1966

Numerical solution of nonlinear equations

P. W. Card
The University of Montana

NUMERICAL SOLUTION OF NONLINEAR EQUATIONS

By

Phillip W. Card

B.A. University of Montana 1964

Presented in partial fulfillment of the requirements

for the degree of

Master of Arts

UNIVERSITY OF MONTANA

1966

Approved by:

Robert P. Banaugh

Chairman, Board of Examiners

Fred S. Honkala

Dean, Graduate School

MAY 20 1966

Date

ACKNOWLEDGEMENTS

I wish to express my appreciation to Professor Robert P. Banaugh for his guidance and instruction throughout the preparation of this thesis. I would also like to thank Professor James Duemmel, Professor C. R. Jeppesen and Professor Krishan K. Gorowara for their reading of the thesis.

P. W. C.

ABSTRACT

In this paper we consider the problem of finding the roots of nonlinear equations, i.e., we summarize some of the techniques for finding the zeros of f(x) where f(x) may be a polynomial, transcendental, or other nonlinear function.

TABLE OF CONTENTS

INTRODUCTION

The problem of finding the real or complex roots of a nonlinear equation is an old problem. This problem is frequently encountered in scientific work. A few typical instances are listed below:

1) in the solution of linear differential equations we must often find the zeros of characteristic polynomials.

2) the stability of a mechanical or electrical system is determined by examining the zeros of an associated polynomial.

3) when finite difference methods are used to solve nonlinear boundary value problems, we must solve simultaneous nonlinear equations.

In this thesis we review several methods of solution of such equations and we also state and prove some theorems that have been found useful in their solution. In addition, to illustrate most of the methods which are presented, we have listed the computer programs, together with the numerical results of typical problems. These results are presented to aid the reader in formulating his own evaluation of the effectiveness of the techniques. The programs are written in the FORTRAN II language for the IBM 1620 computer. The report also contains a rather complete and up to date bibliography.

The equations to be considered are of the form

$$f(x) = 0$$

where f(x) may be a transcendental or a polynomial function. Methods for the determination of both real and complex roots of polynomial equations are reviewed, whereas, only methods for finding the real and separated roots of transcendental equations are studied.

After discussing methods of solution for a single equation we briefly examine simultaneous nonlinear equations. We note here that the solution of simultaneous nonlinear equations is an extremely difficult problem and very few efficient algorithms are available for their solution.

Chapter 0

Let $f(x)$ be a continuous real-valued function with as many derivatives as may be required to permit the operations that may be used in the following development. Let ξ be a root of multiplicity one of $f(x) = 0$ and assume that $y = f(x)$ has an inverse $x = g(y)$ in some neighborhood of ξ.

In chapter 1 we consider functional iteration methods based on n-point inverse interpolation, using polynomials as our interpolation functions. These methods lead to approximate solutions of $f(x) = 0$. It is assumed that the reader is familiar with the theory of inverse interpolation. The theory is discussed in Ostrowski [, pages 1-12] and Ralston [6 , pages 40-75]. The error in using n-point inverse polynomial interpolation as the basis of functional iteration is given by

$$\xi - x_{i+1} = \frac{g^{(n)}(\eta)}{n!}(-1)^n \, y_1 \, y_2 \cdots y_n \qquad (0.1)$$

where η is in the interval spanned by $y_1, y_2, \ldots, y_n$ and 0, $y_i = f(x_i)$ and superscript numbers indicate the order of differentiation.

The derivatives of the inverse function $g(y)$ are calculated in terms of derivatives of $f(x)$, as stated in the following.

THEOREM 0.1 If the first $n + 1$ $(n \geq 0)$ derivatives of $f(x)$ exist and $f'(x) \neq 0$ in some interval $[a, b]$, then the corresponding derivatives of the inverse function $g(y)$ exist in the corresponding y interval. In fact the derivatives are given by:

$$g^{(k)}(y) = \frac{X_k}{(y')^{2k-1}} \quad , \quad k = 1, 2, \ldots, n + 1$$

where X_k is a polynomial in $y', y'', \ldots, y^{(k)}$ and $X_1 = 1$, $X_{m+1} = (\frac{d}{d_x} X_m)y' - (2m-1)X_m\, y''$ $(m = 1, 2, \ldots)$.

