The Mathematics Enthusiast

Volume 1 | Number 2 Article 2

10-2004

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In the last proposition of the *Elements* Euclid proved that there are only five regular polyhedra, namely the tetrahedron, octahedron, icosahedron, cube, and dodecahedron. To show there can be no more than five he used the fact that in a polyhedra, the sum of the interior angles of the faces which meet at each vertex must be less than 360. For if these angles sum to 360 the faces would tile in two dimensions. Since the interior angles of a p-sided polygon are $180 - 360/p$, the only possible polyhedra have the property that $q(180 - 360/p) < 360$ where $q > 2$ is the number of faces which meet at each vertex. With this in mind the only possible regular convex polyhedra are given in table 1.

p	q	$q(180 - 360/p)$	name
3	3	180	tetrahedron
3	4	240	octahedron
3	5	300	icosahedron
4	3	270	cube
5	3	324	dodecahedron

Table 1. Possible regular polyhedra

In order to generalize the idea of this proof to n-dimensional polytopes we need some new terminology. We define the *n-dimensional angle* between two $(n-1)$-dimensional figures X and Y with $(n-2)$-dimensional intersection S, as the angle between the line segments x and y, where $x \in X$, $y \in Y$ and x and y are perpendicular to S with the nonempty intersection of x and y in S. For example the 3-dimensional angle (also referred to as the platonic angle) between any two intersecting faces of an icosahedron as seen in figure 1 is

$$2\arcsin\left(\frac{\sqrt{5}+1}{2\sqrt{3}}\right) \approx 138.2.$$

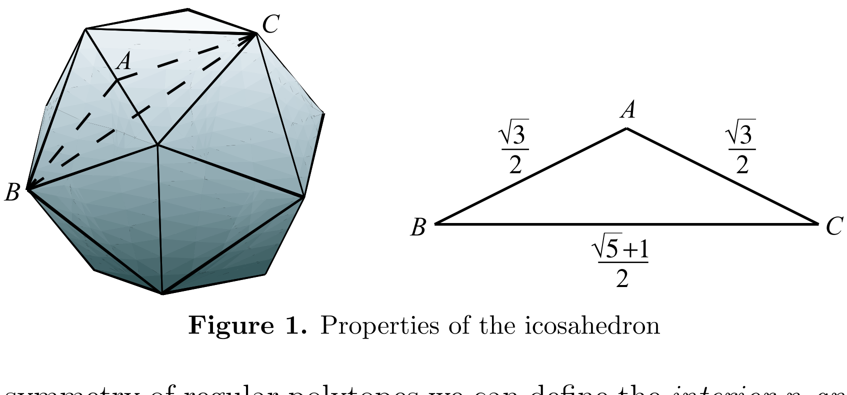

Figure 1. Properties of the icosahedron

Because of the symmetry of regular polytopes we can define the *interior n-angle* of a regular n-dimensional polytope to be the n-dimensional angle between any two intersecting $(n-1)$-dimensional figures in the polytope. Now just as before with 3-dimensional polyhedra, we know that in an n-dimensional polytope the sum of the interior $(n-1)$-angles of the $(n-1)$-dimensional polytopes meeting at an $(n-2)$-polytope must be less than 360. For if the sum of these interior $(n-1)$-angles was 360 then the $(n-1)$-dimensional polytopes would tile in $n-1$ dimensions. An example of this sort of tiling is three hexagons intersecting at one point in two dimensions. Another is four cubes intersecting at on edge as shown in figure 2. This occurs because the interior 3-angle of a cube is easily seen to be 90.

