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Christine Browning

Alden J. Edson

Patrick Kimani

Fatma Aslan-Tutak

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:
A Focus on Geometry and Measurement**

Christine Browning
Western Michigan University

Alden J. Edson
Western Michigan University

Patrick M. Kimani
Glendale Community College

Fatma Aslan-Tutak
Boğaziçi University, Turkey

ABSTRACT: This paper summarizes the extant peer-reviewed research on PTs' understanding of geometry and measurement, focusing on a wide variety of topics within these content domains. When looking across the 26 studies reviewed, findings span a variety of content topics, providing little depth in either the geometry or measurement content domain. However, collective findings do indicate PTs' overall conceptions in geometry and measurement to be limited and weak, with PTs relying on memorized procedural processes. Some evidence indicates that cognitive development, along with spatial visualization skills, plays a greater role in learning geometry than memory skills. In addition, the van Hiele levels of geometric learning provide a helpful framework to think about the development of geometric ideas. Direction of future research is elaborated to address ways to develop PTs' understanding of geometry and measurement. Gaps that still exist in the research literature regarding PTs' mathematical content knowledge in geometry and measurement are identified.

Keywords: geometry, measurement mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

Our Beginnings and Theoretical Perspective

The mathematical content knowledge required for teaching elementary mathematics is not insignificant. Elementary teachers are responsible for laying a mathematical foundation for their students on which they can build their current and future understanding of mathematical content. The quality of this foundation relies to a great extent on the quality of the teachers' own mathematical knowledge. "However, the nature of the knowledge required for successful teaching of mathematics is poorly specified, and the evidence concerning the mathematical knowledge that is needed to improve instructional quality is surprisingly sparse" (Kirby, 2005, p. 2).

Recently, there has been an emphasis in the mathematics education community to describe the needed and desired mathematical content knowledge for teaching, with various descriptions emerging from research (e.g., Hill, Rowan, & Ball, 2005; Ma, 1999; National Research Council, 2001; Shulman, 1986). Hill, Ball, and Shilling (2008) provide a framework for distinguishing the different types of knowledge included in a construct of mathematical knowledge for teaching. This framework distinguishes between subject matter knowledge and pedagogical content knowledge, building on the work of Shulman (1986). This framework serves as a theoretical lens for our summary work.

As mathematics teacher educators, our interest is in examining and summarizing peer-reviewed research related to the understanding of subject matter (mathematical content) knowledge described in the Hill et al. (2008) framework. Further, we are interested in research about elementary prospective teachers (PTs), as the development of the mathematical content knowledge for teaching is initiated in teacher preparation. As elementary teachers lay a learning foundation for mathematics with elementary students,

mathematics teacher educators should lay a similar learning foundation for mathematical content knowledge for teaching with PTs.

Children's Understanding of Geometry and Measurement

The focus of this paper is on mathematical content knowledge for teaching elementary mathematics with particular attention to geometry and measurement. Before discussing what we know about prospective elementary teachers' knowledge of geometry and measurement, we briefly articulate research on children's understanding of these topics.

Children's experiences with geometry start even before school. Geometric thinking levels proposed by van Hiele (1999) indicate that elementary school students' geometric thinking starts from recognizing shapes based on their appearance and proceeds to identifying properties of shapes. Clements and Battista (1992) emphasized school geometry's role as mathematizing objects, relationships, and transformations, in addition to developing skills to construct visual representations via spatial reasoning. Furthermore, van Hiele geometric thinking theory emphasizes the importance of *experience* in geometry learning. Simply growing older does not ensure a growth in geometric understanding; children need to experience and engage in many various activities that allow them to explore and construct geometric ideas (Battista, 2007).

Stephan and Clements (2003) addressed children's understanding of measurement. The authors defined measurement as "assigning a number to continuous quantities" (p. 301) and stressed that as children keep learning about numbers and counting, they get more into measurement. In a sense, measurement is an amalgam of understanding of numbers and geometry. Stephan and Clements presented six categories that emerged from

research on learning linear measurement: partitioning, unit iteration, transitivity, conservation, accumulation of distance, and relations between number and measurement. The authors highlighted particular difficulties children had in partitioning and unit iteration for area measurement and angle measurement, along with challenges in structuring an array and in conservation of area measurement. The authors found children's difficulties in linear measurement transfer into learning area measurement. In the case of learning angle measurement, the authors stated children's difficulty of defining the attribute (angle) adds onto partitioning and unit iteration difficulties.

Thus, in the elementary school years, the type and number of experiences in which schoolteachers engage children to reason about and make sense of geometry and measurement will greatly affect their future learning experiences in higher grades (National Council of Teachers of Mathematics [NCTM], 2006). Since the experiences of children are key elements of learning geometry and measurement, the knowledge of teachers who shape those experiences is very important. However, "teachers are expected to teach geometry when they are likely to have done little geometry themselves since they were in secondary school, and possibly little even then" (Jones, 2000, p. 110). Baturu and Nason (1996) corroborate this concern that PTs who were lacking in knowledge in measurement might transfer it to their students, noting that "The impoverished nature of the students' [PTs'] area measurement subject matter knowledge would extremely limit their ability to help their learners develop integrated and meaningful understandings of mathematical concepts and processes" (p. 263).

As the research suggests that PTs' content knowledge is critical in the development of children's understanding of geometry and measurement, knowledge of what PTs

understand themselves about these areas should be of importance to those who are involved in the content preparation of future elementary teachers. Thus, this summary paper reports on the research conducted (as of 2012) that examines PTs' content knowledge of geometry and measurement. Our goals for the research summary were to (a) identify what we know about PTs' knowledge of geometry and measurement, and (b) identify the gaps in the existing research literature to highlight topics that warrant further research.

Research Methods and Analysis

The authors of this paper were part of a larger group of mathematics teacher educators who participated in a series of Working Groups at the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) (e.g., Thanheiser et al., 2010; for more, see the introductory paper of this Special Issue). We were charged with providing a description of what is known about PTs' geometry and measurement content knowledge from peer-reviewed research articles published prior to 1998—a historical look; an in-depth description of what is known about PTs' geometry and measurement content knowledge from 1998 to 2011—a current perspective; and, finally, a view of the horizon from 2011 to 2012 that builds on the previous time periods. Although the charge spans these three time periods, the work of the group started with the current perspective. For this perspective, common methods were established for each subgroup that focused on different mathematical content and are reported in the introductory paper of this Special Issue. This section reports on the methods for the historical look, methods' modifications made by our subgroup for the current perspective differing from the larger group, and the methods for the view of the horizon.

Methods for the Historical Look

As we began to search for peer-reviewed research journal articles published prior to 1998, we first decided to draw upon any of the cited references from current-perspective articles that focused on elementary PTs' geometry and measurement content knowledge. These studies were included in a list of potential studies for the historical look. Second, a search using the Education Resources Information Center (ERIC) database was conducted to find any additional studies. The ERIC search included various combinations of keywords such as *preservice, prospective, elementary, teacher, education, and content knowledge*, specific content terms such as *geometry, measurement, length, area, volume, and angle*, and the prior-to-1998 publication date requirement. This produced a total of 62 studies that were added to the list of potential studies.

