The Mathematics Enthusiast

Volume 5 | Number 1 Article 14

1-2008

Mathematically gifted elementary students' revisiting of Euler's polyhedron theorem

Jaehoon Yim

Sanghun Song

Jiwon Kim

Recommended Citation

Yim, Jaehoon; Song, Sanghun; and Kim, Jiwon (2008) "Mathematically gifted elementary students' revisiting of Euler's polyhedron theorem," *The Mathematics Enthusiast*: Vol. 5 : No. 1 , Article 14.
Available at: https://scholarworks.umt.edu/tme/vol5/iss1/14

The mathematically gifted elementary students' revisiting of Euler's polyhedron theorem

Jaehoon Yim, Sanghun Song, Jiwon Kim[1]

Gyeongin National University of Education, South Korea

Abstract: This paper explores how the constructions of mathematically gifted fifth and sixth grade students using Euler's polyhedron theorem compare to those of mathematicians as discussed by Lakatos (1976). Eleven mathematically gifted elementary school students were asked to justify the theorem, find counterexamples, and resolve conflicts between the theorem and counterexamples. The students provided two types of justification of the theorem. The solid figures suggested as counterexamples were categorized as 1) solids with curved surfaces, 2) solids made of multiple polyhedra sharing points, lines, or faces, 3) polyhedra with holes, and 4) polyhedra containing polyhedra. In addition to using the monster-barring method, the students suggested two new types of conjectures to resolve the conflicts between counterexamples and the theorem, the exception-baring method and the monster-adjustment method. The students' constructions resembled those presented by mathematicians as discussed by Lakatos.

Key words: counterexample, elementary students, Euler's polyhedron theorem, Lakatos, mathematically gifted

1. Introduction

One perspective on mathematics education states that it is important to analyze and reconstruct the historical development process of mathematical knowledge for improving mathematics teaching and learning. A number of scholars including Clairaut (1741, 1746), Branford (1908), Klein (1948), Toeplitz (1963), Lakatos (1976), Freudenthal (1983, 1991), and Brousseau (1997) share this perspective. This view usually assumes a close relationship between the historical genesis and individual learning process, and supposes that students, with the assistance and guidance of a teacher are capable of constructing knowledge similar to that obtained historically by mathematicians. In particular, Lakatos (1976) demonstrated this view in his book, *Proofs and Refutations*, through an imaginary conversation between a

[1] suhak@ginue.ac.kr

***The Montana Mathematics Enthusiast*, ISSN 1551-3440, Vol. 5, no.1, pp. 125-142**

teacher and pupils. The teacher and pupils support and criticize one another's claims from the perspective of various historical figures. However, the knowledge construction carried out by the teacher and pupils as presented by Lakatos is, in fact, the construction performed by prominent mathematicians including Euler, Legendre, and Cauchy. Lakatos' (1986) quasi-empirical view seems to ask students to learn mathematics by working like mathematicians (Chazan, 1990) prompting the question, "Is it also possible for elementary students to carry out knowledge constructions based on Euler's polyhedron theorem similar to those produced by mathematicians as discussed by Lakatos'?" In seeking a response to this question, this study focuses on (1) the knowledge constructions of mathematically gifted elementary students in comparison to those of mathematicians as discussed by Lakatos (1976), (2) how mathematically gifted fifth and sixth grade students justify Euler's polyhedron theorem, (3) the figures they suggest as counterexamples to Euler's polyhedron theorem, and (4) how they react when presented with counterexamples.

2. Background

2. 1. Literature Review

Sriraman found (2003) that the problem solving behaviors of mathematically gifted high school students' and those of non mathematically gifted students differed significantly. He reported that gifted students invest a considerable amount of time in trying to understand the problem situation, identifying the assumptions clearly, and devising a plan that was global in nature. Previous studies on the cognitive processes of mathematically gifted students have focused on generalization, abstraction, justification, and problem-solving (Krutetskii, 1976; Lee, 2005; Sriraman, 2003; 2004). Lee (2005) also found that mathematically gifted students have a tendency to advance to higher-level reasoning through reflective thinking.

