equilibrium; and the seller, anticipating such equilibrium behavior, can optimize the selling mechanism based on this prior. This poses a challenge in practice as such a prior is not available and it is not clear how the seller's belief and the buyers' beliefs about values should coincide, or how they would be formed correctly. In turn, a fundamental question from practical and theoretical perspectives pertains to how to relax such an assumption and what performance can one expect in its absence. This fundamental need to move beyond mechanisms that rely on priors is often referred

to as the “Wilson doctrine” (Wilson, 1987). Relaxing the assumption on common priors leads to a trade-off between information about the distribution of values and performance which motivate the following questions. What is the maximum fraction of revenues that one can guarantee compared to an oracle that would have access to the underlying distribution of values? How does this fraction vary as a function of the information available about the underlying distribution? These are the central questions that this chapter aims to address.

In the present chapter, we aim to address the above in the canonical private value model of a seller trying to sell a good to two bidders with independent and identically distributed values¹. While mechanism design is very well understood for this classical model under the common prior assumption, it remains challenging in prior-independent environments. (We review shortly in detail related work.) As soon as one relaxes the common prior assumption, a first question is how to formulate the problem. On the one hand, the common prior affected bidding behavior of buyers. On the other hand, it also affects the seller’s mechanism optimization problem. We maintain the fact that values are drawn from an underlying distribution (the true distribution of values), as in the classical framework, but we do not assume knowledge of this distribution by the buyers or the seller. In turn, one needs to specify the information available to the buyers and the resulting equilibrium, as well as the seller’s knowledge and feasible mechanisms, and these two are tightly interconnected. For the buyers’ side, we will adopt a detail-free approach and assume that buyers’ optimal decisions are independent of any information about the other buyers’ values. For that, we will restrict attention to mechanisms for which truth-telling is a dominant strategy, so-called Dominant Strategy Incentive Compatible (DSIC) mechanisms. Against such a mechanism, buyers bidding their values represent a dominant

¹We extend the results to the case in which the number of buyers can be more than two, and is picked adversarially by nature.

strategy Nash Equilibrium. On the seller’s side, we will assume that the seller is free to select among such mechanisms. Given that the seller does not know the true distribution of values, we will adopt a maximin ratio approach. We model our problem as a game between nature and the seller. The seller first selects a mechanism in the class of DSIC mechanisms. Then, nature may counter such a mechanism with *any* distribution for buyers’ values from a given class of admissible distributions. In particular, the resulting equilibrium induced by the mechanism is dominant strategy incentive compatible and the only knowledge the seller is endowed with is the class of admissible distributions. For any distribution and mechanism, we measure the performance of the seller through the ratio of the revenue she garners using this mechanism over the optimal revenue she would have obtained with access to the exact knowledge of the distribution. We refer to the latter as the oracle revenues. The ratio is always between 0 and 1 and the higher the ratio, the better the performance. We focus on a maximin setting in which the seller attempts to maximize the worst-case performance ratio (or competitive ratio) over the class of admissible distributions.

Our results provide a characterization of the maximin ratio across a spectrum of distribution classes. In particular, we consider three main classes of distributions. It is possible to show that against the general class of distributions, no DSIC mechanism can guarantee a positive fraction of oracle revenues and hence there is a need to study how different structures of the underlying distributions affect the type of performance that can be achieved. Beyond the general class of distributions, we will consider a class which is central to mechanism design (including under the common prior assumption), that of so-called regular distributions. These are distributions that admit increasing virtual value function. In addition to the class of regular distributions, we will also analyze the subclass of monotone increasing hazard rate distributions (MHR) (also often referred to as increasing failure rate distributions), which contains many distributions often assumed in practice and in the literature (e.g., uniform,

exponential, ...).

Summary of contributions. Before laying out our main results, it is important to highlight the nature of the problem we study. On the one hand, given a particular mechanism, nature selects the worst possible distributions in the non-parametric classes above. So nature when minimizing the ratio of revenues compared to oracle performance, is solving a non-convex infinite dimensional optimization problem. In turn, fully understanding the worst-case performance of a specific mechanism is highly non-trivial and not necessarily tractable. On the other hand, the seller, when optimizing over DSIC mechanisms, is also solving an infinite dimensional problem (over allocation and payment mappings). An important contribution of the present chapter is to propose an approach to tackle this class of problems and characterize optimal or near-optimal performance.

For regular distributions, it is known that a second price auction² guarantees, in the worst case scenario, 50% of the oracle revenues, as articulated in Dhangwatnotai et al. (2015) through a reinterpretation of the results in Bulow and Klemperer (1996). Notably, Fu et al. (2015) recently establish that a second price auction is not prior-independent optimal. In particular, they exhibit a mechanism that randomizes between a second price auction and an auction that inflates the second value and establish that it ensures a competitive ratio of at least 51.2%. Table 1.1 below summarizes the best known lower bounds on the maximin ratios as well as implications of our results. While there is a lower bound on the maximin ratio against regular distributions, there is no notion of what performance one should aim at, and how good are the prior-independent auctions previously proposed. In the popular subclass of MHR distributions, to the best of our knowledge, no lower or upper bounds were available in the literature.

²Here and throughout the chapter, whenever we refer to a second price auction, unless otherwise noted, it is implicitly assumed that there is no reserve price.

A first significant layer of contribution pertains to the methodological domain and allows to obtain the first impossibility results for *any* mechanism in a broad class of DSIC mechanisms. We mainly focus on the case with two buyers, which intuitively is the case with most tension while relaxing the common prior assumption, and then establish that the bounds obtained for the case of two buyers also apply to the case when the number of buyers is adversarially selected (Section 1.7.1).

We first develop families of tractable upper bounds on the maximin ratio. These are obtained through successive dimensionality reductions on the space of mechanisms and the space of distributions. We show that, under some mild regularity assumption on the mechanisms, an optimal mechanism is scale-free (see Theorem 3.2). In other words, it is sufficient to focus on mechanisms that only rely on the ratio of values of buyers. In turn, leveraging properties of the allocations, we are able to “discretize” the mechanisms without loss of optimality, and reduce the description of mechanisms to a countable set (Proposition 1.1).

Given the result above, we then introduce general subsets of distributions. These abstract subsets are developed in order to, on the one hand, being “hard” for any mechanism, and on the other hand allow to further reduce the complexity of the set of mechanisms under consideration, leading to a new generic upper bound (Theorem 1.2). By customizing this bound through appropriate concrete classes and leveraging additional properties of the classes, we obtain parametric upper bounds for the maximin ratio against regular distributions (Theorem 1.3) and MHR distributions (Theorem 1.4). In turn, these upper bounds lead to the first impossibility results for general randomized mechanisms against these two central classes of distributions. No DSIC mechanism considered can guarantee more than 55.6% of oracle performance against all regular distributions, and no DSIC mechanism considered can guarantee more than 71.53% of oracle performance against all regular distributions.

These results have a significant implication for regular distributions. They imply

that the mechanisms proposed to date in the literature are in fact near-optimal. A second price auction is within 5.6% of optimal and the mechanism proposed in (Fu et al., 2015) is within 4.4% of optimal. These impossibility results allow to quantify the quality of *any* mechanism compared to *optimal* performance in the class of DSIC mechanisms.

As a second layer of contribution, we also develop lower bounds on the maximin ratio. We develop a series of generic parametric lower bounds (Proposition 1.4, Proposition 1.5) and in turn obtain lower bounds on the worst-case performance of specific mechanisms. For the case of regular distributions, we establish that there exists a mechanism that guarantees at least 51.9%, improving the best known lower bound and further closing the gap with the upper bound we have developed. For the case of MHR distributions, we establish that a particular mechanism, a second price auction, guarantees at least 71.53% of oracle performance.

While we improve the lower bound on regular distributions, the significant implication of the lower bounds is for the MHR class. The first implication stems from comparing it to the novel upper bound we derive for regular distributions. In particular, our results show how refined class information (from regular to MHR) translates into improved performance. Against MHR distributions, even with only two buyers, a seller is guaranteed 71.53% of oracle performance. The second implication is even more notable. The conjunction of our upper and lower bounds imply that a second price auction is actually *optimal* against MHR distributions and that we have exactly characterized the maximin ratio for that class. Overall, the results above provide a crisp characterization of the maximin ratio as information regarding distributions is refined.

In addition, the results shed light on the trade-off that an auctioneer might face between running an auction with limited information and the cost of collecting additional information to approach the oracle optimal revenue. Our results highlight how

this trade-off might be affected by the nature of distributions that a decision-maker might face, e.g., if distributions are more “concentrated” (as is the case for MHR).

From a different angle, in practice, there is also often a trade-off between revenue maximization and social efficiency. In the canonical class studied, our results highlight that, in a prior-independent environment, a second price auction is near optimal for the wide class of regular distributions and optimal for the large subclass of MHR distributions. As such, when limited information about the underlying distribution of values is available, a simple, practical and socially efficient mechanism appears “sufficient” from a revenue maximization perspective. Hence, there is a weak trade-off between revenue maximization and social efficiency when facing regular distributions and no trade-off when facing MHR distributions.

Distributions class	Maximin ratio			
	Lower Bounds		Upper Bounds	
	best known	this chapter	this chapter	best known
Regular	51.2%	51.9%	55.6%	n/a
MHR	n/a	71.53%	71.53%	n/a

Table 1.1: **Maximin Performance.** The table contrasts known results in the existing literature with the bounds derived obtained through the analysis in the present chapter.

The remainder of the chapter. After relating our chapter to the existing literature, we formulate our problem and set up our framework for two buyers. In Section 1.3, we establish that one may restrict attention to scale-free mechanisms and characterize the maximin ratio for general distributions. In Section 1.4, we derive a family of upper bounds on the maximin ratio against subsets of regular distributions. In Section 1.5, we investigate the case of regular distributions while the subset of MHR distributions is the focus of Section 1.6. Then, in Section 1.7, we extend our results to the case in which the number of buyers is arbitrary and adversarially selected, and

discuss future directions. All proofs are presented in the appendix in the electronic companion.

1.1.1 Literature review

Our work relates to a rich literature on auction design. Since the seminal work Myerson (1981) that characterized the structure of an optimal revenue maximizing mechanism when the seller has access to the exact distributions of values of buyers, the research community has raised early on the need of designing auctions that do not rely on such informational assumptions, often referred to as the “Wilson doctrine” (Wilson, 1987). Our work belongs to the stream that aims to relax such assumptions. There are different layers of informational assumptions that have been analyzed in the literature. Some layers relate to the seller’s knowledge about the distributions of values of buyers or the number of participating buyers. Other layers relate to the knowledge of buyers about their own values as well as the values or number of competitors.

When relaxing informational assumptions in auction design, there are two implications. On the one hand, the information affects the type of mechanisms that the seller can adopt. On the other hand, the information also affects the type of equilibrium played by the competing buyers.

In terms of the assumption that each buyer makes on the value generating process of his competitors, various alternatives have been analyzed. One extreme is to assume that the buyers’ know their competitors distributions of values. In this case, Caillaud and Robert (2005) show that the seller could exploit this and recover the optimal oracle revenue even if she does not have access to the distributions of values through a dynamic mechanism. A first relaxation is to assume that the buyers know some ambiguity sets characterizing the distributions, see Bose et al. (2006), Chiesa et al. (2015) and Koçyiğit et al. (2017). A further relaxation is to assume that buyers

do not have access to any information about values of other buyers, this is typically done by assuming Dominant Strategy Incentive Compatible (DSIC) mechanisms. We refer the reader to Chung and Ely (2007) that gives a formal foundation of such an assumption by showing that a dominant strategy mechanism always dominates in terms of revenue any other mechanisms when the buyers' beliefs about the distribution of their competitor is selected adversarially. Our work aims to make minimum assumptions on both the seller and the buyer's side, and in turn we focus on DSIC mechanisms. Furthermore, we do not make any assumption on the buyers knowledge on the number of competitors.

Another line of work relaxes the knowledge of the buyers regarding her true value, by assuming that the buyer observes some signal related to the true value. We refer the reader to, e.g., Bergemann et al. (2016) that aims to characterize optimal auctions when there is uncertainty on the information structure of the buyers. See also Bergemann and Morris (2013) for a broader overview. We would like to note that in this line of work, it is typically assumed that the seller knows the distribution of values of buyers. Compared to our work, we assume that the seller does not know the distribution of values of buyers but knows the information structure of the buyers. Furthermore, the DSIC assumption also implies that the equilibrium of buyers does not depend on the underlying distributions of values. The buyers' strategies also does not depend on the number of buyers. In that regard, we also note that another dimension of information on the side of buyers pertains to the number of buyers. Harstad et al. (1990) and Levin and Ozdenoren (2004) relax this, while maintaining knowledge of the distribution of values.

Once information on the buyers' side is formulated, the next dimension relates to the layer of information that the seller has. In that regard, there are at a high level three main classes of information structures assumed on the knowledge of distributions of values of buyers: non-parametric canonical classes of distributions (Dhang-

watnotai et al., 2015), statistics of the distributions (Azar et al., 2013), uncertainty sets on the distributions (Koçyiğit et al., 2017) or the values (Bandi and Bertsimas, 2014).

Finally, a fundamental other dimension pertains to how performance of a mechanism is measured in such an environment. One approach, typically referred to as “robust” is to use the *absolute* worst-case performance based on the information available, see, e.g., Carrasco et al. (2015), Bandi and Bertsimas (2014) and Koçyiğit et al. (2017). Another approach is to measure worst-case performance *relative* to a full information benchmark, see, e.g., Neeman (2003) and Dhangwatnotai et al. (2015). A more detailed discussion on various candidate objectives can be found in Borodin and El-Yaniv (1998). Our work relates to the last branch of literature since we characterize the optimal competitive ratio when the seller has only access to the class of distributions of buyers. The ratio we analyze is unitless and has a physical interpretation in terms of the fraction of oracle performance one can obtain compared to an oracle.³

In this stream, an important set of results pertain to “existence” of mechanisms with good guarantees. Looking at different classes of distributions Neeman (2003) derive an early result and establishes a guarantee for the English auction, compared to the social optimum. In particular, the author characterizes tight lower bounds as a function of some summary statistics, on the performance of an English auction with or without reserve price. The setting we focus on is the independently and identically distributed values case. In this setting, if the seller knows that the distribution of values of buyers belongs to the regular class of distributions, then an implication of classical results of Bulow and Klemperer (1996), based on the interpretation of

³It is worthwhile noting here that against the classes we consider (regular and mhr), a worst-case absolute performance analysis would lead to a value of zero and all feasible mechanisms would be optimal. The relative benchmark approach allows to control for the environment and derive guarantees on broader sets of distributions.

Dhangwatnotai et al. (2015), is that there exists a particular mechanism, namely a second price auction, that extracts 50% of the oracle revenue had one known the true distribution, against *any* regular distribution. Recently, Fu et al. (2015) show that a second price auction is suboptimal against regular distributions by exhibiting a randomized mechanism that has a higher guarantee than a second price auction. In the present work, we focus on optimizing over a very broad class of DSIC mechanisms and in turn establish fundamental impossibility results for any such mechanism. The results complement the literature by not only characterizing what is achievable by a particular mechanism but also characterizing *optimal* performance through upper bounds on the maximin ratio. Furthermore, by focusing on the widely considered subclass of mhr distributions, we establish that a second price auction is actually the exact optimal mechanism in that case. This also sheds light on the role of randomization and its relationship to the class of distributions one faces.

In the case of multiple goods, Goldberg et al. (2006) introduce and analyze the competitive ratio, where the worst case is taken with respect to any possible inputs; and then establish that some auctions are competitive compared to a fixed pricing benchmark. In more general environments Dhangwatnotai et al. (2015) leverages the connection to Bulow and Klemperer (1996) to propose a mechanism that has a non-trivial performance even in general allocation environments. Relatedly, Sivan and Syrgkanis (2013) extend a result of Bulow and Klemperer (1996) to the case in which the distributions of values of buyers are a convex combination of regular distributions.

A related stream of literature focuses on alternative information about the distribution. For instance, Azar and Micali (2012) and Azar et al. (2013) propose mechanisms in cases in which the seller has access to some summary statistics of the distributions of values of buyers (mean or median). They exhibit mechanisms that have performance guarantees compared to an oracle using these. In the present chapter, we do not assume that the seller has access to some summary statistics and we

focus on the optimal mechanism among a broad set of randomized mechanisms.

In our chapter, we focus on a static model with limited information. Other examples of directions analyzed pertain to the amount information available or the dynamics. Cole and Roughgarden (2014) analyze the size of the sample that the seller needs to observe from past data in order to design a near optimal mechanism. Dynamic models have also been considered in the literature; see, e.g., Bose and Daripa (2009) for a dynamic model under ambiguity. We refer the reader to review chapters of Hartline and Roughgarden (2009), Hartline (2013), and Carroll (2018) for a broader overview.

Another information assumption from seller’s perspective that the literature has tried to relax is the knowledge of the exact number of buyers. For instance, while maintaining the common prior assumption, McAfee and McMillan (1987) characterize the optimal auctions when the seller has some prior on the number of buyers and Levin and Ozdenoren (2004) study the seller’s best response when the number of buyers is picked adversarially from some ambiguity set.

Our work also relates to pricing under limited information. Monopoly pricing with unknown demand information was analyzed with various considerations in Bergemann and Schlag (2008) for a minimax regret objective and in Eren and Maglaras (2010) for the competitive ratio. Caldentey et al. (2016) extends this line of work to account for the presence of strategic customers. Cohen et al. (2016) derive performance guarantees for pricing heuristics when the firm has some knowledge about the demand shut-down price. More recently, Chen et al. (2017) study robust single item and bundle pricing based on summary statistics of buyers’ values distribution. Leveraging existing data, Huang et al. (2015) focus on pricing based on a finite sample of values. There is also an extensive body of work on joint learning and pricing with various informational structures. We refer the reader to Kleinberg and Leighton (2003), Keskin and Zeevi (2014) and Besbes and Zeevi (2015) for various informational structures,

as well as to Besbes and Zeevi (2009), Araman and Caldentey (2009) Farias and Van Roy (2010), and Wang et al. (2014) for inventory considerations in such pricing problems. den Boer (2015a) provides a survey of this line of work.

1.2 Problem formulation

We consider a seller offering an indivisible object for sale to two buyers. For now, we focus on the two buyers case since it is the case with minimum competition, and isolates the impact of relaxing the common prior assumption. We return to the case of more than two buyers in Section 1.7. The two buyers have values identically and independently distributed according to a distribution F with support \mathcal{S}_F in $[0, \infty)$. We will denote by $\bar{F}(\cdot) := 1 - F(\cdot)$ the complementary cumulative distribution function (ccdf) of values.

We assume that the seller does not know exactly the distribution of values of buyers, however she knows that it belongs to a particular class. The goal of the seller is to design a mechanism that maximizes her revenue given the limited information about the underlying distribution of values of buyers.

Seller’s problem. We model our problem as a game between the seller and nature, in which the seller selects a prior-independent selling mechanism and then nature may counter such a mechanism with *any* distribution of buyers’ values from an admissible class.

A selling mechanism $m = (\mathbf{x}, \mathbf{t})$ is characterized by an allocation mapping \mathbf{x} and a payment mapping \mathbf{t} , where $\mathbf{x} : \mathbb{R}^2 \rightarrow [0, 1]^2$ and $\mathbf{t} : \mathbb{R}^2 \rightarrow \mathbb{R}$. In particular, given reports b_1, b_2 by buyers 1 and 2, a mechanism would allocate the good to buyer i with probability $x_i(b_i, b_{-i})$ and the expected payment of buyer i is $t_i(b_i, b_{-i})$. Here, and in all that follows, the notation (v_i, v_{-i}) is the vector that has value v_i at position i and v_{-i} at the other position.

We do not make any assumption on the buyer's knowledge of the distribution. Given this, we will restrict attention to dominant strategy incentive compatible (DSIC) mechanisms. For such mechanisms, buyers need not make any assumptions about the underlying distribution of values and will find it optimal to report their true value, independently of the realization of value of the other buyers.⁴

More formally, we focus on the class of mechanisms $m = (\mathbf{x}, \mathbf{t})$ that satisfy the following constraints

$$v_i x_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq 0, \text{ for all } i \text{ and } v_i, v_{-i} \text{ in } \mathbb{R}_+^2 \quad (\text{IR})$$

$$v_i x_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq v_i x_i(\hat{v}_i, v_{-i}) - t_i(\hat{v}_i, v_{-i}), \text{ for all } i \text{ and } v_i, v_{-i}, \hat{v}_i \text{ in } \mathbb{R}_+^3, \quad (\text{IC})$$

$$\sum_{i=1,2} x_i(v_i, v_{-i}) \leq 1, \quad \text{for all } \mathbf{v} \text{ in } \mathbb{R}_+^2. \quad (\text{AC})$$

The first constraint (IR) captures ex-post *individual rationality* and states that buyer i should be willing to participate compared to his outside option, normalized to zero. The second constraint (IC) captures ex-post *incentive compatibility* and imposes that a buyer should always find it optimal to report his true value, independently of the value of the other buyer. Finally, (AC) is a constraint on the allocation probabilities that captures that the seller can allocate at most one good. Note here that we allow for randomized mechanisms by the seller. In addition, we will introduce a regularity assumption on mechanisms. We denote by $TV(x_i, [a, b] \times [c, d])$ the Arzelà total variation of the allocations on the set $[a, b] \times [c, d]$.⁵ We assume that the allocations around zero have finite Arzelà total variation. In particular, we will be focusing on the following set of mechanisms.

$$\mathcal{M} = \left\{ (\mathbf{x}, \mathbf{t}) : (\text{IR}), (\text{IC}), (\text{AC}) \text{ and } \max_{i=1,2} \{TV(x_i, [0, \varepsilon]^2)\} < \infty \text{ for some } \varepsilon > 0 \right\} \quad (1.1)$$

⁴We also refer the reader to Chung and Ely (2007) for an in-depth discussion of DSIC mechanisms.

⁵We recall the definition of Arzelà total variation in Section 1.8.

This class of mechanisms is a rich one, containing for example the second price auction with a deterministic reserve price and most mechanisms typically considered in the literature. The assumption on the boundedness of the total variation of allocations around zero is technical in nature⁶ but could also be seen as a way to avoid potentially overly complex mechanisms that might be hard to implement in practice, given the high burden this would put on the buyers.⁷

The revenue of the seller using a feasible mechanism m in \mathcal{M} , if nature is selecting a distribution F , is given by

$$\mathbb{E}_F \left[\sum_{i=1}^2 t_i(v_i, v_{-i}) \right].$$

We will use the subscript F to emphasize that the expectation is taken with respect to that distribution.

The challenge in the present chapter is that the seller does not know the distribution F and as a result cannot evaluate the objective above to select a “good” or optimal mechanism. We next introduce a performance benchmark and pose a proper objective for the seller for this environment with unknown distribution of values.

Oracle benchmark. The benchmark we will use, $\text{opt}(F)$, is the maximal performance one could achieve *with knowledge of the exact distribution of buyers’ values* when selecting mechanisms in \mathcal{M} . More formally,

$$\text{opt}(F) := \sup_{m \in \mathcal{M}} \mathbb{E}_F \left[\sum_{i=1}^2 t_i(v_i, v_{-i}) \right]. \quad (1.2)$$

Seller’s objective. For an arbitrary mechanism in \mathcal{M} , we define its performance against a distribution F such that $\text{opt}(F) > 0$ as follows

$$R(m, F) = \frac{\mathbb{E}_F \left[\sum_{i=1}^2 t_i(v_i, v_{-i}) \right]}{\text{opt}(F)}.$$

⁶While it is needed for the proofs, we conjecture that it does not imply a loss of optimality.

⁷In recent years, there has been growing literature advocating for simple mechanisms (see for example Hartline and Roughgarden (2009), Daskalakis and Pierrakos (2011)). In that sense, the mechanisms in \mathcal{M} could be thought as a formalization of some broad class of “simple” mechanisms.

In other words, $R(m, F)$ represents the fraction of the oracle benchmark performance the mechanism is able to achieve. The ratio $R(m, F)$ always lies in $[0, 1]$ and the closer the ratio is to 1 the better the performance of the mechanism.

Let \mathcal{G} denote the set of distributions with support included in $[0, \infty)$ with finite and non-zero expectation, i.e.,

$$\mathcal{G} = \{F : [0, \infty) \rightarrow [0, 1] : F \text{ is a cdf and } 0 < \mathbb{E}_F[v] < \infty\}. \quad (1.3)$$

Note that $\mathbb{E}_F[v] > 0$ if and only if $\text{opt}(F) > 0$. Hence the ratio $R(m, F)$ is well defined for any element of the class \mathcal{G} .

The objective of the present chapter is to characterize for classes $\mathcal{F} \subseteq \mathcal{G}$ the maximin ratio

$$\mathcal{R}(\mathcal{M}, \mathcal{F}) = \sup_{m \in \mathcal{M}} \inf_{F \in \mathcal{F}} R(m, F). \quad (1.4)$$

In other words, we are interested in designing mechanisms that admit “good” performance independently of the underlying distribution of values. In particular, the value $\mathcal{R}(\mathcal{M}, \mathcal{F})$ represents the maximal fraction of oracle revenues (obtained with knowledge of the distribution of values) that can be recovered when nature may select any distribution in \mathcal{F} .

Definition 1.1. *A cdf F is said to be regular on its support \mathcal{S}_F if it admits a density f and if the corresponding virtual value function $\phi_F : v \mapsto v - (1 - F(v))/f(v)$ is non-decreasing over \mathcal{S}_F . We will further say that the distribution has monotone hazard rate (MHR) if $v \mapsto f(v)/(1 - F(v))$ is non-decreasing over \mathcal{S}_F .*

The class of regular distributions is very widely used and plays a central role in mechanism design (with knowledge of the distribution of buyers) and the class of monotone hazard rate distributions is a wide subclass of the set of regular distributions that encompasses all distributions with log-concave densities (e.g., uniform,

exponential, ...). In particular, beyond \mathcal{G} , we will analyze the two subclasses of distributions

$$\mathcal{F}_{reg} = \{F \in \mathcal{G} : F \text{ is regular}\}$$

$$\mathcal{F}_{mhr} = \{F \in \mathcal{G} : F \text{ has monotone non-decreasing hazard rate}\}$$

It is clear that we have $\mathcal{F}_{mhr} \subset \mathcal{F}_{reg} \subset \mathcal{G}$ and hence

$$\mathcal{R}(\mathcal{M}, \mathcal{G}) \leq \mathcal{R}(\mathcal{M}, \mathcal{F}_{reg}) \leq \mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}).$$

In the coming sections, we will be interested in quantifying the three quantities above and characterizing optimal or near-optimal mechanisms.

Review of some known results. While, to the best of our knowledge, the problem above has not been addressed in the literature, some mechanisms m have been exhibited and their performance characterized. A classical mechanism in \mathcal{M} is the second price auction m_{spa} defined by

$$x_i(v_i, v_{-i}) = \mathbb{1}\{v_i > v_{-i}\} + .5 \mathbb{1}\{v_i = v_{-i}\}$$

and

$$t_i(v_i, v_{-i}) = v_{-i} \mathbb{1}\{v_i > v_{-i}\} + .5 v_{-i} \mathbb{1}\{v_i = v_{-i}\}.$$

The results of Bulow and Klemperer (1996) and their reinterpretation for the performance of the second price auction (see, e.g., Dhangwatnotai et al. (2015)) imply that

$$\inf_{F \in \mathcal{F}_{reg}} R(m_{spa}, F) = 50\%,$$

Recently, Fu et al. (2015) exhibited a mechanism m that randomizes between the identity and a mapping that inflates the second highest value and established that

$$\inf_{F \in \mathcal{F}_{reg}} R(m, F) \geq 51.2\%.$$

The results above imply a lower bound on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$ through specific mechanisms but leave open the question of *optimal* performance. In the present chapter, we aim at characterizing the maximin ratio (2.2) and corresponding near-optimal solutions for \mathcal{F}_{reg} , but also for \mathcal{G} and \mathcal{F}_{mhr} .

1.3 Optimality of Scale-Free Mechanisms

The goal of this section is to establish that one may reduce the space of mechanisms to a simpler class, without loss of optimality. In particular, we will establish that one may restrict attention to scale-free mechanisms (as defined later in Eq.(1.5)).

We first state a classical result from the mechanism design literature (see Myerson (1981)) that links payments and allocations for any incentive compatible mechanism.

Lemma 1.1. *A mechanism (\mathbf{x}, \mathbf{t}) verifies (IC) if and only if $x_i(\cdot, v_{-i})$ is non-decreasing for any $v_{-i} \geq 0$ and the payment mapping satisfies*

$$t_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(l, v_{-i}) dl + t_i(0, v_{-i}), \quad \text{for all } v_i, v_{-i} \geq 0.$$

Note that by the constraint (IR), $t_i(0, v_{-i}) \leq 0$. Hence, we can restrict attention to mechanisms that set $t_i(0, v_{-i}) = 0$ without loss of optimality. With some abuse of notation, we impose this additional constraint in the class of mechanisms \mathcal{M} . In other words, given (IC), we can restrict attention to allocations that are monotone in own values and payments are fully determined by the allocations.

Before stating the main result of this section, let us now introduce some definitions pertaining to scaled distributions as well as a scale invariant classes of distributions.

Scaled distributions. For any distribution F in \mathcal{G} , and $\theta > 0$, we define $F_\theta(\cdot) := F(\theta \cdot)$ to be the θ -scaled distribution.

Definition 1.2 (scale invariance). *A class of distributions $\mathcal{F} \subseteq \mathcal{G}$ is said to be invariant under scaling if for any element F in \mathcal{F} , the distribution F_θ also belongs to \mathcal{F} for any $\theta > 0$.*

Note that \mathcal{G} , \mathcal{F}_{reg} and \mathcal{F}_{mhr} are all scale invariant. The scale invariance of \mathcal{G} follows from the fact that $\mathbb{E}_{F_\theta}[v] = \theta^{-1} \mathbb{E}_F[v]$. For \mathcal{F}_{mhr} and \mathcal{F}_{reg} , note that for any F , we have for all v in its support,

$$\frac{f_\theta(v)}{1 - F_\theta(v)} = \theta \frac{f(\theta v)}{1 - F(\theta v)} \quad \text{and} \quad v - \frac{1 - F_\theta(v)}{f_\theta(v)} = \frac{1}{\theta} \left(\theta v - \frac{1 - F(\theta v)}{f(\theta v)} \right).$$

Hence, the MHR and regularity properties of any distributions F_θ are inherited from the original distribution F .

Scale-free mechanisms. Recall the class of mechanisms \mathcal{M} introduced in (1.1). We next introduce the subclass of scale-free mechanisms $\mathcal{M}_{sf} \subset \mathcal{M}$ defined as follows

$$\mathcal{M}_{sf} = \{m \in \mathcal{M} : x_i(\theta v_i, \theta v_{-i}) = x_i(v_i, v_{-i}) \quad \text{for all } v_1, v_2 \geq 0, \theta > 0, i = 1, 2\}. \quad (1.5)$$

This subclass of mechanisms have the property that the allocations do not depend on the scale of values. With these definitions in place, we may now state the main result of this section.

Theorem 1.1. *For any class $\mathcal{F} \subseteq \mathcal{G}$ that is invariant under scaling, when solving (2.2), it is sufficient to consider scale-free mechanisms. Namely, we have*

$$\mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}) = \mathcal{R}(\mathcal{M}, \mathcal{F}).$$

This result establishes that we can restrict attention to the scale-free mechanisms without loss of optimality. Intuitively, an optimal prior independent mechanism should not depend on the scale of buyers' values. If that were the case, then nature could leverage it to significantly affect the performance of the seller. The proof builds on this idea by evaluating a mechanism in \mathcal{M} against a particular distribution

and noting that the performance of this mechanism against any scaled version of the distribution serves as an upper bound on the worst-case performance of this mechanism. (For this step, we leverage the boundedness of the total variation of feasible mechanisms around zero.) In turn, by “swapping” the scale from the distribution to the mechanism, we establish that the limiting performance of the mechanism against a scaled version of the distribution as the scale goes to ∞ can be reinterpreted as the performance of a scale-free mechanism against the original distribution. In other words, we obtain that there exists a scale-free mechanism that performs at least as well (in the worst-case) as the original mechanism.

The reduction to scale-free mechanisms significantly simplifies the set of mechanisms under consideration and we will leverage this property to further reduce the space of mechanisms in upcoming sections when we consider regular distributions and its subsets. Before that, we directly leverage Theorem 3.2 to characterize the maximin ratio under arbitrary distributions \mathcal{G} , defined in (1.3).

In the previous literature, it was alluded to that without restrictions the seller cannot have any guarantee, Dhangwatnotai et al. (2015). For completeness, we formalize this here in our specific context.

Lemma 1.2. *No mechanism in \mathcal{M} can achieve a positive max-min ratio against the general class \mathcal{G} , namely,*

$$\mathcal{R}(\mathcal{M}, \mathcal{G}) = 0.$$

Lemma 1.2 shows that it is *impossible* for the seller to design a mechanism that achieves positive worst-case performance against arbitrary distributions. The proof relies on two main ideas. Given Theorem 3.2, one may restrict attention to scale-free mechanisms. In turn, we establish that if the value of a buyer is zero then necessarily, a scale-free mechanism charges zero to the other buyer, independently of its value.

Given this, we establish that the performance of any scale-free mechanism when facing the family of Bernoulli distribution of values can be arbitrarily small.

In the rest of the chapter, we focus on characterizing the maximin ratio for the set of regular distributions \mathcal{F}_{reg} and the set of monotone hazard rate distributions \mathcal{F}_{mhr} .

1.4 Maximin Ratio for Subsets of Regular Distributions

In this section, we focus on the development of a family of upper bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F})$ for any \mathcal{F} that is a subset of the class of regular distributions \mathcal{F}_{reg} . In particular, the analysis of this section applies to both \mathcal{F}_{reg} and \mathcal{F}_{mhr} and we will leverage these results in Sections 1.5 and 1.6, when we specialize the analysis to those classes.

In Section 1.4.1, we establish that one may, without loss of optimality restrict attention to a simpler set of mechanisms that are characterized by a sequence of thresholds. In Section 1.4.2, we focus on a simplification of the set of distributions against which one competes, which leads to a further simplification of the set of mechanisms one needs to consider. The conjunction of results leads to a generic family of upper bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F})$ presented in Theorem 1.2.

Oracle Performance for regular distributions. Note that when the distribution of values F is known and is regular, it is a standard result (cf. Myerson (1981)) that an optimal mechanism is given by a second price auction with reserve price given by $r_F := \phi_F^{-1}(0)$, and in turn

$$\text{opt}(F) = \mathbb{E}_F [\phi_F(\max\{v_1, v_2\}) \mathbb{1}\{\max\{v_1, v_2\} \geq r_F\}].$$

In particular, the optimal oracle mechanism depends on the knowledge of the distribution through the reserve price. In what follows, we denote by $q_F = 1 - F(r_F)$ the

quantile associated with r_F .

1.4.1 From general mechanisms to discrete threshold mechanisms

Our first result consists of a reduction of the set of mechanisms that one needs to focus on when the seller faces a subset of regular distributions. To that end, we introduce the subset of mechanisms \mathcal{M}'_{sf} defined by

$$\begin{aligned} \mathcal{M}'_{sf} &= \{m \in \mathcal{M}_{sf} : \text{for } i = 1, 2, \\ x_i(v_i, v_{-i}) &= \sum_{n=1}^N \frac{1}{N} \mathbb{1}\{v_i > \gamma_n v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\}, \\ &\text{for some } N \geq 1, \gamma \in \mathbb{R}^N \text{ and } c \in [0, 1/2] \}. \end{aligned}$$

Note first that this set \mathcal{M}'_{sf} is nonempty. For example, the second price auction (without reserve price) belongs to this set. (To see that, one can take $N = 1$, $\gamma_1 = 1$ and $c = 1/2$.) This set represents, a subset of the scale-free mechanisms \mathcal{M}_{sf} that consists of mechanisms that are constructed using a randomization over prices to be paid by the buyer that is a linear transformation of the value of the competitor.⁸ The next result characterizes the performance of mechanisms in \mathcal{M}'_{sf} .

Proposition 1.1. *For any subclass \mathcal{F} of the set of regular distributions \mathcal{F}_{reg} , it is sufficient to focus on mechanisms in \mathcal{M}'_{sf} , i.e.,*

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{F}) = \mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}).$$

Proposition 1.1 shows that without loss of optimality we can focus on mechanisms that belong to \mathcal{M}'_{sf} . Furthermore, note that this result allows one to move from a

⁸Note also that this set captures explicitly the probability of allocation to a buyer when the value of buyers are equal. While seemingly unimportant in the class \mathcal{F}_{reg} since ties happen with probability zero, this explicit inclusion of the case of ties will play an important role when we will be dealing with the limiting performance of a mechanism against an appropriate sequence of distributions which converges weakly to a point outside of \mathcal{F}_{reg} (see Proposition 1.3).

(potentially intractable) functional space of mechanisms, \mathcal{M}_{sf} , to the union of finite dimensional vector spaces, \mathcal{M}'_{sf} .

The result relies on three key ingredients. We first leverage the monotonicity of the allocations (cf. Lemma 1.1) to establish that one may approximate those from below by a combination of step functions, where the steps are chosen so that the new allocation stays appropriately close to the original allocation. This leads to a new mechanism in \mathcal{M}'_{sf} . Then, leveraging the scale-free property of mechanisms and the fact that the distributions are regular, we can establish that necessarily the performance (in terms of the ratio of revenues achieved compared the optimal oracle revenues) of the new mechanism is necessarily appropriately close to that of the original mechanism.

1.4.2 Family of Upper Bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F})$

Having reduced the strategies of the seller to a more tractable space by discretizing the allocation function, we next reduce the complexity of the space of distribution functions under consideration $\mathcal{F} \subset \mathcal{F}_{reg}$. To that end, we introduce the subclass of distributions

$$\mathcal{W} := \left\{ F \in \mathcal{G} : \bar{v}_F < \infty, F \text{ admits a density on } [\underline{v}_F, \bar{v}_F] \text{ and } \sup_{v \in [\underline{v}_F, \bar{v}_F]} \phi_F(v) \leq 0 \right\},$$

where for any distribution $F \in \mathcal{G}$, we denote by $\underline{v}_F = \inf\{x : x \in \mathcal{S}_F\}$ and $\bar{v}_F = \sup\{x : x \in \mathcal{S}_F\}$. In particular, \mathcal{W} denotes the class of distributions with bounded support, that have non-positive virtual value function on the interior of the support and a potential mass at the upper limit of the support. Note that this set is clearly non-empty and we will consider explicit examples in Sections 1.5 and 1.6. Moreover, note also that for each element of \mathcal{W} , the expectation of the virtual value function is not necessarily equal to the expected revenue. The expected revenue is given by

$$\mathbb{E}_F [t_i(v_i, v_{-i})] = \int_0^{\bar{v}} \phi_i(v_i) \bar{x}_i(v_i) f(v_i) dv_i + \bar{F}(\bar{v}) \left(\bar{v} \bar{x}_i(\bar{v}) - \int_0^{\bar{v}} \bar{x}_i(s) ds \right).$$

where $\bar{x}_i(v_i) = \int_0^\infty x_i(v_i, u) f(u) du$ is the interim allocation to buyer i . In addition to the “classical” first term on the RHS, a second term, driven by the mass at \bar{v} , is also present.

Note that the payment of any mechanism in \mathcal{M}'_{sf} takes the form (cf. Lemma A.2-5)

$$t_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c' v_{-i} \mathbb{1}\{v_i = v_{-i}\},$$

for some appropriate c' . In particular, when evaluating the expected revenues of a mechanism in \mathcal{M}'_{sf} , one needs to consider terms of the form $\mathbb{E}_F [v_2 \mathbb{1}\{v_1 > \alpha v_2\}]$. The next result establish that one may characterize the performance of terms of $\mathbb{E}_F [v_2 \mathbb{1}\{v_1 > \alpha v_2\}]$, not only for elements of \mathcal{F}_{reg} , but also for limits of such elements.

Lemma 1.3. *Suppose that a sequence $\{F_n : n \geq 1\}$ in \mathcal{F}_{reg} , with $\sup_{n \geq 1} \{\bar{v}_{F_n}\} < \infty$, converges weakly to a distribution F where the latter has at most a discontinuity at $\bar{v}_F < \infty$. Then, for any $\alpha \geq 0$,*

$$\lim_{n \uparrow \infty} \mathbb{E}_{F_n} [v_2 \mathbb{1}\{v_1 > \alpha v_2\}] = \begin{cases} \mathbb{E}_F [v_2 \mathbb{1}\{v_1 > \alpha v_2\}], & \text{if } \alpha \neq 1, \\ \frac{1}{2} \mathbb{E}_F [\min(v_1, v_2)] & \text{if } \alpha = 1, \end{cases} \quad (1.6)$$

This result is established by leveraging the weak convergence in conjunction with the regularity of the distributions F_n 's. This result is a key step in linking the performance against elements of \mathcal{W} to that against \mathcal{F} .

Proposition 1.2. *Fix a non-empty subset \mathcal{F} of \mathcal{F}_{reg} and a non-empty subset \mathcal{W}' of \mathcal{W} . Suppose that for any element of \mathcal{W}' , there exists a sequence of distributions in \mathcal{F} that weakly converges to that element. Then we have*

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{F}) \leq \mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}').$$

In other words, while \mathcal{W}' is not a subset of \mathcal{F}_{reg} , the result states that the maximin ratio against the class of distributions \mathcal{W}' upper bounds the maximin ratio against

the class \mathcal{F}_{reg} . The proof of this result leverages the fact that we are working under the tractable space of mechanisms \mathcal{M}'_{sf} in conjunction with the limits established Lemma 1.3. Indeed, the worst-case performance of any mechanism in \mathcal{M}'_{sf} against \mathcal{F} is upper bounded by that against any element of a sequence F_n that converges weakly to an element F of \mathcal{W}' . In the proof, we characterize an asymptotic upper bound on the performance of any mechanism in \mathcal{M}'_{sf} against F_n . Then, we establish that the asymptotic upper bound may be expressed as the performance of a new mechanism in \mathcal{M}'_{sf} when facing the distribution corresponding to the weak limit F .

Subclass of optimal mechanisms against \mathcal{W} . Next, we exploit the structure of the distributions in \mathcal{W} to further simplify the maximin ratio against subclasses \mathcal{W}' of \mathcal{W} , $\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}')$. Let us introduce the following subset of mechanisms of \mathcal{M}_{sf} :

$$\begin{aligned} \mathcal{M}_{sf}^{max} &= \{m \in \mathcal{M}_{sf} : \text{for } i = 1, 2, \\ &\quad x_i(v_i, v_{-i}) = \sum_{n=1}^N \frac{1}{N} \mathbb{1}\{v_i > \gamma_n v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\}, \\ &\quad \text{for some } N \geq 1, \gamma \in ([1, \infty))^N \text{ and } c \in [0, 1/2] \}. \end{aligned}$$

Note that \mathcal{M}_{sf}^{max} is a subset of \mathcal{M}'_{sf} and is the set of mechanisms in \mathcal{M}'_{sf} that never allocate to the minimum value of buyers (when both values are different).

Proposition 1.3. *For any subset of distributions \mathcal{W}' of \mathcal{W} ,*

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}') = \mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}').$$

This proposition shows that without loss of optimality, when facing distributions in \mathcal{W} , one can focus on mechanisms that never allocate to the minimum value (if the latter is different from the maximum value). The intuition behind the result is that under the class of distributions \mathcal{W} , the seller would like to set a reserve price equal to the upper bound of the support if she would know the distribution (cf. Lemma A.2-4). In addition, allocating to a buyer with value strictly below this reserve price yields

a negative contribution to the revenue of the seller (cf. Myerson (1981)). When the seller sees two values, while she does not know the distribution, she knows that it belongs to \mathcal{W} , and hence she still knows that both values are weakly below the optimal oracle reserve price. In turn, the seller never wants to allocate to the minimum value (if it is different from the maximum value).

We are now ready to put together all earlier results and state the main result of this section.

Theorem 1.2. *Fix a non-empty scale invariant subset \mathcal{F} of \mathcal{F}_{reg} and a non-empty subset \mathcal{W}' of \mathcal{W} . Suppose that for any element of \mathcal{W}' , there exists a sequence of distributions in \mathcal{F} that weakly converges to that element. Then we have*

$$\mathcal{R}(\mathcal{M}, \mathcal{F}) \leq \mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}').$$

This result provides a family of upper bounds on the maximin ratio associated with any subset of the set of regular distributions, and in particular applies to \mathcal{F}_{mhr} and \mathcal{F}_{reg} . In Section 1.5, we apply this upper bound to $\mathcal{F} = \mathcal{F}_{reg}$ and in Section 1.6, we apply it to $\mathcal{F} = \mathcal{F}_{mhr}$, where for each we select a suitable set \mathcal{W}' .

1.5 Maximin Ratio for Regular Distributions

In this section, we develop upper and lower bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$, leading to a narrow interval to which $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$ belongs.

1.5.1 Upper Bound

Theorem 1.3 (Upper bound for regular distributions). *The maximin ratio $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$ is upper bounded as follows*

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg}) \leq \sup_{N \geq 1} \sup_{\gamma \in [1, +\infty)^N} \inf_{q \in (0, 1)} \frac{N - |\mathcal{I}^+|}{N} \frac{1}{2 - q} + \frac{|\mathcal{I}^+|}{N} \frac{q}{2 - q} + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \bar{\psi}(\gamma_k, q),$$

where $\mathcal{I}^+ = \{k \in [1, N] : \gamma_k > 1\}$ and

$$\bar{\psi}(\gamma_k, q) := 2 \frac{\gamma_k}{\gamma_k - 1} \frac{1}{1 - q} \frac{1}{2 - q} \left[\frac{1 - q}{1 - q + \gamma_k q} - \frac{1}{\gamma_k - 1} \ln \left(\frac{\gamma_k}{1 - q + \gamma_k q} \right) \right].$$

Theorem 1.3 provides a fundamental limit on the performance of *any* mechanism in \mathcal{M} . At a high level, the upper bound captures the complexity of the space of mechanisms through a vector $\gamma \in [1, +\infty)^N$ and the space of distributions has been distilled down to a scalar $q \in [0, 1]$. This is in stark contrast with the initial space of mechanisms \mathcal{M} and the space of regular distributions. The sharpness of this upper bound will be apparent in the coming subsections, when we evaluate it and compare it to a lower bound.

The upper bound in Theorem 1.3 also explicitly highlights the tension associated with the design of a prior-free mechanism. On the one hand, one may want to put weight on values $\gamma_k = 1$ to guarantee performance in line with a second price auction, which hedges against deterministic values. This corresponds to the first term in the upper bound, i.e. $1/(2 - q)$. On the other hand, putting weight on terms $\gamma_k > 1$ may yield higher performance if nature selects a distribution with a heavy tail.

Key ideas underlying the proof of Theorem 1.3. The first step in the proof is to derive an upper bound on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$ through Theorem 1.2. Given the latter, the key then is to identify an appropriate subset of distribution \mathcal{W}_{reg} that verifies the conditions of Theorem 1.2 and the rest of the proof is organized around identifying such a subset and explicitly deriving an upper bound on the worst-case performance of any mechanism in \mathcal{M}_{sf}^{max} against \mathcal{W}_{reg} .

The family of distributions for which the revenue curve in the quantile space is a triangle have the following expression

$$F_a(v) = \begin{cases} 1 - \frac{1}{v+1} & , \text{ if } v < a, \\ 1 & , \text{ if } v \geq a, \end{cases} \quad (1.7)$$

for some $a \geq 0$ and has received attention in the literature in various contexts. If we introduce the following class of distribution $\mathcal{W}_{reg} := \{F_a : a > 0\}$, then one can show that each element in this class of distribution \mathcal{W}_{reg} can be approached by a sequence of elements of \mathcal{F}_{reg} (cf. Lemma B.6-2). As a result, $\mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{reg})$ is a valid upper bound for $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$. The proof then relies on deriving an analytical expression for $\mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{reg})$.

1.5.2 Lower Bound

We have just established an upper bound on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$. We next focus on deriving a lower bound.

Proposition 1.4 (Lower bound for regular distributions). *Consider any mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf}^{max} and the corresponding parameters $N \geq 1$, $\gamma \in [1, \infty)^N$ and $c \in [0, 1/2]$. Let $\mathcal{I}^+ = \{k \in [1, N] : \gamma_k > 1\}$. If $|\mathcal{I}^+|/N \leq 1/3$, then the performance of such a mechanism in the presence of two buyers against a distribution F with optimal quantile q_F is lower bounded as follows*

$$R(m, F) \geq \frac{N - |\mathcal{I}^+|}{N} \frac{1}{2 - q_F} + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \underline{\psi}(\gamma_k, q_F),$$

where

$$\underline{\psi}(\gamma, q) := \frac{\gamma}{\gamma - 1} \left(1 - q - \frac{1}{\gamma - 1} \ln \left[\frac{\gamma}{1 + (\gamma - 1)q} \right] \right) \frac{1}{1 - q} - \frac{2\gamma q}{1 + (\gamma - 1)q} \frac{1}{2 - q}.$$

The proposition above gives an explicit lower bound for any mechanism in \mathcal{M}_{sf}^{max} which satisfies $|\mathcal{I}^+|/N \leq 1/3$, i.e., which does not inflate the second price more than a third of the time. In particular, the lower bound admits the same structure as the function characterizing the upper bound up to a correction factor. In particular, it is possible to see that the difference between the upper and lower bounds goes to zero as q approaches zero.

Comparison to the lower bound obtained in Fu et al. (2015). The authors study a mechanism that randomizes between a second price auction and an inflation factor of γ which can be viewed as a special instance of the mechanisms in \mathcal{M}_{sf}^{max} .

For $\gamma = 2$, and using the second price auction with probability $1-p$ and inflation γ with probability p , one may establish that the lower bound obtained in Proposition 1.4 is tighter and higher by a factor of

$$p \frac{2 q_F^2 (1 - q_F)}{(1 + q_F) (2 - q_F)}.$$

The key drivers of the improvement are dual. A first improvement stems from bounding in a dependent fashion the contributions of the second price auction ($\gamma_k = 1$) and that of the inflation mechanisms ($\gamma_k > 1$). A second improvement stems from obtaining a tighter bound on the contributions of high γ_k terms.

1.5.3 Characterization of $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$

We next evaluate numerically values for upper and lower bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg})$. Using Theorem 1.3, we derive an upper bound on the maximin ratio. To that end, we fix $q = 0.17$. For such a value, we have $1/(2 - q) = 54.64\%$. Furthermore, the function $\zeta : (1, +\infty) \rightarrow \mathbb{R}$ defined by

$$\zeta(\gamma) = 2 \gamma \frac{1}{1 - q} \frac{1}{2 - q} \left[\frac{1}{\gamma - 1} \frac{1 - q}{1 - q + \gamma q} - \frac{1}{(\gamma - 1)^2} \ln \left[\frac{\gamma}{1 + (\gamma - 1)q} \right] \right] + \frac{q}{2 - q}$$

reaches its maximum around $\gamma = 1.5$ and its maximal value is 55.59%. From the above, we deduce that maximin ratio is upper bounded by 55.59%.

Applying Proposition 1.4, we evaluate numerically the lower bound by taking $\gamma = (1, 1, 1, 1, 2)$, and a vector q of values from 0 to 1 with a step 0.001. We find that the lower bound is 51.9%. We conclude that

$$51.9\% \leq \mathcal{R}(\mathcal{M}, \mathcal{F}_{reg}) \leq 55.59\%.$$

In other words, we have characterized the maximin ratio up to less than 4%. There is an important implication of the results above. In the face of regular distributions, while randomization is helpful compared to a second price auction (that guarantees 50% of oracle revenues), the extent to which one may improve performance is limited to at most 5.59%. An interpretation of our results is that the second price auction is near-optimal in environments with unknown regular distributions.

1.6 Maximin Ratio for MHR Distributions

In this section, we focus on the maximin ratio when nature can only select distributions in \mathcal{F}_{mhr} , which is a subset of the regular class of distributions \mathcal{F}_{reg} . In other words, the seller now has more information about the distribution of buyers, compared to the setting analyzed in Section 1.5.

1.6.1 Upper Bound

Theorem 1.4 (Upper bound for MHR distributions). *The maximin ratio $\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr})$ is upper bounded as follows*

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}) \leq \inf_{q \in [e^{-1}, 1]} \frac{1 - q^2}{2q(2 - q)\ln(1/q)}.$$

Theorem 1.4 provides a fundamental limit on the performance of any mechanism against distributions in \mathcal{F}_{mhr} . Quite notably, this upper bound comes in quasi-closed form and takes a significantly much simpler form than for the broader class of regular distributions. We next highlight the main ideas in the proof and highlight the role of the MHR knowledge in the derivation of this upper bound.

The proof of this result follows initially the same structure as that of of Theorem 1.3. As earlier, we leverage Theorem 1.2 but now, we use a different family \mathcal{W}_{mhr} , suited to the increasing hazard rate family of distributions \mathcal{F}_{mhr} . In particular, we

define \mathcal{W}_{mhr} to be the set of distribution F parametrized by $a \geq b > 0$ such that

$$F_{a,b}(v) = \begin{cases} 1 - \exp\left(-\frac{v}{a}\right) & , \text{ if } v < b, \\ 1 & , \text{ if } v \geq b. \end{cases}$$

This family is constructed by truncating the exponential family distribution. This family is rich enough to cover the range of all possible optimal oracle quantiles (q_F) of MHR distributions. We establish that any such element can be “approached” by a sequence in \mathcal{F}_{mhr} and in turn, $\mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{mhr})$ is an upper bound on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr})$. The role of the MHR assumption comes into play when we evaluate the performance of any mechanism in \mathcal{M}_{sf}^{max} against \mathcal{W}_{mhr} . In this context, we are able to establish that the optimal performance against \mathcal{W}_{mhr} is given by that of a second price auction. In particular, it is suboptimal to randomize the allocation when facing the family \mathcal{W}_{mhr} .

1.6.2 Lower Bound

We next establish a lower bound on $\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr})$ by lower bounding the performance of a second price auction.

It is worthwhile to note that a first coarse lower bound may be readily obtained from existing results by simply noting that the oracle optimal quantile q_F cannot be less than e^{-1} for MHR distributions; see, e.g., Hartline et al. (2008). Combining this with the lower bound on the performance of a second price auction of $1/(2 - q_F)$ obtained in Fu et al. (2015) for regular distributions, one readily obtains that

$$R(m_{spa}, F) \geq \frac{1}{2 - e^{-1}} \approx 61.2\%.$$

One can already see that a significantly higher performance is possible with the additional knowledge that the distributions belong to the MHR class. Next, we establish a sharp lower bound on $R(m_{spa}, F)$.

Proposition 1.5 (Lower bound for MHR distributions). *For any F in \mathcal{F}_{mhr} , the performance of the second price auction in the presence of two buyers is bounded below as follows*

$$R(m_{spa}, F) \geq \frac{1}{2} \frac{1 - q_F^2}{q_F(2 - q_F)(-\ln(q_F))},$$

where $q_F = 1 - F(r_F)$ is the oracle optimal quantile.

The key idea underlying this result is to leverage the structural properties that the MHR distribution imposes on the structure of the revenue curve in the quantile space. In particular, leveraging a single crossing property between the ccdf of *any* MHR distribution and any exponential tail developed in the reliability theory literature (Barlow and Proschan, 1975), we establish a lower bound on the ccdf of the distribution of any MHR distribution through that of a particular exponential distribution. This leads to a lower bound on the revenue curve in the quantile space, ultimately leading to the bound above.

We discuss the implications of this result next.

1.6.3 Optimality of Second Price Auction and Characterization of $\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr})$

We are now ready to state the main result of Section 1.6, which follows from the two earlier results.

Theorem 1.5 (Optimality of second price auction). *The second price auction is optimal in \mathcal{M} when facing two buyers with MHR distributions. Namely,*

$$\inf_{F \in \mathcal{F}_{mhr}} R(m_{spa}, F) = \mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}).$$

Furthermore

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}) = \inf_{q \in [e^{-1}, 1]} \frac{1 - q^2}{2q(2 - q)\ln(1/q)} \approx 71.53\%.$$

We conclude that the second price auction and an element in \mathcal{W}_{mhr} represent a (quasi) saddle point for the maximin ratio $\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr})$.⁹ Theorem 1.5 provides an exact characterization of the maximin ratio and the corresponding optimal prior-free auction.

Interestingly, while randomization of the allocation helped the seller counter nature when facing regular distributions, such randomization does not help anymore when facing the subclass of monotone hazard rate distributions. It is quite notable that this simple mechanism, a second price auction, which is also efficient, is actually optimal in this environment.

The result above also quantifies the value of additional knowledge about the distributions. If a seller knows that the distribution is MHR, then she gains at least $71.53\% - 55.59\% = 15.94\%$ in guaranteed performance (compared to an oracle). Indeed, MHR distributions have limited variability as measured, e.g., through the coefficient of variation. The latter is bounded by 1 (Barlow and Proschan, 1975) whereas it is unbounded for regular distributions. With such limited variability, a second price auction appears “sufficient.”

1.7 Extensions and Concluding Remarks

We have analyzed the problem of *optimally* selling one indivisible good to two symmetric and independent buyers when one relaxes the common prior assumption. For that, we look at the model where the buyers are not assumed to know any information about the other buyers and the seller does not know the exact distribution. We characterize the maximin ratio for a broad subclass of DSIC mechanisms against the classes of regular and MHR distributions. We refer back the reader to Table 1.1 in Section 1.1 for a summary of some implications of our results. While we have done

⁹Technically speaking, it is not exactly a saddle point given that the elements of \mathcal{W}_{mhr} do not belong to \mathcal{F}_{mhr} .

so while focusing on the case of two buyers, we establish next that the bounds we have derived apply to the case when the number of buyers is selected adversarially.

1.7.1 Extension to the case of an adversarially selected number of buyers

In this section, we will show that our bounds apply to the case in which the number of buyers is arbitrary but adversarially selected.

We assume as earlier that the seller does not know exactly the distribution of values of buyers but knows it belongs to some class of distribution \mathcal{F} in the general class of distributions \mathcal{G} . Moreover, we assume also that the seller does not know the exact the number of buyers $K \geq 2$. We model the seller's problem as a game between the seller and nature, where the seller will first pick a collection of prior-independent mechanisms contingent on the number of bids K , and then nature picks both the number of buyers and their distribution of values from some class.

A seller's mechanism is now a set of allocations and payment functions contingent on the number of bids received $K \geq 2$. The seller will apply a mechanism characterized by an allocation mapping \mathbf{x}^K and a payment mapping \mathbf{t}^K , where $\mathbf{x}^K : \mathbb{R}^K \rightarrow [0, 1]^K$ and $\mathbf{t}^K : \mathbb{R}^K \rightarrow \mathbb{R}^K$. We focus on DSIC mechanisms that verify for any $K \geq 2$

$$v_i x_i^K(v_i, \mathbf{v}_{-i}) - t_i^K(v_i, \mathbf{v}_{-i}) \geq 0, \text{ for all } i \text{ and } v_i, \mathbf{v}_{-i} \text{ in } \mathbb{R}_+^K \quad (\text{IR-K})$$

$$v_i x_i^K(\mathbf{v}) - t_i^K(\mathbf{v}) \geq v_i x_i^K(\hat{v}_i, \mathbf{v}_{-i}) - t_i^K(\hat{v}_i, \mathbf{v}_{-i}), \text{ for all } i \text{ and } \mathbf{v}, \hat{v}_i \text{ in } \mathbb{R}_+^{K+2}, \quad (\text{IC-K})$$

$$\mathbf{x}^K(v_i, \mathbf{v}_{-i}) \text{ belongs to } \Delta^K, \quad \text{for all } v_i, \mathbf{v}_{-i} \text{ in } \mathbb{R}_+^K, \quad (\text{AC-K})$$

where Δ^K is the probability simplex of \mathbb{R}^K . These constraints are similar to those introduced earlier (see (IR), (IC) and (AC)).

More formally, the seller's strategy is a collection of mechanisms from the set $\widetilde{\mathcal{M}}$,

where

$$\begin{aligned} \widetilde{\mathcal{M}} = & \left\{ (\mathbf{x}^K, \mathbf{t}^K)_{K \geq 2} : (\mathbf{x}^K, \mathbf{t}^K) \text{ satisfies (IR-K), (IC-K), (AC-K)} \right. \\ & \left. \text{and } \max_{i=1,2} \{TV(x_i^2, [0, \varepsilon]^2)\} < \infty \text{ for some } \varepsilon > 0 \right\}. \end{aligned}$$

Similarly, we define the oracle benchmark as well as the performance of each mechanism contingent on having K buyers, by

$$\text{opt}^K(F) := \sup_{m \in \mathcal{M}} \mathbb{E}_F \left[\sum_{i=1}^K t_i^K(v_i, \mathbf{v}_{-i}) \right] \text{ and } R^K(m, F) = \frac{\mathbb{E}_F \left[\sum_{i=1}^K t_i^K(v_i, \mathbf{v}_{-i}) \right]}{\text{opt}^K(F)}.$$

In the case of an arbitrary but adversarially selected number of buyers, the objective of the seller is now given by

$$\widetilde{\mathcal{R}}(\widetilde{\mathcal{M}}, \mathcal{F}) = \sup_{m \in \widetilde{\mathcal{M}}} \inf_{K \geq 2} \inf_{F \in \mathcal{F}} R^K(m, F).$$

Next, we state the main result of this section.

Proposition 1.6. *1. The maximin ratio for the regular class of distributions verifies*

$$51.9\% \leq \widetilde{\mathcal{R}}(\widetilde{\mathcal{M}}, \mathcal{F}_{ref}) \leq 55.6\%.$$

2. A second price auction is an optimal mechanism when facing MHR distributions. Furthermore,

$$\widetilde{\mathcal{R}}(\widetilde{\mathcal{M}}, \mathcal{F}_{mhr}) = \inf_{F \in \mathcal{F}_{mhr}} R^2(m_{spa}, F) \approx 71.53\%.$$

Note that a priori it is not clear that the case of two buyers is the worst case, since the oracle benchmark also varies with the number of buyers. In the proof of Proposition 1.6, we show formally that the smallest maximin ratio is achieved when only two buyers participate in the auction. The proof of these results rely fundamentally on the case of two buyers studied earlier in the chapter in conjunction of some known results in the literature. Hence, when nature can pick adversarially any number of buyers $K \geq 2$, a second price auction is still near-optimal against regular distributions and is actually optimal against MHR distributions.

1.7.2 Future directions

A direct and complementary direction would be to characterize the maximin ratio as a function of the number of buyers, when this number cannot be selected adversarially by nature.

Also, in our analysis we have mainly focused on the regular and MHR classes, which are subsets of the α -strongly regular class of distributions; see Ewerhart (2013), Cole and Roughgarden (2014), Cole and Rao (2015) and Schweizer and Szech (2016) for more details about this class of distributions. The results developed in Section 1.4 for the upper bounds have the potential to be applied to any subclass of the α -strongly regular class where α would be a parameter that would capture the degree of knowledge of the seller. As such, an interesting direction is to characterize the maximin ratio as a function of the degree of knowledge of the seller.

Another way to incorporate the knowledge of the seller is to assume that she has access to extra information such as the moments and a potential research question is how one could leverage such additional information to improve the performance and what is the structure of optimal mechanisms in such cases. (We refer the reader to, e.g., Azar et al. (2013) that study deterministic mechanisms that incorporate such information.)

More generally, our work tries to relax the common prior assumption and we have focused on the canonical setting of one indivisible good and symmetric buyers with independent values that are regular. There are various generalizations that naturally emerge. For example, it would be interesting to see if one can develop results of a similar nature when the class of distribution is a “structured” irregular class (see, e.g., Sivan and Syrgkanis (2013) for an example of such a subclass). Similarly, developing parallel lower and upper bounds on the maximin ratio for general environments that would allow, e.g., for correlation among values or asymmetric buyers is a promising direction.