Each of the potential studies was reviewed to determine if the study was published in a peer-reviewed research journal. In this process, titles and abstracts were first used to determine if the focus of the study was on elementary PTs' geometry and measurement content knowledge. If a determination could be made that it clearly was not relevant to elementary PTs' geometry and measurement content knowledge, it was not included in our database of accepted studies. If there were any questions, possibilities, or doubts that an article focused on elementary PTs' geometry and measurement content knowledge, it went through an independent review that identified the research questions, study type and research design, location of study, lens and/or approach used, selection and description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions/implications. If the location of study [country of the population of PTs] was not described or referenced in the study itself, we assumed that the location was the country of

the authors' institution. This information was used to make a determination as to whether an article was excluded or included in our database. If there were any questions or discrepancies from the independent reviews regarding inclusion/exclusion, a mutual consensus was established by subgroup members. Examples of excluded articles were studies that focused on

(a) a general description of content knowledge that lacked specific attention to geometry or measurement, (b) a selection of inservice teachers or college students majoring in mathematics as opposed to mathematics education, (c) a sole focus on perceptions about mathematics not connected to content knowledge needed for teaching, and (d) a focus on describing classroom practice or activities with a lack of attention to research design methods. (Browning, Edson, Kimani, & Aslan-Tutak, 2011, p. 453)

Finally, the research literature cited in any studies included in the database of accepted articles for the historical section was used to determine potential new articles for the database. A total of nine studies were found for the historical look.

Modifications for the Current Perspective

As discussed earlier, a thorough description of the methods for the current perspective are detailed in the introductory paper of this Special Issue. Modifications of the methods for the current perspective section included potential studies suggested by mathematics teacher educators outside of the Working Group. Due to the limited number of studies found in our search, expert mathematics education researchers focusing on elementary PTs' geometry and measurement content knowledge, not part of the Working Group, with several outside of the United States, were contacted to see if they were aware of any additional peer-reviewed research publications, especially in those journals outside of the United States. This produced two additional articles that were included in our

database of accepted studies for the current perspective section, with a total of 12 articles reviewed.

Methods for the View of the Horizon

The view of the horizon section includes peer-reviewed research articles published in 2012, as well as 2011 and 2012 conference proceedings from PME and PME-NA. We examined the proceedings from both the North American Chapter and the International Group of PME to examine current research in geometry and measurement for PTs, to compare with our previous summaries and to note the most recent issues and trends in this area of research. The methods for this section followed a similar process for the other two time periods. Titles and abstracts of research reports, brief research reports, and posters were reviewed to determine whether there was a possibility for inclusion in our work. The inclusion of posters is a modification that differs from other content groups of the larger Working Group. All potential studies were independently reviewed. A total of five papers from the conference proceedings published in 2011 were accepted in our database; no related proceedings papers or research articles were found for 2012.

Analysis

In order to summarize findings across all the studies reported in this paper, we examined the study types, research design, and research questions and characterized each study that dealt with elementary PTs' geometry and measurement content knowledge. All studies reported research results found "in the moment," indicating the *status* of the knowledge of PTs at that time in the study. There were no longitudinal studies, examining the development of content knowledge over an extended period of time. Yet, within this overarching type of study, we found comparison studies that examined *associations and/or*

differences between two entities, aspects, relationships, etc., and then nested within these comparison studies, we found those that experimented with and/or described the *impact of a treatment* in a mathematics content or methods course or lesson. Italicized text emphasizes the key features of the questions in the studies for these three groups. It is important to note that all three groups reference descriptions of elementary PTs' geometry and measurement content knowledge; however, the associations/differences and impact of some treatment categories also contain one or two of these foci in the work, namely, examining to see if there are or are not any connections between two things and describing the outcomes of testing (typically) an instructional intervention.

Classification of a study into one of these groups is to give insight into the various types of research questions that have been investigated in the areas of geometry and measurement. We chose this classification scheme as differences in question type or what the researchers were investigating stood out to us as we read through and summarized the research. As there are a huge variety of topics within the content areas of geometry and measurement and a relatively small amount of studies summarized, using topic themes as a means of classifying the summaries was not possible; many topic themes would have included only one study. We realize there may have been other ways in which to collectively summarize the data, but we chose to systematically examine and highlight the research focus and present findings, allowing the readers of the summaries to sort findings in a manner appropriate for their own future research.

Historical Look: What Was Known About the Geometry and Measurement Content Knowledge of Prospective K–8 Mathematics Teachers Prior to 1998?

A total of nine studies published prior to 1998 focused on geometry and measurement content knowledge of prospective K–8 mathematics teachers (Table 1). The

studies are individually and collectively described below. The collective descriptions are framed around the three broad categories described in the previous methods section, based upon the type of research questions investigated.

Table 1

Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published Prior to 1998

	Author, Year	Content	Location of Study
Status	Baturo & Nason, 1996; Enochs & Gabel, 1984; Mayberry, 1983; Reinke, 1997	Perimeter, area, volume, surface area, and van Hiele levels of geometric thought	Australia and USA
Associations and/or Differences	Battista, Wheatley, & Talsma, 1982, 1989; Bright, 1979	Spatial ability, formal reasoning, geometric problem solving, and embedded figures	USA
Impact of a Treatment	Bright, 1985; Gabel & Enoch, 1987	Estimation of angle and length measurements and spatial ability and volume	Australia and USA

The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement

Four studies (Baturo & Nason, 1996; Enoch & Gabel, 1984; Mayberry, 1983; Reinke, 1997) focused on what our group labeled the *status* of elementary PTs' content knowledge of geometry and measurement.

Mayberry (1983) investigated elementary PTs' van Hiele levels of geometric thinking related to seven concepts: squares, right triangles, isosceles triangles, circles, parallel lines, similarity, and congruence. The results of the study support van Hiele's

(1959) implication that “a student cannot function adequately at a level without having had experiences that enable the student to think intuitively at each preceding level” (p. 67). The results also support the implication that “if the language of instruction is at a higher level than a student’s thought processes are, the student will not understand the instruction” (p. 67). General findings of the study suggest that elementary PTs were at different levels for different concepts and were not ready for a formal deductive geometry course.

In an exploratory study focused on the meaning of volume, Enochs and Gabel (1984) were interested in identifying PTs’ misconceptions of volume and surface area. To this end, the researchers developed, validated, and established reliability of the *Surface Area/Volume Misconception Inventory* (SAVMI) questionnaire instrument. A total of 125 PTs who were enrolled in a science education course for elementary education completed this questionnaire. Interviews were conducted with a small subset of participants, asking them to “think aloud” as they solved different problems and wrote down their calculations. Findings of this exploratory study indicated that “a large percentage of elementary education majors do not understand the concepts of volume and are unable to distinguish volume from surface area” (p. 679). Errors based on misconceptions included concept/definition of volume or surface area; formula memorizing mode; confusion between length, area, and volume; unit memorizing mode; conversion of cm^3 to ml; multiplication of units not correct or units incorrect; and wrong arithmetic. The researchers report that PTs were “found to solve problems using a ‘memorizing mode’ rather than basing their answers on the concept itself” (p. 679). Although the researchers do not indicate how volume and surface area should be taught, they do suggest that an exclusive formula approach is not beneficial for students.

Research related to area and perimeter concepts was conducted by Baturu and Nason (1996) and Reinke (1997), with findings from both studies suggesting struggles in understanding these concepts, some similar to those found in Enochs and Gabel's work (1984). Baturu and Nason investigated 13 teacher education students' subject matter knowledge of area measurement concepts and processes during the first year of their primary prospective program at the Queensland University of Technology. Based upon the work of Ball and McDiarmid, collectively and individually (Ball, 1990, 1991; Ball & McDiarmid, 1989; McDiarmid, 1988), Baturu and Nason viewed subject matter knowledge to be comprised of substantive knowledge, knowledge about the nature and discourse of mathematics, and knowledge about mathematics in culture and society. Results from structured, clinical interviews of area measurement tasks indicate that the PTs' knowledge was "rather impoverished in nature," namely, that their substantive knowledge was incorrect, incomplete, and unconnected, while having a limited ability in transferring from one form of representation to another. PTs had limited meanings for their rule-driven processes for finding area, as these rules were not connected to concrete experiences. For example, they could not explain why one must divide by 2 in the area formula for a triangle. Their knowledge of the nature and discourse of mathematics as well as about mathematics in culture and society appeared to be based on limited assumptions such as:

(1) mathematics is mainly an arbitrary collection of facts and rules . . . ; (2) most mathematical ideas have little or no relationship to real objects and therefore can only be represented symbolically; and (3) the primary purpose of learning area measurement was the utilitarian one of being able to calculate areas of regular shapes. (p. 262)

Baturu and Nason suggest that teaching mathematics without meaning promoted a low self-esteem for many of these PTs, as they failed to remember isolated facts and rules and

attributed their failure to low mathematical ability. These negative dispositions could possibly remain with the PTs and hinder their effectiveness in teaching mathematics to children.

