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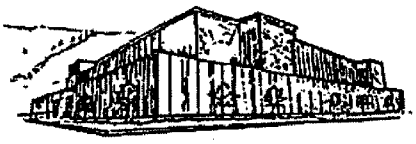

COMPUTATION OF THE FUNCTOR EXT USING GROEBNER BASES

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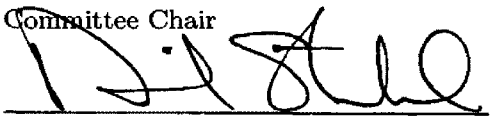

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ABSTRACT

Let R be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ where k is a field, and let M and N be finitely generated R-modules. In this expository paper, we use the theory of Groebner bases to compute the R-modules $\mathrm{Ext}^i_R(M, N)$. We start by computing a presentation of the syzygy module, $\mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$, for $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$. Next we use the syzygy module to compute free resolutions of M and N. Finally, we compute a presentation of $\mathrm{Hom}_R(M, N)$ and we use this to compute a presentation of $\mathrm{Ext}^i_R(M, N)$.

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1. Introduction

The theory of Groebner bases provide algorithms for a variety of important computations regarding modules over a ring. For example, the use of Groebner bases for computations involving ideals in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ where k is a field yields a solution to the ideal membership problem, and plays a prominent role in elimination theory [3]. The purpose of this expository paper is to use the theory of Groebner bases to compute a presentation for the R-module $\mathrm{Ext}^i_R(M, N)$ for finitely generated R-modules M and N. The R-module $\mathrm{Ext}^i_R(M, N)$ has a wide range of applications. For instance $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$. Furthermore $\mathrm{Ext}^1_R(M, N)$ parameterizes the set of all extensions of M by N. More precisely we have the following:

Definition 1.1. For R-modules M and N, an **extension** of M by N is an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Two extensions ξ and ξ' of M by N are **equivalent** if there is a commutative diagram

$$\begin{array}{ccccccccc} \xi: & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle 1_N} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 1_M} & & \\ \xi': & 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where ϕ is an R-module homomorphism. An extension is **split** if it is equivalent to

$$0 \longrightarrow N \xrightarrow{(0,1)} M \oplus N \longrightarrow M \longrightarrow 0.$$

Theorem 1.2.[7, Theorem 7.19, Corollary 7.20] *For R-modules M and N, there is an R-module $Ext^1_R(M, N)$, and a one-to-one correspondence*

$$\{\text{equivelence classes of extensions of } M \text{ by } N\} \xleftrightarrow{1-1} \mathrm{Ext}^1_R(M, N)$$

in which split extensions correspond to $0 \in \mathrm{Ext}^1_R(M, N)$.

The modules $\mathrm{Ext}^i_R(M, N)$ for $i > 1$ encode more subtle properties of the modules M and N [7].

We now give a summary of the contents of this paper. In Section 3 we give the definition of a Groebner basis for submodules of R^m, as well as some properties of Groebner bases which we will need. In Section 4 we define the syzygy module, and given $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$, we find a presentation for $\mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$. We will use the syzygy module to compute a free resolution of certain classes of R-modules in Section 5. In Section 6 we use the syzygy module to compute a presentation of $\mathrm{Hom}_R(M, N)$ for certain R-modules M and N. In the process of computing this presentation of $\mathrm{Hom}_R(M, N)$, we will state two lemmas. We will use these lemmas, along with our knowledge of free resolutions, to compute $\mathrm{Ext}^i_R(M, N)$ in Section 7. Throughout this paper we will give examples of our main results. In Appendix A we show how the computer algebra system *Macaulay 2* has been used in our computations. The results in this paper can be found in [1] and [4], with some passages almost identical. The computations in this paper are new.

Groebner basis over other rings have also been studied. One can define Groebner bases in the ring $A[x_1, x_2, \ldots, x_n]$ where A is a Noetherian commutative ring, see [1, chapter 4]. Furthermore there exists a notion of Groebner bases in noncommutative polynomial rings, see [2] and [6, chapter 8]. The development

of algorithms employing Groebner bases in the latter case are an active area of research, but the description of such algorithms lies beyond the scope of this paper.

2. Preliminaries

Before we define Groebner bases for modules, we give a few definitions which we will need. For the remainder of this paper, unless otherwise noted, we let

$$R = k[x_1, x_2, \ldots, x_n]$$

be the polynomial ring in n variables where k is a field.

Definition 2.1. A monomial order on R is a relation $<$ on the monomials in R which satisfies the following conditions:

- $<$ is a total order.
- If monomials x^α and x^β in R are such that $x^\alpha < x^\beta$, then $x^\alpha x^\gamma < x^\beta x^\gamma$ for any monomial x^γ in R.
- $<$ is a well-ordering.

We now define the monomial orderings we will be using. These orderings can also be found in [3].

Definition 2.2. Let $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$. Also let $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $x^\beta = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be monomials in R. We define the **lex** (lexicographic) order with $x_1 > x_2 > \cdots > x_n$ as follows:

$$x^\alpha <_{lex} x^\beta \iff \text{the leftmost nonzero entry of } \beta - \alpha \text{ is positive.}$$

Definition 2.3. Let $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$. Also let $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $x^\beta = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be monomials in R. We define the **grlex** (graded lexicographic) order with $x_1 > x_2 > \cdots > x_n$ as follows:

$$x^\alpha <_{grlex} x^\beta \iff \begin{cases} \sum_{i=1}^n a_i < \sum_{i=1}^n b_i \\ \text{or} \\ \sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and } x^\alpha <_{lex} x^\beta. \end{cases}$$

Definition 2.4. Let $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$. Also let $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $x^\beta = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be monomials in R. We define the **grevlex** (graded reverse lexicographic) order with $x_1 > x_2 > \cdots > x_n$ as follows:

$$x^\alpha <_{grlex} x^\beta \iff \begin{cases} \sum_{i=1}^n a_i < \sum_{i=1}^n b_i \\ \text{or} \\ \sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and the leftmost nonzero entry of } \beta - \alpha \text{ is negative.} \end{cases}$$

Example 2.5. Let $R = \mathbb{R}[x, y, z]$. Using grlex order on R with $x > y > z$, we have $xy^4z > y^4z^2$, but using grevlex order on R with $x > y > z$, we have $xy^4z < y^4z^2$. Also using lex order on R with $x > y > z$, we have $x > y^5z^{10}$, but using grevlex order on R with $x > y > z$, we have $x < y^5z^{10}$. □

We will be working mainly with submodules of R^m. Although we will think of elements of R^m as columns with m entries from R, we will often write these elements as rows to save space. In symbols, for $\mathbf{f} \in R^m$ such

that

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix},$$

we write $\mathbf{f} = (f_1, f_2, \ldots, f_m)$.

3. Groebner Bases for Modules

In this section we will introduce Groebner bases for modules and give some properties of Groebner bases which we will need. Before introducing Groebner bases we will need the notions of term ordering and division.

Let

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_m = (0, 0, \ldots, 0, 1)$$

be the usual basis of R^m. We define a monomial in R^m to be an element of the form Xe_i for some $1 \leq i \leq m$, where $X = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in R$ for some $a_i \in \{0, 1, 2, \ldots\}$.

Definition 3.1. A **term order** on R^m is a relation $<$ on the monomials in R^m which satisfies the following conditions:

- $<$ is a total order.
- If $\mathbf{X}$ and $\mathbf{Y}$ are monomials in R^m such that $\mathbf{X} < \mathbf{Y}$, then $Z\mathbf{X} < Z\mathbf{Y}$ for every monomial Z in R.
- $<$ is a well-ordering.

Now we define two different term orders on the monomials of R^m.

Definition 3.2. Fix a monomial order $<$ on R, and fix a linear order $<$ on $\{e_1, e_2, \ldots, e_m\}$. Let $\mathbf{X}, \mathbf{Y} \in R^m$ be monomials such that $\mathbf{X} = Xe_i$ and $\mathbf{Y} = Ye_j$. We define the order **TOP** (term over position) as follows:

$$\mathbf{X} <_{TOP} \mathbf{Y} \iff \begin{cases} X < Y \\ \text{or} \\ X = Y \text{ and } e_i < e_j. \end{cases}$$

Definition 3.3. Fix a monomial order $<$ on R, and fix a linear order $<$ on $\{e_1, e_2, \ldots, e_m\}$. Let $\mathbf{X}, \mathbf{Y} \in R^m$ be monomials such that $\mathbf{X} = Xe_i$ and $\mathbf{Y} = Ye_j$. We define the order **POT** (position over term) as follows:

$$\mathbf{X} <_{POT} \mathbf{Y} \iff \begin{cases} e_i < e_j \\ \text{or} \\ e_i = e_j \text{ and } X < Y. \end{cases}$$

Example 3.4. Let $R = \mathbb{Q}[x, y]$. With lex order on R with $x > y$ and $e_1 > e_2$, TOP order on R^2 gives us

$$(0, x^2y) > (xy, 0).$$

Whereas POT order gives us

$$(xy, 0) > (0, x^2y). \qquad \square$$

Now that we have a notion of term order, given $\mathbf{f} \in R^m$, we can define the **leading monomial** of $\mathbf{f}$, $\mathrm{lm}(\mathbf{f})$, the **leading coefficient** of $\mathbf{f}$, $\mathrm{lc}(\mathbf{f})$, and the **leading term** of $\mathbf{f}$, $\mathrm{lt}(\mathbf{f})$.

Definition 3.5. Let $\mathbf{f} \in R^m$ such that $\mathbf{f} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \ldots + a_s\mathbf{X}_s$, where for all $1 \leq i \leq s$, $a_i \in k$ with $a_1 \neq 0$ and $\mathbf{X}_i$ is a monomial such that $\mathbf{X}_i > \mathbf{X}_{i+1}$. We define

$$\mathrm{lm}(\mathbf{f}) = \mathbf{X}_1, \quad \mathrm{lc}(\mathbf{f}) = a_1, \quad \mathrm{lt}(\mathbf{f}) = a_1\mathbf{X}_1.$$

Example 3.6. Let $R = \mathbb{Q}[x, y, z]$ and let

$$\mathbf{f} = (x^2z^2 - 2x, 5x^2y + 2yz, xyz) \in R^3.$$

With lex order on R with $x > y > z$, and with TOP order with $e_1 > e_2 > e_3$, we have

$$\mathrm{lm}(\mathbf{f}) = x^2ye_2, \quad \mathrm{lc}(\mathbf{f}) = 5, \quad \mathrm{lt}(f) = 5x^2ye_2.$$

With lex order on R with $x > y > z$, and with POT order with $e_1 > e_2 > e_3$, we have

$$\mathrm{lm}(\mathbf{f}) = x^2z^2e_1, \quad \mathrm{lc}(\mathbf{f}) = 1, \quad \mathrm{lt}(\mathbf{f}) = x^2z^2e_1. \qquad \square$$

We now define division of monomials in R^m. With this definition, we can formulate a notion of remainders and thus a division algorithm.