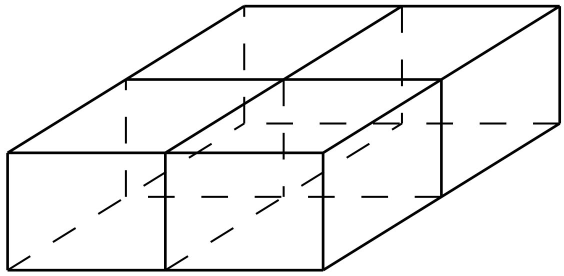

Figure 2. cubes tiling in three dimensions

Now we can find a short list of the possible four dimensional polytopes by simply finding the interior 3-angles of all the three dimensional polyhedra. We already know the interior 3-angles for the icosahedron and the cube. For the dodecahedron, using figure 3, we see the interior 3-angle is

$$2\arcsin\left(\frac{\sqrt{5}+1}{4\sin 72}\right) \approx 116.6.$$

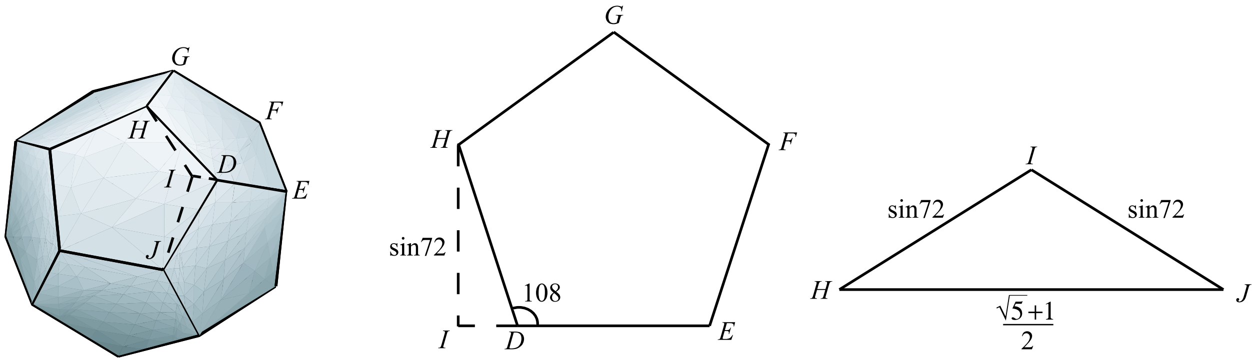

Figure 3. Properties of the dodecahedron

The interior 3-angles for the tetrahedron and octahedron will be calculated later, but it is not hard to show that they are $2\arcsin(\sqrt{3}/3) \approx 70.5$ and $2\arcsin(\sqrt{6}/3) \approx 109.5$ respectively. Instead of naming the four dimensional polytopes, we will use what is known as the "Shläfli symbol." The "Shläfli symbol" can be used to denote regular polytopes of any dimension as follows. The symbol $\{p\}$ denotes the p-sided polygon. The symbol $\{p, q\}$ denotes the polyhedron whose faces are $\{p\}$, and there are q faces meeting at each vertex. Similarly the "Shläfli symbol" $\{a_1, a_2, ..., a_{n-1}\}$ represents an n-dimensional regular polytope made up of $(n-1)$-dimensional regular polytopes $\{a_1, a_2, ..., a_{n-2}\}$ of which there are a_{n-1} meeting at each $(n-2)$-dimensional regular polytope $\{a_1, a_2, ..., a_{n-3}\}$. For example the four dimensional regular polytope known as the hypercube is made of cubes, and three cubes meet at each edge. Therefore it is represented by $\{4, 3, 3\}$. Now if we let $\phi_{\{p,q\}}$ denote the interior 3-angle of the polyhedron $\{p, q\}$, the only possible regular 4-dimensional polytopes have the property that $r\phi_{\{p,q\}} < 360$, where $r > 2$ is the number of polyhedra $\{p, q\}$ meeting at each edge. With this in mind table 2 lists all possible 4-dimensional regular polytopes.