1.8 Additional Notes: Auxiliary Definition

Here, we recall the definition of the Arzelà total variation definition for functions of two variables; see, e.g., Clarkson and Adams (1933).

Definition 1.3. *The Arzelà total variation of a mapping $h : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is given by*

$$TV(h, [a, b] \times [c, d]) := \sup_{N \geq 1} \sup_{\substack{u \in [a, b]^N \\ u_1 \leq \dots \leq u_N}} \sup_{\substack{v \in [c, d]^N \\ v_1 \leq \dots \leq v_N}} \sum_{j=1}^N |h(u_{j+1}, v_{j+1}) - h(u_j, v_j)|.$$

Furthermore, h is said to have finite total variation in $[a, b] \times [c, d]$ if $TV(h, [a, b] \times [c, d]) < \infty$.

Sample-Based Optimal Pricing

2.1 Introduction

Pricing has constituted and continues to be a central decision in a host of industries, ranging from retail to hospitality. In turn, this practical importance has led academic communities in Operations Research, Economics and Computer Science to develop frameworks and algorithm classes for pricing. At the heart of pricing lies a fundamental informational dimension regarding the level of knowledge about customers' willingness-to-pay (WTP).

At an extreme side of the spectrum, if a seller knows the exact WTP of a customer, then pricing is “easy”. If the seller wishes to maximize revenues, the seller may just charge the customer its WTP. Of course, in practice this rarely happens and alternative informational formulations are needed. A first relaxation of the above setting would be that the seller only knows the distribution from which the WTP is drawn. In this case, one may optimize pricing to maximize the expected revenues, leading to the classical setting, which is a building block to a significant number of studies across disciplines. However, such a construct of WTP distribution is rarely available in practice, leading to the basic questions: How should one price without knowledge of the WTP distribution? How should existing data be used to refine pricing decisions? In this regard, there are many levels of information that one could adopt. Ideally, one would like to understand, given a particular amount of data, what should be an *optimal* pricing policy. This is an incredibly challenging problem

from a theoretical perspective. In turn, towards better understanding this question, a first interesting regime is the large data regime in which one may have access to a large number of past observations. In this case, one key question pertains to sample complexity: how many samples would one need to guarantee near-optimal performance. Another regime of interest is the low-data regime, in which the firm has access to very limited information. This regime has dual motivations. On the one hand, in practice, many goods are sold in customer segments with very limited data. On the other hand, from a theoretical perspective, this regime may allow to build from the bottom up towards a general theory for an arbitrary and finite number of samples. The present chapter focuses on this regime.

In more detail, we focus on a seller optimizing her pricing strategy when selling one good to a buyer. The seller does not know the WTP distribution of the buyer. She only knows that it belongs to some broad non-parametric class and in addition, has access to one sample drawn from the same distribution. In particular, the question the seller faces is: what is the optimal pricing strategy given the data available? We adopt a maximin ratio formulation in which performance is measured in comparison to the best the seller could have obtained with complete information on the WTP distribution. The seller optimizes over a broad set of randomized pricing strategies and nature may select any distribution in the class of interest to counter a pricing strategy. We are interested in characterizing the maximin ratio, as well as in deriving insights on the value of the sample and on the structure of near-optimal strategies.

The fundamental problem above, while simple and elegant to state, is seemingly intractable. On the one hand, the set of possible pricing strategies is infinite dimensional and so is the set of possible underlying WTP distributions. Our main contributions lie in developing a novel approach to characterize the structure of near-optimal mechanisms and near-worst case families, and in turn providing crisp upper and lower bounds on the maximin ratio, leading to a characterization of the maximin

ratio over a spectrum of distribution classes. In particular, we focus on α -strongly regular distributions (α -SR) introduced in Cole and Roughgarden (2014). Two special cases are regular distributions ($\alpha = 0$) and non-decreasing monotone hazard rate (MHR) distributions ($\alpha = 1$). The latter two classes include a broad set of distributions that are typically assumed in the pricing and mechanism design literature, including in settings in which the WTP distribution is known.

The present chapter develops a novel and unified approach to characterize jointly the structure of near optimal pricing mechanisms, as well as the structure near-worst case distributions. As a result, the chapter also provides a series of tight upper and lower bounds on the maximin ratio for α -SR distributions. As highlighted earlier, the seller problem is highly intractable. As a first reduction, we show that without loss of optimality, the seller can focus on pricing strategies that randomize over posted prices that are multiplicative of the sample. This reduction enables one to focus on simpler mechanisms, characterized by a single distribution. Leveraging this reduction, we focus on jointly deriving lower bounds as well as impossibility results on the worst-case performance of an optimal mechanism. Note that for a specific mechanism, characterizing worst-case performance is still a priori intractable since nature’s problem is over an infinite dimensional and non-convex set of distributions. One key contribution resides in establishing a lower bound on performance that only depends on a finite number of local quantiles around the optimal one (as opposed to the entire distribution). This is enabled by a succession of structural results on “local” contributions to performance (Proposition 2.2), leading initially to a lower bound that only depends on a countable number of quantiles of the distribution. In turn, a fundamental next step is a dimensionality reduction of the lower bound dependence on the distribution through the development of novel and judiciously constructed dynamic programming recursions (Proposition 2.3 and Proposition 2.4) that capture worst-case performance only through a small number of quantiles. In turn, we derive

an alternative maximin problem that lower bounds the initial problem, and in this alternative problem, nature’s problem is now tractable (Theorem 2.1, Theorem 2.2). Interestingly, the derivation of lower bounds also highlights “hard” cases. Next, by leveraging such families that are meant to counter all mechanisms, we develop impossibility results in the form of an alternative maximin problem that is tractable (Proposition 2.5, Proposition 2.6). Ultimately, as an implication of our results, we obtain for any $\alpha \in [0, 1]$, a lower and upper bound on the maximin ratio but also a near-optimal pricing mechanism and near-worst case distribution. Figure 2.1 depicts the lower and upper bounds we obtain for various values of α . The figure highlights the tightness of our approach. In particular, for all values of α tested, the maximin ratio is characterized up to 1.3%.

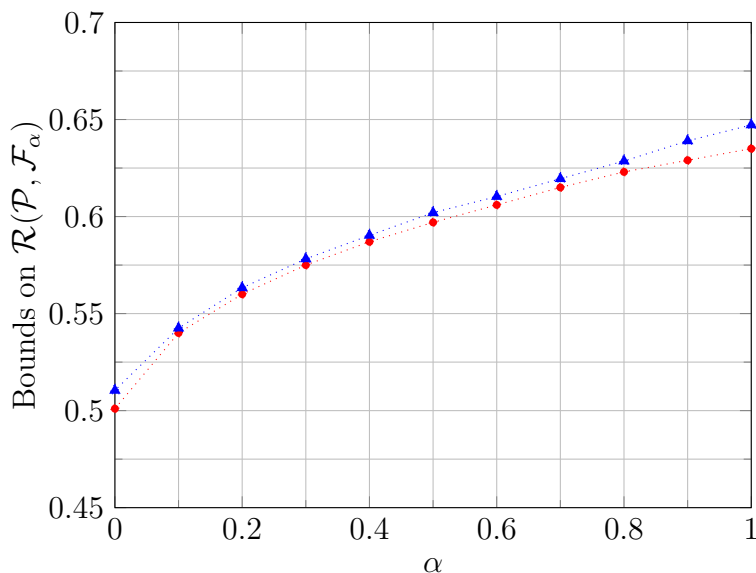


Figure 2.1: **Visualization of the results.** The figure summarizes the upper and lower bounds on the maximin ratio $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ for a grid of values of α . We refer the reader to Table 2.1 for more details.

As a first notable special case for our approach, we develop a new lower bound for regular distributions ($\alpha = 0$). The best known lower bound, established in Fu et al. (2015), is $50\% + 5 \times 10^{-9}$. Our unified approach yields a new lower bound of 50.1%. In addition to improving on the best known lower bound, we provide the

first impossibility results for randomized mechanisms against regular distributions. We establish that no mechanism can achieve a better performance than 51.1%. This impossibility result shows that there is very little room for improvement against regular distributions. In particular, a simple mechanism that just posts the sample is known to achieve a worst-case performance of 50% (Huang et al., 2015). Our impossibility result implies the near-optimality of this simple mechanism against regular distributions.

As a second notable special case of our approach, we develop a lower bound for randomized mechanisms when nature picks from the MHR class of distributions. The best known lower bound for deterministic mechanisms, developed in Huang et al. (2015), is 58.9%. Our approach yields a lower bound of 63.5% through a deterministic mechanism. Also, we provide the first impossibility result for *randomized* mechanisms against MHR distributions. We show that no mechanism can achieve a higher competitive ratio than 64.8%. The conjunction of the lower and upper bounds show how tight the results are.

Furthermore, our approach can be applied to any value of α , leading to the first α -dependent lower bounds and impossibility results for this class of problems. Across all values of α tested, the lower and upper bounds are within 1.3% of each other. This highlights the quality of the bounds developed but also allows to guarantee the near-optimality of the pricing strategies used in the derivation of the lower bounds. In turn, our results suggest that deterministic mechanisms are near-optimal, with losses of at most 1.3%, compared to randomized strategies. For many values of α , the gap is typically smaller than that can be as small as 0.3%.

We believe that the analysis and results may have implications in other applications and also lay some foundation to analyze maximin ratio for the case in which the seller has access to an arbitrary number of samples.

Literature review. While there is a vast literature on pricing across disciplines,

our work relates to the efforts towards relaxing the knowledge of the seller.

A first way to relax informational assumptions is to assume full lack of knowledge of the WTP distribution. Two early studies in that regard are Bergemann and Schlag (2008) and Eren and Maglaras (2010). These study pricing in cases when the seller does not know the underlying distribution of values but only knows the support. In the former, the setting is static, whereas in the latter it is dynamic. These characterize the optimal pricing policy and performance as a function of the support. Building on the framework above in terms of informational assumptions, Caldentey et al. (2016) characterize optimal pricing strategies in a dynamic setting where myopic or strategic customers arrive over time. An intermediary setting is studied in Cohen et al. (2016) in which the seller knows the maximum willingness to pay of customers but does not know the exact demand function. In contrast to these, we study a setting in which the seller has access to some information (a sample), and allow distributions with arbitrary support.

An alternative regime that has received attention is the “large” sample regime, with a first focal question pertaining to sample complexity. How many samples are needed to achieve a particular level of performance compared to an oracle with full information. Huang et al. (2015) have shown that a polynomial number of samples is sufficient to achieve near optimal revenue. In a different line of work, a series of studies look at how to collect and incorporate data on the fly for pricing purposes, in which case an exploration-exploitation trade-off emerges. See Kleinberg and Leighton (2003), Besbes and Zeevi (2009) and Broder and Rusmevichientong (2012), as well as den Boer (2015b) for a review.

In the above, the mode of analysis is asymptotic, understanding performance as the number of samples or time periods grows large. More closely related to our work are recent studies that focus on a regime in between the large sample regime and the no information regime. Huang et al. (2015) and Fu et al. (2015) both analyze the case

when a seller has access to only one sample. Huang et al. (2015) study the performance of deterministic pricing strategies. They establish that post your sample is an optimal strategy (among deterministic ones) against regular distributions and provide lower and upper bounds on the performance of deterministic pricing mechanisms against MHR distributions. In parallel, Fu et al. (2015) have exhibited a randomized pricing mechanism that “beats” any deterministic mechanisms against the regular class of distributions, highlighting the need to expand the set of pricing strategies one focuses on. Building on those chapters, we study the broader class of *randomized* pricing mechanisms, and develop a unified general framework for α -strongly regular distributions, covering as special cases regular and MHR distributions. This general framework allows to develop novel impossibility results, to significantly improve existing lower bounds but also to develop new α -dependent lower bounds. Recently, Babaioff et al. (2018) analyze the setting when the seller has access to two samples. In such a setting, the authors exhibit a deterministic mapping from the two samples to prices that guarantees at least 50.9% of oracle performance.

While we do not assume competition among buyers in the present chapter, our work relates to the growing literature on prior independent auctions in the sense that a seller does not know the exact distribution of buyers’ values. We refer the reader to Hartline (2013) and Roughgarden (2015) for a broader overview of this line of work. In particular, our work relates more closely to Bulow and Klemperer (1996), Dhangwatnotai et al. (2015), Fu et al. (2015) and Allouah and Besbes (2018). At high-level, there are many other problem classes being studied in distribution-free environments; see, e.g., Correa et al. (2019) for a recent contribution in the context of prophet inequalities.

From a methodological perspective, our work builds on the statistical theory of reliability (Barlow and Proschan, 1975) as well as the theory of α -strongly regular distributions introduced in Cole and Roughgarden (2014), and further studied in

Cole and Rao (2015) and Schweizer and Szech (2016) (see also the related class of ρ -concave distributions in Ewerhart (2013)).

2.2 Problem Formulation

We consider a seller trying to sell one indivisible good to one buyer. We assume that the buyer value v is drawn from some distribution F with support included in $[0, \infty)$. We assume that the seller does not know this distribution but has access to one sample s from the same distribution.

The problem we are analyzing is how can the seller leverage the observed sample s to maximize her revenue. More formally, we model the problem as a game between nature and the seller, in which the seller picks a selling mechanism and nature may select a distribution F from which the past sample s and the current value of the buyer v are drawn independently.

The seller selects a randomized posted price mechanism, that is tailored to the sample s observed. In words, a seller's strategy can be interpreted as the set of conditional probability distributions for all possible realizations of s . We denote by $\mathcal{W}(\mathcal{X})$ the set of real-valued functions from a set \mathcal{X} into \mathbb{R} . We denote by \mathcal{D} the set of distributions, i.e., the set of non-decreasing functions from \mathbb{R}^+ into $[0, 1]$ such that the limit at infinity is one. A pricing mechanism Ψ in $\mathcal{W}([0, \infty)^2)$ is a family of distributions $\{\Psi(\cdot|s) \text{ in } \mathcal{D} : s \in [0, \infty)\}$ that specify the cumulative distribution function (cdf) of prices the seller posts conditional on observing sample s .

Definition 2.1. *We say that a pricing mechanism $\Psi(\cdot|\cdot)$ is “well-behaved” around zero if for any $s, p \geq 0$, the function $\Psi(\zeta p|\zeta s)$ admits a limit as $\zeta \downarrow 0$.*

The set of feasible pricing mechanisms \mathcal{P} is defined as follows

$$\mathcal{P} = \{\Psi \text{ in } \mathcal{W}([0, \infty)^2) : \Psi(\cdot|s) \text{ is in } \mathcal{D} \text{ for any } s \in [0, \infty) \\ \text{and } \Psi \text{ is well-behaved around zero}\}.$$

The “well-behaved” condition is purely technical and it is a sufficient condition for the optimization problem we will introduce to be well posed. The expected revenue of the seller using a mechanism Ψ in \mathcal{P} , if nature is selecting a distribution F , is given by

$$\int_0^\infty \int_0^\infty \left[\int_0^\infty p \mathbf{1}\{v \geq p\} dF(v) \right] d\Psi(p|s) dF(s) = \int_0^\infty \left[\int_0^\infty p \bar{F}(p) d\Psi(p|s) \right] dF(s),$$

where $\bar{F} := 1 - F$ is the complementary cumulative distribution function. Note that here, we evaluate the *expected* performance of mechanism Ψ , where the expectation is taken with respect to the value of the buyer v , the sample s as well as the randomization of the pricing strategy.

Oracle benchmark. We define $\text{opt}(F)$ as the maximal performance one could achieve *with knowledge* of the exact distribution of buyer’s values when selecting mechanisms in \mathcal{P} . More formally,

$$\text{opt}(F) := \sup_{p \geq 0} p \bar{F}(p). \quad (2.1)$$

Seller’s objective. For an arbitrary mechanism Ψ in \mathcal{P} , we define its performance against a distribution F such that $\text{opt}(F) > 0$ as follows

$$R(\Psi, F) = \frac{\int_0^\infty \left[\int_0^\infty p \bar{F}(p) d\Psi(p|s) \right] dF(s)}{\text{opt}(F)}.$$

Let \mathcal{G} denote the set of distributions with support included in $[0, \infty)$ with finite and non-zero expectation, i.e.,

$$\mathcal{G} = \{F : [0, \infty) \rightarrow [0, 1] : F \text{ is in } \mathcal{D} \text{ and } 0 < \mathbb{E}_F[v] < \infty\}.$$

Note that $\text{opt}(F)$ is in $(0, \infty)$ for all F in \mathcal{G} , and hence the ratio $R(\Psi, F)$ is well defined for any element of the class \mathcal{G} . The objective of the present chapter is to characterize for classes $\mathcal{F} \subseteq \mathcal{G}$ the maximin ratio

$$\mathcal{R}(\mathcal{P}, \mathcal{F}) = \sup_{\Psi \in \mathcal{P}} \inf_{F \in \mathcal{F}} R(\Psi, F). \quad (2.2)$$

One may easily observe that it is impossible to design any mechanism with a positive competitive ratio when competing against \mathcal{G} (see Cole and Roughgarden (2014)). In turn, we focus on subclasses of \mathcal{G} and we will analyze the maximin ratio against a spectrum of classes of distributions.

Definition 2.2 (α -strong regularity). *Fix α in $[0, 1]$. A cdf F is said to be α -strongly regular (α -SR) on its support \mathcal{S}_F if it admits a density f and if the corresponding virtual value function $\phi_F : v \mapsto v - (1 - F(v))/f(v)$ satisfies $\phi_F(v') - \phi_F(v) \geq \alpha(v' - v)$ for all $v' \geq v$ in \mathcal{S}_F .*

The notion of α -strong regularity was introduced in Cole and Roughgarden (2014). We also refer the reader to Ewerhart (2013) and Schweizer and Szech (2016) for connections to ρ -concavity. Let us introduce the following notation

$$\mathcal{F}_\alpha = \{F \in \mathcal{G} : F \text{ is } \alpha\text{-SR}\}.$$

Two notable special cases are $\alpha = 0$ and $\alpha = 1$. Note that \mathcal{F}_0 is the set of so-called *regular* distributions, i.e., distributions that admit a non-increasing virtual value. The set \mathcal{F}_1 is the set of monotone increasing hazard rate distributions (also referred to as increasing failure rate distributions). Regularity and the monotone increasing hazard rate conditions are very common assumptions across the pricing literature (including in cases in which the distribution is known to the seller). It is worth noting that these non-parametric classes of distributions encompass many widely used and studied parametric classes such as large subsets of Gamma, Beta and other classes of distributions. (see, e.g., Ewerhart (2013)).

It is clear that for any $0 \leq \alpha \leq \alpha' \leq 1$, $\mathcal{F}_{\alpha'} \subseteq \mathcal{F}_\alpha$, and hence $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ is non-decreasing in α . Furthermore, all the classes \mathcal{F}_α are subclasses of the regular class of distributions. In the coming sections, we will be interested in quantifying $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ and characterizing near-optimal mechanisms.

For any distribution F in \mathcal{F}_α , we define $r_F := \arg \max_{p \geq 0} p \bar{F}(p)$. The oracle optimal price r_F is well defined by the regularity assumption of the distribution and the fact that $\mathbb{E}_F[v] < \infty$. Moreover, the optimal revenue is given by

$$\text{opt}(F) = r_F \bar{F}(r_F).$$

Throughout the chapter, whenever a distribution F is defined, we use q_w to denote $\bar{F}(w)$ to lighten the notation. The latter represents the quantile of the distribution at the value w . In most of our analysis, we will leverage working with quantiles (see Bulow and Klemperer (1996) for more detailed economic interpretations of those quantities). We also use the generalized inverse of a distribution F in \mathcal{D} , defined by $F^{-1}(1 - q) := \inf\{v \text{ in } \mathbb{R}^+ \text{ s.t. } F(v) \geq 1 - q\}$ for all q in $[0, 1]$.

2.3 Pricing Mechanisms: Structural Results

2.3.1 Reduction of the Space of Mechanisms

In this section, we show that we can reduce the space of pricing strategies that a seller needs to consider.

Proposition 2.1. *Let $\mathcal{M} \subset \mathcal{P}$ be given by*

$$\mathcal{M} = \left\{ \Psi \in \mathcal{P} : \text{for any } p, s > 0, \Psi(p|s) = \psi \left(\frac{p}{s} \right) \text{ for some } \psi \text{ in } \mathcal{D} \right\}.$$

Then for any α in $[0, 1]$, one can restrict attention to mechanisms in \mathcal{M} without loss of optimality, i.e.,

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) = \mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha),$$

In particular, the result states that while initial pricing mechanisms are characterized by an uncountable collection of distributions (conditional distributions on the sample), it is sufficient to optimize over pricing mechanisms that are characterized by a *single* distribution. To elaborate on what this distribution corresponds to, consider Ψ in \mathcal{M} . The performance against a distribution F in \mathcal{F}_α is given by

$$\begin{aligned}
R(\Psi, F) &= \frac{\int_0^\infty \left[\int_0^\infty p \bar{F}(p) d\Psi(p|s) \right] dF(s)}{\text{opt}(F)} \\
&= \frac{\int_0^\infty \left[\int_0^\infty p \bar{F}(p) d\psi(p/s) \frac{1}{s} \right] dF(s)}{\text{opt}(F)} \\
&= \frac{\int_0^\infty \left[\int_0^\infty \gamma s \bar{F}(\gamma s) d\psi(\gamma) \right] dF(s)}{\text{opt}(F)} \\
&= \frac{\int_0^\infty \left[\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) \right] d\psi(\gamma)}{\text{opt}(F)}. \tag{2.3}
\end{aligned}$$

In other words, a mechanism in \mathcal{M} can be thought of as a mechanism that randomizes, not over the price to post, but over a multiplicative factor to be applied to the observed sample. We use the mnemonic notation \mathcal{M} that refers to the multiplicative nature of the pricing mechanism. Proposition 2.1 above states that restricting attention to such mechanisms is without loss of optimality.

For multiplicative mechanisms, the seller would randomize between different levels of inflation of the sample ($\gamma > 1$) or deflation of the sample ($\gamma \in (0, 1)$) or simply posting the sample ($\gamma = 1$).

Going forward, for any mechanism in \mathcal{M} , with some abuse of notation, we use interchangeably the mechanism $\Psi(\cdot|\cdot)$ and its corresponding object ψ such that $\Psi(p|s) = \psi\left(\frac{p}{s}\right)$. Similarly, we write interchangeably $R(\Psi, F)$ or $R(\psi, F)$.

2.3.2 Approach to bound $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$: Overview

The key challenge in evaluating $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ and designing optimal or near-optimal sample-based pricing policies resides in the fact both \mathcal{P} and \mathcal{F}_α are infinite dimensional spaces. While we have shown that one may reduce attention to mechanisms

in \mathcal{M} , even for any such particular mechanism, it is not clear how to evaluate its worst-case performance. Next, we derive a decomposition of the performance of a mechanism through two steps. First, we decompose the performance of a mechanism through the contributions of deterministic mechanisms in \mathcal{M} . Second, for such mechanisms, we decompose the performance through local contributions stemming from different realizations of the sample. This decomposition will be key in deriving tractable bounds on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$.

For any mechanism ψ in \mathcal{M} , based on Eq.(2.3), one can write

$$R(\psi, F) = \int_0^\infty R(\delta_\gamma, F) d\psi(\gamma), \text{ with } \delta_\gamma(v) := \mathbf{1}\{v \leq \gamma\}, \text{ for all } v \geq 0, \quad (2.4)$$

where δ_γ is a dirac-delta function at γ , in other words, a “deterministic” mechanism that posts the price γs . Furthermore, we have for any $\gamma > 0$,

$$R(\delta_\gamma, F) = \frac{\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s)}{\text{opt}(F)}.$$

Let, for any $0 \leq w \leq w'$,

$$\mathcal{E}(\gamma, w, w'; F) := \int_w^{w'} \gamma s \bar{F}(\gamma s) dF(s). \quad (2.5)$$

The term $\mathcal{E}(\gamma, w, w'; F)$ corresponds to the contribution to the performance stemming from realizations of the sample that are in the interval $[w, w']$ for a deterministic mechanism in \mathcal{M} characterized by the parameter γ . Fix a given non-negative monotone increasing sequence $\{w_i : i \in \mathbb{Z}\}$, such that $\lim_{i \downarrow -\infty} w_i = 0$ and $\lim_{i \uparrow \infty} w_i = +\infty$. Then $\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s)$ may be decomposed into an infinite sum of “local” terms of the form $\mathcal{E}(\gamma, w_j, w_{j+1}; F)$, i.e.,

$$R(\delta_\gamma, F) = \frac{1}{\text{opt}(F)} \sum_{i=-\infty}^{\infty} \mathcal{E}(\gamma, w_i, w_{i+1}; F). \quad (2.6)$$

In what follows, we will derive lower bounds on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ by establishing parametric lower bounds on the worst-case performance of dirac-delta mechanisms $R(\delta_\gamma, F)$. The key to our approach will be a reduction of the complexity of nature’s optimization

problem from an infinite dimensional problem to a finite dimensional problem. This reduction will be enabled by two key steps. On the one-hand a “local” reduction of dimensionality is established in Section 2.4 by focusing on local contributions of the form $\mathcal{C}(\gamma, w, w'; F)$. On the other hand, these local results are leveraged to obtain a global dimensionality reduction through an appropriate decomposition of the form of (2.6) in conjunction with the development of a set of novel judiciously constructed dynamic programs in Section 2.5.

To derive an upper bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ in Section 2.6, we extensively rely on the reduction of mechanisms in \mathcal{M} as well as the intuition gleaned from the lower bound arguments to construct a sufficiently rich family of “hard cases” that yield tractable upper bounds.

2.4 Parametric Lower Bounds on Local Contributions

This section is dedicated to derive lower bounds on local contributions $\mathcal{C}(\gamma, w, w'; F)$ as defined in (2.5), that depend on the distribution F only through a “small” number of local quantile values. To do so, we first establish how local bounds on the tails of distributions translate into bounds on the local performance, and then establish α -dependent local bounds on the tails of distributions in the class \mathcal{F}_α . Finally, combining these two results, we obtain lower bounds on the local contributions.

2.4.1 From tail bounds to lower bounds on local contributions

The next result establishes how one may leverage local bounds on the tails to obtain a lower bound on local performance.

Lemma 2.1. Fix $\alpha \in [0, 1]$ and F in \mathcal{F}_α . Fix $\gamma > 0$ and w, w' such that $0 \leq w < w'$. Suppose that there exists a non-increasing function $H_l : \mathbb{R}_+ \rightarrow [0, 1]$ such that $\bar{F}(v) \geq H_l(v)$ on $[\gamma w, \gamma w']$.

i.) If the revenue function $v\bar{F}(v)$ is non-decreasing on $[\gamma w, \gamma w']$ and there exists a non-increasing function $\tilde{H}_l : \mathbb{R}_+ \rightarrow [0, 1]$ such that $\bar{F}(v) \geq \tilde{H}_l(v)$ on $[w, w']$ and $\tilde{H}_l(w) = \bar{F}(w)$ and $\tilde{H}_l(w') = \bar{F}(w')$, then

$$\mathcal{C}(\gamma, w, w'; F) \geq \int_{q_{w'}}^{q_w} \gamma \tilde{H}_l^{-1}(q) H_l(\gamma \tilde{H}_l^{-1}(q)) dq.$$

ii.) If the function $vH_l(v)$ is non-increasing on $[\gamma w, \gamma w']$ and there exists a non-increasing function $H_u : \mathbb{R}_+ \rightarrow [0, 1]$ such that $\bar{F}(v) \leq H_u(v)$ on $[w, w']$ and $H_u(w) = \bar{F}(w)$, then

$$\mathcal{C}(\gamma, w, w'; F) \geq \int_{q_{w'}}^{q_w} \gamma \min(w', H_u^{-1}(q)) H_l(\gamma \min(w', H_u^{-1}(q))) dq.$$

Part *i.)* of the result establishes that as long as the revenue curve is non-decreasing on the interval $[\gamma w, \gamma w']$ (which, by unimodality, would hold if $w' \leq r_F/\gamma$), then it is sufficient to obtain lower bounds on the tail of the distribution F on the intervals $[w, w']$ and $[\gamma w, \gamma w']$ to obtain a lower bound on the local expected performance.

Part *ii.)* of the result applies to regions where the revenue curve can be lower bounded by a non-increasing function, and intuitively applies to the right of $w \geq r_F/\gamma$. As in the first part, we establish how bounds on the tails translate to bounds on the local expected performance. However, now that the revenue curve can be lower bounded by a non-increasing function, one needs a lower bound on the tail on $[\gamma w, \gamma w']$ and an upper bound on the tail on $[w, w']$.

2.4.2 Local lower and upper bounds on the tail

In the previous section, we have established how lower bounds on local contributions to performance can be derived through suitable *local* lower and upper bounds on the

tails. In this section we establish a systematic way through which such bounds may be derived.

Lemma 2.2. *Fix $\alpha \in [0, 1]$, F in \mathcal{F}_α and a pair of values (w, w') such that $0 \leq w < w'$. Then*

$$\begin{aligned} \bar{F}(v) &\geq q_w \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v-w}{w'-w} \right) && \text{if } w \leq v \leq w', \\ \bar{F}(v) &\leq q_w \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v-w}{w'-w} \right) && \text{if } v \geq w', \end{aligned}$$

where for any $v \geq 0$,

$$\Gamma_\alpha(v) = \begin{cases} (1 + (1 - \alpha) v)^{-1/(1-\alpha)} & \text{if } \alpha \in [0, 1), \\ e^{-v} & \text{if } \alpha = 1. \end{cases}$$

In addition, Γ_α^{-1} is the inverse of Γ_α and we set $\Gamma_\alpha^{-1}(0) := +\infty$ and $\Gamma_\alpha(+\infty) := 0$.

Lemma 2.2 provides a systematic way to obtain local lower and upper bounds on the complementary cumulative distribution function of any distribution in \mathcal{F}_α as a function of $\Gamma_\alpha(\cdot)$. The bound coincides with the original function at the extreme points of the interval $[w, w']$, and provides a lower bound on the interval $[w, w']$ and an upper bound on $[w', +\infty)$ that coincides with the function at w' . Furthermore, the bounds are only parameterized by α and the quantiles at the interval extremes.

The function Γ_α is the complementary cumulative distribution function of a particular Generalized Pareto distribution. The function $\phi_F(v) - \alpha v$ of this distribution is non-decreasing and hence it belongs to \mathcal{F}_α . As a matter of fact, $\phi_F(v) - \alpha v$ is constant, and hence the complementary cumulative distribution function represents in some sense an “extreme” element of the class. In turn, note that these bounds are tight in the sense that there exists an element in \mathcal{F}_α for which all the inequalities are actually equalities.

This result is related to a “single crossing property” studied in the statistical theory of reliability, that is satisfied by the monotone hazard rate class of distributions,

\mathcal{F}_1 . More precisely, (Barlow and Proschan, 1975, Chapter 4, Theorem 2.18) show that for any monotone hazard rate distribution F , if the complementary cumulative distribution function \bar{F} crosses the complementary cumulative distribution function of any exponential distribution then both curves must cross only once. Moreover, if the cross occurs then the exponential tail is always below the tail of F before the cross and above after the cross.

The proof of Lemma 2.2 relies on two steps. The first step of our proof generalizes this result to the α -SR class of distributions by establishing that the family Γ_α plays a similar role for α -SR distributions as the exponential tails do for MHR distributions. In a second step, we apply the single crossing property to a truncated version of the original distribution to the domain $v \geq w$.¹

Figure 2.2 provides a visual illustration of the main implication of Lemma 2.2. Using the same distribution F in \mathcal{F}_α , consider the interval $[w, w']$ such that w corresponds to the quantile $\bar{F}(w) = 0.7$, and w' to the quantile $\bar{F}(w') = 0.4$. An application of Lemma 2.2 with w, w' yields the blue curve and offers a lower bound on $[w, w']$ and an upper bound on $[w', +\infty)$ (dashed line). Furthermore, an application of Lemma 2.2 on $[0, w]$ leads to the red curve and in particular to an upper bound on $[w, w']$ (dashed line).

2.4.3 Parametric lower bounds on local contributions

We are now in a position to develop lower bounds on local contribution $\mathcal{C}(\gamma, w, w'; F)$.

Proposition 2.2. *Fix $\alpha \in [0, 1]$, F in \mathcal{F}_α and $\gamma > 0$. We have*

$$\mathcal{C}(\gamma, w, w'; F) \geq \begin{cases} \gamma w' q_{\gamma w} q_w \mathcal{A}_\alpha \left(\beta_{w/w'}^L \left(\frac{q_{w'}}{q_w}, \frac{q_{\gamma w'}}{q_{\gamma w}} \right) \right), & \text{if } 0 \leq w < w' \leq r_F/\gamma, \\ \gamma w q_{\gamma w} q_w \mathcal{A}_\alpha \left(\beta_{w/w'}^R \left(\frac{q_{w'}}{q_w}, \frac{q_w}{q_{\gamma w}}, \frac{q_{\gamma w'}}{q_{\gamma w}} \right) \right), & \text{if } r_F/\gamma \leq w < w', \end{cases}$$

¹An implication of Lemma 2.2 is that once the function \bar{F} in \mathcal{F}_α is *strictly* below Γ_α then it stays always below. This implication was already noted by Cole and Rao (2015) using a different approach.

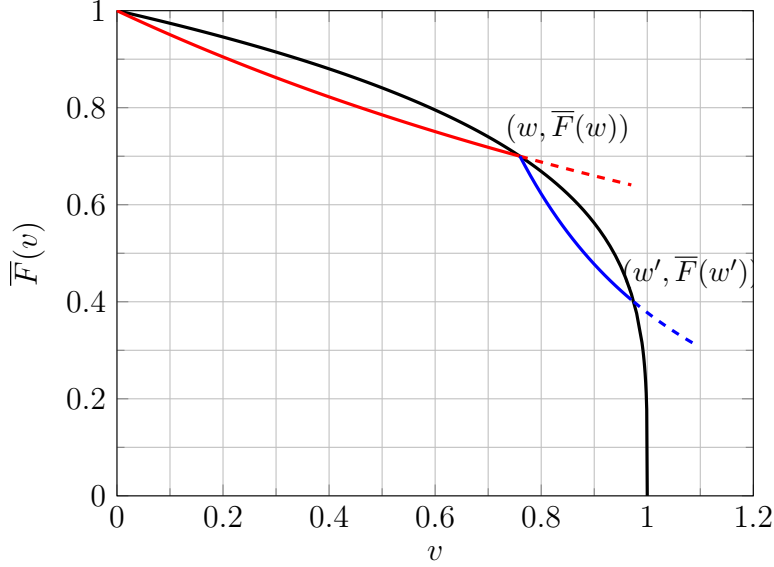


Figure 2.2: **Example of local tail bounds.** The figure depicts some distribution belonging to \mathcal{F}_α with $\alpha = 1/2$, together with local lower (full lines) and upper bounds (dashed lines) obtained through Lemma 2.2.

where $\hat{\gamma} = \frac{w}{w'}$ and for $\sigma > 0$

$$\beta_\sigma^L(\eta_1, \eta_2) := (\eta_1, 1, \eta_1, \sigma, 1, \eta_2, 1, \eta_1, \sigma, 1, \sigma, \sigma), \quad \eta \in \mathbb{R}_+^2$$

$$\beta_\sigma^R(\eta_1, \eta_2, \eta_3) := (\eta_1, \sigma, \eta_2, \sigma, \eta_2, \eta_3, \sigma, \eta_2, \sigma, \eta_2, 1, \sigma), \quad \eta \in \mathbb{R}_+^3$$

$$\mathcal{A}_\alpha(\eta) := \int_{\eta_1}^1 u_\alpha(\eta_2, \eta_3, \eta_4, \eta_5 q) \Gamma_\alpha \left(\Gamma_\alpha^{-1}(\eta_6) \eta_7 \frac{u_\alpha(\eta_2, \eta_8, \eta_9, \eta_{10} q) - \eta_{11}}{1 - \eta_{12}} \right) dq, \quad \eta \in \mathbb{R}_+^{12}$$

$$u_\alpha(\eta) := \min \left(\frac{1}{\eta_1}, \eta_3 + (1 - \eta_3) \frac{\Gamma_\alpha^{-1}(\eta_4)}{\Gamma_\alpha^{-1}(\eta_2)} \right), \quad \eta \in \mathbb{R}_+^4.$$

An important implication of Proposition 2.2 is that the lower bounds developed on the local contributions $\mathcal{C}(\gamma, w, w'; F)$ in both cases only depend on the distribution F through a “small” number of quantiles. Through the lower bound, we have moved from a local contribution that depends on an infinite dimensional object F to a lower bound that only depends on α , as well as the quantiles at the following values $w, w', \gamma w, \gamma w'$ and $\hat{\gamma} w$. The function \mathcal{A}_α coming into play in the lower bound is a tractable function and represents in some sense the “normalized” worst-case contribution driven by the bounds derived in Lemmas 2.1 and 2.2.

We also remark that Proposition 2.2 distinguishes between two cases. Either the

observed sample belongs to a segment which is below r_F/γ or belongs to a segment which is above r_F/γ . Technically, this separation is due to the fact that the revenue curve is non-decreasing on $[\gamma w, \gamma w']$ in case *i*) whereas it is non-increasing in the other case. Intuitively, this separation can be understood as follows: for a deterministic mechanism in \mathcal{M} characterized by a coefficient γ , the value r_F/γ represents the ideal sample that a seller could observe. Hence, if she observed a sample less than r_F/γ then she would be charging a final price under the optimal oracle price. Whereas in the other case if a seller observes a sample higher than r_F/γ then she would be charging over the optimal oracle price. In other words, r_F/γ represents a threshold sample value that separates the types of “errors” the mechanism will, eventually, make. In turn, these two regions require different bounds.

The proof is deferred to Appendix B.3 and leverages the previous results developed in the current section. Concretely, for the first point *i*.), we use the fact that the revenue curve is non-decreasing within the considered interval. Given that, we lower bound the complementary cumulative distribution function \bar{F} using Lemma 2.2 on $[w, w']$ and $[\gamma w, \gamma w']$. In turn, Lemma 2.1 *i*.) leads to the result. The bound given in the latter depends on the inverse of the lower bound on the complementary cumulative distribution function H_l^{-1} . This is exactly what is driving the expression of $u_\alpha(\cdot)$ whereas \mathcal{A}_α^L is driven by the final bound given in Lemma 2.1. The proof of *ii*.) relies on a similar line of arguments. However, we now apply Lemma 2.1 *ii*.) in conjunction with Lemma 2.2.

2.5 Parametric Lower Bounds on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$

Recalling the decomposition for an arbitrary non-negative monotone increasing sequence $\{w_i : i \in \mathbb{Z}\}$, such that $\lim_{i \downarrow -\infty} w_i = 0$ and $\lim_{i \uparrow \infty} w_i = +\infty$ given in (2.6), the performance of any deterministic mechanism characterized by a multiplicative factor

$\gamma > 0$ is given by

$$R(\delta_\gamma, F) = \frac{1}{\text{opt}(F)} \sum_{i=-\infty}^{\infty} \mathcal{C}(\gamma, w_i, w_{i+1}; F),$$

Given the lower bound in Proposition 2.2, we see that if one starts with consecutive elements of the sequence $\{r_F/\tilde{\gamma}^i : i \in \mathbb{Z}\}$ where $\tilde{\gamma} = \min(\gamma, 1/\gamma)$, then all the relevant quantities in the lower bound are always quantiles of elements of the sequence. More formally, we have

$$\begin{aligned} R(\delta_\gamma, F) &= \frac{1}{\text{opt}(F)} \sum_{i: r_F/\tilde{\gamma}^{i+1} \leq r_F/\gamma} \mathcal{C}(\gamma, r_F/\tilde{\gamma}^i, r_F/\tilde{\gamma}^{i+1}; F) \\ &\quad + \frac{1}{\text{opt}(F)} \sum_{i: r_F/\tilde{\gamma}^i \geq r_F/\gamma} \mathcal{C}(\gamma, r_F/\tilde{\gamma}^i, r_F/\tilde{\gamma}^{i+1}; F) \\ &\geq \frac{1}{q_F} \left[\sum_{\substack{i: \\ r_F/\tilde{\gamma}^{i+1} \leq \frac{r_F}{\gamma}}} \frac{\gamma}{\tilde{\gamma}^{i+1}} q_{\gamma w_i} q_{w_i} \mathcal{A}_\alpha \left(\beta_{\tilde{\gamma}}^L \left(\frac{q_{w_{i+1}}}{q_{w_i}}, \frac{q_{\gamma w_{i+1}}}{q_{\gamma w_i}} \right) \right) \right. \\ &\quad \left. + \sum_{\substack{i \\ r_F/\tilde{\gamma}^i \geq \frac{r_F}{\gamma}}} \frac{\gamma}{\tilde{\gamma}^i} q_{\gamma w_i} q_{w_i} \mathcal{A}_\alpha \left(\beta_{\tilde{\gamma}}^R \left(\frac{q_{w_{i+1}}}{q_{w_i}}, \frac{q_{w_i}}{q_{\gamma w_i}}, \frac{q_{\gamma w_{i+1}}}{q_{\gamma w_i}} \right) \right) \right], \end{aligned}$$

where in the last inequality, we have used Proposition 2.2. Under such a decomposition, we now obtain a lower bound on the performance of a deterministic mechanism $R(\delta_\gamma, F)$ that only depends on a countable number of quantiles associated with the underlying distribution F . Next, we develop an approach to obtain a low dimensional lower bound that is based on the construction of judicious dynamic programming recursions and corresponding value functions. We first analyze contributions stemming from “low” realizations of samples, below r_F/γ (Section 2.5.1) and then those stemming from high realizations, above r_F/γ (Section 2.5.2), and then present our main lower bound in Section 2.5.3.

2.5.1 Lower bound for contributions from samples lower than r_F/γ

In this section, we develop a bound that exploits local relationships between a sequence of quantiles associated with values in $[0, r_F/\gamma]$. The key intuition underlying our approach is to “propagate” the local bounds developed in Proposition 2.2, and construct a functional operator such that if one starts from a functional that lower bounds the contribution from samples between 0 and r_F/γ , i.e., $\mathcal{C}(\gamma, 0, r_F/\gamma; F)$, the application of the operator maintains the lower bound structure. In particular, if $\gamma < 1$ (a parallel argument applies if $\gamma > 1$), we assume that we have already developed a lower bound $J(\cdot)$ on the performance on any segment $[0, r_F/\gamma^j]$ with $j \leq 1$. Then we propagate by dividing the interval into a segment $[r_F/\gamma^{j-1}, r_F/\gamma^j]$ and the remainder $[0, r_F/\gamma^{j-1}]$. For the segment $[r_F/\gamma^{j-1}, r_F/\gamma^j]$, we use the bound developed in Proposition 2.2 whereas for the remainder, we use the bound obtained in the previous iteration, i.e., $J(\cdot)$. Then, taking the worst case quantiles around r_F/γ^{j-1} leads a new lower bound defined through a dynamic programming operator.

More formally, let us introduce the following operator $\mathcal{T}_{\alpha, \gamma}^L : \mathcal{W}([0, 1]^2) \rightarrow \mathcal{W}([0, 1]^2)$, that maps bounded functions into bounded functions such that for any (q, ρ^+) in $[0, 1]^2$,

$$\mathcal{T}_{\alpha, \gamma}^L J(q, \rho^+) = \inf_{(q^-, \rho^-) \in \mathcal{B}_{\alpha, \gamma}^L(q, \rho^+)} \{q q^- \mathcal{A}_\alpha(\beta_\gamma^L(\rho^+, \rho^-)) + \tilde{\gamma} J(q^-, \rho^-)\}, \quad (2.7)$$

where $\tilde{\gamma} = \min(\gamma, 1/\gamma)$ and $\mathcal{B}_{\alpha, \gamma}^L$ is defined, for (q, ρ^+) in $[0, 1]^2$,

$$\mathcal{B}_{\alpha, \gamma}^L(q, \rho^+) := \left\{ (q^-, \rho^-) \in [0, 1]^2 : q^- \geq \max \left\{ q, \Gamma_\alpha(\tilde{\gamma} \Gamma_\alpha^{-1}(q)), \Gamma_\alpha(\tilde{\gamma}^2 \Gamma_\alpha^{-1}(\rho^+ q)) \right\} \right. \\ \left. \rho^- \geq \max \left\{ \tilde{\gamma}, \Gamma_\alpha \left(\frac{\tilde{\gamma}}{1 + \tilde{\gamma}} \Gamma_\alpha^{-1}(\rho^+ \rho^-) \right) \right\} \text{ and } \rho^- q^- = q \right\},$$

Intuitively, we operate in the quantile space. The above is a dynamic programming operator that given a reference quantile q and a ratio of quantiles ρ^+ to the left of q , yields a worst-case possible local contribution between the quantile q and 1.

This worst-case possible contribution is obtained from a local contribution between quantile q and a neighboring higher quantile q^- (this contribution is driven by Proposition 2.2 *i.*) and a contribution from q^- to 1, which is driven by our initial input to the operator J .

It is important to note that as worst-case normalized contributions are constructed, the quantiles at play are constrained, and these constraints depend on both γ and the class of distributions that the seller faces through α . We are now in a position to state our result.

Proposition 2.3 (Left normalized contribution). *Fix α in $[0, 1]$, $F \in \mathcal{F}_\alpha$ and $\gamma > 0$ such that $\gamma \neq 1$.*

i.) The functional equation, $\mathcal{T}_{\alpha,\gamma}^L J = J$ admits a unique bounded solution $\mathcal{L}_{\alpha,\gamma}$.

Furthermore, for any bounded function J in $\mathcal{W}([0, 1]^2)$, $(\mathcal{T}_{\alpha,\gamma}^L)^k J$ converges to $\mathcal{L}_{\alpha,\gamma}$ as k grows to ∞ .

ii.) Furthermore

$$\frac{\mathcal{C}(\gamma, 0, r_F/\gamma; F)}{\text{opt}(F)} \geq \begin{cases} \frac{1}{\bar{F}(r_F)} \mathcal{L}_{\alpha,\gamma}(\bar{F}(r_F), \bar{F}(r_F/\gamma)/\bar{F}(r_F)), & \text{if } \gamma < 1 \\ \frac{1}{\bar{F}(r_F)} \mathcal{L}_{\alpha,\gamma}(\bar{F}(r_F/\gamma), \bar{F}(r_F)/\bar{F}(r_F/\gamma)), & \text{if } \gamma > 1. \end{cases}$$

The first point shows that the dynamic program considered admits a fixed point $\mathcal{L}_{\alpha,\gamma}$ and the latter can be obtained through value iteration. The second point establishes that this limit offers a low dimensional functional lower bound on the normalized local contribution $\mathcal{C}(\gamma, 0, r_F/\gamma; F)/\text{opt}(F)$. In particular, quite notably, while the previous object depends on the distribution F , the lower bound depends on F only through two local quantiles, that at the optimal oracle price r_F and that at r_F/γ .

2

²Note that the last proposition provides a lower bound $\mathcal{C}(\gamma, 0, r_F/\gamma; F)$. Given Proposition 2.2 *i.*), another lower bound may be obtained by taking $w = 0$ and $w' = r_F/\gamma$. However, this matches the distribution F only at 0 and r_F/γ , whereas through the local contribution, we are matching the distribution on the sequence r/γ^i , for $i \leq 1$. Hence intuitively, by focusing on the local contributions, the dynamic programming approach is expected to yield a tighter lower bound.

2.5.2 Lower bound for contributions from samples higher than r_F/γ

Focusing on the right side of r_F/γ , to propagate the local lower bound from an interval to $[r_F/\gamma, \infty)$, we apply a similar approach. We assume that we have already developed a lower bound $J(\cdot)$ on the performance on any segment $[r_F/\gamma^j, \infty)$ with $j \geq 1$ then we propagate by dividing the interval into a segment $[r_F/\gamma^j, r_F/\gamma^{j+1})$ and the remainder $[r_F/\gamma^{j+1}, \infty)$ (assuming $\gamma < 1$). For the segment $[r_F/\gamma^j, r_F/\gamma^{j+1})$, we use the bound developed in Proposition 2.2 whereas for the remainder, we use the bound obtained in the previous iteration, i.e., $J(\cdot)$. Then, taking the worst case quantile at r_F/γ^{j+1} leads a new lower bound that is defined through the iteration of a dynamic program.

More formally, let us introduce the following operator $\mathcal{T}_{\alpha, \gamma}^H : \mathcal{W}([0, 1]) \rightarrow \mathcal{W}([0, 1])$, such that for any ρ^- in $[0, 1]$,

$$\mathcal{T}_{\alpha, \gamma}^H J(\rho^-) = \inf_{\rho^+ \in \mathcal{B}_{\alpha, \gamma}^H(\rho^-)} \left\{ \mathbf{1}_{\{\gamma < 1\}} \mathcal{A}_{\alpha}(\beta_{\gamma}^R(\rho^+, \rho^-, \rho^-)) + \frac{1}{\gamma} \rho^+ \rho^- J(\rho^+) \right\}, \quad (2.8)$$

where

$$\mathcal{B}_{\alpha, \gamma}^H(\rho^-) := \{\rho^+ \in [0, 1] : \rho^+ \leq \gamma \text{ and } \rho^+ \rho^- \leq \gamma^2\}.$$

This dynamic program is well defined since the function \mathcal{A}_{α}^R is bounded by $1/\gamma$ and $\rho^+ \rho^- \leq \gamma^2$. This dynamic program formulation captures the intuition outlined above. More precisely, the quantity ρ^+ plays the role of the ratio of quantiles at r_F/γ^{j+1} and r_F/γ^j and ρ^- plays the role of the ratio of quantiles at r_F/γ^j and r_F/γ^{j-1} . Moreover, the first term in the operator $\mathcal{T}_{\alpha, \gamma}^H$ stems from the interval analysis developed in Proposition 2.2 whereas the second stems from the bound of the previous iteration.³ Leveraging this intuition and formulation, we can show the following,

³Note that the operator here is specialized for $\gamma < 1$. For $\gamma > 1$, value iteration converges to zero by construction of the dynamic program. It is possible to construct a recursion tailored to $\gamma > 1$ through a two dimensional dynamic program and such an approach would lead to a lower bound of zero. For exposition purposes, we do not present this here.

Proposition 2.4 (High samples normalized contribution). *Fix α in $[0, 1]$, $F \in \mathcal{F}_\alpha$ and $\gamma > 0$ such that $\gamma \neq 1$.*

i.) The functional equation, $\mathcal{T}_{\alpha, \gamma}^H J = J$ admits a unique bounded solution $\mathcal{H}_{\alpha, \gamma}$ and $(\mathcal{T}_{\alpha, \gamma}^H)^k J$ converges to $\mathcal{H}_{\alpha, \gamma}$ as k grows to ∞ for any bounded J in $\mathcal{W}([0, 1])$.

ii.) We have,

$$\frac{\mathcal{C}(\gamma, r_F/\gamma, \infty; F)}{\text{opt}(F)} \geq \bar{F}(r_F/\gamma) \mathcal{H}_{\alpha, \gamma}(\bar{F}(r_F/\gamma) / \bar{F}(r_F)).$$

Iterations of the dynamic program considered converge to some well defined limit $\mathcal{H}_{\alpha, \gamma}$. Moreover, as in the case of low samples, we are able to derive a lower bound on the worst-case contribution for high values of the sample. Notably, this lower bound captures the dependency of F only through the quantiles at r_F and r_F/γ .

We note here that a related propagation idea appeared in (Huang et al., 2015, Lemma 5.8) where a lower bound is developed for MHR distributions. The propagation there is conducted on constants and the bound derived is not tailored to MHR distributions as the bound applies to any regular distribution. In contrast, the bound we derive propagates functionals on a suitable state-space, and the bound adapts to the value of α under consideration. As we will see, this leads to significantly tighter bounds.

2.5.3 Main lower bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$

We now present the main result for the lower bound. Let us define Δ_N , the simplex of \mathbb{R}^N for a given $N \geq 1$.

Theorem 2.1 (Parametric lower bound). *Fix α in $[0, 1]$, then*

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \geq \sup_{N \geq 1} \sup_{\substack{\zeta \in \Delta_N, \\ \gamma \in (\mathbb{R}^+ \setminus \{1\})^N}} \inf_{\substack{q^* \in Q_\alpha^* \\ \substack{(q^i, \rho^i) \in Q_{\alpha, \gamma_i}(q^*) \\ 1 \leq i \leq N}}} \sum_{i=1}^N \zeta_i \left[\frac{1}{q^*} \mathcal{L}_{\alpha, \gamma_i}(q^i, \rho^i) + q^* \rho^i \mathcal{H}_{\alpha, \gamma_i}(\rho^i) \right],$$

where the functions $\mathcal{L}_{\alpha,\gamma}$ and $\mathcal{H}_{\alpha,\gamma}$ are respectively the unique bounded solutions of the functional equations Eq.(2.7) and Eq.(2.8) and

$$Q_{\alpha}^* := \{q \in [0, 1] \text{ s.t. } q \geq \alpha^{1/(1-\alpha)} \text{ and } q > 0\}$$

$$Q_{\alpha,\gamma}(q^*) := \begin{cases} \{q^*\} \times \mathcal{B}_{\alpha,\gamma}^H(0), & \text{if } \gamma < 1, \\ \mathcal{B}_{\alpha,\gamma}^L(q^*, 0), & \text{if } \gamma > 1. \end{cases}$$

This result has a fundamental implication: the maximin problem $\mathcal{R}(\mathcal{P}, \mathcal{F}_{\alpha})$ can be lower bounded by a sequence of alternative maximin problems parameterized by N . In an alternative maximin problem, the seller selects among multiplicative mechanisms that puts mass on at most N parameters. And nature, rather than selecting from an infinite dimensional space of distributions, now selects from a space with $N + 1$ dimensions. Nature's problem is now one of selecting the quantile at the optimal oracle price r_F as well as N mechanism-specific quantiles.

The bound above is obtained by combining the results of Proposition 2.3 and Proposition 2.4, while also imposing further constraints on the parameters of the functionals in these two results. Such constraints are captured through the sets Q_{α}^* and $Q_{\alpha,\gamma}^*$.

2.6 Parametric Upper Bounds on $\mathcal{R}(\mathcal{P}, \mathcal{F}_{\alpha})$

In this section, we introduce a general class of distributions motivated by the lower bound analysis. In particular, the bounds derived on the tails of any α -SR distribution, see Lemma 2.2, implied that the lower bounds derived were near tight for such piece-wise generalized Pareto distributions.

For fixed $0 \leq \alpha \leq 1$, let us introduce the family of parametric distributions

$F_\alpha(\cdot|q_0, q_1, \bar{v})$ given by

$$\bar{F}(v|q_0, q_1, \bar{v}) = \begin{cases} \Gamma_\alpha(\Gamma_\alpha^{-1}(q_0)v), & \text{if } 0 \leq v < 1, \\ q_0 \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right)(v-1)\right), & \text{if } 1 \leq v < \bar{v}, \\ 0, & \text{if } v \geq \bar{v}. \end{cases} \quad (2.9)$$

with $0 \leq q_0 \leq q_1 \leq 1$ and $\bar{v} > 1$. This family is characterized by two pieces, one from 0 to 1 and one from 1 to \bar{v} and may allow for a mass at \bar{v} . Such piece-wise Γ_α distributions have the shape of the lower bound in solid lines depicted in Figure 2.2. At an intuitive level, the first piece from 0 to 1 counters mechanisms that put too much weight on deflation, while the second piece from 1 to \bar{v} , in conjunction potentially with a mass at \bar{v} counters mechanisms that put too much mass on inflation.

Since such families (and generalizations of such families) were indirectly one of the key building blocks of lower bounding the performance of deterministic mechanism against any α -SR distributions, one would expect that the exact performance of any mechanism against this family would lead to performance that has significant common structure with the lower bounds derived.

Let us introduce two quantities $\bar{\mathcal{L}}_{\alpha,\gamma}$ and $\bar{\mathcal{H}}_{\alpha,\gamma}$ that play similar roles as $\mathcal{L}_{\alpha,\gamma}$ and $\mathcal{H}_{\alpha,\gamma}$ in the lower bound. We define

$$\bar{\mathcal{L}}_{\alpha,\gamma}(q, \rho) := \begin{cases} \gamma \mathcal{A}_\alpha(q, 0^+, q, 0, 1, q, \gamma, q, 0, 1, 0, 0) \\ \quad + 2q\gamma \mathcal{A}_\alpha\left(\frac{\tilde{q}_1(q, \rho, \gamma)}{q}, 0^+, \rho, \frac{1}{2}, 1, q, 2\gamma, \rho, \frac{1}{2}, 1, 0, 0\right), & \text{if } \gamma \leq 1, \\ \frac{\gamma}{q} \mathcal{A}_\alpha(\tilde{q}_1(q, \rho, \gamma), 0^+, q, 0, 1, q, \gamma, q, 0, 1, 0, 0), & \text{if } \gamma > 1, \end{cases}$$

with

$$\tilde{q}_1(q, \rho, \gamma) = \begin{cases} \Gamma_\alpha((1/\gamma) \Gamma_\alpha^{-1}(q)), & \text{if } \gamma > 1, \\ q \Gamma_\alpha\left(\frac{1-\gamma}{\gamma} \Gamma_\alpha^{-1}(\rho)\right), & \text{if } \gamma \leq 1. \end{cases}$$

The quantity $\bar{\mathcal{L}}_{\alpha,\gamma}$ is driven by the same object that was central to the derivation of $\mathcal{L}_{\alpha,\gamma}$ in the lower bound, namely \mathcal{A}_α . In particular, when $\gamma < 1$, the expression

above is characterized by two terms, driven by the contribution from 0 to the optimal oracle price r_F and a second term stemming from the local contribution from r_F to r_F/γ . These two contributions would correspond exactly to the lower bound of the local contribution against any distribution on $[0, r_F]$ and $[r_F, r_F/\gamma]$. In contrast to the lower bound analysis, we do not divide further $[0, r_F]$ into small contributions.

Similarly, we define a functional for the contribution of high values of the sample.

$$\overline{\mathcal{H}}_{\alpha, \gamma}(q, \rho) := \begin{cases} 2\gamma \frac{\tilde{q}_1(q, \rho, \gamma)}{q\rho} \mathcal{A}_\alpha \left(0, 0^+, \rho, \frac{1}{2}, \frac{\tilde{q}_1(q, \rho, \gamma)}{q}, \rho, \gamma, \rho, \frac{1}{2}, \frac{\tilde{q}_1(q, \rho, \gamma)}{q}, \frac{1}{2}\gamma, \frac{1}{2} \right), & \text{if } \gamma \leq 1, \\ \frac{\gamma \tilde{q}_1(q, \rho, \gamma)}{\rho} \mathcal{A}_\alpha \left(\frac{\tilde{q}_1(q, \rho, \gamma)}{q}, 0^+, q, 0, \tilde{q}_1(q, \rho, \gamma), \rho, \frac{\gamma}{2}, q, 0, \tilde{q}_1(q, \rho, \gamma), 1/\gamma, \frac{1}{2} \right) \\ \quad + 2\frac{\gamma}{\rho} \mathcal{A}_\alpha \left(0, 0^+, \rho, \frac{1}{2}, 1, \rho, \gamma, \rho, \frac{1}{2}, 1, \frac{1}{2}\gamma, \frac{1}{2} \right), & \text{if } \gamma > 1. \end{cases}$$

Proposition 2.5. *The maximin ratio is upper bounded as follows*

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \leq \sup_{\psi \in \mathcal{M}} \inf_{(q^*, \rho) \in \bar{\mathcal{Q}}_\alpha} \mathbb{E}_\psi \left[\frac{1}{q^*} \overline{\mathcal{L}}_{\alpha, \gamma}(q^*, \rho) + q^* \rho \overline{\mathcal{H}}_{\alpha, \gamma}(q^*, \rho) \right],$$

with

$$\bar{\mathcal{Q}}_\alpha := \left\{ (q^*, \rho) \in [0, 1]^2 : q^* \geq \alpha^{1/(1-\alpha)}, (1-\alpha) + \frac{1}{\Gamma_\alpha^{-1}(q^*)} \geq \frac{1}{\Gamma_\alpha^{-1}(\rho)} \text{ and } \rho \leq \Gamma_\alpha(1) \right\}.$$

This result implies that one may upper bound the maximin ratio $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ by an alternative maximin ratio and the latter has a very similar structure as the maximin ratio that lower bounded $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ (presented in Theorem 2.1). In particular, in the new maximin problem, nature's problem is now two-dimensional.

The proof of this result consists of three steps. First, we consider a subset of family of distributions at hand where we set the upper support \bar{v} to ∞ and show that under the condition $(1-\alpha) + \frac{1}{\Gamma_\alpha^{-1}(q^*)} \geq \frac{1}{\Gamma_\alpha^{-1}(\rho)}$ the distribution is α -SR. In the second step, leveraging α -SR, we show that under the second condition $\rho \leq \Gamma_\alpha(1)$, the reserve price is necessarily at $v = 1$. In the last step, we compute the lower bound based on the explicit expressions of the distributions considered. Finally, the constraint on q^* stems from the fact that F is in \mathcal{F}_α .

We note here that Cole and Roughgarden (2014) leverage a special case of the family introduced in this section to quantify the worst case in a large sample regime analysis. In the present case, the richer piece-wise structure emerges naturally from our lower bound analysis.

2.7 Maximin ratio characterization: impossibility results and near-optimal mechanisms

2.7.1 Tractable characterization of bounds

As mentioned earlier, the maximin ratio optimization problem is highly intractable. The goal of the previous sections was to reduce the complexity of the problem by bounding the maximin ratio through optimization problems over finite dimensional spaces. Despite the notable complexity reduction, the optimization problems defined are still intractable, involving dynamic programs over continuous spaces. A first goal of the next sections is to develop tractable and provable bounds through a judiciously chosen discretization of the space of quantiles and/or mechanisms. Given these tractable bounds, we are then in a position to derive jointly values for upper and lower bounds on the maximin ratio, as well as near-optimal mechanisms and near-worst case families of distributions.

2.7.1.1 Upper bounds

In this section, we derive readily computable upper bounds based on the family introduced in Section 2.6. For that, we need to ensure a valid upper bound for *all* mechanisms. For fixed $\varepsilon \geq 0$ and $M > 0$, we define a grid

$$\mathcal{G}_\varepsilon^M = \begin{cases} [0, M] & \text{if } \varepsilon = 0, \\ \{k\varepsilon : 1 \leq k \leq \lfloor M/\varepsilon \rfloor\} \cup \{M\} & \text{if } \varepsilon > 0. \end{cases}$$

Proposition 2.6. Fix α in $[0, 1]$. For any $\varepsilon > 0$ and $M > 1$, for any distribution F in the set $\{\bar{F}(\cdot|q_0, q_1, \bar{v}) \text{ s.t. } \bar{v} > 1 \text{ and } (q_0, q_1/q_0) \in \bar{\mathcal{Q}}_\alpha\}$, we have

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \leq \max \left[\max_{\gamma \in \mathcal{G}_\varepsilon^M} \left(\frac{1}{q_0} \mathbb{E}_F [\gamma v \bar{F}((\gamma - \varepsilon)v)] \right), (1 - \bar{F}(1/M)) + R(\delta_M, F) \right].$$

Proposition 2.6 enables one to compute a family of tractable upper bounds. For that, one just needs to select ε , M and a distribution in the parametric family, and evaluate the right-hand-side above. There are three key steps behind this result. First, we leverage the reduction shown in Proposition 2.1. Second, we show that if we take an element from the family $F(\cdot|q_0, q_1, \bar{v})$ introduced in Section 2.6, one can show that it is approachable by a sequence of distributions in \mathcal{F}_α . Third, leveraging the discretization in the space of multiplicative factors using the grid, we bound the performance of any deterministic mechanism by a tractable function.