Definition 3.7. Let $\mathbf{X} = Xe_i$ and $\mathbf{Y} = Ye_j$ be monomials in R^m. We say that $\mathbf{X}$ **divides** $\mathbf{Y}$ if $i = j$ and X divides Y. Furthermore if $\mathbf{X}$ divides $\mathbf{Y}$ we define

$$\frac{\mathbf{Y}}{\mathbf{X}} = \frac{Y}{X} \in R.$$

Example 3.8. Let $R = \mathbb{Q}[x, y, z]$. The monomial $(0, xy^2z, 0) \in R^3$ divides $(0, x^2y^2z^2, 0)$, but does not divide $(x^2y^2z^2, 0, 0)$ or $(0, xyz, 0)$. And

$$\frac{(0, x^2y^2z^2, 0)}{(0, xy^2z, 0)} = \frac{x^2y^2z^2}{xy^2z} = xz. \qquad \square$$

Definition 3.9. Given $\mathbf{f}, \mathbf{g}, \mathbf{h} \in R^m$, we say that $\mathbf{f}$ **reduces to** $\mathbf{h}$ **modulo** $\mathbf{g}$ if there exists a term, $\mathbf{X}$, in $\mathbf{f}$ such that $\mathrm{lm}(\mathbf{g})$ divides $\mathbf{X}$ and $\mathbf{h} = \mathbf{f} - \frac{\mathbf{X}}{\mathrm{lt}(\mathbf{g})}\mathbf{g}$. We write

$$\mathbf{f} \xrightarrow{\mathbf{g}} \mathbf{h}.$$

Example 3.10. Let $R = \mathbb{Q}[x, y]$, and $\mathbf{f}, \mathbf{g}, \mathbf{h} \in R^2$ such that $\mathbf{f} = (x^2y + x, y), \mathbf{g} = (xy, x)$, and $\mathbf{h} = (x, y - x^2)$. Using grlex order on R with $x > y$, and with POT order with $e_1 > e_2$, we have $\mathrm{lm}(\mathbf{g}) = xye_1$. Therefore

$$\mathbf{f} - \frac{x^2y}{xy}(xy, x) = (x, y - x^2) = \mathbf{h},$$

so $\mathbf{f} \xrightarrow{\mathbf{g}} \mathbf{h}$. $\square$

Definition 3.11. Given $F = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_s\} \subset R^m$ and $\mathbf{f}, \mathbf{h} \in R^m$, we say that $\mathbf{f}$ **reduces to** $\mathbf{h}$ **modulo** F if there exist some $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{r-1} \in R^m$ such that for some $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_r} \in F$ the following occurs:

$$\mathbf{f} \xrightarrow{\mathbf{f}_{i_1}} \mathbf{h}_1 \xrightarrow{\mathbf{f}_{i_2}} \mathbf{h}_2 \xrightarrow{\mathbf{f}_{i_3}} \cdots \xrightarrow{\mathbf{f}_{i_{r-1}}} \mathbf{h}_{r-1} \xrightarrow{\mathbf{f}_{i_r}} \mathbf{h}.$$

We write

$$\mathbf{f} \xrightarrow{F} \mathbf{h}.$$

Definition 3.12. Given $\mathbf{r} \in R^m$ and $F \subset R^m$, we say that $\mathbf{r}$ is **reduced with respect to** F if either $\mathbf{r} = \mathbf{0}$ or $\mathbf{r}$ cannot be reduced modulo F. If $\mathbf{h} \in R^m$ is such that

$$\mathbf{h} \xrightarrow{F} \mathbf{r}$$

where $\mathbf{r}$ is reduced with respect to F, then we say that $\mathbf{r}$ is a **remainder** of $\mathbf{h}$ with respect to F and we write

$$\overline{\mathbf{h}}^F = \mathbf{r}.$$

Notice that given $\mathbf{f} \in R^m$ and $F \subset R^m$, a remainder of $\mathbf{f}$ with respect to F is not necessarily unique as can be seen in the following example.

Example 3.13. Let $R = \mathbb{Q}[x, y]$ and let $\mathbf{f} \in R^2$ such that $\mathbf{f} = (x^3 + x, 2xy)$. Also let $F = \{\mathbf{f}_1, \mathbf{f}_2\} \subset R^2$ where $\mathbf{f}_1 = (x^2 + 1, 2y)$ and $\mathbf{f}_2 = (x^2, y)$. Using lex order on R with $x > y$ and with POT order on R^2 with $e_1 > e_2$ we have $\mathrm{lm}(\mathbf{f}_1) = \mathrm{lm}(\mathbf{f}_2) = x^2 e_1$. Thus $\mathbf{f} \xrightarrow{\mathbf{f}_1} \mathbf{0}$ so that $\overline{\mathbf{f}}^F = \mathbf{0}$ since $\mathbf{0}$ is reduced with respect to F. But also we have $\mathbf{f} \xrightarrow{\mathbf{f}_2} (x, xy)$ so that $\overline{\mathbf{f}}^F = (x, xy)$ since (x, xy) is reduced with respect to F. □

Theorem 3.14. *There exists a division algorithm which given $\mathbf{f} \in R^m$, and $F = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_s\} \subset R^m$, will yield $a_1, a_2, \ldots, a_s \in R$ and $\mathbf{r} \in R^m$ such that*

$$\mathbf{f} = a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \cdots + a_s\mathbf{f}_s + \mathbf{r}$$

where $\overline{\mathbf{f}}^F = \mathbf{r}$.

As the example above illustrates, this algorithm may give different remainders. It is natural to ask how helpful this division algorithm is, since it does not give unique remainders. When the set by which we are dividing is a Groebner basis, we get unique remainders. Below is the definition of a Groebner basis for a submodule of R^m followed by some important properties of a Groebner basis. For a proof of Theorem 3.16 we refer the reader to [1, Section 3.5](see also [4, Section 5.2]).

Definition 3.15. Given a submodule $M \subset R^m$, the set $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\} \subset M$ is a **Groebner basis** for M if for every $\mathbf{f} \in M$, $\mathrm{lm}(\mathbf{f})$ is divisible by $\mathrm{lm}(\mathbf{g}_i)$ for some $1 \leq i \leq s$.

Theorem 3.16. *For a submodule $M \subset R^m$ and a set $G = \{g_1, g_2, \ldots, g_s\} \subset M$ the following are equivalent:*

- *G is a Groebner basis for M.*
- *$\mathbf{f} \in M \Leftrightarrow \overline{\mathbf{f}}^G = \mathbf{0}$.*
- *$\forall \mathbf{f} \in R^m$, if $\overline{\mathbf{f}}^G = \mathbf{r}_1$, and $\overline{\mathbf{f}}^G = \mathbf{r}_2$,, we have $\mathbf{r}_1 = \mathbf{r}_2$.*
- *$\langle \mathrm{lt}(\mathbf{g}_1), \mathrm{lt}(\mathbf{g}_2), \ldots, \mathrm{lt}(\mathbf{g}_s) \rangle = \langle \mathrm{lt}(M) \rangle$*
- *$\forall \mathbf{f} \in M : \exists h_1, h_2, \ldots, h_s \in R$ such that $\mathbf{f} = \sum_{i=1}^{s} h_i \mathbf{g}_i$ and $\mathrm{lm}(\mathbf{f}) = \max_i \{\mathrm{lm}(h_i)\mathrm{lm}(\mathbf{g}_i)\}$.*

The theorem above specializes, when $m = 1$, to results about Groebner bases in polynomial rings, see e.g., [3, Section 2.6]. The following Corollaries of Theorem 3.16 are not hard to prove.

Corollary 3.17. *If $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ is a Groebner basis for the submodule $M \subset R^m$, then $M = \langle \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s \rangle$.*

Corollary 3.18. *Every nonzero submodule of R^m has a Groebner basis.*

Given generators for a submodule $M \subset R^m$, there exists an algorithm, known as Buchberger's algorithm for modules, which gives $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s \in R^m$ such that $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ is a Groebner basis for M. The division algorithm and Buchberger's algorithm give us the following relationship between a submodule M of R^m and a Groebner basis for M.

Proposition 3.19. *Given a Groebner basis $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ for the submodule $\langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \rangle \subset R^m$, let F be the $m \times t$ matrix with columns $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t$ and let G be the $m \times s$ matrix with columns $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s$. There*

exist a $t \times s$ matrix S with entries in R and an $s \times t$ matrix T with entries in R such that

$$FS = G \quad \text{and} \quad GT = F.$$

Proposition 3.19 follows easily from Corollary 3.17, so the proof is omitted. Note that the entries of S can be obtained by keeping track of reductions during Buchberger's algorithm, and entries of T are obtained by applying the division algorithm on each $\mathbf{f}_i$ by $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$.

4. The Syzygy Module

In this section we will introduce the syzygy module, and give a method for finding a presentation of certain syzygy modules.

Definition 4.1. Given $F = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t\} \subset R^m$ we say that $h = (h_1, h_2, \ldots, h_t) \in R^t$ is a **syzygy** of F if

$$h_1\mathbf{f}_1 + h_2\mathbf{f}_2 + \cdots + h_t\mathbf{f}_t = \mathbf{0}.$$

We define the set of all such syzygies to be the **syzygy module** of F, and we write

$$\text{Syz}(F) \text{ or } \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t).$$

Notice that $\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$ is the kernel of the R-module homomorphism

$$\begin{aligned} R^t &\longrightarrow R^m \\ (h_1, h_2, \ldots, h_t) &\longmapsto h_1\mathbf{f}_1 + h_2\mathbf{f}_2 + \cdots + h_t\mathbf{f}_t. \end{aligned}$$

It is not hard to show that for any $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$, $\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$ is in fact a submodule of R^t. For all $1 \leq i \leq t$, set $\mathbf{f}_i = (f_{i1}, f_{i2}, \ldots, f_{im}) \in R^m$. Then by the definition of $\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$ we can view the syzygy module as the set of all polynomial solutions $h \in R^t$ to the following system of equations:

$$X_1 f_{1j} + X_2 f_{2j} + \cdots + X_t f_{tj} = 0, \quad \text{for all } 1 \leq j \leq m.$$

Given $F = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t\} \subset R^m$, we wish to find generators for $\text{Syz}(F)$. That is we want to find $h_1, h_2, \ldots, h_s \in R^t$ such that $\langle h_1, h_2, \ldots, h_s \rangle = \text{Syz}(F)$. We will start with a special case of this problem, and find generators for $\text{Syz}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t)$ where $\mathbf{X}_i$ is a monomial in R^m for each $1 \leq i \leq t$. To do so we will need to define the least common multiple of two monomials.

Definition 4.2. Let $\mathbf{X} = Xe_i$ and $\mathbf{Y} = Ye_j$ be monomials in R^m. We define the **least common multiple** of $\mathbf{X}$ and $\mathbf{Y}$ ($\text{LCM}(\mathbf{X}, \mathbf{Y})$) as follows:

$$\text{LCM}(\mathbf{X}, \mathbf{Y}) = \begin{cases} \text{LCM}(X, Y)e_i, & \text{if } i = j; \\ \mathbf{0}, & \text{if } i \neq j. \end{cases}$$

Example 4.3. Let $R = \mathbb{Q}[x, y, z]$ and let $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in R^2$ such that

$$\mathbf{X} = (x^3y, 0), \qquad \mathbf{Y} = (x^2z^2, 0), \qquad \mathbf{Z} = (0, x^2z^2).$$

Then

$$\text{LCM}(\mathbf{X}, \mathbf{Y}) = (x^3yz^2, 0), \quad \text{and} \quad \text{LCM}(\mathbf{Y}, \mathbf{Z}) = \text{LCM}(\mathbf{Z}, \mathbf{X}) = (0, 0). \qquad \square$$

Now we return to the problem of finding generators for the syzygy module of monomials in R^m. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ be monomials in R^m. If for all $i, j \in \{1, 2, \ldots, t\}$ we let

$$\mathbf{X}_{ij} = \mathrm{LCM}(\mathbf{X}_i, \mathbf{X}_j),$$

then we have the following proposition.

Proposition 4.4. *If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ are monomials in R^m, then*

$$\left\{ \frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j : 1 \leq i < j \leq t \right\}$$

is a generating set for $\mathrm{Syz}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t)$.

Proof. First we prove the Proposition in the case $m = 1$. Let $X_1, X_2, \ldots, X_t \in R$. We need to show

$$\mathrm{Syz}(X_1, X_2, \ldots, X_t) = \left\langle \frac{X_{ij}}{X_i} e_i - \frac{X_{ij}}{X_j} e_j : 1 \leq i < j \leq t \right\rangle.$$

It is easy to see

$$\left\langle \frac{X_{ij}}{X_i} e_i - \frac{X_{ij}}{X_j} e_j : 1 \leq i < j \leq t \right\rangle \subset \mathrm{Syz}(X_1, X_2, \ldots, X_t).$$

So we must show the converse. Let $(u_1, u_2, \ldots, u_t) \in \mathrm{Syz}(X_1, X_2, \ldots, X_t)$. If we fix a monomial $X \in R$, we know that the coefficients of X in $u_1 X_1 + u_2 X_2 + \cdots + u_t X_t$ must sum to 0. Therefore it suffices to prove the case where for each $1 \leq i \leq t$, $u_i = c_i X_i'$ where either $c_i = 0$ or $X_i X_i' = X$. Let c_{i_j} denote the nonzero c_i's. Then we have

$$\begin{aligned}
(u_1, u_2, \ldots, u_t) &= c_1 X_1' e_1 + c_2 X_2' e_2 + \cdots + c_t X_t' e_t = c_{i_1} X_{i_1}' e_{i_1} + c_{i_2} X_{i_2}' e_{i_2} + \cdots + c_{i_s} X_{i_s}' e_{i_s} \\
&= c_{i_1} \frac{X}{X_{i_1}} e_{i_1} + c_{i_2} \frac{X}{X_{i_2}} e_{i_2} + \cdots + c_{i_s} \frac{X}{X_{i_s}} e_{i_s} \\
&= c_{i_1} \frac{X}{X_{i_1,i_2}} \left(\frac{X_{i_1,i_2}}{X_{i_1}} e_{i_1} - \frac{X_{i_1,i_2}}{X_{i_2}} e_{i_2} \right) + (c_{i_1} + c_{i_2}) \frac{X}{X_{i_2,i_3}} \left(\frac{X_{i_2,i_3}}{X_{i_2}} e_{i_2} - \frac{X_{i_2,i_3}}{X_{i_3}} e_{i_3} \right) + \cdots \\
&\quad + (c_{i_1} + c_{i_2} + \cdots + c_{i_{s-1}}) \frac{X}{X_{i_{s-1},i_s}} \left(\frac{X_{i_{s-1},i_s}}{X_{i_s}} e_{i_{s-1}} - \frac{X_{i_{s-1},i_s}}{X_{i_s}} e_{i_s} \right) + \\
&\quad (c_{i_1} + c_{i_2} + \cdots + c_{i_s}) \frac{X}{X_{i_s}}.
\end{aligned}$$

But $c_{i_1} + c_{i_2} + \cdots + c_{i_s} = c_1 + c_2 + \cdots + c_t = 0$, so we have written $(u_1, u_2, \ldots, u_t)$ as a linear combination of elements from the set $\{\frac{X_{ij}}{X_i} e_i - \frac{X_{ij}}{X_j} e_j : 1 \leq i < j \leq t\}$. Thus, the proposition is true for $m = 1$.