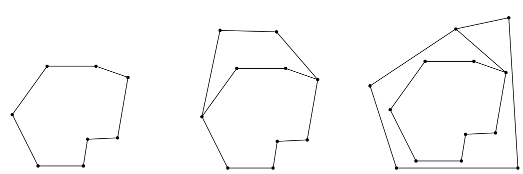
Some researchers have analyzed the knowledge construction of students based on Lakatos' perspective (Athins, 1997; Boats et al., 2003; Borasi, 1992; Cox, 2004; Nunokawa, 1996; Reid, 2002; Sriraman, 2006). For example, Sriraman (2006) reconstructed the quasi-empirical approaches of six above average high school students' attempts to solve a counting problem and present the possibilities for mathematizing during classroom discourse in the spirit of Lakatos. Cox (2004) reported that the ability of high school students to proof improved after introducing them to the process of 'conjecture → proof → critique → accept or reject' in geometry classes. Borasi (1992) described the process where two high school students revised the definition of polygon and concluded that working on polygon "à la Lakatos" provided the context for valuable mathematical thinking and for activities that encourage participants to make use of their mathematical intuition and ability. Reid (2002) analyzed the problem-solving process of fifth-grade students and categorized their process of dealing with counterexamples based on monster-barring and exception-barring into three reasoning patterns. Athins (1997) reported that he observed a case of monster-barring on angles in a fourth grade mathematics class.

2.2. Euler's polyhedron theorem in Lakatos' *Proofs and Refutations*

In Lakatos' (1976) *Proofs and Refutations*, some justifications for Euler's theorem such as Cauchy's proof that appeared in the history of mathematics are shown in the dialogues between the teacher and pupils. For example, Lakatos has pupils Zeta and Sigma say the following explanation (pp.70-72).

Step 1 : For a polygon, $V = E$.

Step 2 : For any polygon $V - E = 0$ (Fig. 1 (a)). If I fit another polygon to it (not necessarily in the same plane), the additional polygon has n_1 edges and n_1 vertices; now by fitting it to the original one along a chain of n'_1 edges and n'_1+1 vertices we shall increase the number of edges by $n_1 - n'_1$ and the number of vertices by $n_1 - (n'_1+1)$; that is, in the new 2-polygonal system there will be an excess in the number of edges over the number of vertices: $E - V = 1$; (Fig. 1 (b)); for an unusual but perfectly proper fitting see Fig. 1 (c). 'Fitting' a new face to the system will always increase this excess by one, or, for an F-polygonal system constructed in this way $E - V = F - 1$.



(a) (b) (c)

Figure 1

Step 3 : I can easily extend my thought-experiment to 'closed' polygonal systems. Such closure can be accomplished by covering an open case-like polygonal system with a polygon-cover: fitting such a covering polygon will increase F by one without changing V or E. Or, for a closed polygonal system – or closed polyhedron – constructed in this way, $V - E + F = 2$.

Following the conjecture and proof, there appear counterexamples that refute the conjecture and proof. Lakatos called a counterexample that refutes lemma or subconjecture a *local* counterexample, and a counterexample that refutes the original conjecture itself a *global* counterexample (pp. 10-11). He suggested six types of counterexamples which appeared in the history of mathematics as described below.

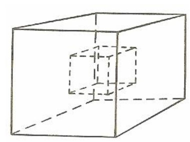


Figure 2. Hollow cube (p.13)

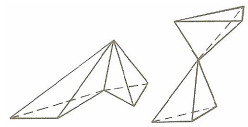


Figure 3. Two tetrahedra with a common edge or vertex (p.15)

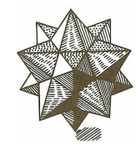


Figure 4 . Star-polyhedron (p.17)

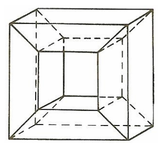


Figure 5. Picture-frame (p.19)

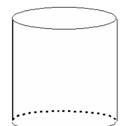


Figure 6. Cylinder (p.22)

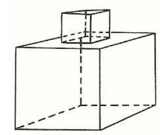


Figure 7. Crested cube (p.34)

When a general counterexample is presented, there are five options. The first option considers the refuted conjecture incorrect and rejects it. The second option is to use the method of monster-barring in which the counterexample is seen as a monster, and the original conjecture is maintained (pp.16-23). This method generates clearer definition, but it is not useful from a heuristic point of view because it does not improve the conjecture. The third option is the method of exception-barring in which the original conjecture is changed into a revised conjecture by adding a conditional clause that mentions an exception (pp.24-27). This method does not guarantee that all exceptions are specified, and leaves the question of what is the range in which the theorem is valid. The fourth option is the method of monster-adjustment where the perspective under which the example was considered as a counterexample is seen as distorted, and the counterexample is interpreted as an example by readjusting the perspective (pp.30-33). The fifth option is the method of lemma-incorporation, where careful analysis of the proof is made to identify the guilty lemma. The lemma can then be incorporated in the conjecture to improve the refuted conjecture (pp.33-42).