2.7.1.2 Lower bounds

While the lower bound presented in Theorem 2.1 provides a theoretical lower bound, it is not a priori possible to evaluate it exactly as it requires computing the value function over a continuous state-space. To obtain computationally tractable lower bounds, we generalize here the bound of Theorem 2.1 in order to obtain a family of tractable lower bounds that only require operating in a finite space. The key idea is to discretize the quantile space *while maintaining the lower bound structure*, which leads to dynamic programs over a finite state-space.

For $\varepsilon > 0$, we define a grid of the quantile space $[0, 1]$ over which we will evaluate the lower bound,

$$\mathcal{G}_\varepsilon = \begin{cases} [0, 1] & \text{if } \varepsilon = 0, \\ \{k \varepsilon : 1 \leq k \leq \lfloor 1/\varepsilon \rfloor\} \cup \{1\} & \text{if } \varepsilon > 0. \end{cases}$$

First, we define counterparts of $\mathcal{T}_{\alpha, \gamma}^L$ and $\mathcal{T}_{\alpha, \gamma}^H$ in a discretized finite state-space.

Normalized contributions of low samples. Let $\gamma \in \mathbb{R}_{++} \setminus \{1\}$. We first define a “generalized version” of β^L to account for values of $\varepsilon > 0$. In particular we define $\hat{\beta}_{\gamma,\varepsilon}^L : \mathbb{R}^2 \rightarrow \mathbb{R}^{12}$ as

$$\hat{\beta}_{\gamma,\varepsilon}^L(\eta_1, \eta_2) := \begin{cases} (\eta_1, \gamma, \eta_1 - \varepsilon, \gamma, 1, \eta_2 - \varepsilon, 1, \eta_1, \gamma, 1, \gamma, \gamma), & \text{if } \gamma \in (0, 1) \\ (\eta_2, \tilde{\gamma}, \eta_2 - \varepsilon, \tilde{\gamma}, 1, \eta_1 - \varepsilon, 1, \eta_2, \tilde{\gamma}, 1, \tilde{\gamma}, \tilde{\gamma}), & \text{if } \gamma > 1, \end{cases}$$

where $\tilde{\gamma} = \min\{\gamma, \gamma^{-1}\}$.

Let us introduce the following operator $\mathcal{T}_{\alpha,\gamma,\varepsilon}^L : \mathcal{W}(\mathcal{G}_\varepsilon^2) \rightarrow \mathcal{W}(\mathcal{G}_\varepsilon^2)$, such that for any (q, ρ^+) in $\mathcal{G}_\varepsilon^2$,

$$\mathcal{T}_{\alpha,\gamma,\varepsilon}^L J(q, \rho^+) = \inf_{(q^-, \rho^-) \in \mathcal{B}_{\alpha,\gamma,\varepsilon}^L(q, \rho^+)} \left\{ (q - \varepsilon)(q^- - \varepsilon) \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma,\varepsilon}^L(\rho^+, \rho^-) \right) + \tilde{\gamma} J(q^-, \rho^-) \right\} \quad (2.10)$$

where for any (q, ρ^+) in $\mathcal{G}_\varepsilon^2$, $\mathcal{B}_{\alpha,\gamma,\varepsilon}^L(q, \rho^+) \subset \mathcal{G}_\varepsilon$ is defined as

$$\begin{aligned} \mathcal{B}_{\alpha,\gamma,\varepsilon}^L(q, \rho^+) := & \left\{ (q^-, \rho^-) \in \mathcal{G}_\varepsilon^2 : \right. \\ & q^- \geq \max \left\{ q, \Gamma_\alpha(\tilde{\gamma} \Gamma_\alpha^{-1}(q - \varepsilon)), \Gamma_\alpha(\tilde{\gamma}^2 \Gamma_\alpha^{-1}((\rho^+ - \varepsilon)(q - \varepsilon))) \right\} \\ & \rho^- \geq \max \left\{ \tilde{\gamma}, \Gamma_\alpha \left(\frac{\tilde{\gamma}}{1 + \tilde{\gamma}} \Gamma_\alpha^{-1}((\rho^+ - \varepsilon)(\rho^- - \varepsilon)) \right) \right\} \\ & \left. \text{and } (\rho^- - \varepsilon)(q^- - \varepsilon) \leq q \leq \rho^- q^- \right\}. \end{aligned}$$

We note here that when $\alpha = 0$, the fact that the optimal quantile can be arbitrarily close to zero leads to a challenge in the evaluation of the lower bound. In particular, the discretized dynamic program for low values of the sample always leads to a value of zero for $\mathcal{T}_{\alpha,\gamma,\varepsilon}^L J$ for the lowest quantile in the grid if one starts with an initial value of zero. To counter this, we will develop an alternative computationally tractable lower bound that does not rely on the operator $\mathcal{T}_{\alpha,\gamma,\varepsilon}^L$. To that end, we define for all $(q, \rho) \in \mathcal{G}_\varepsilon^2$,

$$\tilde{\mathcal{L}}_{\alpha,\gamma,\varepsilon}(q, \rho) = \begin{cases} \frac{1-\alpha}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{q^*}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q)) dq \\ \quad + \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q^*)) \mathcal{A}_\alpha(\hat{\beta}_{\gamma,\varepsilon}^L(\rho, \gamma)), & \text{if } \gamma < 1 \\ \frac{1-\alpha}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{\min\{\gamma q^*, 1\}}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha \left(\frac{\Gamma_\alpha^{-1}(q^*)}{\Gamma_\alpha^{-1}(\gamma q^*)} \Gamma_\alpha^{-1}(q) \right) dq, & \text{if } \gamma > 1 \end{cases}$$

For $\gamma < 1$, the latter function is obtained by loosely bounding the local contributions of the segment $[0, r_F]$ and $[r_F, r_F/\gamma]$. For α very close to 0, especially for the regular family, the bound $\widetilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon}$ leads to a tractable and non-trivial bound.

Normalized contributions of high samples. Define a form of generalization of β_γ^R by $\hat{\beta}_{\gamma, \varepsilon}^R : \mathbb{R}^2 \rightarrow \mathbb{R}^{12}$ as $\hat{\beta}_{\gamma, \varepsilon}^R(\eta_1, \eta_2) := (\eta_1, \gamma, \eta_2 - \varepsilon, \gamma, \eta_2, \eta_2 - \varepsilon, \gamma, \eta_2, \gamma, \eta_2 - \varepsilon, 1, \gamma)$.

We now introduce the following operator $\mathcal{T}_{\alpha, \gamma, \varepsilon}^H : \mathcal{W}(\mathcal{G}_\varepsilon) \rightarrow \mathcal{W}(\mathcal{G}_\varepsilon)$, such that for any ρ^- in \mathcal{G}_ε ,

$$\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J(\rho^-) = \inf_{\rho^+ \in \mathcal{B}_{\alpha, \gamma, \varepsilon}^H(\rho^-)} \left\{ \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R(\rho^+, \rho^-) \right) + \frac{1}{\gamma} (\rho^+ - \varepsilon) (\rho^- - \varepsilon) J(\rho^+) \right\} \quad (2.11)$$

where $\mathcal{B}_{\alpha, \gamma, \varepsilon}^H(\rho^-) := \{\rho^+ \in \mathcal{G}_\varepsilon : \rho^+ \leq \gamma + \varepsilon \text{ and } (\rho^+ - \varepsilon)(\rho^- - \varepsilon) \leq \gamma^2\}$.

Computationally tractable lower bounds. We are now in a position to state our result.

Theorem 2.2. *Fix $\alpha \in [0, 1]$. For any $\varepsilon \in [0, 1)$ and integer $k > 0$. Let J_1 be the zero function in $\mathcal{W}(\mathcal{G}_\varepsilon)$ and J_2 be the zero function in $\mathcal{W}(\mathcal{G}_\varepsilon^2)$. Then, for any multiplicative mechanism defined by $N \geq 1$, $\gamma \in \mathbb{R}_{++}^N$ and ζ in the simplex of \mathbb{R}^N ,*

$$\begin{aligned} \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \geq \inf_{\substack{q^* \in Q_{\alpha, \varepsilon}^* \\ (q^i, \rho^i) \in Q_{\alpha, \gamma_i, \varepsilon}(q^*) \\ 1 \leq i \leq N}} \sum_{i=1}^N \zeta_i \left\{ \max \left\{ \frac{1}{q^*} (\mathcal{T}_{\alpha, \gamma_i, \varepsilon}^L J_2)^k(q^*, \rho), \widetilde{\mathcal{L}}_{\alpha, \gamma_i, \varepsilon}(q^*, \rho) \right\} \right. \\ \left. + (q^* - \varepsilon) (\rho - \varepsilon) (\mathcal{T}_{\alpha, \gamma_i, \varepsilon}^H J_1)^k(\rho) \right\}, \end{aligned}$$

where

$$\begin{aligned} Q_{\alpha, \varepsilon}^* &:= \{q \in \mathcal{G}_\varepsilon \text{ s.t. } q \geq \alpha^{1/(1-\alpha)} \text{ and } q > 0\} \\ Q_{\alpha, \gamma, \varepsilon}(q^*) &:= \begin{cases} \{q^*\} \times \mathcal{B}_{\alpha, \gamma, \varepsilon}^H(\varepsilon), & \text{if } \gamma < 1, \\ \mathcal{B}_{\alpha, \gamma, \varepsilon}^L(q^*, \varepsilon), & \text{if } \gamma > 1. \end{cases} \end{aligned}$$

In other words, the above result provides, for any given α , a family of lower bounds on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ parametrized by k and ε and a mechanism in \mathcal{M} . To obtain a computable and valid lower bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$, it suffices to select a mechanism in

\mathcal{M} , apply the operators $\mathcal{T}_{\alpha,\gamma,\varepsilon}^L$ and $\mathcal{T}_{\alpha,\gamma,\varepsilon}^H$ k times starting from the null functions and then take the minimum over the values of q^* and ρ_i in $Q_{\alpha,\varepsilon}^*$.

We note that Theorem 2.2 provides a generalization of Theorem 2.1. While the bound for $\varepsilon = 0$ is tighter, the result provides a family of tractable bounds through positive values of ε .

2.7.2 Impossibility Results and Near-Optimal Mechanisms

Now we are in position to derive provable lower and upper bounds on the maximin ratio for any value of α in $(0, 1]$. To get the tightest possible bounds as well as the structure of near-optimal mechanisms and near-worst case families, we will jointly evaluate the upper and lower bounds. In particular, we use the following procedure.

1) Using Proposition 2.6, for each value of α , by using a grid of the parameters characterizing the set introduced in the proposition, we find the regions of the parameters that gives the lowest upper bound. This provides an upper bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$. 2) Against this worst case family using Proposition 2.6 we find the best multiplicative factor(s) γ . 3) Using these best multiplicative factors, we evaluate the worst case performance using Theorem 2.2, leading to a lower bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$.⁴

In Table 2.1, we report the provable bounds obtained by following the approach described above as well as the structure of the near-optimal pricing strategies and the near-worst case distributions. For the upper bound, we select $M = 30$ and a discretization parameter $\varepsilon = 5 \times 10^{-4}$ and for the lower bound, we use a discretization parameter $\varepsilon = 2.5 \times 10^{-4}$ and a number of iterations of k such that the value iteration errors are in absolute value less than 10^{-4} .

There are multiple insightful remarks that emerge from Table 2.1. First, quite

⁴Note that for $\alpha = 0$, our impossibility results will imply that a near-optimal randomized strategy is the deterministic strategy of posting the sample that yields 50%. However, we will also explore randomized strategies that put some weight on inflation (in Appendix B.1, we prove that this is necessary against regular distributions) to improve upon the best known randomized performance.

α	Pricing mechanism	Lower bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$	Upper bound on $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$	Distribution parameters		
				\bar{v}	q_0	q_1
1	$\delta_{0.76}$	63.5%	64.8%	∞	0.450	0.077
0.9	$\delta_{0.78}$	62.9%	63.9%	∞	0.440	0.076
0.8	$\delta_{0.80}$	62.3%	62.9%	∞	0.425	0.074
0.7	$\delta_{0.81}$	61.5%	62.0%	∞	0.405	0.070
0.6	$\delta_{0.82}$	60.6%	61.1%	∞	0.385	0.066
0.5	$\delta_{0.84}$	59.7%	60.2%	∞	0.360	0.061
0.4	$\delta_{0.86}$	58.7%	59.1%	∞	0.330	0.055
0.3	$\delta_{0.88}$	57.5%	57.9%	∞	0.290	0.047
0.2	$\delta_{0.91}$	56.0%	56.4%	∞	0.235	0.036
0.1	$\delta_{0.94}$	54.0%	54.3%	∞	0.165	0.022
0	$0.99 \delta_{0.99} + 0.01 \delta_3$	50.1%	51.1%	1.75	0.110	0.009

Table 2.1: **Lower and Upper Bounds** on the maximin ratio $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ for different values of α . We also report the parameters of the mechanisms used for the lower bounds and the distributions used in the upper bounds.

strikingly, our joint approach yields a very tight characterization of the maximin ratio with the difference between the upper and lower bounds ranging from 0.3% up to at most 1.3% across different values of α . In particular, the results demonstrate that the family of distributions that we have constructed yield a near-worst case performance against *all* mechanisms. As mentioned earlier, these “hard” cases emerged naturally in our analysis of the lower bound. To the best of our knowledge, this class of “hard” distributions were not introduced in previous literature.

It is worth noting that, to date, to the best of our knowledge, only upper bounds on *deterministic* mechanisms were known in the literature. In particular, for regular distributions, it is known that no deterministic mechanism can have better competitive ratio than 50% and for MHR distributions, it has been shown that no deterministic mechanism can have a better competitive ratio than 68% (Huang et al., 2015). The results in the present chapter are the first upper bounds on the more general class of *randomized* mechanisms. For the regular class, our bound establishes that while no

deterministic mechanism can do better than 50%, no randomized mechanism can do better than 51.1%. The value of randomization cannot be too large against regular distributions. Against MHR distributions, we show that no randomized mechanism can do better than 64.8%.

It is also worth noting that the best lower bound to date for the case of MHR distributions ($\alpha = 1$) was 58.9% as presented in Huang et al. (2015). Our approach leads to a significantly improved lower bound of 63.5% for MHR distributions. Furthermore, our approach is not tailored to MHR distributions but also leads to a lower bound for an arbitrary value of α in $[0, 1]$. In particular, our bound for the regular case ($\alpha = 0$), is 50.1%, whereas the best lower bound in the literature was $50\% + 5 \times 10^{-9}$ given by Fu et al. (2015).

On the value of randomization. At a higher level, our results also have implications on the value of randomization against α -SR distributions. The conjunction of the lower and upper bounds developed (together with the structure of near-optimal mechanisms) implies that the value of randomization is limited. For all the values of α considered, there always exists a deterministic mechanism in \mathcal{P} that is within 1.3% of the optimal randomized mechanism.

On the value of one sample. Our uniform analysis across different values of α allows to understand the value of a sample as a function of the class of distribution. In particular, our results imply that there is a gain of at least 12.4% from the regular case ($\alpha = 0$) to the MHR case ($\alpha = 1$).

On the structure of near-optimal mechanisms. Furthermore, we are able to characterize the structure of near-optimal mechanisms. As we observe, for all values of $\alpha > 0$, a near-optimal mechanism is one that deflates the sample, and the amount of deflation appears to increase as α increases. This is in line with the fact that the family of distributions contains distribution with lighter tails as α increases. Finally, it is apparent that the structure of near-optimal mechanisms

varies a lot as a function of α . While the approach above leads to sharp results in a constructive fashion, it is possible to derive structural results that prove that fundamentally different mechanisms are needed for the various classes considered. In particular, it is possible to show that against regular distributions, any mechanism that does not put weight on inflation ($\gamma > 1$) is necessarily suboptimal. In other words, inflation is necessary against regular distribution. In stark contrast, against MHR distributions, it is possible to show that one may restrict attention without loss of optimality to mechanisms that do not put any weight on inflation. We provide a formal statement of this result in Proposition B.1-1, along with a discussion on the interplay of inflation and deflation in Appendix B.1.

2.8 Conclusion

In the present chapter, we analyze the fundamental problem of optimal pricing when the seller does not know the exact distribution of values of the buyer but has access to one sample from that unknown distribution. We follow a competitive ratio approach where the seller picks a pricing mechanism, potentially randomized, to maximize the worst case fraction of revenue generated compared to the oracle optimal revenue, when the distribution of values is α -strongly regular. For this problem, we provide a unified tractable approach to analyze the maximin ratio for any value of α in $[0, 1]$. Through this novel approach, we are able to characterize the structure of near-optimal mechanisms as well as near-worst case families of distributions. In turn, we are able to characterize the maximin ratio across different values of α up to 1.3%.

There are different natural avenues of future research. One potential direction is to try to completely close the gap between the lower and upper bounds. While the case of one sample is fundamental, another important direction is to analyze the problem when the seller has access to more samples. The latter direction is very

promising since it will allow to connect the low sample and asymptotic regimes.

Auctions in the Online Display Advertising Chain: Coordinated vs Independent Campaign Management

3.1 Introduction

The online display advertising market has grown rapidly over the last ten years, from less than \$4.3 billion in 2005 to more than \$23 billion in 2015 (Internet Advertising Bureau, 2015). This growth has seen the emergence of a significant new channel through which publishers can sell impressions to advertisers, in addition to the traditional guaranteed contracts. Publishers now auction off in real time impressions allowing advertisers to have increased targeting abilities. This real time market is known as the Real Time Bidding (RTB) market. In the RTB market, while a user is loading a webpage with an advertising slot, the publisher may send information about the user and the characteristics of the slot (e.g., position, length, width, etc.) to an ad exchange (through Supplier Side Platforms). The ad exchange runs an auction in which advertisers bid for the impression through intermediaries called *Demand Side Platforms* (DSPs).¹²

DSPs play an important role for advertisers in providing technology to access the RTB market, with important economies of scale. This role is well recognized and

¹Figure C.5-1 in Appendix C.5 provides a high level illustration of the structure of the different agents and links constituting the value chain in the RTB market.

²The DSPs in RTB have often grown naturally from being intermediaries referred to as Ad Networks in the traditional guaranteed contracts market in which size enabled them to negotiate better deals on behalf of buyers.

understood. The focus of the present chapter is on the *tactical* role that DSPs play in practice and their impact on the online display advertising value chain. Indeed, a DSP typically manages in parallel campaigns of multiple advertisers and accesses the values that the advertisers he represents have for a particular impression, leading to some form of collusion. Depending on the auction mechanism in place, a DSP will potentially have an incentive *not* to transmit multiple bids to the ad exchange, even if many of the advertisers he represents would value the impression. The DSP would want to limit the competition for the item being auctioned off. Consider the following basic motivating example to illustrate some of the incentives at play given the structure described above. A publisher sends an impression to an ad exchange that runs a second price auction with reserve price set to be 3. Assume there is only one DSP who is bidding on behalf of two advertisers. The first advertiser values the impression at 5 whereas the second one values it at 4. How should the DSP bid? Assuming that the DSP maximizes the surplus of his advertisers and coordinates campaigns, he would try to limit competition and it is clear that the best strategy of the DSP is to *only* submit one bid, the highest one, 5 on behalf of the first advertiser. In turn, the DSP would be allocated the item and the buyers's side (DSP plus advertisers) would make a surplus of 2 while the group of sellers (publisher plus ad exchange) would generate revenues of 3. Let us compare this case to what would have happened had the DSP managed the campaigns of the advertisers he represents *independently of each other*. With such a constraint, he would have submitted a bid on behalf of each advertiser equal to the advertiser's value for the impression and the resulting allocation would have been identical. However, the buyers' surplus would have been equal to 1 and the seller's revenues would have amounted to 4. This simple example already brings to the foreground fundamental questions on the DSPs impact on the online display advertising value chain. How does the collusion induced by the tactical role of intermediaries impact the participants in the value chain?

3.1.1 Main questions and contributions

In the absence of collusion through intermediaries, i.e., when DSPs manage campaigns of their advertisers independently of each other, it is well known that under proper regularity assumptions, a second price auction with reserve price is an optimal mechanism for selling a single object (Myerson, 1981). We take this setting as a benchmark, which we refer to as the “multi-bidding” case or the independent campaign management case. With this baseline, we focus on the implications of coordinated campaign management by DSPs in the bidding process on the value chain when the seller uses a second price auction. The joint optimization creates a form of collusion among buyers. The early work of McAfee and McMillan (1992) studied how buyers could collude efficiently under an (first price) auction, and when analyzing the seller’s response, illustrated through a numerical example with uniform valuations that collusion could hurt not only the seller but also the buyers. In the present chapter, we aim to understand and analyze if there exist systematic directional impacts of collusion on the performance of the players in the value chain in a general market with many intermediaries, and isolate the drivers of said impacts. In doing so, we also aim to inform the debate about multi-bidding taking place in the online advertising industry.

To shed light on the questions above, we develop a framework anchored around a model with symmetric intermediaries and buyers for which the multi-bidding benchmark case is well understood. In particular, the main contributions of the present chapter can be summarized as follows.

We analyze the impact of coordinated campaign management by DSPs on three performance measures: the seller’s profit, the buyers’ side surplus (buyers and intermediaries) and the social welfare. We show that the presence of intermediaries leads to an increase in the reserve price, and, as one would expect, always affects negatively the seller’s profit. In addition, coordinated campaign management always

negatively affects social welfare. While it is not possible to obtain a systematic directional conclusion for the impact on buyers across all market scenarios, we are able to characterize the impact in two regimes with and without competition among intermediaries. In those, we establish that *the buyers (together with the intermediaries) are also worse off when intermediaries coordinate campaigns* for a broad set of market characteristics. As result, a fundamental inefficiency is induced by the coordinating role of the intermediaries and there is a potential for a Pareto improvement through multi-bidding in the value chain.

In more detail, to characterize the impact on buyers, we consider two regimes that we establish to be analytically tractable. We first analyze the impact on buyers for the special case in which there is no competition among intermediaries: with a single intermediary representing two buyers. We establish analytically that for the class of Generalized Pareto distributions (that includes uniform and exponential distributions), the buyers (together with the intermediaries) are worse off when intermediaries coordinate campaigns, as long as the coefficient of variation is not too small. Hence, there is a potential for a pareto improvement through multi-bidding.

In the presence of competition among intermediaries, it is not clear if the inefficiency persists as the buyers' surplus is affected by the response of the seller but also by the competitive landscape. To gain tractability in a competitive environment, we analyze the impact on buyers in a large market. In particular, we focus on an appropriate asymptotic regime in which both the number of intermediaries and buyers grow large while maintaining the competition level in the auction constant. In such a regime, we derive an asymptotic upper bound on the impact on the buyers' side surplus for general distributions of values. Leveraging this upper bound, we then establish that for the class of Generalized Pareto distributions, the buyers are again negatively impacted by coordinated campaign management in a very broad range of scenarios. In particular, this holds as long as the average number of advertisers that

participate in the auction is moderate and the coefficient of variation of values is not too small.

We also conduct a robustness analysis through numerical experiments for the family of Beta and Gamma distributions for the buyers' values and find a similar insight to hold.

Implications. The present chapter highlights the two main forces at play for buyers using intermediaries that leverage their tactical role in the bidding process and coordinate bidding strategies as opposed to running all campaigns independently. On the one hand, coordinated campaign management leads to a “myopic benefit of collusion” by decreasing the competitiveness of the auction the buyer participates in. On the other hand, coordinated campaign management leads the seller to react and adjust its selling mechanism. The fundamental insight is that the impact of the latter reaction of the seller, which leads to an increase in the reserve price, dominates, in a very broad set of scenarios, the myopic benefit of collusion. In turn, buyers (together with intermediaries) are worse off in a market with intermediaries that coordinate campaigns. In other words, both the seller and the buyers' side in the RTB value chain suffer due to coordinated campaign management by DSPs. While DSPs play a key technological role in reducing frictions and enabling access to the RTB market for advertisers while providing economies of scale, the present research establishes that coordinated campaign management taken in the bidding process may be detrimental to the online display advertising value chain. Furthermore, the results above establish that independent campaign management and truthful submission of all bids by DSPs, commonly referred to as “multi-bidding”, i.e., independent campaign management, in the online advertising industry, leads to a Pareto improvement in the value chain for a very broad range of environments. Sellers have been concerned about the lack of transparency and visibility of all bids for some time (see, e.g., Mansour et al. (2012), Kaplan (2012)), and the lack of thickness in the market, and have advocated

for multi-bidding. Despite the appearance that multi-bidding might hurt advertisers (as it would increase competition), the present chapter makes a case that, in fact, advertisers may benefit from multi-bidding as an industry norm. More broadly, this chapter exhibits a fundamental inefficiency in the online display advertising value chain. Furthermore, the present chapter indicates that one may not be able to address this inefficiency at the interface between the seller and the intermediaries under a second price auction. One may need to tackle it at the interface between the advertisers and the DSPs and advertisers may want to require as an industry norm to have their campaign managed independently of other buyers. While DSPs should continue to play an important technological role, this chapter provides a rigorous framework for a debate about multi-bidding and the role DSPs should play in the bidding process.

Organization of the chapter. We next discuss the related literature. In Section 3.2, we present the problem formulation. We quantify the impact of coordinated campaign management on the seller’s profit and the social welfare in Section 3.3. Sections 3.4-3.6 focus on the impact of this coordinated campaign management on buyers. We conclude in Section 3.7. All proofs are presented in Appendix A and the online appendix accompanying this chapter.

3.1.2 Literature review

Online advertising. Our chapter relates to the growing online advertising literature. Muthukrishnan (2009) and Korula et al. (2016) review some of the research challenges associated with the industry, such as the optimization of campaign delivery, (Roels and Fridgerisdottir, 2009; Ciocan and Farias, 2012; Balseiro et al., 2014; Hojjat et al., 2016), the design of bidding strategies while learning valuations (Iyer et al., 2014) or the study of contracting in the value chain. The latter has been studied under various angles. For example, Balseiro and Candogan (2016) study the optimal contracting

between intermediaries and budget constrained buyers. Feldman et al. (2010) focus on the relation between the intermediaries and their buyers. Assuming that the seller is running a second price auction with an eventual random reserve price, they characterize the optimal contract between the intermediary and advertisers, given that the intermediary is maximizing the difference between the payment received from the buyers and the payment made to the seller. Balseiro et al. (2015b) study the optimal contract when a network of intermediaries is present and analyze the relation between the position in the network of an intermediary and the profit that can be achieved as a function of the distribution of buyers. An analysis that relates to the present work is the analysis of horizontal mergers of intermediaries. However, the incentives at play are different as intermediaries maximize their own surplus and we focus on a different question, the role that DSPs should play and the impact of multi-bidding on the value chain. Also related in spirit is Hummel et al. (2016) who study, under the assumption that the seller is running a second price auction, if the seller could incentivize the intermediary to reveal his second value. Specifically, they have shown that if the seller shares its revenue with the intermediaries then they might reveal the second highest value in their book if they value the shared revenue from the seller more than the utility of their advertisers. In the present chapter, the intermediaries are assumed to act in order to maximize the surplus of the buyers they represent and we abstract away from how the surplus is shared between buyers and intermediaries. The latter ensures that the findings we obtain are not driven by any misalignment of incentives between the intermediaries and the buyers. Especially, we argue that the fundamental inefficiency, due to the lack of visibility of the second value in the book of intermediaries may be tackled at the interface between the intermediaries and the buyers in the form of an industry norm.

Collusion in auctions. Our work may be seen as studying the implication of collusion when the seller is running a second price auction . In that sense, it

relates to Graham and Marshall (1987) who study how the buyers could collude efficiently when the seller is setting a second price auction. McAfee and McMillan (1992) have also studied how the buyers could collude efficiently under the assumption that the seller is using a first price auction. They have studied different types of cartels: weak and strong; our model is more related to the weak cartels since no side payment is possible, because the assignment set by the seller in our model cannot be changed or manipulated by the intermediary. They considered only the presence of one cartel while in our work we allow for competition among intermediaries. As mentioned earlier, McAfee and McMillan (1992) provide a numerical example for a specific distribution of values (standard uniform) in which collusion through one cartel, under the first price auction, negatively affects its own buyers by the reaction of the seller. In our chapter, we prove analytically, under a second price auction, that such inefficiencies will be systematic in the case of more than one cartel and for a general class of distributions. Furthermore, we elucidate a driver for the presence of these inefficiencies through the coefficient of variation of values of buyers. For more broad analysis of collusion, we also refer the reader to Pavlov (2008) and Che and Kim (2009) who investigate what types of mechanisms could be used to counter collusion when there is asymmetric information between the members of the same cartel (in our model, motivated by online advertising, we do not assume any asymmetry among the members of the same cartel.)

In our model each intermediary might represent many advertisers. In general, since each intermediary could be seen as an agent with a multidimensional vector of private values, a more general version of our problem in which one does not restrict attention to second price auctions falls in the class of multidimensional mechanism design. In particular, the seller's problem would be, e.g., a special case of the model analyzed in Belloni et al. (2010), albeit in a different context. It is well known that in this class of problems, randomization could dominate take it or leave it prices

(Thanassoulis (2004)) and the work of Haghpanah and Hartline (2014) provides sufficient conditions for a posted price to be optimal. The practical appeal and the adoption within the industry (Mansour et al. (2012)) of second price auctions together with the lack of clear understanding of the structure of an optimal solution to the multi-dimensional mechanism design problem, lead to us to limit attention to second price auctions in the present chapter.

Intermediation and efficiency in supply chains. Finally, there has been significant research in supply chain structures and their implications on the chain profits. From that perspective, our work also relates at a high level to this broad stream. A study related in spirit to our study is Hu et al. (2013) that analyzes the impact of pooling purchases on the the buyers’s profits and the strategic reaction this induces on the seller side. See also, e.g., Adida et al. (2016), Yang and Babich (2014), Belavina and Girotra (2012), Wu (2004) for recent studies of the role of intermediaries in supply chains.

3.2 Problem Formulation

We consider one seller (ad exchange) with a single unit to sell (an impression), and J intermediaries (DSPs). Each intermediary is representing K buyers (advertisers), so in total there are $J \times K$ buyers. The k^{th} advertiser of intermediary j has a value \mathbf{v}_j^k for the unit. We assume that $\{\mathbf{v}_j^k : j = 1, \dots, J, k = 1, \dots, K\}$ are independent and identically distributed (i.i.d.).

Each intermediary j is characterized by the vector of values $\mathbf{v}_j = (\mathbf{v}_j^1, \dots, \mathbf{v}_j^K)$ of the buyers he represents. We assume that the buyers provide truthfully their values to the intermediary that represents them.³ Each intermediary is assumed to maximize the surplus of the group of advertisers he represents. This is in line with

³The DSPs are often computing the value their advertisers have for impressions and hence have access to those.

intermediaries having a surplus sharing contract with advertisers. This objective is also in line with the long term objective of DSPs in the marketplace as those aim to retain their customers. Furthermore, with such an objective, the incentives of the DSPs are fully aligned with the group of advertisers they represent. In other words, the results we obtain are not driven by a misalignment of incentives between DSPs and advertisers.

We assume a quasi-linear utility for the buyers, i.e., an advertiser’s utility is his value for the impression (if he receives it) minus his payment. The vector \mathbf{v}_j is the private information of intermediary j . We assume that the intermediaries do not communicate between themselves.

Model of valuations. With many intermediaries and advertisers per intermediaries, the auctions could “degenerate” given the extreme competition that arises when $J \times K$ is large. To avoid this degeneracy and to be able to analyze a meaningful model with large values of J and K (which occurs, e.g., in the online display advertising RTB market), we focus on a two step private value model in which each buyer “matches” with the item with probability $\alpha \in (0, 1]$. If a buyer matches, then his valuation is drawn from a distribution $F(\cdot)$, otherwise his valuation for the item is zero. As a result, the underlying new cumulative distribution of an advertiser’s value is given by

$$G_\alpha(x) = 1 - \alpha + \alpha F(x). \tag{3.1}$$

Intermediaries are ex-ante symmetric. However, an intermediary might represent, ex-post, strictly less than K advertisers in the auction. One could consider K as the total number of clients that an intermediary has in its portfolio. However, for a particular item, only a portion of those clients are interested in bidding for the item.

Related models of valuations have been used recently in studies focusing on different questions in the context of online advertising (Balseiro et al., 2015a; Mirrokni and Nazerzadeh, 2015). These models are meant to capture a key feature of these mar-

kets: there tends to be a large number of advertisers contracting with intermediaries and the latter bids on their behalf; however, when a particular auction takes place, there is only a moderate number of bids submitted. Mathematically, the matching parameter α allows to control how much competition is present for a typical auction.

In the present chapter, we will be looking at two main regimes. First, we analyze a setting without competition among intermediaries, in Section 3.4, where $\alpha = 1$, $J = 1$ and $K = 2$, to highlight the main intuition at play. Then we analyze a general model, with competition among intermediaries in Section 3.5, in a large market. We will take J and K large, α shrinking to zero, while ensuring that αJK is constant. In the latter, there is a large number of players in the market but the number of interested buyers per auction is constant and independent of the scale of the market.

The seller’s optimization problem. The seller can only contract with the intermediaries and does not know the book values list \mathbf{v}_j behind each intermediary. However, the number of advertisers K per intermediary, and the number of intermediaries J and the distribution F are assumed to be common knowledge. Finally, the seller has no value for the item and uses a second price auction⁴ to maximize its expected revenue.

Assumptions. We will assume throughout that the support of the values \mathcal{S} is convex, contained in $[0, \infty)$, and that the values admit a density denoted by f , which is twice continuously differentiable and has increasing failure rate (IFR). These are common assumptions in the auction literature and are satisfied for example for Gamma, Beta and Generalized Pareto families for a wide range of parameters. We will denote the complementary cumulative distribution function by \bar{F} .

Notation. In what follows, we denote by $\mathbf{v}_{1:J}^{[i]}$ the i^{th} highest value of the collection of values $\{\mathbf{v}_j^k : 1 \leq j \leq J; 1 \leq k \leq K\}$. For each intermediary j , we denote the

⁴A second price auction or variations of it is one of the adopted norms in the online display advertising industry Muthukrishnan (2009) and Mansour et al. (2012).

order statistic of $\mathbf{v}_j = (v_j^1, \dots, v_j^K)$ by $v_j^{[1]}, \dots, v_j^{[K]}$, where $v_j^{[k]} \geq v_j^{[k+1]}$. In particular $v_j^{[1]}$ and $v_j^{[2]}$ are the first and second highest values of intermediary j . Furthermore, for v in \mathcal{S} , we let

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} \quad (3.2)$$

denote the virtual value function. In the following sections, we use buyers (respectively, intermediaries) and advertisers (respectively, DSPs) interchangeably given the central application of the chapter.

3.2.1 Performance Metrics and Impact of Active Role of Intermediaries

We track the following performance metrics when intermediaries are coordinating campaigns of their buyers under an optimal second price auction: the *revenue of the seller* $\Pi_{\mathbf{co}}^*$, the *social welfare* $S_{\mathbf{co}}^*$ and the *surplus of the buyers* (along with the intermediaries) $U_{\mathbf{co}}^*$, where the subscript \mathbf{co} is mnemonic for the fact that these metrics are evaluated when the intermediaries are coordinating the campaigns of the buyers they represent (we introduce in a moment the multi-bidding/independent campaign management benchmark). Let us denote by $(\mathbf{w}^{[1]}, \dots, \mathbf{w}^{[J]})$ the order statistic of $(\mathbf{v}_1^{[1]}, \dots, \mathbf{v}_J^{[1]})$. In particular, $\mathbf{w}^{[1]}$ is the highest value of the maximum that each intermediary has and $\mathbf{w}^{[2]}$ is the second highest maximum among the maximum values that each intermediary has. Note that $\mathbf{w}^{[1]} = \mathbf{v}_{1:J}^{[1]}$, i.e., the maximum value among the maximum values that each intermediary represents coincides exactly with the maximum value of all advertisers. If there is only one intermediary, we define $\mathbf{w}^{[2]}=0$. Note that given that the seller is using a second price auction with a reserve price r , it is clear that the best response of an intermediary is to submit only one bid to represent the buyer with the highest value, i.e., intermediary j will only submit one bid equal to $v_j^{[1]}$. Hence, the problem reduces to one in which the seller can be thought

as facing J intermediaries, with each one representing only *one* “virtual buyer.” In turn, we have

$$\begin{aligned}\Pi_{\text{co}}^* &= \mathbb{E}[\max\{r_{\text{co}}, \mathbf{w}^{[2]}\} \mathbb{1}\{\mathbf{w}^{[1]} \geq r_{\text{co}}\}], \\ S_{\text{co}}^* &= \mathbb{E}[\mathbf{w}^{[1]} \mathbb{1}\{\mathbf{w}^{[1]} \geq r_{\text{co}}\}], \\ U_{\text{co}}^* &= \mathbb{E}[(\mathbf{w}^{[1]} - \max\{r_{\text{co}}, \mathbf{w}^{[2]}\})^+],\end{aligned}$$

where r_{co} denotes the optimal reserve price set by the seller when the intermediaries are coordinating the campaigns of the buyers they represent. In Section 3.3, we will characterize explicitly r_{co} .

Multi-bidding benchmark. To understand the impact of tactical campaign coordination, we will consider the benchmark case in which intermediaries manage the campaign of each buyer independently of the other buyers they represent (more formally, the bid that they submit on behalf of a particular buyer cannot depend on the actual values of other buyers). We refer to this benchmark as the *multi-bidding* or *independent campaign management* case (we will use both terms interchangeably).

In this case, the problem of the seller reduces to a case akin to one in which she is dealing directly with the buyers (and intermediaries are simply providing the technology to potentially compute and submit the bids). Then, the problem reduces to selling a single unit to $J \times K$ buyers with i.i.d values drawn according to F . Under the IFR assumption we made on F , it is well known (Myerson, 1981) that an optimal selling mechanism is given by a second price auction with reserve price r_{in} , where r_{in} is the unique solution of $\phi(v) = 0$ with $\phi(\cdot)$ being the virtual value function defined in (3.2). The subscript **in** is mnemonic for the fact the intermediaries are managing campaigns independently. In particular, the good is allocated if and only if at least one buyer has a bid above r_{in} , in which case the buyer with the highest bid is allocated the item and pays the maximum of the reserve price and the second highest bid. Only a buyer who is allocated the item pays anything to the seller. We

recall that the second price auction is a truthful direct mechanism, i.e., it is optimal for each buyer to submit its true value. In turn, the expected revenue of the seller, the social welfare and the surplus of the buyers (advertisers and intermediaries) are given by

$$\begin{aligned}\Pi_{\text{in}}^* &= \mathbb{E} \left[\max\{r_{\text{in}}, \mathbf{v}_{1:J}^{[2]}\} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right], \\ S_{\text{in}}^* &= \mathbb{E} \left[\mathbf{v}_{1:J}^{[1]} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right], \\ U_{\text{in}}^* &= \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{r_{\text{in}}, \mathbf{v}_{1:J}^{[2]}\})^+ \right].\end{aligned}$$

The objective of the chapter is to characterize the impact of coordinated campaign management in the bidding process on the value chain. In particular, we will analyze the three key quantities

$$\begin{aligned}\Pi_{\text{co}}^* - \Pi_{\text{in}}^* &\quad \text{impact of coordinated campaign management on seller revenues,} \\ S_{\text{co}}^* - S_{\text{in}}^* &\quad \text{impact of coordinated campaign management on social welfare,} \\ U_{\text{co}}^* - U_{\text{in}}^* &\quad \text{impact of coordinated campaign management on buyers' surplus.}\end{aligned}$$

3.3 Impact of Intermediaries Coordinating Campaigns of their Buyers

3.3.1 Adjusted reserve price

We first characterize the impact of coordinated campaign management on the reserve price set by the seller.

Lemma 3.1 (Adjusted Reserve Price). *When the intermediaries are coordinating campaigns of their own buyers, an optimal reserve for a second price auction on all bids submitted is given by r_{co} , uniquely defined through*

$$r_{\text{co}} = \frac{1 - (G_{\alpha}(r_{\text{co}}))^K}{\alpha K f(r_{\text{co}}) (G_{\alpha}(r_{\text{co}}))^{K-1}}.$$

Furthermore, this price is strictly greater than the one in the independent campaign management benchmark case, i.e.,

$$r_{\text{co}} > r_{\text{in}}.$$

Hence, when intermediaries are coordinating campaigns of their buyers, the seller is not oblivious to this role and reacts by becoming more “demanding” towards the intermediaries and increasing the minimum payment she is willing to accept. We note that here, we assume that the incentives of intermediaries are perfectly aligned with those of the buyers they represent, so the increase in reserve price does not stem from double-marginalization. Rather, it is driven by the change in the distribution of bids/values that the seller observes due to the coordination of campaigns. Now, the seller sees only the maximum of the values of the buyers that a particular intermediary represents.

3.3.2 Impact on the value chain

Seller’s profit and social welfare. The next result formalizes the fact that the seller is negatively affected when intermediaries coordinate the campaigns of the buyers they represent, and establishes that the latter practice also negatively affects the social welfare.

Corollary 3.1 (impact on seller and social welfare). *The coordinating role of DSPs leads to a strict decrease in the seller’s revenues and in the social welfare, i.e.,*

$$\begin{aligned} \Pi_{\text{co}}^* &< \Pi_{\text{in}}^*, \\ S_{\text{co}}^* &< S_{\text{in}}^*. \end{aligned}$$

The increase of the reserve price only mitigates the revenue losses stemming from the collusion induced by intermediaries but does not allow her to fully recover the revenues she would have obtained in a case with multi-bidding, i.e., $\Pi_{\text{co}}^* < \Pi_{\text{in}}^*$. The

fact that the seller is hurt by the collusion is in line with intuition. Furthermore, since the seller increases its reserve price from r_{in} to r_{co} , i.e., $r_{\text{co}} > r_{\text{in}}$, there is now a lower probability that the item is sold hence the social welfare decreases. In turn, the pie that the seller and buyers (together with the intermediaries) will be sharing shrinks.

Buyers' side surplus. Next, we analyze the effect on buyers's side (buyers together with intermediaries) and analyze the difference $U_{\text{co}}^* - U_{\text{in}}^*$.

To that end, we let $U_{\text{in}}(r)$ denote the expected surplus of the buyers when the seller uses a second price auction with a reserve price r and the intermediaries are multi-bidding, i.e.,

$$U_{\text{in}}(r) = \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{\mathbf{v}_{1:J}^{[2]}, r\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right].$$

Note that $U_{\text{in}}^* = U_{\text{in}}(r_{\text{in}})$.

Similarly, we let $U_{\text{co}}(r)$ denote the expected surplus of the buyers when the seller uses a second price auction with a reserve price r and when the intermediaries are coordinating campaigns of their buyers, i.e.,

$$U_{\text{co}}(r) = \mathbb{E} \left[(\mathbf{w}^{[1]} - \max\{\mathbf{w}^{[2]}, r\}) \mathbb{1}\{\mathbf{w}^{[1]} \geq r\} \right].$$

Note that $U_{\text{co}}^* = U_{\text{co}}(r_{\text{co}})$.

With the notation above, the impact of collusion on buyers may be written as $U_{\text{co}}^* - U_{\text{in}}^*$. In turn, this may be decomposed as follows

$$U_{\text{co}}^* - U_{\text{in}}^* = \mathcal{MBC} + \mathcal{SRI},$$

where

$$\mathcal{MBC} = U_{\text{co}}(r_{\text{in}}) - U_{\text{in}}(r_{\text{in}}),$$

$$\mathcal{SRI} = U_{\text{co}}(r_{\text{co}}) - U_{\text{co}}(r_{\text{in}}).$$

The quantity \mathcal{MBC} captures the benefits that the buyers receive stemming from the intermediaries acting strategically, assuming the seller does not react. We refer

to \mathcal{MBC} as the “myopic benefit of collusion.” Clearly, for any market parameters, one has $\mathcal{MBC} \geq 0$, i.e., if the seller does not respond to the coordinating role of intermediaries and keeps the reserve price at r_{in} , the buyers would benefit from such coordination since competition is softened.

The quantity \mathcal{SRI} represents the impact of the increase in reserve price on the buyers when the intermediaries are coordinating campaigns of their buyers. We refer to \mathcal{SRI} as the “seller reaction” effect. It is also clear that $\mathcal{SRI} \leq 0$, i.e., the buyers are negatively affected by the increase in reserve price from r_{in} to r_{co} .

The following proposition characterizes each quantity as a function of the market primitives.

Proposition 3.1 (impact on buyers). *The myopic benefit of collusion and the seller’s response effect are given by*

$$\begin{aligned} \mathcal{MBC} &= \int_{r_{\text{in}}}^{\infty} \left(JK(1 - (G_{\alpha}(x))^{JK-1}) - (JK - 1)(1 - (G_{\alpha}(x))^{JK}) \right) \\ &\quad - \left(J(1 - (G_{\alpha}(x))^{JK-K}) - (J - 1)(1 - (G_{\alpha}(x))^{JK}) \right) dx, \\ \mathcal{SRI} &= \int_{r_{\text{in}}}^{r_{\text{co}}} \left[J(1 - (G_{\alpha}(x))^{JK-K}) \right. \\ &\quad \left. - (J - 1)(1 - (G_{\alpha}(x))^{JK}) \right] - [(1 - (G_{\alpha}(x))^{JK})] dx. \end{aligned}$$

Quantifying the impact of the coordinating role of DSPs on the buyers’ side surplus is akin to quantifying which of \mathcal{MBC} and \mathcal{SRI} dominates and drives the sign of $U_{\text{co}}^* - U_{\text{in}}^* = \mathcal{MBC} + \mathcal{SRI}$. Analyzing $U_{\text{co}}^* - U_{\text{in}}^*$ based on the expressions in Proposition 3.1 appears analytically intractable in general. In Section 3.4, we analyze the case, without competition among intermediaries (one intermediary representing two advertisers) in detail for a family of distributions, to develop the main intuition and drivers of $U_{\text{co}}^* - U_{\text{in}}^*$. In Section 3.5, we analyze a general model with competition among intermediaries, and characterize $U_{\text{co}}^* - U_{\text{in}}^*$ in a “large market” in an appropriate asymptotic regime.

3.4 Impact on the Buyers's Side: No Competition Among Intermediaries

To highlight the main intuition and the main phenomena at play, we first consider the case of one intermediary ($J = 1$) representing two advertisers ($K = 2$), when the advertisers are always interested by the object ($\alpha = 1$). We denote the order statistic of the values of the two advertisers by $\mathbf{v}^{[1]}$ and $\mathbf{v}^{[2]}$ with $\mathbf{v}^{[1]} \geq \mathbf{v}^{[2]}$. When the intermediary is multi-bidding, meaning that the seller receives two bids $\mathbf{v}^{[1]}$ and $\mathbf{v}^{[2]}$, the optimal mechanism is a second price auction with reserve price r_{in} defined by $r_{\text{in}} = (1 - F(r_{\text{in}}))/f(r_{\text{in}})$. When the intermediary coordinates the bids of its buyers, the optimal mechanism is a second price auction with reserve price r_{co} defined by $r_{\text{co}} = [1 - F^2(r_{\text{co}})]/[2 f(r_{\text{co}}) F(r_{\text{co}})]$ and the seller receives only one bid equal the highest value $\mathbf{v}^{[1]}$.

To quantify the impact of collusion on buyers and the drivers of its sign, we will be analyzing a family of value distributions that includes among other distributions the uniform and exponential ones. In particular, we focus on the Generalized Pareto Distribution parametrized by (σ, ξ) with $\sigma > 0$ and $\xi \leq 0$ with distribution

$$F_{\xi}(z) = \begin{cases} 1 - (1 + \frac{\xi z}{\sigma})^{-1/\xi} & \text{for } \xi < 0 \\ 1 - e^{-z/\sigma} & \text{for } \xi = 0 \end{cases}$$

The support is given by $[0, -\sigma/\xi]$ for $\xi < 0$ and $[0, \infty)$ for $\xi = 0$. In particular, the standard uniform and exponential distributions belong to the family, the former corresponding to $(\sigma, \xi) = (1, -1)$ and the latter to $(\sigma, \xi) = (1, 0)$. Any member of the family satisfies the IFR property (since $\xi \leq 0$). This family allows many possible shapes and some distributions belonging to the family are depicted in Figure C.6.1 in Appendix C.6. This family is attractive given its flexibility but also its tractability in the context of auction design (see, e.g., Balseiro et al. (2015b)).

Using Proposition 3.1, we can derive an expression for $U_{\text{co}}^* - U_{\text{in}}^*$ as a function of ξ . Indeed, we first note that r_{in} admits a simple closed form expression,

$$r_{\text{in}} = \frac{\sigma}{1 - \xi}.$$

On the other hand, the reserve price r_{co} is a solution of the following equation:

$$r_{\text{co}} = \frac{\sigma}{2} \left(1 + \frac{\xi}{\sigma} r_{\text{co}} \right) \left(\frac{1 + F_{\xi}(r_{\text{co}})}{F_{\xi}(r_{\text{co}})} \right).$$

Following Bulow and Roberts (1989), we focus on the quantiles. Let us introduce $q(x) := 1 - F(x)$, so we get that $x = F^{-1}(1 - q)$ and let us denote q_{in} and q_{co} the quantiles corresponding to the reserve prices r_{in} and r_{co} respectively.

Proposition 3.2. *Suppose that the distribution of values is given by a generalized Pareto distribution with parameters (σ, ξ) with $\xi \leq 0$. Then, the impact of campaign coordination by the intermediary on the buyers' side surplus is given by*

$$U_{\text{co}}^* - U_{\text{in}}^* = \sigma \left[\frac{2}{2 - \xi} q_{\text{in}}^{2-\xi} - \frac{2}{1 - \xi} q_{\text{in}}^{1-\xi} + \frac{2}{1 - \xi} q_{\text{co}}^{1-\xi} - \frac{2}{2 - \xi} q_{\text{in}}^{2-\xi} \right],$$

where q_{in} and q_{co} are solutions to the following equations

$$q_{\text{in}} = (1 - \xi)^{\frac{1}{\xi}}, \quad \frac{1}{\xi}(1 - q_{\text{co}}^{\xi}) = \frac{2 - q_{\text{co}}}{2(1 - q_{\text{co}})}.$$

Proposition 3.2 provides a closed-form expression of $U_{\text{co}}^* - U_{\text{in}}^*$ as a function of the quantiles. The key challenge in analyzing $U_{\text{co}}^* - U_{\text{in}}^*$ and its sign is associated with the quantile in the coordinated campaigns case, q_{co} , which does not admit a closed-form expression. Furthermore, the proposition implies that for the generalized Pareto family, the sign of $U_{\text{co}}^* - U_{\text{in}}^*$ does not depend on the value of σ , so the sign depends only on ξ . Furthermore, note that the coefficient of variation of values (the standard deviation normalized by the mean) depends only on ξ , and is given by

$$CV(\xi) = \frac{1}{\sqrt{1 - 2\xi}}.$$

Hence, $CV(\xi)$ is a one-to-one mapping and is increasing in ξ ; as ξ spans $(-\infty, 0]$, $CV(\xi)$ spans $(0, 1]$. We will next analyze the sign of $U_{\text{co}}^* - U_{\text{in}}^*$ as a function of $CV(\xi)$, as the latter is a more “physical” quantity than ξ .

Theorem 3.1. *(impact of coordinated campaigns on buyers’s side) Suppose that the distribution of values is given by a generalized Pareto distribution with parameters (σ, ξ) . Then there exists a threshold ρ in $[0, 1/\sqrt{3}]$ such that buyers are worse-off in a market with coordinated campaigns, i.e., $U_{\text{co}}^* - U_{\text{in}}^* \leq 0$, whenever the coefficient of variation of values $CV(\xi) \geq \rho$.*

The result implies that coordinated campaign management has a negative impact on the buyers’s side (buyers together with the intermediaries) as long as the coefficient of variation of values is not too small. In particular, the uniform and exponential distributions satisfy $CV(\xi) \geq \rho$, ($CV(-1) = 1/\sqrt{3}$ and $CV(0) = 1$). The proof of the previous theorem relies mainly on the fact that when the coefficient of variation increases the seller becomes more “demanding”. In particular, in the proof, we establish structural properties of the ratio of the quantiles $q_{\text{co}}/q_{\text{in}}$ (as depicted in Figure 3.2(b)) and how it depends on the coefficient of variation. Leveraging these properties, we construct a piecewise decreasing concave bound on $U_{\text{co}}^* - U_{\text{in}}^*$ that is negative on $[1/\sqrt{3}, 1]$.

Initially, the motivation for coordinated campaigns is to reduce competition from the advertisers represented by the same intermediary. By hiding the second value (and lower values), the intermediary decreases the second value present in the marketplace. This appears appealing to the advertisers given that the myopic benefit of collusion \mathcal{MBC} is always non-negative. However, the seller is not oblivious to this strategic behavior of intermediaries and adjusts the reserve price given the new distribution of values she faces. Not only the payment for the winner may increase but by increasing its reserve price, the seller also allocates less often the item. These last two effects lead to the negative effect that the seller might have on buyers (\mathcal{SRI}). Theorem 3.1

establishes that this effect dominates the myopic benefit of collusion in a broad range of market scenarios. In particular, as long as there is non-trivial value discovery involved (which corresponds to higher values of $CV(\xi)$), in which case auctions play an important role, then the buyers will benefit from operating in a market in which the intermediary is managing all campaigns independently and is multi-bidding.

Together with Corollary 3.1, Theorem 3.1 implies that coordinated campaigns can have a negative impact *across the value chain*. In other words, multi-bidding by an intermediary can lead to a Pareto improvement in the value chain as long as the coefficient of variation of values is above a threshold. In the next section, we conduct a numerical analysis for the different quantities to support our analytical insights for the Generalized Pareto Distribution. We have also investigated numerically other general classes of distributions (such as Beta or Gamma) and the same central insight holds. There exists a threshold such that if the coefficient of variation exceeds that threshold then both the seller and the buyers are negatively affected by operating in a market with coordinated campaigns (see Section 3.6).

3.4.1 Numerical illustrations and Discussion

In Figure 3.1, we depict the social welfare, the seller's profit and the buyers' surplus as a function of the coefficient of variation $CV(\xi)$ of the values. We normalize the mean of values to 1 by setting $\sigma = 1 - \xi$.

We observe first that when $CV(\xi)$ is close to zero, the role of the intermediary (campaign coordination vs. multi-bidding) has little influence on the performance metrics. Indeed, the values of the buyers are close to deterministic and the seller is able to extract almost all the surplus independently of the nature of the intermediary's role. As $CV(\xi)$ increases, the role of the intermediary becomes much more central and significantly affects the seller and the social welfare. Seller's revenue losses $(\Pi_{co}^* - \Pi_{in}^*)/\Pi_{in}^*$ and social welfare losses $(S_{co}^* - S_{in}^*)/S_{in}^*$ are in the range of 7–8% for $CV(\xi) \geq$

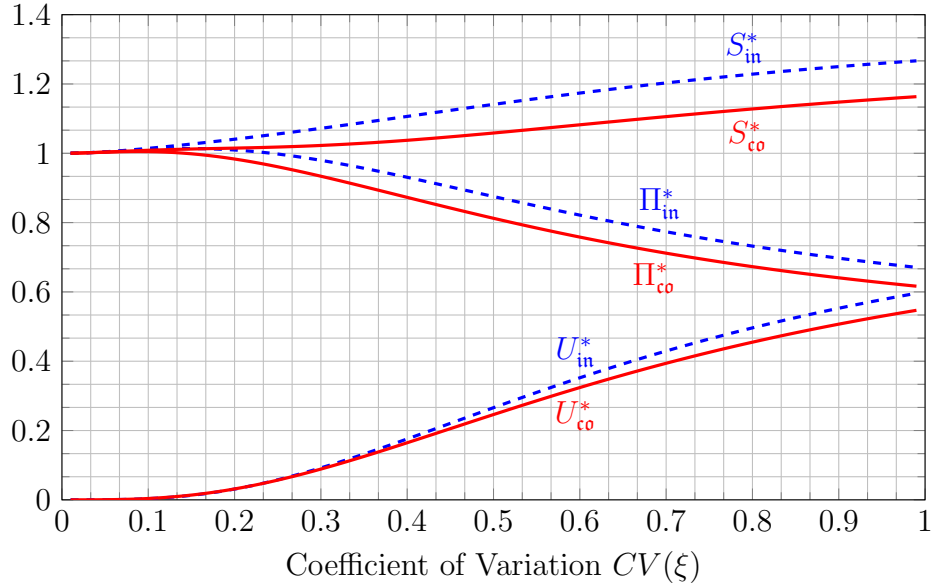
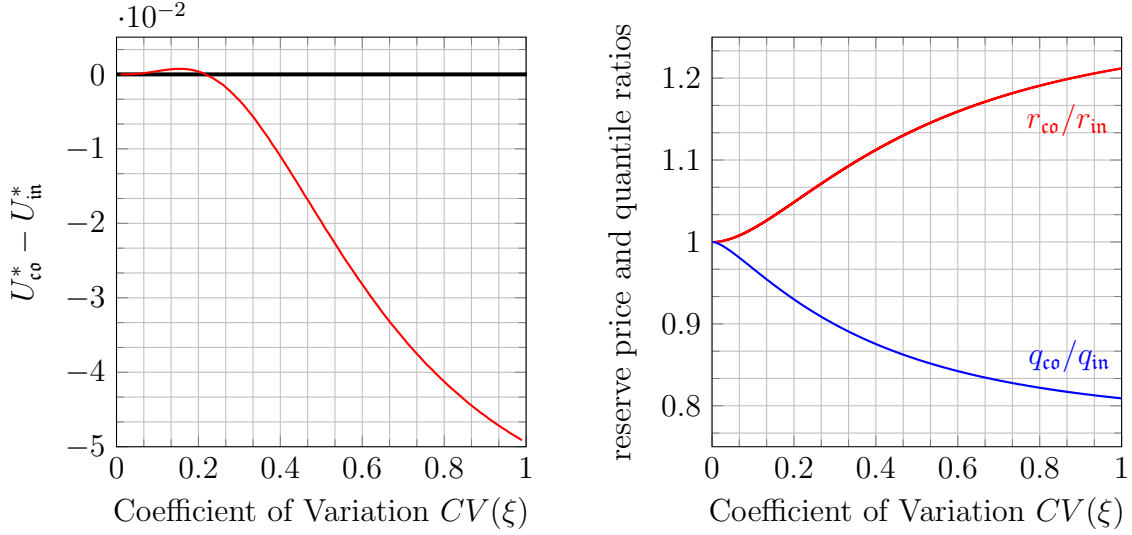


Figure 3.1: **Social welfare, seller's profit and buyers' surplus** as a function of the coefficient of variation of values and the intermediary coordinating or independently managing campaigns. The distribution of values are generalized Pareto distributions with parameters $(\sigma = 1 - \xi, \xi)$, and there is one intermediary and two advertisers.

0.5.

In Figure 3.2, we zoom on the impact on buyers's surplus $U_{co}^* - U_{in}^*$ and analyze in conjunction the optimal reserve prices r_{co} and r_{in} and the corresponding quantiles. Note that here, once the expectation of values is normalized to 1, we have $r_{in} = 1$ for all values of ξ .



(a) **Impact on buyers' surplus** (b) **Optimal reserve prices and quantiles**

Figure 3.2: (a) Impact of campaign coordination on buyers' side surplus (b) Optimal reserve prices and quantiles as a function of the coefficient of variation for generalized Pareto distributions with parameters $(\sigma = 1 - \xi, \xi)$ with one intermediary and two buyers.

We observe that $U_{co}^* - U_{in}^*$ is initially positive and negligible (both U_{co}^* and U_{in}^* are “small” given that the seller extracts almost all the surplus when $CV(\xi)$ is small), crosses the zero axis at $CV(\xi) \approx 0.22$ and then stays negative for all greater values of $CV(\xi)$. This is a picture proof of Theorem 3.1. In addition, we note that the losses for the buyers (together with the intermediaries) $(U_{co}^* - U_{in}^*)/U_{in}^*$ amount to about 7.9% for $CV(\xi) = 1/\sqrt{3} \approx 0.58$, which corresponds to the uniform distribution and about 8.2% for $CV(\xi) = 1$, which corresponds to the exponential distribution. The main driver of these losses is the increase in reserve price from r_{in} to r_{co} which increases by 16% and 21%, respectively, in the latter two cases.

In other words, for the latter two cases of uniform and exponential distributions, the move from coordinated campaigns in the bidding process to independent campaign management, in which the campaign of each buyer is managed independently of other campaigns, would lead to a Pareto improvement in the value chain with both the seller and the buyers obtaining increases in performance in the range of 7 – 8.2%. At an

intuitive level, as the coefficient of variation increases, there is more value discovery involved, which pushes the seller to react more aggressively, and in turn amplifies the fundamental inefficiency associated with campaign coordination in the value chain.

3.5 Impact on the Buyers’s Side: Competition Among Intermediaries

In the previous section, we analyzed the case of one intermediary representing two advertisers to highlight the main intuition and the main phenomena at play. In practice, there would be multiple intermediaries and each will be representing a potentially large number of buyers. The goal of this section is to analyze the impact of coordinated campaigns on the buyers’s side $U_{\text{co}}^* - U_{\text{in}}^*$ in such a general setting.

3.5.1 Large Market Approximation

We will analyze the impact of coordinated campaigns in a regime where both the number of buyers and the number of intermediaries grow large while the probability of any advertiser matching shrinks to zero such that the average number of advertisers who have a positive value for the item auctioned off is constant, equal to some $c > 0$, i.e., $\alpha JK = c$. More precisely, we will assume that α and $K\alpha$ are small, J and K are large, while the average number of buyers having positive value for the item $\alpha JK = c$ is constant.

In other words, in the regime we analyze, the competition in any given auction is always ex-ante identical and fixed, independently of the scale of the network. This ensures that the auctions do not degenerate and that the level of competition in the auctions does not change with the number of intermediaries and advertisers.

From the seller’s perspective, if one fixes the reserve price for a moment, coordinated campaigns leads to inefficiencies as soon as the same intermediary has both

the highest value $v_{1:J}^{[1]}$ and the second highest value $v_{1:J}^{[2]}$ among all advertisers. Given the structure of the network, the probability that such an event occurs is given by

$$\frac{K-1}{KJ-1}.$$

As a result, in a large market, we expect coordinated campaigns to have an impact of order $O(1/J)$ on the various players in the value chain.

The next result shows, through upper and lower bounds that, indeed, the reserve price r_{co} approaches r_{in} in the large market regime at rate $(1/J)$.⁵

Lemma 3.2. *For any α, K such that $\alpha \leq 1/2$ and $\alpha K < 1$, the reserve price r_{co} satisfies*

$$\frac{c}{J} \frac{(\bar{F}(r_{in}))^2}{2f(r_{in})\phi'(r_{in})} - (\alpha + 3\frac{c^2}{J^2})C_1 \leq r_{co} - r_{in} \leq \frac{c}{J} r_{in} e^{\frac{1}{1-\alpha}},$$

where C_1 is a constant depending only on the distribution F .

Next, we analyze the impact on buyers: $U_{co}^* - U_{in}^* = \mathcal{MBC} + \mathcal{SRI}$ by deriving upper bounds on the myopic benefit of collusion \mathcal{MBC} and on the seller's reaction impact \mathcal{SRI} .