Now we prove the result in the case $m > 1$. Notice that for $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t \in R^m$, $\mathrm{Syz}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t)$ is the intersection of the syzygy modules of the lth coordinates of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ for each $1 \leq l \leq m$. Let $J_l = \{i :$ the nonzero entry of $\mathbf{X}_i$ is in the lth coordinate$\}$. Using the result of the proposition when $m = 1$ we know the syzygy module of the lth coordinate of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ is

$$\left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j : i < j : i, j \in J_l \right\rangle \oplus \langle e_i : i \notin J_l \rangle.$$

Therefore the intersection of the syzygy module of the first coordinate of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$, and the syzygy module of the second coordinate of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ is

$$\left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j : i < j : i, j \in J_1 \right\rangle \oplus \left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j : i < j : i, j \in J_2 \right\rangle \oplus \langle e_i : i \notin J_1 \cup J_2 \rangle.$$

For all $i \in J_1, j \in J_2$ we know $\mathbf{X}_{ij} = 0$ which implies $\frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j = 0$. Therefore

$$\left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : i < j : i,j \in J_1 \right\rangle \oplus \left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : i < j : i,j \in J_2 \right\rangle$$

$$= \left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : i < j : i,j \in J_1 \cup J_2 \right\rangle,$$

which implies the intersection of the syzygy module of the first coordinate of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$, and the syzygy module of the second coordinate of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ is

$$\left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : i < j : i,j \in J_1 \cup J_2 \right\rangle \oplus \langle e_i : i \notin J_1 \cup J_2 \rangle.$$

If we continue in this process we see that the intersection of the syzygy modules of the lth coordinates of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t$ for each $1 \leq l \leq m$ is

$$\left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : i < j : i,j \in \bigcup_{l=1}^{m} J_l \right\rangle \oplus \left\langle e_i : i \notin \bigcup_{l=1}^{m} J_l \right\rangle$$

$$= \left\langle \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j : 1 \leq i < j \leq t \right\rangle.$$

□

With the above proposition, given $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$ we can now compute $\mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$. We will do so in two steps. First we compute $\mathrm{Syz}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s)$ where $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ is a Groebner basis for $\langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \rangle$. Let $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$. Note that we can assume $\mathrm{lc}(\mathbf{g}_i) = 1$ for all $1 \leq i \leq s$, for if we divide each element of the Groebner basis by its leading coefficient we will still have a Groebner basis. Let $\mathrm{lm}(\mathbf{g}_i) = \mathbf{X}_i$ and let $\mathrm{LCM}(\mathbf{X}_i, \mathbf{X}_j) = \mathbf{X}_{ij}$. Now for all $\mathbf{g}_i$ and $\mathbf{g}_j$ in G we define

$$S(\mathbf{g}_i, \mathbf{g}_j) = \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}\mathbf{g}_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}\mathbf{g}_j.$$

Since $S(\mathbf{g}_i, \mathbf{g}_j) \in \langle G \rangle$, by Theorem 3.16 we have for some $h_{ij\nu} \in R$

$$S(\mathbf{g}_i, \mathbf{g}_j) = \sum_{\nu=1}^{s} h_{ij\nu}\mathbf{g}_\nu,$$

where

$$\max_{1 \leq \nu \leq s} \{\mathrm{lm}(h_{ij\nu})\mathrm{lm}(\mathbf{g}_\nu)\} = \mathrm{lm}(S(\mathbf{g}_i, \mathbf{g}_j)).$$

Now for all $1 \leq i < j \leq s$ we define

$$\mathbf{s}_{ij} = \frac{\mathbf{X}_{ij}}{\mathbf{X}_i}e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}e_j - (h_{ij1}, h_{ij2}, \ldots, h_{ijs}) \in R^s.$$

Theorem 4.5. *With the notation above, the set* $\{\mathbf{s}_{ij} : 1 \leq i < j \leq s\}$ *is a generating set for* $\mathrm{Syz}(G)$.

Proof. Let G' denote the $m \times s$ matrix with columns $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s$. For all $1 \leq i < j \leq s$ we know $\mathbf{s}_{ij} \in \mathrm{Syz}(G)$ since

$$G'\mathbf{s}_{ij} = S(\mathbf{g}_i, \mathbf{g}_j) - \sum_{\nu=1}^{s} h_{ij\nu}\mathbf{g}_\nu = \mathbf{0}.$$

Therefore $\langle \mathbf{s}_{ij} : 1 \leq i < j \leq s \rangle \subset \mathrm{Syz}(G)$. We prove the converse by contradiction. Assume there exists some

$$(u_1, u_2, \ldots, u_s) \in \mathrm{Syz}(G) \setminus \langle \mathbf{s}_{ij} : 1 \leq i < j \leq s \rangle.$$

Because we are implicitly using a term order on R^m, which is a well ordering, we can choose a $(u_1, u_2, \ldots, u_s)$ so that

$$\mathbf{X} = \max_{1 \leq i \leq s} \{\mathrm{lm}(u_i)\mathrm{lm}(\mathbf{g}_i)\}$$

is least. Let

$$A = \{i : \mathrm{lm}(u_i)\mathrm{lm}(\mathbf{g}_i) = \mathbf{X}\},$$

and set

$$u_i' = \begin{cases} u_i, & \text{if } i \notin A; \\ u_i - \mathrm{lt}(u_i), & \text{if } i \in A. \end{cases}$$

If we let $\mathrm{lt}(u_i) = c_i X_i'$ where $c_i \in k$ and X_i' is a monomial in R, then we can write $(u_1, u_2, \ldots, u_s)$ as follows:

$$(u_1, u_2, \ldots, u_s) = (u_1', u_2', \ldots, u_s') + \sum_{i \in A} c_i X_i' e_i. \tag{1}$$

Now, since $(u_1, u_2, \ldots, u_s) \in \mathrm{Syz}(G)$ we know

$$\sum_{i \in A} c_i X_i' \mathbf{X}_i = \mathbf{0}$$

since $X_i'\mathbf{X}_i = \mathbf{X}$ iff $i \in A$. In other words $\sum_{i \in A} c_i X_i' e_i \in \mathrm{Syz}(\mathbf{X}_i : i \in A)$. So Proposition 4.4 tells us that for some $a_{ij} \in R$ we have

$$\sum_{i \in A} c_i X_i' e_i = \sum_{i,j \in A, i<j} a_{ij} \left(\frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j \right).$$

Notice that each coordinate in the left-hand side of the equation above is homogeneous. Also for all $i \in A$, $X_i'\mathbf{X}_i = \mathbf{X}$, so we can assume each a_{ij} is a constant multiple of $\frac{\mathbf{X}}{\mathbf{X}_{ij}}$. Substituting into equation (1) we get

$$\begin{aligned}(u_1, u_2, \ldots, u_s) &= (u_1', u_2', \ldots, u_s') + \sum_{i,j \in A, i<j} a_{ij} \left(\frac{\mathbf{X}_{ij}}{\mathbf{X}_i} e_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j} e_j \right) \\ &= (u_1', u_2', \ldots, u_s') + \sum_{i,j \in A, i<j} a_{ij} \mathbf{s}_{ij} + \sum_{i,j \in A, i<j} a_{ij}(h_{ij1}, h_{ij2}, \ldots, h_{ijs}).\end{aligned}$$

Now if we let $(v_1, v_2, \ldots, v_s) = (u_1', u_2', \ldots, u_s') + \sum_{i,j \in A, i<j} a_{ij}(h_{ij1}, h_{ij2}, \ldots, h_{ijs})$ then we have

$$(v_1, v_2, \ldots, v_s) = (u_1, u_2, \ldots, u_s) - \sum_{i,j \in A, i<j} a_{ij} \mathbf{s}_{ij}. \tag{2}$$

Since the right side of equation (2) is an element of $\mathrm{Syz}(G) \setminus \langle \mathbf{s}_{ij} : 1 \leq i < j \leq s \rangle$ we have that

$$(v_1, v_2, \ldots, v_s) \in \mathrm{Syz}(G) \setminus \langle \mathbf{s}_{ij} : 1 \leq i < j \leq s \rangle.$$

Also, the definition of $(v_1, v_2, \ldots, v_s)$ tells us

$$\begin{aligned}\mathrm{lm}(v_\nu)\mathrm{lm}(\mathbf{g}_\nu) &= \mathrm{lm}(u_\nu' + \sum_{i,j \in A, i<j} a_{ij} h_{ij\nu})\mathbf{X}_\nu \\ &\leq \max\{\mathrm{lm}(u_\nu')\mathbf{X}_\nu, \max_{i,j \in A, i<j} \{\mathrm{lm}(a_{ij})\mathrm{lm}(h_{ij\nu})\}\mathbf{X}_\nu\}.\end{aligned}$$

By construction of $(u_1', u_2', \ldots, u_s')$ we know that $\mathrm{lm}(u_\nu')\mathbf{X}_\nu < \mathbf{X}$. Also since a_{ij} is a constant multiple of $\frac{\mathbf{X}}{\mathbf{X}_{ij}}$, we have

$$\mathrm{lm}(a_{ij})\mathrm{lm}(h_{ij\nu})\mathbf{X}_\nu = \frac{\mathbf{X}}{\mathbf{X}_{ij}} \mathrm{lm}(h_{ij\nu})\mathbf{X}_\nu \leq \frac{\mathbf{X}}{\mathbf{X}_{ij}} \mathrm{lm}(S(\mathbf{g}_i, \mathbf{g}_j)) < \mathbf{X}.$$

The last inequality follows from $\mathrm{lm}(S(\mathbf{g}_i, \mathbf{g}_j)) = \mathrm{lm}(\frac{\mathbf{X}_{ij}}{\mathbf{X}_i}\mathbf{g}_i - \frac{\mathbf{X}_{ij}}{\mathbf{X}_j}\mathbf{g}_j) < \mathbf{X}_{ij}$. Thus

$$\max_{1 \leq i \leq s} \{\mathrm{lm}(v_i), \mathrm{lm}(\mathbf{g}_i)\} < \mathbf{X}$$

which contradicts our original choice of $\mathbf{X}$. □

Example 4.6. Let $R = \mathbb{Q}[x, y, z]$ and $\langle f_1, f_2 \rangle \subset R$ be such that

$$f_1 = x^3 - y, \quad \text{and} \quad f_2 = xyz^2.$$

Then using grevlex order on R with $x > y > z$ one can show $\{x^3 - y, y^2z^2, xyz^2\}$ is a Groebner basis for $\langle f_1, f_2 \rangle$. So let

$$g_1 = x^3 - y, \quad g_2 = y^2z^2, \quad \text{and} \quad g_3 = xyz^2.$$

Then we find

$$S(g_1, g_2) = \frac{x^3y^2z^2}{x^3}(x^3 - y) - \frac{x^3y^2z^2}{y^2z^2}y^2z^2 = -y^3z^2 = -yg_2,$$

$$S(g_1, g_3) = \frac{x^3yz^2}{x^3}(x^3 - y) - \frac{x^3yz^2}{xyz^2}xyz^2 = -y^2z^2 = -g_2,$$

$$S(g_2, g_3) = \frac{xy^2z^2}{y^2z^2}y^2z^2 - \frac{xy^2z^2}{xyz^2}xyz^2 = 0.$$

Now we compute

$$\mathbf{s}_{12} = \frac{x^3y^2z^2}{x^3}e_1 - \frac{x^3y^2z^2}{y^2z^2}e_2 - (0, -y, 0) = (y^2z^2, -x^3 + y, 0),$$

$$\mathbf{s}_{13} = \frac{x^3yz^2}{x^3}e_1 - \frac{x^3yz^2}{xyz^2}e_3 - (0, -1, 0) = (yz^2, 1, -x^2),$$

$$\mathbf{s}_{23} = \frac{xy^2z^2}{y^2z^2}e_2 - \frac{xy^2z^2}{xyz^2}e_3 - (0, 0, 0) = (0, x, -y).$$

