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Skew Category Algebras

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Abstract We study a new (large) class of algebras (that was introduced in Bavula in Math Comput Sci 11(3–4):253–268, 2017)—the *skew category algebras*. Any such an algebra $\mathcal{C}(\sigma)$ is constructed from a category $\mathcal{C}$ and a functor σ from the category $\mathcal{C}$ to the category of algebras. Criteria are given for the algebra $\mathcal{C}(\sigma)$ to be simple or left Noetherian or right Noetherian or semiprime or have 1.

Keywords A skew category algebra · A simple algebra · A left Noetherian algebra · A semiprime algebra

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1 Skew Category Algebras, Examples and Constructions

In this paper, K is a commutative ring with 1, algebra means a K-algebra. In general, it is not assumed that a K-algebra has an identity element. Module means a left module. Missed definitions can be found in [1].

Let $\mathcal{C}$ be a category, $\mathrm{Ob}(\mathcal{C})$ be the set of its objects and $\mathrm{Mor}(\mathcal{C})$ be the set of its morphisms. For each objects $i, j \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(i, j)$ is the set of morphisms $f : i \to j$, the objects $i = t(f)$ and $j = h(f)$ are called the *tail* and *head* of the morphism f, respectively. For each object $i \in \mathrm{Ob}(\mathcal{C})$, e_i is the identity morphism $i \to i$.

Definition 1.1 ([2]) Let $\mathcal{C}$ be a category and σ be a functor from the category $\mathcal{C}$ to the category of unital K-algebras over a commutative ring K (eg, $K = \mathbb{Z}$ or K is a field). So, for each object $i \in \mathrm{Ob}(\mathcal{C})$, $D_i := \sigma(i)$ is a K-algebra and for each morphism

$$f : i \mapsto j, \quad \sigma_f : D_i \to D_j$$

is a K-algebra homomorphism, and $\sigma_{fg} = \sigma_f \sigma_g$ for all morphisms f and g such that $t(f) = h(g)$. The direct sum of left K-modules

$$\mathcal{C}(\sigma) = \bigoplus_{f \in \mathrm{Mor}(\mathcal{C})} D_{h(f)} f \tag{1}$$

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where $D_{h(f)}f$ is a free left $D_{h(f)}$-module of rank 1, is a K-algebra with multiplication given by the rule: For all $f, g \in \mathrm{Mor}(\mathcal{C})$, $a \in D_{h(f)}$ and $b \in D_{h(g)}$,

$$af \cdot bg = \begin{cases} a\sigma_f(b)fg & \text{if } t(f) = h(g), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is a trivial exercise to verify that the multiplication is associative. The K-algebra $\mathcal{C}(\sigma)$ is called a **skew category K-algebra**. If $K = \mathbb{Z}$, the $\mathbb{Z}$-algebra $\mathcal{C}(\sigma)$ is called a **skew category ring**.

Definition 1.2 If the direct sum (1) admits an associative product which is given by the rule: For all $f, g \in \mathrm{Mor}(\mathcal{C})$, $a \in D_{h(f)}$ and $b \in D_{h(g)}$,

$$af \cdot bg = \begin{cases} a\sigma_f(b)c(f,g)fg & \text{if } t(f) = h(g), \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where

$$c(f,g) \in \begin{cases} D_{h(f)} & \text{if } t(f) = h(g), \\ \{0\} & \text{otherwise,} \end{cases} \tag{4}$$

then it is called the **twisted skew category K-algebra** and is denoted by $\mathcal{C}(\sigma, c)$.

The categorical nature of the above classes of rings especially the categorical/explicit nature of their multiplications makes these classes important as far as various computational aspects are concerned.

Let 1_i be the identity of the algebra D_i. Then $1_i e_i \in D_i e_i \subseteq \mathcal{C}(\sigma)$ where $i \in \mathrm{Ob}(\mathcal{C})$. Abusing the notation, we write e_i for $1_i e_i$. Then $e_i \in \mathcal{C}(\sigma)$.

The $\mathcal{C}$-grading on $\mathcal{C}(\sigma)$. By the very definition, the algebra $\mathcal{C}(\sigma)$ is a $\mathcal{C}$-*graded algebra*, that is

$$D_{h(f)}f \cdot D_{h(g)}g \subseteq D_{h(fg)}fg \quad \text{for all} \quad f, g \in \mathrm{Mor}(\mathcal{C}).$$

The algebra $\mathcal{C}(\sigma)$ is a direct sum

$$\mathcal{C}(\sigma) = \bigoplus_{i,j \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{ij} \quad \text{where} \quad \mathcal{C}(\sigma)_{ij} = \bigoplus_{f \in \mathcal{C}(j,i)} D_i f \tag{5}$$

and for all $i, j, k, l \in \mathrm{Ob}(\mathcal{C})$,

$$\mathcal{C}(\sigma)_{ij}\mathcal{C}(\sigma)_{kl} \subseteq \delta_{jk}\mathcal{C}(\sigma)_{il} \tag{6}$$

where δ_{jk} is the Kronecker delta. In particular, for each $i \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(\sigma)_{ii}$ is a K-algebra without 1, in general. For each $i, j \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(\sigma)_{ij}$ is a $(\mathcal{C}(\sigma)_{ii}, \mathcal{C}(\sigma)_{jj})$-bimodule.