p	q	r	$\phi_{\{p,q\}}$	$r\phi_{\{p,q\}}$	"Shläfli symbol"
3	3	3	70.5	211.5	$\{3, 3, 3\}$
3	3	4	70.5	282	$\{3, 3, 4\}$
3	3	5	70.5	352.5	$\{3, 3, 5\}$
3	4	3	109.5	328.5	$\{3, 4, 3\}$
4	3	3	90	270	$\{4, 3, 3\}$
5	5	3	116.6	349.8	$\{5, 3, 3\}$

Table 2. Possible 4-dimensional regular polytopes

In order to find all the possible 5-dimensional regular polytopes we must calculate the interior 4-angles of all the 4-dimensional regular polytopes. As before we let $\phi_{\{p,q,r\}}$ denote the interior 4-angle of the polytope $\{p, q, r\}$. Then it is easy to see that $\phi_{\{4,3,3\}} = 90$. Also it will be shown later that $\phi_{\{3,3,3\}} \approx 75.5$ and $\phi_{\{3,3,4\}} = 120$. To calculate $\phi_{\{5,3,3\}}$ we need to find the "angle" that two dodecahedrons meet at a pentagonal face. To do this we first let P be a pentagonal face of $\{5, 3, 3\}$. Also let O, N, and M be vertices of $\{5, 3, 3\}$ such that O is on P, N and M are not on P, but NO and MO are edges of $\{5, 3, 3\}$ as in figure 4a. If we let a be the perpendicular distance from M (and therefore N) to P, then

$$a = \sin(72) \sin(180 - \phi_{\{5,3\}})$$

as can be seen in figure 4b. But the distance from M to N is $(\sqrt{5} + 1)/2$ since M and N are vertices of a pentagon. Therefore, as shown in figure 4c, we have

$$\phi_{\{5,3,3\}} = 2 \arcsin(\frac{\sqrt{5} + 1}{4a}) = 144.$$

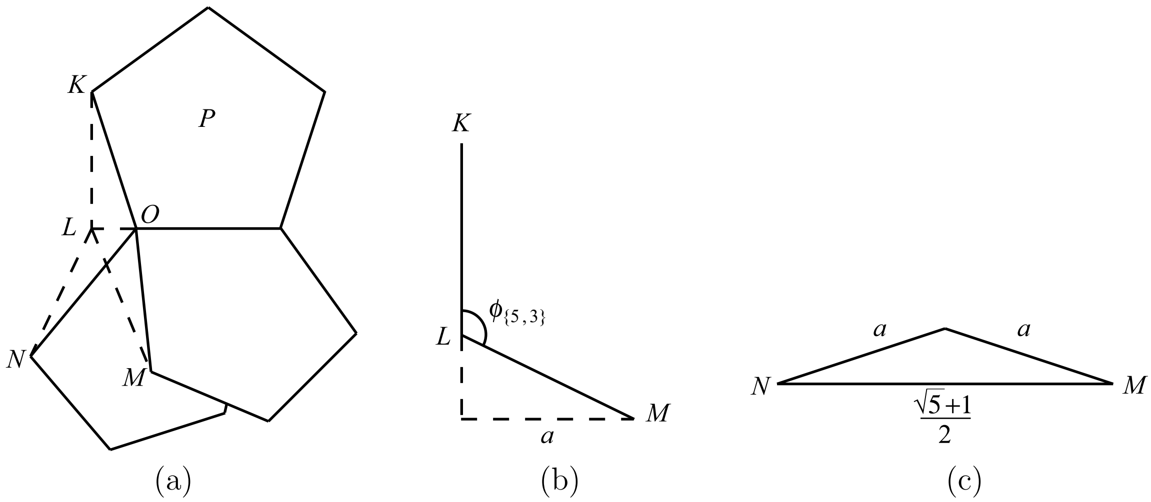

Figure 4. Properties of $\{5, 3, 3\}$

To calculate $\phi_{\{3,3,5\}}$ we need to find the "angle" that two tetrahedrons meet at a triangular face. Let T be a triangular face in $\{3, 3, 5\}$. And let R and S be vertices of $\{3, 3, 5\}$ which are not in T, but are connected by an edge to every vertex of T as in figure 5a. The distance between R and S is $(\sqrt{5} + 1)/2$ since R and S are vertices of a pentagon, and if we let b be the perpendicular distance between R (and therefore S) and T, then

