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ABSTRACT: The present research corroborates K. Sullivan's initial results as stated in her epoch making study about the effectiveness of teaching elementary calculus using Robinson's non standard approach. Our research added to her's results related to the teaching of the elementary integral, with similar positive results. In this essay we propose a definition of the notion of cognitive advantage mentioned by Sullivan in expressing the dramatic differences in understanding of students of non standard calculus as oppossed to those of its standard counterpart. Our proposal is based on ideas of Kitcher and Kuhn and allows us to better understand the didactics of Calculus. Formally K. Sullivan's claim of an observed "advantage" when referring to the improved understanding of non standard calculus students (as opposed to the standard approach of Weierstrass) is a consequence to the accepted fact that mathematical truths remain the same when changes of paradigms ensue, a situation markedly different from that science. While mathematical truths remain, mathematical justifications (proofs) change dramatically and increase in complexity.

Keywords: non standard calculus, standard calculus, elementary integral, calculus education, non standard calculus.

Introduction

In the classical work of Sullivan [27, 1976] data is presented construable as documenting a cognitive advantage for teaching calculus with infinitesimals. In the words of Sullivan [27, 1976; p. 371], "When elementary calculus is developed from this non standard approach, the definitions of the basic concepts become simpler and the arguments more intuitive." As an example of such intuitiveness and simplicity, Sullivan (ibid) mentions the definition of continuity in non standard calculus and, in our view, her remarks apply equally well to the definition of limit.

Perhaps, the treatise that comes closest to being the first textbook of calculus[1] ever written is L'Hospital *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* [21, 1696]. L'Hospital's book (ibid.) was written with the overt intention of *teaching* the new discipline of calculus. In spite of this, it may come as a surprise that it does not include any discussion whatsoever of integration. In L'Hospital [21, 1696, pp. 2-3, I] a basic postulate (*Demande ou Supposition*) is stated that in modern parlance can be taken to assert that all curves are continuous. In his own words (ibid): "two ordinates that are infinitely close to one another, can be taken one for the other." In the original formulation of the calculus "curves" would correspond roughly to our graphs of functions, so that for L'Hospital, all curves were continuous, that is to say, if y is infinitely close to x then $f(y)$ is infinitely close to $f(x)$. This is, in fact, the nonstandard definition of the script "$\lim_{y\to x} f(y) = f(x)$", which defines, as we all know, the notion of continuity at x. In fact, L'Hospital uses the following figure (Figure 1. in his book) to describe the basic infinitesimals identified as important within the context of a curve:

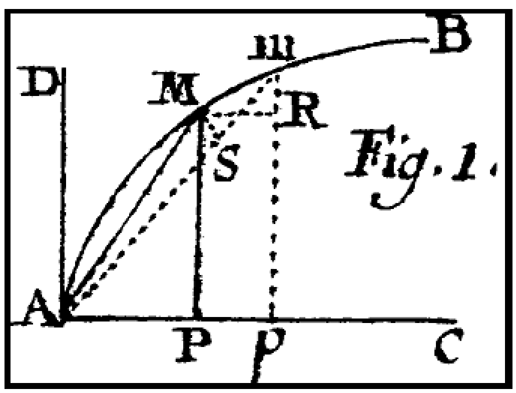


Figure 1: Basic infinitesimals as explained by L'Hospital

L'Hospital's explanation of the figure gets translated by Bradley et al. [4, 2015, Definition II, p. 2] as:

"*Definition II. The infinitely small portion by which a variable quantity continually increases or decreases is called the Differential. For example, let AMB be an arbitrary curved line which has the line AC as its axis or diameter, and has PM as one of its ordinates. Let pm be another ordinate, infinitely close to the first one. Given this, if we also draw MR parallel to AC, and the chords AM Am,and describe the little circular arc MS of the circle with center A and radius AM,then Pp is the differential of AP, Rm the differential of PM, Sm the differential of AM, and Mm the differential of the arc AM. Furthermore, the little triangle MAm, which has the arc Mm as its base is the differential of the segment AM, and the little region MPpm is the differential of the region contained by the straight lines AP and PM, and by the arc AM.*"

In L'Hospital's explanation p is taken to be an abscissa infinitely close to P and (among other things) L'Hospital states (mind you, he does not *deduce*) that the ordinate PM is infinitely close to pm. In more modern terms if $x + dx$ is an abscissa (p) infinitely close to the abscissa x (P) then the ordinate $f(x + dx)$ (pm) is infinitely close to the ordinate $f(x)$ (PM). In modern script, all this means that $f(x + dx) - f(x)$

[1] that is, recognizable by a modern reader as such; another contender to this recognition is Isaac Barrow's [2, 1916] *Geometry Lectures*, but his geometrical approach makes it virtually unrecognizable as calculus to the modern reader.

is an infinitesimal or, equivalently, that $\lim_{y\to x} f(y) = f(x)$. Since for a real number L $\lim_{y\to x} f(y) = L$ means that $f(x)$ is infinitely close to L when y is infinitely close to x, but distinct from x, there is a ready made generalization of the notion of continuity to that of limit. It goes without saying that the non standard definition of limit can be argued, as Sullivan (ibid) did than the present day definition of $\lim_{x\to y} f(x) = L$, namely $(\forall\epsilon > 0)(\exists\delta > 0)(0 < |x - x| < \delta \Rightarrow |f(x) - L| < \epsilon)$. It goes without saying, by the way, that it took mathematics more than one and a half century from the invention of calculus to eliminate all together the traces of infinitesimals in analysis arguments and adopt the standard "epsilon and delta" definition of limits using quantifiers over the real numbers.

Sullivan (ibid) capitalized on the popularity of H. Jerome Keisler's, Elementary Calculus [16, 1976] which, according to her (ibid), was the first textbook to "adapt the ideas of Robinson[2] to a first year calculus course". From September 2002, the second edition of this work (ibid) became freely available online in the form of PDF files made from the printed Second Edition of [16, 1976]. Sullivan (ibid) refers to the expectation of a "considerable pedagogical payoff" stemming from the simplicity and intuitiveness of teaching calculus using infinitesimals. We shall refer to this hypothesis as the *cognitive advantage hypothesis*(CAH) for teaching calculus with infinitesimals. Finally, Sullivan (ibid) expresses many practical concerns related to the use of infinitesimals in the teaching of calculus: will the students buy the idea of infinitely small?; will the instructor need to have background in non standard analysis?; will the students acquire the basic calculus skills?; will they really understand the fundamental concepts any differently?; how difficult will it be for the students in the non standard course to make the transition into standard analysis courses if they want to study more mathematics?; is the standard approach only suitable for gifted mathematics students? All of these concerns are also important for the essay presented here.

The present work reveals new data to document the CAH for teaching calculus with infinitesimals, and concentrates in the areas of the theory of limits (originally considered in the Sullivan (ibid.) study) as well as in the theory of the elementary integral [3], area not included in Sullivan's study. This essay expands on the results of a study carried on at the University of Puerto Rico at Río Piedras (UPRRP) by L. M. Hernández [14], adding new dimensions to Sullivan's important work. As we shall see, the present study further documents the CAH implicit in Sullivan's work (ibid.) regarding the teaching of calculus using the method of infinitesimals. Furthermore, it will be argued that the CAH is consistent with the tenets of Freudenthal's historical and didactic phenomenologies ([8, 1973] and [9, 2002]) when these are applied to the mathematical structures of the second half of the seventeenth century (as exposed, for instance, in the well known treatise of Whiteside [31, 1961]). The latter point is of theoretical interest in so far as it poses a problem related to Freudenthal didactical phenomenologies[4] for calculus and analysis. Finally, as we shall see, the CAH is also consistent with Sausserian semiotics as the latter applies to the interpretations of relations between signs and meanings of the mathematical structures amenable to seventeenth century calculus, specially with regard to the elementary integral. Our findings regarding the elementary integral are new as far as we can tell.