Reinke (1997) investigated elementary PTs' solution strategies for finding the perimeter and area of a shaded geometric figure. A total of 76 PTs, enrolled in a second semester of an elementary mathematics content course, participated in the study. Findings indicated that the most common incorrect strategy by PTs was determining perimeter using the same method for area, suggesting PTs were confused about linear measurement and area measurement. Reinke supports the notion that PTs have been taught to rely on procedural learning and lack comfort with conceptual learning in mathematics, suggesting that PTs need more exposure to problems promoting conceptual understanding.

Summary. Findings across these four studies suggest that PTs enter their mathematics content preparation programs with limited geometry and measurement experiences, experiences chiefly focused on manipulation of formulas. Work using a van Hiele model for geometric learning indicated PTs were at different levels for different concepts, they tended to be at lower levels of geometric understanding, and they were not ready for a formal deductive geometry course (Mayberry, 1983). Other research studies conducted during this time period that focused on the status of PTs' content knowledge cite specific issues. For measurement with perimeter, area, and volume, PTs tend to not understand the concepts behind the measure formulas and confuse the measures, finding surface area instead of volume, or area instead of perimeter, as they rely solely on their memory of disconnected rules and formulas (Baturro & Nason, 1996; Enochs & Gabel, 1984; Reinke, 1997).

Examining Associations and Differences

Three studies (Battista, Wheatley, & Talsma, 1982, 1989; Bright, 1979) had at least one research question that focused on examining associations and differences related to what PTs understood about specific topics in geometry and measurement.

Bright's (1979) work with 145 PTs involved the identification of embedded figures in complex drawings, finding shapes within shapes. Analyzed data were taken from PTs' work with either two triangle figures or two quadrilateral figures, all having embedded shapes within. His findings suggest non-overlapping figures were easier to identify, that non-overlapping figures are generally identified first, and that PTs could identify embedded triangles more easily than quadrilaterals. Noted limitations to the study included the limited types of data analyses and that interviews were not conducted to verify students' thinking on the task. Bright found that only about half of the PTs completely and correctly solved one of the four drawings. Bright indicated that "it is therefore unlikely that as future teachers these people can be expected to teach such problem-solving techniques effectively to students" (p. 326), a somewhat dismal implication.

Battista, Wheatley, and Talsma (1982) investigated the interaction of spatial ability and cognitive development to examine their impact on mathematics learning, specifically that of geometry concepts. Participants for their study were 82 PTs enrolled in an informal geometry course. Instruments for data collection included the Purdue Spatial Visualization Test: Rotations (PSVT) (Guay, 1977) and a modified Longeot Test of cognitive development. Data were summarized on 82 of the enrolled students and included four measures: pre- and posttest means on the PSVT (S_1 and S_2), mean on the modified Longeot

test taken at the end of the semester (C), and the course grade score, which was the total of the student's scores on three course exams (G).

The spatial visualization scores significantly improved by the end of the semester, suggesting to the researchers that the type of activities used in the course may have helped with this improvement. However, research on whether instruction can improve spatial ability was inconclusive at that time. Further, missing from the article was any description of the type of activities used in the course. Examining multiple correlations of course grade (G) on C and S_1 supported the importance of both cognitive development and spatial visualization in learning geometric concepts. The data further suggested that cognitive development is a better predictor of the course grade in geometry than the spatial visualization ability.

In a second study by Battista, Wheatley, and Talsma (1989), they explored the connections between spatial visualization, formal reasoning, and geometric problem-solving abilities of elementary PTs. They worked from research that suggested learning mathematics may depend upon fundamental or "primary" mental abilities; students lacking those primary abilities may not be able to use certain problem-solving processes (Kulm & Bussman, 1980). Building on their previous work described above, Battista and colleagues investigated the relationship between the two primary abilities of spatial visualization and formal reasoning and the strategies used by PTs in geometric problem solving. Using similar instruments from their 1982 study, in addition to a problem-solving strategies test, Battista and colleagues collected data from 83 students enrolled in a geometry course for elementary PTs. From their findings, the researchers suggested an implication for instruction that relates to strategy use and strategy effectiveness, where PTs "should be

taught to identify those strategies that they are most likely to use effectively” (p. 28). “In particular, it seems that all but the brightest of prospective elementary teachers would benefit from learning to use a drawing strategy or some other strategy that would replace the use of pure visualization” (p. 29).

Summary. All three studies examined PTs’ spatial visualization in some way and how those skills connected with some other ability. The work of Battista and colleagues (1982, 1989) connects cognitive development and spatial visualization to geometric problem-solving ability and to the learning of geometry concepts in general, with Bright (1979) finding spatial visualization skills connected to identifying embedded figures in complex drawings. Bright also found the visualization skills of the PTs developed over time.

Describing the Impact of a Treatment

Two studies (Bright, 1985; Gabel & Enochs, 1987) had at least one research question that explored the impact of a treatment related to what PTs understood about specific topics in geometry and measurement.

In 1985, Bright conducted a study to determine the effectiveness of the computer game *Golf Classic* (Kraus, 1982) on PTs’ estimation of length and angle measurements. (Bright’s study also included a probability game, but findings only from the geometry computer game will be discussed in this summary.) Bright, Harvey, and Wheeler (1982) had conducted work with *Geogolf*, a non-computer instructional game, with tenth graders and found that the game effectively taught the students to estimate length and angle measurements. Bright wanted to determine if a computer version of the game would be just as effective in developing estimation skills for length and angle measure with PTs. During a 5-week period of time, each PT ($n = 78$) was randomly assigned to play one of the

two computer games (focused on geometry or probability). PTs played the game twice during this time period, once alone and once with someone else assigned to play the same game. Each time, the game was played for 20 minutes for a total game time of 40 minutes. Pre- and post-measures were taken for both length and angle estimation skills. Findings from these measures showed the computer game, Golf Classic, to have a marginal effect at improving angle estimation skills, with patterns in the performance of length estimation inconsistent. Bright believes his study's findings suggest "expectations should not be too high when attempts are made to translate effective non-computer instructional techniques into computer formats" (p. 522). This raises questions as to how the time length of 40 minutes was determined as sufficient time with the computer game to develop angle and length estimation skills as compared to the length of time on task for the tenth graders when playing the non-computer game.

Gabel and Enochs (1987) examined the research question of "whether spatial-visual skills are related to learning the volume concept, and whether a particular mode of presentation for teaching volume is preferable for students of different spatial ability" (p. 592). In this experimental study, elementary PTs in five sections of an introductory science class in a large Midwestern university used four different instructional sequences for length, area, and volume: length-area-volume ($n = 30$), length-volume-area ($n = 25$), volume-area-length ($n = 38$), and area-volume-length ($n = 37$). Three sections of students were randomly assigned the "length-last" treatment and two sections assigned the "length-first" treatment. Within each section, PTs were assigned "volume-before-area" and "volume-after-area" treatments. To answer the research question, four instruments were administered in this study: the cube-comparison test (French, Ekstrom, & Price, 1963),

surface development test (French et al., 1963), computational volume pretest (Bilbo & Milkent, 1978), and an adapted version of the volume test (Bilbo & Milkent, 1978).