$$b = \frac{\sqrt{3}}{2} \sin(\phi_{\{3,3\}})$$

as can be seen in figure 5b. Therefore, as shown in figure 5c,

$$\phi_{\{3,3,5\}} = 2 \arcsin(\frac{\sqrt{5} + 1}{4b}) \approx 164.5.$$

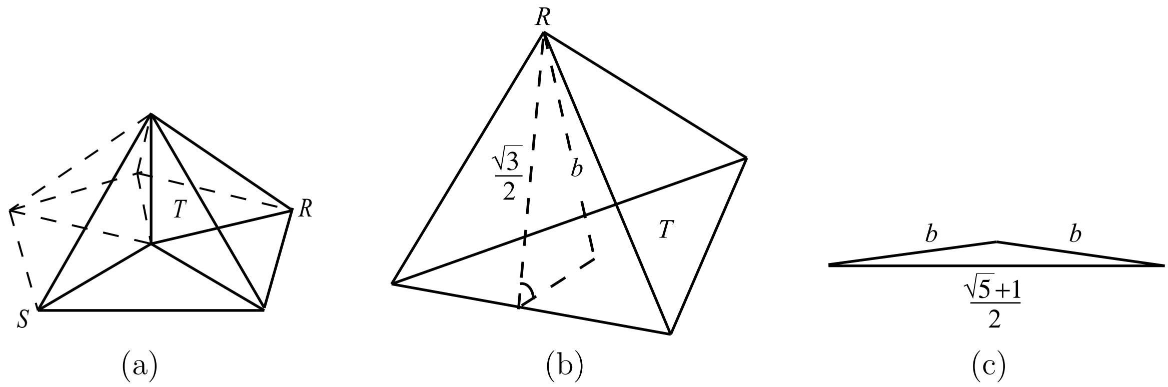

Figure 5. Properties of $\{3, 3, 5\}$

To calculate $\phi_{\{3,4,3\}}$ we need to find the "angle" that two octahedrons meet at a triangular face. So let U be a triangular face in $\{3, 4, 3\}$. And let V and W be vertices of $\{3, 4, 3\}$ which are not contained in U, but are connected by edges to the same two vertices of U as in figure 6a. The distance between U and V is $\sqrt{2}$ since U and V are vertices of a square, and if we let c be the perpendicular distance from V (and therefore W) to T, then