The next two examples show that even for two simplest categories that contain a single object, a single loop or a single invertible loop, the above construction gives apart from a skew polynomial ring or a skew Laurent polynomial ring, new classes of rings.

Example 1 Let $\mathcal{C}$ be a category that contains a single object, say 1, and $\mathrm{Mor}(\mathcal{C}) = \{x^i \mid i \in \mathbb{N}\}$ where $e := x^0$ is the identity morphism. So, $\mathcal{C}(\sigma) = De \oplus Dx \oplus \cdots \oplus Dx^i \oplus \cdots$ where $D = \sigma(1)$ and $ed = \sigma_e(d)e$ and $x^i d = \sigma_x^i(d)x^i$ for all $i \geq 1$ where σ_e and σ_x are ring endomorphisms of D such that $\sigma_e\sigma_x = \sigma_x\sigma_e = \sigma_x$ and $\sigma_e^2 = \sigma_e$.

- If $\sigma_e = \mathrm{id}_D$ then $\mathcal{C}(\sigma) = D[x; \sigma_x]$ is a *skew polynomial ring*.
- If $\sigma_e \neq \mathrm{id}_D$ then $\mathcal{C}(\sigma)$ is *not* a skew polynomial ring since $ed = \sigma_e(d)e$ and, in general, $\sigma_e(d)e \neq de$ for all $d \in D$ (since $\sigma_e \neq \mathrm{id}_D$). For example, let $D = D_1 \times D_2 \times D_3$ and σ_e and σ_x are the projections onto $D_1 \times D_2$ and D_1, respectively. Then $eD_3 = 0$.

Example 2 Let $\mathcal{C}$ be a category that contains a single object, say 1, and $\mathrm{Mor}(\mathcal{C}) = \{x^i \mid i \in \mathbb{Z}\}$ where $e := x^0$ is the identity morphism ($xx^{-1} = x^{-1}x = e$). The functor σ is determined by the algebra $D = \sigma(1)$ and its algebra endomorphisms σ_e, σ_x and $\sigma_{x^{-1}}$ such that

$$\sigma_e^2 = \sigma_e, \quad \sigma_e\sigma_{x^{\pm 1}} = \sigma_{x^{\pm 1}}\sigma_e = \sigma_{x^{\pm 1}} \quad \text{and} \quad \sigma_x\sigma_{x^{-1}} = \sigma_{x^{-1}}\sigma_x = \sigma_e.$$

Then $\mathcal{C}(\sigma) = \oplus_{i \in \mathbb{Z}} Dx^i$.

- If $\sigma_e = \mathrm{id}_D$ then $\sigma_{x^{-1}} = \sigma_x^{-1}$ and $\mathcal{C}(\sigma) = D[x^{\pm 1}; \sigma_x]$ is a skew Laurent polynomial ring.
- If $\sigma_e \neq \mathrm{id}_D$ then $\mathcal{C}(\sigma)$ is *not* a skew Laurent polynomial ring. For example, let $D = D_1 \times D_2$ be a direct product of algebras and $\sigma_e = \sigma_x = \sigma_{x^{-1}}$ be the projection onto D_1. Then $eD_2 = 0$ and $xD_2 = x^{-1}D_2 = 0$.

Example 3 Let $\mathcal{C}$ be a category that contains a single object, say 1, and the monoid $\mathcal{C}(1, 1)$ is generated by elements x and y subject to the defining relation $yx = e$. The functor σ is determined by the algebra $D = \sigma(1)$ and its three algebra endomorphisms σ_x, σ_y and σ_e such that

$$\sigma_y \sigma_x = \sigma_e.$$

The skew category algebra $\mathcal{C}(\sigma)$ is called the **skew semi-Laurent polynomial ring** [2]. It is a new class of rings. Suppose, for simplicity, that $\sigma_e = \mathrm{id}_D$. Then the ring $\mathcal{C}(\sigma)$ is generated by a ring D and elements x and y subject to the defining relations:

$$yx = 1, \quad xd = \sigma_x(d)x \quad \text{and} \quad yd = \sigma_y(d)y \quad \text{for all} \quad d \in D.$$

We denote this ring by $D[x, y; \sigma_x, \sigma_y]$. In particular, $D[x, y; \tau, \tau^{-1}]$ where τ is an automorphism of D.

Example 4 Let $n \geq 1$ be a natural number and $\mathcal{M}_n$ be the **matrix units category**:

$$\mathrm{Ob}(\mathcal{M}_n) = \{1, \ldots, n\}, \quad \mathcal{M}_n(i, j) = \{E_{ji}\} \quad \text{and} \quad E_{ij}E_{jk} = E_{ik} \quad \text{for all} \quad i, j, k.$$

Let D be a ring and $f_1, \ldots, f_n$ be its automorphisms. Define σ by the rule $\sigma(i) = D$ and $\sigma(E_{ij}) = f_i f_j^{-1}$. The skew category algebra

$$\mathcal{M}_n(\sigma) = \oplus_{i,j=1}^n DE_{ij}$$

is called the **skew matrix ring** where the multiplication is given by the rule

$$dE_{ij} \cdot d'E_{kl} = \delta_{jk} d f_i f_j^{-1}(d')E_{jl} \quad \text{for all} \quad d, d' \in D.$$

The skew graph rings and the skew tree rings.