Therefore we have

$$\mathrm{Syz}(g_1, g_2, g_3) = \langle (y^2z^2, -x^3 + y, 0), (yz^2, 1, -x^2), (0, x, -y) \rangle.$$ □

Example 4.7. Let $R = \mathbb{Q}[x, y, z]$ and $\langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle \subset R^3$ be such that

$$\mathbf{f}_1 = (x^2 - y + z, 0, -x^2 + x), \quad \mathbf{f}_2 = (-xyz - yz, y - z - 1, xyz),$$

$$\text{and} \quad \mathbf{f}_3 = (-x^2z + xyz + yz - z^2, x - y + z, x^2z - xyz - xz).$$

Then using lex order in R with $x > y > z$ and using TOP order on R^3 with $e_3 > e_2 > e_1$ one can show $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is a Groebner basis for $\langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle$ where

$$\mathbf{g}_1 = (-yz, x - 1, 0), \quad \mathbf{g}_2 = \mathbf{f}_2, \quad \text{and} \quad \mathbf{g}_3 = -\mathbf{f}_1.$$

Then we find

$$S(\mathbf{g}_1, \mathbf{g}_2) = S(\mathbf{g}_1, \mathbf{g}_3) = 0,$$

$$S(\mathbf{g}_2, \mathbf{g}_3) = \frac{x^2yz}{xyz}(-xyz - yz, y - z - 1, xyz) - \frac{x^2yz}{x^2}(-x^2 + y - z, 0, x^2 - x)$$

$$= (-xyz - y^2z + yz^2, xy - xz - x, xyz) = (y - z - 1)\mathbf{g}_1 + \mathbf{g}_2.$$

Now we compute

$$\mathbf{s}_{12} = \mathbf{s}_{13} = 0,$$

$$\mathbf{s}_{23} = \frac{x^2yz}{xyz}e_2 - \frac{x^2yz}{x^2}e_3 - (y - z - 1, 1, 0) = (-y + z + 1, x - 1, -yz).$$

Therefore we have

$$\mathrm{Syz}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = \langle (-y + z + 1, x - 1, -yz) \rangle.$$ □

Using Theorem 4.5, we can now find generators for $\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$ for any $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$.

Theorem 4.8. *Let $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \in R^m$ and let $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ be a Groebner basis for $\langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \rangle$. Also let the matrices S and T be as in Proposition 3.19. If $\langle \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r \rangle = \text{Syz}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s)$ and if we let $\mathbf{r}_i$ denote the ith column of the matrix $I_t - ST$ where I_t is the $t \times t$ identity matrix, then*

$$\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t) = \langle S\mathbf{s}_1, S\mathbf{s}_2, \ldots, S\mathbf{s}_r, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_t \rangle.$$

Proof. Let matrices F and G have columns $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t$ and $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s$ as in proposition 3.19. Now notice that for each $\mathbf{s}_i \in \text{Syz}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s)$ we have $FS\mathbf{s}_i = G\mathbf{s}_i = \mathbf{0}$. Therefore $\{S\mathbf{s}_1, S\mathbf{s}_2, \ldots, S\mathbf{s}_r\} \subset \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$. Also notice that

$$F(I_t - ST) = F - FST = F - GT = F - F = 0.$$

Therefore $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_t\} \subset \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$. So we have

$$\langle S\mathbf{s}_1, S\mathbf{s}_2, \ldots, S\mathbf{s}_r, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_t \rangle \subset \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t).$$

To show the converse we first notice that for all $\mathbf{h} \in R^t$ such that $\mathbf{h} \in \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$ we know $GT\mathbf{h} = F\mathbf{h} = \mathbf{0}$ which implies $T\mathbf{h} \in \text{Syz}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s)$. Therefore we have $T\mathbf{h} = \sum_{i=1}^{r} h_i \mathbf{s}_i$ for some $h_1, h_2, \ldots, h_r \in R$, and left multiplication by S gives us

$$ST\mathbf{h} = \sum_{i=1}^{r} h_i S\mathbf{s}_i.$$

Now we have

$$\mathbf{h} = \mathbf{h} - ST\mathbf{h} + ST\mathbf{h} = (I_t - ST)\mathbf{h} + \sum_{i=1}^{r} h_i S\mathbf{s}_i.$$

Thus we have shown

$$\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t) \subset \langle S\mathbf{s}_1, S\mathbf{s}_2, \ldots, S\mathbf{s}_r, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_t \rangle.$$

□

Example 4.9. Let $R = \mathbb{Q}[x, y, z]$ and $\langle f_1, f_2 \rangle \subset R$ be such that

$$f_1 = x^3 - y, \quad \text{and} \quad f_2 = xyz^2.$$

Using grevlex order on R with $x > y > z$ one can show $\{x^3 - y, y^2z^2, xyz^2\}$ is a Groebner basis for $\langle f_1, f_2 \rangle$. So let

$$g_1 = x^3 - y, \quad g_2 = y^2z^2, \quad \text{and} \quad g_3 = xyz^2.$$

The desired matrices S and T from Proposition 3.19 can be verified to be

$$S = \begin{pmatrix} 1 & -yz^2 & 0 \\ 0 & x^2 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In Example 4.6 we found $\text{Syz}(g_1, g_2, g_3) = \langle (y^2z^2, -x^3 + y, 0), (yz^2, 1, -x^2), (0, x, -y) \rangle$. So let

$$\mathbf{s}_1 = (y^2z^2, -x^3 + y, 0), \quad \mathbf{s}_2 = (yz^2, 1, -x^2), \quad \text{and} \quad \mathbf{s}_3 = (0, x, -y).$$

Then we have

$$Ss_1 = (x^3yz^2, -x^5 + x^2y), \quad Ss_2 = (0,0), \quad \text{and} \quad Ss_3 = (-xyz^2, x^3 - y).$$

Also we compute $I_2 - ST = 0$. Thus $\text{Syz}(f_1, f_2) = \langle (x^3yz^2, -x^5 + x^2y), (-xyz^2, x^3 - y) \rangle$. Since $(x^3yz^2, -x^5 + x^2y) = -x^2(-xyz^2, x^3 - y)$ we have

$$\text{Syz}(f_1, f_2) = \langle (-xyz^2, x^3 - y) \rangle. \qquad \square$$

Example 4.10. Let $R = \mathbb{Q}[x, y, z]$ and $\langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle \subset R^3$ be such that

$$\mathbf{f}_1 = (x^2 - y + z, 0, -x^2 + x), \quad \mathbf{f}_2 = (-xyz - yz, y - z - 1, xyz),$$

$$\text{and} \quad \mathbf{f}_3 = (-x^2z + xyz + yz - z^2, x - y + z, x^2z - xyz - xz).$$

Using lex order in R with $x > y > z$ and using TOP order on R^3 with $e_3 > e_2 > e_1$ one can show $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is a Groebner basis for $\langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle$ where

$$\mathbf{g}_1 = (-yz, x - 1, 0), \quad \mathbf{g}_2 = \mathbf{f}_2, \quad \text{and} \quad \mathbf{g}_3 = -\mathbf{f}_1.$$

The desired matrices S and T from Proposition 3.19 can be verified to be

$$S = \begin{pmatrix} z & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & z \end{pmatrix}.$$

In Example 4.7 we found

$$\text{Syz}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = \langle (-y + z + 1, x - 1, -yz) \rangle.$$

So let

$$\mathbf{s} = (-y + z + 1, x - 1, -yz).$$

Then we have

$$Ss = (z^2 + z, x - y + z, -y + z + 1).$$

Also we compute $I_3 - ST = 0$. Therefore we have

$$\text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \langle (z^2 + z, x - y + z, -y + z + 1) \rangle. \qquad \square$$

One application of syzygies which we will need involves finding a presentation of M/N where $N \subset M$ are submodules of R^m. That is given generators for N and M, we would like to find a submodule $K \subset R^t$ such that $M/N \cong R^t/K$.

Theorem 4.11. *Let $M = \langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \rangle$ and $N = \langle \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s \rangle$ be submodules of R^m such that $N \subset M$. Also let $L = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t, \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ and suppose $\text{Syz}(L) = \langle \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_r \rangle \subset R^{t+s}$. If we define $\mathbf{k}_i$ for all $1 \leq i \leq r$ to be the first t coordinates of $\mathbf{h}_i$ then we have*

$$M/N \cong R^t/\langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_r \rangle.$$

Proof. We define the homomorphism

$$\begin{array}{ccc} R^t & \xrightarrow{\varphi} & M/N \\ (a_1, a_2, \ldots, a_t) & \longmapsto & a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \cdots + a_t\mathbf{f}_t + N. \end{array}$$

We know $R^t/\ker(\varphi) \cong M/N$, so we need to show the elements of $\ker(\varphi)$ are the first t coordinates of all the elements of $\mathrm{Syz}(L)$. Notice $(b_1, b_2, \ldots, b_t) \in \ker(\varphi)$ if and only if $b_1\mathbf{f}_1 + b_2\mathbf{f}_2 + \cdots + b_t\mathbf{f}_t \in N$. Also $b_1\mathbf{f}_1 + b_2\mathbf{f}_2 + \cdots + b_t\mathbf{f}_t \in N$ if and only if $b_1\mathbf{f}_1 + b_2\mathbf{f}_2 + \cdots + b_t\mathbf{f}_t = c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + \cdots + c_s\mathbf{g}_s$ for some $c_1, c_2, \ldots, c_s \in R$. This last statement is true if and only if $(b_1, b_2, \ldots, b_t, -c_1, -c_2, \ldots, -c_s) \in \mathrm{Syz}(L)$ for some $c_1, c_2, \ldots, c_s \in R$. □

5. Free Resolutions

In this section we will see how syzygies can be used to compute what is known as a free resolution of a module. First we will need the following definitions.

Definition 5.1. In the sequence of R-modules and homomorphisms

$$\cdots \xrightarrow{\phi_{i-2}} M_{i-1} \xrightarrow{\phi_{i-1}} M_i \xrightarrow{\phi_i} M_{i+1} \xrightarrow{\phi_{i+1}} \cdots$$

we say the sequence is **exact at** M_i if $\mathrm{im}(\phi_{i-1}) = \ker(\phi_i)$. We say the sequence is **exact** if it is exact at M_i for all i.

Definition 5.2. Let M be an R-module. An exact sequence

$$\cdots \xrightarrow{\phi_3} R^{t_2} \xrightarrow{\phi_2} R^{t_1} \xrightarrow{\phi_1} R^{t_0} \xrightarrow{\phi_0} M \longrightarrow 0 \quad .$$

is called a **free resolution** of M. If there exists an i such that $R^{t_i} \neq 0$ but $R^{t_{i+1}} = R^{t_{i+2}} = \cdots = 0$, then we say the resolution is finite and of length i.

Proofs of the following theorem can be found in [1, Theorem 3.10.4], and in [4, Theorem 6.2.1].