$$c = \frac{\sqrt{3}\sin(180 - \phi_{\{3,4\}})}{2}$$

as can be seen in figures 6b and 6c. Therefore, as seen in figure 6d,

$$\phi_{\{3,4,3\}} = 2\arcsin(\frac{\sqrt{2}}{2c}) = 120.$$

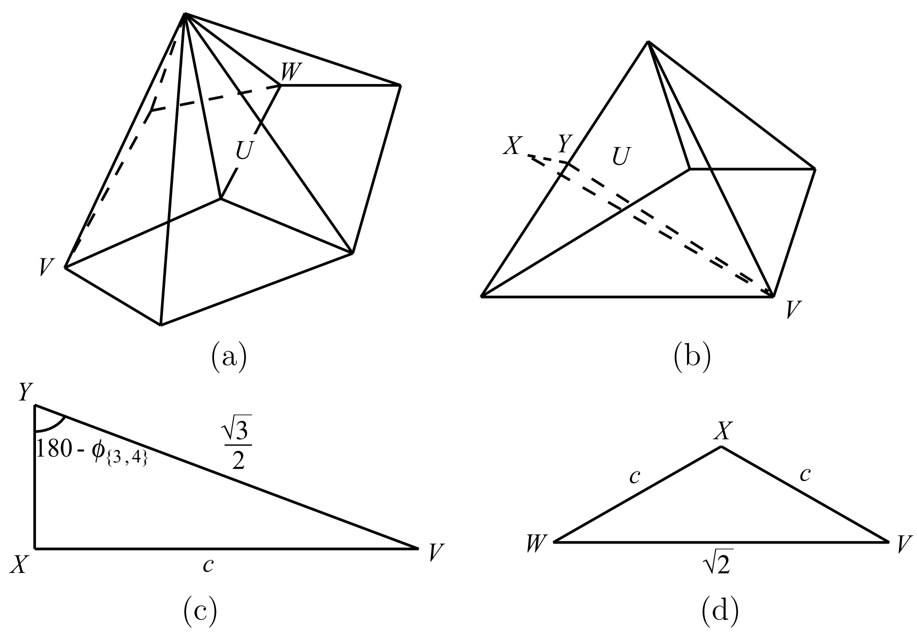

Figure 6. Properties of $\{3, 4, 3\}$

So now we have all the 4-angles of the regular 4-dimensional polytopes. It is interesting to notice that since $\phi_{\{3,3,4\}} = \phi_{\{3,4,3\}} = 120$, the polytopes $\{3, 3, 4\}$ and $\{3, 4, 3\}$ will tile in four dimensions. Now if we let $s > 2$ denote the number of 4-dimensional polytopes of the form $\{p, q, r\}$ meeting at each 3-dimensional polytope $\{p, q\}$, we know that the only possible 5-dimensional regular polytopes have the property that $s\phi_{\{p,q,r\}} < 360$. With this in mind table 3 lists all the possible 5-dimensional polytopes.

p	q	r	s	$\phi_{\{p,q,r\}}$	$s\phi_{\{p,q,r\}}$	"Shläfli symbol"
3	3	3	3	75.5	226.5	$\{3,3,3,3\}$
3	3	3	4	75.5	302	$\{3,3,3,4\}$
4	3	3	3	90	270	$\{4,3,3,3\}$

Table 3. Possible 5-dimensional regular polytopes

Now we will show that for all $n > 4$ there can be no more than three regular polytopes. These polytopes are of the form $\{3,3,\ldots,3\},\{3,3,\ldots,3,4\}$, and $\{4,3,3,\ldots,3\}$. The n-dimensional polytope of the form $\{4,3,3,\ldots,3\}$ is the n-dimensional cube or n-cube, and will always have interior n-angle of 90. Because of this we can never fit four n-cubes about an $(n-1)$-cube without tiling in n-dimensions. Therefore it is not possible for an $(n+1)$-dimensional polytope of the form $\{4,3,3,\ldots,3,4\}$ to exist. The n-dimensional polytope of the form $\{3,3,\ldots,3\}$ is the n-dimensional simplex or n-simplex. Let ϕ_n denote the interior n-angle of of the n-simplex. To find ϕ_n we first look at one of the properties of a simplex. Given an n-simplex we can create an $(n+1)$-simplex by placing a new vertex in our new dimension such that it is at distance one from all the vertices of our n-simplex as shown in figure 7.

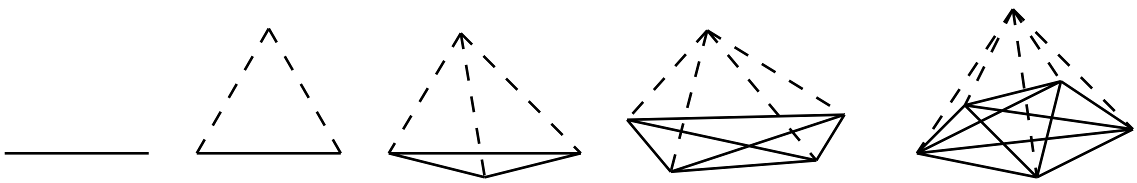

Figure 7. The n-simplex for $1 \leq n \leq 5$

Now if we let A and B be vertices of an n-simplex. And let S denote the $(n-2)$-simplex which is contained in the n-simplex but does not contain A or B. We know that the distance from A to B is 1. If we let t denote the perpendicular distance from A (and therefore B) to S then we know that

$$\phi_n = 2\arcsin(\frac{1}{2t}) \qquad (\star)$$

as can be seen in figure 8.