Definition 1.3 ([2]) Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a non-oriented graph without cycles where Γ_0 is the set of vertices and Γ_1 is the set of edges. If, in addition, Γ is connected then it is called a *tree*. So, any non-oriented graph without cycles is a disjoint union of its connected components which are trees. Let $\mathbf{\Gamma}$ be the category groupoid associated with Γ: $\mathrm{Ob}(\mathbf{\Gamma}) = \Gamma_0$, for each $i \in \mathrm{Ob}(\Gamma)$, $\mathbf{\Gamma}(i, i) = \{e_{ii}\}$, for distinct $i, j \in \mathrm{Ob}(\mathbf{\Gamma})$ such that $(i, j) \in \Gamma_1$, $\mathbf{\Gamma}(i, j) = \{e_{ji}\}$ and $\mathbf{\Gamma}(j, i) = \{e_{ij}\}$, $e_{ij}e_{ji} = e_{ii}$ and $e_{ji}e_{ij} = e_{jj}$. Let σ be a functor from $\mathbf{\Gamma}$ to the category of rings. Then $\mathbf{\Gamma}(\sigma)$ is called the **skew graph ring**. If Γ is a tree then $\mathbf{\Gamma}(\sigma)$ is called the **skew tree ring**. We say that the functor σ is of *isomorphism type* if $\sigma(e_{ij}) : \sigma(i) \to \sigma(j)$ is a unital ring isomorphism for all $(i, j) \in \Gamma_1$.

Theorem 1.4 *Let Γ be a finite tree, $n = |\Gamma_0|$ and the functor σ be of isomorphism type. Suppose that for some $i \in \Gamma_0$ the ring $D_i = \sigma(i)$ is a semiprime, left (resp., right) Goldie ring and $Q_l(D_i)$ (resp., $Q_r(D_i)$) is its left (resp., right) quotient ring. Then $\Gamma(\sigma)$ is a semiprime, left (resp., right) Goldie ring and $Q_l(\Gamma(\sigma)) \simeq M_n(Q_l(D_i))$ (resp., $Q_r(\Gamma(\sigma)) \simeq M_n(Q_r(D_i))$) where $M_n(R)$ is a matrix ring over a ring R. In particular, the left (resp., right) uniform dimension of $\mathbf{\Gamma}(\sigma)$ is nd_l (resp., nd_r) where d_l (resp., d_r) is a left (resp., right) uniform dimension of D_i.*

Proof (Sketch). Let $\mathcal{C}_{D_j}$ be the set of regular elements of the ring $D_j = \sigma(j)$. All the rings D_j are isomorphic. The set of regular elements $S = \oplus_{j=1}^n \mathcal{C}_{D_j} e_{jj}$ is a left Ore set of $\Gamma(\sigma)$ such that $S^{-1}\Gamma(\sigma)$ is a semisimple Artinian ring. Furthermore, $S^{-1}\Gamma(\sigma) \simeq M_n(Q_l(D_i))$. Hence, $Q_l(\Gamma(\sigma)) \simeq M_n(Q_l(D_i))$, and so $\Gamma(\sigma)$ is a semiprime, left Goldie ring. The rest is obvious. □

As a result we have the following corollary.

Corollary 1.5 *Let Γ be a finite non-orientable graph, i.e., $\Gamma = \coprod_{s=1}^{\nu} \Gamma^{(s)}$ is a disjoint union of finite trees $\Gamma^{(s)}$. Then*

1. *The skew graph ring* $\Gamma(\sigma)$ *is a direct product* $\prod_{s=1}^{\nu}\Gamma^{(s)}(\sigma_s)$ *of skew tree rings where* σ_s *is the restriction of the functor* σ *to* $\Gamma^{(s)}(\sigma_s)$.
2. *If the trees* $\Gamma^{(s)}$ $(s=1,\dots,\nu)$ *satisfy the conditions of Theorem 1.4 then* $Q_l(\Gamma(\sigma))\simeq\prod_{s=1}^{\nu}Q_l(\Gamma^{(s)}(\sigma_s))$ *(resp.,* $Q_r(\Gamma(\sigma))\simeq\prod_{s=1}^{\nu}Q_r(\Gamma^{(s)}(\sigma_s))$*) is a direct product of semiprime, left (resp., right) Goldie rings, and so it is a semiprime, left (resp., right) Goldie ring.*

2 Properties of Skew Category Algebras

In this section, criteria are given for a skew category algebra $\mathcal{C}(\sigma)$ to be left/right Noetherian or semiprime or simple.

The ideal $\mathfrak{a}$ and the algebra $\overline{\mathcal{C}(\sigma)}$.