3. Methodology

3.1. Participants

Although there are diverse definitions of mathematical giftedness, there is no one universally accepted definition (e.g., Bluton, 1983; Miller, 1990; Gagne, 1991). In this study, Gagne's (1991) definition of mathematically gifted students as "students who are distinguished by experts to have excellent ability and potential for great achievements" was applied. Eleven fifth and sixth-grade male students (aged 10 -12) from different Korean elementary schools in Gyeonggi province participated in the study. Five students were in the fifth-grade and six were in the sixth-grade. The sixth grade students were attending an advanced program for mathematically gifted students; three (A, B, and C) in a Korean government sponsored university program, and three (D, E, and F) in an office of education program. The fifth-grade students (G, H, I, J, and K), having passed a screening process which included a written test, an in-depth interview, and recommended from their school principal, were scheduled for admission to the university program. All students were motivated and confident of mathematics.

3.2. Tasks

The participants were presented with the following tasks:

Task 1: Explain what you know about the relationship between vertices (V), edges (E) and faces (F) in polyhedra. Explain how the relationship is justified.

Task 2: Is $V-E+F=2$ true in all polyhedra? If not, when is it not true?

Task 3: If you consider a counterexample a polyhedron, how would you revise the theorem?

If you believe a counterexample is not a polyhedron, how would you revise the definition of a polyhedron?

Task 1 was designed to identify the participants' knowledge of the polyhedral theorem and to determine how they justify the theorem. Task 2 was developed to establish the types of counterexamples the participants identified. Task 3 was designed to observe how the participants resolved the disparity between the theorem and the counterexample.

The participants were familiar with the relationship between vertices, edges and faces, $V-E+F=2$, before taking part in this study. However, they had not previously examined whether the theorem was true in all polyhedra, nor had they sought counterexamples to the theorem.

3.3. Data Collection and Analysis

This study was designed based on Yin's (2003) multiple case study methodology. The eleven participants were presented with the tasks in a set order and interviewed between November 2005 and January 2007. Each participant was video-taped by one researcher while they worked on the tasks and later while being interviewed by another researcher. The participants completed the tasks in approximately two hours. The video clips, transcriptions, observation reports and participants' worksheets were analyzed.

The analysis was conducted on three types of data collected: (a) the types of justification, (b) types of counterexamples, and (c) the methods for solving the conflict. The types of justification and counterexamples presented by the participants were analyzed using open coding (Strauss and Corbin, 1998). The types of justification were divided into two categories, and the counterexamples were categorized into four types, three of which were subdivided into two to three subtypes. The analysis of the participants' attempts to deal with the disparity between the counterexamples and the conjectures highlighted by the counterexamples was made using selective coding (Strauss and Corbin, 1998) which was based on "the method of monster-barring," "the method of exception-barring," "the method of monster-adjustment" and "the method of lemma-incorporation" suggested by Lakatos (1976). Cross-tabulation analysis was performed, and the results were examined by peers (Merriam, 1998).

4. Results

4.1. Participants' justification of Euler's polyhedron theorem

The participants' justification of the theorem can be divided in two ways; 1) to classify polyhedra into several categories and justify the theorem for each category of polyhedra, and 2) to attempt general justification without classifying polyhedra. The majority of participants justified the theorem by classifying polyhedra into categories and justifying the theorem. Participant D, in Episode 1 below, demonstrated this by logically explaining that the theorem is justified in prisms, pyramids, and prismoids.

Episode 1:

Participant D: First, in prisms, it seems to be justified in all cases.

Interviewer: Why is that?

Participant D: (Drawing figures) Well, look at an n-angle prism. A rectangular prism, it's called that because the bases are rectangles. So, there are four vertices on the top face and four on the bottom face, so, the number of vertices is $2n$. Also, the number of edges is $3n$ because there are four edges on the top face, four on the bottom face, and four on the lateral sides. And, the number of faces is $n+2$ because there are four faces on the lateral sides plus the top and bottom faces. In the case of a pentagonal prism, also, the number of faces is $n+2$, as there are five lateral faces plus the bases (top and bottom faces). '$V-E+F$' stands for 'number of vertices – number of edges + number of faces,' and in n-angle prisms, it is '$2n-3n+(n+2)$,' so '$V-E+F$' equals 2.

Interviewer: Yes.

Participant D: So, I'm done with prisms... in pyramids, too, it is justified all the time.

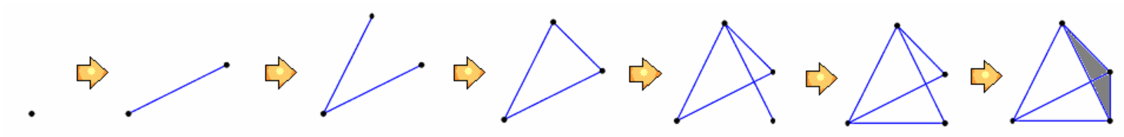
Interviewer: Please explain.

