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THE DERIVATIVE NOTION REVISED: THE FRACTIONAL CASE

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Abstract: In this paper the historical origins of fractional calculus are explored with respect to non-conformable derivatives. Implications are made for the teaching and learning of calculus and for mathematics education.

Keywords: History of Mathematics; Calculus; derivatives; Leibniz rules; mathematics teaching

A historical introduction. The vision of a "fractional calculus" was already evident to the founding fathers of the ordinary calculus (that, because of the double meaning of the word that becomes intolerable in this context, we are tempted to call the "ordinary calculus"). Leibniz, who was the creator of both the notation $\frac{d^n y}{dx^n}$ and of $\int f(x)dx$, wrote to his friend L'Hôpital (on September 30, 1695) in the following way[1]:
"Jean Bernoulli seems to have told him that I mentioned a wonderful analogy that allows us to say that the successive differentials are in geometric progression. You can ask what a differential would be by having a fraction as an exponent. You see that the result can be expressed by an infinite series, Although this seems to be eliminated from Geometry, which does not yet know such fractional exponents, it seems that someday these paradoxes will produce useful consequences, since today is just a paradox without utility. Thoughts that mattered little in themselves can give occasion to more beautiful ones"
Many authors cite this date as the birth of the so-called fractional calculus. Thenceforth, several mathematicians contributed to the development of fractional calculus: Euler, Lacroix, Riemann, Liouville, Caputo, Grunwald, Letnikov, etc. (see [8], [13] and [15]).
Until recently, research on fractional calculus was contemplated within the field of pure mathematics, but in the last two decades, many applications of fractional calculus appeared in various fields of engineering, applied sciences, economics, etc. (see for example [5], [7], [10] and [15]). As a result of this, fractional calculation has become an important issue for researchers in other fields. Other recent works give more details about their applications (see [2], [8] and [14]).
However, the first textbook dedicated to this topic did not appear until 1974, when Oldham and Spanier published "The Fractional Calculus" (see [12]). Although it retains its place as the main reference in the field, this monograph was later joined

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by Miller and Ross "An Introduction to the Fractional Calculus and Fractional Differential Equations" in 1993 (see [10]). Both books are written in a very accessible way, and in terms of design and emphasis they are clearly complementary. So, those who undertake the study of this field must have both books at their fingertips. Both books are written in a very accessible way, and in terms of design and emphasis are clearly complementary, so that those who undertake the study of this field, should have both books at hand because both provide short stories and usefully detailed, the history of fractional calculation, as well as valuable bibliographic data, and it is from them that most of the historical comments that appear in various works on the subject have been taken.
In this paper, we present some of the most recent definitions of fractional derivatives and analyze their impact both in mathematics itself and in mathematics education.

Definitions and limitations. The most famous and popular of these definitions in the world of fractional calculus are the derivatives of Riemann-Liouville and Grunwald-Letnikov. Although there are multiple variations on these definitions there is a detail that we want to make clear: all these refinements and extensions of these definitions are of a global nature, that is, they are not referred to a point, but to an interval, they are not derived in the strict sense of the word, they are integrals!
For example, the derivative of Riemann-Liouville takes the following form:

$$D^{\propto} f(x) = \frac{1}{\Gamma(-\alpha)} \int_b^x \frac{f(t)}{(x-t)^{\alpha-1}} dt, \text{ α<0.}$$

In 1868, Grunwald-Letnikov gave another definition of Fractional Derivative starting from the formal definition of the integer derivative. Luckily it can be proved that the results of Riemann Liouville and Grunwald-Letnikov are equivalent.
Caputo, in 1967, inverts the order of the derivation proposed by Riemann-Liouville and another alternative appears for the fractional derivative:

$${}_bD_x^{\alpha} f(x) = D^{\alpha-n} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_b^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \text{ n-1<α<n.}$$

These two alternatives mark the fundamental research directions in the Global Fractional Calculus. However, there are some details that we would like to clarify. The previous global definitions of D^{α} have some inconsistencies among which are the following:

1) Most fractional derivatives, except those of the Caputo type, do not satisfy $D^{\alpha}(1) = 0$, if α is not a natural number.
2) All fractional derivatives do not satisfy the known Product Rule for two functions $D^{\alpha}(fg) = gD^{\alpha}(f) + fD^{\alpha}(g)$.
3) All fractional derivatives do not satisfy the familiar Quotient Rule for two functions $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) \; - fD^{\alpha}(g)}{g^2}$ with $g \neq 0$.
4) All fractional derivatives do not satisfy the Chain Rule $D^{\alpha}(fog)(x) = D^{\alpha}(fog)(g)D^{\alpha}g(x)$.
5) Fractional derivatives do not have a corresponding "calculus".
6) All fractional derivatives do not satisfy the Index Rule $D^{\alpha}\left(D^{\beta}(f)\right) = D^{\alpha+\beta}(f)$.