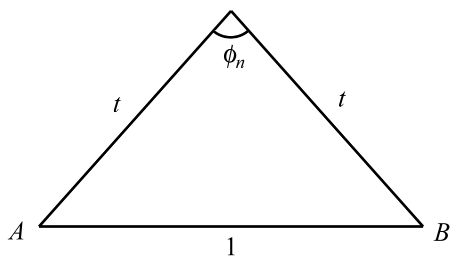

Figure 8. ϕ_n

Since A (and therefore B) is equidistant from all vertices in S, the line through A (and therefore B) perpendicular to S contains the center of S which we denote O_{n-2}. But this line contains the points which are equidistant to all vertices of S it will also contain the center of the $(n-1)$-simplex which we denote O_{n-1}. If we let C be any vertex in S, then figure 9 depicts the relationship described above.

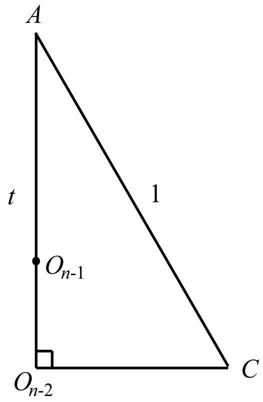

Figure 9.

Now if we let x_n denote the distance from any vertex in an n-simplex to the center of that n-simplex, and let y_n denote the perpendicular distance from the center of an n-simplex to any $(n-1)$-simplex contained in that n-simplex, then we know that $t = x_{n-1} + y_{n-1}$. Also from figure 9 we obtain figure 10 which shows the relationship between x_{n-1}, y_{n-1}, and x_{n-2}.

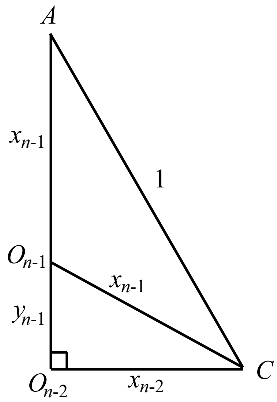

Figure 10.

From $\triangle O_{n-1}O_{n-2}C$ and $\triangle AO_{n-2}C$ we have the following equations:

$$x_{n-2}^2 + y_{n-1}^2 = x_{n-1}^2,$$

$$x_{n-2}^2 + (x_{n-1} + y_{n-1})^2 = 1.$$

Substituting the first equation into the second gives us

$$2x_{n-2}^2 + 2(x_{n-1} + y_{n-1}) = 1$$

$$\Rightarrow x_{n-1} = \frac{1}{2(x_{n-1} + y_{n-1})}.$$

Since $t = x_{n-1} + y_{n-1}$, we can now rewrite $(\star)$ as

$$\phi_n = 2\arcsin(x_{n-1}). \qquad (1)$$

So we have reduced the problem of finding the interior n-angle of the n-simplex to finding x_{n-1}. We will find x_{n-1} recursively in the following way. First we let D be the midpoint of the edge with endpoints A and C. And let s be the distance from D to O_{n-1}. Now since $\triangle AO_{n-1}C$ is isosceles, we have the similar triangles $\triangle AO_{n-1}D$ and $\triangle ACO_{n-2}$ as can be seen in figure 11.

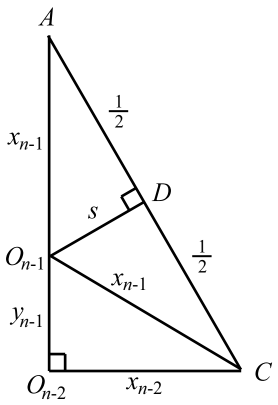

Figure 11.