Proposition 3.3 (Myopic benefit of collusion). *The myopic benefit of collusion \mathcal{MBC} is upper bounded as follows*

$$\begin{aligned} \mathcal{MBC} \leq & \frac{e^{c/J}}{J} \left[\frac{c^2 e^{-c}}{2} \int_{r_{in}}^{\infty} (y - r_{in}) 2\bar{F}(y) f(y) dy \right. \\ & \left. + \int_{r_{in}}^{\infty} \int_x^{\infty} (y - x) \bar{F}(y) f(x) f(y) c^3 e^{-c\bar{F}(x)} dy dx \right] \\ & + O(\ln(JK)/J^2). \end{aligned}$$

This result shows that the \mathcal{MBC} is indeed of order $O(1/J)$ when $\ln(JK) = o(J)$. The main idea behind the proof of the result is to condition on the number of matches

⁵The dependence of the reserve prices on the relevant parameters will be dropped to lighten notation.

and study the myopic benefit of collusion as a function of how the values of the matches are distributed among intermediaries. While the proof deals with the general case, the main ideas can be highlighted in the limiting regime, when K and J increase to infinity. In that case, the distribution of the number of matches converges to a Poisson distribution with mean c . Now, the myopic benefit of collusion is only strictly positive if there are at least two matches for the winning intermediary and the latter has the two highest values among all buyers.

If exactly two matches occur, which happens with probability $c^2 e^{-c}/2$ under the Poisson distribution, then these are at the same intermediary with probability $(K - 1)/(KJ - 1)$. In that case, at the first order, one can show that the market is reduced to the exact setting studied in Section 3.4: one seller with a reserve price r_{in} and one intermediary representing two advertisers having a distribution F , so the myopic benefit of collusion is $\mathbb{E}[(\mathbf{v}^{[2]} - r_{\text{in}})^+] = \int_0^\infty (y - r_{\text{in}})^+ 2\bar{F}(y)f(y)dy$. This is the driver of the first term in the brackets in the bound in Proposition 3.3.

If there are strictly more than two matches, the myopic benefit of collusion is diminished because of competition since a winning intermediary would be charged eventually more than the reserve price r_{in} . One can show that at the first order, if an intermediary has the two highest values of buyers, then it is unlikely that he has the third highest value among buyers. It means that the winning intermediary will be, with high probability, charged the maximum of the third highest price and the reserve price r_{in} . It is this effect that drives the second term in the brackets in the bound in Proposition 3.3.

Next, we derive an upper bound on the seller's reaction impact \mathcal{SRI} .

Proposition 3.4 (seller reaction impact). *Suppose that $\alpha \leq 1/2$, $J \geq 3$ and $JK \geq 16 \max\{1, (c\epsilon)^4\}$, then \mathcal{SRI} is upper bounded as follows*

$$\mathcal{SRI} \leq -cr_{\text{in}}f(r_{\text{in}})e^{-c\bar{F}(r_{\text{in}})} \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} K\alpha + M(\sqrt{\alpha}K\alpha + \alpha + (K\alpha)^2),$$

where $M > 0$ a constant that depends only on the distribution F and the average number of matches c .

As mentioned earlier, the seller's reaction is always non-positive. In the large market regime, the challenge in evaluating $\mathcal{SRI} = U_{\text{co}}(r_{\text{co}}) - U_{\text{co}}(r_{\text{in}})$ stems from two main effects. First, the reserve price r_{co} depends on parameters of the regime, in particular, α and K , which govern the number of buyers that an intermediary might represent in the marketplace. Second, the utility function $U_{\text{co}}(\cdot)$ at any value $r > 0$ is varying as a function of the parameters of the market, J , K , and α and needs to be approximated in the large market regime. The proof leverages the limiting distribution of matches to bound $U_{\text{co}}(\cdot)$ from a suitable approximation, which in conjunction with the characterization of $r_{\text{co}} - r_{\text{in}}$ derived in Lemma 3.2 leads to the bound in the result.

Combining Propositions 3.3 and 3.4 leads to a general upper bound on $U_{\text{co}}^* - U_{\text{in}}^*$.

Theorem 3.2 (Upper bound on impact of coordinated campaigns on buyers's side). *Suppose that $\alpha \leq 1/2$, $J \geq 3$, $JK \geq 16 \max\{1, (ce)^4\}$ and that $\ln(K) = o(J)$. Then the impact of coordinated campaigns on buyers is upper bounded as follows.*

$$\begin{aligned}
U_{\text{co}}^* - U_{\text{in}}^* \leq & \frac{1}{J} \left[\frac{c^2 e^{-c}}{2} \int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) 2\bar{F}(y) f(y) dy \right. \\
& + \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y - x) \bar{F}(y) f(x) f(y) c^3 e^{-c\bar{F}(x)} dy dx \\
& \left. - c^2 \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} r_{\text{in}} f(r_{\text{in}}) e^{-c\bar{F}(r_{\text{in}})} \right] + o(1/J).
\end{aligned}$$

The bound above highlights the tension between the two effects taking place: \mathcal{MBC} and \mathcal{SRI} . The sign of the bound is governed by which effect dominates. The bound can be evaluated for any particular distribution and market parameters.

We denote by $\overline{\Delta U}^*$ the asymptotic value of the upper bound of $J(U_{\text{co}}^* - U_{\text{in}}^*)$, i.e.,

$$\begin{aligned} \overline{\Delta U}^* := & \left[\frac{c^2 e^{-c}}{2} \int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) 2\overline{F}(y) f(y) dy \right. \\ & + \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y - x) \overline{F}(y) f(x) f(y) c^3 e^{-c\overline{F}(x)} dy dx \\ & \left. - c^2 \frac{(\overline{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} r_{\text{in}} f(r_{\text{in}}) e^{-c\overline{F}(r_{\text{in}})} \right]. \end{aligned}$$

In the next section, we will specialize the bound to the case of Generalized Pareto distributions and then explore numerically the bound for other distributions.

3.5.2 Generalized Pareto distributions

In this section, we focus on the Generalized Pareto Distributions introduced in Section 3.4 and analyze the sign of the asymptotic upper bound on $U_{\text{co}}^* - U_{\text{in}}^*$, through $\overline{\Delta U}^*$, as derived in Theorem 3.2. The next result shows that the main insight obtained in the case of one intermediary and two advertisers carries over to this general model.

Theorem 3.3. *(impact of coordinated campaigns on buyers's side) Suppose that the distribution of values is given by a generalized Pareto distribution with parameters (σ, ξ) . Then there exists $\bar{c} > 0$ such that for an average number of matches $c < \bar{c}$, there exists ρ_c in $[0, 1)$ such that if the coefficient of variation of values $CV(\xi) \geq \rho_c$, then $\overline{\Delta U}^* \leq 0$.*

Theorem 3.3 highlights that the coefficient of variation of values plays again a key role in the general model with competition among intermediaries (as it did in the case of one intermediary and two advertisers) to determine which effect dominates \mathcal{SRI} or \mathcal{MBC} . In particular, when the coefficient of variation is sufficiently large, the result implies that in the limiting “large market” regime, $U_{\text{co}}^* - U_{\text{in}}^* \leq 0$, and hence, the buyers's side is negatively affected by coordinated campaigns. Since sellers are always negatively affected by this coordination, independent campaign management leads to a Pareto improvement in the value chain.

Figure 3.3(a) depicts the value of $\overline{\Delta U}^*$ derived in Theorem 3.2, corresponding to an asymptotic upper bound on $U_{\text{co}}^* - U_{\text{in}}^*$ when J grows large. Figure 3.3(b) depicts the limit of $(r_{\text{co}} - r_{\text{in}})/(\alpha K)$ when J grows large, which we denote by $\overline{\Delta r}$.

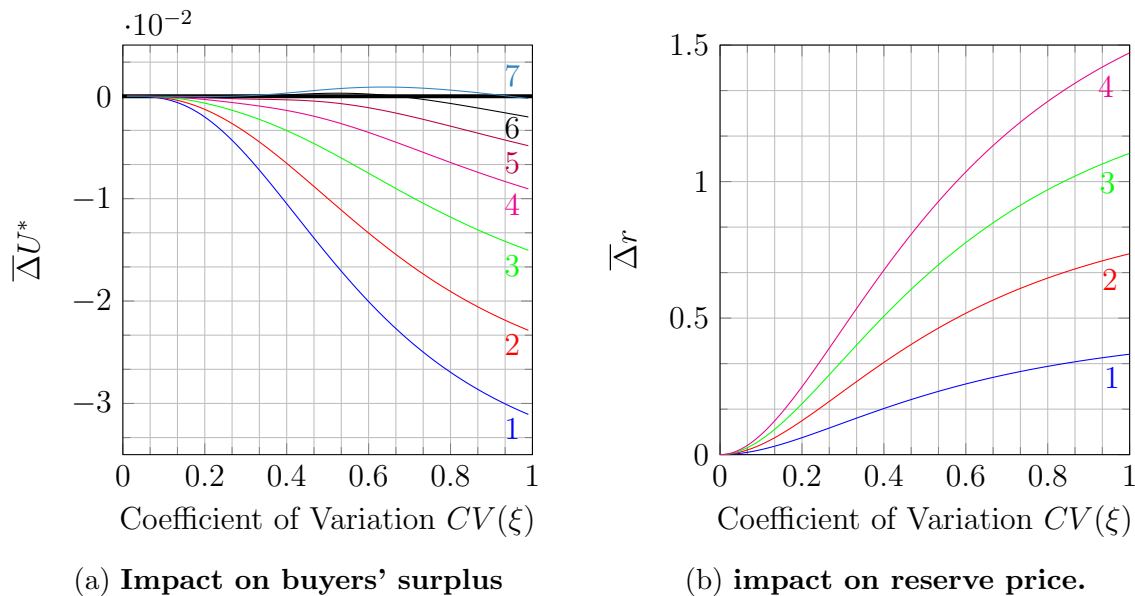


Figure 3.3: Impact of coordinated campaigns on the buyers' side surplus and reserve price as a function of the coefficient of variation and values of $c \in \{1, \dots, 7\}$ for generalized Pareto distributions with parameters $(\sigma = 1 - \xi, \xi)$ in the limiting regime.

Figure 3.3(a) provides a picture-proof of Theorem 3.3 but also enriches the result with numerical values for the thresholds. As seen in the one intermediary and two buyers case in Section 3.4, the negative impact on buyers is mainly due to the fact that the seller reaction dominates the myopic benefit. This is mitigated under the general model due to the competition among intermediaries. However the seller's impact can still be high compared to the myopic benefit of collusion. This reaction of the seller is pictured in Figure 3.3(b). It shows that the seller increases his reserve price with the coefficient of variation and with the average number of matched buyers. Furthermore, if we focus only on markets where the competition is moderate, meaning the average number of interested advertisers is less than four ($c \leq 4$), then for almost any coefficient of variation, coordinated campaigns in the market is always detrimental to buyers' side. This emphasizes the robustness of the insight obtained in the case

of one intermediary and two advertisers. In the next section, we will see that this finding is also robust across different distributions of values for buyers.

3.6 Robustness Analysis

In Sections 3.4 and 3.5, we have shown formally that a Pareto improvement in the value chain is possible under the class of Generalized Pareto distributions as long as the coefficient of variation of values is not too small. In this section, we explore numerically the same question across classes of distributions. In particular, we analyze the Gamma and Beta distributions.

Each of these distributions are parametrized by two positive parameters. The Gamma distribution, parameterized by $a > 0$, $b > 0$, has support in $[0, \infty[$ and its density for $x \geq 0$ is given by

$$f(x) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-\frac{x}{b}},$$

where $\Gamma(\cdot)$ is the gamma function. The Beta distribution, parameterized by $a > 0$ and $b > 0$, has support in $[0, 1]$ and its density is given by for $0 \leq x \leq 1$,

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}.$$

In general, both classes of function belong to the IFR class if their parameters (a, b) are both greater than one. For each of these classes, we sample 1,000 pair of parameters from $[1, 10]^2$. We then compute the corresponding coefficient of variation for that pair and the impact of coordinated campaigns on buyers $U_{\text{co}}^* - U_{\text{in}}^*$ (or the asymptotic upper bound $\bar{\Delta}U^*$). At the end of the procedure, we plot the 1,000 pairs $(CV, U_{\text{co}}^* - U_{\text{in}}^*)$ obtained in the process.

In Figure 3.4, we plot $U_{\text{co}}^* - U_{\text{in}}^*$ for the case of one intermediary and two advertisers for both the Beta and Gamma distributions.

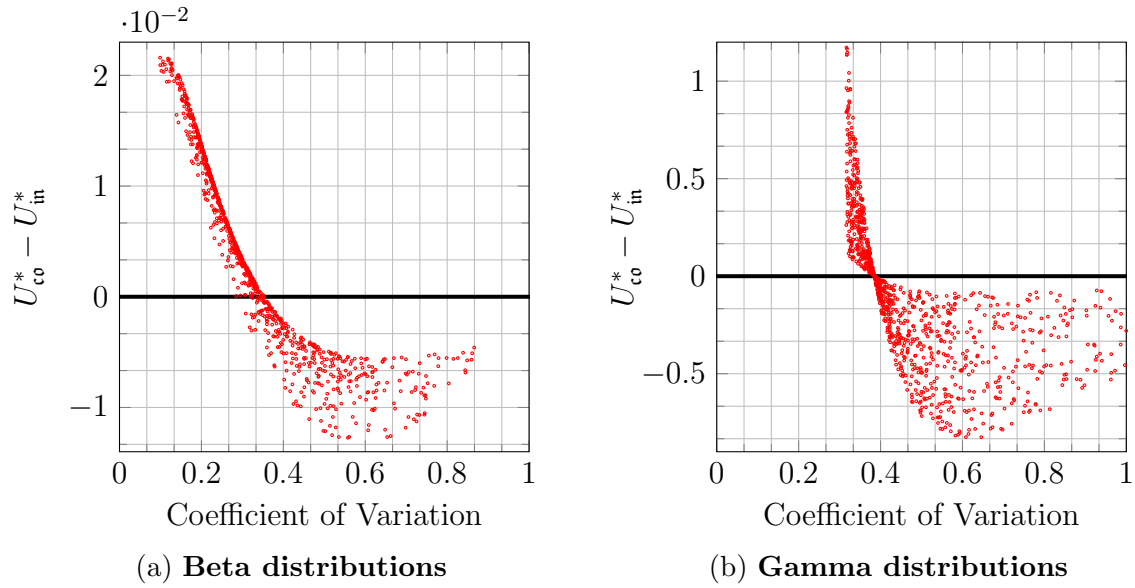


Figure 3.4: Impact of coordinated campaigns on the buyers' side surplus as a function of the coefficient of variation for Beta and Gamma distributions, with one intermediary and two buyers.

We observe that a similar result to Theorem 3.1 appears to hold for the Beta and Gamma distributions. When the coefficient of variation is greater than a threshold (in this case 0.4), then coordinated campaign management is detrimental for the buyers.

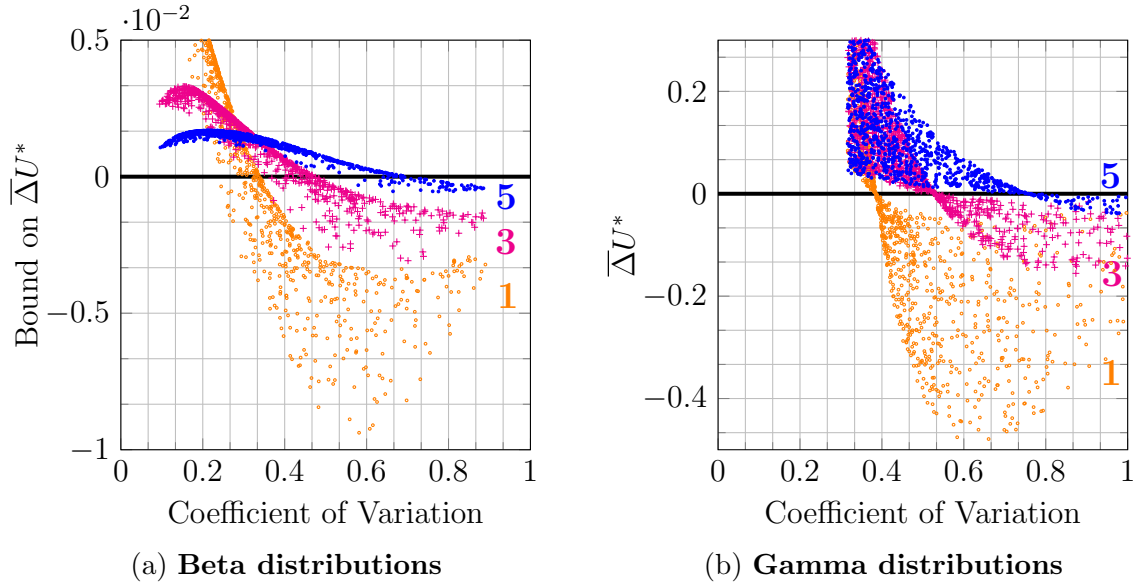


Figure 3.5: Impact of coordinated campaigns on the buyers' side surplus, under the general model, as a function of the coefficient of variation and values of $c \in \{1, 3, 5\}$ for Beta and Gamma distributions.

In Figure 3.5, we plot the asymptotic upper bound $\bar{\Delta U}^*$ derived for the general model analyzed in Section 3.5 for both the Beta and Gamma distributions. Figure 3.5 further emphasizes the robustness of the insights obtained in the previous sections across a broad set of distributions. Indeed, we observe the coefficient of variation is a key driver of the sign of $U_{co}^* - U_{in}^*$ and that for moderate levels competition (as measured by c), there exists a threshold such that the buyers' side is always negatively affected by the practice of coordinated campaign management when the coefficient of variation exceeds this threshold.

3.7 Conclusion

The present chapter has analyzed the implications of the coordination of campaigns implemented by intermediaries in the bidding process when these represent multiple buyers. In particular, we have characterized the impact on the different parties in the value chain. The seller's revenues and the social welfare are always negatively

affected. Notably, the buyers's side is also negatively affected in a broad set of market scenarios, which we characterize to be those with moderate competition and medium to high values of the coefficient of variation of values.

In the online display advertising market, we emphasize here that we do not challenge the value of DSPs in the market. DSPs provide value to advertisers in different ways. What we challenge in this chapter is the tactical role they play when coordinating bids in the auctions, that leads to collusion among subsets of advertisers. This chapter provides a framework to better understand and quantify the implications of this coordination and anchor the debate about multi-bidding.

Many possible extensions can be considered, from the consideration of more general models of values for buyers to include, e.g., common values, to the potential analysis of markets with asymmetric players. Additionally, in the presence of intermediaries, the seller faces a multi-dimensional mechanism design problem and the question of understanding the performance of second price auctions compared to an optimal mechanism is still open. More broadly, this chapter raises the important question of whether it is possible to resolve the inefficiency identified in the online display advertising value chain. The present chapter shows that advertisers themselves might have an incentive to advocate for independent campaign management as an industry norm, to ensure that advertisers' campaigns are always managed independently of other advertisers represented by the same DSP. From that perspective, a possible approach to this inefficiency is to tackle it at the interface between advertisers and DSPs. Another possible avenue, which is also interesting from a research perspective, is to investigate if this inefficiency can be mitigated at the interface between the seller and the DSPs through an adjustment of the selling mechanism to, e.g., dynamic mechanisms across multiple items.

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Appendices

Appendix A

Appendix for Chapter 1

A.1 Proofs for Section 1.3

Proof of Theorem 3.2. The goal of this proof is to show that we can focus on the scale-free mechanisms without loss of optimality. More precisely, we will establish that for any mechanism in $m \in \mathcal{M}$ there exists a mechanism $\check{m} \in \mathcal{M}_{sf}$ such that

$$\inf_{F \in \mathcal{F}} R(\check{m}, F) \geq \inf_{F \in \mathcal{F}} R(m, F).$$

Consider a mechanism $(\boldsymbol{x}, \boldsymbol{t})$ in \mathcal{M} . Consider $\tilde{F} \in \mathcal{F}$, with corresponding density \tilde{f} , and let $\theta > 0$. By the scale invariance assumption on \mathcal{F} , the distribution $F_{\theta^{-1}}$ also belongs to \mathcal{F} . This implies that for any $\theta > 0$, by definition of the infimum, we have

$$R(m, \tilde{F}_{\theta^{-1}}) \geq \inf_{F \in \mathcal{F}} R(m, F). \tag{A.1-1}$$

We first analyze $R(m, \tilde{F}_{\theta^{-1}})$. By Lemma A.1-2, stated and proved after this result, we have that the denominator of $R(m, \tilde{F}_{\theta^{-1}})$ is given by

$$\text{opt}(\tilde{F}_{\theta^{-1}}) = \theta \text{opt}(\tilde{F}). \tag{A.1-2}$$

We next analyze the numerator of $R(m, \tilde{F}_{\theta^{-1}})$. Using Lemma 1.1, we have

$$\begin{aligned}
& \mathbb{E}_{\tilde{F}_{\theta^{-1}}} \left[\sum_{i=1}^2 t_i(v_1, v_2) \right] \\
&= \sum_{i=1}^2 \mathbb{E}_{\tilde{F}_{\theta^{-1}}} \left[v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(l, v_{-i}) dl \right] \\
&= \sum_{i=1}^2 \int_0^\infty \int_0^\infty \left[v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(l, v_{-i}) dl \right] d\tilde{F}(\theta^{-1}v_i) d\tilde{F}(\theta^{-1}v_{-i}) \\
&= \sum_{i=1}^2 \int_0^\infty \int_0^\infty \left[\theta v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{\theta v_i} x_i(l, \theta v_{-i}) dl \right] d\tilde{F}(u_i) d\tilde{F}(u_{-i}) \\
&= \sum_{i=1}^2 \int_0^\infty \int_0^\infty \theta \left[v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \right] d\tilde{F}(u_i) d\tilde{F}(u_{-i}).
\end{aligned}$$

Hence, we have

$$\mathbb{E}_{\tilde{F}_{\theta^{-1}}} \left[\sum_{i=1}^2 t_i(v_1, v_2) \right] = \theta \sum_{i=1}^2 \mathbb{E}_{\tilde{F}} \left[v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \right].$$

and

$$R(m, \tilde{F}_{\theta^{-1}}) = \frac{\sum_{i=1}^2 \mathbb{E}_{\tilde{F}} \left[v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \right]}{\text{opt}(\tilde{F})}$$

We show in Lemma A.1-1 (stated and proved after this proof) that for any \mathbf{x} in \mathcal{M} , there exists $\check{x}_i(\cdot, \cdot)$, such that for any $v_1, v_2 \geq 0$,

$$\check{x}_i(v_1, v_2) := \lim_{\theta \downarrow 0} x_i(\theta v_1, \theta v_2)$$

Note that $x_i(\theta s, \theta v_{-i}) \leq 1$ and hence by an application of the dominated convergence theorem, we have

$$\int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \rightarrow \int_0^{v_i} \check{x}_i(s, v_2) ds \quad \text{as } \theta \downarrow 0.$$

Now note that

$$0 \leq v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \leq v_i,$$

and $\mathbb{E}_{\tilde{F}}[v_i] < \infty$, since \tilde{F} belongs to \mathcal{G} . By another application of the dominated convergence theorem, we have

$$\begin{aligned} \lim_{\theta \downarrow 0} R(m, \tilde{F}_{\theta^{-1}}) &= \lim_{\theta \downarrow 0} \frac{\sum_{i=1}^2 \mathbb{E}_{\tilde{F}} \left[v_i x_i(\theta v_i, \theta v_{-i}) - \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \right]}{\text{opt}(\tilde{F})} \\ &= \frac{\sum_{i=1}^2 \mathbb{E}_{\tilde{F}} \left[v_i \check{x}_i(v_i, v_{-i}) - \int_0^{v_i} \check{x}_i(s, v_{-i}) ds \right]}{\text{opt}(\tilde{F})} \end{aligned}$$

Note that $\check{x}_i(v_1, v_2)$ and the payments $\check{t}_i(v_1, v_2) = v_i \check{x}_i(v_i, v_{-i}) - \int_0^{v_i} \check{x}_i(s, v_{-i}) ds$ is a feasible mechanism. Indeed,

$$v_i \check{x}_i(v_1, v_2) - \check{t}_i(v_1, v_2) = \int_0^{v_i} \check{x}_i(s, v_{-i}) ds \geq 0,$$

and hence (IR) is satisfied.

$$\begin{aligned} v_i \check{x}_i(v_i, v_{-i}) - \check{t}_i(v_i, v_{-i}) &= \lim_{\theta \downarrow 0} \int_0^{v_i} x_i(\theta s, \theta v_{-i}) ds \\ &= \lim_{\theta \downarrow 0} \left[v_i x_i(\theta v_i, \theta v_{-i}) - \theta^{-1} t_i(\theta v_i, \theta v_{-i}) \right] \\ &\geq \lim_{\theta \downarrow 0} \theta^{-1} \left[\theta v'_i x_i(\theta v'_i, \theta v_{-i}) - t_i(\theta v'_i, \theta v_{-i}) \right] \\ &= v'_i \check{x}_i(v'_i, v_{-i}) - \check{t}_i(v'_i, v_{-i}), \end{aligned}$$

where the last inequality follows from the fact that the initial mechanism $m = (\mathbf{x}, \mathbf{t})$ is incentive compatible. Hence, we deduce that (IC) is satisfied for the the new mechanism $\check{m} = (\check{\mathbf{x}}, \check{\mathbf{t}})$. The probability constraints (AC) is clearly satisfied by $\check{\mathbf{x}}$. Hence, we have established that

$$\lim_{\theta \downarrow 0} R(m, \tilde{F}_{\theta^{-1}}) = R(\check{m}, \tilde{F}).$$

Taking the limit on the left-hand-side of (A.1-1) as $\theta \downarrow 0$, we obtain

$$R(\check{m}, \tilde{F}) = \lim_{\theta \downarrow 0} R(m, \tilde{F}_{\theta^{-1}}) \geq \inf_{F \in \mathcal{F}} R(m, F).$$

Taking the infimum over all \tilde{F} in \mathcal{F} , we obtain

$$\inf_{F \in \mathcal{F}} R(\check{m}, F) \geq \inf_{F \in \mathcal{F}} R(m, F). \quad (\text{A.1-3})$$

Note that \tilde{m} belongs to \mathcal{M}_{sf} . Indeed,

$$\tilde{x}_i(\lambda v_1, \lambda v_2) = \lim_{\theta \downarrow 0} x_i(\theta \lambda v_1, \theta \lambda v_2) = \lim_{\gamma \downarrow 0} x_i(\gamma v_1, \gamma v_2) = \tilde{x}_i(v_1, v_2).$$

Taking successively the supremum over all mechanisms \tilde{m} in \mathcal{M}_{sf} , and then over all mechanisms m in \mathcal{M} in (A.1-3), we obtain

$$\mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}) \geq \mathcal{R}(\mathcal{M}, \mathcal{F}). \quad (\text{A.1-4})$$

Since the other inequality is trivial, we conclude that

$$\mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}) = \mathcal{R}(\mathcal{M}, \mathcal{F}).$$

This finalizes the proof. □

Lemma A.1-1. *For any mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M} , for any $v_1, v_2 \geq 0$, and $i = 1, 2$, $x_i(\lambda v_i, \lambda v_{-i})$ admits a limit as $\lambda \downarrow 0$.*

Proof of Lemma A.1-1. Fix $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M} , there exists $\varepsilon > 0$ such that $\max_{i=1,2} \{TV(x_i, [0, \varepsilon]^2)\} < \infty$.

Let us fix (v_i, v_{-i}) . To show the result, we will first analyze the ‘‘variation’’ of the function $J(\lambda) := x_i(\lambda v_i, \lambda v_{-i})$ as a function of λ on $[0, \varepsilon / \max(v_i, v_{-i})]$. For that let us fix an integer $N \geq 1$ and a sequence of non-negative integer $(\lambda_j)_{1 \leq j \leq N}$ such that $\lambda_j \leq \lambda_{j+1} \leq \varepsilon / \max(v_i, v_{-i})$ for all j in $1, \dots, N-1$.

If we denote $\mathbf{u}_j = (\lambda_j v_i, \lambda_j v_{-i})$ then we have $\mathbf{u}_j \leq \mathbf{u}_{j+1} \leq (\varepsilon, \varepsilon)$. Since x_i has a bounded variation then

$$\sum_{j=1}^{N-1} |J(\lambda_{j+1}) - J(\lambda_j)| = \sum_{j=1}^{N-1} |x_i(\mathbf{u}_{j+1}) - x_i(\mathbf{u}_j)| \leq TV(x_i, [0, \varepsilon]^2) < \infty,$$

Hence the function $J(\cdot)$ has a bounded total variation on $[0, \varepsilon / \max(v_i, v_{-i})]$. By Jordan’s decomposition, see (Cohn, 2013, Proposition 4.4.2.), one can write J as the difference between two monotone functions. In turn, the fact that a monotone function admits a right limit at each point implies that $x_i(\lambda v_i, \lambda v_{-i})$ admits a limit as $\lambda \downarrow 0$. This concludes the proof. □

Lemma A.1-2. For any distribution F in \mathcal{G} , and $\theta > 0$, we have $\text{opt}(F_\theta) = \theta^{-1} \text{opt}(F)$.

Proof of Lemma A.1-2. The goal of this proof is to show that the optimal revenue scales as we we scale the distribution.

Fix $\theta > 0$ and $F \in \mathcal{G}$. Let $m = (\mathbf{x}, \mathbf{t})$ denote a mechanism in \mathcal{M} that is ε away from optimal for problem (2.1), i.e., such that

$$\mathbb{E}_F \left[\sum_{i=1}^2 t_i(v_1, v_2) \right] \geq \text{opt}(F) - \varepsilon.$$

Denote by m^θ the mechanism characterized by the allocations and payments given by

$$\begin{aligned} \mathbf{x}^\theta(v_1, v_2) &= \mathbf{x}(\theta v_1, \theta v_2), \\ \mathbf{t}^\theta(v_1, v_2) &= \theta^{-1} \mathbf{t}(\theta v_1, \theta v_2). \end{aligned}$$

Note that for any $i = 1, 2$, $v_1, v_2, \hat{v}_1, \hat{v}_2$, we have

$$v_i x_i^\theta(v_1, v_2) - t_i^\theta(v_1, v_2) = \theta^{-1} [\theta v_i x_i(\theta v_1, \theta v_2) - t_i(v_1, v_2)] \geq 0,$$

where the last inequality follows from the fact that m satisfies (IR). Hence m^θ also satisfies (IR). In addition, we have

$$\begin{aligned} v_i x_i^\theta(v_i, v_{-i}) - t_i^\theta(v_i, v_{-i}) &= \theta^{-1} [\theta v_i x_i(\theta v_i, \theta v_{-i}) - t_i(\theta v_i, \theta v_{-i})] \\ &\geq \theta^{-1} [\theta v_i x_i(\theta \hat{v}_i, \theta v_{-i}) - t_i(\theta \hat{v}_i, \theta v_{-i})] \\ &= v_i x_i^\theta(\hat{v}_i, v_{-i}) - t_i^\theta(\hat{v}_i, v_{-i}), \end{aligned}$$

where the inequality is consequence of the fact that m satisfies (IR). Hence, (IC) is also satisfied. Finally, (AC) follows directly from the feasibility of the mechanism m .

We deduce that m^θ is feasible for problem (2.1). Furthermore, its performance when the distribution of values is F_θ satisfies

$$\begin{aligned}
\mathbb{E}_{F_\theta} \left[\sum_{i=1}^2 t_i^\theta(v_1, v_2) \right] &= \theta^{-1} \mathbb{E}_{F_\theta} \left[\sum_{i=1}^2 t_i(\theta v_1, \theta v_2) \right] \\
&= \theta^{-1} \sum_{i=1}^2 \int_0^\infty \int_0^\infty t_i(\theta v_1, \theta v_2) dF(\theta v_1) dF(\theta v_2) \\
&= \theta^{-1} \sum_{i=1}^2 \int_0^\infty \int_0^\infty t_i(u_1, u_2) dF(u_1) dF(u_2) \\
&= \theta^{-1} \mathbb{E}_F \left[\sum_{i=1}^2 t_i(v_1, v_2) \right] \\
&\geq \theta^{-1} [\text{opt}(F) - \varepsilon].
\end{aligned}$$

We deduce that

$$\text{opt}(F_\theta) \geq \theta^{-1} \text{opt}(F).$$

Through a symmetric argument, we have that

$$\text{opt}(F) \geq \theta \text{opt}(F_\theta).$$

The result of the lemma follows and the proof is complete. \square

Proof of Lemma 1.2. We will exhibit a distribution, F_B , for which the performance of any scale-free mechanism can be arbitrarily close to zero. We will consider a Bernoulli distribution, F_B that put mass $q > 0$ at $v = 1$ and the remaining $1 - q$ at 0, i.e.

$$\mathbb{P}(v = 1) = 1 - \mathbb{P}(v = 0) = q.$$

Let us analyze the ratio for this distribution. Let us start by the denominator, i.e. the optimal revenue of F_B .

It is clear that given the structure of the distribution, it is suboptimal to charge the winner a price different from 0 or 1. Hence the optimal mechanism is a posted price equal to 1, hence the optimal revenue is given by

$$\text{opt}(F_B) = 1 \mathbb{P}(v_{[1]} = 1) = q (2 - q).$$

Now let us analyze the performance of a prior independent mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf} . We have

$$\mathbb{E}[t_i(v_i, v_{-i})] = q^2 t_i(1, 1) + q(1 - q)(t_i(1, 0) + t_i(0, 1)) + (1 - q)^2 t_i(0, 0).$$

From the incentive rational constraints, we have that an optimal mechanism necessary verifies $t_i(0, 0) = t_i(0, 1) = 0$. From Lemma A.1-3, stated and proved after this proof, we have that $t_i(1, 0) = 0$. Hence we get that

$$\sum_{i=1}^2 \mathbb{E}[t_i(v_i, v_{-i})] = q^2 (t_1(1, 1) + t_2(1, 1)).$$

Using the coupling and incentive rational constraints, we have that $t_1(1, 1) + t_2(1, 1) \leq 1$. Hence, we conclude that

$$\sum_{i=1}^2 \mathbb{E}[t_i(v_i, v_{-i})] \leq q^2.$$

So from the previous analysis, we conclude that the performance of any mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf} is upper bounded as follows

$$R(m, F_B) \leq \frac{q^2}{q(2 - q)} = \frac{q}{2 - q}.$$

Hence, by taking the limit as q goes to 0, we conclude

$$\mathcal{R}(\mathcal{M}, \mathcal{G}) = \mathcal{R}(\mathcal{M}_{sf}, \mathcal{G}) = 0.$$

□

Lemma A.1-3. *Consider any mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf} . Then it must satisfy that*

$$t_i(v_i, 0) = 0 \quad \text{for any } v_i \geq 0.$$

Proof of Lemma A.1-3. Let $m = (\mathbf{x}, \mathbf{t})$ a mechanism in \mathcal{M}_{sf} . Let $v_i \geq 0$, then we have that $t_i(v_i, 0) = v_i t_i(1, 0)$. So it is sufficient to show that $t_i(1, 0) = 0$ to conclude the result.

Using the DSIC, we have that for any $a > 0$,

$$x_i(1, 0) - t_i(1, 0) \geq x_i(a, 0) - t_i(a, 0).$$

Since $m = (\mathbf{x}, \mathbf{t})$ is a mechanism in \mathcal{M}_{sf} , then $x_i(1, 0) = x_i(a, 0)$ and $t_i(a, 0) = at_i(1, 0)$, so we conclude that for any $a > 0$ that

$$(a - 1) t_i(1, 0) \geq 0,$$

so necessary, we conclude that $t_i(1, 0) = 0$, hence $t_i(v_i, 0) = v_i t_i(1, 0) = 0$. \square

A.2 Proofs for Section 1.4

Proof of Proposition 1.1. Let us fix subclass \mathcal{F} of the set of regular distributions \mathcal{F}_{reg} . It is clear that since $\mathcal{M}'_{sf} \subset \mathcal{M}_{sf}$, then

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{F}) \leq \mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}).$$

Let us show the reverse inequality. For that, we will proceed in two steps. We start from an initial mechanism in \mathcal{M}_{sf} and approximate the allocations by a combination of step functions (with corresponding payments). In second step, we will compare the performance of the approximation to that of the original mechanism to conclude.

Step 1: Let us pick a symmetric mechanism (\mathbf{x}, \mathbf{t}) in \mathcal{M}_{sf} .

Let us fix an integer $N \geq 2$. Define

$$\alpha_n := \inf \left\{ r \geq 0 : x_1(r, 1) \geq \frac{n}{N} \right\}, \text{ for } n \in \{1, \dots, N\}.$$

Note that by symmetry, we have that $\alpha_n = \inf \{ r \geq 0 : x_2(1, r) \geq \frac{n}{N} \}$. Note also that by the monotonicity of $x_1(\cdot, 1)$, the sequence $\{\alpha_n\}_{1 \leq n \leq N}$ is non-decreasing. We define $\alpha_0 := 0$ and $\alpha_{N+1} := \infty$.

For any $v_i, v_{-i} \geq 0$, define

$$\tilde{x}_i(v_i, v_{-i}) = \begin{cases} \sum_{n=1}^N \frac{1}{N} \mathbf{1}\{v_i > \alpha_n v_{-i}\}, & \text{if } v_i \neq v_{-i}, \\ \tilde{x}_i(1^-, 1), & \text{if } v_i = v_{-i}, \end{cases}$$

where $\tilde{x}_i(1^-, 1) := \lim_{r \uparrow 1} \sum_{n=1}^N \frac{1}{N} \mathbb{1}\{r > \alpha_n\}$.

The allocation \tilde{x} clearly has the scale-free property. Moreover by construction, we have that for any $v_i \geq 0$,

$$0 \leq \tilde{x}_i(v_i, v_{-i}) \leq 1, \quad \sum_{i=1}^2 \tilde{x}_i(v_i, v_{-i}) \leq \sum_{i=1}^2 x_i(v_i, v_{-i}) \leq 1 \text{ and } \tilde{x}_i(\cdot, v_{-i}) \text{ is non-decreasing.}$$

and if we introduce the payment function $\tilde{\mathbf{t}}$ corresponding to the allocation $\tilde{\mathbf{x}}$, defined by Lemma 1.1, then the mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$ belongs to \mathcal{M}_{sf} . Moreover note that, by construction and monotonicity of $x_i(\cdot, 1)$, we have

$$\tilde{x}_i(1^-, 1) = \lim_{r \uparrow 1} \sum_{n=1}^N \frac{1}{N} \mathbb{1}\{r > \alpha_n\} \leq x_i(1^-, 1) \leq x_i(1, 1).$$

Since (\mathbf{x}, \mathbf{t}) is in \mathcal{M}_{sf} then by constraint (AC), we have that $x_i(1, 1) \leq 1/2$. Hence, by construction, we have that $(\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$ belongs to \mathcal{M}'_{sf} .

Furthermore, note that for any $0 \leq n \leq N$, we have for any l in $(\alpha_n v_{-i}, \alpha_{n+1} v_{-i})$ such that $l \neq v_{-i}$,

$$\tilde{x}_i(l, v_{-i}) \leq x_i(l, v_{-i}) \leq \frac{1}{N} + \tilde{x}_i(l, v_{-i}). \quad (\text{A.2-5})$$

Indeed, if l in $(\alpha_n v_{-i}, \alpha_{n+1} v_{-i}) \setminus \{v_{-i}\}$, then we have by construction

$$\tilde{x}_i(l, v_{-i}) = \frac{n}{N} \text{ and } \frac{n}{N} \leq x_i(l, v_{-i}) \leq \frac{n+1}{N},$$

where the inequalities follows from the definition of the sequence α_n .

Step 2: Next, we compute the revenue of the original mechanism and that of the new mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$.

For any $F \in \mathcal{F} \subset \mathcal{F}_{reg}$, we have that

$$\begin{aligned} \mathbb{E}_F [t_i(v_i, v_{-i})] &= \mathbb{E}_F [x_i(v_i, v_{-i})\phi(v_i)] \\ &= \sum_{n=0}^N \mathbb{E}_F [x_i(v_i, v_{-i}) \phi(v_i) \mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\} \mathbb{1}\{v_{-i} \neq v_i\}]. \end{aligned}$$

The first equality is a classical result in the literature, see Myerson (1981). In the second equality, we have used that the set

$$B_N = \{(v_i, v_{-i}) \text{ in } \mathbb{R}^+ \times \mathbb{R}^+ : v_{-i} = v_i\} \cup (\cup_{1 \leq n \leq N} \{(v_i, v_{-i}) \text{ in } \mathbb{R}^+ \times \mathbb{R}^+ : \alpha_n v_{-i} = v_i\})$$

is such that $\mathbb{P}(B_N) = 0$ given that F admits a density. Hence we get that (recalling that r_F denotes the optimal reserve price when the distribution is known to be F)

$$\begin{aligned}
& \mathbb{E}_F [t_i(v_i, v_{-i})] \\
&= \sum_{n=0}^N \mathbb{E}_F [x_i(v_i, v_{-i})\phi(v_i)\mathbb{1}\{v_i < r_F\}\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\}] \\
&\quad + \mathbb{E}_F [x_i(v_i, v_{-i})\phi(v_i)\mathbb{1}\{v_i \geq r_F\}\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\}] \\
&\stackrel{(a)}{\leq} \sum_{n=0}^N \mathbb{E}_F [\tilde{x}_i(v_i, v_{-i})\phi(v_i)\mathbb{1}\{v_i < r_F\}\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\}] \\
&\quad + \mathbb{E}_F \left[\left(\tilde{x}_i(v_i, v_{-i}) + \frac{1}{N} \right) \phi(v_i)\mathbb{1}\{v_i \geq r_F\}\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\} \right] \\
&= \sum_{n=0}^N \mathbb{E}_F [\tilde{x}_i(v_i, v_{-i})\phi(v_i)\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\}] \\
&\quad + \frac{1}{N} \mathbb{E}_F [\phi(v_i)\mathbb{1}\{v_i \geq r_F\}\mathbb{1}\{\alpha_n v_{-i} < v_i < \alpha_{n+1} v_{-i}\}\mathbb{1}\{v_{-i} \neq v_i\}].
\end{aligned}$$

In (a) we have used that $\phi(v_i)$ is non-negative for $v_i \geq r_F$, non-positive for $v_i < r_F$ and the inequalities in (A.2-5).

Now using again the fact that $\mathbb{P}(B_N) = 0$, we have that

$$\begin{aligned}
\mathbb{E}_F [t_i(v_i, v_{-i})] &\leq \mathbb{E}_F [\tilde{x}_i(v_i, v_{-i})\phi(v_i)] + \frac{1}{N} \mathbb{E}_F [\phi(v_i)\mathbb{1}\{v_i \geq r_F\}] \\
&\leq \mathbb{E}_F [\tilde{x}_i(v_i, v_{-i})\phi(v_i)] + \frac{1}{N} \mathbb{E}_F [\phi(v_{[1]})\mathbb{1}\{v_{[1]} \geq r_F\}] \\
&= \mathbb{E}_F [\tilde{t}_i(v_i, v_{-i})] + \frac{1}{N} \text{opt}(F),
\end{aligned}$$

in the last inequality, we have used the regularity of the distribution F and we recall that $v_{[1]} := \max\{v_1, v_2\}$. Hence, we conclude that for any distribution F in \mathcal{F} ,

$$\frac{\mathbb{E}_F [\sum_{i=1}^2 t_i(v_i, v_{-i})]}{\text{opt}(F)} \leq \frac{\mathbb{E}_F [\sum_{i=1}^2 \tilde{t}_i(v_i, v_{-i})]}{\text{opt}(F)} + \frac{2}{N}$$

By taking the infimum over all distribution F in \mathcal{F} , we get that for all $N \geq 2$,

$$\inf_{F \in \mathcal{F}} R(m, F) \leq \inf_{F \in \mathcal{F}} R(\tilde{m}, F) + \frac{2}{N}$$

Taking the supremum over mechanisms in \mathcal{M}'_{sf} and then \mathcal{M}_{sf} , and letting $N \uparrow \infty$ leads to the result. \square

Proof of Lemma 1.3. Fix $\alpha \geq 0$. We will mainly use the Portmanteau Theorem to show the limits.

Suppose first that $\alpha \neq 1$. Let $M = \max\{\sup_{n \geq 1} \bar{v}_{F_n}, \bar{v}_F + 1\}$. Consider the function $h(v_1, v_2) = v_2 \mathbf{1}\{v_1 > \alpha v_2\} \mathbf{1}\{v_1 \leq M\} \mathbf{1}\{v_2 \leq M\}$. The function h is bounded and continuous $F \times F$ -almost surely, where $F \times F$ is the product measure. Indeed, h is discontinuous on the lines $v_1 = \alpha v_2$, $v_1 = M$ and $v_2 = M$, which have $F \times F$ measure zero since $\alpha \neq 1$ and $M > \bar{v}_F$. Hence by Corollary 1 and Theorem 25.12 in Billingsley (2008), we conclude that

$$\lim_{n \uparrow \infty} \mathbb{E}_{F_n} [v_2 \mathbf{1}\{v_1 > v_2\}] = \mathbb{E}_{F_a} [v_2 \mathbf{1}\{v_1 > \alpha v_2\}].$$

Suppose now that $\alpha = 1$.

Since F_n admits a density, we have that

$$\mathbb{E}_{F_n} [v_2 \mathbf{1}\{v_1 > v_2\}] = \frac{1}{2} (\mathbb{E}_{F_n} [v_2 \mathbf{1}\{v_1 > v_2\}] + \mathbb{E}_{F_n} [v_1 \mathbf{1}\{v_1 \leq v_2\}]) = \frac{1}{2} \mathbb{E}_{F_n} [\min(v_1, v_2)].$$

Note that the minimum of two independent values with cdf F_n admits a cdf given by $F_n(2 - F_n)$. the fact that F_n converges weakly to F implies that $F_n(2 - F_n)$ converges weakly to $F(2 - F)$, which is the cdf of the minimum of two independent values with cdf F . By Corollary 1 and Theorem 25.12 in Billingsley (2008) applied to the function $\frac{1}{2} w \mathbf{1}\{w \leq M\}$, we have that $\lim_{n \uparrow \infty} \frac{1}{2} \mathbb{E}_{F_n} [\min(v_1, v_2)] = \frac{1}{2} \mathbb{E}_F [\min(v_1, v_2)]$, and we conclude that

$$\lim_{n \uparrow \infty} \mathbb{E}_{F_n} [v_2 \mathbf{1}\{v_1 > \alpha v_2\}] = \frac{1}{2} \mathbb{E}_F [\min(v_1, v_2)].$$

This concludes the proof. □

Proof of Proposition 1.2. Let us fix a mechanism in $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}'_{sf} . Then, for $i = 1, 2$,

$$x_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \mathbf{1}\{v_i > \gamma_k v_{-i}\} \mathbf{1}\{v_i \neq v_{-i}\} + c \mathbf{1}\{v_i = v_{-i}\},$$

for some $N \geq 1$, $\gamma \in \mathbb{R}^N$ and $c \in [0, \frac{1}{2}]$.

We will establish that the worst-case performance of this mechanism against \mathcal{F} is upper bounded by the performance of an alternate mechanism $\tilde{m} \in \mathcal{M}'_{sf}$ against \mathcal{W}' . Fix an element $F \in \mathcal{W}'$. By assumption, there exists a sequence F_n from elements in \mathcal{F} such that F_n weakly converges to F .

Step 1. We first establish that

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq \text{opt}(F).$$

Let $a = \bar{v}_F < \infty$. Note that F_n is regular and hence

$$\text{opt}(F_n) = \max_{y \geq 0} \mathbb{E}_{F_n} [\max\{v_{[2]}, y\} \mathbf{1}\{v_{[2]} \geq y\}].$$

In particular, we have for any $y < a$,

$$\begin{aligned} \text{opt}(F_n) &\geq \mathbb{E}_{F_n} [v_{[2]} \mathbf{1}\{v_{[2]} \geq y\}] + 2y \bar{F}_n(y) (1 - \bar{F}_n(y)) \\ &= \int_y^\infty x 2 f_n(x) \bar{F}_n(x) dx + 2y \bar{F}_n(y) (1 - \bar{F}_n(y)) \\ &= \left[-x \bar{F}_n^2(x) \right]_y^\infty + \int_y^\infty \bar{F}_n^2(x) dx + 2y \bar{F}_n(y) (1 - \bar{F}_n(y)) \\ &= \int_y^\infty \bar{F}_n^2(x) dx + y \bar{F}_n(y) (2 - \bar{F}_n(y)) \\ &\geq y \bar{F}_n(y) (2 - \bar{F}_n(y)). \end{aligned}$$

By taking the \liminf we get that for any $y < a$,

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq y \bar{F}(y) (2 - \bar{F}(y)).$$

by taking the limit as $y \uparrow a$

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq a \bar{F}(a^-) (2 - \bar{F}(a^-)) = \text{opt}(F).$$

where the last equality follows from Lemma A.2-4 (stated and proved after this proof).

Step 2. Next, we derive an asymptotic upper bound on the ratio $R(m, F_n)$. Let us define

$$\begin{aligned} \mathcal{I} &:= \{k \in [1, N] : \gamma_k = 1\}, \quad \mathcal{I}^- := \{k \in [1, N] : \gamma_k < 1\}, \\ \mathcal{I}^+ &:= \{k \in [1, N] : \gamma_k > 1\}. \end{aligned}$$

Note that since the distribution F_n admits a density, we have

$$R(m, F_n) = \sum_{k=1}^N \frac{1}{N} \sum_{i=1}^2 \frac{\mathbb{E}_{F_n} [\gamma_k v_{-i} \mathbf{1}\{v_i > \gamma_k v_{-i}\}]}{\text{opt}(F_n)}.$$

Using Step 1 in conjunction with Lemma 1.3, we have

$$\begin{aligned} & \limsup_{n \uparrow \infty} R(m, F_n) \\ &= \frac{1}{\liminf_{n \uparrow \infty} \text{opt}(F_n)} \sum_{k=1}^N \frac{1}{N} \sum_{i=1}^2 \lim_{n \uparrow \infty} \mathbb{E}_{F_n} [\gamma_k v_{-i} \mathbf{1}\{v_i > \gamma_k v_{-i}\}] \\ &\leq \frac{1}{\text{opt}(F)} \left[\sum_{k \in \mathcal{I}} \frac{1}{N} \mathbb{E}_F [\min\{v_1, v_2\}] + \sum_{k \in \mathcal{I}^- \cup \mathcal{I}^+} \frac{1}{N} \sum_{i=1}^2 \mathbb{E}_F [\gamma_k v_{-i} \mathbf{1}\{v_i > \gamma_k v_{-i}\}] \right]. \end{aligned}$$

Noting that

$$\mathbb{E}_F [\min\{v_1, v_2\}] = \mathbb{E}_F \left[\sum_{i=1}^2 v_{-i} \mathbf{1}\{v_i > v_{-i}\} + \frac{1}{2} v_{-i} \mathbf{1}\{v_{-i} = v_i\} \right],$$

we deduce that

$$\begin{aligned} & \limsup_{n \uparrow \infty} R(m, F_n) \\ &\leq \frac{1}{\text{opt}(F)} \sum_{i=1}^2 \mathbb{E}_F \left[\sum_{k \in [1, N]} \frac{1}{N} \gamma_k v_{-i} \mathbf{1}\{v_i > \gamma_k v_{-i}\} + \frac{|\mathcal{I}|}{2N} v_{-i} \mathbf{1}\{v_{-i} = v_i\} \right], \end{aligned}$$

where $|S|$ represents the cardinality of a finite set S .

For any $v_i, v_{-i} \geq 0$, we have that

$$\mathbf{1}\{v_i > \gamma_k v_{-i}\} \mathbf{1}\{v_{-i} = v_i\} = \mathbf{1}\{\gamma_k < 1\} \mathbf{1}\{v_{-i} = v_i\}.$$

This implies that

$$\begin{aligned} & \limsup_{n \uparrow \infty} R(m, F_n) \\ &\leq \frac{1}{\text{opt}(F)} \sum_{i=1}^2 \mathbb{E}_F \left[\sum_{k \in [1, N]} \frac{\gamma_k v_{-i}}{N} \mathbf{1}\{v_i > \gamma_k v_{-i}\} \mathbf{1}\{v_{-i} \neq v_i\} \right. \\ &\quad \left. + \left(\sum_{k \in \mathcal{I}^-} \frac{1}{N} \gamma_k + \frac{|\mathcal{I}|/2}{N} \right) v_{-i} \mathbf{1}\{v_{-i} = v_i\} \right]. \end{aligned}$$

Step 3: We now show that the upper bound above can be expressed as the performance of mechanism against F . Let us define the allocation $\tilde{\mathbf{x}}$ for each $v_i, v_{-i} \geq 0$ as

$$\tilde{x}_i(v_i, v_{-i}) = \sum_{k \in [1, N]} \frac{1}{N} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_{-i} \neq v_i\} + \frac{1}{N} \left(|\mathcal{I}^-| + \frac{|\mathcal{I}|}{2} \right) \mathbb{1}\{v_{-i} = v_i\},$$

and $\tilde{\mathbf{t}}$ its corresponding payment using Lemma A.2-5, and let $\tilde{m} = (\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$. We may rewrite the previous inequalities as

$$\limsup_{n \uparrow \infty} R(m, F_n) \leq \frac{\sum_{i=1}^2 \mathbb{E}_F [\tilde{t}_i(v_i, v_{-i})]}{\text{opt}(F_a)} = R(\tilde{m}, F_a).$$

Since $\inf_{F \in \mathcal{F}} R(m, F) \leq R(m, F_n)$ for any n , we deduce that

$$\inf_{F \in \mathcal{F}} R(m, F) \leq \inf_{F \in \mathcal{W}'} R(\tilde{m}, F). \quad (\text{A.2-6})$$

To conclude, we need to show that $\tilde{m} = (\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$ belongs to \mathcal{M}'_{sf} . Let us first start by showing that it belongs to \mathcal{M}_{sf} . Note that for any $v_i \neq v_{-i}$, we have $\tilde{x}_i(v_i, v_{-i}) = x_i(v_i, v_{-i})$ and that the allocation $\tilde{x}_i(\cdot, v_{-i})$ is non decreasing on $[0, v_{-i}]$ and on $[v_{-i}, \infty)$ for fixed $v_{-i} \geq 0$. Furthermore,

$$\begin{aligned} \lim_{v_i \uparrow v_{-i}} \tilde{x}_i(v_i, v_{-i}) &= \lim_{v_i \uparrow v_{-i}} \sum_{k \in [1, N]} \frac{1}{N} \mathbb{1}\{\gamma_k v_{-i} < v_i\} = \frac{|\mathcal{I}^-|}{N} \\ \lim_{v_i \downarrow v_{-i}} \tilde{x}_i(v_i, v_{-i}) &= \lim_{v_i \downarrow v_{-i}} \sum_{k \in [1, N]} \frac{1}{N} \mathbb{1}\{\gamma_k v_{-i} < v_i\} = \frac{|\mathcal{I}^-| + |\mathcal{I}|}{N}. \end{aligned}$$

In turn, this implies that

$$\tilde{x}_i(v_i, v_i) = \frac{1}{2} \left(\lim_{v_i \uparrow v_{-i}} \tilde{x}_i(v_i, v_{-i}) + \lim_{v_i \downarrow v_{-i}} \tilde{x}_i(v_i, v_{-i}) \right).$$

We conclude that $x_i(\cdot, v_{-i})$ is non decreasing, for fixed $v_{-i} \geq 0$ and that $\tilde{x}_i(v_i, v_i) \leq 1$. Let us show that $\tilde{x}(v_i, v_i) \leq 1/2$. For that, let us introduce the following indices

$$\begin{aligned} \bar{k} &:= \arg \max\{k \in [1, N] : \gamma_k < 1\}, \underline{k} := \arg \min\{k \in [1, N] : \gamma_k > 1\}, \\ \underline{\gamma} &:= \frac{1}{2} \left(1 + \min(\gamma_{\underline{k}}, \frac{1}{\gamma_{\bar{k}}}) \right). \end{aligned}$$

By construction, we have that $\underline{\gamma} > 1$, $\underline{\gamma} < \gamma_k$, for all $k \in \mathcal{I}^+$, $\underline{\gamma} \gamma_j < 1$, for all $j \in \mathcal{I}^-$.

This implies that

$$x_1(\underline{\gamma}, 1) + x_2(\underline{\gamma}, 1) = \sum_{k \in [1, N]} \frac{1}{N} (\mathbf{1}\{\underline{\gamma} > \gamma_k\} + \mathbf{1}\{1 > \gamma_k \underline{\gamma}\}) = \frac{1}{N} (|\mathcal{I}| + 2|\mathcal{I}^-|).$$

Given that m belongs to \mathcal{M}_{sf} , we have that $x_1(\underline{\gamma}, 1) + x_2(\underline{\gamma}, 1) \leq 1$. In turn, we have

$$\tilde{x}_i(v_i, v_i) = \frac{1}{N} \left(|\mathcal{I}^-| + \frac{|\mathcal{I}|}{2} \right) \leq \frac{1}{2}.$$

Hence, we have established that for all $v_i, v_{-i} \geq 0$,

$$0 \leq \tilde{x}_i(v_i, v_{-i}) \leq 1, \quad \sum_{i=1}^2 \tilde{x}_i(v_i, v_{-i}) \leq 1, \quad \tilde{x}_i(\cdot, v_{-i}) \text{ is non-decreasing,}$$

$$\frac{1}{N} \left(|\mathcal{I}^-| + \frac{|\mathcal{I}|}{2} \right) \leq \frac{1}{2}.$$

We conclude that \tilde{m} belongs to \mathcal{M}'_{sf} . Returning to (A.2-6), and taking the supremum over mechanisms m and \tilde{m} , we obtain

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{F}) \leq \mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}').$$

This concludes the proof. □

Lemma A.2-4. *Fix a distribution F in \mathcal{W} . Then,*

- *For any mechanism $m = (\mathbf{x}, \mathbf{t}) \in \mathcal{M}$, the revenues generated by m are given by*

$$\begin{aligned} \sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] &= a \bar{F}(a^-) \sum_{i=1}^2 \int_0^a x_i(a, v_{-i}) dF(v_{-i}) \\ &\quad + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_F(v_i) f(v_i) dv_i dF(v_{-i}), \end{aligned}$$

where $a = \bar{v}_F$, i.e. the upper support of F .

- *The optimal revenue that the seller could achieve is given by*

$$\text{opt}(F) = a \bar{F}(a^-) (2 - \bar{F}(a^-)).$$

Proof of Lemma A.2-4. We show the first point then we leverage it to show the second point.

Recall that by the envelope theorem, Lemma 1.1, we have that

$$\sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] = \sum_{i=1}^2 \mathbb{E}_F [v_i x_i(v_i, v_{-i})] - \mathbb{E}_F \left[\int_0^{v_i} x_i(l, v_{-i}) dl \right].$$

Let us start by analyzing the last term. For that, using integration by parts, we have

$$\begin{aligned} & \mathbb{E}_F \left[\int_0^{v_i} x_i(l, v_{-i}) dl \right] \\ &= \int_0^a \int_0^a \int_0^{v_i} x_i(l, v_{-i}) dl dF(v_i) dF(v_{-i}) \\ &= \int_0^a \int_0^{a^-} \int_0^{v_i} x_i(l, v_{-i}) dl f(v_i) dv_i dF(v_{-i}) + \bar{F}(a^-) \int_0^a \int_0^a x_i(l, v_{-i}) dl dF(v_{-i}) \\ &= \int_0^a \int_0^{a^-} \left(\int_0^{v_i} x_i(l, v_{-i}) dl \right) [-\bar{F}]'(v_i) dv_i dF(v_{-i}) \\ & \quad + \bar{F}(a^-) \int_0^a \int_0^a x_i(l, v_{-i}) dl dF(v_{-i}). \end{aligned}$$

Let us focus on the first term. By integration by parts, we have

$$\begin{aligned} & \int_0^a \int_0^{a^-} \left(\int_0^{v_i} x_i(l, v_{-i}) dl \right) [-\bar{F}]'(v_i) dv_i dF(v_{-i}) \\ &= \int_0^a \left[\left(\int_0^{v_i} x_i(l, v_{-i}) dl \right) (-\bar{F})(v_i) \right]_{v_i=0}^{v_i=a^-} dF(v_{-i}) \\ & \quad + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}) \\ &= - \int_0^a \int_0^a x_i(l, v_{-i}) dl \bar{F}(a^-) dF(v_{-i}) + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}) \\ &= -\bar{F}(a^-) \int_0^a \int_0^a x_i(l, v_{-i}) dl dF(v_{-i}) + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}). \end{aligned}$$

Hence by combining the last result with the previous one, we get that,

$$\mathbb{E}_F \left[\int_0^{v_i} x_i(l, v_{-i}) dl \right] = \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}).$$

So the total revenue is given by

$$\begin{aligned}
& \sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] \\
&= \sum_{i=1}^2 \mathbb{E}_F [v_i x_i(v_i, v_{-i})] - \mathbb{E}_F \left[\int_0^{v_i} x_i(l, v_{-i}) dl \right], \\
&= \sum_{i=1}^2 \mathbb{E}_{F_a} [v_i x_i(v_i, v_{-i})] - \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}), \\
&= \sum_{i=1}^2 \int_0^a \int_0^a v_i x_i(v_i, v_{-i}) dF(v_i) dF(v_{-i}) - \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \bar{F}(v_i) dv_i dF(v_{-i}), \\
&= \sum_{i=1}^2 \bar{F}(a^-) \int_0^a a x_i(a, v_{-i}) dF(v_{-i}) + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_{F_a}(v_i) f(v_i) dv_i dF(v_{-i}).
\end{aligned}$$

Hence we conclude the first point, i.e.

$$\begin{aligned}
\sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] &= \sum_{i=1}^2 \bar{F}(a^-) \int_0^a a x_i(a, v_{-i}) dF(v_{-i}) \\
&\quad + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_F(v_i) f(v_i) dv_i dF(v_{-i}). \quad (\text{A.2-7})
\end{aligned}$$

Now let us try to characterize the optimal revenue leveraging the previous expression.

Since F is in \mathcal{W} , then for all $v_i < a$, $\phi_{F_a}(v_i) \leq 0$. In turn, setting $x_i(v_i, v_{-i}) = 0$ for all $v_i < a$ and for all $v_{-i} < a$, $x_i(a, v_{-i}) = 1$ and $x_i(a, a) = 1/2$ maximizes point-wise the expressions in the integrals in (A.2-7). This mechanism is clearly feasible (corresponding to a posted price of a to both buyers and when both buyers are willing to

buy, the seller allocates uniformly). Moreover, the corresponding revenue is given by

$$\begin{aligned}
& \sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] \\
&= \sum_{i=1}^2 \bar{F}(a^-) \int_0^a a x_i(a, v_{-i}) dF(v_{-i}) + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_{F_a}(v_i) f(v_i) dv_i dF(v_{-i}) \\
&= \sum_{i=1}^2 a \bar{F}(a^-) \left(\int_0^{a^-} 1 dF(v_{-i}) + \frac{1}{2} \bar{F}(a^-) \right) + 0 \\
&= \sum_{i=1}^2 a \bar{F}(a^-) \left(1 - \bar{F}(a^-) + \frac{1}{2} \bar{F}(a^-) \right) \\
&= a \bar{F}(a^-) (2 - \bar{F}(a^-)).
\end{aligned}$$

Hence, we get the second result. □

Proof of Proposition 1.3. Fix a mechanism $m = (\mathbf{x}, \mathbf{t})$ in $\mathcal{M}'_{sf} \subset \mathcal{M}_{sf}$. Let $\tilde{m} = (\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$ with $\tilde{x}_i(v_i, v_{-i}) = x_i(v_i, v_{-i}) \mathbb{1}\{v_i \geq v_{-i}\}$, for $i = 1, 2$ and $v_i, v_{-i} \geq 0$ (and $\tilde{\mathbf{t}}$ its corresponding payment using Lemma 1.1). The proof is organized as follows. In step 1, we establish that \tilde{m} belongs to \mathcal{M}_{sf}^{max} . In step 2, we establish that \tilde{m} dominates m against any distribution in \mathcal{W} and deduce the result.