Theorem 5.3 (Hilbert Syzygy Theorem). *Let $R = k[x_1, x_2, \ldots, x_n]$. Every finitely generated R-module has a finite free resolution of length less than or equal to n.*

Using syzygies we can find a free resolution of a finitely generated submodule of R^m in the following way. Let $M = \langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{t_0} \rangle \subset R^m$ and let $L_0 = \mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{t_0})$. Notice that the homomorphism

$$\begin{array}{ccc} R^{t_0} & \xrightarrow{\psi_0} & M \\ (a_1, a_2, \ldots, a_{t_0}) & \longmapsto & a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \cdots + a_{t_0}\mathbf{f}_{t_0} \end{array}$$

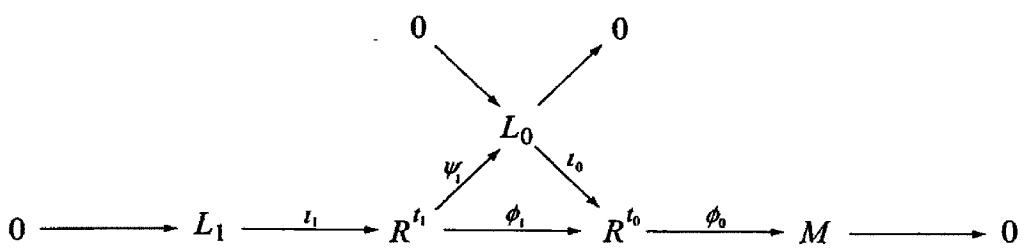
has kernel L_0. So if we let $\iota_0 : L_0 \longrightarrow R^{t_0}$ be the inclusion map, we have the following short exact sequence:

$$0 \longrightarrow L_0 \xrightarrow{\iota_0} R^{t_0} \xrightarrow{\psi_0} M \longrightarrow 0.$$

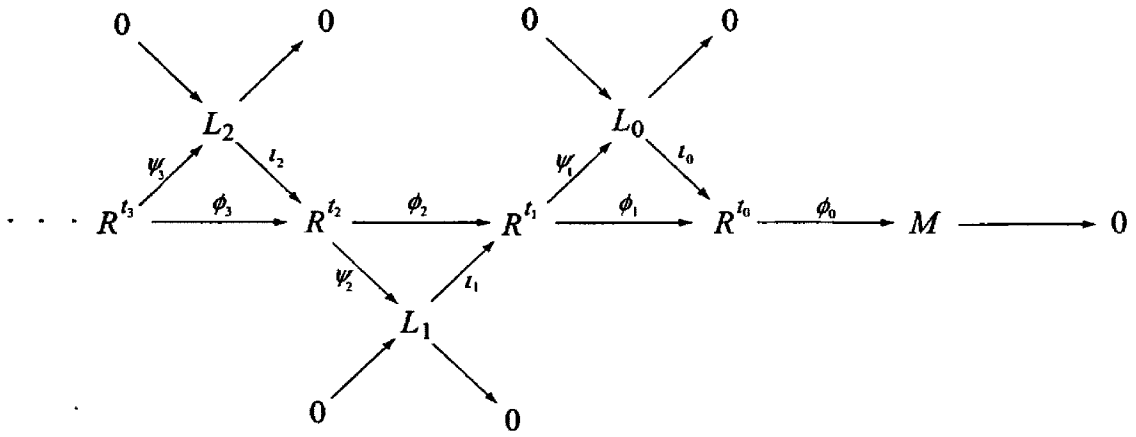
Now we repeat this process for L_0 and obtain another short exact sequence

$$0 \longrightarrow L_1 \xrightarrow{\iota_1} R^{t_1} \xrightarrow{\psi_1} L_0 \longrightarrow 0.$$

Putting these sequences together we have



where $\phi_0 = \psi_0$ and $\phi_1 = \iota_0 \circ \psi_1$. If we continue this process we will obtain the following sequence of R-modules and homomorphisms:



So we have the following free resolution of M:

$$\cdots \xrightarrow{\phi_4} R^{t_3} \xrightarrow{\phi_3} R^{t_2} \xrightarrow{\phi_2} R^{t_1} \xrightarrow{\phi_1} R^{t_0} \xrightarrow{\phi_0} M \longrightarrow 0.$$

Example 5.4. Let $R = \mathbb{Q}[x, y, z]$ and let $N = \langle x^3 - y, xyz^2 \rangle \subset R$. Now define the homomorphism

$$\begin{array}{ccc} R^2 & \xrightarrow{\psi_0} & N \\ (a, b) & \longmapsto & (x^3 - y)a + (xyz^2)b. \end{array}$$

Then we know $\ker(\psi_0) = \mathrm{Syz}(x^3 - y, xyz^2)$. In Example 4.9 we found $\mathrm{Syz}(x^3 - y, xyz^2) = \langle (-xyz^2, x^3 - y) \rangle$, so let $L_0 = \langle (-xyz^2, x^3 - y) \rangle$ and let $\iota_0 : L_0 \longrightarrow R^2$ be the inclusion map. We now have the short exact sequence

$$0 \longrightarrow L_0 \xrightarrow{\iota_0} R^2 \xrightarrow{\psi_0} N \longrightarrow 0.$$

Now we repeat this process for L_0. But since L_0 is generated by one element, we have $\mathrm{Syz}((-xyz^2, x^3 - y)) = 0$. So if we define

$$\begin{array}{ccc} R & \xrightarrow{\psi_1} & L_0 \\ a & \longmapsto & (-xyz^2, x^3 - y)a, \end{array}$$

then we have the short exact sequence

$$0 \longrightarrow R \xrightarrow{\psi_1} L_0 \longrightarrow 0.$$

Putting these two exact sequences together and letting $\phi_0 = \psi_0$ and $\phi_1 = \iota_0 \circ \psi_1$ we get the following free resolution of N.

$$0 \longrightarrow R \xrightarrow{\phi_1} R^2 \xrightarrow{\phi_0} N \longrightarrow 0.$$

Notice that this resolution is finite of length 1. Also notice that since N is not a free module, there does not exist a free resolution of length 0. Thus we have found a free resolution of N of minimal length. □

Example 5.5. Let $R = \mathbb{Q}[x, y, z]$ and let $M = \langle x^2 - x, xyz, x^2 - y + z \rangle \subset R$. Now define the homomorphism

$$\begin{array}{ccc} R^3 & \xrightarrow{\psi_0} & M \\ (a, b, c) & \longmapsto & (x^2 - x)a + (xyz)b + (x^2 - y + z)c. \end{array}$$

Then we know $\ker(\psi_0) = \text{Syz}(x^2 - x, xyz, x^2 - y + z)$. Using Theorems 4.5 and 4.8 one can compute $\text{Syz}(x^2 - x, xyz, x^2 - y + z) = \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle \subset R^3$ where

$$\mathbf{f}_1 = (x^2 - y + z, 0, -x^2 + x), \quad \mathbf{f}_2 = (-xyz - yz, y - z - 1, xyz),$$

$$\text{and} \quad \mathbf{f}_3 = (-x^2z + xyz + yz - z^2, x - y + z, x^2z - xyz - xz).$$

Let $L_0 = \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle$ and let $\iota_0 : L_0 \longrightarrow R^3$ be the inclusion map. We now have the short exact sequence

$$0 \longrightarrow L_0 \xrightarrow{\iota_0} R^3 \xrightarrow{\psi_0} M \longrightarrow 0.$$

Now we repeat this process for L_0. Define

$$\begin{array}{ccc} R^3 & \xrightarrow{\psi_1} & L_0 \\ (a,b,c) & \longmapsto & a\mathbf{f}_1 + b\mathbf{f}_2 + c\mathbf{f}_3. \end{array}$$

In Example 4.10 we found $\langle (z^2 + z, x - y + z, -y + z + 1) \rangle = \text{Syz}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \ker(\psi_1)$. So if we let $L_1 = \langle (z^2 + z, x - y + z, -y + z + 1) \rangle$ and let $\iota_1 : L_1 \longrightarrow R^3$ be the inclusion map, we have the short exact sequence

$$0 \longrightarrow L_1 \xrightarrow{\iota_1} R^3 \xrightarrow{\psi_1} L_0 \longrightarrow 0.$$

Now we repeat this process for L_1. But since L_1 is generated by one element, we have $\text{Syz}((z^2 + z, x - y + z, -y + z + 1)) = 0$. So if we define

$$\begin{array}{ccc} R & \xrightarrow{\psi_2} & L_1 \\ a & \longmapsto & (z^2 + z, x - y + z, -y + z + 1)a, \end{array}$$

then we have the short exact sequence

$$0 \longrightarrow R \xrightarrow{\psi_2} L_1 \longrightarrow 0.$$

Now we let $\phi_0 = \psi_0, \phi_1 = \psi_1 \circ \iota_0$, and $\phi_2 = \psi_2 \circ \iota_1$. Then we have the following free resolution of M.

$$0 \longrightarrow R \xrightarrow{\phi_2} R^3 \xrightarrow{\phi_1} R^3 \xrightarrow{\phi_0} M \longrightarrow 0.$$

Notice that this resolution is finite of length 2. □

6. Computing Hom

In this section we will introduce $\text{Hom}_R(M, N)$, and compute a presentation of $\text{Hom}_R(M, N)$ for given R-modules M and N. We will need some new notation for this section which we will introduce first. For an $s \times t$ matrix M with entries in R we let $\langle M \rangle$ denote the submodule of R^t which is generated by the columns of M. Given matrices $M_1, M_2, \ldots, M_r$ where M_i is an $s_i \times t_i$ matrix, we let $M_1 \oplus M_2 \oplus \cdots \oplus M_r$ denote the $(s_1 + s_2 + \cdots + s_r) \times (t_1 + t_2 + \cdots + t_r)$ matrix with matrices $M_1, M_2, \ldots, M_r$ down the diagonal and zeros everywhere else.

Example 6.1. If

$$M = \begin{pmatrix} x & 0 \\ xy & y \end{pmatrix}, \quad N = \begin{pmatrix} x + y \\ x^2y \\ 1 \end{pmatrix}, \text{ and } \quad L = \begin{pmatrix} y + xy & 2x & 1 \\ y^3 & 0 & x \end{pmatrix},$$

then

$$M \oplus N \oplus L = \left(\begin{array}{cc:c:ccc} x & 0 & 0 & 0 & 0 & 0 \\ xy & y & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & x+y & 0 & 0 & 0 \\ 0 & 0 & x^2y & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & y+xy & 2x & 1 \\ 0 & 0 & 0 & y^3 & 0 & x \end{array}\right).$$

□

For an $s \times t$ matrix M and the $r \times r$ identity matrix I_r, we define the tensor product $M \otimes I_r$ to be the $rs \times rt$ matrix obtained by replacing each entry m_{ij} of M by the matrix $m_{ij}I_r$.

Example 6.2. If

$$M = \begin{pmatrix} x & y & z \\ xy & yz & zx \end{pmatrix}, \text{ then } M \otimes I_2 = \left(\begin{array}{cc:cc:cc} x & 0 & y & 0 & z & 0 \\ 0 & x & 0 & y & 0 & z \\ \hdashline xy & 0 & yz & 0 & zx & 0 \\ 0 & xy & 0 & yz & 0 & zx \end{array}\right).$$

□

Definition 6.3. For R-modules M and N, we define

$$\mathrm{Hom}_R(M,N) = \{\phi : M \longrightarrow N \mid \phi \text{ is an } R\text{-module homomorphism}\}.$$

We give $\mathrm{Hom}_R(M,N)$ an R-module structure by defining for all $\phi, \psi \in \mathrm{Hom}_R(M,N)$ and for all $r \in R$, $(\phi+\psi)(\mathbf{m}) = \phi(\mathbf{m}) + \psi(\mathbf{m})$ for all $\mathbf{m} \in M$ and $(r\phi)(\mathbf{m}) = r(\phi(\mathbf{m})) = \phi(r\mathbf{m})$ for all $\mathbf{m} \in M$.

Our goal is to compute a presentation of $\mathrm{Hom}_R(M,N)$ for any R-modules M and N. In other words we want to find a submodule $K \subset R^l$ such that $\mathrm{Hom}_R(M,N)$ is isomorphic as an R-module to R^l/K. To do so we will first need a few properties of the functor Hom. Given R-modules L, M and N and a homomorphism $\phi : L \longrightarrow M$, we define the two homomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_R(N,L) & \xrightarrow{\phi\circ} & \mathrm{Hom}_R(N,M) \\ \psi & \longmapsto & \phi \circ \psi \end{array},$$

and

$$\begin{array}{ccc} \mathrm{Hom}_R(M,N) & \xrightarrow{\circ\phi} & \mathrm{Hom}_R(L,N) \\ \psi & \longmapsto & \psi \circ \phi \end{array}.$$

Lemma 6.4. *Given an exact sequence of R-modules and homomorphisms*

$$M_3 \xrightarrow{\psi} M_2 \xrightarrow{\phi} M_1 \longrightarrow 0$$

and an R-module P, the sequence

$$\mathrm{Hom}_R(M_3,P) \xleftarrow{\circ\psi} \mathrm{Hom}_R(M_2,P) \xleftarrow{\circ\phi} \mathrm{Hom}_R(M_1,P) \longleftarrow 0$$

is exact. Furthermore if P is projective, the sequence

$$\mathrm{Hom}_R(P,M_3) \xrightarrow{\psi\circ} \mathrm{Hom}_R(P,M_2) \xrightarrow{\phi\circ} \mathrm{Hom}_R(P,M_1) \longrightarrow 0$$

is exact. This yields

$$\mathrm{Hom}_R(P, M_1) \cong \mathrm{Hom}_R(P, M_2)/\ker(\phi_\circ) = \mathrm{Hom}_R(P, M_2)/\mathrm{im}(\psi_\circ),$$

and

$$\mathrm{Hom}_R(M_1, P) \cong \ker({}_\circ\psi).$$

The proof of Lemma 6.4 is left to the reader, it follows from the definition of Hom. Now we return to finding a presentation of $\mathrm{Hom}_R(M, N)$ for given R-modules M and N. For the case when M and N are the free modules R^t and R^s respectively we have the following result.