1 Description of the Present Study

The mathematics department of UPRRP has collected statistical data relating to the performance of students who register in the introductory calculus course. A few years before this study the department started developing tests designed to ascertain the formal and informal student understanding of the line models associated with the rational and the real number systems. The tests also attempted to measure student

[2]Abraham Robinson, a noted logician, invented non standard analysis (analysis includes calculus) based on the theory of ultrafilters; see Robinson [26, 1974].

[3]We use this name to refer to the modern rendition of the methods originally developed by Barrow, Newton, Leibniz and others to calculate areas bounded by curves.The modern version of their approach, which has become rather popular in recent years, is presented and discussed in Gillman [10, 1993], Lang [20, 1986] and Hernández and López [13, 2012].

[4]Didactical phenomenologies for the calculus are, to a great extent, deduced from the corresponding historical phenomenologies when the latter are applied to the mathematical structures associated with the development of analysis during the second half of the seventeenth century. Robinson's formulation of the calculus brought about a newer and correct foundation for the calculus in the context of a non standard number line (a non Archimedian ordered field containing the real numbers in coexistence with infinite and infinitesimal numbers), which essentially rendered correct, upon revision, the infinitesimal arguments of Barrow, Newton, Leibniz, Euler and others. The work of Robinson opened the possibility of developing non standard didactical phenomenologies for the calculus.

understanding of the discrete (integer) "number line". In this section we document the claim that students of elementary calculus at UPRRP lack the most basic understanding of the various number line models of mathematics. This fact significantly restricts the possibilities of students attaining the expected mastery of college level mathematics thinking and limits their likelihood of pursuing upper divisional courses in mathematics. And, of course, these cognitive limitations also curtail the possibilities of students succeeding in the academic pursuit of other intellectual areas closely related to mathematics. We preface the following discussion by remarking that the College of Natural Sciences (CNS) of UPRRP is main academic unit of the UPR system[5], offering careers in the sciences, mathematics, computer science, environmental sciences and other areas of support for the bio-medical sciences. With the exception of students applying for academic careers related to specific areas of engineering (like computer architecture or mechanical engineering, offered at the engineering school campus of UPR at Mayagüez), entering students at the UPRRP are the most able of all university students in the UPR system. The latter system has the highest academic demand of all institutions of higher learning in Puerto Rico, thus attracting the most able students in science, mathematics and other related academic fields [6]. Virtually, all graduating high school students in Puerto Rico take standardized tests developed by the Assessment and College Admission Program (PEAU for its acronym in Spanish: *Pruebas de Evaluación y Admisión Universitaria*) of the Office of The College Board in Puerto Rico[7], that include the areas of Verbal Reasoning and Mathematical Reasoning, as well as other tests related to acquired advanced knowledge in several areas during the high school studies[8]. The PEAU tests are graded on a scale of 200-800 (mean of 500 and standard deviation of 100) and are calibrated with respect to a predetermined population of students tested some years back, in order to be able to make inferences across years of administration of the tests. These tests are combined with high school grade point averages of applying students, scaled from 1 to 4, to give a an application index (igs by its Spanish acronym) that in principle can fluctuate with values in a 200-800 scale. The different programs of the University have their own igs the higher the igs the more competitive is admission to the program. The possibility of admission, is also affected by maximum admissions quotas established by each academic program. The igs established by the academic units depend on variables reflecting the ability of the programs to render adequately the corresponding academic services. Our study involved students of the CNS who registered for introductory calculus (Math 3153) during the summer session of the of the 2011-2012 academic year. Over 90% of the students of the study ware admitted to the UPR system in 2010 and a few of them, even before. That year, the average scores in Verbal Reasoning and Mathematics Reasoning for all students applying for college in Puerto Rico[9] were 472 and 479 respectively, of a population shy of 111,000 applicants. These students ranked in approximately the 75th percentile for that year, and students admitted to the mathematically intensive concentrations of the CNS rank in approximately the 5th percentile. In 2011, the average igs of the CNS concentrations was 321 as compared to an average igs of 317 for entering students at UPRRP and an average igs of 298 of entering students for the whole UPR system; see [24, 2010-2014], A. Magriñá [23, 2015] and M. de los A. Ródríguez [25, 2013].

1.1 Some educational realities of present day in Puerto Rico

In pondering the results of the present study it should be mentioned that, traditionally, the DEPR has unsuccessfully placed great amounts of effort in raising the standardized tests indicators of student's proficiency in mathematics to respectable levels when compared to results in the United States. For instance, in the present day standardized testing administered to eleven graders by the DEPR for the school year 2011-2012, students in six of the seven educational regions of Puerto Rico[10] attained scores that show more than fifty percent of the student population performing at or below the "basic level[11]"; see DEPR [6, 2010]. The

[5]consisting of 11 autonomous campuses -including the schools of engineering and medicine- and having an enrollment shy of 60,000 students

[6]With the exception, perhaps, of those students who applied directly to top rated universities in the United States and abroad.

[7]The Puerto Rico and Latin America Office of the College Board, a subsidiary of Educational Testing Services (ETS)

[8]These tests correspond to the Advanced Placement Program tests of ETS .

[9]We are indebted to Antonio Magriñá, Executive Director of Research and Measurement of the Office of College Board Office of Puerto Rico and Latin America.

[10]as officially established by the DEPR

[11]The exception is the MayagÃ$\frac{1}{4}$ez region, performing at a level of 47 percent below the basic level. In Puerto Rico the DEPR has created the "pre-basic level", even below the basic level, to be able to differentiate degrees of lack of mathematical

standardized 2011-2012 school year examination included testing in the content standards of Numeration and Operations, Algebra, Geometry, Measurement and Probability and Statistics. On the other hand, as required by federal law, Puerto Rico and all jurisdictions of the United States receiving Title I funds must participate in the National Assessment of Educational Progress (NAEP) fourth and eight grade testing every other year, beginning in 2003. NAEP included Puerto Rico in trial mathematics assessments of 2003 and 2005, with disturbing results. According to NAEP [3] executive summary, preliminary analysis of the 2003 data "raised concerns that the items were not functioning as they did in other jurisdictions", and also that in Puerto Rico larger amounts of missing data and fewer correct responses than expected were observed for every content area, including dramatic mismatches between expected and actual student performance on items (item misfit). Furthermore more than 90% of all Puertorrican students tested, both fourth graders as well as eighth graders ibid [3, p. 19] performed below the basic level in the full scale[12] when compared to 24% and 33% in the U.S. for the fourth and eighth grades respectively. A good argument can be made to explain these dismal results in terms of the sociopolitical practices associated with the administration of such testing[13] by the DEPR. To this it should be added that there are no nationwide examinations administered by the DEPR that serve as quality control indicators to ascertain the levels of mathematical knowledge attained by students completing specific school grades. In fact, statistics relating to the dropout rates by grade and the grade to grade progression rates are hard to get in Puerto Rico. In spite of this, according to Allison et al. [1, 2005], Puerto Rico's grade-to-grade progression rates in high school are far more comparable to those for the United States. In spite of this, when one considers other standardized testing, like the College Board's College Entrance Examination Exam, designed for students seeking college admission in Puerto Rico, the evidence that surfaces adds to the generally accepted view that Puertorrican students have lower levels of mathematical knowledge than those expected for college careers. In fact, in Puerto Rico as in many universities in the United States, the entry level mathematics course is precalculus, rather than calculus.