Findings of the experimental study (Gabel & Enochs, 1987) indicated that spatial orientation is a key factor for volume test performance. Further, “the sequence in which the metric system is taught to PTs is an important factor in teaching the metric system if the visual-orientation ability of the student is considered” (p. 596). For elementary PTs of low visual orientation, teaching volume before area and length is beneficial, whereas those with high visualization skills can “use them to logically construct volume from area and height” (p. 596). Findings indicated that the order in which length, area, and volume were presented to PTs did not have a significant effect on how well they performed on the volume test. However, the researchers found that volume-area-length is preferable for students of low spatial orientation, whereas students of high spatial orientation prefer length-area-volume. It is important to note that elementary PTs likely experienced length, area, and volume sequence in school mathematics and that the study was limited by examining only volume of the metric system. In addition, the researchers emphasized they did not compare the effectiveness of the instruction on PTs’ understanding of length, area, and volume, and that “other sequences might be preferable for teaching these other concepts [length and area], and this needs to be considered in teaching the entire unit on the metric system” (p. 597).

Summary. The two treatments under consideration were the use of computer games in instruction and the sequence of instruction for measurement topics in a geometry course. Results indicated that PTs’ use of computer technology software, Golf Classic, positively impacted their estimation of angle measurements, a similar finding of the

software used with Grade 10 high school students. In terms of sequencing the measurement topics of length, area, and volume, results indicated that the various sequence tests did not have a significant effect; however, the researchers noted that the sequence did matter in terms of PTs' spatial orientation. PTs' with low spatial orientation prefer the volume-area-length sequence, as challenges occurred when constructing volume concepts from area and height.

Key findings across the nine studies support the importance of PTs developing their spatial abilities as related to geometric problem solving, finding embedded shapes, and developing concepts of measure; but spatial ability alone is not sufficient for success in geometric learning. PTs need to move away from focusing on memorization of formulas and focus on making meaning of concepts. Most findings, in general, support the importance of having numerous geometric experiences to advance geometric understanding, as noted by van Hiele (1959); it appears the importance of experience is true for both children and for PTs.

Current Perspective: What Was Known About the Geometry and Measurement Content Knowledge of Prospective K–8 Mathematics Teachers From 1998 to 2011?

Twelve studies focused on geometry and measurement in the 1998 to 2011 timeframe (Table 2). Topics explored in these studies include shape and shape properties; measurement topics of area, perimeter, and volume; use of dynamic geometry environments; and van Hiele levels of understanding. We note the range of topics is fairly similar to those in the previous historical section. Differences include a lack of any studies examining measurement estimation skills, and the technology focus has shifted from computer games to dynamic learning environments. Again, we present the studies

individually and collectively, where the collective framework revolves around the research focus of the study, described in our analysis section.

Table 2

Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published From 1998 to 2011

	Author, Year	Content	Location of Study
Status	Fujita, 2011; Fujita & Jones, 2007; Gutierrez & Jaime, 1999; Menon, 1998; Pickreign, 2007; Zevenbergen, 2005	Quadrilaterals, triangle altitudes, volume, perimeter, and area	Australia, UK, USA, Scotland, and Spain
Associations and/or Differences	Halat, 2008; Pitta-Pantazi & Christou, 2009; Lin, Luo, Lo, & Yang, 2011; Tsamir & Pitta-Pantazi, 2008	Geometry thinking using the van Hiele levels of understanding, intuitive rules theory applied to geometric tasks (median, bisector, perimeter and area), and the relationship between cognitive styles and mathematical performance in measurement and spatial tasks	Cyprus, USA, Taiwan, Turkey
Impact of a Treatment	Cunningham & Roberts 2010; Gerretson, 2004; Zevenbergen, 2005	Altitudes of triangles and diagonals of polygons, the impact of using dynamic geometry software on understanding of similarity, and the impact of various learning dispositions on the understanding of volume	Australia and USA

The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement

Six studies focused on the status of what PTs understand about specific topics in geometry and measurement.

Menon (1998) investigated PTs' understanding of perimeter and area. The participants of the study were 54 students who had completed one semester of their teacher preparation program prior to their enrollment in an elementary mathematics methods course taught by the researcher. Data came in the form of the participants' responses to four tasks. The study found that PTs' conceptions of mathematical ideas, particularly in geometry, were not fully developed, with most of the participants lacking the ability to articulate complete descriptions of rectangle and rhombus. In addition, Menon stated, "Yet, even with an apparently better foundation in mathematics, the students seemed to have poor conceptual understanding (*in perimeter and area*)" (p. 365). In sum, Menon decried the lack of conceptual understanding despite satisfactory performance on paper-and-pencil assessments, as the implication was that these PTs were less likely to offer their students opportunities to explore problems that require conceptual reasoning.

Further research on PTs' knowledge of rectangles and rhombi was conducted by Pickreign (2007). In particular, this study examined what is revealed about PTs' understanding of the properties and relationships among parallelograms through their articulation of the meaning of rectangle and rhombus. Participants of the study were 40 PTs taking the first course in a two-course sequence randomly selected from four sections of the course taught by the same instructor. Data came from the PTs' written personal definitions of rectangle and rhombus.

Nine of the 40 participants (22.5%) gave a definition of a rectangle that was classified as complete—inclusive of squares and excluding any parallelogram that did not have a right angle. Only 1 of the 40 (2.5%) defined a rhombus in a way that was inclusive of squares and excluded parallelograms that did not have equal adjacent sides. Pickreign added, “It is not the complete definitions that are most interesting, nor even how few of the participants got them correct; it is the misconceptions regarding these shapes that seem to be indicated by the other responses” (p. 3). Pickreign concluded that irrespective of the experiences the students in this study had with rectangles and rhombi, PTs lacked the ability to articulate these two types of quadrilaterals.

PTs’ conceptual understanding of quadrilaterals in general was studied by Fujita and Jones (2007). The research reported was part of a larger study focusing on PT education in the UK and Japan. The researchers explored the nature of PTs’ *personal figural concepts* and *formal figural concepts*, building on the work of Tall and Vinner (1981) with respect to concept image and concept definition and Fischbein’s (1993) figural concept. The study examined data from 158 PTs in order to investigate the gap between their formal and personal concept images of quadrilaterals. The results indicated that PTs rely on their personal concept images of shapes to construct definitions rather than examining and using properties of shapes.

Fujita (2011) continued to investigate learners’ understanding of quadrilaterals by developing a questionnaire that focused on inclusion relations. The questionnaire was piloted with 19 PTs in the UK, and then with 85 Japanese lower secondary school students. Results from the PTs’ answers to geometry questions revealed that the majority of them hold a prototype definition for quadrilaterals based on limited personal figural concepts of

the shape, and they have difficulty in understanding the inclusion relationships between quadrilaterals. For example, even though PTs stated a definition of parallelogram, they could not use it to show that a square is a parallelogram. The author suggested that participants' literal use of definitions may cause deficiencies in understanding the inclusion relationships. Fujita suggested carefully integrating visual and conceptual aspects of quadrilaterals to create an effective learning environment to help overcome the prototype definition phenomenon. Further, "a careful use of dynamic geometry environments . . . might encourage learners to develop their dynamic images of shapes and to pay attention to what properties are changed/unchanged between the different shapes (Leung, 2008).

Similar to the work of Fujita and Jones (2007) and Fujita (2011), Gutierrez and Jaime (1999) used a theoretical framework of *concept image* and *concept definition* (Vinner, 1991; Vinner & Hershkowitz, 1980, 1983) to investigate primary PTs' understanding of the concept of altitude of a triangle. The study identified the students' reasoning process and the effect of variables such as the students' "previous knowledge, the presence of a formal definition in the test, or the influence of learning activities that dealt with altitudes of triangles as part of the content of a course on mathematics education" (p. 259). The researchers reported evidence of PTs holding onto certain concept images that are not helpful. Specific student misconceptions included poor concept images, with students (a) relying more on visual cues for defining shapes, (b) believing that altitudes of triangles must exist within the shape, (c) mixing definitions of medians and altitudes, and (d) mixing perpendicular bisectors and altitudes. In sum, this study found PTs had poor concept images that are comparable to those of primary or secondary students and offered that this

situation can provide a platform for the PTs to examine their and their classmates' concept images and concurrently learn what types of concept images children are likely to have.