Proposition 6.5.[1, pages 183, 184] $\mathrm{Hom}_R(R^t, R^s) \cong R^{st}$.

More generally we wish to compute $\mathrm{Hom}_R(M, N)$ when M and N are any R-modules such that $M \cong R^t/L$ and $N \cong R^s/H$ for some submodules $L \subset R^t$ and $H \subset R^s$. To do so we first find free resolutions of M and N respectively. Notice that if $M = \langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t \rangle$, then $L = \mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_t)$, and likewise for N. Suppose we obtain the following free resolutions of M and N:

$$\cdots \longrightarrow R^{t_1} \xrightarrow{\Gamma} R^t \xrightarrow{\pi} M \longrightarrow 0,$$

$$\cdots \longrightarrow R^{s_1} \xrightarrow{\Delta} R^s \xrightarrow{\pi'} N \longrightarrow 0.$$

To compute $\mathrm{Hom}_R(M, N)$ we will look at the exact sequences

$$R^{t_1} \xrightarrow{\Gamma} R^t \xrightarrow{\pi} M \longrightarrow 0, \tag{3}$$

$$R^{s_1} \xrightarrow{\Delta} R^s \xrightarrow{\pi'} N \longrightarrow 0. \tag{4}$$

Notice, as in Section 4, that the map Γ is given by a $t \times t_1$ matrix whose columns are the generators of L. Similarly Δ is given by a $s \times s_1$ matrix whose columns are the generators of H. We will also abuse notation by letting Γ and Δ represent these matrices. Now Lemma 6.4 along with sequence (3) give us the exact sequence

$$\mathrm{Hom}_R(R^{t_1}, N) \xleftarrow{{}^\circ\Gamma} \mathrm{Hom}_R(R^t, N) \xleftarrow{{}^\circ\pi} \mathrm{Hom}_R(M, N) \longleftarrow 0$$

and the isomorphism

$$\mathrm{Hom}_R(M, N) \cong \ker({}_\circ\Gamma).$$

To find a presentation of $\mathrm{Hom}_R(M, N)$ it suffices to find a presentation of $\ker({}_\circ\Gamma)$. To this end we first find presentations of $\mathrm{Hom}_R(R^t, N)$ and $\mathrm{Hom}_R(R^{t_1}, N)$. Using Lemma 6.4 along with sequence (4) we obtain the exact sequences

$$\mathrm{Hom}_R(R^t, R^{s_1}) \xrightarrow{\delta} \mathrm{Hom}_R(R^t, R^s) \xrightarrow{\pi'_\circ} \mathrm{Hom}_R(R^t, N) \longrightarrow 0$$

where $\delta = \Delta_\circ$, and

$$\mathrm{Hom}_R(R^{t_1}, R^{s_1}) \xrightarrow{\delta'} \mathrm{Hom}_R(R^{t_1}, R^s) \xrightarrow{\pi'_\circ} \mathrm{Hom}_R(R^{t_1}, N) \longrightarrow 0$$

where $\delta' = \Delta_\circ$, along with the isomorphisms

$$\mathrm{Hom}_R(R^t, N) \cong \mathrm{Hom}_R(R^t, R^s)/\mathrm{im}(\delta)$$

and

$$\mathrm{Hom}_R(R^{t_1}, N) \cong \mathrm{Hom}_R(R^{t_1}, R^s)/\mathrm{im}(\delta').$$

Now Proposition 6.5 tells us $\mathrm{Hom}_R(R^t, R^s) \cong R^{st}$ and $\mathrm{Hom}_R(R^{t_1}, R^s) \cong R^{st_1}$, so to find presentations of $\mathrm{Hom}_R(R^t, N)$ and $\mathrm{Hom}_R(R^{t_1}, N)$ we need only describe $\mathrm{im}(\delta)$ and $\mathrm{im}(\delta')$. Now by Proposition 6.5, $\delta : \mathrm{Hom}_R(R^t, R^{s_1}) \longrightarrow \mathrm{Hom}_R(R^t, R^s)$ corresponds to a map $\delta^* : R^{s_1 t} \longrightarrow R^{st}$. And δ^* is given by a $st \times s_1 t$ matrix S, in other words $\delta^*(\mathbf{a}) = S\mathbf{a}$ for all $\mathbf{a} \in R^{s_1 t}$. The following lemma gives us the matrix S and thus a presentation of $\mathrm{Hom}_R(R^t, N)$.

Lemma 6.6.[1, Lemma 3.9.3] *The matrix S is given by $\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{ copies}}$. And therefore*

$$\mathrm{Hom}_R(R^t, N) \cong R^{st}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{ copies}} \rangle.$$

Likewise $\mathrm{Hom}_R(R^{t_1}, N) \cong R^{st_1}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}} \rangle$. So now we know the map

$$_\circ\Gamma : \mathrm{Hom}_R(R^t, N) \longrightarrow \mathrm{Hom}_R(R^{t_1}, N)$$

corresponds to a map

$$_\circ\Gamma^* : R^{st}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{ copies}} \rangle \longrightarrow R^{st_1}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}} \rangle.$$

Lemma 6.7.[1, Lemma 3.9.4] *Let T be the transpose of the matrix $\Gamma \otimes I_s$. Then for any $\mathbf{b} \in R^{st}$,*

$$_\circ\Gamma^*(\mathbf{b} + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{ copies}} \rangle) = T\mathbf{b} + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}} \rangle.$$

Thus we know

$$\mathrm{Hom}_R(M, N) \cong \ker(_\circ\Gamma) \cong \ker(_\circ\Gamma^*).$$

Using Lemma 6.7 we can find $\ker(_\circ\Gamma^*)$ as follows. First we compute the kernel of the map

$$\begin{aligned} R^{st} &\xrightarrow{\chi} R^{st_1}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}} \rangle \\ \mathbf{b} &\longmapsto T\mathbf{b} + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}} \rangle. \end{aligned}$$

By Theorem 4.11, $\ker(\chi) = \langle U \rangle$, where U is the matrix whose columns are the first st coordinates of the generators of the syzygy module of the columns of T and $\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}}$. Thus we have

$$\mathrm{Hom}_R(M, N) \cong \langle U \rangle/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{ copies}} \rangle.$$

We summarize our result in the following theorem.

Theorem 6.8. *Suppose M and N are R-modules with $M \cong R^t/L$ and $N \cong R^s/H$ where $L \subset R^t$ and $H \subset R^s$ are submodules, and $L \cong R^{t_1}/L_1$ for some submodule $L_1 \subset R^{t_1}$.*

- *Let Γ and Δ be the matrices whose columns are generators of L and H respectively,*
- *let T be the transpose of the matrix $\Gamma \otimes I_s$, and*

- *let U be the matrix whose columns are the first st coordinates of generators of the syzygy module of the columns of T and $\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_1 \text{ copies}}$.*

Then $\mathrm{Hom}_R(M,N) \cong \langle U \rangle / (\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t \text{copies}})$, *and we can use Theorem 4.11 to compute a presentation of* $\mathrm{Hom}_R(M,N)$.

Example 6.9. Let $R = \mathbb{Q}[x,y,z]$ and let

$$M = \langle x^2 - x, xyz, x^2 - y + z \rangle \subset R \quad \text{and} \quad N = \langle x^3 - y, xyz^2 \rangle \subset R.$$

In Example 5.5 we stated $\mathrm{Syz}(x^2 - x, xyz, x^2 - y + z) = \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle$ where

$$\mathbf{f}_1 = (x^2 - y + z, 0, -x^2 + x), \quad \mathbf{f}_2 = (-xyz - yz, y - z - 1, xyz),$$

$$\text{and} \quad \mathbf{f}_3 = (-x^2z + xyz + yz - z^2, x - y + z, x^2z - xyz - xz).$$

In Example 4.9 we found $\mathrm{Syz}(x^3 - y, xyz^2) = \langle (-xyz^2, x^3 - y) \rangle$. So we let

$$\Gamma = \begin{pmatrix} x^2 - y + z & -xyz - yz & -x^2z + xyz + yz - z^2 \\ 0 & y - z - 1 & x - y + z \\ -x^2 + x & xyz & x^2z - xyz - xz \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} -xyz^2 \\ x^3 - y \end{pmatrix}.$$

Then the matrix T, which is the transpose of the matrix $\Gamma \otimes I_2$, is

$$\left(\begin{array}{cc:cc:cc} x^2 - y + z & 0 & 0 & 0 & -x^2 + x & 0 \\ 0 & x^2 - y + z & 0 & 0 & 0 & -x^2 + x \\ \hdashline -xyz - yz & 0 & y - z - 1 & 0 & xyz & 0 \\ 0 & -xyz - yz & 0 & y - z - 1 & 0 & xyz \\ \hdashline -x^2z + xyz + yz - z^2 & 0 & x - y + z & 0 & x^2z - xyz - xz & 0 \\ 0 & -x^2z + xyz + yz - z^2 & 0 & x - y + z & 0 & x^2z - xyz - xz \end{array}\right).$$

Now we compute the syzygy module of the columns of T and the columns of

$$\Delta \oplus \Delta \oplus \Delta = \left(\begin{array}{c:c:c} -xyz^2 & 0 & 0 \\ x^3 - y & 0 & 0 \\ \hdashline 0 & -xyz^2 & 0 \\ 0 & x^3 - y & 0 \\ \hdashline 0 & 0 & -xyz^2 \\ 0 & 0 & x^3 - y \end{array}\right)$$

The first 6 entries of the generators of this module are

$$\mathbf{u}_1 = \begin{pmatrix} -x^2 + x \\ 0 \\ -xyz \\ 0 \\ -x^2 + y - z \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ -x^2 + x \\ 0 \\ -xyz \\ 0 \\ -x^2 + y - z \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x^2y + x^2 - xy - x \\ -3x^2 + 3x \\ xy^2z - xyz^2 + xyz \\ x^3 - 3xyz - y \\ x^2y + x^2 - y^2 + yz - y + z \\ -3x^2 + 3y - 3z \end{pmatrix},$$

$$\mathbf{u}_4 = \begin{pmatrix} xy^2z^2 - xyz^3 \\ x^2y^2 - 2x^2yz + x^2z^2 - x^2y - xy^2 + x^2z + 2xyz - xz^2 + xy + y^2 - yz - y \\ 0 \\ x^2y^2z + xy^3z - x^2yz^2 - 2xy^2z^2 + xyz^3 + y^2z \\ xy^2z^2 - xyz^3 + y^2z^2 - yz^3 \\ x^2y^2 - 2x^2yz + x^2z^2 - x^2y - xy^2 - y^3 + x^2z + 2xyz + 3y^2z - xz^2 - 3yz^2 + z^3 + xy + y^2 - yz \end{pmatrix},$$

$$\mathbf{u}_5 = \begin{pmatrix} x^2yz^2 - xy^2z^2 + xyz^3 \\ x^3y - x^2y^2 - x^3z + 2x^2yz - x^2z^2 - x^3 + xy^2 - 2xyz + xz^2 + x^2 - y^2 + yz - x + y \\ 0 \\ x^3yz - xy^3z + 2xy^2z^2 - xyz^3 + xyz - y^2z \\ x^2yz^2 - xy^2z^2 + xyz^3 + xyz^2 - y^2z^2 + yz^3 \\ x^3y - x^2y^2 - x^3z + 2x^2yz - x^2z^2 - x^3 + y^3 - 3y^2z + 3yz^2 - z^3 + x^2 - y^2 + yz + z \end{pmatrix},$$

$$\text{and} \quad \mathbf{u}_6 = \begin{pmatrix} -xyz^2 \\ x^3 - y \\ -xyz^3 \\ x^3z - yz \\ -xyz^2 \\ x^3 - y \end{pmatrix}.$$