1.2 Academic profile of students participating in the project

Some particular characteristics of the entering students to the College of Natural Sciences during the year of our study are as follows: the average scores for the igs, verbal reasoning and mathematical reasoning parts of the PEAU are 317, 597 and 611 respectively. Also, 62% of all incoming students were graduates of private schools. It has become progressively clear that public education provided by the government is grossly inadequate for the attainment of the levels of academic excellence as described in the normative documents defining the expected outcomes of mathematics education. The interested reader is invited to examine the essay by Hernández et al. [13, 2014] as well as the newspaper articles by I. Torres [29, April,28,2016] and E. Vélez Lloréns [30, April 28, 2016]. Also, 78% of the students participating in our study are at least second generation college students and 62% hold jobs while studying.

In addition to the College Board battery of tests mentioned before, student participating in the project were give a test of their "mathematical sense". The process of understanding thoroughly the details of the mathematical structures characteristic of a given mathematical endeavor depends on the student being able to appropriate and internalize progressively larger areas of those structures. At some point, in our pursuit of academic knowledge learning gets to be easier to the extent that we do not always need to be in pursuit of details, as these become natural and more obvious and need not be elucidated every time they are encountered. For instance, if a, b and x are real or complex numbers so that for some positive real number ϵ we have $\max\{|a-x|, |b-x|\} < \frac{\epsilon}{2}$, then, necessarily, $|a-b| < \epsilon$. This is a direct consequence of the triangle inequality and, for the trained mind, a little reflection is enough to ascertain its validity. Or similarly if a, b are real or complex numbers then, for any real or complex number x, either $|a-x| \geq \frac{|a-b|}{2}$ or $|b-x| \geq \frac{|a-b|}{2}$. Again, this is clear from the simple geometric fact that a point cannot be in in both of two non intersecting circles (or intervals). These are obvious from the implicit geometry, and advanced students will not stop significantly when elucidating the truths of these statements, but neophyte students will stop to ascertain their

achievement.

[12]NAEP designed two scales to rate Puertorrican performance in the tests. The "reduced scale" reported only two levels, "basic" and "proficient", because so few students performed at the "advanced" level in Puerto Rico.

[13]For instance, such testing has no effect in student's grade promotion or in his/her school performance, and thus they tend to be considered inconsequential.

validity. In our attempt to measure "mathematical sense" acquired by students before their introductory calculus course we were able to corroborate the little knowledge of the geometry of the line or the plane that students, in general, have. It is extaneous for students to understand, for instance, that the decimal 0.132 describes a number that lies in the second interval obtained in dividing the unit interval in 10 congruent subintervals, in the fourth interval if the latter interval is divided, again, in ten congruent subintervals and in the third interval when the interval just mentioned is divided, again, in ten congruent intervals. We shall give her some of the sample problems of the "mathematical sense" exam so that the reader can gain a clearer view of the situation we are trying to describe.

Problem A:
Point $(1, a)$ is in the graph of the funcion $y = f(x)$. Then,

a. $f(1) = a$.

b. $f(1) = a^2$.

c. $f(1) = f(a)$.

d. $(a, 0)$ is in the graph of $y = f(x)$.

73% of all students answered this question correctly. The fact that 27% of the students failed the question may seem surprising, specially since this is one of the central ideas of precalculus, a subject matter that everyone approved by taking the course or opting out of the course by taking the College Board's advanced placement exam on precalculus.

In the area of "variational thinking" the following question was included:

Problem B:
If x and y are positive real numbers and $x > y > 0$, then

1. $\dfrac{1}{x} < \dfrac{1}{y}$.

2. $\dfrac{1}{x} < -\dfrac{1}{y}$.

3. $\dfrac{1}{1+\dfrac{1}{x}} < \dfrac{1}{1+\dfrac{1}{y}}$.

4. $x = y$.

67% of all students answered Problem B correctly. Most students who solved this problem considered it as a formal problem on inequalities. Student work on the sheets of the exam showed a lot of "experimentation" with specific numbers in order to eliminate possible answers. Subsequent interviews with students showed that the 33% who failed to answer correctly the item thought of this problem as a "hard problem" on the solution of inequalities. This problem has an alternative easily discarded and in spite of this fact the results show some lack of algebraic skills that put some students at a serious disadvantage in studying calculus.

For "proportional reasoning" the following question was included:

Problem C:

Let $\dfrac{x}{17} = \dfrac{37}{y} = \dfrac{1}{5}$. Which of the following statements is not necessarily true.

1. $\dfrac{37x}{17y} = \dfrac{1}{5}$.
2. $\dfrac{37+x}{17+y} = \dfrac{1}{5}$.
3. $\dfrac{x-37}{17-y} = \dfrac{1}{5}$.
4. $\dfrac{2x+37}{y+34} = \dfrac{1}{5}$.

Only 25% of all students answered this question correctly. Again, most students cross multiplied and tried to handle the resulting equations, but very few thought of ratio tables or proportions.
Finally, a typical algebraic problem of precalculus included was:

Problem D:

The center of an interval is $a/2$ and its length is $|a|$, for a real number a. If $a < 0$, the interval is:

1. $(0, a)$.
2. $(-a, a)$.
3. $(a, -a)$.
4. $(a, 0)$.

70% of all students failed this question. Given the ability of participating students it is surprising that students lack the algebraic skills to be able to solve the item. This seems to be a general situation prevalent in our times.

The original study of Sullivan [27, 1976] was carried out during the 1973-74 school year and, as we shall presently see, involved a population of students of 58 students in the control group and 55 students in the experimental group. The groups were in charge of five instructors, at the time, teaching standard calculus courses in the Chicago-Milwaukee area. The Sullivan (ibid) study used H. Jerome Keisler textbook [16, 1976] and referred to (ibid) as the "first textbook to adapt the ideas of Robinson to a first year calculus course". Sullivan (ibid) does not indicate if there were any sort of theoretical preparation for instructors regarding the theoretical aspects of the non standard course. Our study used class notes produced by one of the authors; see López [22].

1.3 Background

This study exposed two groups of college students of very close mathematical abilities to two versions of the introductory calculus course[14].

1.4 The standard calculus course

The planning for the non-standard calculus course was tailored after the regular course to accommodate common schedules for examinations and homework gathering in order to facilitate comparison of results. As it happens with most introductory college calculus courses, the standard course offered by the Faculty of Natural Sciences at UPRRP is a hybrid mix of analysis results in the corresponding topological settings. The topological "pillars" of the course are the Intermediate Value Theorem (IVT)[15] and the Extreme Value

[14]The course, codified as Math 3151, is a five credit-hour introductory course characterized by its high student attrition. The course is actually required for all subject concentrations in the Natural Sciences Faculty of the UPRRP.

[15]That states that any real valued continuous function that changes parity over an interval of the real line must have a zero on the interval.