Participant D: ..... an n-angle pyramid. It's justified because the number of its vertices is $n+1$, and it has $2n$ edges and $n+1$ faces. If you add the number of vertices and the number of faces, and then subtract the number of edges, you get 2.

Participant D provided explanations using polyhedra such as rectangular prism in the case of prisms, pyramids, and prismoids. Rectangular prism is a generic example (Mason and Pimm, 1984) which represents general n-angle prism. In the case of regular polyhedra or a polyhedron like the soccer ball, D investigate the theorem application by counting the numbers of points, edges, and faces of specific solids.

Participant B did not categorize solid figures but instead attempted generalized justification. He started with a point and verified $V-E+F$ as the number of points, lines, and faces gradually increased. According to him there is only one V at first, but V and E or E and F increases by 1 respectively as procession is made from (a) to (g) and,

$V-E+F$ is maintained at 1. In the last stage, when one face is covered in (g), he proved that $V-E+F=2$, based on the fact that the number of F increases by 1. This justification is similar to the explanation of pupils Zeta and Sigma in Lakatos (1976, pp. 70-72).



(a) (b) (c) (d) (e) (f) (g)

Figure 8

After justifying Euler's theorem, all the participants expressed the view that there might be a polyhedron with which Euler's theorem was not true. For example, participant D, as indicated in Episode 2, thought that the theorem would not hold in all polyhedra.

Episode 2:

Participant D: Well... first, it is justified in regular polyhedra without exception, because there are only five kinds of regular polyhedra. I think it is justified in all of the five, and then, it is justified, first, in prisms and pyramids. So, I think it is justified in the majority of general polyhedra...

Interviewer: Then, do you think there are some cases in which it doesn't apply?

Participant D : *In some cases... I think it won't apply in all cases.* (Starts drawing figures to find solids with which the polyhedral theorem is not true)

Although participant B justified the theorem using a general method, he tried to find a counterexample, thinking that there still might be one. All the participants express the view that there had to be an example in which the theorem does not apply.

4.2. Solid figures suggested by participants as counterexamples

Participants suggested various types of solid figures as counterexamples to the theorem. The solid figures suggested by the participants were categorized into the four groups below.

4.2.1. Solids with curved surfaces

Six participants (B, C, E, F, H and I) suggested solids with curved surfaces such as a cone (Fig. 9), a cylinder (Fig. 10) and a sphere (Fig. 11) as counterexamples. Each participant had a different reason for suggesting the cone as a counterexample. Participant F drew the net of a cone in order to count the points, lines, and faces. He claimed that the circle in the net was not counted as an edge because it was a curve, but the radius of the sector had to be counted as an edge because it was a straight line ($V=1$, $E=2$, $F=2$, $V-E+F=1$).

Participant E insisted that the radius of the sector in the net could not be counted as an edge because it was not actually seen in the solid, and thus, $V=1$, $E=0$, $F=2$, $V-E+F=3$. Participant H said that a cone provided a counterexample, "Because you can't say how many edges there are in a circle."

Figure 9

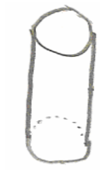


Figure 10

Figure 11

4.2.2. Solids made of multiple polyhedron sharing points, lines, or faces

Nine participants (A, B, C, D, E, F, G, H and I) cited solids made of two polyhedra sharing points, lines, and faces as counterexamples. These solids can be divided into (1) solids that completely share some points, lines, or faces (Fig. 12 through Fig. 15), and (2) solids that only partially share lines or faces (Fig. 16 through Fig. 19).

4.2.2.1. Solids that completely share points, lines, or faces

In solids that share one point as shown in Fig. 12, the theorem holds in each polyhedron and two polyhedra share a point, $V-E+F=3$. Participants also suggested solids that share an edge (Fig. 13) and those that share a face completely (Fig. 14 or Fig. 15) are counterexamples.

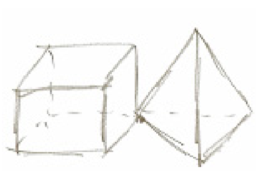


Figure 12

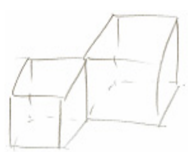


Figure 13

Figure 14

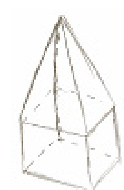


Figure 15

4.2.2.2. Solids that partially share lines or faces

Solids such as in Fig. 14 and Fig. 15 raised the issue with participants of whether it is appropriate to consider shared faces as separated faces. Participants suggested that modified solids that partially share lines or faces were counterexamples.