Lemma 2.1 *Let D be a ring and σ' be its ring endomorphism such that $\sigma'^2=\sigma'$. Then $D=\sigma'(D)\oplus\ker(\sigma')$ and the restriction homomorphism $\sigma'|_{\sigma'(D)}:\sigma'(D)\to\sigma'(D)$, $d\mapsto d$ is the identity automorphism.*

Proof Straightforward. □

By (5), the formal sum

$$e=\sum_{i\in \mathrm{Ob}(\mathcal{C})}e_i$$

determines two well-defined maps:

$$e\cdot:\mathcal{C}(\sigma)\to\mathcal{C}(\sigma),\quad a\mapsto ea\quad\text{and}\quad \cdot e:\mathcal{C}(\sigma)\to\mathcal{C}(\sigma),\quad a\mapsto ae.$$

Clearly, the map $\cdot e$ is the identity map id on $\mathcal{C}(\sigma)$ but the kernel $\mathfrak{a}$ of the map $e\cdot$ is equal to

$$\mathfrak{a}(\mathcal{C}(\sigma)):=\mathfrak{a}:=\bigoplus_{f\in \mathrm{Mor}(\mathcal{C})}\mathfrak{a}_{h(f)}f$$

where $\mathfrak{a}_i:=\ker(\sigma_{e_i})$ and $\sigma_i:=\sigma_{e_i}:D_i\to D_i$ is a K-algebra endomorphism, and $(e\cdot)^2=e\cdot$. Since $\sigma_i^2=\sigma_i$,

$$D_i=\sigma_i(D)\oplus\mathfrak{a}_i\quad\text{for all}\quad i\in \mathrm{Ob}(\mathcal{C}),\tag{7}$$

by Lemma 2.1.

$$\mathcal{C}(\sigma)=\overline{\mathcal{C}(\sigma)}\oplus\mathfrak{a}\quad\text{where}\quad\overline{\mathcal{C}(\sigma)}:=\bigoplus_{f\in \mathrm{Mor}(\mathcal{C})}\sigma_{h(f)}(D_{h(f)})f\tag{8}$$

is a K-subalgebra of $\mathcal{C}(\sigma)$ such that the maps $(e\cdot)|_{\overline{\mathcal{C}(\sigma)}}:\overline{\mathcal{C}(\sigma)}\to\overline{\mathcal{C}(\sigma)}$, $c\mapsto c$ and $(\cdot e)|_{\overline{\mathcal{C}(\sigma)}}:\overline{\mathcal{C}(\sigma)}\to\overline{\mathcal{C}(\sigma)}$, $c\mapsto c$ are the identity map on $\overline{\mathcal{C}(\sigma)}$.

Lemma 2.2 *The set $\mathfrak{a}$ is an ideal of the algebra $\mathcal{C}(\sigma)$ such that $\mathcal{C}(\sigma)\mathfrak{a}=0$, $\mathfrak{a}\mathcal{C}(\sigma)=\mathfrak{a}$ and $\mathfrak{a}^2=0$.*

Proof $\mathcal{C}(\sigma)\mathfrak{a}=\mathcal{C}(\sigma)\cdot e\cdot\mathfrak{a}=0$, the rest is obvious. □

The next theorem shows that the algebra $\overline{\mathcal{C}(\sigma)}$ is also a skew category algebra.

Theorem 2.3 1. *The subalgebra $\overline{\mathcal{C}(\sigma)}$ of $\mathcal{C}(\sigma)$ is also a skew category algebra $\overline{\mathcal{C}(\sigma)}=\mathcal{C}(\overline{\sigma})$ where for each $i\in \mathrm{Ob}(\mathcal{C})$, $\overline{\sigma}(i):=\sigma_i(D_i)$ and for each $f\in\mathcal{C}(i,j)$, $\overline{\sigma}_f:=\sigma_f|_{\sigma_i(D_i)}:\sigma_i(D_i)\to\sigma_i(D_i)$, $d\mapsto\sigma_f(d)$.*
2. *For all $i\in \mathrm{Ob}(\mathcal{C})$, $\overline{\sigma}_i=\mathrm{id}_{\overline{\sigma}(i)}$.*
3. $\mathfrak{a}(\mathcal{C}(\overline{\sigma}))=0$.
4. *The maps $e\cdot$ and $\cdot e$ are the identity maps on $\mathcal{C}(\overline{\sigma})$.*

Proof 1. Statement 1 follows from (8) and the fact that $\sigma_j\sigma_f=\sigma_f=\sigma_f\sigma_i$ for all elements $f\in\mathcal{C}(i,j)$.
2–4. Statement 2 is obvious. Then statements 3 and 4 follow from statement 2. □

The ideal $\mathfrak{a}$ is a $\mathcal{C}$-graded ideal of the algebra $\mathcal{C}(\sigma)$. Furthermore,

$$\mathfrak{a} = \bigoplus_{i,j\in \mathrm{Ob}(\mathcal{C})} \mathfrak{a}_{ij}$$

where $\mathfrak{a}_{ij} = \bigoplus_{f\in\mathcal{C}(j,i)} \mathfrak{a}_i f \subseteq \mathcal{C}(\sigma)_{ij}$, $0 = \mathfrak{a}_{ij}\mathfrak{a}_{kl} \subseteq \delta_{jk}\mathfrak{a}_{il}$ for all $i, j, k, l \in \mathrm{Ob}(\mathcal{C})$. Since $\overline{\mathcal{C}(\sigma)} = \mathcal{C}(\overline{\sigma})$ (Theorem 2.3.(1)), the factor algebra

$$\overline{\mathcal{C}(\sigma)} = \mathcal{C}(\sigma)/\mathfrak{a} = \bigoplus_{f\in \mathrm{Mor}(\mathcal{C})} \overline{D}_{h(f)} f \subseteq \mathcal{C}(\sigma)$$