So we have $\mathrm{Hom}_R(M,N) \cong \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6 \rangle / \langle \Delta \oplus \Delta \oplus \Delta \rangle$. Using Theorem 4.11 we have

$$\mathrm{Hom}_R(M,N) \cong R^6 / \langle \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6 \rangle$$

where

$$\mathbf{k}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} y+1 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} 0 \\ z \\ 0 \\ x-y+z \\ -y+z \\ 0 \end{pmatrix}, \quad \mathbf{k}_4 = \begin{pmatrix} z^2 \\ y^2 - 2yz - 2z^2 - 1 \\ z^2 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{k}_5 = \begin{pmatrix} z^2 \\ x^2 + xy + y^2 - xz - 2yz + z^2 \\ z^2 \\ 1 \\ 0 \\ x+y-z \end{pmatrix}, \quad \mathbf{k}_6 = \begin{pmatrix} x^2z^2 - xz^3 - xz^2 + z^2 \\ 3xz^2 \\ x^2z^2 + xyz^2 - xz_x^3z^2 + z^2 \\ -y^2 + 2yz + 2z^2 + 1 \\ -x^2 - xy - y^2 + xz + 2yz + 2z^2 \\ 0 \end{pmatrix}.$$

For more detail on the computations in this example see Appendix A. □

7. Computing Ext

In this section we will be interested in computing $\mathrm{Ext}^i_R(M,N)$ for R-modules M and N. To do this we start with a free resolution of an R-module M

$$\cdots \longrightarrow R^{t_{i+1}} \xrightarrow{\Gamma_{i+1}} R^{t_i} \xrightarrow{\Gamma_i} R^{t_{i-1}} \longrightarrow \cdots \longrightarrow R^{t_2} \xrightarrow{\Gamma_2} R^{t_1} \xrightarrow{\Gamma_1} R^{t_0} \xrightarrow{\Gamma_0} M \longrightarrow 0. \qquad (*)$$

For an R-module N we now obtain the new sequence

$$\cdots \longleftarrow \mathrm{Hom}_R(R^{t_2},N) \xleftarrow{{}_\circ\Gamma_2} \mathrm{Hom}_R(R^{t_1},N) \xleftarrow{{}_\circ\Gamma_1} \mathrm{Hom}_R(R^{t_0},N) \longleftarrow 0$$

which at the ith position (for $i>0$) looks like

$$\cdots \longleftarrow \mathrm{Hom}_R(R^{t_{i+1}},N) \xleftarrow{{}_\circ\Gamma_{i+1}} \mathrm{Hom}_R(R^{t_i},N) \xleftarrow{{}_\circ\Gamma_i} \mathrm{Hom}_R(R^{t_{i-1}},N) \longleftarrow \cdots$$

Definition 7.1. With the sequence above, we define

$$\mathrm{Ext}^i_R(M,N) \cong \ker({}_\circ\Gamma_{i+1})/\mathrm{im}({}_\circ\Gamma_i).$$

We wish to compute a presentation of $\mathrm{Ext}^i_R(M,N)$, and to do this we find a free resolution of N

$$\cdots \longrightarrow R^{s_1} \xrightarrow{\Delta} R^s \xrightarrow{\pi'} N \longrightarrow 0.$$

Now Lemma 6.6 tells us

$$\mathrm{Hom}_R(R^{t_i},N) \cong R^{st_i}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle,$$

and so ${}_\circ\Gamma_i : \mathrm{Hom}_R(R^{t_{i-1}},N) \longrightarrow \mathrm{Hom}_R(R^{t_i},N)$ corresponds to a map

$${}_\circ\Gamma_i^* : R^{st_{i-1}}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_{i-1} \text{ copies}} \rangle \longrightarrow R^{st_i}/\langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle.$$

Lemma 6.7 tells us for any $\mathbf{b} \in R^{st_{i-1}}$,

$$(5) \qquad {}_\circ\Gamma_i^*(\mathbf{b} + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_{i-1} \text{ copies}} \rangle) = T_i\mathbf{b} + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle,$$

where T_i is the transpose of the matrix $\Gamma_i \otimes I_s$. As in the Section 6, we let U_{i+1} be the matrix whose columns are the first st_i coordinates of the generators of the syzygy module of the columns of T_{i+1} and $\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_{i+1} \text{ copies}}$. Then we have

$$\ker({}_\circ\Gamma_{i+1}) \cong \langle U_{i+1} \rangle / \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle.$$

Equation (5) gives us

$$\mathrm{im}({}_\circ\Gamma_i) \cong (\langle T_i \rangle + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle) / \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle.$$

Thus

$$\mathrm{Ext}^i_R(M,N) \cong \langle U_{i+1} \rangle / (\langle T_i \rangle + \langle \underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{ copies}} \rangle). \qquad (**)$$

We summarize our result in the following theorem.

Theorem 7.2. *Suppose M and N are R-modules with M as in $(*)$, and $N \cong R^s/H$ where $H \subset R^s$ is a submodule.*

- *Let Δ be the matrix whose columns are generators of H, and let Γ_i be the matrix associated with the map Γ_i in $(*)$,*
- *let T_i be the transpose of the matrix $\Gamma_i \otimes I_s$, and*
- *let U_{i+1} be the matrix whose columns are the first st_i coordinates of generators of the syzygy module of the columns of T_{i+1} and $\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_{i+1}\ copies}$.*

Then $\mathrm{Ext}_R(M,N) \cong \langle U_{i+1}\rangle/(\langle T_i\rangle + \langle\underbrace{\Delta \oplus \Delta \oplus \cdots \oplus \Delta}_{t_i \text{copies}}\rangle)$, *and we can use Theorem 4.11 to compute a presentation of* $\mathrm{Ext}_R(M,N)$.

Example 7.3. We now compute $\mathrm{Ext}^1_R(M,N)$ for M and N as in Example 6.9. Let $R, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \Gamma_1 := \Gamma, \Delta$, and $T_1 := T$ be as in Example 6.9. Then $\langle T_1\rangle + \langle \Delta \oplus \Delta \oplus \Delta\rangle = \langle \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_9\rangle$, where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_6$ are the columns of T_1, and $\mathbf{r}_7, \mathbf{r}_8, \mathbf{r}_9$ are the columns of $\Delta \oplus \Delta \oplus \Delta$. Now in Example 4.10 we found $\mathrm{Syz}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \langle (z^2+z, x-y+z, -y+z+1)\rangle$. So we let

$$\Gamma_2 = \begin{pmatrix} z^2+z \\ x-y+z \\ -y+z+1 \end{pmatrix}.$$

Then the matrix T_2, which is the transpose of the matrix $\Gamma_2 \otimes I_2$, is

$$\left(\begin{array}{cc:cc:cc} z^2+z & 0 & x-y+z & 0 & -y+z+1 & 0 \\ 0 & z^2+z & 0 & x-y+z & 0 & -y+z+1 \end{array}\right).$$

Now we compute the syzygy module of the columns of T_2 and the columns of Δ. The first 6 entries of the generators of this module are the columns of U_2. These columns are

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \\ y-z-1 \\ 0 \\ x-y+z \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y-z-1 \\ 0 \\ x-y+z \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} y-z-1 \\ 0 \\ 0 \\ 0 \\ z^2+z \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 0 \\ y-z-1 \\ 0 \\ 0 \\ 0 \\ z^2+z \end{pmatrix},$$

$$\mathbf{u}_5 = \begin{pmatrix} x-1 \\ 0 \\ -z^2-z \\ 0 \\ z^2+z \\ 0 \end{pmatrix}, \quad \mathbf{u}_6 = \begin{pmatrix} 0 \\ x-1 \\ 0 \\ -z^2-z \\ 0 \\ z^2+z \end{pmatrix}, \quad \mathbf{u}_7 = \begin{pmatrix} -z^2-z \\ -1 \\ -z^4-2z^3-z^2 \\ x^2z+x^2+xz+x+z+1 \\ xz^3+z^4+xz^2+2z^3+z^2 \\ -x^2z-x^2-xz-x \end{pmatrix},$$

$$\mathbf{u}_8 = \begin{pmatrix} -yz+z^2+z \\ 0 \\ -yz^3+z^4-yz^2+2z^3+z^2 \\ x^2y-x^2z-x^2+xy-xz-x+y-z-1 \\ xyz^2-xz^3+yz^3-z^4-xz^2+yz^2-2z^3-z^2 \\ -x^2y+x^2z+x^2-xy+xz+x+z \end{pmatrix}, \quad \mathbf{u}_9 = \begin{pmatrix} -xz+z \\ 0 \\ -xz^3-xz^2+z^3+z^2 \\ x^3-z-1 \\ x^2z^2+xz^3-z^3-z^2 \\ -x^3+x+z \end{pmatrix}.$$

By (**), $\mathrm{Ext}^1_R(M,N) \cong \langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_9 \rangle / \langle \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_9 \rangle$, and

$$\mathrm{Ext}^1_R(M,N) \cong R^9 / \langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{16} \rangle$$

using Theorem 4.11 where

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{k}_6 = \begin{pmatrix} 0 \\ 0 \\ z \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{k}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_8 = \begin{pmatrix} 0 \\ 0 \\ y-z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_9 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y-z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{10} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ y-z-1 \\ -z-1 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{11} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y-z \\ 0 \end{pmatrix},$$

$$\mathbf{k}_{12} = \begin{pmatrix} 0 \\ 0 \\ x-1 \\ 0 \\ -y+z+1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{13} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x-1 \\ 0 \\ -y+z+1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{14} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ x-1 \\ -z-1 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{15} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x \\ 0 \end{pmatrix}, \quad \mathbf{k}_{16} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ z \\ z^2+z \\ 0 \end{pmatrix}.$$

For more detail on the computations in this example see Appendix A. □

Appendix A. Examples in Macaulay 2

In examples 6.9 and 7.3 we made use of the computer algebra system *Macaulay 2*. This appendix contains those computer sessions. First we say a few words about this computer algebra system. *Macaulay 2* does not wrap input and output lines. Alternatively the input and output lines in *Macaulay 2* grow horizontally without bound. In this paper ellipses are used to indicate the cases where the lines exceed the width of the

paper. The default orderings in *Macaulay 2* are grevlex with $x_1 > x_2 > x_3 > \cdots$ on R, and TOP with $e_1 < e_2 < e_3 < \cdots$ on R^m. For more information on this program we refer the reader to [5].

***Macaulay 2* Session for Example 6.9.** In *Macaulay 2* we must first define the ring, R which we are working over.

```
i1 : R = QQ[x,y,z]

o1 = R

o1 : PolynomialRing
```

In our example we need to compute the syzygy module of the columns of the matrices T and $\Delta \oplus \Delta \oplus \Delta$. We define these matrices in *Macaulay 2* as follows:

```
i2 : T = matrix{{x^2-y+z,0,0,0,-x^2+x,0},{0,x^2-y+z,0,0,0,-x^2+x},{-x*y*z-y*z,0,y-z-1,0,x*y*z,0},{0,-x*y*z-y*z, . . .

o2 = | x2-y+z          0              0       0       -x2+x       0          |
     | 0               x2-y+z         0       0       0           -x2+x      |
     | -xyz-yz         0              y-z-1   0       xyz         0          |
     | 0               -xyz-yz        0       y-z-1   0           xyz        |
     | -x2z+xyz+yz-z2  0              x-y+z   0       x2z-xyz-xz  0          |
     | 0               -x2z+xyz+yz-z2 0       x-y+z   0           x2z-xyz-xz |

             6      6
o2 : Matrix R <--- R

i3 : DDD = matrix{{-x*y*z^2,0,0},{x^3-y,0,0},{0,-x*y*z^2,0},{0,x^3-y,0},{0,0,-x*y*z^2},{0,0,x^3-y}}

o3 = | -xyz2  0      0     |
     | x3-y   0      0     |
     | 0      -xyz2  0     |
     | 0      x3-y   0     |
     | 0      0      -xyz2 |
     | 0      0      x3-y  |

             6      3
o3 : Matrix R <--- R
```

Macaulay 2 will compute the syzygy module of columns of a matrix. We form one matrix with the columns of T and $\Delta \oplus \Delta \oplus \Delta$ using the command |, and compute the syzygy module of this matrix. The output which we call U' is a matrix whose columns generate this syzygy module.

```
i4 : U' = syz(T|DDD)

o4 = {3} | -x2+x     0         x2y+x2-xy-x                          xy2z2-xyz3                          ...
     {3} | 0         -x2+x     -3x2+3x                              x2y2-2x2yz+x2z2-x2y-xy2+x2z         ...
     {1} | -xyz      0         xy2z-xyz2+xyz                        0                                   ...
     {1} | 0         -xyz      x3-3xyz-y                            x2y2z+xy3z-x2yz2-2xy2z2+xyz         ...
     {3} | -x2+y-z   0         x2y+x2-y2+yz-y+z xy2z2-xyz3+y2z2-yz3 x2yz2-xy2z2+xyz3+xyz2-y2z2+         ...
     {3} | 0         -x2+y-z   -3x2+3y-3z                           x2y2-2x2yz+x2z2-x2y-xy2-y3+x        ...
     {4} | 0         0         0                                    -y2+2yz-z2+y-z                      ...
     {4} | 0         0         -y+z+1                               0                                   ...
     {4} | 0         0         -x+y-z                               -yz2+z3-yz+z2                       ...

              9     6
o4 : Matrix R  <--- R
```