Figure 16

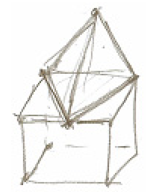


Figure 17

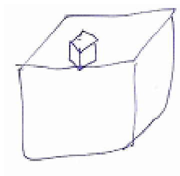


Figure 18

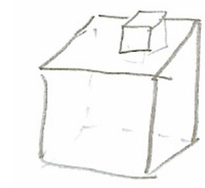


Figure 19

Where edges are partially shared (Fig. 16) and where an edge is divided (Fig. 17) the participants reflected on how to count the number of edges. And a counterexample such as Fig. 19 led the participants to contemplate the question, "Is it appropriate to consider the face created by joining two faces a face?" Lakatos (1976, p.74) called this a "ring-shaped face."

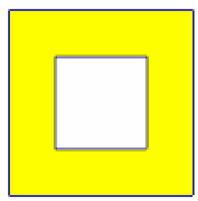


Figure 20. Ring-shape face

4.2.2.3. Polyhedra with holes

The third type of solids that the participants (A, B, C, G, J and K) suggested as counterexamples is solids with holes as shown in Fig. 21 through Fig. 23. These counterexamples also prompted the participants to rethink the definition of face.

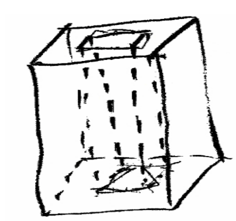


Figure 21

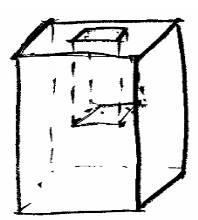


Figure 22

Figure 23

4.2.2.4. Polyhedra containing other polyhedra

Eight participants (A, B, C, D, F, G, J and K) also suggested solids that are polyhedra containing other polyhedra are counterexamples. Counterexamples of this type can be subdivided into three subtypes. The first is the type in which other solids -not sharing any face, point or line- exist in certain solids (Fig. 24). The second is the type in which two solids completely share a face (Fig. 25). The third type is one in which a figure exists inside another and the two figures share part of a face.

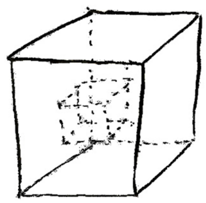


Figure 24

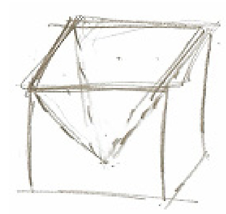


Figure 25

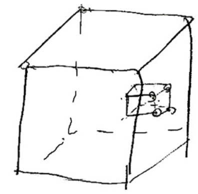


Figure 26

4.2.3. Participants' responses to the disparity created by counterexamples

The participants' responses to the disparity between counterexamples and the theorem are divided into four categories; the method of monster-barring, the method of exception barring; the method of monster adjustments and new conjectures.

4.2.3.1. The method of monster-barring

Participants D and E used the method of monster-barring. In Episode 3, participant E, suggested cones, cylinders, and spheres as counterexamples, and wondered how to determine the numbers of points, lines, and faces in these figures. He then stated, "*A polyhedron is a solid figure made of multiple polygons*", and that the curved surface is not a polygon, and thus, solids with curved surfaces are not polyhedra but monsters.

Episode 3:

Participant E: Cones have curved surfaces, so I think they will not work.

Interviewer: What's wrong with curved surfaces?

Participant E: Because in a curved surface, you can't count the number of edges, and faces... Can you count the number of faces? But the number of vertices is one... I think there is no edge, in the definition that I think of.

Interviewer: Can you say a cone is a polyhedron? Euler's theorem is about polyhedra.

Participant E: When you talk about curved surfaces, a sphere has a curved surface, and a sphere has one face, ... but no distinguishable edge or point, I guess there are none.

Interviewer: What do you think is the definition of a polyhedron?

Participant E: I think it is made of faces that have angles. (Writing down the definition) "Polyhedron = solid figure made of multiple polygons"

Participant D also used the method of monster-barring where polyhedra existed in other polyhedra. He used the method of monster-barring stating that a polyhedron signified "one" solid figure, and that the polyhedron in which there is another polyhedron meant two different solid figures. He modified the definition of polyhedron as "one solid figure surrounded by multiple polygons."

4.2.3.2. The method of exception-barring

Participants A, D, F, G, and I were observed attempting the method of exception-barring. Participant F defined a polyhedron as a "figure made of faces." So the solid figures with curved surfaces are polyhedra because curved surfaces are faces. To exclude cones, cylinders, and spheres as exceptions, participant F modified the original conjecture to "*In all polyhedra excluding those made of curved faces,* $V - E + F = 2$." Participant I, Episode 4, also used the method of exception-barring by modifying the theorem to "*In polyhedra that do not include a circle,* $V - E + F = 2$."