Zevenbergen (2005) explored the understandings of volume among primary PTs. This study set out to critically explore the reactions and learnings of PTs in a course in which discipline knowledge was taught in tandem with pedagogical content. The participants in this study were 98 PTs enrolled in a third-year course in which students were expected to "develop a strong understanding of mathematics discipline knowledge" (p. 8). Data came from the PTs' responses to a quiz item requesting the amount of concrete needed to fill a barbeque area with dimensions of $8.5\text{m} \times 3.2\text{m} \times 30\text{cm}$. PTs were asked to express their answer "in the way you would if you were to phone the concrete company to place the order" (p. 8). Follow-up interviews were conducted with 30 of the PTs.

From the quiz data, the researcher reported "only 32 out of 98 students were able to calculate a result and transfer the result into an appropriately communicable form (i.e., approximately 8 cubic meters) and concluded the data suggested "students have the esoteric knowledge of school mathematics but have not transferred it to the practical context, and that there has been a prioritizing of school mathematical knowledge over practical mathematical knowledge (or numeracy)" (p. 10). The interviews were performed after quizzes were corrected and aimed to offer insights into the students' thinking as well as an appreciation of the diversity of the responses. Data from the interviews offered more evidence of incomplete concepts about volume among the PTs. Zevenbergen stated, "The interview data suggested that there was heavy reliance on procedural knowledge, that is, algorithmic methods in which lock-step strategies were used to solve the task. These

strategies suggest that the students relied on particular ways of knowing in mathematics" (p. 11).

Summary. The researchers described PTs' understandings of geometry and measurement as not fully developed, based on unproductive concept images and/or concept definitions, and lacking in their ability to articulate their reasoning with geometry and measurement. The prospective teachers in these studies relied on visual examinations to define shapes, relied on procedural knowledge, and lacked conceptual understandings of geometry and measurement concepts. Their understandings were compared to those of primary grade or secondary school students, which raises questions about their preparedness to teach geometry and measurement concepts with fidelity to the standards expected for elementary grades. For example, given prospective teachers' superficial understandings of geometry and measurement concepts and their deficient concept images and concept definitions, it is difficult to imagine their ability to see the structure of the geometry and measurement concepts they will be asked to develop in their students, especially under the new standards in the United States that ask for making connections among mathematics concepts.

Examining Associations and Differences

Four studies included research questions that examined relationships and/or differences related to PTs' geometry content knowledge; these are described below.

Halat (2008) administered a van Hiele Geometry Test (VHGT), based upon the work of Usiskin (1982), to compare two groups of PTs' (elementary and secondary) geometric thinking levels while investigating for differences in terms of gender. The researcher used data from 281 Turkish PTs (125 elementary and 156 secondary). There were 68 female

and 57 male elementary PTs who took the test after completion of a geometry course at a Turkish university. Also, 72 female and 84 male secondary PTs answered van Hiele test questions after they completed advanced level mathematics and geometry courses. Halat found

no statistically significant difference in regard to the reasoning stages between the pre-service elementary school and secondary mathematics teachers, and that although there was a difference with reference to van Hiele levels between male and female pre-service secondary mathematics teachers favoring males, there was no sex-related difference found between male and female pre-service elementary school teachers. (p. 1)

Further work using the van Hiele levels of understanding was conducted by Lin, Luo, Lo, and Yang (2011) involving a comparative study to investigate and compare the geometry knowledge and levels of PTs from the United States and Taiwan. Data were collected from 48 U.S. PTs and 40 Taiwanese PTs, with both groups enrolled in a mathematics methods course. Two instruments (the VHGT and the Entering Geometry Test [EGT] also created by Usiskin, 1982), were used to collect data regarding PTs' knowledge and their levels of geometric thinking. The 20-item multiple-choice EGT was used to measure the PTs' content knowledge. The 25-item multiple-choice VHGT is divided into five levels with five questions in each level that focuses "not only on content knowledge but also on the sophistication levels of geometric thought including proof" (p. 9).

The PTs' performance on the EGT showed a statistically significant difference between the two groups, suggesting that Taiwanese PTs entered their teacher education program with a better understanding of geometry than their U.S. counterparts. The Taiwanese PTs also outperformed the U.S. students on each item on the EGT. The VHGT data also showed significant differences between the U.S. and Taiwanese students. While 77.5% of Taiwanese PTs achieved at least the third van Hiele level, only 27% of their U.S.

counterparts achieved at least level three on the VHGT. However, unlike on the EGT where Taiwanese students outperformed the U.S. students on every item, they did not outperform their U.S. counterparts on every VHGT item. The data indicated no significant associations between the EGT and VHGT scores for Taiwanese students, while there was evidence of a positive weak relationship among the U.S. participants.

The authors noted that despite the importance of teachers' mathematical content knowledge, it is not known what minimal van Hiele level of understanding elementary teachers should achieve so they can provide a sufficient quality of geometric teaching for their students. They argue that a satisfactory level of achievement for PTs needs to be justified prior to making suggestions for change in geometry expectations for elementary teacher preparation.

Shifting from a focus on levels of geometric understanding to analyzing errors in PTs' geometric thinking, Tsamir and Pitta-Pantazi (2008) focused on the intuitive rules theory posited by Stavy and Tirosh (2000) with 98 PTs in a mathematics education course from the University of Cyprus. The intuitive rules theory was designed by Stavy and Tirosh for analyzing and predicting inappropriate responses to a wide variety of mathematical and scientific tasks. Tsamir and Pitta-Pantazi used the framework to help interpret errors made by the PTs in solving a variety of geometric tasks, specifically tasks related to geometric ideas of median, bisector, perimeter, and area.

The intuitive rule *more A–more B* was identified in tasks in which there are two objects or systems where one quality or quantity A, fulfills the condition $A1 > A2$ and this inequality is either perceptually or directly given, or alternatively, it can be logically derived through the schemes of conservation or proportion. However, participants are asked to compare the two objects or systems with regard to another quantity B, for which the two given objects or systems fulfill either $B1 = B2$ or $B1 < B2$. A common incorrect response to such tasks, regardless of the content domain, takes the form: “ $B1 > B2$ because $A1 > A2$, or more A–more B. (pp. 72–73)

Tsamir and Pitta-Pantazi (2008) found that PTs' solutions were consistent with the intuitive, more A-more B or same A-same B lines of reasoning, where most PTs based their arguments on their visual grasp of the data in illustrations provided in the tasks. Further, as the authors found comparable findings to the Cypriot data in an earlier study done in Israel with secondary school mathematics PTs, they have provided extended data regarding PTs' ways of thinking about perimeter and area. Tsamir and Pitta-Pantazi suggest that similar findings across two different countries provide the mathematics education community a better picture of intuitive pitfalls hidden in these topics and suggest possible reasons for the PTs' difficulties.

Pitta-Pantazi and Christou (2009) investigated the relationship between cognitive styles and mathematical performance in measurement and spatial tasks of 116 elementary prospective kindergarten teachers with varying mathematical ability. Given that there are many different ways to define cognitive styles (Riding & Cheema, 1991), the researchers used two dimensions of cognitive styles that grouped most definitions: Verbal-Imagery and Wholistic-Analytic. The first dimension, Verbal-Imagery, refers "principally to mental representations, i.e., to the way individuals represent knowledge in mental pictures or words" (p. 132). The second dimension, Wholistic-Analytic, refers to "individuals' typical methods for organizing and processing information, either in parts or as a whole" (p. 133).