Theorem[16] (EVT). On the other hand, the analytical pillars of the course are both of the Fundamental Theorems of the Calculus (FTC)[17] and the Mean Value Theorem. The course begins with a review of the structure of the real number system as a complete ordered field and although mention is made of the completeness axiom, no attempt is made at this juncture to develop a profound working understanding of the completeness axiom. In spite of this fact, tradition dictates that the rudiments of the Weierstrassian $\epsilon - \delta$ theory of limits be developed and, and from there, the notion of continuity discussed. Limits of functions are discussed and students can get some practice in actually working $\epsilon - \delta$ arguments for the limits of *carefully chosen* functions like linear, quadratic and rational. As the reader can suspect, this task can turn out to be a rather trying one, specially for students with poor algebra and estimation skills. Derivatives, the chain rule, the solution of optimization problems and some basic techniques of integration within the context of rational, logarithmic and exponential functions rounds up the standard course.

1.5 The non-standard course

This study involved two groups of students in the Natural Sciences Faculty of the University of Puerto Rico at Ró Piedras (UPRRP). The students participating in the study, upon admittance to the College of Natural Sciences ranked in the mid and upper 90th percentiles of the population applying for admission to the UPR system. The College of Natural Sciences offers the academic programs having the highest demand among all applicants to the UPR system; [25, 2013, p. 6] and [23, 2015]. For instance, according to [25, 2013], the concentration of biology with an igs of 333, ranks first in student demand in the UPR system[18]. The students in the study are only surpassed in their academic ability (as evidenced by the standardized college entrance examination and their high school grades[19]) by students in the UPR system who wish to pursue careers in in even more contested academic concentrations, such as some computer engineering and computer science related careers, and some particular concentrations in engineering. These students typically aim to complete academic degrees in science, mathematics, biological sciences and medicine. In short, both the control groups and the experimental group in the present study were populated by students of high academic potential and superior intellectual abilities. It should be stated that the experimental and the control groups had students of very close academic abilities; the experimental group (the non standard course) had an average igs of 341.90, as shown in Figure 6, while the average igs score in the control group (the standard course) was slightly lower, that is, 334.12, as shown by Figure 6.

The students who participated in the experiment were admitted to the UPR at different years to academic programs requiring different entrance igs scores. Furthermore, some students from UPR campuses other than the RÃo Piedras campus of the UPR were allowed to register in the introductory calculus course of the College of Natural Sciences. Also, some students from academic programs of the UPR outside the College of Natural Sciences (like nursing, nutrition and dietetics and psychology) were able to register in the course and, still, a few students came from universities outside the UPR system[20]. Students were admitted from the following UPR academic programs having the indicated entrance igs: Architecture, 339; Mechanical Engineering, 335; Biology, 333; Computer Engineering, 333; General Science, 330; Electrical Engineering, 328; Chemistry, 325; Physics, 321; Nutrition and Dietetics, 320; Mathematics, 310; Computer Science, 310; Nursing 307. In the experimental group there were 2 students coming from outside the UPR system and 9 students who repeated

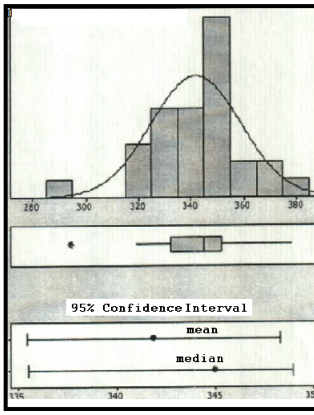
[16] Which asserts that every real valued continuous function on a closed bounded interval is bounded and assumes both its maximum and its minimum values.

[17] The First Fundamental Theorem of Calculus (FTCI) stating that if $f : [a, b] \to \mathbb{R}$ is continuous $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ for every $x \in [a, b]$, and The Second Fundamental Theorem of Calculus (FTC2) which states that $\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F of f on $[a, b]$ (f as before).

[18] Taking a weighted average over the years the students in the study were admitted to the UPR system. The most able students in Puerto Rico, typically go to study abroad or are admitted to the UPR system. The UPR system has the highest admission requirements of all universities in Puerto Rico, particularly in the areas of science, engineering and mathematics.

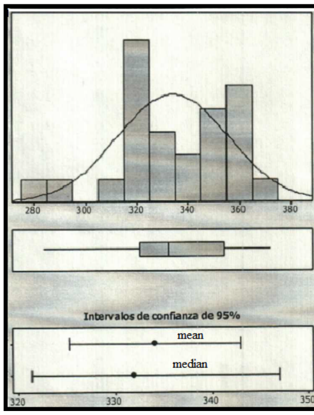
[19] The UPR system computes a "minimum entrance index" to determine admission to particular Faculties of the UPR system. Admissions are determined by this minimum entrance index and availability of space for new students. The highest minimum admissions indices are typically those pertaining to computer sciences, pure sciences, engineering and mathematics related careers.

[20] The UPR allows some students from other Universities in Puerto Rico to be considered to register in the introductory calculus course. This is only allowed in the Summer Session; our study was carried out in the Summer Session of 2012.



Anderson-Darling 95% Test	
A square	0.726
P value	0.726
Mean	341.90
Std. Dev.	16.98
Var.	288.17
Skewness	-0.40859
Kurtosis	1.23973
N	29
Min.	294.00
1st quartile	332.00
Median	345.00
3rd quartile	351.50
Max.	378.00
95% Confidence Interval for the **mean**	
335.44	348.35
95% Confidence Interval for the **median**	
335.50	349.00
95% Confidence Interval for the **std. dev.**	
13.47	22.96

Figure 2: Distrubution of IGS values for students in the experimental group



Anderson-Darling 95% Test	
A square	0.33
P value	0.506
Mean	334.12
Std. Dev.	21.55
Var.	464.19
Skewness	-4.03432
Kurtosis	-0.080138
N	25
Min.	284.00
1st quartile	321.00
Median	332.00
3rd quartile	354.00
Max.	372.00
95% Confidence Interval for the **mean**	
325.23	343.01
95% Confidence Interval for the **median**	
321.40	347.00
95% Confidence Interval for the **std. dev.**	
16.82	29.97

Figure 3: Distrubution of IGS values for students in the control group

the course after having registered in it during the 2012 Summer Session[21] (of a total of 34 students). In the control group there were 7 students coming from outside the UPR system, 9 students repeating the course after registering for it in the 2012 Summer Session (of a total of 36 students).

The experimental design accounted for both population of students taking, at the beginning of the summer session, the "mathematical sense" test mentioned before. Also, during the course, on the regular examinations of both groups, items on the theory of limits and the elementary theory of the integral were included. These items were, either identical or required, as expected, the formulation of problems in the language of standard or non standard calculus, as needed, to accommodate needs of the experimental or the control groups. We analyze here the results of a sample of these problems. Here are the items used with the corresponding statistics: LP refers to items relating to limits and IP refers to integration items.

1.6 Limit exercises

LE-A Find the value of

$$\lim_{x\to -3} \frac{x^2-9}{x+3}.$$

a) 6 b) 0 c) -6 d) 3

[21] This do not indicate necessarily that students failed the first course; some students take the course as a refresher course.

This exercise gives a straightforward limit, easy to compute for both the control and the experimental group. In spite of this, 87% of the control group answered the item correctly as opposed to 48% of the control group. This accounts for a difference of 39%. The control group had the usual epsilon delta definition of limit and students take some time in completely understanding that the limit theorems are desined in fact to make automatic problems whose details may become somewhat complicated. As opposed to this, the non standard definition of limit is a notion which students somehow readily capture. In absolute numbers, 31 students of the experimental group took the exam with this item as opposed to 33 students of the control group.