However, in [7] the authors define a new fractional derivative, of local character, called conformable fractional derivative, which depends only on the classical definition of derivative (see also [1], [5] and [6]). Namely, for a function $f:(0;+\infty)\rightarrow R$ the conformal fractional derivative of order $0<\alpha\leq 1$ of f at $t>0$ was defined by

$$T_{\alpha}f(x):=\lim_{h\to 0}\frac{f(1+ht^{1-\alpha})-f(t)}{h}$$

If f is α-differentiable in some $(0,\mathrm{a}),\mathrm{a}>0$ and lim $\lim_{x\to 0^{+}}f^{(\alpha)}(x)$ exists, then we define $f^{(\alpha)}(0)=\lim_{x\to 0^{+}}f^{(\alpha)}(x)$.

As a consequence of this definition, the authors proved that many of the previous insufficiencies are overcome. The conformable adjective may not be very appropriate here, since initially it was called a conformable fractional derivative to those that satisfy $\lim_{\alpha\to 1}D^{\alpha}f(x)=f'(x)$, that is, when $\alpha\rightarrow 1$, $D^{\alpha}f(x)$ preserves the angle of the line tangent to the curve, while in the definition of [11], as we shall see later, this angle is not conserved.

A non-conformable local fractional derivative. In [11] the following definition of a non-conformal local fractional derivative is presented.

Definition. Given a function $f:[0;+\infty)\rightarrow\mathbb{R}$. Then the N-derivative of f of order α is defined by

$$N_{1}^{\alpha}f(t):=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon e^{t^{-\alpha}}\right)-f(t)}{\varepsilon}$$

For $all\ t>0,\alpha\in(0;1)$. If f is α-differentiable in some $(0,\mathrm{a})$, $\mathrm{a}>0$ and if $\lim_{x\to 0^{+}}N_{1}^{\alpha}f(x)$ exists, then we define $N_{1}^{\alpha}f(0)=\lim_{x\to 0^{+}}N_{1}^{\alpha}f(x)$.

Remark 1. This fractional derivative is local by definition, therefore, any comparison with the classical fractional derivatives is erroneous. We are considering mathematical objects of different types! However, in [4] they make a comparison of this type.

The following two theorems proved in [11] come to correct the inconsistency 5) indicated at the beginning of this paper.

Theorem 1. If a function $f:[0;+\infty)\rightarrow R$ is N-differentiable at $t_{0}>0$, $\alpha\in(0;1)$ then f is continuous at t_{0}.

Theorem 2. Let f and g N-differentiable at $t_{0}>0$, $\alpha\in(0;1)$ then:

a) ${N_{1}}^{\alpha}(\mathrm{af}+\mathrm{bg})(\mathrm{t})=\mathrm{a}{N_{1}}^{\alpha}(\mathrm{f})(\mathrm{t})+\mathrm{b}{N_{1}}^{\alpha}(\mathrm{g})(\mathrm{t})$.

b) ${N_{1}}^{\alpha}(\mathrm{t}^{p})=\exp(t^{-\alpha})\mathrm{p}t^{\mathrm{p}-1};\ \mathrm{p}\in\mathbb{R}$.

c) ${N_{1}}^{\alpha}(\lambda)=0;\ \lambda\in\mathbb{R}$.

d) ${N_{1}}^{\alpha}(\mathrm{fg})(\mathrm{t})=\mathrm{f}{N_{1}}^{\alpha}(\mathrm{g})(\mathrm{t})+\mathrm{g}{N_{1}}^{\alpha}(\mathrm{f})(\mathrm{t})$.

e) ${N_{1}}^{\alpha}(\mathrm{f}/\mathrm{g})(\mathrm{t})=\frac{\mathrm{g}{N_{1}}^{\alpha}(\mathrm{f})(\mathrm{t})-\mathrm{f}{N_{1}}^{\alpha}(\mathrm{g})(\mathrm{t})}{g^{2}(t)}$.

f) If, in addition, f is differentiable then ${N_{1}}^{\alpha}(\mathrm{f})(\mathrm{t})=\exp(t^{-\alpha})f'(\mathrm{t})$.

g) If f is differentiable and $\alpha = n,$ n positive integer, we have that $N_1^n(\mathrm{f})(\mathrm{t}) = \exp(\mathrm{t}^{-n}) f'(\mathrm{t})$.