This study used two cognitive style tests, Verbal-Imagery Cognitive Style test (VICS test) and the Extended Cognitive Style Analysis Wholistic-Analytic test (CSA-WA) (Peterson, 2005), and a mathematics test with six spatial and six measurement tasks. The findings of the study suggest that there are "no performance differences between spatial and measurement tasks across the various cognitive styles of the participants . . . however,

... the impact of cognitive style is significant for some groups of participants (the low achievers)” for the measurement pictorial tasks (p. 146). Low-achieving thinkers who consider information they read, see, or listen to in words (verbalizers) and those who deconstruct information to its components (analytic thinkers) performed much better than those who use mental pictures (imagers) in all pictorial measurement tasks. Therefore, the results from the study suggest “verbalisers and analytic low achievers perform best when given an instructional format enhanced with graphical features” (p. 146) and raised the importance that classroom material should be presented in various formats.

Summary. PTs’ geometry thinking was examined using the van Hiele levels of understanding, with some studies comparing PTs’ understanding across different groups. While Halat (2008) found no significant differences in van Hiele levels of understanding between Turkish elementary and secondary PTs, significant differences were found between Taiwanese and U.S. PTs, with a majority of Taiwanese PTs reaching level 3 understanding based on the VHGT compared to only 27% of U.S. PTs (Lin et al., 2011). These results suggest some possible international differences in the content preparation of PTs prior to their entrance in a teacher preparation program. Other work made use of Stavy and Tirosh’s (2000) intuitive rules theory. Tsamir and Pitta-Pantazi (2008) compared findings of elementary PTs’ thinking of area and perimeter with triangles to that of secondary mathematics PTs, finding the intuitive reasoning of more A–more B or same A–same B prevalent in both groups. Pitta-Pantazi and Christou (2009) examined the relationship between cognitive styles and mathematical performance in measurement and spatial tasks finding that for low achievers, the cognitive style was significant on the measurement pictorial task, with the low performance from those PTs who used mental

pictures (imagers). A study of Battista et al. (1989), summarized in the earlier historical look, found similar results, suggesting that the majority of PTs would benefit from replacing their singular visualization strategy (similar to imagers) by learning to use some other problem-solving strategy, such as drawing.

Describing the Impact of a Treatment

There are three research peer-reviewed studies published in journals that had questions of an investigative nature, exploring the impact of a treatment. Each of these studies is described in detail below.

Gerretson (2004) examined whether there was a difference in elementary PTs' performance on similarity tasks when using dynamic geometry software as compared to a paper-and-pencil learning environment using traditional tools (e.g., compass, ruler). There were 52 PTs who were enrolled in an introductory course that addressed content, methods, material development, and assessment in mathematics teaching. Using a pre- and posttest control group experiment using randomized blocks controlling for initial performance, she found a statistically significant difference in learning environments between the two treatment groups. "Fundamentally, software users outperformed non-software users even when prior knowledge variability was taken into consideration" (p. 18). Analysis suggested elementary PTs using a paper-and-pencil learning environment encountered more difficulties particularly situated around similarity properties of unfamiliar shapes, whereas PTs using dynamic geometry software had "acquired a greater knowledge base to access, network, and apply" (p. 19).

In an exploratory study, Zevenbergen (2005) investigated the impact of various learning dispositions emphasized within a mathematics course module on volume. These

dispositions included developing mathematical meaning of volume as opposed to using only algorithmic methods, measurement and spatial sense, and the capacity within the PTs to identify errors in children's mathematical thinking. However, despite the various methods used in the course to develop these dispositions, there were a "worrying number of students" who had not achieved them (p. 21), with some students quite resistant to alter their thinking about how to learn mathematics. Responses to interviews of students in the course highlighted the power of the teaching practicum, with PTs rejecting the nature of the work done in their mathematics course due to their experiences in the schools. "Ideally, it would be useful to expose students to schools and classrooms that demonstrate the values embedded within teacher education courses if such courses are to effectively change teaching practice" (p. 21).

Cunningham and Roberts (2010) used the theoretical framework of *concept image* and *concept definition* (Vinner & Hershkowitz, 1980) to explore the impact of a treatment lesson involving instructional strategies designed to assist the development of PTs' concept images and concept definitions related to altitudes of triangles and diagonals of polygons. They used a one-group pre- and posttest design with 57 primary school PTs enrolled in a content course. For this study, the researchers investigated instructional strategies that included the use of graphic organizers (e.g., Frayer, Frederick, & Klausmeier, 1969) along with the concept attainment model (Eggen, Kauchak, & Harder, 1979; Joyce, Weil, & Calhoun, 2004) in the development of definitions. Pretest results showed PTs' weak conceptual understanding. A combination of the teaching strategies resulted in some posttest improvement in their understanding of triangle altitudes and diagonals of polygons. The researchers posited, "This study advocates that teaching challenging

geometry concepts to PTs needs careful attention so that the mismatch between concept definitions and their concept images may be minimized” (p. 10). This study concluded that owing to the weak conceptual understanding for some PTs, there is a need for mathematics teacher educators to utilize more than “passive” or traditional teaching approaches, going beyond memorizing concept definitions.

Summary. These studies explored the impact of instruction using graphic organizers and concept attainment strategies on the understanding of altitudes of triangles and diagonals of polygons, the impact of using dynamic geometry software on the understanding of similarity, and the impact of various learning dispositions on the understanding of volume. While these studies reported some gains in the development of PTs’ conceptual understandings of geometry and measurement concepts, the gains were somewhat tempered by the PTs’ perceptions of the nature of mathematics as a body of knowledge that can be developed through the memorization of formulas. These perceptions may be due, in part, to traditional teaching approaches that focus on memorization of formulas. Also highlighted in these results was the challenge of implementing change in teacher education. For example, the PTs’ teaching practicum experiences need to support the productive ways of reasoning developed in prospective teacher training; otherwise, the gains achieved during teacher training are eroded.

Similar to key findings from the historical look, status research in the current perspective still shows that PTs’ understanding of geometry and measurement is not fully developed, with deficient concept images and concept definitions, and PTs performing at low van Hiele levels of understanding. Research suggests that PTs who are low-achieving need to tap into other cognitive styles beyond mental imagery or visualization, such as

those that make more use of reading or listening to information (verbal) or those styles that focus on deconstructing given information into components (analytic). Further, the finding that teaching and learning geometry and measurement concepts need to move away from a focus on memorizing formulas remains constant in this timeframe as well. Teachers need to consider the use of alternative instruction strategies that engage PTs more in developing their own geometric definitions through problem-solving experiences that may also improve their concept images. With geometry dynamic software becoming more accessible, learning experiences can be more exploratory and investigative and can enhance geometric understanding, as Gerretson's (2004) work shows.

View of the Horizon: What Is Known About the Geometry and Measurement Content Knowledge of Prospective K-8 Mathematics Teachers Since 2011

For the view of the horizon, peer-reviewed research articles published in 2012 as well as relevant research from 2011 and 2012 conference proceedings from PME and PME-NA were examined. A total of five studies were found, all published in 2011, that focused on geometry and measurement content knowledge of prospective mathematics teachers (Table 3). No related proceedings or research articles were found for 2012. (As writing for this current Special Issue took place during the majority of 2013, searches for related research ended in December 2012.)

Table 3

Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published Since 2011

	Author, Year	Content	Location of Study
Status	İymen, Pakmak, & Paksu, 2011; Patton & Parker, 2011	Parallelogram and geometric terms	Turkey and USA
Associations and/or Differences	Köse & Özen, 2011	Problem solving in paper-pencil and dynamic geometry environments	Turkey
Impact of a Treatment	Morgan & Sack, 2011; Schnorenberg & Chamberlin, 2011	Progression through the van Hiele levels of geometric thinking based on the use of triangles and area and volume	USA

The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement

Two papers from PME and PME-NA conference proceedings (İymen, Pakmak, & Paksu, 2011; Patton & Parker, 2011) had at least one research question that focused on the status of PTs' content knowledge of geometry and measurement. As both papers are based on poster presentations, details are minimal and brief. However, both examine knowledge of shape and geometric terms, and findings still show PTs struggling with definitions of geometric shapes.