Episode 4:

Interviewer: (Pointing to the sphere and cylinder.) Then, can we call them polyhedra, too?
Participant I: It has one or more faces... We can call them polyhedra.
Interviewer: Then, don't we need to modify this ($V - E + F = 2$) ?
Participant I: Yeah...
Interviewer: How can we change it?
Participant I: (Thinking hard) So, if a circle is included... I guess only the polyhedra without any circles belong to this category ($V - E + F = 2$), don't they?

Participant I suggested two rectangular solids that share one edge (Fig. 27) are another counterexample. Then he redefined the theorem to "In polyhedra that do not include a circle and are not attached to other polyhedra, $V - E + F = 2$." Participant G found solid figures with holes as counterexamples, and modified the theorem to "In polyhedra which are not completely penetrated by a hole, $V - E + F = 2$."

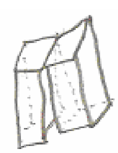
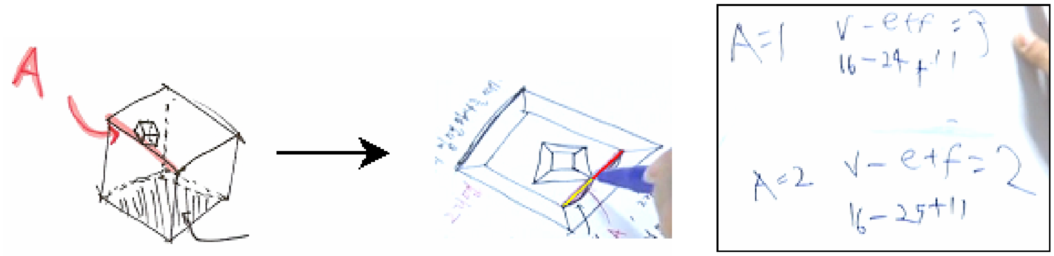


Figure 27

4.2.3.3. The method of monster-adjustment

Participants B, D, E, F, and G tried the method of monster-adjustment to convert a counterexample into an example. Participant B thought, after finding the counterexample in which part of a face was shared by two figures, that the justification of Euler's theorem depended on whether to consider the edge divided by a point as one or two.



(a) (b) (c)

Figure 28

Participant B compared the results when the edge (line A in Fig. 28(a)) divided by a point was counted as one (A=1) and when it was counted as two (A=2). Then, B, Episode 5, explained the reason why the edge divided by a point in this solid figure should be counted as two.

Episode 5:

Interviewer: Is the solid a polyhedron?

Participant B: It is a polyhedron.

Interviewer: Then, what can we do?

(Participant B is writing)

Participant B: If there is a vertex in the middle of an edge (even when it is not on the exact center), the left and right sides of the vertex should be separately counted… It is absolutely necessary to separately count this part (left part of line A) and that part (right part of line A). In the case of plane figures, we count any line between two points separately… In solids, to make it (the value of $V-E+F$) become 2, you need to count the left and right sides of the point separately.

Two polyhedra that completely share a face, with one inside the other (Fig. 25) were also considered not to be a counterexample by one of the participants after the method of monster-adjustment was used. Participant D claimed that the figure was not a counterexample because it was considered a sunken solid without a lid, rather than two solids sharing one face.

For the ring-shaped face (Fig. 20), some participants preferred to use the method of monster-barring by not considering it as a face, and subsequently, employed the method of monster-adjustment by not considering the solid figures with ring-shaped faces as counterexamples (e.g. $V=16$, $E=24$, $F=10$, and thus, $V-E+F=2$ in Fig. 19). Participant I, who used the method of exception-barring for cylinders and spheres, used the method of monster-adjustment for the cone, considering the polyhedral theorem to be justified under the condition of $V=1$, $E=1$, and $F=2$.

4.2.3.4. New Conjectures

The participants' approaches were not limited to monster-barring, exception-barring, and monster-adjustment which are similar in the sense that they are used to support the formula of $V-E+F=2$. Participants also suggested two types of new conjectures. One involved the participants searching for a new formula about the value of $V-E+F$ with which to express the relationships between the points, lines, and faces in solid figures including the counterexamples they found. New conjectures suggested by participants are summarized in Table 1.