is a $\mathcal{C}$-graded algebra where $\overline{D}_i = D_i/\mathfrak{a}_i = \mathrm{im}(\sigma_i)$. Furthermore,

$$\overline{\mathcal{C}(\sigma)} = \bigoplus_{\mathbf{i},j\in \mathrm{Ob}(\mathcal{C})} \overline{\mathcal{C}(\sigma)}_{ij} \quad \text{where} \quad \overline{\mathcal{C}(\sigma)}_{ij} = \mathcal{C}(\sigma)_{ij}/\mathfrak{a}_{ij} \tag{9}$$

and $\overline{\mathcal{C}(\sigma)}_{ij}\overline{\mathcal{C}(\sigma)}_{kl} \subseteq \delta_{jk}\overline{\mathcal{C}(\sigma)}_{il}$ for all $i, j, k, l \in \mathrm{Ob}(\mathcal{C})$.

Theorem 2.4 (Criterion for $\mathcal{C}(\sigma)$ to be a left Noetherian algebra) *The algebra $\mathcal{C}(\sigma)$ is a left Noetherian algebra iff the following conditions hold*

1. *the set* $\mathrm{Ob}(\mathcal{C})$ *is a finite set,*
2. *the ideal* $\mathfrak{a}$ *is a finitely generated abelian group,*
3. *for every object $i \in \mathrm{Ob}(\mathcal{C})$, the K-algebra $\overline{\mathcal{C}(\sigma)}_{ii}$ is a left Noetherian algebra, and*
4. *for all objects $i, j \in \mathrm{Ob}(\mathcal{C})$ such that $i \neq j$, the left $\overline{\mathcal{C}(\sigma)}_{ii}$-module $\overline{\mathcal{C}(\sigma)}_{ij}$ is finitely generated.*

Proof The algebra $\mathcal{C}(\sigma) = \bigoplus_{j\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{*j}$ is a direct sum of nonzero left ideals where

$$\mathcal{C}(\sigma)_{*j} := \bigoplus_{i\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{ij}.$$

So, the algebra $\mathcal{C}(\sigma)$ is a left Noetherian algebra iff the set $\mathrm{Ob}(\mathcal{C})$ is a finite set and all the left ideals $\mathcal{C}(\sigma)_{*j}$ are Noetherian left $\mathcal{C}(\sigma)$-modules iff $|\mathrm{Ob}(\mathcal{C})| < \infty$, the left $\mathcal{C}(\sigma)$-module $\mathfrak{a}$ is Noetherian and all the left $\overline{\mathcal{C}(\sigma)}$-modules

$$\overline{\mathcal{C}(\sigma)}_{*j} = \bigoplus_{i\in \mathrm{Ob}(\mathcal{C})} \overline{\mathcal{C}(\sigma)}_{ij}$$

are Noetherian (since $\mathcal{C}(\sigma) = \overline{\mathcal{C}(\sigma)} \oplus \mathfrak{a}$ is a direct sum of left $\mathcal{C}(\sigma)$-modules) iff conditions 1 and 2 hold (since $\mathcal{C}(\sigma)\mathfrak{a} = 0$, Lemma 2.4) and the left $\overline{\mathcal{C}(\sigma)}_{ii}$-module $\overline{\mathcal{C}(\sigma)}_{ij}$ is Noetherian for all $i, j \in \mathrm{Ob}(\mathcal{C})$ (since each left $\overline{\mathcal{C}(\sigma)}$-submodule M of $\overline{\mathcal{C}(\sigma)}_{*j}$ is a direct sum

$$M = eM = \bigoplus_{i\in \mathrm{Ob}(\mathcal{C})} e_i M$$

where $e_i M$ is a left $\overline{\mathcal{C}(\sigma)}_{ii}$-submodule of $\overline{\mathcal{C}(\sigma)}_{ij}$ and the functor from the category of all $\overline{\mathcal{C}(\sigma)}_{ii}$-submodules of $\overline{\mathcal{C}(\sigma)}_{ij}$ to the category of all $\overline{\mathcal{C}(\sigma)}$-submodules of $\overline{\mathcal{C}(\sigma)}_{*j}$,

$$N \to \overline{\mathcal{C}(\sigma)}N = \bigoplus_{k\in \mathrm{Ob}(\mathcal{C})} \overline{\mathcal{C}(\sigma)}_{ki} N$$

is faithful since $e_i\overline{\mathcal{C}(\sigma)}N = \overline{\mathcal{C}(\sigma)}_{ii}N = N$) iff statements 1–4 hold. □

Proposition 2.5 (Criterion for $\mathcal{C}(\sigma)$ to be a right Noetherian algebra) *The algebra $\mathcal{C}(\sigma)$ is a right Noetherian algebra iff the following conditions hold*

1. *the set* $\mathrm{Ob}(\mathcal{C})$ *is a finite set,*
2. *for every object $i \in \mathrm{Ob}(\mathcal{C})$, the K-algebra $\mathcal{C}(\sigma)_{ii}$ is a right Noetherian algebra, and*