Step 1. We first establish that $\tilde{m} \in \mathcal{M}_{sf}^{max}$. We have that

$$\begin{aligned}
\tilde{x}_i(v_i, v_{-i}) &= x_i(v_i, v_{-i}) \mathbb{1}\{v_i \geq v_{-i}\} \\
&= \sum_{k=1}^N \frac{1}{N} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} \mathbb{1}\{v_i \geq v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\} \mathbb{1}\{v_i \geq v_{-i}\} \\
&= \sum_{k=1}^N \frac{1}{N} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i > v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\} \\
&= \sum_{k=1}^N \frac{1}{N} \mathbb{1}\{v_i > \max(\gamma_k, 1) v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\}.
\end{aligned}$$

It follows that $\tilde{m} \in \mathcal{M}_{sf}^{max}$ by setting $\alpha_n := \max(\gamma_n, 1)$.

Step 2. We next establish that for any $F \in \mathcal{W}$,

$$R(m, F) \leq R(\tilde{m}, F). \tag{A.2-8}$$

By Lemma A.2-4, stated and proved right before this proof, the total revenue of the mechanism m when nature picks the distribution F_a is given by

$$\begin{aligned} \sum_{i=1}^2 \mathbb{E}_F [t_i(v_i, v_{-i})] &= a\bar{F}(a^-) \sum_{i=1}^2 \int_0^a x_i(a, v_{-i}) dF(v_{-i}) \\ &\quad + \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_F(v_i) f(v_i) dv_i dF(v_{-i}). \end{aligned}$$

Similarly, the revenues of \tilde{m} against F are given by

$$\begin{aligned} \sum_{i=1}^2 \mathbb{E}_F [\tilde{t}_i(v_i, v_{-i})] &= a\bar{F}(a^-) \sum_{i=1}^2 \int_0^a \tilde{x}_i(a, v_{-i}) dF(v_{-i}) \\ &\quad + \int_0^a \int_0^{a^-} \tilde{x}_i(v_i, v_{-i}) \phi_{F_a}(v_i) f(v_i) dv_i dF(v_{-i}). \end{aligned}$$

Note that

$$\begin{aligned} a\bar{F}(a^-) \int_0^a x_i(a, v_{-i}) dF(v_{-i}) &= a\bar{F}(a^-) \int_0^a \tilde{x}_i(a, v_{-i}) dF(v_{-i}) \\ \int_0^a \int_0^{a^-} x_i(v_i, v_{-i}) \phi_F(v_i) f(v_i) dv_i dF(v_{-i}) &\leq \int_0^a \int_0^{a^-} \tilde{x}_i(v_i, v_{-i}) \phi_F(v_i) f(v_i) dv_i dF(v_{-i}), \end{aligned}$$

where the first equality follows from the fact that by construction for any $v_i \geq v_{-i}$, we have that $x_i(v_i, v_{-i}) = \tilde{x}_i(v_i, v_{-i})$; the second inequality follows from the fact that $\tilde{x}_i(\cdot, \cdot) \leq x_i(\cdot, \cdot)$ and that for any $v_i < a$, we have $\phi_F(v_i) \leq 0$ since F belongs to \mathcal{W}' . In turn, we conclude that (A.2-8) holds. This implies that

$$\mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}') \leq \mathcal{R}(\mathcal{M}^{max}_{sf}, \mathcal{W}').$$

Since $\mathcal{M}^{max}_{sf} \subset \mathcal{M}'_{sf}$, the proof is complete. \square

Lemma A.2-5. *If $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}'_{sf} , then there exist $N \geq 1$, $\gamma \in [0, \infty)^N$ and $c \in [0, 1/2]$ such that for all $v_i, v_{-i} \geq 0$, we have*

- *The allocation is given by*

$$x_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \mathbf{1}\{v_i > \gamma_k v_{-i}\} \mathbf{1}\{v_i \neq v_{-i}\} + c \mathbf{1}\{v_i = v_{-i}\}$$

- The payment is given by

$$\begin{aligned}
t_i(v_i, v_{-i}) &= \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} \\
&\quad + \left(c + \frac{1}{N} \sum_{k \in \mathcal{J}^-} \gamma_k - \frac{1}{N} |\mathcal{J}^-| \right) v_{-i} \mathbb{1}\{v_i = v_{-i}\},
\end{aligned}$$

where $\mathcal{J}^- = \{k : \gamma_k < 1\}$ and $|\mathcal{J}^-|$ represents the cardinality of \mathcal{J}^- .

Note that if $m = (\mathbf{x}, \mathbf{t})$ is in \mathcal{M}_{sf}^{max} then $|\mathcal{J}^-| = 0$

Proof of Lemma A.2-5. We will show the result for \mathcal{M}'_{sf} , the proof for \mathcal{M}_{sf}^{max} is very similar.

Fix $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}'_{sf} , then by definition, there exist $N \geq 1$, $\gamma \in [0, \infty)^N$ and $c \in [0, 1/2]$ such that for all $v_i, v_{-i} \geq 0$, we have

$$x_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c \mathbb{1}\{v_i = v_{-i}\}.$$

By Lemma 1.1, we have that for all $v_i \neq v_{-i}$,

$$\begin{aligned}
t_i(v_i, v_{-i}) &= v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(l, v_{-i}) dl \\
&= \sum_{k=1}^N \frac{1}{N} v_i \mathbb{1}\{v_i > \gamma_k v_{-i}\} - \int_0^{v_i} \mathbb{1}\{l > \gamma_k v_{-i}\} dl \\
&= \sum_{k=1}^N \frac{1}{N} v_i \mathbb{1}\{v_i > \gamma_k v_{-i}\} - (v_i - \gamma_k v_{-i}) \mathbb{1}\{v_i > \gamma_k v_{-i}\} \\
&= \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\},
\end{aligned}$$

hence we conclude that for all $v_i \neq v_{-i}$, that the payment is given by

$$t_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\},$$

Moreover for $v_i = v_{-i}$, we have by Lemma 1.1

$$t_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(l, v_{-i}) dl = c v_i - \frac{1}{N} \sum_{k \in \mathcal{J}^-} \int_0^{v_i} \mathbb{1}\{l > \gamma_k v_i\} dl$$

Hence we get that

$$t_i(v_i, v_{-i}) = c v_i - \frac{1}{N} \sum_{k \in \mathcal{I}^-} (1 - \gamma_k) v_i = v_i \left(c - \frac{|\mathcal{I}^-|}{N} + \frac{1}{N} \sum_{k \in \mathcal{I}^-} \gamma_k \right).$$

This concludes the proof. \square

Proof of Theorem 1.2. The result is consequence of earlier results. Indeed, we have

$$\mathcal{R}(\mathcal{M}, \mathcal{F}) \stackrel{(a)}{=} \mathcal{R}(\mathcal{M}_{sf}, \mathcal{F}) \stackrel{(b)}{=} \mathcal{R}(\mathcal{M}'_{sf}, \mathcal{F}) \stackrel{(c)}{\leq} \mathcal{R}(\mathcal{M}'_{sf}, \mathcal{W}') \stackrel{(d)}{=} \mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}'),$$

where (a) follows from Theorem 3.2, (b) from Proposition 1.1, (c) from Proposition 1.2 in conjunction with the assumption that any element in \mathcal{W}' admits a sequence in \mathcal{F} that weakly converges to it; and (d) from Proposition 1.3. \square

A.3 Proofs of Section 1.5

Proof of Theorem 1.3. In this proof, we will use the family of distributions $\mathcal{W}_{reg} \subset \mathcal{W}$ defined as follows:

$$\mathcal{W}_{reg} := \{F_a : a > 0\},$$

where

$$F_a(v) = \begin{cases} 1 - \frac{1}{v+1} & , \text{ if } v < a, \\ 1 & , \text{ if } v \geq a, \end{cases}$$

In Lemma B.6-2, stated and proved following this proof, we establish that any element in \mathcal{W}_{reg} can be “approached” by a sequence in \mathcal{F}_{reg} , i.e., a sequence of elements in \mathcal{F}_{reg} converges weakly to the element. In turn, using Theorem 1.2 in conjunction with Lemma B.6-2, we have that

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_{reg}) \leq \mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{reg}).$$

We next bound $\mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{reg})$. Let us fix $a > 0$ and a mechanism $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf}^{max} . We will bound $R(m, F_a)$. For some $N \geq 1$, $\gamma \in [1, \infty)^N$ and $c \in [0, 1/2]$, by Lemma A.2-5, stated proved at the end of this section, the payments are given by

$$t_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c v_{-i} \mathbb{1}\{v_i = v_{-i}\},$$

for $i = 1, 2$ and $v_i, v_{-i} \geq 0$.

Note that since $\gamma_k \geq 1$, we have $\mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} = \mathbb{1}\{v_i > \gamma_k v_{-i}\}$.

We first derive a close form expression for $\sum_{i=1}^2 \mathbb{E}_{F_a} [t_i(v_i, v_{-i})]$. We have

$$\sum_{i=1}^2 \mathbb{E}_{F_a} [t_i(v_i, v_{-i})] = \sum_{i=1}^2 \sum_{k=1}^N \frac{1}{N} \mathbb{E}_{F_a} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] + 2caq_a^2,$$

where $q_a = 1/(1+a)$.

Note that

$$\begin{aligned} \mathbb{E}_{F_a} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] &= \int_0^a \int_0^a \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} dF_a(v_i) dF_a(v_{-i}) \\ &= \int_0^{(a/\gamma_k)^-} \gamma_k v_{-i} \bar{F}_a(\gamma_k v_{-i}) dF_a(v_{-i}) \\ &= \int_{\bar{F}((a/\gamma_k)^-)}^1 \frac{\gamma_k(1-q)}{q + \gamma_k(1-q)} dq, \end{aligned}$$

where the last equality follows from a change of variable $q = \bar{F}_a(v_{-i}) = 1/(1+v_{-i})$.

Suppose $\gamma_k = 1$. Note that $\bar{F}(a^-) = 1/(1+a) = q_a$. Then we have that

$$\mathbb{E}_{F_a} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] = \int_{q_a}^1 (1-q) dq = \frac{1}{2}(1-q_a)^2.$$

Suppose $\gamma_k > 1$. Let $\bar{q} = \bar{F}(a/\gamma_k)$. We have that

$$\begin{aligned} \mathbb{E}_{F_a} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] &= \gamma_k \int_{\bar{q}}^1 \frac{1-q}{\gamma_k(1-q) + q} dq \\ &= \gamma_k \int_{\bar{q}}^1 \left(\frac{1}{\gamma_k - 1} - \frac{1}{\gamma_k - 1} \frac{1}{q(1-\gamma_k) + \gamma_k} \right) dq \\ &= \gamma_k \int_{\bar{q}}^1 \left(\frac{1}{\gamma_k - 1} + \frac{1}{(\gamma_k - 1)^2} \frac{1-\gamma_k}{q(1-\gamma_k) + \gamma_k} \right) dq \\ &= \gamma_k \left[\frac{1-\bar{q}}{\gamma_k - 1} - \frac{1}{(\gamma_k - 1)^2} \ln(\bar{q}(1-\gamma_k) + \gamma_k) \right]. \end{aligned}$$

Let us now compute \bar{q} . We have

$$\bar{q} = \bar{F}(a/\gamma_k) = \frac{1}{1 + a/\gamma_k} = \frac{\gamma_k}{\gamma_k + a} = \frac{\gamma_k q_a}{1 - q_a + \gamma_k q_a},$$

where we have used the fact $a q_a = a/(a + 1) = 1 - q_a$. So we conclude from the previous computation that when $\gamma_k > 1$,

$$\begin{aligned} & \mathbb{E}_{F_a} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] \\ = & \gamma_k \left[\frac{1 - \bar{q}}{\gamma_k - 1} - \frac{1}{(\gamma_k - 1)^2} \ln(\bar{q}(1 - \gamma_k) + \gamma_k) \right] \\ = & \gamma_k \left[\frac{1}{\gamma_k - 1} \frac{1 - q_a}{1 - q_a + \gamma_k q_a} - \frac{1}{(\gamma_k - 1)^2} \ln\left(\frac{\gamma_k}{1 - q_a + \gamma_k q_a}\right) \right]. \end{aligned}$$

Recall that by Lemma A.2-4, $\text{opt}(F_a) = a q_a (2 - q_a) = (1 - q_a)(2 - q_a)$, hence

$$\begin{aligned} & \frac{\mathbb{E}_{F_a} [\gamma v_{-i} \mathbb{1}\{v_i > \gamma v_{-i}\}]}{\text{opt}(F_a)} \\ = & \begin{cases} \gamma \frac{1}{1 - q_a} \frac{1}{2 - q_a} \left[\frac{1}{\gamma - 1} \frac{1 - q_a}{1 - q_a + \gamma q_a} - \frac{1}{(\gamma - 1)^2} \ln\left(\frac{\gamma}{1 - q_a + \gamma q_a}\right) \right], & \text{if } \gamma > 1 \\ \frac{1}{2} \frac{1 - q_a}{2 - q_a}, & \text{if } \gamma = 1. \end{cases} \end{aligned}$$

Let us denote $\mathcal{I} = \{k : \gamma_k > 1\}$. We have

$$\begin{aligned} & R(m, F_a) \\ = & \frac{\sum_{i=1}^2 \mathbb{E}_{F_a} [t_i(v_i, v_{-i})]}{\text{opt}(F_a)} \\ = & \frac{N - |\mathcal{I}|}{N} \frac{1 - q_a}{2 - q_a} \\ & + 2 \sum_{k \in \mathcal{I}} \frac{1}{N} \gamma_k \frac{1}{1 - q_a} \frac{1}{2 - q_a} \left[\frac{1}{\gamma_k - 1} \frac{1 - q_a}{1 - q_a + \gamma_k q_a} - \frac{1}{(\gamma_k - 1)^2} \ln\left(\frac{\gamma_k}{1 - q_a + \gamma_k q_a}\right) \right] \\ & + 2 c \frac{q_a}{2 - q_a} \\ = & \frac{N - |\mathcal{I}|}{N} \frac{1}{2 - q_a} \\ & + 2 \sum_{k \in \mathcal{I}} \frac{1}{N} \gamma_k \frac{1}{1 - q_a} \frac{1}{2 - q_a} \left[\frac{1}{\gamma_k - 1} \frac{1 - q_a}{1 - q_a + \gamma_k q_a} - \frac{1}{(\gamma_k - 1)^2} \ln\left(\frac{\gamma_k}{1 - q_a + \gamma_k q_a}\right) \right] \\ & + \left(2 c - \frac{N - |\mathcal{I}|}{N} \right) \frac{q_a}{2 - q_a}. \end{aligned}$$

Noting that $c \leq 1/2$, we have

$$\begin{aligned}
& R(m, F_a) \\
\leq & \frac{N - |\mathcal{I}|}{N} \frac{1}{2 - q_a} \\
& + 2 \sum_{k \in \mathcal{I}} \frac{1}{N} \gamma_k \frac{1}{1 - q_a} \frac{1}{2 - q_a} \left[\frac{1}{\gamma_k - 1} \frac{1 - q_a}{1 - q_a + \gamma_k q_a} - \frac{1}{(\gamma_k - 1)^2} \ln \left(\frac{\gamma_k}{1 - q_a + \gamma_k q_a} \right) \right] \\
& + \frac{|\mathcal{I}|}{N} \frac{q_a}{2 - q_a},
\end{aligned}$$

The bound in the theorem follows and the proof is complete. \square

Lemma A.3-1. *For each $a > 0$, there is a sequence $\{F_n : n \geq 1\}$ in \mathcal{F}_{reg} that converges weakly to F_a defined in (1.7), such that there exists $M_a > a$ such that for all n , $F_n(M_a) = 1$.*

Proof of Lemma B.6-2. Fix $a > 0$. The proof will be constructive in that we will explicitly exhibit a sequence F_n that satisfies the properties in the result.

Step 1. We first construct the sequence and characterize its weak limit.

a) Let $n \geq 2$, and define for $x \geq 0$,

$$g_n(x) = 1 + x - \left(\frac{x}{a}\right)^n.$$

Note that there exists a unique $x_n \geq 0$ such that $g_n(x_n) = 0$. Indeed, g_n is differentiable with derivative given by $g'_n(x) = 1 - n x^{n-1}/a^n$, hence g_n is strictly increasing on $[0, (a^n/n)^{1/(n-1)})$ and strictly decreasing on $((a^n/n)^{1/(n-1)}, +\infty[$. Since $g_n(0) = 1$ and $\lim_{x \uparrow \infty} g_n(x) = -\infty$ there exists a unique x_n s.t. $g_n(x_n) = 0$. Furthermore, noting that $g_n(a) = a > 0$, we have that

$$x_n > a, \quad n \geq 2.$$

Define the sequence of cumulative distribution functions F_n

$$F_n(x) = \begin{cases} 1 - \exp\left(-\int_0^x \frac{1}{g_n(t)} dt\right) & , \text{ if } x < x_n, \\ 1 & , \text{ if } x \geq x_n. \end{cases} \quad (\text{A.3-1})$$

Step 2. We next establish that F_n belongs to \mathcal{F}_{reg} and that the sequence converges weakly to F_a .

We first show that the sequence x_n is decreasing and is lower bounded by a . The fact that $x_n > a$ implies that

$$g_{n+1}(x_n) = 1 + x_n - (x_n/a)^{n+1} < 1 + x_n - (x_n/a)^n = g_n(x_n) = 0.$$

In turn, by definition of x_{n+1} , we get that

$$x_n < x_{n+1}.$$

Hence for all $n \geq 2$, $x_n < x_2$. Setting $M_a = x_2$, we have $M_a > a$ and $F_n(M_a) = 1$ for all $n \geq 2$.

Since x_n is decreasing and lower bounded by a , it necessarily converges to some limit $l \geq a$. If $l > a$ then for n sufficiently large, we would have $x_n \geq (1/2)(l + a)$ implying that $g_n((1/2)(l + a)) \geq 0$. However $\lim_{n \uparrow \infty} g_n((1/2)(l + a)) = -\infty$, which is a contradiction. We conclude that necessarily $\lim_{n \uparrow \infty} x_n = a$.

Note also that for $x \geq 0$, g_n is a polynomial with root x_n , and no root in $[0, x_n]$. Since $g'_n(x_n) \neq 0$, then necessarily the multiplicity of x_n is one, so we can find a polynomial function Q_n such that $g_n(x) = (x_n - x)Q(x)$ and for all $x \in [0, x_n]$ we have $Q_n(x) > 0$. In turn, by the Weierstrass extreme value theorem, we have $A_n := \inf_{x \in [0, x_n]} Q_n(x) \in (0, \infty)$ and $B_n := \sup_{x \in [0, x_n]} Q_n(x) \in (0, \infty)$. and we can find $B_n, A_n > 0$ such that for all $x \in [0, x_n]$, we have $A_n \leq Q_n(x) \leq B_n$.

For $x \in [0, x_n)$, we have

$$\frac{1}{A_n} \ln \left(\frac{x_n - x}{x_n} \right) \leq - \int_0^x \frac{1}{g_n(t)} dt \leq \frac{1}{B_n} \ln \left(\frac{x_n - x}{x_n} \right),$$

and we deduce that

$$\lim_{x \uparrow x_n} \exp \left(- \int_0^x \frac{1}{g_n(t)} dt \right) = 0. \tag{A.3-2}$$

We deduce that F_n defined in (B.6-7) has no atoms. Furthermore, its virtual value function is given by

$$\phi_n(x) = \left(\frac{x}{a}\right)^n - 1.$$

which is clearly non-decreasing on $[0, x_n)$. Hence, F_n belongs to \mathcal{F}_{reg} .

Step 3. Let us establish that F_n converges weakly to F_a . The points of continuity of F_a are $\mathbb{R}^+ \setminus \{a\}$. Fix $x < a$. we have $\lim_{n \uparrow \infty} (x/a)^n = 0$. Hence for all $x < a$, using the Dominated Convergence Theorem, we have $\lim_{n \uparrow \infty} F_n(x) = F_a(x)$. Fix $x > a$. Since x_n converges to a , there exists N such that for all $n \geq N$, we have $x > x_n$, hence for all $n \geq N$, $F_n(x) = F_a(x) = 1$. We conclude that F_n weakly converges to F_a . \square

Proof of Proposition 1.4. Let F be a distribution in \mathcal{F}_{reg} and $m = (\mathbf{x}, \mathbf{t})$ be a mechanism \mathcal{M}_{sf}^{max} . By Lemma A.2-5, the revenue of the mechanism m when nature picks F is given by

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^2 t_i(v_i, v_{-i}) \right] &= \mathbb{E} \left[\sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c v_{-i} \mathbb{1}\{v_i = v_{-i}\} \right] \\ &= \frac{N - |\mathcal{I}|}{N} \mathbb{E}_F [v_{[2]}] + 2 \sum_{k \in \mathcal{I}^+} \frac{1}{N} \int_0^\infty Rev(\gamma_k v) f(v) dv, \end{aligned}$$

where $Rev(v) := v\bar{F}(v)$, the last equality follows from symmetry and the fact that the set $\{(v_i, v_{-i}) : v_i = v_{-i}\}$ has measure zero with respect to the F -measure, since F has a density.

Let us start by analyzing the contribution of any of the terms in sum. We have

$$\begin{aligned} \int_0^\infty Rev(\gamma_k v) f(v) dv &\stackrel{(a)}{=} \left[-Rev(\gamma_k v) \bar{F}(v) \right]_{v=0}^{v=\infty} + \int_0^\infty \gamma_k Rev'(\gamma_k v) \bar{F}(v) dv \\ &\stackrel{(b)}{=} \int_0^\infty \gamma_k Rev'(\gamma_k v) \bar{F}(v) dv \\ &\stackrel{(c)}{=} \int_0^\infty Rev'(v) \bar{F}(v/\gamma_k) dv, \end{aligned}$$

where (a) follows by integration by parts; (b) follows from the fact that $Rev(\gamma_k 0) \bar{F}(0) = 0$ (since $Rev(0) = 0$) and $\lim_{v \rightarrow \infty} Rev(\gamma_k v) \bar{F}(v) = 0$ (since $Rev(\gamma_k v) \bar{F}(v) \leq Rev(r_F) \bar{F}(v)$)

and $\lim_{v \rightarrow \infty} \bar{F}(v) = 0$; (c) follows from a change of variable. In turn, by decomposing the last integral, we get that

$$\begin{aligned} \int_0^\infty Rev(\gamma_k v) f(v) dv &= \int_0^{r_F} Rev'(v) \bar{F}(v/\gamma) dv + \int_{r_F}^\infty Rev'(v) \bar{F}(v/\gamma) dv \\ &= \int_{q_F}^1 (-Rev'(q)) \bar{F}\left(\frac{F^{-1}(1-q)}{\gamma_k}\right) dq + \int_{r_F}^\infty Rev'(v) \bar{F}(v/\gamma) dv, \end{aligned}$$

where the last equality follows from the change of variable $q = \bar{F}(v)$ and noting that that $Rev'(v) = f(v)$ ($-Rev'(q)$), where here we abuse notation and denote $Rev(q) = qF^{-1}(1-q)$.

For $v \geq r_F$, we have $Rev'(v) \leq 0$ and $\bar{F}(v/\gamma_k) \leq \bar{F}(r_F/\gamma_k)$. Hence, we have

$$\int_{r_F}^\infty Rev'(v) \bar{F}(v/\gamma) dv \geq -\bar{F}(r_F/\gamma) Rev(q_F) \geq -\frac{\gamma_k q_F}{1 + (\gamma_k - 1) q_F} \mathbb{E}[v_{[2]}],$$

where for the last inequality, we have used Lemma A.3-2, stated and proved after this proof.

For $q \geq q_F$, we have $Rev'(q) \leq 0$ since F belongs to \mathcal{F}_{reg} and by (Fu et al., 2015, Lemma 4), $\bar{F}(F^{-1}(1-q)/\gamma_k) \geq \gamma_k q / (1 + (\gamma_k - 1) q)$ and hence

$$\begin{aligned} &\int_{q_F}^1 (-Rev'(q)) \bar{F}\left(\frac{F^{-1}(1-q)}{\gamma_k}\right) dq \\ &\geq \int_{q_F}^1 (-Rev'(q)) \frac{\gamma_k q}{1 + (\gamma_k - 1) q} dq \\ &= (1 - q_F) \left(\frac{\gamma_k}{\gamma_k - 1}\right) \int_{q_F}^1 (-Rev'(q)) \left[1 - \frac{1}{1 + (\gamma_k - 1) q}\right] \frac{dq}{1 - q_F}. \end{aligned}$$

Since F is regular, $Rev(q)$ is concave, so $Rev'(q)$ is decreasing. Noting that

$$(1 - 1/(1 + (\gamma_k - 1) q))$$

is increasing, then by (Ross, 1996, Proposition 7.2.1), we have

$$\begin{aligned} &\int_{q_F}^1 (-Rev'(q)) \left[1 - \frac{1}{1 + (\gamma_k - 1) q}\right] \frac{dq}{1 - q_F} \\ &\geq \left(\int_{q_F}^1 \left[1 - \frac{1}{1 + (\gamma_k - 1) q}\right] \frac{dq}{1 - q_F}\right) \left(\int_{q_F}^1 (-Rev'(q)) \frac{dq}{1 - q_F}\right) \\ &= \frac{1}{(1 - q_F)^2} \left[1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1) q_F}\right]\right] Rev(q_F). \end{aligned}$$

In turn, we deduce that

$$\begin{aligned}
& \int_0^\infty Rev(\gamma_k v) f(v) dv \\
& \geq \frac{\gamma_k}{\gamma_k - 1} \frac{1}{1 - q_F} \left[1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1)q_F} \right] \right] Rev(q_F) \\
& \quad - \frac{\gamma_k q_F}{1 + (\gamma_k - 1)q_F} \mathbb{E} [v_{[2]}] \\
& \geq \frac{\gamma_k}{\gamma_k - 1} \frac{1}{1 - q_F} \left[1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1)q_F} \right] \right] \frac{\text{opt}(F)}{2} \\
& \quad - \frac{\gamma_k q_F}{1 + (\gamma_k - 1)q_F} \mathbb{E} [v_{[2]}],
\end{aligned}$$

where for the last inequality, we used that $Rev(q_F) \geq \text{opt}(F)/2$. and the fact that

$$1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1)q_F} \right] \geq 0.$$

The performance of mechanism m against the distribution F is given by

$$\begin{aligned}
& R(m, F) \\
& = \frac{N - |\mathcal{I}^+|}{N} \frac{\mathbb{E}_F [v_{[2]}]}{\text{opt}(F)} + 2 \sum_{k \in \mathcal{I}^+} \frac{1}{N} \frac{\int_0^\infty Rev(\gamma_k v) f(v) dv}{\text{opt}(F)} \\
& \geq \frac{N - |\mathcal{I}^+|}{N} \frac{\mathbb{E}_F [v_{[2]}]}{\text{opt}(F)} \\
& \quad + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left[\frac{\gamma_k}{\gamma_k - 1} \left(1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1)q_F} \right] \right) \frac{1}{1 - q_F} \right. \\
& \quad \quad \left. - \frac{\gamma_k q_F}{1 + (\gamma_k - 1)q_F} \frac{2\mathbb{E} [v_{[2]}]}{\text{opt}(F)} \right] \\
& \geq \left[1 - \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left(1 + \frac{2 \gamma_k q_F}{1 + (\gamma_k - 1)q_F} \right) \right] \frac{\mathbb{E}_F [v_{[2]}]}{\text{opt}(F)} \\
& \quad + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left[\frac{\gamma_k}{\gamma_k - 1} \left(1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1)q_F} \right] \right) \frac{1}{1 - q_F} \right].
\end{aligned}$$

Since $2 \gamma_k q_F / (1 + (\gamma_k - 1)q_F) \leq 2$, and $|\mathcal{I}^+|/N \leq 1/3$ by assumption, we have

$$1 - \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left(1 + \frac{2 \gamma_k q_F}{1 + (\gamma_k - 1)q_F} \right) \geq 1 - \frac{3|\mathcal{I}^+|}{N} \geq 0.$$

On the other side, using (Fu et al., 2015, Corollary 1), we also have that

$$\frac{\mathbb{E}_F [v_{[2]}]}{\text{opt}(F)} \geq \frac{1}{2 - q_F}.$$

In turn, we have

$$\begin{aligned}
& R(m, F) \\
& \geq \left[1 - \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left(1 + \frac{2 \gamma_k q_F}{1 + (\gamma_k - 1) q_F} \right) \right] \frac{1}{2 - q_F} \\
& \quad + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left[\frac{\gamma_k}{\gamma_k - 1} \left(1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1) q_F} \right] \right) \frac{1}{1 - q_F} \right] \\
& = \frac{N - |\mathcal{I}^+|}{N} \frac{1}{2 - q_F} \\
& \quad + \sum_{k \in \mathcal{I}^+} \frac{1}{N} \left[\frac{\gamma_k}{\gamma_k - 1} \left(1 - q_F - \frac{1}{\gamma_k - 1} \ln \left[\frac{\gamma_k}{1 + (\gamma_k - 1) q_F} \right] \right) \frac{1}{1 - q_F} \right. \\
& \quad \left. - \frac{2 \gamma_k q_F}{1 + (\gamma_k - 1) q_F} \frac{1}{2 - q_F} \right].
\end{aligned}$$

Rearranging terms, one obtains the bound in the result and this concludes the proof. \square

Lemma A.3-2. *For any distribution F in \mathcal{F}_{reg} , we have*

$$\mathbb{E}_F[v_{[2]}] = 2 \int_0^1 Rev(q) dq, \quad (\text{A.3-3})$$

$$Rev(q_F) \bar{F} \left(\frac{r_F}{\alpha} \right) \leq \frac{\alpha q_F}{1 + (\alpha - 1) q_F} \mathbb{E}[v_{[2]}] \quad \text{for any } \alpha \geq 1. \quad (\text{A.3-4})$$

Proof of Lemma A.3-2. We will show each claim separately. Let us fix F in \mathcal{F}_{reg} .

We first prove (A.3-3). We have,

$$\mathbb{E}_F[v_{[2]}] = \int_0^\infty 2 Rev(v) f(v) dv = \int_0^1 2 Rev(q) dq,$$

where the last equality follows from the the change of variable $q = \bar{F}(v)$, i.e. $v = F^{-1}(1 - q)$ and noting that $f(v) dv = -dq$.

We next establish (A.3-4). Let us denote $\bar{q} = \bar{F} \left(\frac{r_F}{\alpha} \right)$, we have

$$Rev(\bar{q}) = \bar{q} \frac{r_F}{\alpha} = \bar{q} \frac{Rev(q_F)}{\alpha q_F},$$

now using the concavity of $Rev(q)$, since F is in \mathcal{F}_{reg} , the area below the curve of $Rev(q)$ when $q \in [q_F, 1]$ is greater than the area of the quadrilateral defined by the

following points $\{(1, 0), (\bar{q}, Rev(\bar{q})), (q_F, Rev(q_F)), (q_F, 0)\}$,

$$\begin{aligned}
& \int_{q_F}^1 Rev(q) dq \\
& \geq (\bar{q} - q_F) Rev(\bar{q}) + \frac{1}{2}(1 - \bar{q}) Rev(\bar{q}) + \frac{1}{2}(\bar{q} - q_F) (Rev(q_F) - Rev(\bar{q})) \\
& = \bar{q} Rev(\bar{q}) - q_F Rev(\bar{q}) + \frac{1}{2} Rev(\bar{q}) - \frac{\bar{q}}{2} Rev(\bar{q}) \\
& \quad + \frac{\bar{q}}{2} Rev(q_F) - \frac{\bar{q}}{2} Rev(\bar{q}) - \frac{q_F}{2} Rev(q_F) + \frac{q_F}{2} Rev(\bar{q}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{q_F}^1 Rev(q) dq & \geq -q_F Rev(\bar{q}) + \frac{1}{2} Rev(\bar{q}) + \frac{\bar{q}}{2} Rev(q_F) - \frac{q_F}{2} Rev(q_F) + \frac{q_F}{2} Rev(\bar{q}) \\
& = -\frac{q_F}{2} Rev(\bar{q}) + \frac{1}{2} Rev(\bar{q}) + \frac{\bar{q}}{2} Rev(q_F) - \frac{q_F}{2} Rev(q_F) \\
& = -\frac{\bar{q}}{2\alpha} Rev(q_F) + \frac{\bar{q}}{2\alpha q_F} Rev(q_F) + \frac{\bar{q}}{2} Rev(q_F) - \frac{q_F}{2} Rev(q_F).
\end{aligned}$$

So we conclude that

$$\begin{aligned}
\int_{q_F}^1 Rev(q) dq & \geq \bar{q} Rev(q_F) \frac{1}{2} \left[-\frac{1}{2\alpha} + \frac{1}{2\alpha q_F} + \frac{1}{2} \right] - \frac{q_F}{2} Rev(q_F) \\
& = \bar{q} Rev(q_F) \frac{(-q_F + 1 + \alpha q_F)}{2\alpha q_F} - \frac{q_F}{2} Rev(q_F) \\
& = \bar{q} Rev(q_F) \frac{(1 + (\alpha - 1) q_F)}{2\alpha q_F} - \frac{q_F}{2} Rev(q_F).
\end{aligned}$$

Hence,

$$\begin{aligned}
\bar{q} Rev(q_F) & \leq \frac{\alpha q_F}{1 + (\alpha - 1) q_F} \left[2 \int_{q_F}^1 Rev(q) dq + q_F Rev(q_F) \right] \\
& = \frac{\alpha q_F}{1 + (\alpha - 1) q_F} \left[\mathbb{E}[v_{[2]}] + q_F Rev(q_F) - 2 \int_0^{q_F} Rev(q) dq \right].
\end{aligned}$$

Moreover, using the concavity of $Rev(q)$, the area of the triangle of vertices $\{(0, 0), (q_F, Rev(q_F)), (q_F, 0)\}$ is less the area than of the curve $Rev(q)$ between $[0, q_F]$.

So we get that

$$\int_0^{q_F} Rev(q) dq \geq \frac{1}{2} q_F Rev(q_F),$$

hence

$$q_F Rev(q_F) - 2 \int_0^{q_F} Rev(q) dq \leq 0,$$

so we conclude that

$$\text{Rev}(q_F)\bar{F}\left(\frac{r_F}{\alpha}\right) \leq \frac{\alpha q_F}{1 + (\alpha - 1)q_F} \mathbb{E}[v_{[2]}].$$

The proof is complete. \square

A.4 Proofs for Section 1.6

Proof of Theorem 1.4. The proof follows initially the same structure of the proof of Theorem 1.3. However, there are two main differences. We will be considering a different limiting worst case family, \mathcal{W}_{mhr} . Furthermore, we will be able to upper bound the performance by establishing that the best achievable performance against \mathcal{W}_{mhr} is that of a second price auction.

Let us introduce, $\mathcal{W}_{mhr} \subset \mathcal{W}$ defined as follows,

$$\mathcal{W}_{mhr} := \{F_{a,b}, : a \geq b > 0\},$$

where

$$F_{a,b}(v) = \begin{cases} 1 - \exp\left(-\frac{v}{a}\right) & , \text{ if } v < b, \\ 1 & , \text{ if } v \geq b, \end{cases}$$

In Lemma A.4-1, stated and proved following this proof, we establish that any element in \mathcal{W}_{mhr} can be “approached” by a sequence in \mathcal{F}_{mhr} , i.e., a sequence of elements in \mathcal{F}_{mhr} converges weakly to the element. In turn, using Theorem 1.2 in conjunction with Lemma A.4-1, we have

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}) \leq \mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{mhr}).$$

We next bound $\mathcal{R}(\mathcal{M}_{sf}^{max}, \mathcal{W}_{mhr})$. Let us fix $a \geq b > 0$ and a mechanism in $m = (\mathbf{x}, \mathbf{t})$ in \mathcal{M}_{sf}^{max} . We will bound $R(m, F_{a,b})$. For some $N \geq 1$, $\gamma \in [1, \infty)^N$ and $c \in [0, 1/2]$, using Lemma A.2-5, the payments are given by

$$t_i(v_i, v_{-i}) = \sum_{k=1}^N \frac{1}{N} \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} + c v_{-i} \mathbb{1}\{v_i = v_{-i}\},$$

for $i = 1, 2$, and $v_i, v_{-i} \geq 0$. Note that since $\gamma_k \geq 1$, we have $\mathbb{1}\{v_i > \gamma_k v_{-i}\} \mathbb{1}\{v_i \neq v_{-i}\} = \mathbb{1}\{v_i > \gamma_k v_{-i}\}$.

We first derive a close form expression for $\sum_{i=1}^2 \mathbb{E}_{F_{a,b}} [t_i(v_i, v_{-i})]$. We have

$$\sum_{i=1}^2 \mathbb{E}_{F_{a,b}} [t_i(v_i, v_{-i})] = \sum_{i=1}^2 \sum_{k=1}^N \frac{1}{N} \mathbb{E}_{F_{a,b}} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] + 2 c b q_{a,b}^2,$$

where $q_{a,b} = e^{-b/a}$. Recall that by Lemma A.2-4, $\text{opt}(F_{a,b}) = b q_{a,b} (2 - q_{a,b})$, hence we have that

$$R(m, F_{a,b}) = \sum_{i=1}^2 \sum_{k=1}^N \frac{1}{N} \frac{1}{b q_{a,b} (2 - q_{a,b})} \mathbb{E}_{F_{a,b}} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] + 2 c \frac{q_{a,b}}{2 - q_{a,b}}.$$

Note that

$$\begin{aligned} \mathbb{E}_{F_{a,b}} [\gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\}] &= \int_0^b \int_0^b \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} dF_{a,b}(v_i) dF_{a,b}(v_{-i}) \\ &= \int_0^{(b/\gamma_k)^-} \int_0^b \gamma_k v_{-i} \mathbb{1}\{v_i > \gamma_k v_{-i}\} dF_{a,b}(v_i) dF_{a,b}(v_{-i}) \\ &= \int_0^{(b/\gamma_k)^-} \gamma_k v_{-i} \bar{F}_{a,b}(\gamma_k v_{-i}) dF_{a,b}(v_{-i}) \\ &= \int_0^{(b/\gamma_k)^-} \gamma_k \frac{v}{a} \exp(-\gamma_k v/a) \exp(-v/a) dv \\ &= \int_0^{(b/\gamma_k)^-} \gamma_k \frac{v}{a} \exp(-(\gamma_k + 1)v/a) dv, \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} &\mathbb{E}_{F_{a,b}} [\gamma_k v_{-i} \mathbb{1}\{\gamma_k v_{-i} < v_i\}] \\ &= \left[\gamma_k v \frac{-1}{\gamma_k + 1} \exp(-(\gamma_k + 1)\frac{v}{a}) \right]_{v=0}^{v=b/\gamma_k} + \frac{\gamma_k}{\gamma_k + 1} \int_0^{(b/\gamma_k)^-} \exp(-(\gamma_k + 1)\frac{v}{a}) dv \\ &= \frac{-b}{\gamma_k + 1} \exp\left(-\frac{\gamma_k + 1}{\gamma_k} \frac{b}{a}\right) - \frac{\gamma_k}{(\gamma_k + 1)^2} a \left[\exp(-(\gamma_k + 1)\frac{v}{a}) \right]_{v=0}^{v=b/\gamma_k} \\ &= a \left(\frac{\gamma_k}{(\gamma_k + 1)^2} - \frac{1}{\gamma_k + 1} \left(\frac{\gamma_k}{\gamma_k + 1} + \frac{b}{a} \right) \exp\left(-\frac{\gamma_k + 1}{\gamma_k} \frac{b}{a}\right) \right). \end{aligned}$$

Since $b/a = -\ln(q_{a,b})$, and $c \leq 1/2$, we obtain

$$R(m, F_{a,b}) \leq \sum_{i=1}^2 \sum_{k=1}^N \frac{1}{N} \frac{1}{q_{a,b}(2 - q_{a,b}) \ln(1/q_{a,b})} \Psi(\gamma_k, q_{a,b}) + \frac{q_{a,b}}{2 - q_{a,b}}, \quad (\text{A.4-1})$$

where $\psi : [1, \infty) \times [e^{-1}, 1] \rightarrow \mathbb{R}$ is defined as

$$\Psi(\alpha, q) := \frac{\alpha}{(\alpha + 1)^2} - \frac{1}{\alpha + 1} \left(\frac{\alpha}{\alpha + 1} + \ln(1/q) \right) q^{1+\frac{1}{\alpha}}. \quad (\text{A.4-2})$$

Next we further simplify the bound. To that end, we will establish that for any q in $[e^{-1}, 1]$ $\Psi(\alpha, q)$ is non-increasing in α on $[1, \infty)$. Note first that $\Psi(\alpha, 1) = 0$, which is non-decreasing for $\alpha \geq 1$. Furthermore, note that

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial \alpha \partial q}(\alpha, q) &= \frac{\partial}{\partial \alpha} \left(-\frac{1}{\alpha} q^{1/\alpha} \ln(1/q) \right) \\ &= \frac{1}{\alpha^2} q^{1/\alpha} \left(1 + \frac{1}{\alpha} \ln(q) \right) \ln(1/q) \geq \frac{1}{\alpha^2} q^{1/\alpha} \left(1 - \frac{1}{\alpha} \right) \ln(1/q) \geq 0, \end{aligned}$$

where in the previous to last inequality, we used that $q \geq e^{-1}$ and in the last inequality, we used that $\alpha \geq 1$. By the Schwarz's theorem, we conclude that $\frac{\partial \Psi}{\partial \alpha}(\alpha, \cdot)$ is non-decreasing, and hence for any q in $[e^{-1}, 1]$, we have that

$$\frac{\partial \Psi}{\partial \alpha}(\alpha, q) \leq \frac{\partial \Psi}{\partial \alpha}(\alpha, 1) = 0.$$

In turn, we obtain that Ψ is non-increasing in α on $[1, \infty)$.

Returning to Eq. (A.4-1) and leveraging the above, we have that

$$\Psi(\gamma_k, q_{a,b}) \leq \Psi(1, q_{a,b}),$$

and hence the performance of any mechanism in \mathcal{M}_{sf}^{max} is bounded as follows

$$\begin{aligned} R(m, F_{a,b}) &\leq \frac{1}{q_{a,b}(2 - q_{a,b}) \ln(1/q_{a,b})} \left(\frac{1}{2} - \left(\frac{1}{2} + \ln(1/q_{a,b}) \right) q_{a,b}^2 \right) + \frac{q_{a,b}}{2 - q_{a,b}} \\ &= \frac{1}{2} \frac{1 - q_{a,b}^2}{q_{a,b}(2 - q_{a,b}) \ln(1/q_{a,b})} - \frac{q_{a,b}^2 \ln(1/q_{a,b})}{q_{a,b}(2 - q_{a,b}) \ln(1/q_{a,b})} + \frac{q_{a,b}}{2 - q_{a,b}} \\ &= \frac{1}{2} \frac{1 - q_{a,b}^2}{q_{a,b}(2 - q_{a,b}) \ln(1/q_{a,b})}. \end{aligned}$$

Recalling that $q_{a,b}$ spans $[e^{-1}, 1]$, this concludes the proof. \square

Lemma A.4-1. *For each distribution F in \mathcal{W}_{mhr} , there exists a sequence F_n in \mathcal{F}_{mhr} that converges weakly to F , such that there exists $M > 0$ such that for all n , $F_n(M) = 1$.*

Proof of Lemma A.4-1. Fix F in \mathcal{W}_{mhr} . Then there exists $a \geq b > 0$, such that

$$F(v) = \begin{cases} 1 - \exp(-\frac{v}{a}) & \text{if } v < b, \\ 1 & \text{if } v \geq b. \end{cases}$$

The proof will be constructive in that we will explicitly exhibit a sequence F_n that satisfies the properties in the result.

Step 1. We first construct the sequence and analyze it. Let $n \geq 1$, and define for $v \geq 0$

$$g_n(v) = a \left(1 - \left(\frac{v}{b}\right)^n\right), \text{ for } 0 \leq v \leq b.$$

Define the sequence of cumulative distribution functions F_n

$$F_n(v) = \begin{cases} 1 - \exp\left(-\int_0^v \frac{1}{g_n(u)} du\right) & , \text{ if } v < b, \\ 1 & , \text{ if } v \geq b. \end{cases} \quad (\text{A.4-3})$$

Note that $F_n(a) = 1$, for all $n \geq 1$, so we can set $M := a$.

Step 2. We next establish that F_n belongs to \mathcal{F}_{mhr} and that the sequence converges weakly to F . For that let us show that F_n has no mass at $v = b$.

Since for any $0 \leq v < b$ and $0 \leq u \leq v$, we have that

$$\frac{1}{g_n(u)} = \frac{1}{a} \frac{1}{1 - u/b} \frac{1}{\sum_{i=0}^{n-1} (u/b)^i} \geq \frac{1}{na} \frac{1}{1 - u/b},$$

where the last inequality follows from the fact that $0 \leq u \leq b$. Hence, we conclude that for all $v < b$,

$$-\int_0^v \frac{1}{g_n(u)} du \leq -\int_0^v \frac{1}{an} \frac{1}{1 - u/b} du = -\frac{b}{an} \ln(1 - v/b),$$

thus for all $v < b$, the cumulative distribution F_n is lower bounded as follows

$$F_n(v) \geq 1 - \left(1 - \frac{v}{b}\right)^{\frac{b}{an}},$$

by taking the limit, we conclude that $\lim_{v \uparrow b} F_n(v) = 1$, hence F_n does not have any mass. Furthermore its hazard rate is given by

$$a^{-1} / \left(1 - \left(\frac{v}{b}\right)^n\right), \text{ for } v < b.$$

which is clearly non-decreasing on the support of F_n . Hence, F_n belongs to \mathcal{F}_{mhr} .

Let us check that F_n converges weakly to F . The points of continuity of F are $\mathbb{R}^+ \setminus \{b\}$. For all $v < b$, we have $\lim_{n \uparrow \infty} (v/b)^n = 0$. Hence, by the dominated convergence theorem, for all $v < b$, we have $\lim_{n \uparrow \infty} F_n(v) = F(v)$. For $v > b$, we have that for all $n \geq 1$, $F_n(v) = F(v) = 1$. The result follows. □

Proof of Proposition 1.5. The goal of this proof is to lower bound the performance of the second price auction. For that, we will bound the difference between the optimal revenue and the expected value of the second highest value leveraging the MHR property. Fix any F in \mathcal{F}_{mhr} . We have

$$\begin{aligned} & \text{opt}(F) - \mathbb{E}[v_{[2]}] \\ &= \mathbb{E}[v_{[2]} \mathbf{1}\{v_{[2]} \geq r_F\}] + 2r_F q_F (1 - q_F) - [\mathbb{E}[v_{[2]} \mathbf{1}\{v_{[2]} \geq r_F\}] + \mathbb{E}[v_{[2]} \mathbf{1}\{v_{[2]} < r_F\}]] \\ &= 2r_F q_F (1 - q_F) - \mathbb{E}[v_{[2]} \mathbf{1}\{v_{[2]} < r_F\}] \\ &= 2r_F q_F (1 - q_F) - 2 \int_0^{r_F} v \bar{F}(v) f(v) dv \\ &= 2r_F q_F (1 - q_F) - 2 \int_{q_F}^1 \text{Rev}(q) dq, \end{aligned}$$

where the last equality follows from a change of variable $q = \bar{F}(v)$. We next lower bound $\text{Rev}(q) = qF^{-1}(1-q)$ on $[q_F, 1]$. For that, we use a classical result in reliability theory: (Barlow and Proschan, 1975, Chapter 4, Theorem 2.18). Namely, for any distribution with increasing hazard rate, we have that for each $\lambda > 0$, $\bar{F}(v) - \exp\{-\lambda v\}$ has at most one change of sign and if one change of sign occurs, it occurs from + to -. In particular, letting $\lambda = -\ln(q_F)/r_F$, we have that $\bar{F}(r_F) = \exp(-\lambda r_F)$, i.e., the

crossing occurs at r_F and hence

$$\bar{F}(v) \geq \exp\{-\lambda v\}, \quad \text{for all } v \leq r_F.$$

In turn, moving to the quantile space, with $q = \bar{F}(v)$, we have

$$q \geq \exp\{(\ln(q_F)/r_F)F^{-1}(1-q)\} \quad \text{for all } q \geq q_F.$$

Taking the log, and multiplying q , we obtain

$$q \ln(q) \frac{r_F}{\ln(q_F)} \leq qF^{-1}(1-q) = Rev(q) \quad \text{for all } q \geq q_F.$$

In turn, we have

$$\begin{aligned} \int_{q_F}^1 Rev(q) dq &\geq \frac{r_F}{\ln(q_F)} \int_{q_F}^1 q \ln(q) dq = \frac{r_F}{\ln(q_F)} \left[\frac{1}{2} [q^2 \ln(q)]_{q_F}^1 - \frac{1}{2} \int_{q_F}^1 q dq \right] \\ &= \frac{r_F}{\ln(q_F)} \left[-\frac{1}{2} q_F^2 \ln(q_F) - \frac{1}{4} (1 - q_F^2) \right] \\ &= -\frac{1}{2} r_F q_F^2 - \frac{1}{4} \frac{r_F (1 - q_F^2)}{\ln(q_F)}. \end{aligned}$$

We deduce that

$$\begin{aligned} \text{opt}(F) - \mathbb{E}[v_{[2]}] &\leq 2r_F q_F (1 - q_F) + r_F q_F^2 + \frac{1}{2} \frac{(1 - q_F^2)}{\ln(q_F)} \\ &= r_F q_F (2 - q_F) + \frac{1}{2} \frac{r_F (1 - q_F^2)}{\ln(q_F)}, \end{aligned}$$

and hence

$$\frac{\mathbb{E}[v_{[2]}]}{\text{opt}(F)} \geq 1 - \frac{r_F q_F (2 - q_F) + \frac{1}{2} \frac{r_F (1 - q_F^2)}{\ln(q_F)}}{\text{opt}(F)}.$$

Using the argument in the proof of (Fu et al., 2015, Corollary 1), we have that $\text{opt}(F) \geq Rev(q_F)(2 - q_F) = r_F q_F (2 - q_F)$. Note that the latter is the revenue of the mechanism that offers a posted price of r_F to the maximum value buyer. In turn, this implies that

$$\frac{\mathbb{E}[v_{[2]}]}{\text{opt}(F)} \geq -\frac{1}{2} \frac{1 - q_F^2}{q_F (2 - q_F) \ln(q_F)}.$$

□

Proof of Theorem 1.5. This is a direct consequence of Theorem 1.4 and Proposition 1.5. Indeed, we have

$$\begin{aligned} \inf_{q \in [e^{-1}, 1]} \frac{1 - q^2}{2q(2 - q)\ln(1/q)} &\stackrel{(a)}{\leq} \inf_{F \in \mathcal{F}_{mhr}} R(m_{spa}, F) \\ &\leq \mathcal{R}(\mathcal{M}, \mathcal{F}_{mhr}) \stackrel{(b)}{\leq} \inf_{q \in [e^{-1}, 1]} \frac{1 - q^2}{2q(2 - q)\ln(1/q)}, \end{aligned}$$

where (a) follows from Proposition 1.5 and (b) follows from Theorem 1.4. Hence, we conclude the result. For the numerical evaluation of the infimum, we took a grid of the interval $[e^{-1}, 1]$ with step size 10^{-6} . □

A.5 Proofs for Section 1.7.1

Proof of Proposition 1.6. In this proof, we analyze the upper bound then we analyze the lower bound.

Upper bound We will first show the upper bounds, for that note the following that we can decompose the worst case performance as either two buyer case, i.e. $K = 2$ or at least three buyers, i.e. $K \geq 3$, formally

$$\tilde{\mathcal{R}}(\tilde{\mathcal{M}}, \mathcal{F}) = \sup_{m \in \tilde{\mathcal{M}}} \min \left(\inf_{F \in \mathcal{F}} R^2(m, F), \inf_{K \geq 3} \inf_{F \in \mathcal{F}} R^K(m, F) \right). \quad (\text{A.5-1})$$

Hence, we conclude that

$$\tilde{\mathcal{R}}(\tilde{\mathcal{M}}, \mathcal{F}) \leq \mathcal{R}(\mathcal{M}, \mathcal{F}). \quad (\text{A.5-2})$$

We conclude that all the upper bounds developed in the previous sections can still be applied, hence we get the upper bounds.

Lower bound For the lower bound, note that from (A.5-1) that we can control the case of two buyers using our previous results whereas now we need some lower bound

performance on a mechanisms when the number of players is greater than 2. For that we use we will use the implication of a result shown in (Fu et al., 2015, Corollary 1).

Lemma A.5-1. *For any number of buyers $K \geq 2$ and a distribution F in \mathcal{F}_{reg} then the performance of second price auction without a reserve price is bounded as follows,*

$$R^K(m_{spa}, F) \geq \max\left(\frac{1 - (1 - q_F)^{K-1}}{1 - (1 - q_F)^K}, \frac{K - 1}{K}\right),$$

where $q_F = \bar{F}(r_F)$ is the oracle optimal quantile.

Now we will treat the case of regular and mhr class of distributions separately. Let us start by the latter.

One can show that the function $\frac{1 - (1 - q_F)^{K-1}}{1 - (1 - q_F)^K}$ is non-decreasing in K and q_F . By Hartline et al. (2008), we know that for any distribution F in \mathcal{F}_{mhr} , we have that $q_F \geq e^{-1}$. Hence we conclude that for any F in \mathcal{F}_{mhr} and $K \geq 3$ we have

$$R^K(m_{spa}, F) \geq \frac{1 - (1 - e^{-1})^2}{1 - (1 - e^{-1})^3} \geq 80\%,$$

By Theorem 1.5, we have that

$$\inf_{F \in \mathcal{F}_{mhr}} R^2(m_{spa}, F) \approx 71.53\%,$$

Hence by (A.5-1), we get that

$$\tilde{\mathcal{R}}(\tilde{\mathcal{M}}, \mathcal{F}) \geq \inf_{F \in \mathcal{F}_{mhr}} R^2(m_{spa}, F).$$

and the analysis of the upper bound in particular, (A.5-2), we get that

$$\tilde{\mathcal{R}}(\tilde{\mathcal{M}}, \mathcal{F}) = \inf_{F \in \mathcal{F}_{mhr}} R^2(m_{spa}, F) \approx 71.53\%.$$

Now let us now analyse the performance against the regular class of distributions. For that let us consider the following mechanism if there are two buyers, i.e. $K = 2$, then the seller uses the mechanism, we proposed in Section 1.5.3 and if $K \geq 3$ the seller uses a second price auction. For such mechanism the seller would make at least

51.9% when there are two buyers and get $2/3$ when there are at least three buyers based on Lemma A.5-1. Using (A.5-1), we conclude the result.

□

Appendix B

Appendix for Chapter 2

B.1 The Interplay of Inflation and Deflation

A simple heuristic. A natural heuristic analyzed in the literature (see Huang et al. (2015) and Fu et al. (2015)) is simply to “post your sample”, namely not use any deflation or inflation and just use $\gamma = 1$ with probability one. It is well known that for regular and MHR distributions, the competitive ratio of such a heuristic is 50%. The key idea is that the sample is equally likely to be above or below the willingness to pay of a new customer and for a random variable with support $[\underline{s}, \bar{s}]$, with $\bar{s} = \underline{s} + o(\underline{s})$, i.e., with a very small variability, the optimal revenue is at least \underline{s} . For such a family, post your sample collects half of $\underline{s} + o(\underline{s})$ and the ratio of performances is of order $1/2 + o(1)$. Hence relying only on post your sample may not lead to optimal performance, and the need for inflation and deflation arises.

The necessity of deflation. Against the family of distributions described above (with support $[\underline{s}, \bar{s}]$, with $\bar{s} = \underline{s} + o(\underline{s})$), as a matter of fact any mechanism that puts all mass on multiplicative factors $\gamma > 1$ would actually perform arbitrarily poorly. This failure could be mitigated by putting some weight on deflation for some $\gamma < 1$. Indeed, in such a case, if $\gamma\bar{s} \leq \underline{s}$ and $\gamma > 1/2$, then the deflation mechanism guarantees a fraction $\gamma > 1/2$ of the optimal revenues. In particular, the argument above implies that any mechanism that does not have some weight on deflation will be suboptimal against \mathcal{F}_α (as any such family contains distributions with the properties outlined).

The selective need for inflation. While we have seen above that some positive weight on deflation is necessary (for all values of α), the role of inflation is more subtle. Let $\mathcal{M}^{\leq 1} := \{\Psi \in \mathcal{M} : \Psi(s|s) = 1\}$ denote mechanisms in \mathcal{M} that do not put any mass on inflation.

Proposition B.1-1 (the role of inflation). *i.) Against regular distributions ($\alpha = 0$), any mechanism that does not put any mass on inflation is suboptimal, i.e.,*

$$\mathcal{R}(\mathcal{M}^{\leq 1}, \mathcal{F}_0) < \mathcal{R}(\mathcal{P}, \mathcal{F}_0).$$

ii.) Let $\tilde{\mathcal{F}}_1$ denote the set of MHR distributions ($\alpha = 1$) with bounded support and continuous density f on the support \mathcal{S} . Against such distributions, it is sufficient to focus on mechanisms that do not put any mass on inflation, i.e.,

$$\mathcal{R}(\mathcal{M}^{\leq 1}, \tilde{\mathcal{F}}_1) = \mathcal{R}(\mathcal{P}, \tilde{\mathcal{F}}_1).$$

Part *i.*) follows from Fu et al. (2015) in conjunction with formalizing that a mechanism with no mass on inflation cannot do better than 50%. In particular, this implies that while inflation will lead to arbitrarily poor performance against distributions that have little variation (such as those exhibited in the post your sample discussion above), any mechanism that does not put some mass on inflation will be suboptimal against regular distributions.

Part *ii.*) establishes that against a large subclass of MHR distributions, inflation is actually not necessary and one may focus on the class $\mathcal{M}^{\leq 1}$ without loss of optimality. This highlights the subtle and different role of inflation across different values of α . In contrast to deflation, inflation is not needed “across the board”. The proof of part *ii.*) of Proposition B.1-1 is based on establishing that the strategy that posts the sample s always dominates any strategy that inflates by any amount. In particular, this implies that we can always exclude inflation and restrict to deflation or post your sample without loss of optimality. We conjecture that this is also true for the entire

MHR class \mathcal{F}_1 ; the assumptions of continuity of the density and boundedness of the support are made for technical reasons to allow for an exchange of the integral and derivative operators in the proof.

Intuitively, MHR distributions have little variation, since their coefficient of variation is always less than one (Barlow and Proschan, 1975), and their optimal oracle price is sufficiently far from the upper support. Indeed, for any distribution F in the MHR class \mathcal{F}_1 , it has been established that $\bar{F}(r_F) \geq e^{-1}$ (Hartline et al., 2008). This means that there is significant mass above the optimal oracle posted price, which suggests that the benefits of any mass on inflation may be limited. Proposition B.1-1ii.) establishes that there is actually *no benefit* for a large subclass of MHR distributions.

B.2 Proofs for Section 2.3

Proof of Proposition 2.1. Let us fix α in $[0, 1]$. Clearly, $\mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha) \leq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$ since $\mathcal{M} \subset \mathcal{P}$. We next establish that $\mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$. Fix $\varepsilon > 0$. We will establish that $\mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon$. By definition of the supremum, there exists $\Psi \in \mathcal{P}$ such that

$$\inf_{G \in \mathcal{F}_\alpha} R(\Psi, G) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon.$$

Fix an arbitrary $F \in \mathcal{F}_\alpha$ and for any integer n , let $F_n(\cdot) = F(n\cdot)$ denote the scaled version of F . Note that F_n belongs to \mathcal{F}_α since $\phi_{F_n}(v) - \alpha v = (1/n) (\phi_F(nv) - \alpha nv)$, for all $v \geq 0$. By definition of the infimum, we have that,

$$R(\Psi, F_n) \geq \inf_{G \in \mathcal{F}_\alpha} R(\Psi, G) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon. \quad \text{for all } n \geq 1. \quad (\text{B.2-1})$$

Next, we re-express $R(\Psi, F_n)$.

$$\begin{aligned}
R(\Psi, F_n) &= \frac{1}{\text{opt}(F_n)} \int_0^\infty \int_0^\infty p \bar{F}_n(p) d\Psi(p|s) dF_n(s) \\
&= \frac{1}{\text{opt}(F_n)} \int_0^\infty \int_0^\infty p \bar{F}(np) d\Psi\left(p \mid \frac{u}{n}\right) dF(u) \\
&= \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty np \bar{F}(np) d\Psi\left(p \mid \frac{u}{n}\right) dF(u) \\
&= \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \zeta \bar{F}(\zeta) d\Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) dF(u).
\end{aligned}$$

The second equality follows from a change of variable $u = ns$, the third follows from the fact that $\text{opt}(F_n) = n^{-1}\text{opt}(F)$; and the fourth equality follows from another change of variable $\zeta = np$.

Hence, plugging into Eq. (B.2-1) and taking the limit over n , we get that

$$\liminf_{n \rightarrow \infty} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \zeta \bar{F}(\zeta) d\Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) dF(u) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon.$$

Let us analyze the limiting term. For each $u \geq 0$, we have, by integrations by parts,

$$\begin{aligned}
\int_0^\infty \zeta \bar{F}(\zeta) d\Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) &\stackrel{(a)}{=} \left[\zeta \bar{F}(\zeta) \Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) \right]_{\zeta=0}^{\zeta=\infty} - \int_0^\infty (\zeta \bar{F}(\zeta))' \Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) d\zeta \\
&\stackrel{(a')}{=} - \int_0^\infty (\zeta \bar{F}(\zeta))' \Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right) d\zeta,
\end{aligned}$$

where in (a'), we have used the fact that $\Psi\left(\frac{\zeta}{n} \mid \frac{u}{n}\right)$ is in $[0, 1]$ and the fact that $0 \bar{F}(0) = \lim_{\zeta \rightarrow \infty} \zeta \bar{F}(\zeta) = 0$. The latter follows from the fact that for any $a > 0$, $a \bar{F}(a) = \int_0^a \bar{F}(v) dv - \mathbb{E}[v \mathbb{1}\{v \leq a\}]$. On the one hand, by the dominated convergence theorem $\lim_{a \uparrow \infty} \mathbb{E}[v \mathbb{1}\{v \leq a\}] = \mathbb{E}[v]$ and on the other hand, we have that $\int_0^\infty \bar{F}(v) dv = \mathbb{E}[v]$ (see Ross (1996)), since $\mathbb{E}[v]$ is finite because F is in \mathcal{G} . We conclude that $\lim_{a \uparrow \infty} a \bar{F}(a) = 0$.

By the well-behaved assumption around zero, c.f. Definition 2.1, we have that there exists a mechanism Ψ^∞ such that for all $u, q \geq 0$

$$\lim_{n \rightarrow \infty} \Psi\left(\frac{q}{n} \mid \frac{u}{n}\right) = \Psi^\infty(q|u),$$

Furthermore, note that

$$\int_0^\infty |(\zeta \bar{F}(\zeta))'| dq = \int_0^{r_F} (\zeta \bar{F}(\zeta))' dq - \int_{r_F}^\infty (\zeta \bar{F}(\zeta))' dq = 2 \text{opt}(F).$$

By the dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \int_0^\infty \zeta \bar{F}(\zeta) d\Psi \left(\frac{\zeta}{n} \mid \frac{u}{n} \right) \stackrel{(b)}{=} - \int_0^\infty (\zeta \bar{F}(\zeta))' \Psi^\infty(\zeta \mid u) d\zeta \stackrel{(c)}{=} \int_0^\infty \zeta \bar{F}(\zeta) d\Psi^\infty(\zeta \mid u),$$

where for (c), we used the same integration by parts as in the two previous equalities (a) and (a').