Remark 2. From the point of view of Mathematics Education we would like to stop on the last points f) and g) above. The first one is telling us, geometrically, that if there is the limit of the secant of the points $(\mathrm{t}, \mathrm{f}(\mathrm{t}))$ and $((\mathrm{t}+\varepsilon), \mathrm{f}(\mathrm{t}+\varepsilon)),$ then the limit of the points will exist $(\mathrm{t}, \mathrm{f}(\mathrm{t}))$ and $((\mathrm{t}+\varepsilon), \mathrm{f}(\mathrm{t} + \varepsilon\exp(t^{-\alpha}) f'))),$ since the latter is contained in the former, since $\mathrm{t} + \varepsilon\exp(t^{-\alpha}) < t + \varepsilon$. Point g) teach us something very interesting and it is the fact that to calculate the N-derivative of order n, it is not necessary to calculate the "successive" derivatives, it is enough to know the first of them. For the students, this can be very striking given that, in the courses of calculus, the derivatives of higher order are defined recursively according to the preceding ones.

Remark 3. If $N_1^\alpha(f)(t)$ exists, for t>0 then f is differentiable in t and $f'(t) = e\mathrm{xp}(-t^\alpha) N_1^\alpha(f)(t)$.

Remark 4. As in the case of ordinary calculation, the sign of derivative ${N_1}^\alpha$ gives us information about the behavior of the function in the vicinity of a certain point, the change of sign of the N-derivative allows us to establish the same conclusions as in the ordinary calculus, with respect to the existence or not of an end of the function in that point, since the factor $\exp(t^{-\alpha})$ is positive for t>0. Obviously it is impossible to draw conclusions in the case t = 0. Then the behavior of a function, studied in the courses of ordinary calculus, continues to be maintained in any interval of the form (a,+∞) with a>0, although it will suffer a "distortion" due to the presence of the afore mentioned factor.

Unfortunately, inconsistency 6) cannot be overcome with the definition of N-derived, but this fact will see that it has a very important significance.

Theorem 3. Let α and β be positive constants such that $0 < \alpha, \beta < 1$ and let f be a function (non-constant) twice differentiable over in the interval $(0; +\infty)$. So

$$N_1^\alpha[N_1^\beta f(t)] \neq N_1^{\alpha+\beta} f(t) \tag{1}$$

Remark 5. Although (1) deviates from the behavior of whole-order derivatives, non-commutativity offers a richness that is interesting to explore. If f is a derivable function, we have the following differential equations $(\lambda \in R)$:

I) $N_1^\alpha y(t)] + \lambda y(t) = 0$ with solution $y(t) = e^{\lambda \int^t \frac{ds}{e^{s-\alpha}}}$.

II) $N_1^\alpha[N_1^\beta y(t)] = 0$, then $y(t) = C_1 \int^t \frac{ds}{e^{s-\alpha}} + C_2$. That is, the solution is independent of β; i.e., the order of the last derivation does not influence the general solution¡¡¡¡¡

III) $N_1^\alpha y(t) + p(t) y(t) = q(t)$ from where we have $y(t) = e^{\lambda \int^t \frac{p(s)ds}{e^{s-\alpha}}} \left[\int^t \frac{q(s)}{e^{s-\alpha}} e^{\int^t \frac{p(s)ds}{e^{s-\alpha}}} ds + C\right]$.

While there are cases very similar to the classical theory of solving ordinary differential equations, that is, to the case where α and β are integers, the second gives us a different significant with the known theory.

Remark 6. Failure to comply with the Semigroup Law (Rule of Indices) may seem disappointing, but it is one of the essential characteristics of the fractional derivatives (see [16] and [17]).