3. *for all objects* $i, j \in \mathrm{Ob}(\mathcal{C})$ *such that* $i \neq j$, *the right* $\mathcal{C}(\sigma)_{jj}$*-module* $\mathcal{C}(\sigma)_{ij}$ *is finitely generated.*

Proof The algebra $\mathcal{C}(\sigma) = \bigoplus_{i\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{i*}$ is a direct sum of nonzero right ideals where

$$\mathcal{C}(\sigma)_{i*} = \bigoplus_{j\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{ij}.$$

So, the algebra $\mathcal{C}(\sigma)$ is a right Noetherian algebra iff the set $\mathrm{Ob}(\mathcal{C})$ is a finite set and all right ideals $\mathcal{C}(\sigma)_{i*}$ are Noetherian right $\mathcal{C}(\sigma)$-modules iff $|\mathrm{Ob}(\mathcal{C})| < \infty$ and the right $\mathcal{C}(\sigma)_{jj}$-module $\mathcal{C}(\sigma)_{ij}$ is Noetherian for all $i, j \in \mathrm{Ob}(\mathcal{C})$ iff $|\mathrm{Ob}(\mathcal{C})| < \infty$, the rings $\mathcal{C}(\sigma)_{ii}$ are right Noetherian and the right $\mathcal{C}(\sigma)_{jj}$-modules $\mathcal{C}(\sigma)_{ij}$ are finitely generated for all $i \neq j$. □

Example 5 Let $\mathcal{C}$: $1 \xrightarrow{f} 2$ and the functor σ is as follows: $\sigma(1) = \mathbb{Q}$, $\sigma(2) = \mathbb{R}$, $\sigma_{e_1} = \mathrm{id}_\mathbb{Q}$, $\sigma_{e_2} = \mathrm{id}_\mathbb{R}$ and $\sigma_f : \mathbb{Q} \to \mathbb{R}$, $q \mapsto q$. Then the algebra $\mathcal{C}(\sigma)$ is isomorphic to the lower triangular matrix algebra $\begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$. By Theorem 2.4, the algebra $\mathcal{C}(\sigma)$ is left Noetherian but not right Noetherian, by Proposition 2.5 (since $\mathbb{R}_\mathbb{Q}$ is not a finitely generated right $\mathbb{Q}$-module).

Example 6 Let $\mathcal{C}$: $1 \xrightarrow{f} 2$ and the functor σ is as follows: $\sigma(1) = K[t]$ is a polynomial algebra in the variable t over K, $\sigma(2) = K$, $\sigma_{e_1} : K[t] \to K[t]$, $t \mapsto 0$; $\sigma_{e_2} = \mathrm{id}_K : K \to K$ and $\sigma_f : K[t] \to K$, $t \mapsto 0$. Then $\mathfrak{a} = tK[t]e_1$ is not a finitely generated $\mathbb{Z}$-module. So, the algebra $\mathcal{C}(\sigma)$ is not a left Noetherian algebra, by Theorem 2.4. Since the algebra $\mathcal{C}(\sigma)_{11} = K[t]e_1$ is not a right Noetherian algebra, the ring $\mathcal{C}(\sigma)$ is not a right Noetherian ring, by Proposition 2.5.

Lemma 2.6 (Existence of 1 in $\mathcal{C}(\sigma)$) *The algebra* $\mathcal{C}(\sigma)$ *has 1 iff the set* $\mathrm{Ob}(\mathcal{C})$ *is a finite set and* $\sigma_{e_i} = \mathrm{id}_{D_i}$ *for all* $i \in \mathrm{Ob}(\mathcal{C})$. *In this case,* $e = \sum_{i\in \mathrm{Ob}(\mathcal{C})} e_i$ *is the identity of the algebra* $\mathcal{C}(\sigma)$.

Proof ($\Rightarrow$) Suppose that 1 is an identity of $\mathcal{C}(\sigma)$. Then necessarily the set $\mathrm{Ob}(\mathcal{C})$ is a finite set, otherwise $1a = 0$ for some nonzero element a of $\mathcal{C}(\sigma)$. The $1 = \sum_{i,j} 1_{ij}$ where $1_{ij} \in \mathcal{C}(\sigma)_{ij}$. The equalities $1e_j = e_j = e_j1$ for all $j \in \mathrm{Ob}(\mathcal{C})$ imply that $1 = \sum_{i\in \mathrm{Ob}(\mathcal{C})} e_i = e$. Then, necessarily $\sigma_{e_i} = \mathrm{id}_{D_i}$ for all $i \in \mathrm{Ob}(\mathcal{C})$.

($\Leftarrow$) Clearly, e is the identity of the algebra $\mathcal{C}(\sigma)$. □

Lemma 2.7 *Suppose that* $n = |\mathrm{Ob}(\mathcal{C})| < \infty$. *If* I *is an ideal of* $\mathcal{C}(\sigma)$ *such that* $e_iIe_i = 0$ *for all* $i \in \mathrm{Ob}(\mathcal{C})$ *then* $I^{n+1} = 0$.