İymen, Pakmak, and Paksu (2011) investigated PTs' geometry content knowledge with a focus on their understanding of parallelogram. Forty-five PTs were interviewed using a parallelogram task, with 82% responding correctly. However, the interview

revealed weaknesses in the PTs' understanding of the shape and relationships between parallelograms and trapezoids.

Rather than assessing PTs' knowledge of key measurement and geometry terms, Patton and Parker (2011) investigated if PTs were, perhaps, not able to apply their knowledge of vocabulary in a measurement application test. Fifty-two PTs were given the test consisting of 12 multiple-choice questions in the first month of a mathematics methods course. The results indicated 33% of PTs scored at the mastery level (scoring 90–100%), 33% passed (scoring 75–90%), and 33% failed (scoring below 75%). A follow-up vocabulary test of 24 sixth-grade-level terms was given in the following semester. Scores indicated that approximately 60% of the PTs scored mastery, 35% passed, and 5% failed. No additional information was provided to gain further insight into these results.

Examining Associations and Differences

Köse and Özen's (2011) PME conference proceeding paper had at least one research question that focused on examining associations and/or differences related to what is understood about specific topics in geometry and measurement. Their qualitative work with a sample of three PTs compared the PTs' problem-solving process in a paper-and-pencil situation with that done in a dynamic geometry environment (DGE). They found that the PTs attempted to solve a given problem using similar processes, yet could not find a solution in the paper-and-pencil environment. Further, in DGE, all students used different problem-solving processes. (No clear reference was made to the correctness of the PTs' solutions in the proceedings summary of the poster.) It was found that the PTs have essentially two stages in problem solving, that of constructions and investigations, with the PTs having no difficulties at the construction stage. With the investigation stage, PTs used a

helical process of seeking a relationship, finding a relationship, testing the relationship, seeking for a new relationship, and justifying. In general, the findings suggest that future work in DGE is promising, in that students were more willing to seek, find, and test relationships prior to justifying, as compared to students using a paper-and-pencil approach.

Describing the Impact of a Treatment

Two papers from PME and PME-NA conference proceedings (Morgan & Sack, 2011; Schnorenberg & Chamberlin, 2011) had at least one research question that explored the impact of a treatment related to what PTs understand about specific topics in geometry and measurement. The treatments explored in these two studies included the use of “giant triangles” in the development of a variety of geometric ideas, and the effect of the use of differentiated instruction on PTs’ understanding of area and volume.

Morgan and Sack (2011) present a learning trajectory, based upon the van Hiele levels of geometric thinking, that make use of “giant triangles,” flexible manipulatives that are 1-meter in edge length, which can be assembled to make a variety of polyhedra with triangular faces. (The use of the triangles was thought of as an “instructional treatment” and thus placed in this category of studies.) The trajectory presented describes activities that took place during a single 160-minute class session in the semester-long course. The activities are intended to move PTs through the van Hiele levels, visual to descriptive to relational. The PTs in this study were enrolled in an elementary/middle school mathematics methods course at a mid-Southwestern university in the United States, federally designated a Minority Serving Institution. The authors state that “substantial, deep and interconnected mathematics” is made available quickly and effectively using the

triangles. It is further stated that “no entry-level content knowledge is required and transfer from prior content courses has generally not been observed” (p. 255). We interpret this was meant to be a positive finding, namely, that weak conceptual understanding did not interfere with the learners in engaging with the triangle activities and making sense of the geometry concepts involved. However, it raises a question as to why PTs retain limited knowledge from high school geometry experiences and why the understandings they do retain tend to be weak and fragmented. A third finding presented suggested that “high levels of student engagement and collaboration are achieved associated with hands-on play and figuring out activities, in a positive affective social context” (p. 255). And finally, the authors indicated that the “use of these manipulatives may avoid some of the affective pitfalls that occur when introducing challenging mathematical problems” (p. 255).

Schnorenberg and Chamberlin (2011) investigated how differentiated instruction impacts elementary PTs’ mathematical understanding of area and volume. In this lesson experiment, instruction was differentiated by the use of several formative assessments, flexible heterogeneous and homogeneous groups, various activities with multiple modalities (e.g., visual, audio, kinesthetic mediums), and tracking of student progress on learning goals. Specifically, two groups of PTs were formed based upon their pre-assessment results. Each group focused on a series of activities designed for either area or volume. To examine the impact on PTs’ understanding of area and volume in a geometry and measurement course for elementary teachers, data sources on students’ work of pre-assessments, group activities, and post-assessments were collected for nine elementary PTs. In addition, audio recordings captured each group’s discussion and video recordings

captured the instructors' teaching. Although analysis was ongoing in this study, the findings indicated that PTs improved their knowledge and skills related to area and volume. For example, PTs gained understanding in area as covering a two-dimensional shape and volume as filling a three-dimensional shape, understanding that surface area and volume are independent, and understanding that measuring objects with different unit sizes may lead to different measures. Thus, in conclusion, the researchers state that "differentiating by area and volume in this lesson enhanced the students' understandings, allowed us to maximize the use of class time, and possibly provided a model of differentiated instruction for the students" (p. 1499).

Summary. The summary of research from a view of the horizon, based upon minimal insights and research findings garnered from conference proceedings and posters, suggests PTs continue to struggle with meanings of content vocabulary terms from specific geometry and measurement topics. However, these same findings did provide some evidence that future work in dynamic geometry environments, learning trajectories grounded in van Hiele levels of geometry thinking, and differentiated instruction can positively improve PTs' content knowledge of geometry and measurement.

Discussion and Conclusion

The *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000), the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM, 2006), and the recent *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) all include content expectations specific to geometry and measurement; thus, minimally, PTs would need to have a solid understanding of these

same expectations. In addition, two reports released from the Conference Board of the Mathematical Sciences (CBMS) discuss recommendations for the mathematical preparation of teachers at all grade levels. *The Mathematical Education of Teachers* (CBMS Report I; CBMS, 2001) based recommendations at that time upon the PSSM. Similarly, *The Mathematical Education of Teachers II* (CBMS Report II; CBMS, 2010) uses the CCSSM “as a framework for outlining the mathematical ideas that elementary teachers, both prospective and practicing, should study and know” (p. 25).

Table 4 shows a correlation between the recommendations from CBMS Reports I and II and the research focusing on elementary PTs’ geometry and measurement content knowledge. We initially sorted research articles into the CBMS Report II recommendations. Only 15 of the 26 studies’ content emphases could be matched to content recommendations from the CBMS Report II. We were curious if perhaps some of the remaining studies focused their research on recommendations found in the earlier CBMS Report. So we sorted these remaining studies using CBMS Report I recommendations. Five more studies were then classified. We noticed that the remaining six research studies focused on general PTs’ geometry and measurement content, such as examining van Hiele levels of understanding for geometry. This new category is also shown in Table 4.