This exercise gives a straightforward limit, easy to compute for both the control and the experimental group. In spite of this the 87% of the control group answered the item correctly while only the 48% of the control group were able to answer the item correctly, for a difference of 39%. Remembering the the groups are nearly homogeneous, in absolute numbers, 31 students of the experimental group took the exam with this item as opposed to 33 students of the control group. Figure 4 shows the distribution of answers for this item by the participating students.

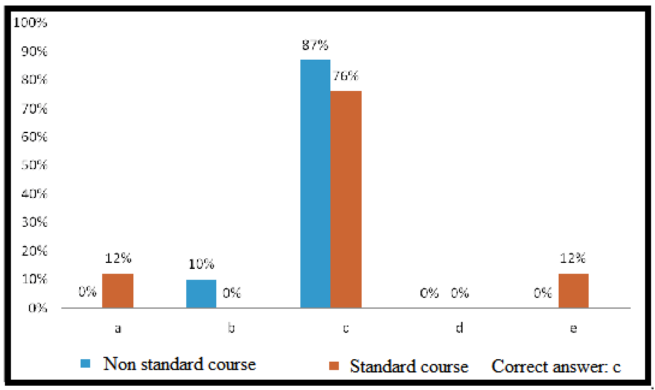


Figure 4: Determine the value of $\lim_{x\to 3} \frac{x^2-9}{x+3}$

LE-B For the non standard calculus group:

If $H > 0$ is an infinite number, find the value of

$$\mathbf{st}\left(\frac{2H^3 - 5H^2 + \pi H - 5}{H^3 - 1}\right).$$

a) does not exist; b) $-\infty$ c) ∞ d) -1

For the standard calculus group:

Find the value of

$$\lim_{x\to\infty} \frac{2x^3 - 5x^2 + \pi x - 5}{x^3 - 1}.$$

a) does not exist; b) $-\infty$ c) ∞ d) -1

In this item 68% of the experimental group answered the item correctly compared to 39% of the control group. This dramatic difference in performace is surprising and, perhaps unexpected, but adds credibility to the hypothesis that there is a cognitive advantage to studying limits in the setting of non standard calculus. Furthermore, this is an infinite limit and as such presents some difficulties of definition and corresponding class discussions. In fact, in the non standard calculus course the scrip "$\lim_{x\to a} f(x) = L$ is elucidated beginning with the case where a and L are real numbers. In non standard calculus any thorough discussion of this script would by necessity take a significant amount of time. In non standard calculus, where the script "$\lim_{x\to\ a} f(x) = L$" is taken to mean that $f(x)$ is infinitely close to L when x is close to a but distinct to a, the property of limits come as a result of the properties of the function "**st**" (standard part) and in this case, the properties constitute an important structural feature of the real numbers within the hiperreal number line. Once this case was discussed in the non standard calculus course, then the script "$\lim_{x\to\ a} f(x) = L$" was elucidated for the cases where L is any real number or any infinite hyperreal and a is either real or an infinite hyperreal. And this is very easily done including one sided limits. Definitions are very natural and students are asked to produce them. For instance, it is not at all surprising to see students define correctly, for instance, $\lim_{x\to\ a+} f(x) = \infty$ to mean that $f(x)$ is an infinite positive hyperreal number whenever x is infinitely close to a and $a < x$.

The summary of the performance of the control and the experimental group on this item appears on Figure 5

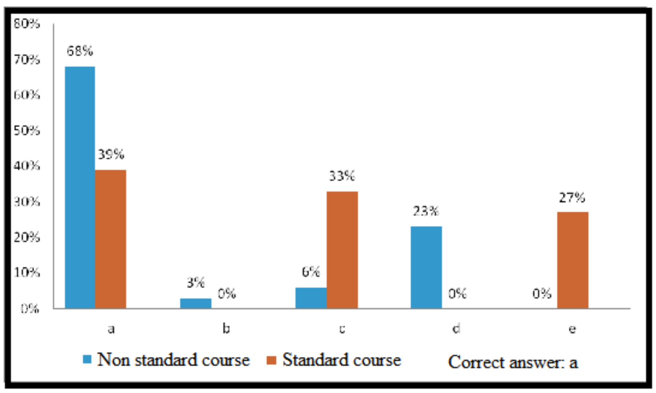


Figure 5: Determine the value of $\lim_{x\to\infty} \dfrac{2x^3 - 5x^2 + \pi x - 5}{x^3 - 1}$

LE-C This exercise, also, has to versions:

Non standard version: If $H < 0$ is an infinite number, the value of

$$\mathbf{st}\left(\frac{\sqrt[3]{H}}{H^5 + 1}\right) \text{ is:}$$

a) there is no value b) 1 c) -1 d) 0

Standard version: The value of

$$\lim_{x\to\ -\infty} \frac{\sqrt[3]{x}}{x^5 + 1} \text{ is:}$$

a) there is no value   b) 1   c) -1 d) 0

This problem is another infinite limit and 45% of the students in the experimental group answered the item correctly as opposed of 33% in the control group. In general this exercise was a difficult one for both groups as can be expected by the presence of a cube root and the matter of the parity of the expression for negative numbers of large absolute values. Figure 6 has information regarding the behavior of the two groups with respect to this item.

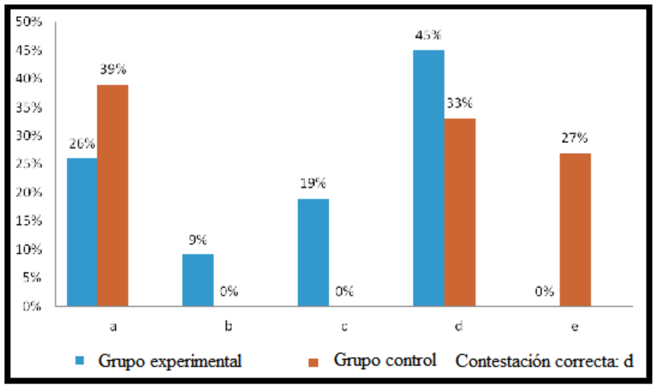


Figure 6: Determine the value of $\lim_{x\to\ -\infty} \dfrac{\sqrt[3]{x}}{x^5+1}$

LE-D $\lim_{x\to 1-} \dfrac{x^2}{x^3-x^2}$:

This is another straightforward exercise which shows a significant difference in results between the experimiental and the control group of our study. As shown in Figure 7 36% of the experimental group were successful in answering the item correctly as opposed to 27% of the control group. In Figure 7 it can be noticed that a sizable group (37%) of the experimental group chose the incorrect alternative c and an even larger group of the control group (54%) choose the incorrect alternative e. Figure 7 shows how the two population of students handled this particular item.