Proof By (8), $\mathcal{C}(\sigma) = \overline{\mathcal{C}(\sigma)} \oplus \mathfrak{a}$. Hence, $I \subseteq \overline{I} \oplus \mathfrak{a}$ where $\overline{I} = (I + \mathfrak{a})/\mathfrak{a} = \sum_{i,j\in \mathrm{Ob}(\mathcal{C})} e_iIe_j \subseteq \overline{\mathcal{C}(\sigma)}$. Notice that $\overline{I}^n = 0$ since $e_iIe_i = 0$ for all $i \in \mathrm{Ob}(\mathcal{C})$. Now,

$$I^{n+1} \subseteq (\overline{I} + \mathfrak{a})^{n+1} \subseteq \overline{I}^{n+1} + \mathfrak{a}\overline{I}^n = 0$$

since $\mathfrak{a}^2 = 0$ and $\mathcal{C}(\sigma)\mathfrak{a} = 0$ (Lemma 2.2). □

Recall that a ring is a *semiprime ring* if the zero ideal is the only nilpotent ideal.

Theorem 2.8 (Criterion for $\mathcal{C}(\sigma)$ to be a semiprime algebra) *Suppose that* $n := |\mathrm{Ob}(\mathcal{C})| < \infty$. *Then the following statements are equivalent.*

1. *The algebra* $\mathcal{C}(\sigma)$ *is a semiprime algebra.*
2. *The algebras* $\mathcal{C}(\sigma)_{ii}$ *are semiprime where* $i \in \mathrm{Ob}(\mathcal{C})$ *and, for all distinct* $i, j \in \mathrm{Ob}(\mathcal{C})$, $a_{ij}\mathcal{C}(\sigma)_{ji} \neq 0$ *and* $\mathcal{C}(\sigma)_{ji}a_{ij} \neq 0$ *for all nonzero elements* $a_{ij} \in \mathcal{C}(\sigma)_{ij}$.
3. *The algebras* $\mathcal{C}(\sigma)_{ii}$ *are semiprime where* $i \in \mathrm{Ob}(\mathcal{C})$ *and each ideal* I *of* $\mathcal{C}(\sigma)$ *such that* $e_iIe_i = 0$ *for all* $i \in \mathrm{Ob}(\mathcal{C})$ *is equal to zero.*

Proof Since $|\mathrm{Ob}(\mathcal{C})| < \infty$, the direct product of algebras $\mathcal{D} := \prod_{i\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(\sigma)_{ii}$ is a semiprime algebra iff all the algebras $\mathcal{C}(\sigma)_{ii}$ are semiprime.

(1 $\Rightarrow$ 2) If $\mathfrak{b}$ is a nonzero nilpotent ideal of the ring $\mathcal{D}$ and $(\mathfrak{b}) = \mathcal{C}(\sigma)\mathfrak{b}\mathcal{C}(\sigma)$ is the ideal of $\mathcal{C}(\sigma)$ generated by $\mathfrak{b}$ then

$$(\mathfrak{b})^k \subseteq (\mathfrak{b}^{\lfloor \frac{k}{n^2} \rfloor}) \quad \text{for all} \quad k \geq 1$$

where for a real number r, $\lfloor r \rfloor := \max\{z \in \mathbb{Z} \,|\, z \leq r\}$, and so the ideal $(\mathfrak{b})$ of the algebra $\mathcal{D}$ is a nilpotent ideal. Therefore, the ring $\mathcal{C}(\sigma)_{ii}$ must be semiprime for all $i \in \mathrm{Ob}(\mathcal{C})$.

Suppose that there exists a nonzero element $a_{ij} \in \mathcal{C}(\sigma)_{ij}$ for some distinct objects i and j such that either $a_{ij}\mathcal{C}(\sigma)_{ji} = 0$ or $\mathcal{C}(\sigma)_{ji}a_{ij} = 0$. Then $(a_{ij})^2 = (a_{ij}\mathcal{C}(\sigma)_{ji}a_{ij}) = 0$, a contradiction.

(2 $\Rightarrow$ 1) Since all rings $\mathcal{C}(\sigma)_{ii}$ are semiprime, the ideal $\mathfrak{a}$ is equal to zero, by Lemma 2.2. Therefore, if J is a nilpotent ideal of $\mathcal{C}(\sigma)$ then necessarily $J = \bigoplus_{i,j\in \mathrm{Ob}(\mathcal{C})} J_{ij}$ where $J_{ij} = e_i J e_j$. Furthermore, all $J_{ii} = 0$ since the rings $\mathcal{C}(\sigma)_{ii}$ are semiprime (and $J_{ii}^m \subseteq J^m$ for all $m \geq 1$). Suppose that $J \neq 0$. We seek a contradiction. Then $J_{ij} \neq 0$ for some $i \neq j$. Then, by the assumption, either $\mathcal{C}(\sigma)_{ji}J_{ij}$ is a nonzero nilpotent ideal of the algebra $\mathcal{C}(\sigma)_{jj}$ or $J_{ij}\mathcal{C}(\sigma)_{ji}$ is a nonzero nilpotent ideal of the algebra $\mathcal{C}(\sigma)_{ii}$, a contradiction.

(1 $\Rightarrow$ 3) The algebras $\mathcal{C}(\sigma)_{ii}$ are semiprime for all $i \in \mathrm{Ob}(\mathcal{C})$, by the implication (1 $\Rightarrow$ 2). By Lemma 2.7, each ideal I of $\mathcal{C}(\sigma)$ such that $e_i I e_i = 0$ for all $i \in \mathrm{Ob}(\mathcal{C})$ is a nilpotent ideal, so it must be zero (since $\mathcal{C}(\sigma)$ is a semiprime ring).