Now using the dominated convergence theorem again, we get that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \zeta \bar{F}(\zeta) d\Psi \left(\frac{\zeta}{n} \mid \frac{u}{n} \right) dF(u) \\ &= \frac{1}{\text{opt}(F)} \int_0^\infty \lim_{n \rightarrow \infty} \int_0^\infty \zeta \bar{F}(\zeta) d\Psi \left(\frac{\zeta}{n} \mid \frac{u}{n} \right) dF(u) \\ &= \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \zeta \bar{F}(\zeta) d\Psi^\infty(\zeta \mid u) dF(u) \\ &= R(\Psi^\infty, F). \end{aligned}$$

In turn, we have for all $F \in \mathcal{F}_\alpha$,

$$R(\Psi^\infty, F) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon.$$

Fix $s, p > 0$ and $\theta > 0$. It is clear that $\Psi^\infty(\cdot \mid s)$ is non-decreasing and in $[0, 1]$, for any s in $[0, \infty)$. Moreover, we have that

$$\Psi^\infty(p \mid s) = \lim_{n \rightarrow \infty} \Psi \left(\frac{p}{n} \mid \frac{s}{n} \right) = \lim_{n \rightarrow \infty} \Psi \left(\frac{\theta p}{n} \mid \frac{\theta s}{n} \right) = \Psi^\infty(\theta p \mid \theta s), \quad (\text{B.2-2})$$

Now let us further reduce the expression of the performance of Ψ^∞ . For F in \mathcal{F}_α ,

we have

$$\begin{aligned}
R(\Psi^\infty, F) &= \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty p \bar{F}(p) d\Psi^\infty(p|s) dF(s) \\
&\stackrel{(a)}{=} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \gamma s \bar{F}(\gamma s) d\Psi^\infty(\gamma s|s) dF(s) \\
&\stackrel{(b)}{=} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \gamma s \bar{F}(\gamma s) d\Psi^\infty(\gamma|1) dF(s) \\
&\stackrel{(c)}{=} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) d\Psi^\infty(\gamma|1) \\
&\stackrel{(d)}{\leq} \frac{1}{\text{opt}(F)} \int_0^\infty \int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) \frac{d\Psi^\infty(\gamma|1)}{\int_0^\infty d\Psi^\infty(\zeta|1)},
\end{aligned}$$

where in (a) we have used a change of variable $p = \gamma s$; in (b), we have used Eq. (B.2-2); in (c), we have used Tonelli's Theorem; in (d) we have used the fact that $\Psi^\infty(\cdot|1)$ is in $[0, 1]$.

By taking the infimum over all distributions then we conclude that for all $\varepsilon > 0$

$$\inf_{F \in \mathcal{F}_\alpha} \mathcal{R}(\tilde{\Psi}^\infty, F) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) - \varepsilon,$$

with $\tilde{\Psi}^\infty(p|s) = \Psi^\infty(\frac{p}{s}|1) / \int_0^\infty d\Psi^\infty(\gamma|1)$. Since $\tilde{\Psi}^\infty \in \mathcal{M}$ and ε was arbitrary, we obtain that $\mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha) \geq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha)$. The proof is complete. \square

B.3 Proofs for Section 2.4

Proof of Lemma 2.1. We will show each point separately.

i). Noting that \tilde{H}_l^{-1} is non-increasing, the fact that \tilde{H}_l lower bounds \bar{F} on $[w, w']$ implies that for all v in $[w, w']$,

$$v \geq \tilde{H}_l^{-1}(\bar{F}(v)). \tag{B.3-1}$$

Furthermore note that by the assumption that $\tilde{H}_l(w) = \bar{F}(w)$ and $\tilde{H}_l(w') = \bar{F}(w')$, hence the composition of functions $\tilde{H}_l^{-1}(\bar{F}(\cdot))$ maps $[w, w']$ into $[w, w']$. So the left-hand side above always belongs to $[w, w']$ when $v \in [w, w']$.

Now, by assumption, the revenue $u\bar{F}(u)$ curve is non-decreasing for u in $[\gamma w, \gamma w']$ and hence $\gamma v\bar{F}(\gamma v)$ is non-decreasing for v in $[w, w']$. Leveraging the latter point and the inequality (B.3-1), we deduce that

$$\gamma v \bar{F}(\gamma v) \geq \gamma \tilde{H}_l^{-1}(\bar{F}(v)) \bar{F}\left(\gamma \tilde{H}_l^{-1}(\bar{F}(v))\right).$$

Using now the fact that $\tilde{H}_l^{-1}(\bar{F}(\cdot))$ belongs to $[w, w']$ and that H_l lower bounds \bar{F} on $[\gamma w, \gamma w']$, we obtain

$$\gamma \tilde{H}_l^{-1}(\bar{F}(v)) \bar{F}\left(\gamma \tilde{H}_l^{-1}(\bar{F}(v))\right) \geq \gamma \tilde{H}_l^{-1}(\bar{F}(v)) H_l\left(\gamma \tilde{H}_l^{-1}(\bar{F}(v))\right).$$

In turn, we have

$$\begin{aligned} \int_w^{\gamma w'} \gamma v \bar{F}(\gamma v) f(v) dv &\geq \int_w^{\gamma w'} \gamma \tilde{H}_l^{-1}(\bar{F}(v)) H_l\left(\gamma \tilde{H}_l^{-1}(\bar{F}(v))\right) f(v) dv \\ &= \int_{q_w}^{q_{w'}} \gamma \tilde{H}_l^{-1}(q) H_l\left(\gamma \tilde{H}_l^{-1}(q)\right) dq. \end{aligned}$$

where the last equality follows from change of variable $q = \bar{F}(v)$. This completes the proof of the first point.

ii.) Let us now show the second point. By assumption, we have

$$\bar{F}(v) \geq H_l(v) \quad \text{if } \gamma w \leq v \leq \gamma w', \quad (\text{B.3-2})$$

$$\bar{F}(v) \leq H_u(v) \quad \text{if } w \leq v \leq w'. \quad (\text{B.3-3})$$

In particular, Eq.(B.3-2) implies that

$$\int_w^{\gamma w'} \gamma v \bar{F}(\gamma v) f(v) dv \geq \int_w^{\gamma w'} \gamma v H_l(\gamma v) f(v) dv.$$

In addition, Eq.(B.3-3) implies that

$$v \leq \min(w', H_u^{-1}(\bar{F}(v))), \quad w \leq v \leq w'. \quad (\text{B.3-4})$$

By assumption *iii.*), $H_u^{-1}(\bar{F}(v))$ maps $[w, +\infty)$ to $[w, +\infty)$ and by assumption *ii.*), $vH_l(v)$ is non-increasing on $[\gamma w, +\infty)$ and hence

$$\gamma v H(\gamma v) \geq \gamma \min(w', H_u^{-1}(\bar{F}(v))) H_l(\gamma \min(w', H_u^{-1}(\bar{F}(v)))).$$

We deduce that

$$\begin{aligned} \int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv &\geq \int_w^{w'} \gamma \min(w', H_u^{-1}(\bar{F}(v))) H_l(\gamma \min(w', H_u^{-1}(\bar{F}(v)))) f(v) dv \\ &= \int_{q_{w'}}^{q_w} \gamma \min(w', H_u^{-1}(q)) H_l(\gamma \min(w', H_u^{-1}(q))) dq, \end{aligned}$$

where in the last equality, we performed the change of variable $q = \bar{F}(v)$. This completes the proof. \square

Proof of Lemma 2.2 . Fix $\alpha \in [0, 1]$, and F in \mathcal{F}_α .

Let us define introduce the inverse function of Γ_α , denoted by $\Gamma_\alpha^{-1}(\cdot)$, is well defined and for q in $(0, 1]$ is given by

$$\Gamma_\alpha^{-1}(q) = \begin{cases} (q^{-(1-\alpha)} - 1) (1 - \alpha)^{-1} & \text{if } \alpha \in (0, 1] \\ -\ln(q) & \text{if } \alpha = 1. \end{cases}$$

Step 1. Fix $\xi > 0$, We first establish the following holds.

- If the set $\mathcal{S} := \{v : \bar{F}(v) = \Gamma_\alpha(\xi v)\}$ is empty then for all $v \geq 0$, we have

$$\bar{F}(v) \geq \Gamma_\alpha(\xi v).$$

- If the set \mathcal{S} is non-empty then for all u in \mathcal{S} and $v \geq 0$, we have

$$\bar{F}(v) \geq \Gamma_\alpha(\xi v), \quad \text{if } v \leq u,$$

$$\bar{F}(v) \leq \Gamma_\alpha(\xi v), \quad \text{if } v \geq u.$$

In other words, the difference $\bar{F}(v) - \Gamma_\alpha(\xi v)$ has at most one change of sign and if one change of sign occurs, it occurs from $+$ to $-$. If $\alpha = 1$, the result follows from (Barlow and Proschan, 1975, Chapter 4, Theorem 2.18), since the MHR class is a subclass of the increasing failure rate in average (IFRA) class of distributions.

We next generalize the line of arguments used for $\alpha = 1$. Note that $\Gamma_\alpha(\cdot)$ is a bijection, hence its inverse is well defined $\Gamma_\alpha^{-1}(\cdot)$ and moreover by (Schweizer and

Szech, 2016, Proposition 1 and Lemma 1), we have that $f(u)/\overline{F}^{1+(1-\alpha)}(u)$ is non decreasing in u and that

$$\Gamma_\alpha^{-1}(\overline{F}(v)) = \int_0^v \frac{f(u)}{\overline{F}^{1+(1-\alpha)}(u)} du,$$

Hence $\Gamma_\alpha^{-1}(\overline{F}(\cdot))$ is convex. Since $\Gamma_\alpha^{-1}(\overline{F}(0)) = 0$, then by convexity we get that

$$\frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v}$$

is non-decreasing in v . Define for each $\xi > 0$ the sets

$$\begin{aligned} \mathcal{E}_\xi &= \left\{ v : \frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v} = \xi \right\} \\ \mathcal{S} &= \left\{ v : \overline{F}(v) = \Gamma_\alpha(\xi v) \right\}. \end{aligned}$$

Note that \mathcal{S} is non-empty if and only if \mathcal{E}_ξ is also non-empty.

If the set \mathcal{E}_ξ is empty then since $\frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v}$ is non-decreasing, then necessarily we \mathcal{S} is empty and we get the result.

If the set \mathcal{E}_ξ is non-empty, then, since $\frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v}$ is non-decreasing, then for all u in \mathcal{E}_ξ , we have

$$\begin{aligned} \frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v} &\leq \xi, \text{ if } v \leq u \\ \frac{\Gamma_\alpha^{-1}(\overline{F}(v))}{v} &\geq \xi, \text{ if } v \geq u. \end{aligned}$$

based on the latter inequalities and the fact that Γ_α is decreasing we conclude the first step.

Step 2. Define the ccdf \overline{G} as follows

$$\overline{G}(u) = \frac{\overline{F}(u+w)}{q_w}, \quad u \geq 0.$$

Note that $G = 1 - \overline{G}$ is an element \mathcal{F}_α . Note also $\overline{G}(u) - \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) u / (w' - w) \right)$ equals zero when $u = w' - w$, hence the set $\mathcal{S} = \{v : \overline{G}(v) = \Gamma_\alpha(\tilde{\xi}v)\}$ is non-empty.

Using step 1, we deduce that

$$\begin{aligned}\bar{G}(u) &\geq \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) u / (w' - w) \right), \quad 0 \leq u \leq w' - w, \\ \bar{G}(u) &\leq \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) u / (w' - w) \right) \quad u \geq w' - w.\end{aligned}$$

Conducting a change of variable $v = u + w$, we obtain the desired result. \square

Proof of Proposition 2.2. *i.)* We first note that an application of Lemma 2.2 yields that

$$\bar{F}(v) \geq q_w \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v - w}{w' - w} \right), \quad v \in [w, w'], \quad (\text{B.3-5})$$

$$\bar{F}(u) \geq q_{\gamma w} \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{v - \gamma w}{\gamma w' - \gamma w} \right), \quad v \in [\gamma w, \gamma w']. \quad (\text{B.3-6})$$

Since $w' \leq r_F / \gamma$, the revenue curve $v\bar{F}(v)$ is non-decreasing on $[0, \gamma w']$. Hence the first condition of Lemma 2.1 is satisfied. On the other hand, the remaining conditions of Lemma 2.1 are satisfied by using (B.3-5) and (B.3-6) and setting

$$\begin{aligned}\tilde{H}_l(v) &:= \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v - w}{w' - w} \right), \\ H_l(v) &:= q_{\gamma w} \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{v - \gamma w}{\gamma w' - \gamma w} \right).\end{aligned}$$

Lemma 2.1 implies that

$$\int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv \geq \int_{q_w}^{q_{w'}} \gamma \tilde{H}_l^{-1}(q) H_l(\gamma \tilde{H}_l^{-1}(q)) dq.$$

Note that

$$\begin{aligned}\tilde{H}_l^{-1}(q) &= w + (w' - w) \frac{\Gamma_\alpha^{-1} \left(\frac{q}{q_w} \right)}{\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right)} \\ &= w' \left(\frac{w}{w'} + \left(1 - \frac{w}{w'} \right) \frac{\Gamma_\alpha^{-1} \left(\frac{q}{q_w} \right)}{\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right)} \right) = w' u_\alpha \left(1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, \frac{q}{q_w} \right),\end{aligned}$$

since for any $q \leq q_w$ and θ in $[0, 1]$, we have

$$\theta + (1 - \theta) \frac{\Gamma_\alpha^{-1} \left(\frac{q}{q_w} \right)}{\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right)} \leq 1.$$

By substituting all functions explicitly, we get that

$$\begin{aligned}
& \int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv \\
& \geq \gamma w' q_{\gamma w} \int_{q_{w'}}^{q_w} u_\alpha \left(1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, \frac{q}{q_w} \right) \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{\gamma w' u_\alpha \left(1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, \frac{q}{q_w} \right) - \gamma w}{\gamma w' - \gamma w} \right) dq \\
& = \gamma w' q_{\gamma w} q_w \int_{q_{w'}/q_w}^1 u_\alpha \left(1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, s \right) \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{u_\alpha \left(1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, s \right) - \frac{w}{w'}}{1 - \frac{w}{w'}} \right) ds \\
& = \gamma w' q_{\gamma w} q_w \mathcal{A}_\alpha \left(\frac{q_{w'}}{q_w}, 1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, 1, \frac{q_{\gamma w'}}{q_{\gamma w}}, 1, \frac{q_{w'}}{q_w}, \frac{w}{w'}, 1, \frac{w}{w'}, \frac{w}{w'} \right).
\end{aligned}$$

ii.) Next we analyze the case when $r_F/\gamma \leq w < w'$. The proof follows a similar structure, but now we apply Lemma 2.2 by exhibiting appropriate functions that verify the required conditions. Using Lemma 2.2, we have for any $\underline{w} \in [0, w)$,

$$\begin{aligned}
\bar{F}(v) & \leq q_{\underline{w}} \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_w}{q_{\underline{w}}} \right) \frac{v - \underline{w}}{w - \underline{w}} \right) =: H_u(v), \quad v \in [\underline{w}, w], \\
\bar{F}(v) & \geq q_{\gamma w} \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{v - \gamma w}{\gamma w' - \gamma w} \right) =: H_l(v), \quad v \in [\gamma w, \gamma w'].
\end{aligned}$$

By Lemma B.7-1, that is stated and proved in Appendix B.7, the function $vH_l(v)$ is non-increasing on $[\gamma w, \infty)$. In turn, Lemma 2.2 implies that

$$\int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv \geq \int_{q_{w'}}^{q_w} \gamma H_u^{-1}(q) H_l(\gamma H_u^{-1}(q)) dq.$$

Note that

$$H_u^{-1}(q) = w \left(\frac{w}{w} + \left(1 - \frac{w}{w} \right) \frac{\Gamma_\alpha^{-1} \left(\frac{q}{q_w} \right)}{\Gamma_\alpha^{-1} \left(\frac{q_w}{q_w} \right)} \right) =: w \tilde{u}_\alpha \left(\frac{q_w}{q_w}, \frac{w}{w}, \frac{q}{q_w} \right).$$

So we deduce that

$$\begin{aligned}
& \int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv \\
& \geq \gamma q_{\gamma w} \int_{q_{w'}}^{q_w} \min \left(w', w \tilde{u}_\alpha \left(\frac{q_w}{q_w}, \frac{w}{w}, \frac{q_w}{q_w} s \right) \right) \\
& \quad \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{\gamma \min \left(w', w \tilde{u}_\alpha \left(\frac{q_w}{q_w}, \frac{w}{w}, \frac{q_w}{q_w} s \right) \right) - \gamma w}{\gamma w' - \gamma w} \right) dq \\
& = \gamma w q_{\gamma w} q_w \int_{q_{w'}/q_w}^1 u_\alpha \left(\frac{w'}{w}, \frac{q_w}{q_w}, \frac{w}{w}, \frac{q_w}{q_w} s \right) \\
& \quad \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{w}{w'} \frac{u_\alpha \left(\frac{w'}{w}, \frac{q_w}{q_w}, \frac{w}{w}, \frac{q_w}{q_w} s \right) - 1}{1 - \frac{w}{w'}} \right) ds.
\end{aligned}$$

Setting $\underline{w} = \tilde{\gamma} w$, with $\tilde{\gamma} = \frac{w}{w'}$, and $w' = \tilde{\gamma} w$, we obtain

$$\begin{aligned}
& \int_w^{w'} \gamma v \bar{F}(\gamma v) f(v) dv \\
& \geq \gamma w q_{\gamma w} q_w \int_{q_{w'}/q_w}^1 u_\alpha \left(\frac{w'}{w}, \frac{q_w}{q_{\tilde{\gamma} w}}, \tilde{\gamma}, \frac{q_w}{q_{\tilde{\gamma} w}} s \right) \\
& \quad \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{\gamma w'}}{q_{\gamma w}} \right) \frac{w}{w'} \frac{u_\alpha \left(\frac{w'}{w}, \frac{q_w}{q_{\tilde{\gamma} w}}, \tilde{\gamma}, \frac{q_w}{q_{\tilde{\gamma} w}} s \right) - 1}{1 - \frac{w}{w'}} \right) ds \\
& = \gamma w q_{\gamma w} q_w \mathcal{A}_\alpha \left(\frac{q_{w'}}{q_w}, \frac{w}{w'}, \frac{q_w}{q_{\tilde{\gamma} w}}, \tilde{\gamma}, \frac{q_w}{q_{\tilde{\gamma} w}}, \frac{q_{\gamma w'}}{q_{\gamma w}}, \frac{w}{w'}, \frac{q_w}{q_{\tilde{\gamma} w}}, \tilde{\gamma}, \frac{q_w}{q_{\tilde{\gamma} w}}, 1, \frac{w}{w'} \right).
\end{aligned}$$

□

B.4 Proofs for Section 2.5

The proof of Proposition 2.3 follows from taking $\varepsilon = 0$ and applying Proposition B.6-1 and Proposition B.6-2.

The proof of Proposition 2.4 follows from taking $\varepsilon = 0$ and applying Proposition B.6-3 and Proposition B.6-4.

The proof of Theorem 2.1 follows from taking $\varepsilon = 0$ in Theorem 2.2 and letting $k \uparrow \infty$.

B.5 Proofs for Section 2.6

Proof of Proposition 2.5. The proof is divided into three steps:

1. In the first step, we show that the maximin ratio is upper bounded as follows

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \leq \mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha^W),$$

where

$$\mathcal{F}_\alpha^W = \left\{ \overline{F}(\cdot | q_0, q_1, \infty) : (q_0, q_1) \in [0, 1]^2 : q_1 \leq q_0 \leq 1 \right. \\ \left. \text{and } (1 - \alpha) + \frac{1}{\Gamma_\alpha^{-1}(q_0)} \geq \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)} \right\}.$$

Note that we focus only on the unbounded support distributions of the family introduced in Section 2.6.

2. Second, we show that under the conditions stated in the proposition for any F in \mathcal{F}_α^W , we have the optimal reserve price is achieved at $v = 1$ and $\text{opt}(F) = q_0$.
3. Finally, we show that the main inequality.

Step 1. Let us start by showing the first point. Fix (q_0, q_1) in \mathcal{Q}_α and $F(\cdot | q_0, q_1)$ in \mathcal{F}^W , its $(1 - \alpha)$ -virtual value function is given by

$$\phi_F^\alpha(v) = \begin{cases} -\frac{1}{\Gamma_\alpha^{-1}(q_0)} & , \text{ if } v < 1, \\ (1 - \alpha) - \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)} & , \text{ if } v > 1. \end{cases}$$

Since F in \mathcal{F}^W then $(1 - \alpha) - \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)} \geq -\frac{1}{\Gamma_\alpha^{-1}(q_0)}$ is clearly non-decreasing on \mathbb{R}^+ . Hence, F belongs to \mathcal{F}_α . So we conclude that $\mathcal{F}_\alpha^W \subset \mathcal{F}_\alpha$. In turn, we get

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) = \mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha) \leq \mathcal{R}(\mathcal{M}, \mathcal{F}_\alpha^W).$$

Step 2.

Let us now move to the second point. Fix (q_0, q_1) in \mathcal{Q}_α and $F(\cdot | q_0, q_1)$ in \mathcal{F}^W , $v = 1$ is the optimal reserve price of F , i.e. $\text{opt}(F) = \overline{F}(1) = q_0$. The proof is a direct implication of the following Lemma stated here and shown below.

Lemma B.5-1. Fix two scalars $\beta \geq 0$ and $w \geq 0$. The revenue function $v\Gamma_\alpha(\beta(v-w))$ for $v \geq w$ attains its maximum at

$$r = \max\left(\frac{1 - (1 - \alpha)\beta w}{\beta\alpha}, w\right).$$

Given this Lemma, then

$$1 \leq \frac{1}{\alpha\Gamma_\alpha^{-1}(q_0)}$$

makes sure that $v = 1$ is below the reserve price of the first piece. This is equivalent to

$$q_0 \geq \alpha^{1/(1-\alpha)}.$$

Whereas on the side, based on Lemma B.5-1, the condition

$$1 \geq \frac{1 - (1 - \alpha)\beta_1}{\beta_1\alpha}.$$

makes sure that $v = 1$ is above the reserve price of the second piece, with $\beta_1 = \Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right)$. This is equivalent to the fact that

$$q_1 \leq q_0\Gamma_\alpha(1).$$

As a conclusion and by α -SR, we get that $v = 1$ is necessarily the reserve price.

Step 3. Let us now move to the third point. For that note that we can compute the inverse of the CCDF which is given by

$$v(q) = \begin{cases} \Gamma_\alpha^{-1}(q)/\Gamma_\alpha^{-1}(q_0), & \text{if } q > q_0, \\ \Gamma_\alpha^{-1}\left(\frac{q}{q_0}\right)/\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right) + 1, & \text{if } q \leq q_0, \end{cases}$$

which can be rewritten as

$$v(q) = \begin{cases} u_\alpha(0^+, q_0, 0, q), & \text{if } q > q_0, \\ 2 u_\alpha(0^+, q_1/q_0, 1/2, q/q_0), & \text{if } q \leq q_0, \end{cases}$$

Let us fix $\gamma > 0$, let us denote $\tilde{q}_1 = \bar{F}(1/\gamma)$. We will analyze the case of deflation, i.e. $\gamma \leq 1$ then the case of inflation, i.e. $\gamma > 1$ separately.

Deflation. Let us assume that $\gamma \leq 1$. We first decompose the bound into three terms

$$\begin{aligned} R(\delta_\gamma, F) &= \frac{1}{\text{opt}(F)} [\mathcal{C}(\gamma, 0, r_F; F) + \mathcal{C}(\gamma, r_F, r_F/\gamma; F) + \mathcal{C}(\gamma, r_F/\gamma, \infty; F)] \\ &= \frac{1}{q_0} [\mathcal{C}(\gamma, 0, 1; F) + \mathcal{C}(\gamma, 1, 1/\gamma; F) + \mathcal{C}(\gamma, 1/\gamma, \infty; F)], \end{aligned}$$

where δ_γ was defined in Eq.(2.4).

Let us analyze each term separately. Let us start by the first term, by analyzing the quantity in the quantile space, we get,

$$\begin{aligned} \mathcal{C}(\gamma, 0, 1; F) &= \int_{q_0}^1 \gamma u_\alpha(0^+, q_0, 0, q) \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_0) \gamma u_\alpha(0^+, q_0, 0, q) \right) dq \\ &= \gamma \mathcal{A}_\alpha \left(q_0, 0^+, q_0, 0, 1, q_0, \gamma, q_0, 0, 1, 0, 0 \right) \end{aligned}$$

Now let us move to the second term, we have

$$\begin{aligned} &\mathcal{C}(\gamma, 1, 1/\gamma; F) \\ &= \int_{\tilde{q}_1}^{q_0} 2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_0) \gamma 2 u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \right) dq \\ &= q_0 \int_{\tilde{q}_1/q_0}^1 \gamma 2 u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q) \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_0) \gamma 2 u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q) \right) dq \\ &= 2\gamma q_0 \mathcal{A}_\alpha \left(\tilde{q}_1/q_0, 0^+, \frac{q_1}{q_0}, 1/2, 1, q_0, 2\gamma, \frac{q_1}{q_0}, 1/2, 1, 0, 0 \right) \end{aligned}$$

where in the second equality follows from change of variable.

Now let us move to the third and last term,

$$\begin{aligned} &\mathcal{C}(\gamma, 1/\gamma, \infty; F) \\ &= 2 \int_0^{\tilde{q}_1} \gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_1}{q_0} \right) \left(2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) - 1 \right) \right) dq \\ &= \int_0^{\tilde{q}_1} 2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_1}{q_0} \right) \gamma \frac{u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) - 1/2\gamma}{1 - 1/2} \right) dq \\ &= 2\gamma \tilde{q}_1 \mathcal{A}_\alpha \left(0, 0^+, \frac{q_1}{q_0}, 1/2, \tilde{q}_1/q_0, \frac{q_1}{q_0}, \gamma, \frac{q_1}{q_0}, 1/2, \tilde{q}_1/q_0, 1/2\gamma, 1/2 \right) \end{aligned}$$

Inflation. Let us now analyze the case $\gamma > 1$. Using a similar decomposition of the performance as in the deflation case, we have

$$\begin{aligned} R(\delta_\gamma, F) &= \frac{1}{\text{opt}(F)} [\mathcal{C}(\gamma, 0, r_F/\gamma; F) + \mathcal{C}(\gamma, r_F/\gamma, r_F; F) + \mathcal{C}(\gamma, r_F, \infty; F)] \\ &= \frac{1}{q_0} [\mathcal{C}(\gamma, 0, 1/\gamma; F) + \mathcal{C}(\gamma, 1/\gamma, 1; F) + \mathcal{C}(\gamma, 1, \infty; F)]. \end{aligned}$$

As in the deflation case, let us analyze each term separately. Let us start by the first term, by analyzing the quantity in the quantile space, we get,

$$\begin{aligned} \mathcal{C}(\gamma, 0, 1/\gamma; F) &= \int_{\tilde{q}_1}^1 \gamma u_\alpha(0^+, q_0, 0, q) \Gamma_\alpha(\Gamma_\alpha^{-1}(q_0) \gamma u_\alpha(0^+, q_0, 0, q)) dq \\ &= \gamma \mathcal{A}_\alpha(\tilde{q}_1, 0^+, q_0, 1, q_0, \gamma, q_0, 0, 1, 0, 0) \end{aligned}$$

Now let us move to the second term, we have

$$\begin{aligned} &\mathcal{C}(\gamma, 1/\gamma, 1; F) \\ &= \int_{q_0}^{\tilde{q}_1} \gamma u_\alpha(0^+, q_0, 0, q) q_0 \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right) (\gamma u_\alpha(0^+, q_0, 0, q) - 1)\right) dq \\ &= \gamma q_0 \tilde{q}_1 \int_{q_0/\tilde{q}_1}^1 u_\alpha(0^+, q_0, 0, \tilde{q}_1 q) \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right) (\gamma u_\alpha(0^+, q_0, 0, \tilde{q}_1 q) - 1)\right) dq \\ &= \gamma q_0 \tilde{q}_1 \mathcal{A}_\alpha(q_0/\tilde{q}_1, 0^+, q_0, 0, \tilde{q}_1, q_1/q_0, \gamma/2, q_0, 0, \tilde{q}_1, 1/\gamma, 1/2) \end{aligned}$$

where the first equality follows from a change of variable.

Now let us move to the third and last term,

$$\begin{aligned} &\mathcal{C}(\gamma, 1, \infty; F) \\ &= \int_0^{q_0} 2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right) \left(2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) - 1\right)\right) dq \\ &= \int_0^{q_0} 2\gamma u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q_1}{q_0}\right) \gamma \frac{u_\alpha(0^+, \frac{q_1}{q_0}, 1/2, q/q_0) - 1/2\gamma}{1 - 1/2}\right) dq \\ &= 2\gamma q_0 \mathcal{A}_\alpha\left(0, 0^+, \frac{q_1}{q_0}, 1/2, 1, \frac{q_1}{q_0}, \gamma, \frac{q_1}{q_0}, 1/2, 1, 1/2\gamma, 1/2\right) \end{aligned}$$

where the third equality follows from a change of variable.

Finally, we get the result by combining all the terms and making the following change of variable $\rho = q_1/q_0$.

□

Proof of Lemma B.5-1. First, note that the distribution $\Gamma_\alpha(\beta(v-w))$ for $v \geq w$ is α -SR. Its virtual value function is given by

$$\psi(v) = v - \frac{\Gamma_\alpha(\beta(v-w))}{-\Gamma'_\alpha(\beta(v-w))} = v - \frac{1 + (1-\alpha)\beta(v-w)}{\beta} = \alpha v - \frac{1 - (1-\alpha)\beta w}{\beta}.$$

Since the virtual value function achieves its maximum at $v = (1 - (1-\alpha)\beta w)/\beta\alpha$, then we get the result since necessarily $v \geq w$. \square

B.6 Proofs of Section 2.7

Throughout, we fix $\varepsilon \geq 0$. We first remind the definition of a grid of quantile values

$$\mathcal{G}_\varepsilon = \begin{cases} [0, 1] & \text{if } \varepsilon = 0, \\ \{k\varepsilon : 1 \leq k \leq \lfloor 1/\varepsilon \rfloor\} \cup \{1\} & \text{if } \varepsilon > 0. \end{cases}$$

For any x in $[0, 1]$, we define $\pi(x)$ to be the right-projection of x on the grid \mathcal{G}_ε , i.e.,

$$\pi(x) := \inf\{y \in \mathcal{G}_\varepsilon : x \leq y\}.$$

B.6.1 General tractable lower bound

B.6.1.1 Tractable Lower bound for contributions from samples lower than r_F/γ

Proposition B.6-1. Fix $\varepsilon \geq 0, \alpha \in [0, 1]$ and $\gamma \in \mathbb{R}_{++} \setminus \{1\}$. The functional equation, $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J = J$ introduced in (2.10) admits a unique bounded solution $\mathcal{L}_{\alpha, \gamma, \varepsilon}$. Furthermore, for any bounded function J in $\mathcal{W}(\mathcal{G}_\varepsilon^2)$, $(\mathcal{T}_{\alpha, \gamma, \varepsilon}^L)^k J$ converges to $\mathcal{L}_{\alpha, \gamma, \varepsilon}$ as k grows to ∞ .

Proof of Proposition B.6-1. We focus on the properties of recursion (2.10). Fix an admissible policy $\mu : \mathcal{G}_\varepsilon^2 \rightarrow \mathcal{G}_\varepsilon^2$ such that $(\mu_1(q, \rho^+), \mu_2(q, \rho^+)) \in \mathcal{B}_{\alpha, \gamma, \varepsilon}^L(q, \rho^+)$ for all $(q, \rho^+) \in \mathcal{G}_\varepsilon^2$. Define the operator $\mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} : \mathcal{W}(\mathcal{G}_\varepsilon^2) \rightarrow \mathcal{W}(\mathcal{G}_\varepsilon^2)$ as follows. For

$(q, q^+) \in \mathcal{G}_\varepsilon^2$,

$$\begin{aligned} & (\mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J)(q, q^+) \\ &= (q - \varepsilon) (\mu_1(q, \rho^+) - \varepsilon) \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L (\rho^+, \mu_2(q, \rho^+)) \right) + \tilde{\gamma} J (\mu_1(q, \rho^+), \mu_2(q, \rho^+)). \end{aligned}$$

Note that for any $(q, q^+) \in \mathcal{G}_\varepsilon^2$, by Lemma B.7-5, we have that $\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L (\rho^+, \mu_2(q, \rho^+)) \right)$ is bounded above by 1. Moreover, q , ρ^+ and $\mu_1(q, \rho^+)$ as well as $\mu_2(q, \rho^+)$ belong to $[0, 1]$. We deduce that for any bounded functions J in $\mathcal{W}(\mathcal{G}_\varepsilon^2)$, $\mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J$ is also bounded. Furthermore we have for any J, J' and $\mu(\cdot, \cdot)$,

$$\mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J(q, \rho^+) - \mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J'(q, \rho^+) = \tilde{\gamma} [J(\mu_1(q, \rho^+), \mu_2(q, \rho^+)) - J'(\mu_1(q, \rho^+), \mu_2(q, \rho^+))].$$

We deduce that

$$\|\mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J - \mathcal{T}_{\alpha, \gamma, \varepsilon}^{L, \mu} J'\| \leq \tilde{\gamma} \|J - J'\|,$$

where the norm denotes the sup norm in the space of bounded functions. Since $\tilde{\gamma} < 1$, $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L$ is a contraction operator. (Bertsekas, 2013, Proposition 2.1.1) implies that $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J = J$ admits a unique solution $\mathcal{L}_{\alpha, \gamma, \varepsilon}$. Furthermore, $\mathcal{L}_{\alpha, \gamma, \varepsilon}$ can be computed through value iteration. We have, starting with any bounded J ,

$$\mathcal{L}_{\alpha, \gamma, \varepsilon} = \lim_{k \rightarrow \infty} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^L)^k J.$$

This completes the proof. \square

Proposition B.6-2. Fix $\varepsilon \geq 0$, $\alpha \in [0, 1]$, $\gamma \in \mathbb{R}_{++} \setminus \{1\}$, and $F \in \mathcal{F}_\alpha$. Furthermore, let

$$r_j := \frac{r_F}{\tilde{\gamma}^j}, \quad q_j := \bar{F}(r_j), \quad \text{and} \quad \hat{q}_j := \pi(q_j), \quad j \in \mathbb{Z}. \quad (\text{B.6-1})$$

i.) The operator $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L$ preserves the following property

$$\frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{\gamma}{q_0 \tilde{\gamma}^j} J \left(\hat{q}_{j+\tilde{\alpha}}, \pi \left(\frac{q_{j+1+\tilde{\alpha}}}{q_{j+\tilde{\alpha}}} \right) \right), \quad (\text{B.6-2})$$

where $j \leq 1$ and $\tilde{\alpha} = -\mathbf{1}\{\gamma \in [0, 1]\}$.

ii.) Furthermore, we have

$$\frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{\gamma}{q_0 \tilde{\gamma}^j} \mathcal{L}_{\alpha, \gamma, \varepsilon} \left(\hat{q}_{j+\tilde{\alpha}}, \pi \left(\frac{q_{j+1+\tilde{\alpha}}}{q_{j+\tilde{\alpha}}} \right) \right), \quad j \leq 1,$$

where $\mathcal{L}_{\alpha, \gamma, \varepsilon}$ is the unique fixed point of $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J = J$.

Proof of Proposition B.6-2. i.) We first establish a property preserved by the operator $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L$ defined in (2.10). Suppose that a bounded function J satisfies Eq. (B.6-2) and Fix $j \leq 1$. We have, by decomposing the integral,

$$\begin{aligned} & \frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \\ = & \frac{1}{\text{opt}(F)} \int_{r_{j-1}}^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv + \frac{1}{\text{opt}(F)} \int_0^{r_{j-1}} \gamma v \bar{F}(\gamma v) f(v) dv. \end{aligned}$$

We have

$$\begin{aligned} & \frac{1}{\text{opt}(F)} \int_{r_{j-1}}^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \\ \stackrel{(a)}{\geq} & \gamma \frac{q_{j+\tilde{\alpha}} q_{j-1+\tilde{\alpha}}}{q_0 \tilde{\gamma}^j} \mathcal{A}_\alpha \left(\beta_{\tilde{\gamma}}^L \left(\frac{q_j}{q_{j-1}}, \frac{q_{j+1-2\tilde{\alpha}}}{q_{j-2\tilde{\alpha}}} \right) \right) \\ \stackrel{(b)}{\geq} & \gamma \frac{q_{j+\tilde{\alpha}} q_{j-1+\tilde{\alpha}}}{q_0 \tilde{\gamma}^j} \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1+\tilde{\alpha}}}{q_{j+\tilde{\alpha}}} \right), \pi \left(\frac{q_{j+\tilde{\alpha}}}{q_{j-1+\tilde{\alpha}}} \right) \right) \right), \end{aligned}$$

where (a) follows from an application of Proposition 2.2i.) with $w = r_{j-1}$ and $w' = r_j$ and (b) follows from the monotonicity established in Lemma B.7-2.

From the assumption that (B.6-2) is true for the function J , we also have

$$\frac{1}{\text{opt}(F)} \int_0^{r_{j-1}} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{\gamma}{q_0 \tilde{\gamma}^{j-1}} J \left(\hat{q}_{j-1+\tilde{\alpha}}, \pi \left(\frac{q_{j+\tilde{\alpha}}}{q_{j-1+\tilde{\alpha}}} \right) \right).$$

In turn, we get

$$\begin{aligned}
& \frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \\
& \geq \gamma \frac{q_{j+\bar{\alpha}} q_{j-1+\bar{\alpha}}}{q_0 \tilde{\gamma}^j} \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right), \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \right) \\
& \quad + \frac{\gamma}{q_0 \tilde{\gamma}^{j-1}} J \left(\hat{q}_{j-1+\bar{\alpha}}, \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \\
& \geq \gamma \frac{(\hat{q}_{j+\bar{\alpha}} - \varepsilon) (\hat{q}_{j-1+\bar{\alpha}} - \varepsilon)}{q_0 \tilde{\gamma}^j} \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right), \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \right) \\
& \quad + \frac{\gamma}{q_0 \tilde{\gamma}^{j-1}} J \left(\hat{q}_{j-1+\bar{\alpha}}, \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \\
& = \frac{\gamma}{q_0 \tilde{\gamma}^j} \left[(\hat{q}_{j+\bar{\alpha}} - \varepsilon) (\hat{q}_{j-1+\bar{\alpha}} - \varepsilon) \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right), \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \right) \right. \\
& \quad \left. + \tilde{\gamma} J \left(\hat{q}_{j-1+\bar{\alpha}}, \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right) \right] \\
& \geq \frac{\gamma}{q_0 \tilde{\gamma}^j} \inf_{\substack{q^-, \rho^- \in \\ \mathcal{B}_{\alpha, \gamma, \varepsilon}^L \left(\hat{q}_{j+\bar{\alpha}}, \pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right) \right)}} \left[(\hat{q}_{j+\bar{\alpha}} - \varepsilon) (q^- - \varepsilon) \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right), \rho^- \right) \right) \right. \\
& \quad \left. + \tilde{\gamma} J(q^-, \rho^-) \right] \\
& = \frac{\gamma}{q_0 \tilde{\gamma}^j} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J) \left(\hat{q}_{j+\bar{\alpha}}, \pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right) \right),
\end{aligned}$$

where the last inequality is a consequence of the fact that $\left(\hat{q}_{j-1+\bar{\alpha}}, \pi \left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}} \right) \right)$ belongs to $\mathcal{B}_{\alpha, \gamma, \varepsilon}^L \left(\hat{q}_{j+\bar{\alpha}}, \pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right) \right)$, a fact established in Lemma B.7-4. Since the above was true for any $j \leq 1$, we have established that Eq. (B.6-2) holds for $\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J$.

ii.) Now starting with $J = 0$, and applying repeatedly the argument above leads to the following: For any $k \geq 1$,

$$\frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{\gamma}{q_0 \tilde{\gamma}^j} \left((\mathcal{T}_{\alpha, \gamma, \varepsilon}^L)^k J \right) \left(\hat{q}_{j+\bar{\alpha}}, \pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right) \right), \quad j \leq 1.$$

Furthermore, by Proposition B.6-1, $(\mathcal{T}_{\alpha, \gamma, \varepsilon}^L)^k J$ converges to $\mathcal{L}_{\alpha, \gamma}$ as k grows to ∞ .

Hence, for any $j \leq 1$,

$$\frac{1}{\text{opt}(F)} \int_0^{r_j} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{\gamma}{q_0 \tilde{\gamma}^j} \mathcal{L}_{\alpha, \gamma, \varepsilon} J \left(\hat{q}_{j+\bar{\alpha}}, \pi \left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}} \right) \right).$$

This concludes the proof. \square

As mentioned in the main text, for $\alpha = 0$, the discretized lower bound leads to the trivial bound of zero. This is due to the fact the quantile at the the optimal price can be arbitrarily small. Hence there is a need to develop a bound for this case that do not use the a dynamic program and is still gives good performance. More formally, we show the following,

Corollary B.1. *Fix $\varepsilon \in [0, 1)$ and a distribution F in \mathcal{F}_α , we have*

$$\frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) \geq \widetilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon}(q^*, \rho),$$

where $q^* = \pi(\bar{F}(r_F))$ and $\rho = \pi(\bar{F}(r_F/\gamma)/\bar{F}(r_F))$ and

$$\widetilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon}(q, \rho) = \begin{cases} \frac{(1-\alpha)}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{q^*}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q)) dq \\ \quad + \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q^*)) \mathcal{A}_\alpha(\hat{\beta}_{\gamma, \varepsilon}^L(\rho, \gamma)), & \text{if } \gamma < 1 \\ \frac{(1-\alpha)}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{\min\{\gamma q^*, 1\}}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha\left(\frac{\Gamma_\alpha^{-1}(q^*)}{\Gamma_\alpha^{-1}(\gamma q^*)} \Gamma_\alpha^{-1}(q)\right) dq, & \text{if } \gamma > 1. \end{cases}$$

Proof of Corollary B.1. Let us start by the deflation, i.e. the first case. For that, let us fix $\gamma < 1$, we have

$$\frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) = \frac{1}{\text{opt}(F)} [\mathcal{C}(\gamma, 0, r_F; F) + \mathcal{C}(\gamma, r_F, r_F/\gamma; F)],$$

Let us lower bound each term. Let us start by the first term. by Proposition 2.2, we have

$$\frac{\mathcal{C}(\gamma, 0, r_F; F)}{\text{opt}(F)} \geq \gamma \frac{r_F}{\text{opt}(F)} \mathcal{A}_\alpha(\beta_\gamma^L(\bar{F}(r_F), \bar{F}(\gamma r_F))).$$

By Lemma B.7-4, we have

$$\bar{F}(\gamma r_F) \geq \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(\bar{F}(r_F))) \text{ and } \bar{F}(r_F)/\bar{F}(\gamma r_F) \geq \gamma. \quad (\text{B.6-3})$$

then we get that using the monotonicity of $\Gamma_\alpha(\cdot)$ that

$$\begin{aligned} \frac{\mathcal{C}(\gamma, 0, r_F; F)}{\text{opt}(F)} &\geq \gamma \frac{r_F}{\text{opt}(F)} \mathcal{A}_\alpha(\beta_\gamma^L(\bar{F}(r_F), \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(\bar{F}(r_F)))))) \\ &= \gamma \frac{1}{\bar{F}(r_F) \Gamma_\alpha^{-1}(\bar{F}(r_F))} \int_{\bar{F}(r_F)}^1 \Gamma_\alpha^{-1}(q) \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q)) dq, \end{aligned}$$

since F in \mathcal{F}_α , by Cole and Roughgarden (2014), we have that $\bar{F}(r_F) \geq \alpha^{1/(1-\alpha)}$ and also since the function $q\Gamma_\alpha^{-1}(q)$ is non-increasing on $[\alpha^{1/(1-\alpha)}, 1]$ then we conclude that

$$(q^* - \varepsilon)\Gamma_\alpha^{-1}(q^* - \varepsilon) \geq \bar{F}(r_F)\Gamma_\alpha^{-1}(\bar{F}(r_F)),$$

hence, we get that

$$\begin{aligned} \frac{\mathcal{C}(\gamma, 0, r_F; F)}{\text{opt}(F)} &\geq \gamma \frac{1}{(q^* - \varepsilon)\Gamma_\alpha^{-1}(q^* - \varepsilon)} \int_{\bar{F}(r_F)}^1 \Gamma_\alpha^{-1}(q) \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q)) dq \\ &= \gamma \frac{1 - \alpha}{(q^* - \varepsilon)^\alpha - (q^* - \varepsilon)} \int_{q^*}^1 \Gamma_\alpha^{-1}(q) \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q)) dq, \end{aligned}$$

For the second term, by Proposition 2.2, we have that

$$\frac{\mathcal{C}(\gamma, r_F, r_F/\gamma; F)}{\text{opt}(F)} \geq \bar{F}(\gamma r_F) \mathcal{A}_\alpha(\beta_\gamma^L(\bar{F}(r_F/\gamma)/\bar{F}(r_F), \bar{F}(r_F)/\bar{F}(\gamma r_F))).$$

Now using Eq.(B.6-3) and Lemma B.7-3, we get

$$\begin{aligned} \frac{\mathcal{C}(\gamma, r_F, r_F/\gamma; F)}{\text{opt}(F)} &\geq \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(\bar{F}(r_F))) \mathcal{A}_\alpha(\beta_\gamma^L(\bar{F}(r_F/\gamma)/\bar{F}(r_F), \gamma)) \\ &= \Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q^*)) \mathcal{A}_\alpha(\hat{\beta}_{\gamma, \varepsilon}^L(\rho, \gamma)). \end{aligned}$$

Hence we conclude the first point.

For $\gamma > 1$, by Proposition 2.2, we have

$$\frac{\mathcal{C}(\gamma, 0, r_F/\gamma; F)}{\text{opt}(F)} \geq \frac{r_F}{\text{opt}(F)} \mathcal{A}_\alpha(\beta_0^L(\bar{F}(r_F/\gamma), \bar{F}(r_F))).$$

By Lemma B.7-4, we have

$$\bar{F}(r_F/\gamma) \geq \Gamma_\alpha(\gamma^{-1} \Gamma_\alpha^{-1}(\bar{F}(r_F))) \text{ and } \bar{F}(r_F)/\bar{F}(\gamma^{-1} r_F) \geq \gamma^{-1},$$

and using the monotonicity of the function $\Gamma_\alpha^{-1}(\cdot)$, we get if $\gamma \bar{F}(r_F) \leq 1$,

$$\begin{aligned} &\mathcal{A}_\alpha(\beta_0^L(\bar{F}(r_F/\gamma), \bar{F}(r_F))) \\ &\geq \int_{\gamma \bar{F}(r_F)}^1 \gamma \frac{\Gamma_\alpha^{-1}(q)}{\Gamma_\alpha^{-1}(\bar{F}(r_F))} \Gamma_\alpha\left(\frac{\Gamma_\alpha^{-1}(\bar{F}(r_F))}{\Gamma_\alpha^{-1}(\gamma \bar{F}(r_F))} \Gamma_\alpha^{-1}(q)\right) dq. \end{aligned}$$

Using the same arguments as in the deflation, we get that

$$\begin{aligned} & \frac{\mathcal{C}(\gamma, 0, r_F/\gamma; F)}{\text{opt}(F)} \\ & \geq \frac{(1-\alpha)}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{\gamma \bar{F}(r_F)}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha \left(\frac{\Gamma_\alpha^{-1}(\bar{F}(r_F))}{\Gamma_\alpha^{-1}(\gamma \bar{F}(r_F))} \Gamma_\alpha^{-1}(q) \right) dq. \end{aligned}$$

Moreover the function $q \rightarrow \frac{\Gamma_\alpha^{-1}(q)}{\Gamma_\alpha^{-1}(\gamma q)}$ is non-decreasing and since Γ_α is non increasing we conclude that

$$\frac{\mathcal{C}(\gamma, 0, r_F/\gamma; F)}{\text{opt}(F)} \geq \frac{1-\alpha}{(q^*-\varepsilon)^\alpha - (q^*-\varepsilon)} \int_{\gamma q^*}^1 \gamma \Gamma_\alpha^{-1}(q) \Gamma_\alpha \left(\frac{\Gamma_\alpha^{-1}(q^*)}{\Gamma_\alpha^{-1}(\gamma q^*)} \Gamma_\alpha^{-1}(q) \right) dq.$$

This concludes the proof. □

B.6.1.2 Tractable Lower bound for contributions from samples higher than r_F/γ

Proposition B.6-3. *Fix $\varepsilon \geq 0, \alpha \in [0, 1]$ and $\gamma \in (0, 1)$. The functional equation, $\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J = J$ admits a unique bounded solution $\mathcal{H}_{\alpha, \gamma, \varepsilon}$ and $(\mathcal{T}_{\alpha, \gamma, \varepsilon}^H)^k J$ converges to $\mathcal{H}_{\alpha, \gamma, \varepsilon}$ as k grows to ∞ for any bounded J in $\mathcal{W}(\mathcal{G}_\varepsilon)$.*

Proof of Proposition B.6-3. Fix an admissible policy $\mu : \mathcal{G}_\varepsilon^2 \rightarrow \mathcal{G}_\varepsilon$ such that $\mu(\rho^-) \in \mathcal{B}_{\alpha, \gamma, \varepsilon}^H(\rho^-)$ for all $\rho^- \in \mathcal{G}_\varepsilon$. We define the mapping $\mathcal{T}_{\alpha, \gamma, \varepsilon}^{H, \mu} : \mathcal{W}(\mathcal{G}_\varepsilon) \rightarrow \mathcal{W}(\mathcal{G}_\varepsilon)$ such that for all $\rho^- \in \mathcal{G}_\varepsilon$,

$$(\mathcal{T}_{\alpha, \gamma, \varepsilon}^{H, \mu} J)(\rho^-) = \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R(\mu(\rho^-), \rho^-) \right) + \frac{1}{\gamma} (\mu(\rho^-) - \varepsilon) (\rho^- - \varepsilon) J(\mu(\rho^-)).$$

Note that $\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R(\cdot, \cdot) \right)$ is bounded above by $1/\gamma$ by Lemma B.7-5 and that the quantity $(\mu(\rho^-) - \varepsilon) (\rho^- - \varepsilon)$ is also bounded by γ^2 since $\mu(\rho^-) \in \mathcal{B}_{\alpha, \gamma, \varepsilon}^H(\rho^-)$. We deduce that for all bounded functions J in $\mathcal{W}(\mathcal{G}_\varepsilon)$, both $\mathcal{T}_{\alpha, \gamma, \varepsilon}^{H, \mu} J$ and $\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J$ are also

bounded. Furthermore we have for any J, J' bounded functions and $\mu(\cdot)$,

$$\begin{aligned} \mathcal{T}_{\alpha,\gamma,\varepsilon}^{H,\mu} J(\rho^-) - \mathcal{T}_{\alpha,\gamma,\varepsilon}^{H,\mu} J'(\rho^-) &= \frac{1}{\gamma} (\mu(\rho^-) - \varepsilon) (\rho^- - \varepsilon) [J(\mu(\rho^-)) - J'(\mu(\rho^-))] \\ &\leq \frac{1}{\gamma} \gamma^2 [J(\mu(\rho^-)) - J'(\mu(\rho^-))] \\ &\leq \gamma [J(\mu(\rho^-)) - J'(\mu(\rho^-))], \end{aligned}$$

where we have used that $\mu(\rho^-)$ belongs to $\mathcal{B}_{\alpha,\gamma,\varepsilon}^H(\rho^-)$. We deduce (where the norm denotes the sup norm in the space of bounded functions)

$$\|\mathcal{T}_{\alpha,\gamma,\varepsilon}^{H,\mu} J - \mathcal{T}_{\alpha,\gamma,\varepsilon}^{H,\mu} J'\| \leq \gamma \|J - J'\|.$$

Since $\gamma < 1$, $\mathcal{T}_{\alpha,\gamma,\varepsilon}^{H,\mu}$ is a contraction operator. Using (Bertsekas, 2013, Proposition 2.1.1), $\mathcal{T}_{\alpha,\gamma,\varepsilon}^H J = J$ admits a unique solution $\mathcal{H}_{\alpha,\gamma,\varepsilon}$. Furthermore, $\mathcal{H}_{\alpha,\gamma,\varepsilon}$ can be computed through value iteration. We have, starting with any bounded J ,

$$\mathcal{H}_{\alpha,\gamma,\varepsilon} = \lim_{k \rightarrow \infty} (\mathcal{T}_{\varepsilon}^R)^k J.$$

This completes the proof. □

Proposition B.6-4. *Fix $\varepsilon \geq 0, \alpha \in [0, 1], \gamma \in (0, 1)$ and $F \in \mathcal{F}_\alpha$. Furthermore, let*

$$r_j := r_F / \gamma^j \text{ and } q_j := \bar{F}(r_j), \quad j \in \mathbb{Z}.$$

i.) *The operator $\mathcal{T}_{\alpha,\gamma,\varepsilon}^H J$ preserves the following property*

$$\frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} J \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right), \quad (\text{B.6-4})$$

for $j \geq 1$.

ii.) *Furthermore, we have*

$$\frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \mathcal{H}_{\alpha,\gamma,\varepsilon} \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right), \quad j \geq 1,$$

where $\mathcal{H}_{\alpha,\gamma,\varepsilon}$ is the unique fixed point of $\mathcal{T}_{\alpha,\gamma,\varepsilon}^H J = J$.

Proof of Proposition B.6-4. This proof follows the same structure as that of Proposition B.6-2.

i.) Suppose Eq. (B.6-4) is true for some bounded function J and fix $j \geq 1$. We have, by decomposing the integral,

$$\begin{aligned} & \frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \\ &= \frac{1}{\text{opt}(F)} \int_{r_j}^{r_{j+1}} \gamma v \bar{F}(\gamma v) f(v) dv + \frac{1}{\text{opt}(F)} \int_{r_{j+1}}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv. \end{aligned}$$

In turn, focusing on the first term, we have

$$\begin{aligned} \frac{1}{\text{opt}(F)} \int_{r_j}^{r_{j+1}} \gamma v \bar{F}(\gamma v) f(v) dv &\stackrel{(a)}{\geq} \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \mathcal{A}_\alpha \left(\beta_\gamma^R \left(\frac{q_{j+1}}{q_j}, \frac{q_j}{q_{j-1}}, \frac{q_j}{q_{j-1}} \right) \right) \\ &\stackrel{(b)}{\geq} \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right). \end{aligned}$$

where (a) follows from an application of Proposition 2.2ii.) with $w = r_j$ and $w' = r_{j+1}$; and (b) follows from the monotonicity properties established in Lemma B.7-2.

Now combining the above and (B.6-4), we have

$$\begin{aligned} & \frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \\ &\geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right) + \frac{q_{j+1} q_j}{q_0 \gamma^j} J \left(\pi \left(\frac{q_{j+1}}{q_j} \right) \right) \\ &= \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \left[\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right) + \frac{1}{\gamma} \frac{q_{j+1}}{q_{j-1}} J \left(\pi \left(\frac{q_{j+1}}{q_j} \right) \right) \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned}
& \frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \\
& \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \left[\mathcal{A}_{\alpha} \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right) + \frac{1}{\gamma} \frac{q_{j+1}}{q_j} \frac{q_j}{q_{j-1}} J \left(\pi \left(\frac{q_{j+1}}{q_j} \right) \right) \right], \\
& \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \left[\mathcal{A}_{\alpha} \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right) \right. \\
& \quad \left. + \frac{1}{\gamma} \left(\pi \left(\frac{q_{j+1}}{q_j} \right) - \varepsilon \right) \left(\pi \left(\frac{q_j}{q_{j-1}} \right) - \varepsilon \right) J \left(\pi \left(\frac{q_{j+1}}{q_j} \right) \right) \right], \\
& \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \inf_{\rho^+ \in \mathcal{B}_{\alpha, \gamma, \varepsilon}^H \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right)} \left\{ \mathcal{A}_{\alpha} \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\rho^+, \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right) \right. \\
& \quad \left. + \frac{1}{\gamma} (\rho^+ - \varepsilon) \left(\pi \left(\frac{q_j}{q_{j-1}} \right) - \varepsilon \right) J(\rho^+) \right\} \\
& = \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J) \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right),
\end{aligned}$$

where for the second inequality, we used the definition of the projection operator π ; and for the last inequality, we used the fact established in Lemma B.7-4 that $\pi \left(\frac{q_{j+1}}{q_j} \right)$ belongs to $\mathcal{B}_{\alpha, \gamma, \varepsilon}^H \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right)$. Since the above was for any $j \geq 1$, we have hence established that $\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J$ satisfies Eq. (B.6-4).

ii.) Now starting with $J = 0$, and applying repeatedly the argument above leads to the following. For any $k \geq 0$,

$$\frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \left((\mathcal{T}_{\alpha, \gamma, \varepsilon}^H)^k J \right) \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right)$$

Furthermore, by Proposition B.6-3, $\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J = J$ admits a unique bounded solution $\mathcal{H}_{\alpha, \gamma, \varepsilon}$ and $(\mathcal{T}_{\alpha, \gamma, \varepsilon}^H)^k J$ converges to $\mathcal{H}_{\alpha, \gamma, \varepsilon}$ as k grows to ∞ . Hence, we conclude that for any $j \geq 1$,

$$\frac{1}{\text{opt}(F)} \int_{r_j}^{\infty} \gamma v \bar{F}(\gamma v) f(v) dv \geq \frac{q_j q_{j-1}}{q_0 \gamma^{j-1}} \mathcal{H}_{\alpha, \gamma, \varepsilon} \left(\pi \left(\frac{q_j}{q_{j-1}} \right) \right).$$

This concludes the proof. \square

B.6.1.3 Main lower bound

Proof of Theorem 2.2. Fix $\gamma > 0$, such that $\gamma \neq 1$. We have for any distribution in \mathcal{F}_α ,

$$R(\delta_\gamma, F) = \frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) + \frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, r_F/\gamma, \infty; F).$$

We have by Proposition B.6-2, in particular the first point, Eq.(B.6-2), with $j = -1 - 2\tilde{\alpha}$ and iterate k times,

$$\begin{aligned} \frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) &= \frac{1}{\text{opt}(F)} \int_0^{r_F/\gamma} \gamma v \bar{F}(\gamma v) f(v) dv \\ &\geq \frac{1}{q_0} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J_2)^k \left(\hat{q}_{-1-\tilde{\alpha}}, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right), \end{aligned}$$

where $\tilde{\alpha} = -\mathbf{1}\{\gamma \in [0, 1]\}$.

By Corollary B.1, we have

$$\frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) \geq \tilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon} \left(q_0, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right)$$

By combining the last two results, we get that

$$\begin{aligned} &\frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, 0, r_F/\gamma; F) \\ &\geq \max \left\{ \frac{1}{q_0} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J_2)^k \left(q_0, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right), \tilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon} \left(q_0, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right) \right\}. \end{aligned}$$

Now let us analyze the remaining term, we have by Proposition B.6-4,

$$\frac{1}{\text{opt}(F)} \mathcal{C}(\gamma, r_F/\gamma, \infty; F) \geq (q_0 - \varepsilon) \left(\pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) - \varepsilon \right) (\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J_1)^k \left(\pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right).$$

Hence, we conclude that

$$\begin{aligned} R(\delta_\gamma, F) &\geq \max \left\{ \frac{1}{q_0} (\mathcal{T}_{\alpha, \gamma, \varepsilon}^L J_2)^k \left(q_0, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right), \tilde{\mathcal{L}}_{\alpha, \gamma, \varepsilon} \left(q_0, \pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right) \right\} \\ &\quad + (q_0 - \varepsilon) \left(\pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) - \varepsilon \right) (\mathcal{T}_{\alpha, \gamma, \varepsilon}^H J_1)^k \left(\pi \left(\frac{q-\tilde{\alpha}}{q-1-\tilde{\alpha}} \right) \right). \end{aligned}$$

By taking average over randomized values of γ and by Lemma B.7-4, we have

$$\left(\hat{q}_{-1-\tilde{\alpha}}, \pi \left(\frac{q-\tilde{\alpha}}{q_{-1-\tilde{\alpha}}} \right) \right) \in Q_{\alpha, \gamma, \varepsilon}(q_0),$$

and by Cole and Roughgarden (2014), we have $q_0 \geq \alpha^{1/(1-\alpha)}$ then, we conclude the result. \square

B.6.2 Evaluation of the upper bound

We fix $\varepsilon \geq 0$ and $M > 0$. We define a grid of $[0, M]$

$$\mathcal{G}_\varepsilon^M = \begin{cases} [0, M] & \text{if } \varepsilon = 0, \\ \{k\varepsilon : 1 \leq k \leq \lfloor M/\varepsilon \rfloor\} \cup \{M\} & \text{if } \varepsilon > 0. \end{cases}$$

For any x in $[0, M]$, we define $\pi(x)$ to be the right-projection of x on the grid \mathcal{G}_ε , i.e.,

$$\pi^M(x) := \inf\{y \in \mathcal{G}_\varepsilon^M : x \leq y\}.$$

Proof of Proposition 2.6. . We show the result in two main steps:

1. First we show the following Lemma,

Lemma B.6-1. *We have*

$$\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \leq \mathcal{R}(\mathcal{M}, \bar{\mathcal{F}}_\alpha^W),$$

where

$$\bar{\mathcal{F}}_\alpha^W = \left\{ \bar{F}(\cdot | q_0, q_1, \bar{v}) \text{ s.t. } \bar{v} > 1 \text{ and } (q_0, q_1) \in [0, 1]^2 \text{ with } q_1 \leq q_0 \leq 1 \right. \\ \left. \text{and } 1 - \alpha + \frac{1}{\Gamma_\alpha^{-1}(q_0)} \geq \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)} \right\}.$$

In particular, for any F in $\bar{\mathcal{F}}_\alpha^W$ and ψ in \mathcal{M} , we have

$$\inf_{\tilde{F} \in \bar{\mathcal{F}}_\alpha} R(\psi, \tilde{F}) \leq R(\psi, F)$$

2. Then we leverage and the projection to derive the upper bound.

The proof of Lemma B.6-1 is deferred at the end of this proof. Now, let us show the second point for that. Fix a distribution F in

$$\left\{ \bar{F}(\cdot | q_0, q_1, \bar{v}) \text{ s.t. } \bar{v} > 1 \text{ and } (q_0, q_1) \in [0, 1]^2 \text{ with } q_1 \leq q_0 \leq 1, \right. \\ \left. 1 - \alpha + \frac{1}{\Gamma_\alpha^{-1}(q_0)} \geq \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)} \text{ and } q_1/q_0 \leq \Gamma_\alpha(1) \right\}.$$

In this case, we can show as in step 2 of Proposition 2.5, the optimal price is $r_F = 1$ and the optimal revenue is given by $\text{opt}(F) = q_0$.

Let us fix $\gamma > 0$. The goal here is to upper bound the performance of the pricing strategy δ_γ , i.e. $R(\delta_\gamma, F)$. To do so, there are two cases either $\gamma \leq M$ or $\gamma > M$.