In our example we need the first six coordinates of these generators. We obtain these coordinates in the following way.

```
i5 : U = U'^{0,1,2,3,4,5}

o5 = {3} | -x2+x     0         x2y+x2-xy-x                          xy2z2-xyz3                          ...
     {3} | 0         -x2+x     -3x2+3x                              x2y2-2x2yz+x2z2-x2y-xy2+x2z         ...
     {1} | -xyz      0         xy2z-xyz2+xyz                        0                                   ...
     {1} | 0         -xyz      x3-3xyz-y                            x2y2z+xy3z-x2yz2-2xy2z2+xyz         ...
     {3} | -x2+y-z   0         x2y+x2-y2+yz-y+z xy2z2-xyz3+y2z2-yz3 x2yz2-xy2z2+xyz3+xyz2-y2z2+         ...
     {3} | 0         -x2+y-z   -3x2+3y-3z                           x2y2-2x2yz+x2z2-x2y-xy2-y3+x        ...

              6     6
o5 : Matrix R  <--- R
```

We now use Theorem 4.11 to compute a presentation of $\langle U \rangle / \langle \Delta \oplus \Delta \oplus \Delta \rangle$. To do so we first compute the syzygy module of the columns of U and $\Delta \oplus \Delta \oplus \Delta$.

```
i6 : K' = syz(U|DDD)

o6 = {5} | y+1   y2z2-yz3+yz2                          yz2                    -yz2                   y2  ...
     {5} | -3    -y3+3y2z-3yz2+z3-y2+2yz-z2+y+1        -x2-xy-y2+xz+2yz-z2    x2+xy+y2-xz-2yz+z2      -y3 ...
     {6} | 1     0                                    0                      0                      0   ...
     {8} | 0     x-y+z                                -1                     1                      x-y ...
     {8} | 0     1                                    0                      0                      0   ...
     {7} | 0     -y2+2yz-z2+1                         -x-y+z                 x+y-z-1                -y2 ...
     {4} | 0     0                                    1                      0                      0   ...
     {4} | -1    -yz2+z3-yz-z                         xz-z2                  -xz+z2+z               -yz ...
     {4} | 0     0                                    0                      1                      0   ...

              9     5
o6 : Matrix R  <--- R
```

We now Let K'' denote the matrix whose columns are the first six coordinates of the generators of this syzygy module.

```
i7 : K'' = K'^{0,1,2,3,4,5}

o7 = {5}|y+1   y2z2-yz3+yz2                    yz2                    -yz2                   y2 ...
     {5}|-3    -y3+3y2z-3yz2+z3-y2+2yz-z2+y+1 -x2-xy-y2+xz+2yz-z2    x2+xy+y2-xz-2yz+z2     -y3 ...
     {6}|1     0                              0                      0                      0  ...
     {8}|0     x-y+z                          -1                     1                      x-y ...
     {8}|0     1                              0                      0                      0  ...
     {7}|0     -y2+2yz-z2+1                   -x-y+z                 x+y-z-1                -y2 ...

                6     5
o7 : Matrix  R <--- R
```

Although not necessary, we compute a Groebner basis for this submodule since the elements of the Groebner basis are less complicated than the generators for the submodule above.

```
i8 : K = gb K''

o8 = {5}|0   y+1   0      z2          z2                    x2z2-xz3-xz2+z2      |
     {5}|0   -3    z      y2-2yz-2z2-1 x2+xy+y2-xz-2yz-2z2 3xz2                 |
     {6}|0   1     0      z2          z2                    x2z2+xyz2-xz3-xz2+z2 |
     {8}|0   0     x-y+z  0           1                     -y2+2yz+2z2+1        |
     {8}|0   0     -y+z   -1          0                     -x2-xy-y2+xz+2yz+2z2 |
     {7}|1   0     0      0           x+y-z                 0                    |

o8 : GroebnerBasis
```

The columns of K are labelled $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_6$ in example 6.9. □

***Macaulay 2* Session for Example 7.3.** In *Macaulay 2* we must first define the ring, R which we are working over.

```
i1 : R = QQ[x,y,z]

o1 = R

o1 : PolynomialRing
```

We first define the matrices Δ, $\Delta \oplus \Delta \oplus \Delta$, T_1, and T_2.

```
i2 : D = matrix{{-x*y*z^2},{x^3-y}}

o2 = | -xyz2 |
     | x3-y  |

               2     1
o2 : Matrix  R <--- R
```

```
i3 : DDD = matrix{{-x*y*z^2,0,0},{x^3-y,0,0},{0,-x*y*z^2,0},{0,x^3-y,0},{0,0,-x*y*z^2},{0,0,x^3-y}}

o3 = | -xyz2  0      0     |
     | x3-y   0      0     |
     | 0      -xyz2  0     |
     | 0      x3-y   0     |
     | 0      0      -xyz2 |
     | 0      0      x3-y  |

             6     3
o3 : Matrix R <--- R

i4 : T1 = matrix{{x^2-y+z,0,0,0,-x^2+x,0},{0,x^2-y+z,0,0,0,-x^2+x},{-x*y*z-y*z,0,y-z-1,0,x*y*z,0},{0,-x*y*z-y*z, . . .

o4 = | x2-y+z          0               0       0       -x2+x       0          |
     | 0               x2-y+z          0       0       0           -x2+x      |
     | -xyz-yz         0               y-z-1   0       xyz         0          |
     | 0               -xyz-yz         0       y-z-1   0           xyz        |
     | -x2z+xyz+yz-z2  0               x-y+z   0       x2z-xyz-xz  0          |
     | 0               -x2z+xyz+yz-z2  0       x-y+z   0           x2z-xyz-xz |

             6     6
o4 : Matrix R <--- R

i5 : T2 = matrix{{z^2+z,0,x-y+z,0,-y+z+1,0},{0,z^2+z,0,x-y+z,0,-y+z+1}}

o5 = | z2+z  0     x-y+z  0      -y+z+1  0       |
     | 0     z2+z  0      x-y+z  0       -y+z+1  |

             2     6
o5 : Matrix R <--- R
```

We now compute the syzygy module of the columns of T_2 and and the columns of Δ.

```
i6 : U2' = syz(T2|D)

o6 = {2} | 0     0     y-z-1  0     x-1   0     -z2-z  -yz+z2+z  . . .
     {2} | 0     0     0      y-z-1 0     x-1   -1     0         . . .
     {1} | y-z-1 0     0      0     -z2-z 0     -z4-2z3-z2  -yz3+z4-yz2+2z3+z2 . . .
     {1} | 0     y-z-1 0      0     0     -z2-z x2z+x2+xz+x+z+1  x2y-x2z-x2+xy-xz-x+ . . .
     {1} | x-y+z 0     z2+z   0     z2+z  0     xz3+z4+xz2+2z3+z2  xyz2-xz3+yz3-z4-xz2 . . .
     {1} | 0     x-y+z 0      z2+z  0     z2+z -x2z-x2-xz-x  -x2y+x2z+x2-xy+xz+ . . .
     {7} | 0     0     0      0     0     0     -z-1   -y+z+1  . . .

             7     9
o6 : Matrix R <--- R
```

In our example we need the first six coordinates of these generators. We obtain these coordinates in the following way.

```
i7 : U2 = U2'^{0,1,2,3,4,5}

o7 = {2}|0     0     y-z-1  0     x-1   0     -z2-z                -yz+z2+z                ...
     {2}|0     0     0      y-z-1 0     x-1   -1                   0                       ...
     {1}|y-z-1 0     0      0     -z2-z 0     -z4-2z3-z2           -yz3+z4-yz2+2z3+z2      ...
     {1}|0     y-z-1 0      0     0     -z2-z x2z+x2+xz+x+z+1      x2y-x2z-x2+xy-xz-x+     ...
     {1}|x-y+z 0     z2+z   0     z2+z  0     xz3+z4+xz2+2z3+z2    xyz2-xz3+yz3-z4-xz2     ...
     {1}|0     x-y+z 0      z2+z  0     z2+z -x2z-x2-xz-x         -x2y+x2z+x2-xy+xz+      ...

              6     9
o7 : Matrix R <--- R
```

We now use Theorem 4.11 to compute a presentation of $\langle U_2 \rangle / (\langle T_1 \rangle + \langle \Delta \oplus \Delta \oplus \Delta \rangle)$. To do so we first compute the syzygy module of the columns of U_2, T_1, and $\Delta \oplus \Delta \oplus \Delta$.

```
i8 : K' = syz(U2|T1|DDD)

o8 = {2}|-1 0  -z2-z  0      xz-z2 0     -xz+z2+z 0       z3+z2   -xz2-z3-z2 ...
     {2}|0  -1 0      -z2-z  0     xz-z2 0       -xz+z2+z -x2-x-1 x2+x       ...
     {3}|0  0  x-1    0      x     0     -x+1    0        z       -z         ...
     {3}|0  0  0      x-1    0     x     0       -x+1     0       0          ...
     {3}|0  0  -y+z+1 0      -x-y+z 0     x+y-z-1 0        0       z          ...
     {3}|0  0  0      -y+z+1 0     -x-y+z 0     x+y-z-1  0       0          ...
     {5}|0  0  0      0      0     0     0       0        0       0          ...
     {5}|0  0  0      0      0     0     0       0        1       -1         ...
     {5}|0  0  0      0      0     0     0       0        0       1          ...
     {3}|0  0  0      0      1     0     0       0        0       0          ...
     {3}|0  0  0      0      0     1     0       0        0       0          ...
     {1}|1  0  0      0      0     0     0       0        0       0          ...
     {1}|0  1  0      0      0     0     0       0        0       0          ...
     {3}|0  0  0      0      0     0     1       0        0       0          ...
     {3}|0  0  0      0      0     0     0       1        0       0          ...
     {4}|0  0  0      0      0     0     0       0        0       0          ...
     {4}|0  0  0      0      0     0     0       0        0       -1         ...
     {4}|0  0  0      0      0     0     0       0        1       0          ...

              18    14
o8 : Matrix R <--- R
```

We now Let K'' denote the matrix whose columns are the first nine coordinates of the generators of this syzygy module.

```
i9 : K'' = K'^{0,1,2,3,4,5,6,7,8}

o9 = {2}|-1 0  -z2-z  0      xz-z2 0     -xz+z2+z 0       z3+z2   -xz2-z3-z2 ...
     {2}|0  -1 0      -z2-z  0     xz-z2 0       -xz+z2+z -x2-x-1 x2+x       ...
     {3}|0  0  x-1    0      x     0     -x+1    0        z       -z         ...
     {3}|0  0  0      x-1    0     x     0       -x+1     0       0          ...
     {3}|0  0  -y+z+1 0      -x-y+z 0     x+y-z-1 0        0       z          ...
     {3}|0  0  0      -y+z+1 0     -x-y+z 0     x+y-z-1  0       0          ...
     {5}|0  0  0      0      0     0     0       0        0       0          ...
     {5}|0  0  0      0      0     0     0       0        1       -1         ...
     {5}|0  0  0      0      0     0     0       0        0       1          ...

              9     14
o9 : Matrix R <--- R
```

Although not necessary, we compute a Groebner basis for this submodule since the elements of the Groebner basis are less complicated than the generators for the submodule above.

```
i10 : K = gb K"

o10 = {2} | 1   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0  |
      {2} | 0   1   0   0   0   0   0   0   0   0   0   0   0   0   0   0  |
      {3} | 0   0   -1  0   0   z   0   y-z 0   0   0   x-1 0   0   0   0  |
      {3} | 0   0   0   -1  0   0   z   0   y-z 1   0   0   x-1 1   0   0  |
      {3} | 0   0   1   0   0   0   0   0   0   0   0   -y+z+1 0   0   0   0  |
      {3} | 0   0   0   1   0   0   0   0   0   0   0   0   -y+z+1 0   0   0  |
      {5} | 0   0   0   0   0   0   0   0   0   y-z-1 0   0   0   x-1 0   z  |
      {5} | 0   0   0   0   -1  1   0   0   0   -z-1  y-z 0   0   -z-1 x   z2+z |
      {5} | 0   0   0   0   1   0   0   0   0   0   0   0   0   0   0   0  |

o10 : GroebnerBasis
```

The columns of K are labelled $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{16}$ in example 7.3. □

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