(3 $\Rightarrow$ 1) If I is a nilpotent ideal of $\mathcal{C}(\sigma)$ then for each $i \in \mathrm{Ob}(\mathcal{C})$, I_{ii} is a nilpotent ideals of the semiprime ring $\mathcal{C}(\sigma)_{ii}$, and so $I_{ii} = 0$. Then, we must have $I = 0$, by the second assumption of statement 3. □

Theorem 2.9 (Simplicity criterion for $\mathcal{C}(\sigma)$) *The algebra $\mathcal{C}(\sigma)$ is a simple algebra iff the following conditions hold*

1. $\mathfrak{a} = 0$,
2. *for every* $i \in \mathrm{Ob}(\mathcal{C})$, *the ring* $\mathcal{C}(\sigma)_{ii}$ *is simple,*
3. *for all distinct* $i, j \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(\sigma)_{ij}$ *is a simple* $(\mathcal{C}(\sigma)_{ii}, \mathcal{C}(\sigma)_{jj})$*-bimodule (in particular,* $\mathcal{C}(\sigma)_{ij} \neq 0$*), and*
4. $\mathcal{C}(\sigma)_{ij}\mathcal{C}(\sigma)_{jk} \neq 0$ *for all* $i, j, k \in \mathrm{Ob}(\mathcal{C})$.

Proof ($\Rightarrow$) Let $\mathcal{C}_{ij} = \mathcal{C}(\sigma)_{ij}$.

- (i) $\mathfrak{a} = 0$, by Lemma 2.2.
- (ii) *For every* $i \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}_{ii}$ *is a simple ring*: Suppose that $\mathfrak{b}$ is a proper ideal of the ring $\mathcal{C}_{ii}$ then $(\mathfrak{b})$ is a proper ideal of $\mathcal{C}(\sigma)$ since $(\mathfrak{b}) \cap \mathcal{C}_{ii} = \mathfrak{b}$, a contradiction.
- (iii) *For all distinct objects* $i, j \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}_{ij} \neq 0$: Suppose that $\mathcal{C}_{ij} = 0$ for some distinct objects i and j. Then the ideal $(\mathcal{C}_{ii})$ of $\mathcal{C}(\sigma)$ is a proper ideal since $(\mathcal{C}_{ii}) \cap \mathcal{C}_{jj} = \mathcal{C}_{ji}\mathcal{C}_{ii}\mathcal{C}_{ij} = 0$, a contradiction.
- (iv) *For all distinct objects* $i, j \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}_{ij}$ *is a simple* $(\mathcal{C}_{ii}, \mathcal{C}_{jj})$*-bimodule*: Suppose that $\mathfrak{b}$ is a proper $(\mathcal{C}_{ii}, \mathcal{C}_{jj})$-sub-bimodule of $\mathcal{C}_{ij}$ then $(\mathfrak{b})$ is a proper ideal of the algebra $\mathcal{C}(\sigma)$ since $(\mathfrak{b}) \cap \mathcal{C}_{ij} = \mathfrak{b}$, a contradiction.
- (v) $\mathcal{C}_{ij}\mathcal{C}_{jk} \neq 0$ *for all objects* $i, j, k \in \mathrm{Ob}(\mathcal{C})$: The statement (v) holds in the following cases $i = j = k$ (by (ii)), $i = j$ or $j = k$ (by (iii)). Suppose that $i = k$ and $\mathcal{C}_{ij}\mathcal{C}_{ji} = 0$, we seek a contradiction. Then the ideal $(\mathcal{C}_{ij})$ of $\mathcal{C}(\sigma)$ is a proper ideal since $(\mathcal{C}_{ij}) \cap \mathcal{C}_{ii} = \mathcal{C}_{ij}\mathcal{C}_{ji} = 0$, a contradiction. Suppose that $\mathcal{C}_{ij}\mathcal{C}_{jk} = 0$ for some distinct i, j and k. Then the ideal $(\mathcal{C}_{ij})$ of $\mathcal{C}(\sigma)$ is a proper ideal since $(\mathcal{C}_{ij}) \cap \mathcal{C}_{kk} = \mathcal{C}_{ki}\mathcal{C}_{ij}\mathcal{C}_{jk} = 0$, a contradiction.

($\Leftarrow$) Suppose that conditions 1–4 hold. By conditions 1–3, condition 4 can be replaced by condition 4$'$: $\mathcal{C}_{ij}\mathcal{C}_{jk} = \mathcal{C}_{ik}$ for all $i, j, k \in \mathrm{Ob}(\mathcal{C})$. Let J be a nonzero ideal of $\mathcal{C}(\sigma)$. We have to show that $J = \mathcal{C}(\sigma)$. By condition 1, $e_i J e_j \neq 0$ for some i and j. By conditions 2 and 3, $J_{ij} = J \cap \mathcal{C}_{ij} = \mathcal{C}_{ij}$. By condition 4$'$, $\mathcal{C}_{st} = \mathcal{C}_{si}\mathcal{C}_{ij}\mathcal{C}_{jt} \subseteq J$ for all s, t. This means that $J = \mathcal{C}(\sigma)$, as required. □

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