Methodological remarks by way of conclusion. To conclude our work, we could ask ourselves some questions related to the definition of this new derivative that we will see below.

Does it make sense to study these non-conformable derivatives, when there are conformable local fractional derivatives?

To answer this question, we prefer to use the words of other researchers on the subject[2]. Before we must clarify some concepts involved.

Definition 2. Let $\mathrm{k}:[\mathrm{a}\,;\mathrm{b}]\to\mathrm{R}$ be a non-negative continuous application such that $\mathrm{k}(\mathrm{t})\neq 0$, and $\mathrm{t}>a$. Given a function $\mathrm{f}\,[\mathrm{a}\,;\mathrm{b}]\to\mathrm{R}$ and $\alpha\in(0{,}1)$ a real number, we say that f is α-differentiable in $\mathrm{t}>a$, with respect to the kernel k, if the limit

$$f^{\alpha}(x) := \lim_{h\to 0}\frac{f\left(1+hk(t)^{1-\alpha}\right)-f(t)}{h}, \qquad (2)$$

exists. The α-derivative at t = a is defined by $f^{\alpha}(a)=\lim\limits_{t\to a^{+}} f^{\alpha}(t)$, if the limit exists. Then they affirm and they do not show the Theorem 2.2 because they point out that it is trivial, where they obtain:

$$f^{\alpha}(t) = k(t)^{1-\alpha}f'(\mathrm{t}) \qquad (3)$$

and they conclude:

"some of the existent notions about local fractional derivative are very close related to the usual derivative function, in fact, the α-derivative of a function is equal to the first derivative, multiplied by a continuous function. Also using formula (3), most of the results concerning α-differentiation can be deduced trivially from the ordinary ones. In the author's opinion, local fractional calculus is an interesting idea and deserves further research, but definitions like (2) are not the best ones and a different path should be followed".

In other words, the most interesting path is not the one of conformable fractional derivatives, but on the contrary, the no conformable ones are those that can provide a panorama that differs from the theory known in the classical calculus.

The consideration of these non-conformable local fractional derivatives, *What can you offer our students?*

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[2] Cf. [3], p. 1124.

Let's go back to Theorem 2f). From here it is clear that $\lim_{t\to\infty} N_1^{\alpha} f(t) = f'(t)$.

With what we are getting that the asymptotic behavior of the N-derivative is similar to that of the ordinary derivative and this is of vital importance in the Qualitative Theory, one of the fundamental tools in applied research, mainly in the Stability Theory.
On the other hand, to complete the answer, we would like to abound more in the formation of our students.

I) What underlies this work, is a dynamic conception of Mathematics, coined in the famous phrase that Philip E. Jourdain made in the introduction to his book "The Nature of Mathematics", when declaring the central objective of the The book pointed: *"I hope that I will be able to show that the process of mathematical discovery is something alive and developing"*. If students do not conceive of Mathematics as a science in constant growth, how can we separate them from Platonic positions that have done so much damage to Mathematics itself and to Mathematics Education. In other words, we have tried to convey to our students a dynamic conception of Mathematics, a position compatible with the "Problem Solving" advocated by the most accepted theories of learning.

II) It is not always understood that the mathematical object in consideration for teaching is structurally, but not qualitatively, the same as in Mathematics hence that most mathematicians make the mistake of believing that the education of mathematics is affected only by problems of the type: How to transmit the important mathematical facts to the students? From the notion of the meaning of accepted mathematical objects, we consider the socio-anthropological approach of how mathematical knowledge is produced and what is framed within the broader ethnomathematics line. Based on the fact that there is no universal agreement on what constitutes a "good teaching of Mathematics", we accept that what everyone considers as desirable ways of learning and teaching Mathematics is influenced by their conceptions of Mathematics. It is unlikely that disagreements about what constitutes good mathematics teaching can be resolved without addressing important issues about the nature of Mathematics. One of those basic topics is the notion of derivative.

III) On the other hand, it is natural that in the current courses of Calculus (or Mathematical Analysis) both the traditional classical results and methods are exposed, as well as the models that have emerged over the last decades, but despite the efforts by applying these in the teaching-learning process, there is an insufficient linkage with the mathematical modeling of processes, particularly those that present discontinuities, sudden jumps, divergences, etc., procedures and examples that new notions of derivative allow to link, each Once again, with the knowledge and skills that the Calculus deals and develops.

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