Case: $\gamma \leq M$. Then the performance of the mechanism δ_γ can be rewritten as follows,

$$R(\delta_\gamma, F) = \frac{1}{q_0} \mathbb{E}_F [\gamma v \bar{F}(\gamma v)] \leq \frac{1}{q_0} \mathbb{E}_F [\pi^M(\gamma) v \bar{F}((\pi^M(\gamma) - \varepsilon) v)],$$

where the last inequality is a direct implication of the fact that $\gamma \leq \pi^M(\gamma)$.

Case: $\gamma > M$. In this case, we have,

$$R(\delta_\gamma, F) = \frac{1}{q_0} \int_0^\infty \gamma v \bar{F}(\gamma v) f(v) dv = \frac{1}{q_0} \int_0^{1/M} \gamma v \bar{F}(\gamma v) f(v) dv + \frac{1}{q_0} \int_{1/M}^\infty \gamma v \bar{F}(\gamma v) f(v) dv.$$

Let us now bound each term in the expression above. Let us start by the first one. We have $\gamma v \bar{F}(\gamma v) \leq \text{opt}(F) = q_0$ hence we conclude that

$$\frac{1}{q_0} \int_0^{1/M} \gamma v \bar{F}(\gamma v) f(v) dv \leq (1 - \bar{F}(1/M)).$$

Now, let us bound the second term. For any $v \geq 1/M$, we have $M v \geq 1 = r_F$ and since $\gamma \geq M$, then $\gamma v \geq M v \geq r_F$. By the monotonicity of the revenue curve we have,

$$\frac{1}{q_0} \int_{1/M}^\infty \gamma v \bar{F}(\gamma v) f(v) dv \leq \frac{1}{q_0} \int_{1/M}^\infty M v \bar{F}(M v) f(v) dv \leq R(\delta_M, F)$$

Hence we conclude that for any $\gamma > M$, we have

$$R(\delta_\gamma, F) \leq (1 - \bar{F}(1/M)) + R(\delta_M, F).$$

Based on the two cases, we get that

$$\sup_{\gamma > 0} R(\delta_\gamma, F) \leq \max \left[\max_{\gamma \in \mathcal{G}_\varepsilon^M} \left(\frac{1}{q_0} \mathbb{E}_F [\gamma v \bar{F}((\gamma - \varepsilon) v)] \right), (1 - \bar{F}(1/M)) + R(\delta_M, F) \right].$$

Since F is in $\bar{\mathcal{F}}_\alpha^W$, by the first step of the proof in particular Lemma B.6-1 we get that

$$\begin{aligned} \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha) \leq \mathcal{R}(\mathcal{M}, \bar{\mathcal{F}}_\alpha^W) &= \sup_{\psi \in \mathcal{M}} \inf_{G \in \bar{\mathcal{F}}_\alpha^W} \mathbb{E}_\psi R(\delta_\gamma, G) \leq \sup_{\psi \in \mathcal{M}} \mathbb{E}_\psi R(\delta_\gamma, F) \\ &\leq \sup_{\gamma \geq 0} R(\delta_\gamma, F), \end{aligned}$$

hence, we get the result. □

Proof of Lemma B.6-1. The proof is organized around multiple steps. In a first step, we show through Lemma B.6-2 that we can approach the family $\bar{\mathcal{F}}_\alpha^W$ by a sequence in \mathcal{F}_α . In second step, we show that we can rate the limit of the performance as the performance of the limit. Then we conclude the result in the last step.

Step 1. We first establish that any element \mathcal{F}_α^W can be approached by a sequence of elements of \mathcal{F}_α .

Lemma B.6-2. Fix $0 \leq \alpha \leq 1$ and F_α^W in \mathcal{F}_α^W . If

$$(1 - \alpha) + \frac{1}{\Gamma_\alpha^{-1}(q_0)} \geq \frac{1}{\Gamma_\alpha^{-1}(q_1/q_0)}, \quad (\text{B.6-5})$$

then there exists a sequence $\{F_n : n \geq 1\}$ in \mathcal{F}_α that convergences weakly to F_α^W , such that there exists $M > v_1$ such that for all n , $F_n(M) = 1$.

Step 2. Fix $\gamma > 0$ We next establish that any mechanism δ_γ in \mathcal{M} , the worst-case performance of this mechanism against \mathcal{F}_α is upper bounded by the performance of an alternate mechanism $\tilde{m} \in \mathcal{M}$ against \mathcal{F}_α^W .

For any distribution F in $\mathcal{F} \subset \mathcal{F}_\alpha$, we have

$$R(\delta_\gamma, F) = \frac{\mathbb{E}_F [\gamma s \mathbf{1}\{v \geq \gamma s\}]}{\text{opt}(F)}.$$

Fix an element $F \in \mathcal{F}_\alpha^W$. By assumption, there exists a sequence F_n from elements in \mathcal{F}_α such that F_n weakly converges to F .

Step 2a). We first establish that

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq \text{opt}(F).$$

Let $a = \bar{v}_F < \infty$. Note that $\text{opt}(F_n) = \max_{v \geq 0} v \bar{F}_n(v)$. In particular, we have for any $y < a$,

$$\text{opt}(F_n) \geq y \bar{F}_n(y).$$

By taking the lim inf we get that for any $y < a$,

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq y \bar{F}(y).$$

by taking the limit as $y \uparrow a$, we conclude that that for any $v \leq a$

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq v \bar{F}(v),$$

hence, we conclude that

$$\liminf_{n \uparrow \infty} \text{opt}(F_n) \geq \text{opt}(F) = \max_{v \geq 0} v \bar{F}(v),$$

Step 2b). Next, we derive an asymptotic upper bound on the ratio $R(\delta_\gamma, F_n)$.

Note that (Allouah and Besbes, 2018, Lemma 2) implies that for any $\zeta \geq 0$,

$$\lim_{n \uparrow \infty} \mathbb{E}_{F_n} [v_2 \mathbf{1}\{v_1 > \zeta v_2\}] \leq \begin{cases} \mathbb{E}_F [v_2 \mathbf{1}\{v_1 > \zeta v_2\}], & \text{if } \zeta \neq 1, \\ \mathbb{E}_F [\min(v_1, v_2)] & \text{if } \zeta = 1. \end{cases} \quad (\text{B.6-6})$$

Using Step 1 in conjunction with (B.6-6), we have

$$\begin{aligned} \limsup_{n \uparrow \infty} R(\delta_\gamma, F_n) &= \frac{1}{\liminf_{n \uparrow \infty} \text{opt}(F_n)} \lim_{n \uparrow \infty} \mathbb{E}_{F_n} [\gamma s \mathbf{1}\{v > \gamma s\}] \\ &\leq \frac{1}{\text{opt}(F)} [\mathbf{1}\{\gamma = 1\} \mathbb{E}_F [\min\{s, v\}] + \mathbf{1}\{\gamma \neq 1\} \mathbb{E}_F [\gamma s \mathbf{1}\{v > \gamma s\}]]. \end{aligned}$$

Noting that

$$\mathbb{E}_F [\min\{s, v\}] = \mathbb{E}_F \left[s \mathbb{1}\{v > s\} + \frac{1}{2} s \mathbb{1}\{s = v\} \right],$$

we deduce that

$$\begin{aligned} \limsup_{n \uparrow \infty} R(\delta_\gamma, F_n) &\leq \frac{1}{\text{opt}(F)} \mathbb{E}_F \left[\gamma s \mathbb{1}\{v > \gamma s\} + \frac{\mathbb{1}\{\gamma = 1\}}{2} s \mathbb{1}\{s = v\} \right] \\ &\leq \frac{1}{\text{opt}(F)} \mathbb{E}_F [\gamma s \mathbb{1}\{v \geq \gamma s\}] \\ &= R(\delta_\gamma, F). \end{aligned}$$

Hence, we get that for all $\gamma > 0$

$$\limsup_{n \uparrow \infty} R(\delta_\gamma, F_n) \leq R(\delta_\gamma, F).$$

Using Reverse Fatou's Lemma (since $R(\delta_\gamma, F_n) \leq 1$) in conjunction with Eq.(2.4), we get that for all ψ in \mathcal{M} that

$$\limsup_{n \uparrow \infty} R(\psi, F_n) \leq R(\psi, F).$$

By taking the infimum over distribution and supremum over mechanisms, we conclude the proof. \square

Proof of Lemma B.6-2. Let us fix q_0, q_1 and $\bar{v} < \infty$ such that $\bar{F}_\alpha^W = F(\cdot | q_0, q_1, \bar{v})$ is in $\bar{\mathcal{F}}_\alpha^W$. Let us denote $\beta_0 = \Gamma_\alpha^{-1}(q_0)$ and $\beta_1 = \Gamma_\alpha^{-1}(q_1/q_0)$.

The proof will be constructive in that we will explicitly exhibit a sequence F_n that satisfies the properties in the result.

Step 1. We first construct the sequence and characterize its weak limit.

a) Let $n \geq 2$, and define for $x \geq 1$,

$$g_n(x) = (1 - \alpha)(x - 1) + \frac{1}{\beta_1} - \frac{1}{2\beta_1} \left(\frac{x - 1}{\bar{v} - 1} \right)^n.$$

Note that there exists a unique $x_n \geq 0$ such that $g_n(x_n) = 0$. Indeed, g_n is differentiable with derivative given by $g'_n(x) = 1 - \alpha - n(x - 1)^{n-1}/2\beta_1(\bar{v} - 1)^n$, hence g_n is strictly increasing on $[1, 1 + (2(1 - \alpha)\beta_1(\bar{v} - 1)^n/n)^{1/(n-1)}]$ and strictly

decreasing on $(1 + (2(1 - \alpha)\beta_1(\bar{v} - 1)^n/n)^{1/(n-1)}, +\infty)$. Since $g_n(1) = 1/\beta_1 > 0$ and $\lim_{x \uparrow \infty} g_n(x) = -\infty$ there exists a unique x_n in $[1, \infty)$ s.t. $g_n(x_n) = 0$. Furthermore, noting that $g_n(\bar{v}) \geq 1/2\beta_1 > 0$, we have that

$$x_n > \bar{v}, \quad n \geq 2.$$

Define the sequence of cumulative distribution functions F_n

$$\bar{F}_n(x) = \begin{cases} \Gamma_\alpha(\beta_0 x), & \text{if } x < 1, \\ q_0 \exp\left(-\int_0^x \frac{1}{g_n(t)} dt\right), & \text{if } 1 < x < x_n, \\ 0 & \text{if } x \geq x_n. \end{cases} \quad (\text{B.6-7})$$

Step 2. We next establish that F_n belongs to \mathcal{F}_α and that the sequence converges weakly to \bar{F}_α^W .

We first show that the sequence x_n is decreasing and is lower bounded by \bar{v} . The fact that $x_n > \bar{v}$ implies that

$$\begin{aligned} g_{n+1}(x_n) &= (1 - \alpha)(x_n - 1) + \frac{1}{\beta_1} - \frac{1}{2\beta_1} \left(\frac{x_n - 1}{\bar{v} - 1}\right)^{n+1} \\ &< (1 - \alpha)(x_n - 1) + \frac{1}{\beta_1} - \frac{1}{2\beta_1} \left(\frac{x_n - 1}{\bar{v} - 1}\right)^n \\ &= g_n(x_n) = 0. \end{aligned}$$

In turn, by definition of x_{n+1} , we get that

$$x_n < x_{n+1}.$$

Hence for all $n \geq 2$, $x_n < x_2$. Setting $M = x_2$, we have $M > \bar{v}$ and $F_n(M) = 1$ for all $n \geq 2$.

Since x_n is decreasing and lower bounded by \bar{v} , it necessarily converges to some limit $l \geq \bar{v}$. If $l > \bar{v}$ then for n sufficiently large, we would have $x_n \geq (1/2)(l + \bar{v})$ implying that $g_n((1/2)(l + \bar{v})) \geq 0$. However $\lim_{n \uparrow \infty} g_n((1/2)(l + \bar{v})) = -\infty$, which is a contradiction. We conclude that necessarily $\lim_{n \uparrow \infty} x_n = \bar{v}$.

Note also that for $x \geq 1$, g_n is a polynomial with root x_n , and no root in $[1, x_n]$. Since $g'_n(x_n) \neq 0$, then necessarily the multiplicity of x_n is one, so we can find a polynomial function Q_n such that $g_n(x) = (x_n - x)Q(x)$ and for all $x \in [1, x_n]$ we have $Q_n(x) > 0$. In turn, by the Weierstrass extreme value theorem, we have $A_n := \inf_{x \in [1, x_n]} Q_n(x) \in (0, \infty)$ and $B_n := \sup_{x \in [1, x_n]} Q_n(x) \in (0, \infty)$. and and we can find $B_n, A_n > 0$ such that for all $x \in [0, x_n]$, we have $A_n \leq Q_n(x) \leq B_n$.

For $x \in [0, x_n)$, we have

$$\frac{1}{A_n} \ln \left(\frac{x_n - x}{x_n} \right) \leq - \int_0^x \frac{1}{g_n(t)} dt \leq \frac{1}{B_n} \ln \left(\frac{x_n - x}{x_n} \right),$$

and we deduce that

$$\lim_{x \uparrow x_n} \exp \left(- \int_0^x \frac{1}{g_n(t)} dt \right) = 0. \quad (\text{B.6-8})$$

We deduce that F_n defined in (B.6-7) has no atoms. Furthermore, its $(1 - \alpha)$ -virtual value function is given by

$$\phi_n^{1-\alpha}(x) = \begin{cases} -1/\beta_0, & \text{if } x < 1, \\ 1 - \alpha - 1/\beta_1, & \text{if } 1 < x < x_n. \end{cases}$$

Since \bar{F}_α^W is in $\bar{\mathcal{F}}_\alpha^W$ then $1 - \alpha - 1/\beta_1 \geq -1/\beta_0$ is clearly non-decreasing on $[0, x_n)$. Hence, F_n belongs to \mathcal{F}_α .

Step 3. Let us establish that \bar{F}_n converges weakly to \bar{F}_α^W . The points of continuity of \bar{F}_α^W are $\mathbb{R}^+ \setminus \{\bar{v}\}$. Fix $1 \leq x < \bar{v}$, we have $\lim_{n \uparrow \infty} \frac{1}{2\beta_1} \left(\frac{x-1}{\bar{v}-1} \right)^n = 0$. Hence for all $x < \bar{v}$, using the Dominated Convergence Theorem, we have $\lim_{n \uparrow \infty} \bar{F}_n(x) = \bar{F}_\alpha^W(x)$. Fix $x > \bar{v}$. Since x_n converges to \bar{v} , there exists N such that for all $n \geq N$, we have $x > x_n$, hence for all $n \geq N$, $\bar{F}_n(x) = \bar{F}_\alpha^W(x) = 1$. We conclude that \bar{F}_n weakly converges to \bar{F}_α^W .

□

B.7 Proofs of Auxiliary Results

Proof of Proposition B.1-1. We show each point separately.

i). First note that the results of Fu et al. (2015) imply that $\mathcal{R}(\mathcal{M}, \mathcal{F}_0) > 1/2$. We next establish that $\mathcal{R}(\mathcal{M}^{\leq 1}, \mathcal{F}_0) \leq 1/2$.

Let us consider the following distribution

$$\bar{F}(v) = \begin{cases} 1/(1+v), & \text{if } 0 \leq v \leq 1, \\ 1, & \text{if } v > 1. \end{cases}$$

Note that $\text{opt}(F) = 1/2$. Furthermore, remark that $F(\cdot) = F(\cdot|q_0, q_1, \bar{v})$, belongs to the family introduced in (2.9), with $q_0 = 1/2$, $q_1 = 0$ and $\bar{v} = 1$. In turn, by applying Lemma B.6-1 for any ψ in $\mathcal{M}^{\leq 1}$, we have

$$\begin{aligned} \inf_{\tilde{F} \in \mathcal{F}_0} R(\psi, \tilde{F}) &\leq R(\psi, F) \\ &= 2 \int_0^1 \int_0^1 \gamma s \bar{F}(\gamma s) dF(s) d\psi(\gamma) \\ &= 2 \int_0^1 \int_0^1 \frac{\gamma s}{1 + \gamma s} dF(s) d\psi(\gamma) \\ &\leq 2 \mathbb{E}_F \left[\frac{s}{1 + s} \right] \\ &= \frac{1}{2}, \end{aligned}$$

where the last inequality follows from the monotonicity of $v/(1+v)$. Hence against this specific distribution F , post the sample or the identity dominates weakly any level of deflation. This concludes the proof of *i*).

ii). Let us now move to the second point. For that, let F be a distribution in $\tilde{\mathcal{F}}_1$, which is a subset of the MHR class of distribution \mathcal{F}_1 , and \mathcal{S} its support and we let $\underline{s} = \inf\{\mathcal{S}\}$ and $\bar{s} = \sup\{\mathcal{S}\}$.

Let us fix a pricing mechanism $\Psi(p|s)$ in \mathcal{M} . Then there exists $\psi(\cdot)$ in \mathcal{D} , such that $\Psi(p|s) = \psi(p/s)$. According to (2.3), the revenue generated by such a mechanism is

given by

$$\int_0^\infty \left(\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) \right) d\psi(\gamma).$$

Define for all $\gamma \geq 0$,

$$g(\gamma) := \int_0^\infty \gamma v \bar{F}(\gamma v) f(v) dv.$$

We will establish that g is differentiable and $g'(\gamma) \leq 0$, for all $\gamma \geq 1$. In other words, any mechanism that puts positive mass on inflation levels $\gamma > 1$ is dominated by an alternative mechanism that transfers this mass to $\gamma = 1$. We have by assumption that $\bar{s} := \sup\{\mathcal{S}\} < \infty$. Defining

$$h(\gamma, v) = \gamma v \bar{F}(\gamma v) f(v),$$

we get that

$$g(\gamma) = \int_{\underline{s}}^{\bar{s}/\gamma} h(\gamma, v) dv.$$

Note that, for $\gamma \geq 1$, $h(\gamma, v)$ is differentiable with respect to γ for all $v \in [\underline{s}, \bar{s}/\gamma]$ and its partial derivative with respect to γ is continuous with respect to both arguments, given by

$$\frac{\partial}{\partial \gamma} h(\gamma, v) = (v \bar{F}(\gamma v) - \gamma v^2 f(\gamma v)) f(v).$$

In turn, using Leibniz integral rule, the function g is differentiable on $[1, \infty)$ and its derivative is given

$$g'(\gamma) = \int_{\underline{s}}^{\bar{s}/\gamma} \frac{\partial}{\partial \gamma} h(\gamma, v) f(v) dv - \frac{1}{\gamma^2} \bar{s} \bar{F}(\bar{s}) f(\bar{s}/\gamma).$$

Hence, we get that for all $\gamma \geq 1$

$$\begin{aligned} g'(\gamma) &\leq \int_0^\infty (v \bar{F}(\gamma v) - \gamma v^2 f(\gamma v)) f(v) dv \\ &= \int_0^\infty f(v) v (\bar{F}(\gamma v) - \gamma v f(\gamma v)) dv \\ &= \int_{\underline{s}}^{\bar{s}/\gamma} \frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} v \bar{F}(\gamma v) (v \bar{F}(\gamma v))' dv. \end{aligned}$$

Note that the function f/\bar{F} is monotone non-decreasing on $(0, \infty)$ by the assumption that $F \in \mathcal{F}_1$ and hence is differentiable almost everywhere. In turn, integration by parts leads to

$$\begin{aligned} g'(\gamma) &\leq \frac{1}{2} \left[\frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} (v \bar{F}(\gamma v))^2 \right]_{\underline{s}}^{\bar{s}/\gamma} - \frac{1}{2} \int_{\underline{s}}^{\bar{s}/\gamma} \left(\frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} \right)' (v \bar{F}(\gamma v))^2 dv \\ &= -\frac{1}{2} \int_{\underline{s}}^{\bar{s}/\gamma} \left(\frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} \right)' (v \bar{F}(\gamma v))^2 dv. \end{aligned} \quad (\text{B.7-1})$$

We next analyze $\bar{F}(v)/\bar{F}(\gamma v)$. For that let us introduce the following function, for $z, t \geq 0$,

$$\phi(t, z) := \frac{\bar{F}(z+t)}{\bar{F}(t)}.$$

It is clear that $\phi(t, \cdot)$ is differentiable almost everywhere and $\frac{\partial \phi}{\partial z} \leq 0$. Furthermore, since $F \in \mathcal{F}_1$, $\phi(\cdot, z)$ is monotone non-increasing (see, e.g., Ross (1996)) and almost everywhere we have $\frac{\partial \phi}{\partial t} \leq 0$. Noting that

$$\bar{F}(\gamma v)/\bar{F}(v) = \phi(v, (\gamma-1)v),$$

we have have, almost everywhere on $[0, \bar{s}/\gamma]$,

$$\frac{\partial}{\partial v} \left(\frac{\bar{F}(\gamma v)}{\bar{F}(v)} \right) = \frac{\partial \phi}{\partial t}(v, (\gamma-1)v) + (\gamma-1) \frac{\partial \phi}{\partial z}(v, (\gamma-1)v).$$

The latter is non-positive for all $\gamma \geq 1$ given that the partial derivatives are non-positive. So $\bar{F}(\gamma v)/\bar{F}(v)$ is non-increasing in v . Hence, $\bar{F}(v)/\bar{F}(\gamma v)$ is non-decreasing in v .

Thus, the function $(f(v)/\bar{F}(v)) (\bar{F}(v)/\bar{F}(\gamma v))$ is non-decreasing and

$$\left(\frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} \right)' \geq 0.$$

Since

$$g'(\gamma) \leq - \int_0^\infty \left(\frac{f(v)}{\bar{F}(v)} \frac{\bar{F}(v)}{\bar{F}(\gamma v)} \right)' (v \bar{F}(\gamma v))^2 dv.$$

Using (B.7-1), we conclude that $g'(\gamma) \leq 0$ for all $\gamma \geq 1$. In turn, we conclude that $g(1) \geq g(\gamma)$ for all $\gamma \geq 1$. In particular, we get that for all F in $\tilde{\mathcal{F}}_1$,

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) \right) d\psi(\gamma) \\ & \leq \int_0^1 \left(\int_0^\infty \gamma s \bar{F}(\gamma s) dF(s) \right) d\psi(\gamma) + \int_1^\infty \left(\int_0^\infty s \bar{F}(s) dF(s) \right) d\psi(\gamma). \end{aligned}$$

In other words, that for all F in $\tilde{\mathcal{F}}_1$,

$$R(\psi, F) \leq R(\tilde{\psi}, F),$$

where for all $\gamma \geq 0$,

$$\tilde{\psi}(\gamma) = \begin{cases} \psi(\gamma), & \text{if } \gamma < 1, \\ 1, & \text{if } \gamma \geq 1, \end{cases}$$

Since $\tilde{\Psi}(p|s) = \tilde{\psi}(p/s)$ is in $\mathcal{M}^{\leq 1}$, we conclude the result. \square

Lemma B.7-1. Fix a distribution F in \mathcal{F}_α and $w < w'$. Suppose that $w > r_F$. Then the function $v \mapsto v \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v-w}{w'-w} \right)$ is non-increasing on $[w, \infty)$.

Proof of Lemma B.7-1. Fix a distribution F in \mathcal{F}_α , let us define the following distribution through its cdf, for all $v \in [w, \infty)$

$$\bar{G}(v) = \begin{cases} 1, & \text{if } v \in [0, w), \\ \Gamma_\alpha \left(\Gamma_\alpha^{-1} \left(\frac{q_{w'}}{q_w} \right) \frac{v-w}{w'-w} \right), & \text{if } v \geq w. \end{cases}$$

Note that \bar{G} belongs to \mathcal{F}_α , hence its revenue curve is unimodal. We also note that

$$w\bar{F}(w) = q_w w \bar{G}(w)$$

and moreover by Lemma 2.2, we have that for all v in $[w, w']$,

$$v\bar{F}(v) \geq q_w v \bar{G}(v).$$

Using the fact that the revenue curve of F is non-increasing on $[w, \infty)$ since $w \geq r_F$, we have that for all v in $[w, w']$,

$$q_w v \bar{G}(v) \leq v \bar{F}(v) \leq w \bar{F}(w) = q_w w \bar{G}(w).$$

On the other side, since the revenue curve of \bar{G} is unimodal, then necessarily its revenue curve is non-increasing on $[w, \infty)$. \square

Lemma B.7-2. *For η in $[0, 1]^{12}$. The function $\mathcal{A}_\alpha(\eta)$ defined in Eq. (2.7) has the following monotonicity properties*

- $\mathcal{A}_\alpha(\eta)$ is non-increasing in η_1, η_5 and η_8 .
- $\mathcal{A}_\alpha(\eta)$ is non-decreasing in η_3, η_6 and η_{10} .

Proof of Lemma B.7-2. We verify each property separately. We analyze only our case of interest which is η in $[0, 1]^{12}$.

Since the integrand is non-negative, the function $\mathcal{A}_\alpha(\eta)$ is non-increasing in η_1 .

Note now that the function $\Gamma_\alpha(\cdot)$ is continuous and non-decreasing. A direct implication is that $\mathcal{A}_\alpha(\eta)$ is non-decreasing in η_6 .

Also given that the function $\Gamma_\alpha(\cdot)$ is non-decreasing, the function $u_\alpha(\beta_0, \beta_1, \beta_2, \beta_3 q)$ (defined in Eq. (2.7)) is non-decreasing in β_1 and non-increasing in β_3 . Hence $\mathcal{A}_\alpha(\eta)$ is non-decreasing in η_3 and η_{10} and non-increasing in η_5 and η_8 . This completes the proof. \square

Lemma B.7-3. *The following inequalities hold*

$$\mathcal{A}_\alpha \left(\beta_{\tilde{\gamma}}^L \left(\frac{q_j}{q_{j-1}}, \frac{q_{j+1-2\tilde{\alpha}}}{q_{j-2\tilde{\alpha}}} \right) \right) \geq \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right),$$

and

$$\mathcal{A}_\alpha \left(\beta_R \left(\frac{q_{j+1}}{q_j}, \frac{q_j}{q_{j-1}}, \frac{q_j}{q_{j-1}} \right) \right) \geq \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R \left(\pi \left(\frac{q_{j+1}}{q_j} \right), \pi \left(\frac{q_j}{q_{j-1}} \right) \right) \right).$$

Proof of Lemma B.7-3. We first note that from the definition of the projector operator, we have for i in $\{j-2, j-1, j\}$,

$$\pi\left(\frac{q_{i+1}}{q_i}\right) - \varepsilon \leq \frac{q_{i+1}}{q_i} \leq \pi\left(\frac{q_{i+1}}{q_i}\right).$$

For the left term, we have

$$\begin{aligned} & \mathcal{A}_\alpha\left(\beta_{\hat{\gamma}}^L\left(\frac{q_j}{q_{j-1}}, \frac{q_{j+1-2\bar{\alpha}}}{q_{j-2\bar{\alpha}}}\right)\right) \\ &= \mathcal{A}_\alpha\left(\frac{q_j}{q_{j-1}}, 1, \frac{q_j}{q_{j-1}}, \gamma, 1, \frac{q_{j+1-2\bar{\alpha}}}{q_{j-2\bar{\alpha}}}, 1, \frac{q_j}{q_{j-1}}, \gamma, 1, \gamma, \gamma\right) \\ &\geq \mathcal{A}_\alpha\left(\pi\left(\frac{q_j}{q_{j-1}}\right), 1, \pi\left(\frac{q_j}{q_{j-1}}\right) - \varepsilon, \gamma, 1, \pi\left(\frac{q_{j+1-2\bar{\alpha}}}{q_{j-2\bar{\alpha}}}\right) - \varepsilon, 1, \pi\left(\frac{q_j}{q_{j-1}}\right), \gamma, 1, \gamma, \gamma\right) \\ &= \mathcal{A}_\alpha\left(\hat{\beta}_{\gamma, \varepsilon}^L\left(\pi\left(\frac{q_{j+1+\bar{\alpha}}}{q_{j+\bar{\alpha}}}\right), \pi\left(\frac{q_{j+\bar{\alpha}}}{q_{j-1+\bar{\alpha}}}\right)\right)\right), \end{aligned}$$

where the last inequality follows from the monotonicity properties established in Lemma B.7-2.

Similarly, we have

$$\begin{aligned} & \mathcal{A}_\alpha\left(\beta_R\left(\frac{q_{j+1}}{q_j}, \frac{q_j}{q_{j-1}}, \frac{q_j}{q_{j-1}}\right)\right) \\ &= \mathcal{A}_\alpha\left(\frac{q_{j+1}}{q_j}, \gamma, \frac{q_j}{q_{j-1}}, \gamma, \frac{q_j}{q_{j-1}}, \frac{q_j}{q_{j-1}}, \gamma, \frac{q_j}{q_{j-1}}, \gamma, \frac{q_j}{q_{j-1}}, 1, \gamma\right) \\ &\geq \mathcal{A}_\alpha\left(\hat{\beta}_{\gamma, \varepsilon}^R\left(\pi\left(\frac{q_{j+1}}{q_j}\right), \pi\left(\frac{q_j}{q_{j-1}}\right)\right)\right), \end{aligned}$$

where the last inequality follows from the monotonicity properties established in Lemma B.7-2. \square

Lemma B.7-4. *i.) If $j \leq 1$ and $\gamma < 1$, then for any $(\hat{q}_{j-1}, \pi(q_j/q_{j-1}))$ is in*

$$\mathcal{B}_{\alpha, \gamma, \varepsilon}^L(\hat{q}_j, \pi(q_{j+1}/q_j)).$$

ii.) If $j \geq 1$ and $\gamma < 1$, then $\pi\left(\frac{q_{j+1}}{q_j}\right)$ belongs to $\mathcal{B}_{\alpha, \gamma, \varepsilon}^H\left(\pi\left(\frac{q_j}{q_{j-1}}\right)\right)$.

Proof of Lemma B.7-4. *i.)* First note that by the unimodality of the revenue curve and the optimality of $r_F = r_0$, we have $r_{j-1} q_{j-1} \leq r_j q_j$ for $j \leq 0$, since

$r_{j-1} \leq r_j \leq r_F$. Hence, $q_{j-1} \leq q_j/\gamma$, hence $q_j/q_{j-1} \geq \gamma$. By definition $\pi(\cdot)$, we get $\pi(q_j/q_{j-1}) \geq \gamma$.

For the left side of the second point, by Lemma 2.2 and picking consecutively $(w, w') = (0, r_j)$ and $(w, w') = (0, r_{j+1})$ we get that

$$q_{j-1} \geq \max(\Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(q_j)), \Gamma_\alpha(\gamma^2 \Gamma_\alpha^{-1}(q_{j+1}))),$$

So by definition of $\pi(\cdot)$ and monotonicity of Γ_α , we get

$$\hat{q}_{j-1} \geq \max\left(\Gamma_\alpha(\gamma \Gamma_\alpha^{-1}(\hat{q}_j - \varepsilon)), \Gamma_\alpha\left(\gamma^2 \Gamma_\alpha^{-1}(\hat{q}_j - \varepsilon) \left(\pi\left(\frac{q_{j+1}}{q_j}\right) - \varepsilon\right)\right)\right).$$

The last condition is direct consequence of the fact that $q_{j+1} = q_j \frac{q_{j+1}}{q_j}$ and the monotonicity of $\pi(\cdot)$.

By using Lemma 2.2, with $w = r_{j-1}$, $w' = r_{j+1}$ and $u = r_j$ we get that

$$q_j \geq q_{j-1} \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_{j+1}/q_{j-1}) \frac{r_j - r_{j-1}}{r_{j+1} - r_{j-1}}\right),$$

hence we get

$$\frac{q_j}{q_{j-1}} \geq \Gamma_\alpha\left(\frac{\gamma}{1+\gamma} \Gamma_\alpha^{-1}((q_{j+1}/q_j)(q_j/q_{j-1}))\right),$$

by projecting on the grid and using the monotonicity of Γ_α , we get that

$$\pi\left(\frac{q_j}{q_{j-1}}\right) \geq \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{\gamma}{1+\gamma} (\pi(q_{j+1}/q_j) - \varepsilon) (\pi(q_j/q_{j-1}) - \varepsilon)\right)\right),$$

This concludes the first case when $j \leq 0$.

Let us now move to the second case which is $j \geq 1$. In this case using the monotonicity of the revenue curve, we have $r_{j+1} q_{j+1} \leq r_j q_j \leq r_{j-1} q_{j-1}$ since $r_{j+1} \geq r_j \geq r_{j-1} \geq r_F$, for $j \geq 1$. So, we get that, $q_{j+1}/q_j \leq \gamma$ and $q_{j+1}/q_j \leq \gamma^2$, hence we conclude that

$$\pi(q_{j+1}/q_j) \leq \gamma + \varepsilon \text{ and } (\pi(q_{j+1}/q_j) - \varepsilon) (\pi(q_j/q_{j-1}) - \varepsilon) \leq \gamma^2,$$

hence we get the result. □

Lemma B.7-5. Fix $\alpha \in [0, 1]$, $\gamma \in (0, 1)$ and $\varepsilon \geq 0$. For any ρ^- and ρ^+ in \mathcal{G}_ε , we have

$$\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L (\rho^+, \rho^-) \right) \leq 1, \quad \text{and} \quad \mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R (\rho^+, \rho^-) \right) \leq \frac{1}{\gamma}.$$

Proof of Lemma B.7-5. We will show each point separately. By using the explicit expression of \mathcal{A} and the fact that cdf Γ_α is at most 1, we get that

$$\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L (\rho^+, \rho^-) \right) \leq \int_{\rho^+}^1 u_\alpha(0^+, \rho^+ - \varepsilon, \gamma, t) dt.$$

Moreover since $u_\alpha(0^+, \rho^+ - \varepsilon, \gamma, t) \leq 1$, for any $t \geq \rho^+$ then we conclude that for any $\rho^+ \in \mathcal{G}_\varepsilon$,

$$\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^L (\rho^+, \rho^-) \right) \leq 1.$$

By using the expression of \mathcal{A} and $\hat{\beta}_{\gamma, \varepsilon}^R$ and the definition of the minimum also that that Γ_α is a cdf, we have

$$\mathcal{A}_\alpha \left(\hat{\beta}_{\gamma, \varepsilon}^R (\rho^+, \rho^-) \right) \leq \int_{\rho^-}^1 \frac{1}{\gamma} dq \leq \frac{1}{\gamma}.$$

□

Appendix for Chapter 3

C.1 Proofs of Section 3.3

Proof of Lemma 3.1

The distribution of the value of the virtual buyer corresponding to an intermediary is the distribution of the maximum of the values of the buyers that the intermediary was representing initially, i.e., $(G_\alpha)^K$. We first ensure that the distribution G_α has an increasing failure rate (IFR) and that the distribution of the maximum has also an increasing failure rate. Then we characterize the optimal reserve price r_{co} .

It is clear that G_α admits a density on $\mathcal{S} \setminus \{0\}$. Let $x \geq 0$, for all $t > 0$ we have

$$\frac{\bar{G}_\alpha(x+t)}{\bar{G}_\alpha(t)} = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

Since F has an increasing failure rate, then $\bar{F}(x+t)/\bar{F}(t)$ is decreasing in t , following (Barlow and Proschan, 1975, Definition 1.1, Section 3). We conclude that $\bar{G}_\alpha(x+t)/\bar{G}_\alpha(t)$ is also decreasing in t implying that G_α is IFR.

Lemma C.1-1 (Preservation of IFR property by maximum operator). *Consider a set of n i.i.d. random variables drawn from a distribution with increasing failure rate. Then the distribution of the maximum of these n variables has also an increasing failure rate.*

The proof of the latter result is deferred to Appendix C.4. In turn, the virtual

value function of the maximum value of an intermediary

$$v - \frac{1 - G_\alpha^K(v)}{K\alpha f(v)G_\alpha^{K-1}(v)}$$

is increasing. We deduce that the optimal reserve price r_{co} is uniquely defined as

$$r_{\text{co}} = \frac{1 - G_\alpha^K(r_{\text{co}})}{K\alpha f(v)G_\alpha^{K-1}(r_{\text{co}})}.$$

We next establish that $r_{\text{co}} > r_{\text{in}}$. We have that for any $v \in \mathcal{S}$,

$$\frac{1 - G_\alpha^K(v)}{K\alpha f(v)G_\alpha^{K-1}(v)} = \frac{1 - F(v)}{f(v)} \frac{\sum_{i=0}^{K-1} G_\alpha^i(v)}{KG_\alpha^{K-1}(v)},$$

and hence,

$$\frac{1 - G_\alpha^K(v)}{K\alpha f(v)G_\alpha^{K-1}(v)} \geq \frac{1 - F(v)}{f(v)},$$

where the last inequality is strict if $F(v) < 1$. Recall that r_{co} and r_{in} are the unique solution of the following equations

$$\begin{aligned} r_{\text{co}} &= \frac{1 - G_\alpha^K(r_{\text{co}})}{K\alpha f(v)G_\alpha^{K-1}(r_{\text{co}})}, \\ r_{\text{in}} &= \frac{1 - F(r_{\text{in}})}{f(r_{\text{in}})}, \end{aligned}$$

Using the fact the distribution F has an increasing failure rate, we deduce that r_{co} is always such that $r_{\text{co}} > r_{\text{in}}$. In particular, the strict inequality is a consequence of the fact we always have $F(r_{\text{in}}) < 1$. (Indeed, it is clear that $r_{\text{in}} > 0$ and $f(r_{\text{in}}) > 0$ and hence $F(r_{\text{in}}) = 1 - r_{\text{in}}f(r_{\text{in}}) < 1$.) This completes the proof. \square

Proof of Corollary 3.1

We first establish that $\Pi_{\text{co}}^* < \Pi_{\text{in}}^*$.

$$\Pi_{\text{co}}^* = \mathbb{E}[\max\{r_{\text{co}}, \mathbf{w}_2\} \mathbb{1}\{\mathbf{w}_1 \geq r_{\text{co}}\}] = \mathbb{E}\left[\max\{r_{\text{co}}, \mathbf{w}_2\} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{co}}\}\right]$$

since $\mathbf{w}_1 = \mathbf{v}_{1:J}^{[1]}$. In turn, we have

$$\begin{aligned}
\Pi_{\text{co}}^* &\stackrel{(a)}{<} \mathbb{E} \left[\max\{r_{\text{co}}, \mathbf{v}_{1:J}^{[2]}\} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{co}}\} \right] \\
&\leq \max_{r \geq 0} \mathbb{E} \left[\max\{r, \mathbf{v}_{1:J}^{[2]}\} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right] \\
&\stackrel{(b)}{=} \mathbb{E} \left[\max\{r_{\text{in}}, \mathbf{v}_{1:J}^{[2]}\} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right] \\
&= \Pi_{\text{in}}^*,
\end{aligned}$$

where (a) follows since $\mathbf{v}_{1:J}^{[2]} \geq \mathbf{w}_2$ and the strict equality follows from the fact that $\mathbf{v}_{1:J}^{[2]} > \mathbf{w}_2 > r_{\text{co}}$ occurs with positive probability. Finally, (b) follows from the fact that r_{in} is the optimal reserve price when the intermediaries are multi-bidding.

Next, we establish that $S_{\text{co}}^* < S_{\text{in}}^*$.

$$S_{\text{co}}^* = \mathbb{E}[\mathbf{w}^{[1]} \mathbb{1}\{\mathbf{w}^{[1]} \geq r_{\text{co}}\}] = \mathbb{E}[\mathbf{v}_{1:J}^{[1]} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{co}}\}] < \mathbb{E}[\mathbf{v}_{1:J}^{[1]} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\}],$$

where the second equality follows from $\mathbf{w}_1 = \mathbf{v}_{1:J}^{[1]}$ and the inequality follows from the fact that $r_{\text{co}} > r_{\text{in}}$. This concludes the proof. \square

Proof of Proposition 3.1

We first characterize $S_{\text{co}}^*, \Pi_{\text{co}}^*, S_{\text{in}}^*, \Pi_{\text{in}}^*$. Second, we derive explicitly the myopic benefit of collusion \mathcal{MBC} , then we characterize $U_{\text{co}}^* - U_{\text{in}}^*$, then we conclude with a derivation for \mathcal{SRI} .

The social welfare when the intermediary is coordinating campaigns is given by

$$\begin{aligned}
S_{\text{co}}^* &= \mathbb{E}[\mathbf{v}_{1:J}^{[1]} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{co}}\}] \\
&= \int_{r_{\text{co}}}^{\infty} x JK \alpha f(x) (G_{\alpha}(x))^{JK-1} dx \\
&= - \left[x [1 - (G_{\alpha}(x))^{JK}] \right]_{r_{\text{co}}}^{\infty} + \int_{r_{\text{co}}}^{\infty} [1 - (G_{\alpha}(x))^{JK}] dx \\
&= r_{\text{co}} [1 - (G_{\alpha}(r_{\text{co}}))^{JK}] + \int_{r_{\text{co}}}^{\infty} [1 - (G_{\alpha}(x))^{JK}] dx.
\end{aligned}$$

Similarly, the social welfare when the intermediary is multi-bidding is given by

$$S_{\text{in}}^* = \mathbb{E}[\mathbf{v}_{1:J}^{[1]} \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\}] = r_{\text{in}} [1 - (G_{\alpha}(r_{\text{in}}))^{JK}] + \int_{r_{\text{in}}}^{\infty} [1 - (G_{\alpha}(x))^{JK}] dx.$$

Next we compute the revenue of the seller.

$$\begin{aligned}
\Pi_{\text{in}}^* &= \mathbb{E} \left[\max(\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right] \\
&= r_{\text{in}} \mathbb{P}(\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}} \geq \mathbf{v}_{1:J}^{[2]}) + \mathbb{E} \left[\mathbf{v}_{1:J}^{[2]} \mathbb{1}\{\mathbf{v}_{1:J}^{[2]} \geq r_{\text{in}}\} \right] \\
&= r_{\text{in}} JK \bar{G}_\alpha(r_{\text{in}}) (G_\alpha(r_{\text{in}}))^{JK-1} + \int_{r_{\text{in}}}^{\infty} x JK (JK-1) (G_\alpha(x))^{JK-2} \bar{G}_\alpha(x) \alpha f(x) dx.
\end{aligned}$$

Through integration by parts applied to the last term we obtain

$$\begin{aligned}
& \int_{r_{\text{in}}}^{\infty} x JK (JK-1) (G_\alpha(x))^{JK-2} \bar{G}_\alpha(x) \alpha f(x) dx \\
&= - \left[x \left(JK (1 - (G_\alpha(x))^{JK-1}) - (JK-1) (1 - (G_\alpha(x))^{JK}) \right) \right]_{r_{\text{in}}}^{\infty} \\
&\quad + \int_{r_{\text{in}}}^{\infty} \left(JK (1 - (G_\alpha(x))^{JK-1}) - (JK-1) (1 - (G_\alpha(x))^{JK}) \right) dx \\
&= r_{\text{in}} \left(JK (1 - (G_\alpha(r_{\text{in}}))^{JK-1}) - (JK-1) (1 - (G_\alpha(r_{\text{in}}))^{JK}) \right) \\
&\quad + \int_{r_{\text{in}}}^{\infty} \left(JK (1 - (G_\alpha(x))^{JK-1}) - (JK-1) (1 - (G_\alpha(x))^{JK}) \right) dx.
\end{aligned}$$

We deduce that

$$\begin{aligned}
& \Pi_{\text{in}}^* \\
&= r_{\text{in}} [1 - (G_\alpha(r_{\text{in}}))^{JK}] + \int_{r_{\text{in}}}^{\infty} \left(JK (1 - (G_\alpha(x))^{JK-1}) - (JK-1) (1 - (G_\alpha(x))^{JK}) \right) dx.
\end{aligned}$$

Let us define $H(x) := (G_\alpha(x))^K$ and $h(x) = K \alpha f(x) (G_\alpha(x))^{K-1}$. Let us compute $\Pi_{\text{co}}(r)$ the seller's revenue at any point r when the intermediary is coordinating campaigns.

$$\begin{aligned}
\Pi_{\text{co}}(r) &= \mathbb{E} \left[\max(\mathbf{w}^{[2]}, r) \mathbb{1}\{\mathbf{w}^{[1]} \geq r\} \right] \\
&= r \mathbb{P}(\mathbf{w}^{[1]} \geq r \geq \mathbf{w}^{[2]}) + \mathbb{E} \left[\mathbf{w}^{[2]} \mathbb{1}\{\mathbf{w}^{[2]} \geq r\} \right] \\
&= r J (1 - H(r)) (H(r))^{J-1} + \int_r^{\infty} x J (J-1) (H(x))^{J-2} (1 - H(x)) h(x) dx.
\end{aligned}$$

Applying a similar reasoning as earlier through integration by parts for the last term leads to

$$\begin{aligned}
& \int_r^\infty xJ(J-1)(H(x))^{J-2}(1-H(x)) \\
&= r \left(J(1-(H(r))^{J-1}) - (J-1)(1-(H(r))^J) \right) \\
& \quad + \int_r^\infty \left(J(1-(H(x))^{J-1}) - (J-1)(1-(H(x))^J) \right) dx,
\end{aligned}$$

So we conclude that

$$\begin{aligned}
& \Pi_{\text{co}}(r) \\
&= r[1-(H(r))^J] + \int_r^\infty \left(J(1-(H(x))^{J-1}) - (J-1)(1-(H(x))^J) \right) dx \\
&= r[1-(G_\alpha(r))^{JK}] + \int_r^\infty \left(J(1-(G_\alpha(x))^{JK-K}) - (J-1)(1-(G_\alpha(x))^{JK}) \right) dx.
\end{aligned}$$

Thus the optimal seller's revenue is given by

$$\Pi_{\text{co}}^* = r_{\text{co}}[1-(G_\alpha(r_{\text{co}}))^{JK}] + \int_{r_{\text{co}}}^\infty \left(J(1-(G_\alpha(x))^{JK-K}) - (J-1)(1-(G_\alpha(x))^{JK}) \right) dx.$$

Given the above, we can derive the myopic benefit.

$$\begin{aligned}
\mathcal{MBC} &= U_{\text{co}}(r_{\text{in}}) - U_{\text{in}}(r_{\text{in}}) \\
&= \mathbb{E}[(\max(r_{\text{in}}, \mathbf{v}_{1:J}^{[2]}) - \max(r_{\text{in}}, \mathbf{w}^{[2]})) \mathbb{1}\{\mathbf{w}^{[1]} \geq r_{\text{in}}\}] \\
&= \Pi_{\text{in}}^* - \Pi_{\text{co}}(r_{\text{in}}) \\
&= \int_{r_{\text{in}}}^\infty \left(JK(1-(G_\alpha(x))^{JK-1}) - (JK-1)(1-(G_\alpha(x))^{JK}) \right) \\
& \quad - \left(J(1-(G_\alpha(x))^{JK-K}) - (J-1)(1-(G_\alpha(x))^{JK}) \right) dx.
\end{aligned}$$

Using the fact that the impact on the buyers is given by $U_{\text{co}}^* - U_{\text{in}}^* = S_{\text{co}}^* - \Pi_{\text{co}}^* -$

$(S_{\text{in}}^* - \Pi_{\text{in}}^*)$, we obtain

$$\begin{aligned}
& U_{\text{co}}^* - U_{\text{in}}^* \\
&= S_{\text{co}}^* - \Pi_{\text{co}}^* - (S_{\text{in}}^* - \Pi_{\text{in}}^*) \\
&= \int_{r_{\text{in}}}^{\infty} \left[JK(1 - (G_{\alpha}(x))^{JK-1}) - (JK - 1)(1 - (G_{\alpha}(x))^{JK}) \right] dx \\
&\quad - \int_{r_{\text{co}}}^{\infty} \left[J(1 - (G_{\alpha}(x))^{JK-K}) - (J - 1)(1 - (G_{\alpha}(x))^{JK}) \right] dx - \int_{r_{\text{in}}}^{r_{\text{co}}} (1 - (G_{\alpha}(x))^{JK}) dx
\end{aligned}$$

So we obtain the seller's reaction

$$-\mathcal{SRI} = \int_{r_{\text{in}}}^{r_{\text{co}}} (1 - (G_{\alpha}(x))^{JK}) - \left[J(1 - (G_{\alpha}(x))^{JK-K}) - (J - 1)(1 - (G_{\alpha}(x))^{JK}) \right] dx.$$

This concludes the proof. \square

C.2 Proofs of Section 3.4

Proof of Proposition 3.2

Fix $\xi < 0$. For $0 \leq q \leq 1$, the corresponding value $z(q)$ corresponding to the quantile q and its derivative are given by:

$$\begin{aligned}
z(q) &= -\frac{\sigma}{\xi}(1 - q^{-\xi}), \\
z'(q) &= -\sigma q^{-\xi-1}.
\end{aligned} \tag{C.2-1}$$

Let us now characterize the optimal reserve prices and their corresponding quantiles. As mentioned earlier r_{in} verifies $r_{\text{in}} = (1 - F_{\xi}(r_{\text{in}}))/f_{\xi}(r_{\text{in}}) = \sigma(1 + \frac{\xi}{\sigma}r_{\text{in}})$. Hence $r_{\text{in}} = \frac{\sigma}{1-\xi}$ and using (C.2-1), we get

$$q_{\text{in}} = (1 - \xi)^{\frac{1}{\xi}}.$$

On the other hand, r_{co} verifies the following $r_{\text{co}} = (1 - F^2(r_{\text{co}}))/(2 f(r_{\text{co}}) F(r_{\text{co}}))$.

Using (C.2-1), we get

$$\frac{\sigma}{\xi}(1 - q_{\text{co}}^{-\xi}) = \frac{q_{\text{co}}(2 + q_{\text{co}})}{2 \left(\frac{1}{\sigma}q_{\text{co}}^{\xi+1}\right) (1 - q_{\text{co}})},$$

so we obtain

$$\frac{1}{\xi}(1 - q_{\text{co}}^\xi) = \frac{2 - q_{\text{co}}}{2(1 - q_{\text{co}})}.$$

Using Proposition 3.1 with $J = 1$ and $K = 2$, we get that

$$\begin{aligned} U_{\text{co}}^* - U_{\text{in}}^* &= \int_{r_{\text{in}}}^{\infty} \overline{F}(x)^2 dx - \int_{r_{\text{in}}}^{r_{\text{co}}} (1 - F^2(x)) dx \\ &\stackrel{\text{(C.2-1)}}{=} \int_0^{q_{\text{in}}} q^2 (\sigma q^{-\xi-1}) dq - \int_{q_{\text{co}}}^{q_{\text{in}}} (1 - (1 - q)^2) (\sigma q^{-\xi-1}) dq \\ &= \sigma \left(\left[\frac{q^{2-\xi}}{2-\xi} \right]_0^{q_{\text{in}}} - 2 \left[\frac{q^{1-\xi}}{1-\xi} \right]_{q_{\text{co}}}^{q_{\text{in}}} + \left[\frac{q^{2-\xi}}{2-\xi} \right]_{q_{\text{co}}}^{q_{\text{in}}} \right) \\ &= \sigma \left[\frac{2}{2-\xi} q_{\text{in}}^{2-\xi} - \frac{2}{1-\xi} q_{\text{in}}^{1-\xi} + \frac{2}{1-\xi} q_{\text{co}}^{1-\xi} - \frac{1}{2-\xi} q_{\text{co}}^{2-\xi} \right]. \end{aligned}$$

This completes the proof. \square

Proof of Theorem 3.1

Here we assume that $\xi \in [-1, 0]$. We will construct an upper bound on $U_{\text{co}}^* - U_{\text{in}}^*$, which we will show is piecewise-concave and decreasing on the intervals $[-1, -0.5]$ and $[-0.5, 0]$. We then establish that this upper bound is negative on those intervals. In the proof, we will sometimes drop the dependence of some quantities on ξ to lighten the notation.

We have

$$\begin{aligned} U_{\text{co}}^* - U_{\text{in}}^* &= \sigma \left[\frac{2}{2-\xi} q_{\text{in}}^{2-\xi} - \frac{1}{2-\xi} q_{\text{co}}^{2-\xi} - \frac{2}{1-\xi} q_{\text{in}}^{1-\xi} + \frac{2}{1-\xi} q_{\text{co}}^{1-\xi} \right] \\ &= \sigma q_{\text{in}}^{1-\xi} \left[\frac{q_{\text{in}}}{2-\xi} \left(2 - \left(\frac{q_{\text{co}}}{q_{\text{in}}} \right)^{2-\xi} \right) - \frac{2}{1-\xi} \left(1 - \left(\frac{q_{\text{co}}}{q_{\text{in}}} \right)^{1-\xi} \right) \right] \\ &\leq \sigma q_{\text{in}}^{1-\xi} \left[\frac{e^{(-\frac{2}{2-\xi} + \gamma(\xi))}}{2-\xi} \left(2 - \left(\frac{q_{\text{co}}}{q_{\text{in}}} \right)^{2-\xi} \right) - \frac{2}{1-\xi} \left(1 - \left(\frac{q_{\text{co}}}{q_{\text{in}}} \right)^{1-\xi} \right) \right]. \end{aligned}$$

In the last inequality, we have used for all $\xi < 0$, $q_{\text{in}} = \exp\{\xi^{-1} \ln(1 - \xi)\} \leq \exp\{-2/(2 - \xi) + \gamma(\xi)\}$, with $\gamma(\xi) := (2/5 - \ln(3/2)) \mathbf{1}_{\{\xi \leq -0.5\}}$. (We note that $\gamma(\xi) \leq 0$.) The inequality follows from the fact that $\ln(1 + x) \geq 2x/(2 + x) - \gamma(-x)$

for $x \geq 0$ and the fact that $-1 \leq \xi < 0$. We introduce the following function $h(\cdot, \cdot)$,

$$h(\xi, x) = \frac{e^{-\frac{2}{2-\xi} + \gamma(\xi)}}{2-\xi} (2 - x^{2-\xi}) - \frac{2}{1-\xi} (1 - x^{1-\xi}) \quad \text{for } -1 \leq \xi \leq 0 \text{ and } 0 \leq x \leq 1.$$

In turn, we have

$$U_{\text{co}}^* - U_{\text{in}}^* \leq \sigma q_{\text{in}}^{1-\xi} h\left(\xi, \frac{q_{\text{co}}(\xi)}{q_{\text{in}}(\xi)}\right). \quad (\text{C.2-2})$$

The upper bound on $U_{\text{co}}^* - U_{\text{in}}^*$ is driven by the function h and the ratio $q_{\text{co}}/q_{\text{in}}$. The next two results present properties of h and the quantiles and their ratio.

Lemma C.2-2. *For any $x \in [0, 1]$, $h(\cdot, x)$ is concave on $[-1, -0.5]$ and on $(-0.5, 0]$; for any $\xi \in [-1, 0]$, $h(\xi, \cdot)$ is increasing.*

Lemma C.2-3. *The following properties hold.*

i.) *The quantiles $q_{\text{in}}(\xi)$ and $q_{\text{co}}(\xi)$ are decreasing with ξ on $(-\infty, 0)$.*

ii.) *For any $\xi \leq 0$, the ratio of quantiles satisfies*

$$\frac{q_{\text{co}}(\xi)}{q_{\text{in}}(\xi)} \leq \alpha(\xi, q_{\text{co}}),$$

where for all non-positive ξ, y , $\alpha(\xi, y) = \exp\{-(((2/y) - 1)(1 - \xi) - 1)^{-1}\}$.

Moreover $\alpha(\xi, y)$ is decreasing with respect to ξ and decreasing with respect to y .

This lemma characterizes how the quantiles change with the respect to the coefficient of variation (which admits a one-to-one relationship with ξ). Furthermore, it characterizes the relation ratio of the quantiles and suggests through an upper bound that as the coefficient of variation increases, the seller's reaction becomes more pronounced in the sense that (the bound on) $q_{\text{co}}/q_{\text{in}}$ decreases.

Using the bound above in conjunction with (C.2-2), we obtain that on any interval $[\underline{\xi}, \bar{\xi}]$, we have for all $\xi \in [\underline{\xi}, \bar{\xi}]$,

$$U_{\text{co}}^* - U_{\text{in}}^* \leq \sigma q_{\text{in}}^{1-\xi} h(\xi, \alpha(\xi, q_{\text{co}}(\xi))) \leq h(\xi, \alpha(\xi, q_{\text{co}}(\xi))) \leq h(\xi, \alpha(\xi, q_{\text{co}}(\bar{\xi}))). \quad (\text{C.2-3})$$

where the second inequality follows from the fact that h is nondecreasing with respect to its second argument and α is nonincreasing with respect to its first argument; in the third inequality, we used the fact that α is nonincreasing with respect to its second argument and $q_{\text{co}}(\xi)$ is nonincreasing in ξ . In the next result, we analyze $h(\xi, \alpha(-1, q_{\text{co}}(-0.5)))$ and $h(\xi, \alpha(-0.5, q_{\text{co}}(0)))$.

Lemma C.2-4. *We have that $h(\xi, \alpha(-1, q_{\text{co}}(-0.5))) < 0$ on all $\xi \in [-1, -0.5]$ and $h(\xi, \alpha(-0.5, q_{\text{co}}(0))) < 0$ on all $\xi \in (-0.5, 1]$.*

Using this result and the bound (C.2-3) for $[\underline{\xi}, \bar{\xi}] = [-1, -0.5]$ and $[\underline{\xi}, \bar{\xi}] = [-0.5, 0]$, we obtain

$$U_{\text{co}}^* - U_{\text{in}}^* < 0 \quad \text{for all } \xi \text{ in } [-1, 0].$$

This completes the proof. □

C.3 Proofs of Section 3.5

Proof of Proposition 3.3

The proof is organized into two main steps. We first bound the \mathcal{MBC} by conditioning on the number of matches and how they are distributed between intermediaries. This leads to a bound composed of two terms, which we bound separately to obtain a final bound on \mathcal{MBC} .

Step 1. For $i = 1, \dots, K$, let

$$E_i = \{\omega : \mathbf{w}^{[2]} = \mathbf{v}_{1:J}^{[i+1]}\} \tag{C.3-4}$$

denote the event that the intermediary with the highest value has all the i^{th} highest values but does not uniquely have the $(i+1)^{\text{st}}$ value. Let M denote the number of

matches. Note that conditional on $M = m$ matches, $\{E_i : 1 \leq i \leq \min\{K, m\}\}$ is a partition of the probability space. We have

$$\begin{aligned}
& \mathcal{MBC} \tag{C.3-5} \\
&= U_{\text{co}}(r_{\text{in}}) - U_{\text{in}}(r_{\text{in}}) \\
&= \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{\mathbf{w}^{[2]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right] - \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right] \\
&= \mathbb{E} \left[(\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{w}^{[2]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} \right].
\end{aligned}$$

Conditioning on the events $\{E_i : 1 \leq i \leq \min\{K, m\}\}$, we obtain

$$\begin{aligned}
& \mathcal{MBC} \tag{C.3-6} \\
&= \sum_{m=0}^{KJ} \sum_{i=1}^{\min(K,m)} \mathbb{E} \left[(\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{w}^{[2]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} | E_i, M = m \right] \\
&\quad \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
&= \sum_{m=2}^{KJ} \sum_{i=2}^{\min(K,m)} \mathbb{E} \left[(\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} | E_i, M = m \right] \\
&\quad \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
&= \sum_{m=2}^{KJ} \sum_{i=2}^{\min(K,m)} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_i | M = m) \mathbb{P}(M = m),
\end{aligned}$$

where the last equality follows from the fact that if $\mathbf{v}_{1:J}^{[2]} \geq r_{\text{in}}$, then $\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}$ by definition, and $(\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\}) \mathbb{1}\{\mathbf{v}_{1:J}^{[1]} \geq r_{\text{in}}\} = \max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\}$; if $\mathbf{v}_{1:J}^{[2]} \leq r_{\text{in}}$, then $\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} = 0$. In turn, we have

$$\begin{aligned}
& \mathcal{MBC} \tag{C.3-7} \\
&= \sum_{i=2}^K \sum_{m=i}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
&= \sum_{m=2}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[3]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_2 | M = m) \mathbb{P}(M = m) \\
&\quad + \sum_{i=3}^K \sum_{m=i}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_i | M = m) \mathbb{P}(M = m),
\end{aligned}$$

Step 2. In this step we analyze each term in the RHS of (C.3-7). Let us first analyze the probability of events E_1 and E_2 . For $m \geq 2$, the quantity $1 - \mathbb{P}(E_1|M = m)$ represents the probability that the buyers that have the two highest values are represented by the same intermediary. Given the ex-ante symmetry across intermediaries, we have:

$$\mathbb{P}(E_1|M = m) = 1 - \frac{K-1}{JK-1}. \quad (\text{C.3-8})$$

For $m \geq 3$, given that the buyers that have the two highest values are represented by the same intermediary, the probability that the latter has also the third value is given by $\frac{K-2}{JK-2}$, then we can conclude that

$$\mathbb{P}(E_2|M = m) = \frac{K-1}{JK-1} \left(1 - \frac{K-2}{JK-2} \right). \quad (\text{C.3-9})$$

Since for $M = 2$ matches, $\{E_1, E_2\}$ is a partition of the probability space. we have $\mathbb{P}(E_2|M = 2) = 1 - \mathbb{P}(E_1|M = 2)$, so using (C.3-8), we get

$$\mathbb{P}(E_2|M = 2) = \frac{K-1}{JK-1}. \quad (\text{C.3-10})$$

Step 2.1 We first analyze the second sum in the RHS of (C.3-7). We have

$$\begin{aligned} & \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \\ & \leq \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - r_{\text{in}})^+ | M = m \right] \\ & \leq \mathbb{E} \left[\mathbf{v}_{1:J}^{[1]} | M = m \right] \\ & \stackrel{(a)}{\leq} \mathbb{E}_F[V] \sum_{k=1}^m \frac{1}{k} \\ & \leq \mathbb{E}_F[V] \sum_{k=1}^{JK} \frac{1}{k} \\ & \stackrel{(b)}{\leq} \mathbb{E}_F[V] (1 + \ln(JK)), \end{aligned}$$

where in (a) we used a result from Barlow (1965) for IFR random variables and in (b) we used that $\sum_{k=1}^n 1/k \leq (1 + \ln(n))$ for any $n \geq 1$.

In turn, we have that

$$\begin{aligned}
& \sum_{i=3}^K \sum_{m=i}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
& \leq (\ln(JK) + 1) \mathbb{E}_F[V] \sum_{i=3}^K \sum_{m=i}^{KJ} \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
& = (\ln(JK) + 1) \mathbb{E}_F[V] \sum_{m=3}^{KJ} \sum_{i=3}^{\min\{K, m\}} \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
& = (\ln(JK) + 1) \mathbb{E}_F[V] \sum_{m=3}^{KJ} \sum_{i=3}^{\min\{K, m\}} \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
& = (\ln(JK) + 1) \mathbb{E}_F[V] \sum_{m=3}^{KJ} (1 - \mathbb{P}(E_1 | M = m) - \mathbb{P}(E_2 | M = m)) \mathbb{P}(M = m).
\end{aligned}$$

Using (C.3-8) and (C.3-9), we have

$$1 - \mathbb{P}(E_1 | M = m) - \mathbb{P}(E_2 | M = m) = \frac{K-1}{JK-1} \frac{K-2}{JK-2} \leq \frac{1}{J^2}.$$

We hence have

$$\begin{aligned}
& \sum_{i=3}^K \sum_{m=i}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[i+1]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_i | M = m) \mathbb{P}(M = m) \\
& \leq \frac{1}{J^2} (\ln(JK) + 1) \mathbb{E}_F[V].
\end{aligned}$$

Step 2.2 Now we analyze the first sum in the RHS (C.3-7). We have

$$\begin{aligned}
& \sum_{m=2}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[3]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(E_2 | M = m) \mathbb{P}(M = m) \\
& = \frac{K-1}{JK-1} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - r_{\text{in}} | M = 2 \right] \mathbb{P}(M = 2) + \frac{K-1}{JK-1} \\
& \quad \left(1 - \frac{K-2}{JK-2} \right) \sum_{m=3}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[3]}, r_{\text{in}}\} | M = m \right] \mathbb{P}(M = m),
\end{aligned}$$

where we have used (C.3-9) and (C.3-10).