These similar triangle give us $s = x_{n-1}x_{n-2}$. So $\triangle ADO_{n-1}$ gives us the following relation between x_{n-1} and x_{n-2}:

$$x_{n-1}^2 = (x_{n-1}x_{n-2})^2 + \frac{1}{4}. \qquad (\star\star)$$

To find a recursive formula for x_{n-1} in terms of x_{n-2} we look at $\triangle AO_{n-2}C$ to see that the angle at vertex A is $\arcsin(x_{n-2})$. And $\triangle ADO_{n-1}$ gives us

$$\cos(\arcsin(x_{n-2})) = \frac{1}{2x_{n-1}} \Rightarrow$$

$$x_{n-1} = \frac{1}{2\cos(\arcsin(x_{n-2}))}. \qquad (2)$$

From $(\star\star)$ we can find the limit of x_n as n approaches infinity as follows.

$$x^2 = x^4 + \frac{1}{4} \Rightarrow x = \frac{\sqrt{2}}{2} \Rightarrow \lim_{n\to\infty} x_n = \frac{\sqrt{2}}{2}.$$

And since we know $\phi_n = 2\arcsin(x_{n-1})$ we have

$$\lim_{n\to\infty} \phi_n = 90.$$

But x_n (and therefore ϕ_n) is a strictly increasing sequence, so $\phi_n < 90$ for every n. Thus we can always fit three or four n-simplexes about an $(n-2)$-simplex without tiling in n dimensions. Therefore it is still possible for an $(n+1)$-simplex and an $(n+1)$-dimensional polytope of the form $\{3, 3, \ldots, 3, 4\}$ to exist. But since ϕ_n is increasing, it is not possible for an n-dimensional polytope of the form $\{3, 3, \ldots, 3, 5\}$ to exist when $n > 4$. Also equations (1) and (2) give us a way to compute the interior n-angle for the n-simplex. Because the one dimensional simplex is a line segment, we know $x_1 = 1/2$. From this we can recursively find ϕ_n for any n. Table 4 lists the interior n-angle of the n-simplex for $2 \le n \le 8$.

n	x_{n-1}	ϕ_n	"Shläfli symbol"
2	$1/2$	60	$\{3\}$
3	$\sqrt{3}/3$	70.5	$\{3,3\}$
4	$\sqrt{6}/4$	75.5	$\{3,3,3\}$
5	$\sqrt{10}/5$	78.5	$\{3,3,3,3\}$
6	$\sqrt{15}/6$	80.4	$\{3,3,3,3,3\}$
7	$\sqrt{21}/7$	81.8	$\{3,3,3,3,3,3\}$
8	$\sqrt{7}/4$	82.8	$\{3,3,3,3,3,3,3\}$

Table 4. interior n-angles for the n-simplex

If we let ϕ'_n denote the interior n-angle for the n-dimensional polytope $\{3, 3, \ldots, 3, 4\}$, then one can similarly show

$$\phi'_n = 2 \arcsin(\sqrt{2} x_{n-1}).$$

Using this equation along with (2) we can find ϕ'_n for all n. Table 5 lists ϕ'_n for $2 \leq n \leq 5$.

n	x_{n-1}	ϕ'_n	"Shläfli symbol"
2	$1/2$	90	$\{4\}$
3	$\sqrt{3}/3$	109.5	$\{3, 4\}$
4	$\sqrt{6}/4$	120	$\{3, 3, 4\}$
5	$\sqrt{10}/5$	126.9	$\{3, 3, 3, 4\}$

Table 5. interior n-angles for the n-dimensional polytope $\{3, 3, \ldots, 3, 4\}$

Since ϕ'_n is an increasing sequence, $\phi'_n \geq 120$ for all $n > 3$. So the n-dimensional polytopes of the form $\{3, 3, \ldots, 3, 4, 3\}$ can not exist for $n > 4$. Thus there can be no more than three regular n-dimensional polytopes for $n > 4$.

References

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