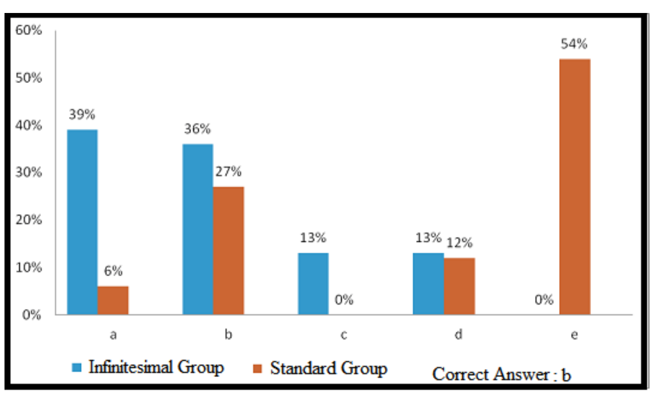


Figure 7: Determine the value of $\lim_{x\to 1-} \dfrac{x^2}{x^3-x^2}$

1.7 Integration exercises

Integration in our version of non standard calculus is interesting and adds credence to Sullivan's CAH. In fact, we used a single axiom, namely that given any continuous function $f : [a, b] \to \mathbb{R}$ there is a function mapping the subintervals $[u, v] \subseteq [a, b]$ (where $a \leq u < v \leq b$), $[u, v] \mapsto I_u^v(f)$ with the property that:

I-a. (additivity over intervals): For all u, v, w with $a \leq u < v < w \leq b$,

$$I_u^v(f) + I_v^w(f) = I_u^w(f), \text{ and} \tag{1.1}$$

I-b. (boundedness): For all u, v with $a \leq u < v \leq b$

$$\min x \in [u, v](u - v) \leq I_u^v(f) \leq \max_{x \in [u,v]} (u - v) \tag{1.2}$$

These "axioms" for integration were used by Newton and Barrow; see Hernández and López [13, 2012]. In fact, relation (1.1) is taken to be esseantial for integration and relation 1.2 can be interpreted as saying that the area under the graph of $y = f(x)$ between u and v lies between the area of the inscribed rectangle and the area of the cirumscribed rectangle. For instance, Newton's proof of the First Fundamental Theorem of the Calculus (FFTC) uses this fact for an infinitesimal interval, and then argues that the area can be realized as the area of some intermediate rectangle between the inscribed and the cirumscribed one; see (ibid.) [13, 2012; p. 3]. The first to suggest the axiomatic treatment of the theory of the elementary integral was Gillman [10, 1993] and the technique was popularized in texbooks by Gillman [11, 1978] and also by Lang [20, 1986]. We used these two axioms in the setting of non standard calculus to get an efficient presentation of the theory of the integral for continuous functions, in which many arguments of Newton and Barrow regarding the integral get written in the modern mathematical lingo of non standard calculus. No use are made of Riemann sums as a basis for the presentation of the theory of integration. In fact, there is mounting evidence in the mathematics education literature that shows the inveterate difficulties that students have in understanding complex limits like those used in showing, for example, that all Riemann sums of the function over the intervals converge to the integral of the function as the number of intervals increase indefinitely; see Tall & Vinner [28, 1981]. In spite of all these remarks, it is possible to show, using our axioms that the elementary integral is the limit of Rimemann sums[22]; see Hernández & López [13, 1912, Theorem 4].

Since the experimental group had a more thorough introduction to the elementary integral using based on the discussion of the geometric points just discussed, it comes at no surprise that the students of the experimental course performed at an advange when answering the items related to integration. We discuss five of these items.

IE-A If $y = f(x)$ and $y = g(x)$ are integrable in $[1, 10]$ and

$$\int_1^2 f(x)\,dx = -4,$$
$$\int_1^5 f(x)\,dx = 6,$$
$$\int_1^5 g(x)\,dx = 8,$$

This item was one of extremely good performance for the experimental group. Of the students of the experimental group 89% answered the item correctly as opposed to 71% of the students in the control group. In Figure 8 we find the details of how the population of both groups choose their answers.

[22] Cauchy's original result about the integral as a sum of Riemann sums is proved today using the uniform continuity of of continuous functions on closed intervals; see Cauchy [5, 1899] and Hernández & López [13, 1912, Theorem 4]

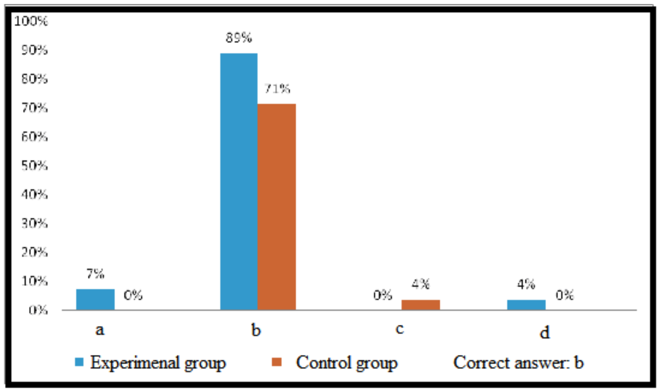
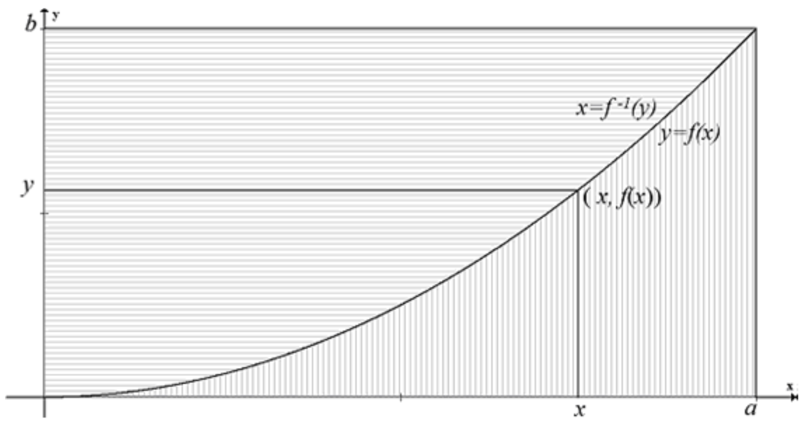


Figure 8: Integration Problem A



IE-B The figure shows the graph of an invertible function $y = f(x)$. By observing the two regions with shaded areas determine which of the following relations are likely to be true:

a. $ab = \int_a^b f(x)\,dx + \int_0^a [b - f(x)]\,dx$

b. $ab = \int_0^b f^{-1}(y)\,dy + \int_0^b [a - f^{-1}(y)]\,dy$

c. $ab = \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)]\,dy$

d. all previous alternatives

e. none of the previous alternatives

This question presented some problems insofar as it moved students to make a choice they ascertained as correct and not bother to check other possible choices. About half of each group chose the correct answer to the item. None of the students of the experimental group chose alternative d while a fourth of the student in the control group chose this alternative. This alternative was attractive to students with poor understanding of the subjacent geometry implicit in the theory of the elementary integral.

IE-C If $\int_1^2 f(x)\,dx = 5$ find the value of $\int_1^2 [1 - f(x)]\,dx$. This question required the students to solve the problem on a sheet of paper and a rubric was prepared for grading the exercise. Student's were also

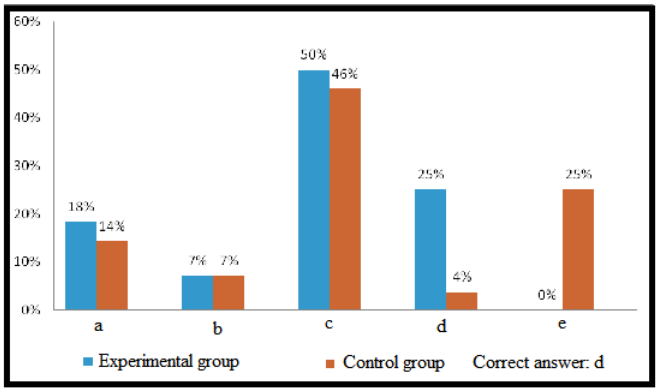


Figure 9: Integration Problem B

interviewed following a protocol to discuss their solutions and their approach to the problem. The rubric accounted for 6 total points to be accumulated as follows: 1 point for applying the axiom of additivity and its consequences to the expression $\int_1^2[1-f(x)]\,dx$; 2 points for carrying out the resulting integration; 3 points for using the hypothesis of the problem and write the answer correctly.