Table 4

Correlation of Recommendations from CBMS Reports I and II and Geometry and Measurement Research

	Historical Look	Current Perspective	View of the Horizon
Geometry			
Understanding geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; describing and reasoning about spatial locations (including the coordinate plane). (CBMS Report II)		Gutierrez & Jaime (1999); Cunningham & Roberts (2010)	Morgan & Sack (2011)
Classifying shapes into categories and reasoning to explain relationships among the categories. (CBMS Report II)		Fujita & Jones (2007); Pickreign (2007); Fujita (2011)	İymen, Pakmak, & Paksu (2011)
Basic shapes, their properties, and relationships among them: developing an understanding of angles, transformations (reflections, rotations, and translations), congruence and similarity. (CBMS Report I)			
Reason about proportional relationships in scaling shapes up and down. (CBMS Report II)		Gerretson (2004)	Morgan & Sack (2011)
Visualization skills: becoming familiar with projections, cross-sections, and decompositions of common two- and three-dimensional shapes; representing three-dimensional objects in two dimensions and constructing three-dimensional objects from two-dimensional representations. (CBMS Report I)	Bright (1979); Battista, Wheatley, & Talsma (1982, 1989)		
Communicating geometric ideas: learning technical vocabulary and understanding the role of mathematical definition. (CBMS Report I)			Patton & Parker (2011)

(continued)

Table 4—continued

	Historical Look	Current Perspective	View of the Horizon
General understanding of geometry	Mayberry (1983)	Halat (2008); Lin, Luo, Lo, & Yang (2011)	Köse & Özen (2011)
Measurement			
The general principles of measurement, the process of iterations, and the central role of units: that measurement requires a choice of measurable attribute, that measurement is comparison with a unit and how the size of a unit affects measurements, and the iteration, additivity, and invariance used in determining measurements. (CBMS Report II)	Bright (1985)		
The process of measurement: understanding the idea of a unit and the need to select a unit appropriate to the attribute being measured, knowing the standard (English and metric) systems of units, understanding that measurements are approximate and that different units affect precision, being able to compare units and convert measurements from one unit to another. (CBMS Report I)			
How the number line connects measurement with number through length. (CBMS Report II)			

(continued)

Table 4—continued

	Historical Look	Current Perspective	View of the Horizon
Understanding what area and volume are and by giving rationales for area and volume formulas that can be obtained by finitely many compositions and decompositions of unit squares or unit cubes, including formulas for the areas of rectangles, triangles, and parallelograms, and volumes of rectangular prisms. (CBMS Report II)	Enochs & Gabel (1984); Gabel & Enoch (1987); Batur & Nason (1996); Reinke (1997)	Menon (1998); Zevenbergen (2005)	Schnorenberg & Chamberlin (2011); Morgan & Sack (2011)
Length, area, and volume: seeing rectangles as arrays of squares, rectangular solids as arrays of cubes; recognizing the behavior of measure (length, area, and volume) under uniform dilations; devising area formulas for basic shapes; understanding the independence of perimeter and area, of surface area and volume. (CBMS Report I)			
General understanding of measurement		Tsamir & Pitta-Pantazi (2008); Pitta-Pantazi & Christou (2009)	

Given the content recommendations from the CBMS Reports I and II, what have we learned from our summary of research? What have we learned about PTs' understanding of these topics across the years that would give us insights into the nature of this understanding? Examining the table, we do note gaps in the research literature of topics identified for the preparation of elementary mathematics teachers. There was no peer-reviewed published research found that specifically addressed the general principles of measurement or how the number line connects measurement with number. Not all components within each recommendation were addressed, leaving much to be investigated

regarding what we know about PTs' understanding of geometry and measurement. Yet many cells in the table show related work to the recommendations.

The total of 26 studies spans across a wide variety of topics within the content areas of geometry and measurement. Although it is encouraging that a variety of topics exist in the research literature, concentrated research effort is needed for targeted topics in order to have a better picture of PTs' understanding in geometry and measurement. Across the historical look, the current perspective, and the view of the horizon, PTs' general understanding of core ideas in geometry and measurement is limited and weak (Baturó & Nason, 1996; Cunningham & Roberts, 2010; Enochs & Gabel, 1984; Fujita, 2011; Fujita & Jones, 2007; Gutierrez & Jaime, 1999; Lin et al., 2001; Menon, 1998; Pickreign, 2007; Reinke, 1997; Zevenbergen, 2005), with PTs relying on procedural processes, recalled from the depth of their memory (Baturó & Nason, 1996; Enochs & Gabel, 1984).

Work from Battista et al. (1982, 1989) suggests cognitive development, along with spatial visualization skills, plays a greater role in learning geometry than memory skills, as many PTs purport. The van Hiele levels of geometric learning (van Hiele, 1959) provide a framework for helping those teaching geometry to think about the development of geometric ideas through stages and to provide experiences for the learners that engage them in thinking and reasoning (Mayberry, 1983; Morgan & Sack, 2011). Also, the use of dynamic geometry environments (DGE) might foster a more dynamic image of shapes and allow for visual and conceptual aspects of shapes to meaningfully coalesce when forming concept images and definitions, thus helping create a more effective learning environment for learners. Gerretson's (2004) study supports this suggestion, with PTs acquiring a greater knowledge base when using a DGE. Köse and Özen (2011) also found PTs engaging

in different problem-solving strategies in a DGE as compared to using paper and pencil and not being able to find solutions in that environment.

Yet even if PTs' content knowledge is strengthened, does it ensure successful teaching? Our research review did not examine work related to teaching geometry and measurement. However, Zevenbergen (2005) noted the importance of classrooms that demonstrate the values of the teacher preparation program in order to help sustain the dispositions developed in the program, as well as dispositions that put an emphasis on conceptual understanding and a developmental approach to learning that doesn't rush to a procedural rule.

Pickreign (2007) questions if there is sufficient time in a teacher preparation program for PTs to have the needed experiences to advance their learning to satisfactory levels of understanding. Further, Lin et al. (2011) note there is no definitive van Hiele level of understanding set as an expectation for all elementary PTs. So, what is a "satisfactory level of understanding"? If we do set minimal expectations, we return to our general question of how do we help PTs attain these expectations in the geometry and measurement courses? Fujita's work (Fujita, 2011; Fujita & Jones, 2007), based upon a synthesis of learning theories from van Hiele (1959), Tall and Vinner's (1981) concept definition, Fischbein's (1993) figural concepts, personal and formal figural concepts (Fujita & Jones, 2007), dynamic figural concepts (Walcott, Mohr, & Kastberg, 2009) and Hershkowitz's (1990) prototype phenomenon of geometrical figures, provided some suggestions for learning opportunities, specifically for understanding inclusion relations for quadrilaterals, to help PTs move beyond simply memorizing procedures and relying solely on personal figural concepts. It involves helping learners identify their

misconceptions, clarifying definitions of shapes, applying relationships between shapes that they understand to other situations, and using definitions to further reflect on properties of shapes. Others (Cunningham & Roberts, 2010; Gutierrez & Jaime, 1999) have similar findings from their collective work, highlighting the importance of the appropriate development of concept images and concept definitions. As the CBMS (2001) observes,

The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they do know—the mathematical ideas they hold, the skills they possess, and the context in which these are understood—so they can move from where they are to where they need to go. . . . And this is where the mathematics courses for elementary school teachers must begin. (p. 17)

This quote from CBMS highlights how readers can use information from our summary of PTs' geometry and measurement content knowledge based upon peer-reviewed research published over the past 20 years. With respect to the design of curriculum, we see a need for well-designed, engaging geometric and measurement experiences that (a) further the content understanding of PTs, moving them beyond a focus on procedural and memorization skills; (b) further develop PTs' spatial visualization but also help PTs develop other geometric problem-solving skills, such as drawing; (c) focus on developing PTs' concept definitions of shapes and their properties; (d) still engage the PTs at beginning van Hiele levels of understanding, rather than assuming all PTs can initially engage with thinking at advanced levels; and (e) incorporate the use of dynamic geometry software to further develop reasoning skills.

Our summary work also has indicated areas of more and needed future research, such as research focusing on (a) *how* PTs develop their content knowledge using technology; (b) determining a satisfactory level of geometry (and measurement) understanding for PTs; and (c) addressing the gaps to the content expectations from the

MET I and II documents (CBMS, 2001, 2010) as shown in Table 4, with measurement showing the greatest need for further study. (See the final paper of this Special Issue for several areas of future research common across the other content areas summarized.) Such future work that builds upon what we know regarding the geometry and measurement content knowledge of PTs can help us strengthen our existing content preparation programs to develop the independent mathematical thinkers future elementary teachers need to be.

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