$V-E+F$	Conditions	Participants
0	If the ring-shaped face is not a face, in polyhedra with hole(s)	G
1	In polyhedra including a circle	I
3	In polyhedra that completely share either a point or a line with other polyhedra	E and F
	If solid figures are attached at a vertex, edge, or face	H
4	In polyhedra which contained other polyhedra such as a hollow cube	F

Table 1 Summary of Participants Conjectures

The other type of conjecture, suggested by participant A, relates to the necessity of considering new elements other than points, lines, and faces. He proposed, Episode 6, that a formula that including three-dimensional elements be developed.

Episode 6:

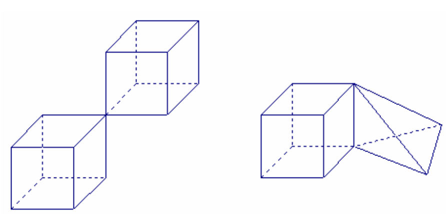
Participant A: In the two-dimensional circumstance, a rule can be easily found using just V, E and F, but in the three dimensions, a new element of space is added. So, if Euler's theorem is a formula established using two-dimensional elements, I guess we can make a new formula that exclusively applies to the third dimension including space, can't we?

Interviewer: The new element of three dimensions. Can we really do it if we consider that?

Participant A: Yes, I think so.

Interviewer: Then how can we determine the numbers in the three dimensions?

Participant A: Space.



(a) (b)

Figure 29

After that, $V-E+F-S=1$ and $V-E+F+S=3$ were proposed as new conjectures, and it was confirmed that $V-E+F-S=1$ is justified with the type of solid figures in Fig. 29 ((a) : $V=15$, $E=24$, $F=12$, $S=2$, $V-E+F-S=1$, (b) : $V=10$, $E=17$, $F=10$, $S=2$, $V-E+F-S=1$). This conjecture led participant A to think that the polyhedral theorem could be expanded to four-dimensional solids.

5. Discussion

Polyhedra, which the participants studied prior to the research, were limited to the category of regular polyhedra, prisms, pyramids, prismoids, and semi-regular polyhedra such as soccer balls, all satisfied Euler's theorem. Nevertheless, the participants thought that there must be some polyhedra for which the theorem was not valid. This belief appears to resulted from the method of justification that the majority of participants used. The value of $V-E+F$ can be obtained by counting the numbers of points, edges, and faces in the case of prisms, pyramids, and prismoids (e.g. in n-angle prism, $V=2n$, $E=3n$, $F=n+2$, and thus, $V-E+F=2$). However, this justification fails to provide information about new kinds of solids that have yet to pass this test. The participants' view that there must be polyhedra with which the polyhedral theorem was not valid indicates they belive that the scope of polyhedra is extensive. This view is supported by the various types of solids that the participants presented as counterexamples.

A strong similarity exists between solid figures suggested by the participants as counterexamples and those discussed by Lakatos (1976). The first type of counterexample that participants found, solids with curved surfaces, appeared as cylinders in Lakatos (p.22). The second type, two or more polyhedra that shares points, lines, or faces, was discovered by mathematicians Hessel (figures that share points or lines) and Lhulier (cube with crest) in 1832 and 1813, respectively (p.15, p.34). The third type of counterexample was first discovered by Lhuilier (p.19). In addition to the tunnel and picture frame mentioned in Lakatos, participants also found a polyhedron which is not completely penetrated. The fourth type, polyhedra within polyhedra, was discovered by Lhuilier and Hessel based on the idea obtained by observing the crystalloid of mineralogic collection enclosed within a translucent crystalloid (p.13).

Counterexamples can be used to help students develop their mathematical reasoning (Lakatos, 1976; Boats, et al., 2003). In this study participants examined concepts such as polyhedron and face and created new definitions. The counterexamples discovered by participants also encouraged them to examine more closely the definition of terms. The ring-shaped face in particular prompted some participants to reconsider the definition of polygon. They asserted that it could not be called a polygon, because the figure did not comply with the sum of interior angles of n-polygon $180\times(n-2)$. This suggests that the formula for the sum of interior angles of a polygon was seen as a definitive property that determines whether the figure was a polygon or not. This method of defining a polygon is similar to the definition of polyhedron stated by Baltzer (Lakatos, 1976, p.16), "polygon system with which the equation of $V-E+F=2$"

In Reid (2002) and Athins (1997), the method of monster-barring and the method of exception-barring were observed among elementary students. The method of monster-barring, the method of exception-barring, the method of monster-adjustment, and new conjectures were observed among the participants in this study. The participants did not reject the original theorem and attempted to develop new conjectures that comprised counterexamples, and of the five participants who used the method of exception-barring, four developed new conjectures. In the past, there have been cases in which counterexamples were first recognized as monsters and excluded, but later reintroduced and accepted as examples (e.g. Lakatos, 1976, p.31). This ability to review and change a position was also demonstrated by the participants. Initially, they used the monster-barring method or exception-barring method for the counterexamples they identified, but they attempted to include the counterexamples within the scope of examples through monster adjustment or new conjecture. Krutetskii, (1976) Sriraman, (2004) point out that this flexibility of thinking is an attribute of mathematically gifted.