Noting that M has a Binomial distribution with success probability $\alpha = c/(JK)$,

we have

$$\begin{aligned}
\mathbb{P}(M = m) &= \binom{JK}{m} \alpha^m (1 - \alpha)^{JK-m} \\
&= \frac{JK \dots (JK - m + 1)}{m!} \frac{c^m}{(JK)^m} \left(1 - \frac{c}{JK}\right)^{JK-m} \\
&= \frac{JK \dots (JK - m + 1)}{(JK)^m} e^{(JK-m) \ln(1 - \frac{c}{JK})} \frac{c^m}{m!}.
\end{aligned}$$

Note that $(JK - m) \ln(1 - c/JK) \leq -\frac{c}{JK}(JK - m) = -c + \frac{mc}{KJ} \leq -c + \frac{c}{J}$, since $m \leq K \leq JK$. Hence one may bound $\mathbb{P}(M = m)$ as follows

$$\mathbb{P}(M = m) \leq \frac{c^m}{m!} e^{-c + \frac{c}{J}}. \quad (\text{C.3-11})$$

Hence, we have that

$$\begin{aligned}
&\sum_{m=3}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[3]}, r_{\text{in}}\} \mid M = m \right] \mathbb{P}(M = m) \\
&= \sum_{m=3}^{KJ} \int_{r_{\text{in}}}^{\infty} \int_{r_{\text{in}}}^y (y - \max\{x, r_{\text{in}}\}) \frac{m!}{(m-3)!} (F(x))^{m-3} \bar{F}(y) f(x) f(y) dx dy \mathbb{P}(M = m) \\
&\leq \sum_{m=3}^{KJ} \int_{r_{\text{in}}}^{\infty} \int_{r_{\text{in}}}^y (y - \max\{x, r_{\text{in}}\}) \frac{m!}{(m-3)!} (F(x))^{m-3} \bar{F}(y) f(x) f(y) dx dy \frac{c^m}{m!} e^{-c + \frac{c}{J}} \\
&= e^{-c + \frac{c}{J}} c^3 \int_{r_{\text{in}}}^{\infty} \int_{r_{\text{in}}}^y (y - \max\{x, r_{\text{in}}\}) \left(\sum_{m=3}^{KJ} \frac{c^{m-3}}{(m-3)!} (F(x))^{m-3} \right) \bar{F}(y) f(x) f(y) dx dy \\
&\leq e^{-c + \frac{c}{J}} c^3 \int_{r_{\text{in}}}^{\infty} \int_{r_{\text{in}}}^y (y - \max\{x, r_{\text{in}}\}) e^{cF(x)} \bar{F}(y) f(x) f(y) dx dy \\
&= e^{-c + \frac{c}{J}} c^3 \int_{r_{\text{in}}}^{\infty} e^{cF(x)} f(x) \left(\int_x^{\infty} (y - x) \bar{F}(y) f(y) dy \right) dx.
\end{aligned}$$

On another hand, we have

$$\begin{aligned}
\mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - r_{\text{in}} \mid M = 2 \right] &\leq \int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) 2\bar{F}(y) f(y) dy \frac{c^2}{2} e^{-c + c/J} \\
&\leq e^{-c + c/J} c^2 \int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) \bar{F}(y) f(y) dy.
\end{aligned}$$

So we conclude that

$$\begin{aligned} & \sum_{m=2}^{KJ} \mathbb{E} \left[\max\{\mathbf{v}_{1:J}^{[2]}, r_{\text{in}}\} - \max\{\mathbf{v}_{1:J}^{[3]}, r_{\text{in}}\} \mid M = m \right] \mathbb{P}(E_2 \mid M = m) \mathbb{P}(M = m) \\ & \leq \frac{e^{-c+c/J}}{J} \left[c^2 \int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) \bar{F}(y) f(y) dy \right. \\ & \quad \left. + c^3 \int_{r_{\text{in}}}^{\infty} e^{cF(x)} f(x) \left(\int_x^{\infty} (y - x) \bar{F}(y) f(y) dy \right) dx \right]. \end{aligned}$$

Step 3 Combining all the previous steps we obtain

$$\begin{aligned} \mathcal{MBC} & \leq \frac{c^2 e^{c/J}}{J} \left[\int_{r_{\text{in}}}^{\infty} (y - r_{\text{in}}) \bar{F}(y) f(y) e^{-c} dy \right. \\ & \quad \left. + c \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y - x) e^{-c\bar{F}(x)} f(x) \bar{F}(y) f(y) dy dx \right] \\ & \quad + O(\ln(JK)/J^2). \end{aligned}$$

This completes the proof of Proposition 3.3. □

Proof of Proposition 3.4

We aim to lower bound

$$-\mathcal{SRI} = U_{\text{co}}(r_{\text{in}}) - U_{\text{co}}(r_{\text{co}}(\alpha, K)).$$

In the proof, we will sometimes use the shorthand notation r_{co} for $r_{\text{co}}(\alpha, K)$ to avoid cluttering the exposition. The proof is organized around four steps. First, we show that $U_{\text{co}}(r)$ is differentiable and characterize its derivative. Second, we investigate the derivative of the utility at r_{co} . Third, we bound the difference in reserve prices $r_{\text{co}}(\alpha, K) - r_{\text{in}}$. In a last step, we conclude by combining the various bounds to obtain a lower bound on $-\mathcal{SRI}$. The proofs of some auxiliary results are deferred to Appendix C.4.

We denote $F_{\text{co}}^{(j)}$ the distribution of the j^{th} highest value among intermediaries and by $f_{\text{co}}^{(j)}$ the corresponding density on $(0, +\infty)$. Similarly, we denote by $F_{\text{in}}^{(j)}$ and $f_{\text{in}}^{(j)}$ the corresponding functions for the j^{th} highest value among all buyers.

Step 1. We first establish that the buyers' surplus $U_{\text{co}}(\cdot)$ is differentiable and characterize its derivative.

Lemma C.3-5 (derivative of U_{co}). *The buyers' surplus $U_{\text{co}}(\cdot)$ is twice differentiable on the interior of the support with derivative given by*

$$U'_{\text{co}}(r) = -\mathbb{P}\{\mathbf{w}^{[2]} \leq r \leq \mathbf{v}_{1:J}^{[1]}\} \quad \text{for any } r > 0.$$

Furthermore, if the probability of a match $\alpha \leq 1$ and the number of intermediaries $J \geq 3$ then

$$|U''_{\text{co}}(r)| \leq 3cf(r) \text{ for any } r > 0.$$

Note that the derivations in the proof of Lemma C.3-5 also allow to conclude that

$$U'_{\text{co}}(r_{\text{co}}) = S'_{\text{co}}(r_{\text{co}}) - \Pi'_{\text{co}}(r_{\text{co}}) = S'_{\text{co}}(r_{\text{co}}) = -r_{\text{co}}f_{\text{co}}^{(1)}(r_{\text{co}}).$$

In turn, one may apply Taylor's theorem to deduce that for any $r \in \mathcal{S} \cap [r_{\text{in}}, \infty[$, there exist $\tilde{r} \in \mathcal{S} \cap [r_{\text{in}}, \infty[$ such that

$$\begin{aligned} U_{\text{co}}(r) - U_{\text{co}}(r_{\text{co}}) &= U'_{\text{co}}(r_{\text{co}})(r - r_{\text{co}}) + \frac{U''_{\text{co}}(\tilde{r})}{2}(r - r_{\text{co}})^2 \\ &= -r_{\text{co}}f_{\text{co}}^{(1)}(r_{\text{co}})(r - r_{\text{co}}) + \frac{U''_{\text{co}}(\tilde{r})}{2}(r - r_{\text{co}})^2. \end{aligned} \quad (\text{C.3-12})$$

Step 2. We next analyze $f_{\text{co}}^{(1)}(\cdot)$. We first note that $f_{\text{in}}^{(1)}(r) = f_{\text{co}}^{(1)}(r)$ for any $r \in \mathcal{S} \setminus \{0\}$ since the maximum value among all buyers $\mathbf{v}_{1:J}^{[1]}$ coincides with the maximum value among all intermediaries $\mathbf{w}^{[1]}$. The following lemma derives a lower bound on $f_{\text{in}}^{(1)}(r)$.

Lemma C.3-6. *For any $r \in \mathcal{S} \setminus \{0\}$, the density of $\mathbf{v}_{1:J}^{[1]}$ is bounded below as follows*

$$f_{\text{co}}^{(1)}(r) = f_{\text{in}}^{(1)}(r) \geq cf(r)e^{-c(\bar{F}(r))} - cf(r) \left(\frac{18c^2 + 2}{2\sqrt{JK} - 1} \right),$$

if we assume that $\sqrt[4]{JK} \geq 2ce$ and $\sqrt[4]{JK} \geq 2$.

So the derivative of the utility is bounded by

$$U'_{\text{co}}(r_{\text{co}}) \leq -cr_{\text{co}}f(r_{\text{co}})e^{-c\bar{F}(r_{\text{co}})} + cr_{\text{co}}f(r_{\text{co}}) \left(\frac{18c^2 + 2}{2\sqrt{JK} - 1} \right), \quad (\text{C.3-13})$$

Step 3. Next, we analyze $r_{\text{co}}(\alpha, K)$.

Proposition C.3-1 (reserve price - Lemma 3.2). *For any α, K such that $\alpha \leq 1/2$ and $\alpha K < 1$, the reserve price $r_{\text{co}}(\alpha, K)$ satisfies*

$$K\alpha \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} - (\alpha + 3(K\alpha)^2)C_1 \leq r_{\text{co}}(\alpha, K) - r_{\text{in}} \leq K\alpha r_{\text{in}}e^{\frac{1}{1-\alpha}},$$

where C_1 is a constant depending only on the distribution F .

Step 4. Combining (C.3-12), Lemma C.3-5, (C.3-13) and Proposition C.3-1, we have

$$\begin{aligned} & -\mathcal{S}\mathcal{R}\mathcal{I} \\ &= -r_{\text{co}}f_{\text{co}}^{(1)}(r_{\text{co}})(r_{\text{in}} - r_{\text{co}}) + \frac{1}{2}U''_{\text{co}}(\tilde{r})(r_{\text{in}} - r_{\text{co}})^2 \\ &\geq cr_{\text{co}}f(r_{\text{co}})e^{-c\bar{F}(r_{\text{co}})} \left[1 - \frac{18c^2 + 2}{2\sqrt{JK} - 1} e^{c\bar{F}(r_{\text{co}})} \right] \left[\frac{\alpha K (\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} - (\alpha + 3(K\alpha)^2)C_1 \right] \\ &\quad - \frac{3c}{2}(K\alpha)^2 r_{\text{in}}^2 f(\tilde{r}) e^{\frac{2}{1-\alpha}} \\ &\geq cr_{\text{co}}f(r_{\text{co}})e^{-c\bar{F}(r_{\text{co}})} \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} \alpha K - R_1, \end{aligned}$$

where

$$\begin{aligned} & R_1 \\ &= \frac{18c^2 + 2}{2\sqrt{JK} - 1} e^c \left[\alpha K \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} \right] + (\alpha + 3(K\alpha)^2)C_1 + \frac{3c}{2}(K\alpha)^2 r_{\text{in}}^2 f(\tilde{r}) e^{\frac{2}{1-\alpha}} \\ &= \left(\frac{18c^2 + 2}{2\sqrt{JK} - 1} e^c \left[\frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} \right] \right) \alpha K + (\alpha + 3(K\alpha)^2)C_1 \\ &\quad + \left(\frac{3c}{2} r_{\text{in}}^2 f(\tilde{r}) e^{\frac{2}{1-\alpha}} \right) (K\alpha)^2 \\ &= \left(\frac{(18c^2 + 2)\sqrt{JK}}{2\sqrt{JK} - 1} e^c \left[\frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} \right] \right) \sqrt{c\alpha} K\alpha \\ &\quad + (\alpha + 3(K\alpha)^2)C_1 + \left(\frac{3c}{2} r_{\text{in}}^2 f(\tilde{r}) e^{\frac{2}{1-\alpha}} \right) (K\alpha)^2. \end{aligned}$$

We consider the following function $rf(r) \exp\{-c\bar{F}(r)\}$, given that f has a continuous derivative bounded around r_{in} then the latter has a continuous derivative bounded at r_{in} using Taylor expansion, we get that there exist constants C_2 and C_3 such that

$$\begin{aligned} cr_{\text{co}}f(r_{\text{co}})e^{-c\bar{F}(r_{\text{co}})} &\geq cr_{\text{in}}f(r_{\text{in}})e^{-c\bar{F}(r_{\text{in}})} - C_2(r_{\text{co}} - r_{\text{in}}) \\ &\stackrel{(a)}{\geq} cr_{\text{in}}f(r_{\text{in}})e^{-c\bar{F}(r_{\text{in}})} - C_2K\alpha r_{\text{in}}e^{\frac{1}{1-\alpha}} \\ f(\tilde{r}) &\leq C_3 \end{aligned}$$

where in (a), we have used the RHS inequality in Proposition C.3-1. So we conclude that

$$-\mathcal{S}\mathcal{R}\mathcal{I} \geq cr_{\text{in}}f(r_{\text{in}})e^{-c\bar{F}(r_{\text{in}})} \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} K\alpha - R_2,$$

with

$$R_2 = C_2r_{\text{in}}e^{\frac{1}{1-\alpha}} \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} (\alpha K)^2 + R_1.$$

If we denote

$$\begin{aligned} &C_4 \\ := &5 \max \left\{ \frac{C_2e^2r_{\text{in}}(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})}, \left(\sqrt{c} (18c^2 + 2) e^c \left[\frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} \right] \right), C_1, \left(\frac{3c}{2} r_{\text{in}}^2 C_3 e^4 \right) \right\}, \end{aligned}$$

then if $\alpha \leq 1/2$ and $\sqrt[4]{JK} \geq 2ce$ and $\sqrt[4]{JK} \geq 2$ and $J \geq 3$, we get

$$|R_2| \leq C_4 (\sqrt{\alpha}K\alpha + \alpha + (K\alpha)^2).$$

This completes the proof of Proposition 3.4. □

Proof of Theorem 3.3

For this proof, we simplify first the expression of the bound in Theorem 3.2 and then compute in closed each term of the simplified bound for the case of Generalized Pareto distributions. In turn, we find explicitly the root of the simplified bound (i.e. where

the bound is 0). Then we study the relation between this root and the coefficient of variation to conclude the result.

Let us first analyze the main terms in the upper bound derived in Theorem 3.2, we have

$$\begin{aligned}
& \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y-x) \bar{F}(y) f(x) f(y) c^3 e^{-c\bar{F}(x)} dy dx \\
& \leq \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y-r_{\text{in}}) \bar{F}(y) f(x) f(y) c^3 e^{-c\bar{F}(x)} dy dx \\
& = \int_{r_{\text{in}}}^{\infty} (y-r_{\text{in}}) \bar{F}(y) f(y) c^2 \left[e^{-c\bar{F}(x)} \right]_{x=r_{\text{in}}}^{x=y} dy \\
& = \int_{r_{\text{in}}}^{\infty} (y-r_{\text{in}}) \bar{F}(y) f(y) c^2 (e^{-c\bar{F}(y)} - e^{-c\bar{F}(r_{\text{in}})}) dy.
\end{aligned}$$

So we get that

$$\begin{aligned}
& \frac{c^2 e^{-c}}{2} \int_{r_{\text{in}}}^{\infty} (y-r_{\text{in}}) 2\bar{F}(y) f(y) dy + \int_{r_{\text{in}}}^{\infty} \int_x^{\infty} (y-x) \bar{F}(y) f(x) f(y) c^3 e^{-c\bar{F}(x)} dy dx \\
& \leq c^2 \int_{r_{\text{in}}}^{\infty} (y-r_{\text{in}}) \bar{F}(y) f(y) (e^{-c} + e^{-c\bar{F}(y)} - e^{-c\bar{F}(r_{\text{in}})}) dy \\
& \stackrel{(a)}{\leq} c^2 \int_{r_{\text{in}}}^{\infty} (y-r_{\text{in}}) \bar{F}(y) f(y) dy \\
& = \frac{c^2}{2} \left[-(y-r_{\text{in}}) \bar{F}(y)^2 \right]_{y=r_{\text{in}}}^{y=\infty} + \frac{c^2}{2} \int_{r_{\text{in}}}^{\infty} \bar{F}(x)^2 dx \\
& \stackrel{(b)}{=} \frac{c^2}{2} \int_{r_{\text{in}}}^{\infty} \bar{F}(x)^2 dx,
\end{aligned}$$

where in (a), we have used that $(e^{-c} + e^{-c\bar{F}(y)} - e^{-c\bar{F}(r_{\text{in}})}) \leq 1$ for all $y \geq r_{\text{in}}$ and in (b), we have used the Markov inequality, since for $x \geq r_{\text{in}}$

$$\left| \frac{1}{2} ((x-r_{\text{in}})^+) \bar{F}(x)^2 \right| \leq \frac{((x-r_{\text{in}})^+) (\mathbb{E}[V])^2}{x^2} \leq \frac{(\mathbb{E}[V])^2}{x} \rightarrow_{x \rightarrow \infty} 0.$$

Hence, we have

$$U_{\text{co}}^* - U_{\text{in}}^* \leq \frac{1}{J} \frac{c^2}{2} \left[\int_{r_{\text{in}}}^{\infty} \bar{F}(x)^2 dx - \frac{(\bar{F}(r_{\text{in}}))^2}{\phi'(r_{\text{in}})} r_{\text{in}} e^{-c\bar{F}(r_{\text{in}})} \right] + o(1/J).$$

Let us analyze the main term of the previous bound as a function of ξ . As computed in Section 3.4, we have $q_{\text{in}} = (1-\xi)^{\frac{1}{\xi}}$. Let us now compute explicitly each term in

the bound.

$$\begin{aligned}
\int_{r_{\text{in}}}^{\infty} \bar{F}(x)^2 dx &= \int_0^{q_{\text{in}}} q^2 \frac{dq}{f(F^{-1}(1-q))} \\
&= \int_0^{q_{\text{in}}} q^2 (q^{-\xi-1}) dq \\
&= \int_0^{q_{\text{in}}} q^{1-\xi} dq \\
&= \frac{q_{\text{in}}^{2-\xi}}{2-\xi} = \frac{(1-\xi)^{\frac{2-\xi}{\xi}}}{2-\xi},
\end{aligned}$$

and

$$\frac{r_{\text{in}} \bar{F}(r_{\text{in}})^2}{\phi'(r_{\text{in}})} = \frac{\frac{1}{1-\xi} ((1-\xi)^{\frac{1}{\xi}})^2}{(1-\xi)} = (1-\xi)^{\frac{2-2\xi}{\xi}}.$$

By combining the two previous equations, we get that the main term of the previous bound is

$$\frac{c^2}{2} \left[\int_{r_{\text{in}}}^{\infty} \bar{F}(x)^2 dx - \frac{(\bar{F}(r_{\text{in}}))^2}{\phi'(r_{\text{in}})} r_{\text{in}} e^{-c\bar{F}(r_{\text{in}})} \right] = \frac{1}{(2-\xi)(1-\xi)} \left((1-\xi) - (2-\xi)e^{-c(1-\xi)^{\frac{1}{\xi}}} \right).$$

Since $\xi \leq 0$ then the sign of the RHS is the sign of the following quantity

$$\gamma(\xi, c) := (1-\xi) - (2-\xi)e^{-c(1-\xi)^{\frac{1}{\xi}}}.$$

The function $\gamma(\xi, c)$ is increasing in c and the value at $c = 0$ is -1 . This implies that for an $\xi \leq 0$, there exists a threshold c_ξ such that for all $c \leq c_\xi$, $U_{\text{co}}^* - U_{\text{in}}^* \leq 0$. Let us look in details to the threshold c_ξ for which $\gamma(\xi, c) = 0$, i.e., $\exp\{-c_\xi(1-\xi)^{1/\xi}\} = (1-\xi)(2-\xi)$. We can solve for c_ξ to obtain

$$c_\xi = -(1-\xi)^{\frac{-1}{\xi}} \ln \left(\frac{1-\xi}{2-\xi} \right).$$

Since $(1-\xi)^{\frac{-1}{\xi}}$ and $-\ln(\frac{1-\xi}{2-\xi})$ are increasing positive functions in ξ then c_ξ is increasing with ξ . Hence, the maximum value \bar{c} of the threshold c_ξ is reached when $\xi \rightarrow 0^-$.

Fix any $c < \bar{c}$, since the function that maps the ξ to the threshold c_ξ is strictly increasing and continuous in ξ , there exists ξ_c such that $c_{\xi_c} = c$. In turn, we have for all $\xi \geq \xi_c$, we have $\gamma(\xi, c) \leq 0$. Recall that $\xi = (CV^2 - 1)/(2CV^2)$ is increasing in

the coefficient of variation CV , so we conclude that there is corresponding threshold ($\tau_c := 1/\sqrt{1-\xi_c}$) on the coefficient of variation, such that for all $CV \geq \tau_c$, we have $U_{co}^* - U_{in}^* \leq 0$. This completes the proof. \square

C.4 Proofs of auxiliary results

Proof of Lemma C.1-1

Let $n \geq 2$ and X_1, X_2, \dots, X_n be i.i.d. random variables drawn from an IFR distribution F and define $M = \max(X_1, X_2, \dots, X_n)$. The cumulative distribution function of M is $(F)^n$. If $F(x) = 0$, then the hazard rate is given by zero and if $F(x) > 0$, the hazard rate is given by

$$\begin{aligned} \lambda_M(x) &= \frac{n f(x) (F(x))^{n-1}}{1 - (F(x))^n} = \frac{n f(x) (F(x))^{n-1}}{\overline{F}(x) (1 + F(x) + \dots + (F(x))^{n-1})} \\ &= n \frac{f(x)}{1 - F(x)} \frac{1}{1 + \frac{1}{F(x)} + \dots + \frac{1}{(F(x))^{n-1}}}. \end{aligned}$$

Note that $f(x)/\overline{F}(x)$ is non-decreasing since by assumption F is IFR. In addition the fact that F is non-decreasing and non-negative implies that $1 + 1/F(x) + \dots + 1/(F(x))^{n-1}$ is non-increasing. In turn, $(1 + 1/F(x) + \dots + 1/(F(x))^{n-1})^{-1}$ is seen to be non-decreasing. We deduce that $\lambda_M(\cdot)$ is non-decreasing. This concludes the proof. \square

Proof of Lemma C.2-2

Fix ξ in $[-1, 0]$. We first prove that $h(\xi, \cdot)$ is increasing for $x \in [0, 1]$. Indeed,

$$\frac{\partial h(\xi, x)}{\partial x} = -e^{-\frac{2}{2-\xi} + \gamma(\xi)} x^{1-\xi} + 2x^{-\xi} = x^{-\xi} (2 - e^{-\frac{2}{2-\xi} + \gamma(\xi)} x) \geq 0,$$

where in the last inequality, we used the fact that $\xi \leq 0$, $x \in [0, 1]$ and $\gamma(\xi) \leq 0$. Next, we prove that h is concave with respect to the first component on the intervals $[-1, -0.5]$ and $(-0.5, 0)$. For that we will show that the second partial

derivative w.r.t to ξ of h is increasing w.r.t x , and then show that its maximum value at $x = 1$ is negative. It is clear that h is infinitely differentiable in the considered intervals $[-1, -0.5]$ and $(-0.5, 0)$. Applying Schwartz's theorem to switch the order of derivatives yields

$$\begin{aligned}
& \frac{\partial^3 h(\xi, x)}{\partial x \partial^2 \xi} \\
&= \frac{\partial^3 h(\xi, x)}{\partial^2 \xi \partial x} \\
&= \frac{\partial^2}{\partial^2 \xi} \left(x^{-\xi} (2 - e^{-\frac{2}{2-\xi} + \gamma(\xi)} x) \right) \\
&= \ln^2(x) x^{-\xi} (2 - e^{-\frac{2}{2-\xi} + \gamma(\xi)} x) - \ln(x) \frac{4x}{(2-\xi)^2} x^{-\xi} e^{-\frac{2}{2-\xi} + \gamma(\xi)} + \frac{4x(1-\xi)}{(2-\xi)^4} x^{-\xi} e^{-\frac{2}{2-\xi} + \gamma(\xi)}.
\end{aligned}$$

Since $\xi \leq 0$ and $0 \leq x \leq 1$ and $\gamma(\xi) \leq 0$, we have that

$$\frac{\partial^3 h(\xi, x)}{\partial x \partial^2 \xi} \geq 0.$$

In turn, this implies that

$$\frac{\partial^2 h(\xi, x)}{\partial^2 \xi} \leq \frac{\partial^2 h(\xi, 1)}{\partial^2 \xi} = \frac{2e^{-\frac{2}{2-\xi} + \gamma(\xi)}}{(2-\xi)^5} (\xi^2 - 2).$$

The right-hand-term above is non-positive for all $\xi \in [-1, 0]$ so we conclude that $h(\cdot, x)$ is concave on the intervals $[-1, -0.5]$ and $(-0.5, 0)$. This completes the proof. \square

Proof of Lemma C.2-3

i.) Let us show that the quantiles q_{in} and q_{co} are decreasing with ξ .

For all $\xi < 0$, $q_{\text{in}} = (1 - \xi)^{1/\xi}$ is differentiable and its derivative is given by

$$\frac{\partial q_{\text{in}}}{\partial \xi} = \frac{\partial}{\partial \xi} \left(e^{\frac{\ln(1-\xi)}{\xi}} \right) = \frac{q_{\text{in}}}{\xi^2} \left(-\frac{\xi}{1-\xi} - \ln(1-\xi) \right) = \frac{q_{\text{in}}}{\xi^2} \left(1 - \frac{1}{1-\xi} + \ln\left(\frac{1}{1-\xi}\right) \right).$$

Since for all $x > 0$ we have $\ln(x) \leq x - 1$ then we conclude that $\partial q_{\text{in}} / \partial \xi \leq 0$.

Let us now show that q_{co} is decreasing. We consider the following function for all $\xi \leq 0$ and $0 < q < 1$.

$$G(q, \xi) = 1 - q^\xi - \frac{\xi(2-q)}{2(1-q)}.$$

We have $G(q_{co}, \xi) = 0$ and $G(\cdot, \cdot)$ is differentiable so by the implicit function theorem, q_{co} is differentiable with respect to ξ with derivative given by

$$\frac{\partial q_{co}}{\partial \xi} = -\frac{\frac{\partial G}{\partial \xi}}{\frac{\partial G}{\partial q}}.$$

We have

$$\begin{aligned} \frac{\partial G}{\partial q}(q_{co}, \xi) &= -\xi \left(q_{co}^{\xi-1} + \frac{1}{2(1-q_{co})^2} \right) \\ \frac{\partial G}{\partial \xi}(q_{co}, \xi) &= -\ln(q_{co})q_{co}^\xi - \frac{1}{2} \frac{2-q_{co}}{1-q_{co}} \\ &\stackrel{(a)}{=} -\ln(q_{co})q_{co}^\xi - \frac{1}{\xi}(1-q_{co}^\xi) \\ &= \frac{q_{co}^\xi}{\xi} \left(\ln(q_{co}^{-\xi}) - (q_{co}^{-\xi} - 1) \right), \end{aligned}$$

where in (a), we used the fact that $G(q_{co}, \xi) = 0$. It is clear that $\partial G/\partial q > 0$. Since $\xi < 0$ and since $\ln(x) \leq x - 1$ for all $x > 0$, we also observe that $\partial G/\partial \xi > 0$. We conclude that $\partial q_{co}/\partial \xi < 0$.

ii.) Recalling that $q_{in} = (1 - \xi)^{1/\xi}$ and $(1/\xi)(1 - q_{co}^\xi) = (2 - q_{co})/(2(1 - q_{co}))$, we have

$$[q_{in}(\xi)]^\xi - [q_{co}(\xi)]^\xi = 1 - \xi - \left(1 - \frac{\xi}{2} \frac{2 - q_{co}}{1 - q_{co}} \right) = \frac{\xi}{2} \frac{q_{co}}{1 - q_{co}}.$$

Note that for $\xi \in [-1, 0]$,

$$\begin{aligned} \frac{\xi (q_{co}(\xi))^{1-\xi}}{2(1-q_{co}(\xi))} &= \frac{(1 - (q_{co}(\xi))^\xi)(q_{co}(\xi))^{1-\xi}}{2 - q_{co}(\xi)} \\ &= \frac{((q_{co}(\xi))^{-\xi} - 1)q_{co}(\xi)}{2 - q_{co}(\xi)} \\ &\geq -q_{co}(\xi) \geq -q_{in}(\xi) \\ &\geq -q_{in}(-1) > -1, \end{aligned}$$

so we conclude that

$$q_{in}(\xi) = q_{co}(\xi) \left(1 + \frac{\xi (q_{co}(\xi))^{1-\xi}}{2(1-q_{co}(\xi))} \right)^{\frac{1}{\xi}} \geq q_{co}(\xi) \exp \left\{ \frac{(q_{co}(\xi))^{1-\xi}}{2(1-q_{co}(\xi))} \right\},$$

where in the last inequality we used that $\xi < 0$ and that $\ln(1+x) \leq x$ for all $x > -1$.

In turn, we obtain

$$\frac{q_{\text{co}}(\xi)}{q_{\text{in}}(\xi)} \leq \exp \left\{ (q_{\text{co}}(\xi))^{-\xi} \frac{-q_{\text{co}}(\xi)}{2(1-q_{\text{co}}(\xi))} \right\}.$$

We recall that $(1/\xi)(1-q_{\text{co}}^\xi) = (2-q_{\text{co}})/(2(1-q_{\text{co}}))$, so

$$q_{\text{co}}^{-\xi} = \left(1 + \left(-\frac{\xi}{2} \right) \left(1 + \frac{1}{1-q_{\text{co}}} \right) \right)^{-1},$$

Hence

$$\begin{aligned} (q_{\text{co}})^{-\xi} \frac{-q_{\text{co}}}{2(1-q_{\text{co}})} &= \left(1 + \left(-\frac{\xi}{2} \right) \left(1 + \frac{1}{1-q_{\text{co}}} \right) \right)^{-1} \left(\frac{-q_{\text{co}}}{2(1-q_{\text{co}})} \right) \\ &= -\frac{1}{2} \left(\frac{1-q_{\text{co}}}{q_{\text{co}}} + \left(-\frac{\xi}{2} \right) \left(\frac{1-q_{\text{co}}}{q_{\text{co}}} + \frac{1}{q_{\text{co}}} \right) \right)^{-1} \\ &= -\frac{1}{2} \left(\left(\frac{2}{q_{\text{co}}} - 1 \right) \left(\frac{1-\xi}{2} \right) - \frac{1}{2} \right)^{-1} \\ &= - \left(\left(\frac{2}{q_{\text{co}}} - 1 \right) (1-\xi) - 1 \right)^{-1}. \end{aligned}$$

So we conclude that

$$\frac{q_{\text{co}}(\xi)}{q_{\text{in}}(\xi)} \leq \alpha(\xi, q_{\text{co}}),$$

where

$$\alpha(\xi, y) := \exp \left\{ - \left(\left(\frac{2}{y} - 1 \right) (1-\xi) - 1 \right)^{-1} \right\}.$$

Since $0 \leq q_{\text{co}} \leq 1$ it is clear that $\alpha(\xi, y)$ is decreasing with respect to ξ . Let us now show that α is nonincreasing with respect to y . The function

$$- \left((2z-1)(1-\xi) - 1 \right)^{-1} \text{ for } z \geq 1,$$

is nondecreasing in z for all ξ . Since $1/y$ is non-increasing in y , we conclude that $\alpha(\xi, y)$ is nonincreasing with respect to y . This concludes the proof. \square

Proof of Lemma C.2-4

We first find approximations for $q_{\text{co}}(0)$ and $q_{\text{co}}(-0.5)$.

Consider $\xi = 0$. The virtual value function of the maximum of two exponential random variables in terms of quantiles is given by

$$\phi(q) = -\ln(q) - \frac{2-q}{2(1-q)}.$$

We recall that the latter is non increasing and the optimal quantile q_{co} verifies $\phi(q_{\text{co}}) = 0$ and we have $\phi(0.29) > 0$ which means that $q_{\text{co}}(0) > 0.29$.

For $\xi = -\frac{1}{2}$, the quantile verifies that

$$\frac{1 - q^{-\frac{1}{2}}}{-\frac{1}{2}} = \frac{2 - q}{2(1 - q)},$$

which can be re written as

$$-5q\sqrt{q} + 4q + 6\sqrt{q} - 4 = 0,$$

so $\sqrt{q_{\text{co}}}$ verifies a third degree equation. The latter can be solved exactly using Cardano's method. It has three roots, and only one belongs to the interval $[0, 1]$.

The latter is bounded as follows

$$0.369 \leq q_{\text{co}}(-0.5) \leq 0.37.$$

The derivative of $h(\cdot, x)$ on $[-1, 0] \setminus \{-0.5\}$ is given by

$$\begin{aligned} \frac{\partial h(\xi, x)}{\partial \xi} &= \frac{e^{\frac{2}{\xi-2} + \gamma(\xi)} (2 - x^{2-\xi})}{(\xi - 2)^2} + \frac{2x^{1-\xi} \log(x)}{\xi - 1} \\ &\quad - \frac{e^{\frac{2}{\xi-2} + \gamma(\xi)} x^{2-\xi} \log(x)}{\xi - 2} - \frac{2(x^\xi - x)x^{-\xi}}{(\xi - 1)^2} + \frac{e^{\frac{2}{\xi-2} + \gamma(\xi)} (4x^\xi - 2x^2)x^{-\xi}}{(\xi - 2)^3}. \end{aligned}$$

Now we have all ingredients to evaluate the right partial derivatives at $\xi = -0.5, x = \alpha(-0.5, 0.29)$ and $\xi = -1, x = \alpha(-1, 0.369)$, we find that both are negative. Given that $h(\cdot, x)$ is concave on the intervals $[-1, -0.5]$ and $(-0.5, -1]$, we deduce

that $h(\cdot, \alpha(-0.5, 0.29))$ is decreasing on $[-0.5, 0]$ and $h(\cdot, \alpha(-1, 0.369))$ is decreasing on $[-0.5, -1]$.

Hence we get that for any $\xi \in (-0.5, 0]$,

$$h(\xi, \alpha(-0.5, q_{\text{co}}(0))) \leq h(\xi, \alpha(-0.5, 0.29)) \leq \lim_{\xi \downarrow -0.5^+} h(\xi, \alpha(-0.5, 0.29)) < 0.$$

Similarly, for any $\xi \in [-1, -0.5]$,

$$h(\xi, \alpha(-1, q_{\text{co}}(-0.5))) \leq h(\xi, \alpha(-1, 0.369)) \leq h(-1, \alpha(-1, 0.369)) < 0.$$

The proof is complete. \square

Proof of Lemma C.3-5

Recall the expression for $G_\alpha(\cdot)$ given in (3.1). For any $x \geq 0$, we denote the distribution function of the maximum of each intermediary by $H(x)$, i.e., $H(x) := (G_\alpha(x))^K$.

For any $r > 0$,

$$\begin{aligned} U_{\text{co}}(r) &= \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{\mathbf{w}^{[2]}, r\})^+ \right] \\ &= \mathbb{E} \left[(\mathbf{v}_{1:J}^{[1]} - \max\{\mathbf{w}^{[2]}, r\}) \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right] \\ &= \mathbb{E} \left[\mathbf{v}_{1:J}^{[1]} \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right] - \mathbb{E} \left[\mathbf{w}^{[2]} \mathbf{1}\{\mathbf{w}^{[2]} \geq r\} \right] - r \mathbb{E} \left[\mathbf{1}\{\mathbf{w}^{[2]} < r\} \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right] \\ &= h_1(r) - h_2(r) - h_3(r), \end{aligned}$$

where

$$\begin{aligned} h_1(r) &:= \mathbb{E} \left[\mathbf{v}_{1:J}^{[1]} \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right], \\ h_2(r) &:= \mathbb{E} \left[\mathbf{w}^{[2]} \mathbf{1}\{\mathbf{w}^{[2]} \geq r\} \right], \\ h_3(r) &:= r \mathbb{E} \left[\mathbf{1}\{\mathbf{w}^{[2]} < r\} \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right]. \end{aligned}$$

We next compute the derivative of the functions $h_i(r)$, $i = 1, 2, 3$. Note that for $i = 1, 2$,

$$h_i(r) = \int_r^1 v dF_{\text{co}}^{(i)}(v),$$

and hence $h_i(r)$ is differentiable at any $r > 0$ and

$$h'_i(r) = -r f_{\mathbf{co}}^{(i)}(r).$$

Note that

$$\begin{aligned} F_{\mathbf{co}}^{(1)}(r) &= [H(r)]^J, & f_{\mathbf{co}}^{(1)}(r) &= JH'(r)[H(r)]^{J-1}, \\ F_{\mathbf{co}}^{(2)}(r) &= [H(r)]^J + J[H(r)]^{J-1}(1 - H(r)), \\ f_{\mathbf{co}}^{(2)}(r) &= J(J - 1)H'(r)(1 - H(r))[H(r)]^{J-2}. \end{aligned}$$

We now turn to $h_3(r)$.

$$h_3(r) = r \mathbb{E} \left[\mathbf{1}\{\mathbf{w}^{[2]} < r\} \mathbf{1}\{\mathbf{v}_{1:J}^{[1]} \geq r\} \right] = r J H(r)^{J-1} (1 - H(r)).$$

$h_3(r)$ is differentiable since f is twice differentiable and furthermore,

$$\begin{aligned} &h'_3(r) \\ &= J H(r)^{J-1} (1 - H(r)) + r J (J - 1) H'(r) H(r)^{J-2} (1 - H(r)) - r J H'(r) H(r)^{J-1} \\ &= J H(r)^{J-1} (1 - H(r)) + r f_{\mathbf{co}}^{(2)}(r) - r f_{\mathbf{co}}^{(1)}(r) \\ &= J H(r)^{J-1} (1 - H(r)) - h'_2(r) + h'_1(r). \end{aligned}$$

By combining the result of the three previous terms, we obtain that $U_{\mathbf{co}}(r)$ is differentiable for $r > 0$ with derivative given by

$$U'_{\mathbf{co}}(r) = h'_1(r) - h'_2(r) - h'_3(r) = -J H(r)^{J-1} (1 - H(r)) = -\mathbb{P} \left\{ \mathbf{w}^{[2]} \leq r \leq \mathbf{v}_{1:J}^{[1]} \right\}.$$

Next, we compute $U''_{\mathbf{co}}(r)$ then try to bound it. $U'_{\mathbf{co}}$ is differentiable for $r > 0$ and the second derivative is given by

$$\begin{aligned} U''_{\mathbf{co}}(r) &= JH'(r)[H(r)]^{J-1} - J(J - 1)H'(r)(1 - H(r))[H(r)]^{J-2} \\ &= JH'(r)[H(r)]^{J-2} [H(r) - (J - 1)(1 - H(r))] \\ &= JH'(r)[H(r)]^{J-2} [1 - J(1 - H(r))] \end{aligned}$$

Since $H'(r) = K\alpha f(r)(G_\alpha(r))^{K-1}$, and $\alpha JK = c$, we have

$$U''_{\text{co}}(r) = cf(r)[G_\alpha(r)]^{KJ-K-1} [1 - J(1 - H(r))].$$

We get that

$$|U''_{\text{co}}(r)| \leq cf(r) [1 + [G_\alpha(r)]^{KJ-K-1} J(1 - [G_\alpha(r)]^K)].$$

Since $(1 - [G_\alpha(r)]^K) \leq -K \ln(1 - \alpha(1 - F(r)))$ and $-x \ln(x) \leq e^{-1}$ for all $x \geq 0$ then we get that

$$\begin{aligned} & [G_\alpha(r)]^{KJ-K-1} J(1 - [G_\alpha(r)]^K) \\ \leq & \frac{JK}{JK - K - 1} ((1 - \alpha(1 - F(r)))^{JK-K-1}) (-\ln((1 - F(r)))^{JK-K-1}) \\ \leq & \frac{JK}{JK - K - 1} e^{-1} \\ = & \frac{c}{c - K\alpha - \alpha} e^{-1}. \end{aligned}$$

So we conclude that if $J \geq 3$

$$\begin{aligned} |U''_{\text{co}}(r)| & \leq cf(r) \left[1 + \frac{c}{c - K\alpha - \alpha} e^{-1} \right] \\ & \leq cf(r) \left[1 + \frac{c}{c - 2K\alpha} e^{-1} \right] \\ & \stackrel{(a)}{=} cf(r) \left[1 + \frac{J}{J-2} e^{-1} \right] \\ & \stackrel{(b)}{\leq} cf(r) [1 + 3e^{-1}] \\ & \leq 3cf(r), \end{aligned}$$

where in (a) we have used that $c = JK\alpha$ and in (b) we have used that $J \geq 3$. This completes the proof. \square

Proof of Lemma C.3-6

Note that the distribution of the maximum $\mathbf{v}_{1:J}^{[1]}$ can be written as follows

$$f_{\text{in}}^{(1)}(r) = \sum_{m=1}^{JK} f_{\text{in}}^{(1)}(r|M=m)\mathbb{P}(M=m) = f(r) \sum_{m=1}^{JK} mF^{m-1}(r)\mathbb{P}(M=m).$$

The focus of this proof is to bound the probability of matching $\mathbb{P}(M = m)$. Since M has a Binomial distribution with success probability $\alpha = c/(JK)$, then

$$\begin{aligned}\mathbb{P}(M = m) &= \binom{JK}{m} \alpha^m (1 - \alpha)^{JK-m} = \frac{JK!}{m!(JK-m)!} \frac{c^m}{(JK)^m} \left(1 - \frac{c}{JK}\right)^{JK-m} \\ &= \frac{JK!}{(JK-m)!(JK)^m} \left(1 - \frac{c}{JK}\right)^{JK-m} \frac{c^m}{m!}.\end{aligned}$$

On one hand, using the fact that $c/JK = \alpha < 1$ and the fact that $\ln(1+x) \geq x/(1+x)$ for all $x > -1$, we have

$$\left(1 - \frac{c}{JK}\right)^{JK-m} \geq \left(1 - \frac{c}{JK}\right)^{JK} = e^{JK \ln(1-c/(JK))} \geq e^{-c/(1+c/JK)} \geq e^{-c}.$$

Let us note $d = \sqrt[4]{JK}$. For any $m \leq d$ we have

$$\frac{JK \cdots (JK - m + 1)}{(JK)^m} \geq \left(\frac{JK - m + 1}{JK}\right)^m \geq \left(\frac{JK - m + 1}{JK}\right)^d \geq \left(\frac{JK - d + 1}{JK}\right)^d.$$

Furthermore using the fact that $\ln(1+x) \geq x/(1+x)$ for all $x > -1$ and $e^x \geq 1+x$ for all x , we have

$$\left(\frac{JK - d + 1}{JK}\right)^d = e^{d \ln(1 - \frac{d-1}{JK})} \geq e^{\frac{-d(d-1)}{JK-d+1}} \geq 1 - \frac{d(d-1)}{JK-d+1} \geq 1 - \frac{d^2}{d^4 - d + 1}.$$

So we conclude that the probability of matching is lower bounded as follows for any $m \leq d$:

$$\mathbb{P}(M = m) \geq \frac{c^m}{m!} e^{-c} \left(1 - \frac{d^2}{d^4 - d + 1}\right).$$

Using the above, we get a lower bound on the density of the maximum

$$\begin{aligned}f_{\text{in}}^{(1)}(r) &= \sum_{m=1}^{JK} f_{\text{in}}^{(1)}(r|M = m) \mathbb{P}(M = m) \\ &= f(r) \sum_{m=1}^{JK} m F^{m-1}(r) \mathbb{P}(M = m) \\ &\geq f(r) \sum_{m=1}^d m F^{m-1}(r) \mathbb{P}(M = m) \\ &\geq \left(1 - \frac{d^2}{d^4 - d + 1}\right) (c f(r) e^{-c}) \sum_{m=1}^d F^{m-1}(r) \frac{c^{m-1}}{(m-1)!} \\ &= \left(1 - \frac{d^2}{d^4 - d + 1}\right) (c f(r) e^{-c}) \left(e^{cF(r)} - \sum_{m=d}^{\infty} \frac{(F(r)c)^m}{m!}\right).\end{aligned}$$

To conclude the proof, now we will bound the remainder $\sum_{m=d}^{\infty} \frac{(F(r)c)^m}{m!}$.

Let Z be a random variable with Poisson distribution with mean $cF(r)$ and let $t = \ln(\frac{d}{c\bar{F}(r)})$, then by Chernoff bound we get

$$\begin{aligned} \sum_{m=d}^{+\infty} \frac{(cF(r))^m}{m!} &= e^{cF(r)} \mathbb{P}\{Z \geq d\} \leq e^{cF(r)} e^{cF(r)(e^t-1)-td} \\ &= e^{cF(r)e^t - td} \\ &= e^{d - d \ln(\frac{d}{c\bar{F}(r)})} \\ &= \left(\frac{ecF(r)}{d}\right)^d \\ &\leq \left(\frac{ce}{d}\right)^d. \end{aligned}$$

We conclude that

$$\begin{aligned} f_{\text{in}}^{(1)}(r) &\geq \left(1 - \frac{d^2}{d^4 - d + 1}\right) (cf(r)e^{-c}) \left(e^{cF(r)} - \left(\frac{ce}{d}\right)^d\right) \\ &= cf(r)e^{-c\bar{F}(r)} - \left[cf(r)e^{-c}\left(\frac{ce}{d}\right)^d + \frac{d^2 cf(r)e^{-c} \left(e^{cF(r)} - \left(\frac{ce}{d}\right)^d\right)}{d^4 - d + 1}\right] \\ &\geq cf(r)e^{-c\bar{F}(r)} - \left[\left(\frac{ce}{d}\right)^d cf(r)e^{-c} + \frac{d^2}{d^4 - d + 1} cf(r)e^{-c\bar{F}(r)}\right] \\ &\geq cf(r)e^{-c\bar{F}(r)} - cf(r) \left[\left(\frac{ce}{d}\right)^d + \frac{d^2}{d^4 - d + 1}\right]. \end{aligned}$$

If we assume that $d \geq 2ce$ and $d \geq 2$, then we get that

$$\begin{aligned} \left(\frac{ce}{d}\right)^d &\leq \frac{(ce)^2}{d^2} \left(\frac{1}{2}\right)^{d-2} \leq \frac{(ce)^2}{d^2} \leq \frac{18c^2}{2d^2 - 1}, \\ \frac{d^2}{d^4 - d + 1} &\leq \frac{1}{d^2 - \frac{1}{d}} \leq \frac{2}{2d^2 - 1}. \end{aligned}$$

We conclude that

$$f_{\text{in}}^{(1)}(r) \geq cf(r)e^{-c\bar{F}(r)} - cf(r) \left(\frac{18c^2 + 2}{2d^2 - 1}\right).$$

This completes the proof. □

Proof of Proposition C.3-1

The goal of this proof is to bound the difference $r_{\text{co}} - r_{\text{in}}$ from both sides. In a first step, we will bound the difference from above then in a second step, we will bound it from below. In the whole proof, we assume that $\alpha < 1$ and $\alpha K < 1$.

Step 1. Upper bound $r_{\text{co}} - r_{\text{in}}$

Let

$$\phi_{\alpha,K}(v) := v - \frac{1 - (G(v))^K}{\alpha K f(v) (G(v))^{K-1}}.$$

r_{in} is the unique solution to $\phi_F(v) = v - (1 - F(v))/f(v)$. Note that for any $v > 0 \in \mathcal{S}$

$$\frac{1 - G^K(v)}{\alpha K f(v) G^{K-1}(v)} = \frac{1 - F(v)}{f(v)} \frac{\sum_{i=0}^{K-1} G^i(v)}{K G^{K-1}(v)} \geq \frac{1 - F(v)}{f(v)},$$

where the last inequality follows from the fact that $G(v) \leq 1$. On another hand, we have

$$\begin{aligned} \frac{1 - G^K(v)}{\alpha K f(v) G^{K-1}(v)} &= \frac{1 - F(v)}{f(v)} \frac{\sum_{i=0}^{K-1} G^i(v)}{K G^{K-1}(v)} \\ &\leq \frac{1 - F(v)}{f(v)} \frac{1}{G^{K-1}(v)} \leq \frac{1 - F(v)}{f(v)} \frac{1}{(1 - \alpha)^K}. \end{aligned}$$

We deduce that

$$\phi_F(v) + \left(1 - \frac{1}{(1 - \alpha)^K}\right) \frac{1 - F(v)}{f(v)} \leq \phi_{\alpha,K}(v) \leq \phi_F(v).$$

Both the right-hand-side and the left-hand-side are monotonically increasing by the IFR assumption on F . Let \tilde{r}_{in} denote the unique solution to

$$\tilde{\phi}(v) := \phi_F(v) + \left(1 - \frac{1}{(1 - \alpha)^K}\right) \frac{1 - F(v)}{f(v)} = 0.$$

We have that

$$r_{\text{in}} \leq r_{\text{co}} \leq \tilde{r}_{\text{in}}.$$

Furthermore, we have

$$\begin{aligned}
& 0 - \left(1 - \frac{1}{(1-\alpha)^K}\right) r_{\text{in}} = \tilde{\phi}(\tilde{r}_{\text{in}}) - \tilde{\phi}(r_{\text{in}}) \\
& = \tilde{r}_{\text{in}} - r_{\text{in}} - \frac{1}{(1-\alpha)^K} \left(\frac{1 - F(\tilde{r}_{\text{in}})}{f(\tilde{r}_{\text{in}})} - \frac{1 - F(r_{\text{in}})}{f(r_{\text{in}})} \right) \\
& \geq \tilde{r}_{\text{in}} - r_{\text{in}},
\end{aligned}$$

and hence

$$0 \leq r_{\text{co}} - r_{\text{in}} \leq \left(\frac{1}{(1-\alpha)^K} - 1 \right) r_{\text{in}}.$$

We have for all $\alpha < 1$,

$$(1-\alpha)^{-K} = e^{-K \ln(1-\alpha)} \stackrel{(a)}{\leq} e^{\frac{K\alpha}{1-\alpha}} \leq e^{\frac{K\alpha}{1-\alpha}} \stackrel{(b)}{=} 1 + \frac{K\alpha}{1-\alpha} e^{\tilde{x}} \leq 1 + \frac{K\alpha}{1-\alpha} e^{\frac{1}{1-\alpha}},$$

where in (a), we have used the fact that for all $x > -1$ we have $\frac{x}{1+x} \leq \ln(1+x)$ and $\alpha < 1$. While in (b), we have used the fact by the Taylor expansion of e^x around 0 at the point $\frac{K\alpha}{1-\alpha}$, there exists $0 \leq \tilde{x} \leq \frac{K\alpha}{1-\alpha} \leq \frac{1}{1-\alpha}$ -(because $\alpha K < 1$)- such that $e^{\frac{K\alpha}{1-\alpha}} = 1 + \frac{K\alpha}{1-\alpha} e^{\tilde{x}}$.

We conclude that that for $\alpha < 1$ and $K\alpha < 1$

$$0 \leq r_{\text{co}} - r_{\text{in}} \leq \frac{K\alpha}{1-\alpha} r_{\text{in}} e^{\frac{1}{1-\alpha}}. \quad (\text{C.4-14})$$

This concludes the first step.

Step 2. The goal of this step is to find a lower bound of order $K\alpha$ on the difference $r_{\text{co}} - r_{\text{in}}$. By definition, r_{co} is the unique solution to the following equation

$$r_{\text{co}} = \frac{1 - (G_\alpha(r_{\text{co}}))^K}{\alpha K f(v) (G_\alpha(r_{\text{co}}))^{K-1}}.$$

Note that for any $k \geq 0$, we have $G_\alpha^k(v) = \exp \{k \ln(1 - \alpha \bar{F}(v))\}$ and hence, using the fact that for all $0 \geq x > -1$, $\ln(1+x) \leq x$, we have

$$G_\alpha^k(v) \leq \exp \{-k\alpha \bar{F}(v)\}.$$

In turn, using the fact that for all $x \leq 0$, $\exp(x) \leq 1 + x + x^2/2$, we have

$$G_\alpha^k(v) \leq 1 - k\alpha\bar{F}(v) + (k\alpha)^2(\bar{F}(v))^2/2.$$

We hence have for $\alpha K < 1$,

$$\begin{aligned} r_{\text{co}} &\geq \frac{K\alpha\bar{F}(r_{\text{co}}) - K^2(\alpha\bar{F}(r_{\text{co}}))^2/2}{\alpha K f(r_{\text{co}}) [1 - (K-1)\alpha\bar{F}(r_{\text{co}}) + ((K-1)\alpha\bar{F}(r_{\text{co}}))^2/2]} \\ &= \frac{\bar{F}(r_{\text{co}}) - K\alpha(\bar{F}(r_{\text{co}}))^2/2}{f(r_{\text{co}}) [1 - (K-1)\alpha\bar{F}(r_{\text{co}}) + ((K-1)\alpha\bar{F}(r_{\text{co}}))^2/2]} \\ &\stackrel{(a)}{\geq} \frac{\bar{F}(r_{\text{co}}) - K\alpha(\bar{F}(r_{\text{co}}))^2/2}{f(r_{\text{co}})} \left[1 + \alpha(K-1)\bar{F}(r_{\text{co}}) - \frac{((K-1)\alpha\bar{F}(r_{\text{co}}))^2}{2} \right] \\ &= \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} [1 + \alpha(K-1)\bar{F}(r_{\text{co}})] - \frac{\alpha K(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} [1 + \alpha(K-1)\bar{F}(r_{\text{co}})] + R_1, \end{aligned}$$

where

$$R_1 = -\frac{((K-1)\alpha\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} [\bar{F}(r_{\text{co}}) - K\alpha(\bar{F}(r_{\text{co}}))^2/2].$$

The inequality (a) falls from the fact that $(K-1)\alpha\bar{F}(r_{\text{co}}) < 1$ and for all $0 \leq x \leq 1$, we have $x(1-x/2) \leq 1/2$ and for $x \leq \frac{1}{2}$, we have $\frac{1}{1-x} \geq 1+x$.

Using the fact that $\alpha \leq \alpha K < 1$ and $\bar{F}(r_{\text{co}}) \leq 1$ and the distribution is IFR then the remainder verifies $|R_1| \leq 2\frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})}(\alpha K)^2$.

So far we have shown that r_{co} verifies the following:

$$\begin{aligned} &r_{\text{co}} \\ &\geq \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} [1 + \alpha(K-1)\bar{F}(r_{\text{co}})] - \frac{\alpha K(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} [1 + \alpha(K-1)\bar{F}(r_{\text{co}})] - \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})}(\alpha K)^2 \\ &= \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} + \alpha K \frac{(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} - \alpha \frac{(\bar{F}(r_{\text{co}}))^2}{f(r_{\text{co}})} - \alpha^2 K(K-1) \frac{(\bar{F}(r_{\text{co}}))^3}{2(f(r_{\text{co}}))^2} - \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})}(\alpha K)^2 \\ &\geq \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} + \alpha K \frac{(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} - \alpha \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} - (K\alpha)^2 \frac{(\bar{F}(r_{\text{co}}))^2}{2(f(r_{\text{co}}))^2} - \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})}(\alpha K)^2 \\ &\geq \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} + \alpha K \frac{(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} - \alpha \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} - (K\alpha)^2 \frac{(\bar{F}(r_{\text{in}}))^2}{2(f(r_{\text{in}}))^2} - \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})}(\alpha K)^2, \end{aligned}$$

where in the last two inequalities, we have used the fact that $\bar{F}(r_{\text{co}}) \leq 1$ and the monotonicity of the hazard rate and that $r_{\text{co}} \geq r_{\text{in}}$.

Noting that $\bar{F}/f = v - \phi(v)$, we have, using a Taylor expansion that there exists $\tilde{r} \in [r_{\text{in}}, r_{\text{co}}]$ such that

$$\frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} = \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} + (1 - \phi'(r_{\text{in}}))(r_{\text{co}} - r_{\text{in}}) - (1/2)\phi''(\tilde{r})(r_{\text{co}} - r_{\text{in}})^2$$

There exists C_1 such that $\phi'' \leq C_1$ because f is twice continuously differentiable, and $f(r_{\text{in}}) > 0$, so $\phi(\cdot)$ is twice continuously differentiable so its second derivative is bounded around r_{in} . In turn, we get that

$$\frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} \geq \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} + (1 - \phi'(r_{\text{in}}))(r_{\text{co}} - r_{\text{in}}) - (1/2)C_1(r_{\text{co}} - r_{\text{in}})^2.$$

Similarly, we have for some other $\tilde{r} \in [r_{\text{in}}, r_{\text{co}}]$

$$\frac{(\bar{F}(r_{\text{co}}))^2}{f(r_{\text{co}})} = \frac{(\bar{F}(r_{\text{in}}))^2}{f(r_{\text{in}})} + (r_{\text{co}} - r_{\text{in}})H'(\tilde{r}) \geq \frac{(\bar{F}(r_{\text{in}}))^2}{f(r_{\text{in}})} - C_2(r_{\text{co}} - r_{\text{in}}),$$

where $H(r) = \frac{(\bar{F}(r))^2}{f(r)}$ and $|H'(\tilde{r})| \leq C_2$, such C_2 exists since, F is twice differentiable and $f(r_{\text{in}}) > 0$ so $H'(\cdot)$ is twice differentiable and bounded around r_{in} .

By combining the last two bounds, we get that r_{co} verifies the following:

$$\begin{aligned} r_{\text{co}} &\geq \frac{\bar{F}(r_{\text{co}})}{f(r_{\text{co}})} + \alpha K \frac{(\bar{F}(r_{\text{co}}))^2}{2f(r_{\text{co}})} - \alpha \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} - (K\alpha)^2 \frac{(\bar{F}(r_{\text{in}}))^2}{(f(r_{\text{in}}))^2} - \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} (\alpha K)^2 \\ &\geq \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} + (1 - \phi'(r_{\text{in}}))(r_{\text{co}} - r_{\text{in}}) + \alpha K \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})} + R_2 \\ &= \phi'(r_{\text{in}})r_{\text{in}} + r_{\text{co}}(1 - \phi'(r_{\text{in}})) + \alpha K \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})} + R_2, \end{aligned}$$

where

$$R_2 = -\frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} (\alpha K)^2 - \alpha K \frac{C_2}{2} (r_{\text{co}} - r_{\text{in}}) - (1/2)C_1 (r_{\text{co}} - r_{\text{in}})^2 - \alpha \frac{\bar{F}(r_{\text{in}})}{f(r_{\text{in}})} - (K\alpha)^2 \frac{(\bar{F}(r_{\text{in}}))^2}{(f(r_{\text{in}}))^2}.$$

Let us assume that $\alpha \leq 1/2$. If we denote $M_1 := \frac{r_{\text{in}}}{\phi'(r_{\text{in}})} \max(2, r_{\text{in}}, 2C_1 e^4, C_2 e^2)$ and using (C.4-14), we get that $|R_2| \leq (\alpha + 3(K\alpha)^2)M_1\phi'(r_{\text{in}})$. So we conclude that

$$\phi'(r_{\text{in}})r_{\text{co}} \geq \phi'(r_{\text{in}})r_{\text{in}} + \alpha K \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})} - (\alpha + 3(K\alpha)^2)M_1\phi'(r_{\text{in}}).$$

So

$$r_{\text{co}} \geq r_{\text{in}} + \alpha K \frac{(\bar{F}(r_{\text{in}}))^2}{2f(r_{\text{in}})\phi'(r_{\text{in}})} - (\alpha + 3(K\alpha)^2)M_1,$$

The result follows. \square

C.5 Illustration of the Online Display

Advertising Value Chain

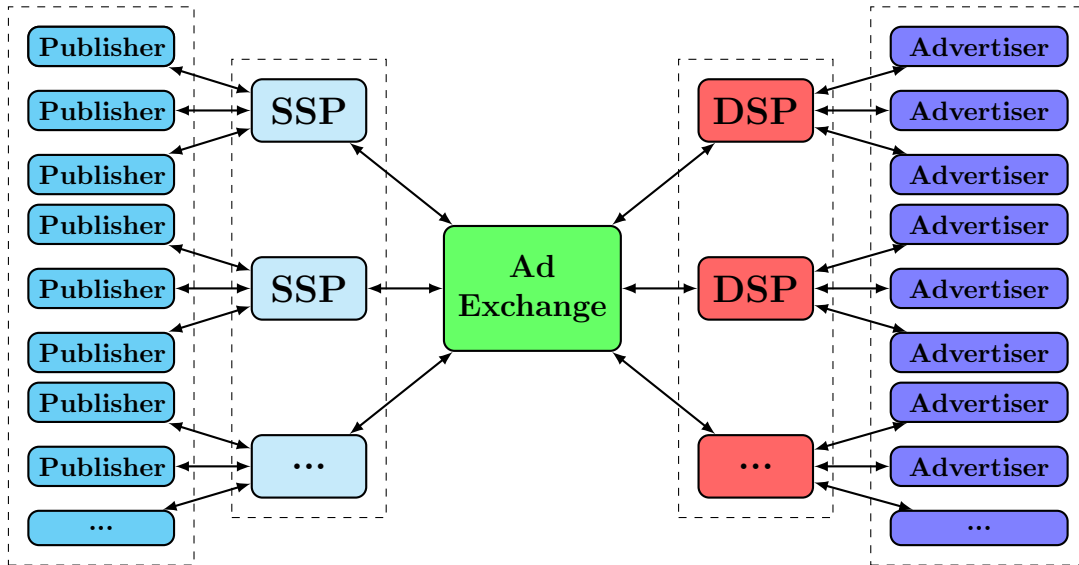


Figure C.5-1: High level overview of basic actors and communication links in the real time bidding market.

C.6 Generalized Pareto distributions

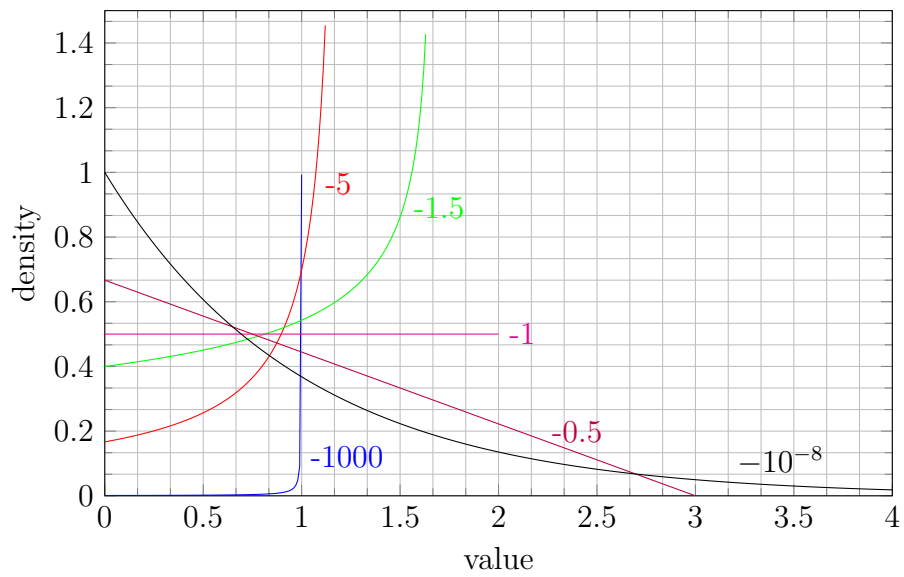


Figure C.6.1: The density function for generalized Pareto distributions for different parameters of ξ when $\sigma = 1 - \xi$.