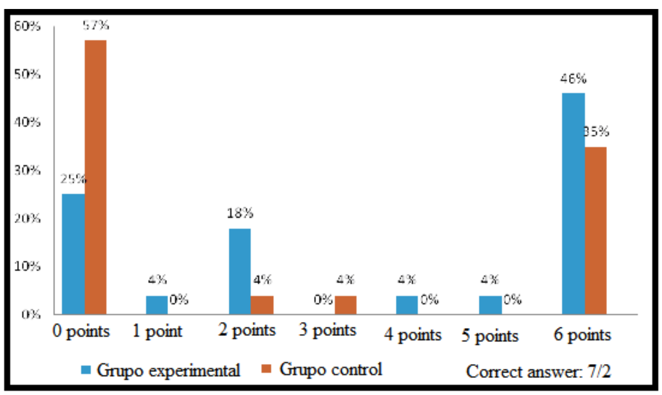


Figure 10: Integration Problem C

IE-D Express the shaded area in the figure as in integral.

This item was one that required the direct student production as opposed to the choice answer from a set of proposed possible answers. It was graded as an all or none valued question. The correct answer is

$$\int_0^2 [f(x) - g(x)]\,dx.$$

Of course, there are other possible expressions for the answer, in one assumes that the functions have inverses, but students were instructed to answer and assume only the stated hypothesis. The population behavior with respect to this item is depicted in Figure 11. It is evident that the experimental group students obtained a significantly higher percentage of correct answers.

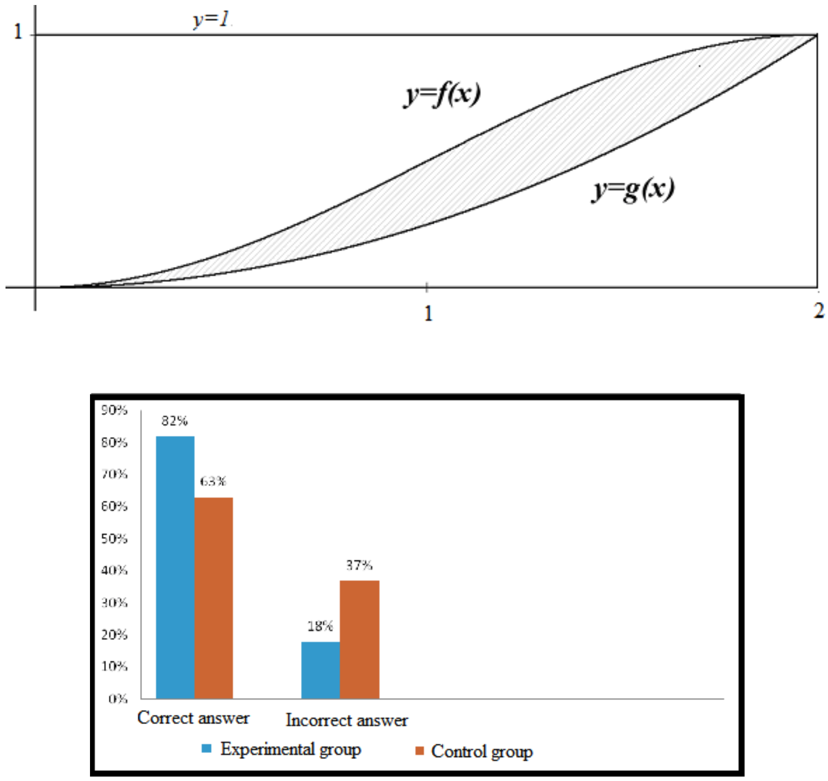


Figure 11: Integration Problem D

2 Calculus history and Sullivan's cognitive advantage hypothesis (CAH)

Calculus, a creation of the second half of the seventeenth century, is regarded as a monumental creation of the human mind. It is no exaggeration to assert that the mathematical tradition started with the creation of the calculus changed the way we do and think about mathematics, what we regard as *interesting* in our mathematical endeavours, shaped our mathematical tastes and and beliefs, our conception of *rigor* for times to follow and, certainly, ended up changing the way we all carry on our persoal lives; see Kitcher [19, 1984, §10, pp. 229-271]. Given the depth and scope of the calculus developed during that time, the mathematical knowledge generated during second half of the seventeenth century[23] was dramatic and awesome, specially in light of the cryptic knowledge about the "structure of number" characteristic of the times. It is no exaggeration to claim that most of the calculus available today to students in modern high schools and universities was already known in the times of Newton and Leibniz. The Geometrical Lectures of Isaac Barrow the first treatise ever written about calculus, although it is hardly recognizable as such to the modern student of mathematics[24]. In the writings of Newton, Leibniz, L'Hospital and Euler we recognize the algebraic version of calculus that was passed onto us; its intuition largely relied on the notions of movement and change while its argumentation rested on the notion of an infinitesimal. Right from the beginning, calculus was a very successful form of mathematics that yielded an unprecedented plethora of solutions to problems that had resisted the efforts of previous mathematics (or that were only solved in special cases[25]); see [19, 1984, §10, p. 230]. But, right from its onset, calculus was also plagued with inconsistencies and logical flaws that were quickly challenged by George Berkeley, whose criticisms, according to [19, 1984,p.

[23]by Barrow, Newton, Leibniz, L'Hospital, the Bernoulli brothers, Euler and others

[24]Barrow attempted an *axiomatic* approach to calculus in the geometric tradition of Euclid, in which the geometric objects of study were called "curves" and correspond, roughly, to today's graphs of functions.

[25]like the quadrature of the parabola solved by Archimedes for the parabolic sector

239], were presented clearly and displayed a competent reading of Newton's mathematics.

In our view, the CAH can be read implicitly in the account Kitcher [19, 1984, §10, p. 229] gives on the historical development of the calculus. Although Berkeley's criticisms were taken seriously as remarked by Kitcher (ibid), the point is that calculus was being unprecedentedly successful in resolving open problems of mathematics as never before or after, and although Newton attempted formal justifications to his use of infinitesimals (ibid.) [p. 237], the efforts required were to be directed towards the generation of new mathematics rather than to ascertain that the methods employed had the blessings of what was at the time regarded as formally correct. Leibniz' approach to the misgivings of calculus pointed out by Berkeley, on the other hand, was more nonchalant, supposing that mathematics in the long run would take care of appropriately explaining the method of infinitesimals (as it actually occurred). Kitcher [19, 1984, p.163] discusses the work of Kuhn [17, 1967] and contraposes Kuhn's idea of a revolutionary change of paradigms in science to the changes of paradigms in mathematics. In Kitcher own words: "One of Kuhn's major insights about scientific change is to view the history of a scientific field as a sequence of practices. I propose to adopt an analogous thesis about mathematical change. I suggest that we focus on the development of *mathematical practice*, and that we view a mathematical practice as consisting of five components: a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important and a set of mathematical views (including standards for proof and definition and claims about the scope and structure of mathematics)." In science the changes of paradigms are drastic (revolutionary) and, in principle, paradigms cannot coexist and, invariably, the new paradigms refute the old ones. For example, given that the theory of special relativity that requires the laws of physics to be invariant under the action of the Lorenz transformations, electromagnetism (but not Newton's equations of mechanics), turn out to be consistent with the new theory. In principle Newton's equations of mechanics are good for a first approximation to the mechanics of macro bodies but are blatantly off target for bodies with large de Broglie wavelengths. In mathematics, however, something quite different (although completely expected) happens. All results of the calculus as developed by Newton and Leibniz with the method of infinitesimals, as it happens, continue to be consistent with the new paradigm of the weierstrassian epsilon-delta definition of limit, and thus remain integrated to the formal corpus of mathematical knowledge. Mathematical statements formally deducible from the paradigms of superseded mathematical theories, remain deductible from the new paradigms but, somehow, they lose their cognitive advantage in the generation of new knowledge. One would be surprised to imagine that the weierstrassian "static" definition of limit would be conducive to the great effervescence of the discovery of the results of the initial calculus, which so much relied on intuitions stemming on kinematic metaphors and the idea of change (fluxions and flows). Hence the conclusions based on the paradigm of weierstrassian or standard analysis and the results obtained from Robinson's non standard analysis, both remain true (as deductions from first principles) in the corpus of mathematical facts; they, in fact, coexist harmoniously as opposed to the analogous case in physics. In mathematics statements remain formally deductible, but lose, somehow, their cognitive advantage and their ability to generate new deductions, and this gets dramatically reflected in the didactical and historical phenomenologies of the mathematical structures associated with the the corresponding statements.