Lakatos (1976) argues that the method of lemma-incorporation is a productive way of refining conjecture based on the proof. Proof-analysis is a prerequisite to this method and, as Nunokawa points out (1996) proof-analysis is an important component of proofs and refutations. However, in this study, the method of lemma-incorporation and proof-analysis was not observed. When participant B, provided proof of increasing the elements of polyhedra, was encouraged considering the validity of his proof for a counterexample (Fig. 18), he provided a monster adjustment solution stating, "It's not the proof that's wrong, but there is a problem with this solid."

6. Conclusion

This study focuses on the constructions of mathematically gifted fifth or sixth-grade students in solving tasks related to Euler's polyhedron theorem and compares them to those of mathematicians discussed by Lakatos (1976). By analyzing ninth grader students notion of proof, Sriraman (2004) reports that the processes used by gifted students demonstrate remarkable isomorphism to those employed by professional mathematicians, This study also shows parallels in constructions of mathematically gifted fifth and sixth grade student and mathematicians discussed by Lakatos. With the exception of the method of lemma incorporation and proof-analysis, counterexamples and the method for solving conflicts between the theorem and counterexamples suggested by the participants demonstrated remarkable similarities to those presented in the history of mathematics.

References

Athins, S. (1997). Lakatos' proofs and refutations comes alive in an elementary classroom. *School Science and Mathematics*, 97(3). 150-154.

Bluton, C. (1983). Science talent: The elusive gift. *School Science and Mathematics*, 83(8). 654-664.

Boats, J. J., Dwyer, N. K., Laing, S. & Fratella, M. P. (2003). Geometric conjectures: The importance of counterexamples. *Mathematics Teaching in the Middle School*, 9(4). 210-215.

Borasi, R. (1992). *Learning mathematics through inquiry*. Portsmouth, NH : Heinemann.

Branford, B. (1908). *A study of mathematical education*. Oxford: Clarendon Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield , Ed. and Trans.). Dordrecht: Kluwer academic publishers.

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. *Interchange*, 20(1), 14-23.

Clairaut, A. C. (1741). *Éléments de géométrie*. Paris : Gauthier-Villars.

Clairaut, A. C. (1746). *Éléments de algébre*. Paris : Rue Saint Jacques.

Cox, R. (2004). Using conjectures to teach sutdents the role of proof. *Mathematics Teacher*, 97(1). 48-52.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publishers.

Gagne, F. (1991). Toward a differentiated model of gifted and talent. In N. Colangelo & G.A. Davis (Eds.). *Handbook of gifted education*. Boston: Allyn and Bacon, 65-80.

Klein, F. (1948). *Elementary mathematics from an advanced standpoint: Arithmatic, algebra, analysis*. (E. R. Hedrick & C. A.. Noble, Trans.). New York: Dover publications. (Original work published 1924)

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago: The University of Chicago Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge university press.

Lakatos, I. (1986). A renaissance of empiricism in the recent philosophy of mathematics? In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 29-48). Boston: Birkhauser.

Lee, K. H. (2005). Mathematically Gifted Students' Geometrical Reasoning and Informal Proof, In Helen L. C. & Jill, L. V. (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education,* (Vol 3. pp.241-248), Melbourn, Australia : PME.

Mason, J. & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics,* 15, 277-289.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. John Wiley & Sons, Inc.

Miller, R. C. (1990). *Discovering mathematical talent.* (ERIC Digest No. E482).

Nunokawa, K. (1996). Applying Lakatos' theory to the theory of mathematical problem solving. *Educational Studies in Mathematics*, 31, 269-293.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, 33(1), 5-29.

Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations: The problem-solving experiences of four gifted students. *Journal of Secondary Gifted Education*, 14(3), 151-165.

Sriraman, B. (2004). Gifted ninth graders' notions of proof: Investigating parallels in approaches of mathematically gifted students and professional mathematicians. *Journal for the Education of the Gifted*, 27(4), 267-292.

Sriraman, B. (2006). An ode to Imre Lakatos: Quasi-thought experiments to bridge the Ideal and actual mathematics classrooms. *Interchange*, 37(1-2), 151-178.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Toeplitz, O. (1963). *The calculus – A genetic approach* (L. Lange, Trans.). Chicago: The University of Chicago Press. (Original work published 1949)

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage Publications.