Thus, in our view, this work adds valuable information to Sullivan's original results and sheds some revealing connections between the history of mathematics and its teaching, both from the perspective of Freudenthal's phenomenological studies of mathematical structures and also, from the framework of Sausserian semiotics. In the case of the integral, we propose an additional semiotic argument to substantiate how the economy of thought characteristic of Newton's and Leibniz approach to the elementary integral translates into a streamlined and direct approach to integration theory, based on simple properties of the integral and not directly related to the existence of complicated limits of Riemann or Darboux sums. This constitutes a clear cognitive advantage in the teaching of elementary integration.

3 Theoretical framework: some semiotic and phenomenological considerations

The study of the mathematical structures of the second half of the seventeenth century (at the time calculus was invented) reveals that, as far as the calculus is concerned, these structures were build on a cryptic and incomplete conception of "number" that allowed for the existence of infinitely large and infinitely small numbers (see, for instance, Whiteside [31, 1961]). This conception lead to contradictions and to many formal inconsistencies (see Kitcher [19, 1984, p.229]). Although the understanding of "number" was at best cryptic during that time period, it can be argued, nevertheless, that infinitesimals constituted a very useful cognitive framework for the development of the calculus. The efficient use of infinitesimals for reasoning in calculus is amply illustrated in the works of Newton and Leibniz (ibid.) and effectively subsumed in L'Hôspital [15] famous treatise *Analise de Infiniment Petits*[26]. The "method of infinitesimals" reaches truly inspirational levels with the work of Euler; see, for example, his discussion of the natural logarithm in Euler [7, pp. 17-37]. In Newton's time Bishop George Berkeley raised incisive criticismagainst the use of infinitesimals, pointing to instances where infinitesimals were conveniently used, sometimes as non zero quantities while at other times they were discarded for being "essentially" zero. Today, no one doubts the correctness of Berkeley's objections to the use of infinitesimals in the arguments of calculus, and yet no one relinquished their use as it yielded new and interesting results in mathematics. Leibniz himself pressed for the use of infinitesimals in his calculus while at the same time recognizing the problems they entailed, as he hoped for their resolution as further enquiries developed. Mathematicians only gave serious attention to infinitesimals only when they openly interfered with their ability to do mathematics. It was not until Cauchy's famous "proof" that a converging series of continuous functions converges to a continuous limit[27] (Cours d'Analyse, p120) that mathematics became concerned with the use of infinitesimals. Anyone reading Cauchy's proof will certainly coincide with Kitcher (pag. 254) in that the proof flows "easily in the language of infinitesimals". Kitcher (ibid) remarks that it took mathematics a great effort to understand the nature of the flaw in Cauchy's argument, and its resolution changed the nature of analysis forever after. Weierstrass succeeded in eliminating the infinitesimal language used by a long tradition of mathematicians, beginning in Newton's and Leibniz' times end effectively ending with Cauchy. Weierstrass thus endowed analysis with the rigor it was lacking, thus effectively answering Berkeley's objections regarding lack of rigor (Kitcher, p.258-259). It should be remarked that Weierstrass profound influence in mathematics is consistent with what we call the "pragmatic productivity criterion" proposed by him (pag. 268), by which mathematical revision and renovation is not guided by "exalted epistemological aims" but rather by "an attempt to respond to the needs of mathematical research". Infinitesimals were not discarded until it became evident that Weierstrass alternative algebraic method for analysis yielded abundant new, interesting and, sometimes, unexpected results[28]. The weierstrassian epsilon arguments with inequalities and quantifiers became the substitute of "the method of infinitesimals". In short, mathematicians of the 19th century adhered to Weierstrass algebraic analysis precisely for the same reason that mathematicians adhered to the infinitesimal analysis of the seventeenth century: they both promoted mathematical research and discovery.

3.1 A cognitive disadvantage implied?

From the very beginning, motion is at the heart of the development of Calculus. Barrow (1570) [2, Lectures II-IV, pp. 42-59] discusses in some detail how "magnitudes" are generated my " local motions" and discusses as geometric transformations some motions such as translations and rotations. Points in motion, for instance, generate curves, lines (or segments) in motion generate surfaces. and surfaces in motion generate solids. For instance, integrals are generated by moving ordinates[29], that is, by a line segment with one endpoint in the horizontal axis and the other one on a curve[30]. Derivatives or differences also entailed the study of quantities that changed[31]. Thus the initial calculus was conceived and explicitly noted[32] to reflect a discipline intimately tied with motion and change. It is appropriate to remark at this point that with the consolidation of the weierstrassian method for analysis, calculus gained a rigor it never had before, but,

[26] at least for the case of derivatives; and, of course, we know today that the mathematical ideas in L'Hospital book were mainly those of Jacob Bernoulli.

[27] Abel had given an example in 1826 of a trigonometric series which converges to a function with discontinuities.

[28] Between 1857 and 1887 Weierstrass delivered thirty six sets of lectures on elliptic function theory. In this work Weierstrass finished crystalizing the details of his algebraic approach to analysis.

[29] which Barrow represented graphically as repeated positions of a given ordinate as in Ibid. [**?**, Lecture 10, p.117, Fig. 109]

[30] or, similarly, one endpoint in one curve and the other one in another.

[31] In the language of Newton derivatives were fluxions that showed the rate of change of fluents which were the rate of change of quantities that were constantly changing, usually with respect to time. This brought about the coining of language to reinforce these conceptions, amenable to movement and change, that have remained to this day (like "dependent" versus "independent" variables in the setting of the modern notion function).

[32] This is amply evidenced by Leibniz notation for differences and derivatives.

in our view, lost its "dynamic character" which it had since it beginnings. In terms of the teaching of the calculus, it can be argued that it ended up being formally correct but less intuitive the theory of limits, the cohesive element of Weierstrassian analysis, somehow failed to convey the dynamic nature of calculus and rendered the metaphors of change and motion of the initial calculus more removed from its theoretical formulation.

4 Conclusions

The present research corroborates K. Sullivan's initial results as stated in her epoch making article. We added to her research by including the elementary integral in the research, with similar positive results. In this essay we define the notion of cognitive advantage, using ideas of Kitcher and Kuhn as an attempt to justify formally K. Sullivan's claim of an observed "advantage" when referring to the improved understanding of calculus students when the course is presented in the language of infinitesimals as opposed to the standard approach of Weierstrass.

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