Proof: Clearly since $f'(x) \neq 0$ in [a, b] then

$$g'(y) = \frac{dx}{dy} = \frac{1}{y'} = \frac{1}{f'}$$

and $$g''(y) = \frac{-f''}{[f']^2} \frac{dx}{dy} = \frac{-f''}{[f']^3} .$$

Let $$g^{(k)}(y) = \frac{X_k}{(y')^{2k-1}} \qquad k = 1, 2, \ldots, n + 1 \qquad (0.2)$$

Here X_k is a polynomial in $y', y'', \ldots, y^{(k)}$. This is true for $k = 1, 2$ for in particular $X_1 = 1$, $X_2 = - y''$. Assume the truth of our assertion for the first n derivatives of $g(y)$. We write (0.2) with $k = n$

$$g^{(n)}(y) = \frac{X_n}{y'^{2n-1}}$$

and get by differentiation, since $\frac{dy'}{dy} = \frac{y''}{y'}$

$$g^{(n+1)}(y) = \frac{d}{dx}(X_n) \frac{1}{y'^{2n}} - (2n-1)\, X_n \frac{y''}{y'} (y')^{-2n}.$$

Multiply the right hand side of the above equation by $\frac{(y')^{2n+1}}{(y')^{2n+1}}$ to obtain

$$g^{(n+1)}(y) = \frac{\frac{d}{dx}(X_n)y' - (2n-1)\, X_n\, y''}{(y')^{2n+1}}$$

so that

$$X_{n+1} = \frac{d}{dx} (X_n)\, y' - (2n-1)\, X_n\, y'', \; n = 1, 2, \ldots ,$$

$X_1 = 1$ and

$$g^{(n+1)}(y) = \frac{X_{n+1}}{(y')^{2n+1}} .$$

An n-point functional iteration method has the general form

$$x_{i+1} = F(x_i, x_{i-1}, \ldots, x_{i-n+1}) \qquad (0.3)$$

The iteration function F may involve not only the points $x_i, x_{i-1}, \ldots, x_{i-n+1}$, but also values of f(x) and some of its derivatives at one or more of the points $x_i, \ldots, x_{i-n+1}$.

We will want to determine when an iteration method converges, and, if it does converge, how fast it converges. The convergence or non-convergence will in general depend upon the choice of the initial approximation(s) to the root. We will see that if the initial approximation(s) are "close enough" to ξ then convergence is usually assured. The problem of obtaining a "close enough" initial approximation to a root is a very difficult one about which very little is known. Usually the initial approximation is obtained from the investigators "intuition" which was derived from his "feel" of how the real system (from whence the original nonlinear equation was derived) should behave. Some methods will converge independently of the initial approximation. In practice we often begin our computation with a guess at the root and just hope that the iteration process will converge.

For comparative purposes we will use the concept of order. Order is a measure of how fast the method in question converges. To define the order of an iterative method we first define the error in the i^{th} iterate to be

$$\epsilon_{i+1} = \xi - x_{i+1} \tag{0.4}$$

Under the assumption that the method will converge we have

DEFINITION 0.1 If there exists a real number $p \geq 1$ such that

$$\lim_{i \to \infty} \frac{|\xi - x_{i+1}|}{|\xi - x_i|^p} = \lim_{i \to \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|^p} = C \neq 0 \text{ and } |C| < \infty ,$$

we say the method is of order p at ξ .

If a method has order 2 for example, then the error of any iterate is approximately proportional to the square of the error of the previous iterate. The concept of order is illustrated in Problem 3, Chapt. 1.

We now have

THEOREM 0.2 The order of a method is unique.

Proof. Suppose p is the order, i.e.,

$$\lim_{i \to \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|^p} = C \neq 0$$

Then $\lim_{i \to \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|^{p+\delta}} = C \lim_{i \to \infty} \frac{1}{|\epsilon_i|^{\delta}}$. If $\delta > 0$ the latter limit diverges to infinity. If $\delta < 0$, this limit converges to zero. Thus $\delta = 0$ and p is unique.

Chapter 1

In this chapter we consider some numerical methods for the solution of transcendental equations whose roots are real and separated.

One of the oldest known methods is the method of false position (regula falsi), in which we are given two interpolation points $x_1 \neq x_2$. Let $y_i = f(x_i)$ and we assume $f(x_i) \neq f(x_j)$, $i \neq j$. We interpolate the inverse function $g(y)$ by a linear function which assumes the values x_1, x_2 for y_1 and y_2, i.e.,

$$g(y) \approx \frac{(y-y_1)x_2 - (y-y_2)x_1}{y_2-y_1}$$

Let $x_3 = g(0)$, the first approximation to the root of $f(x) = 0$.

Thus $x_3 = \dfrac{x_1y_2 - y_1x_2}{y_2-y_1}$ which may be rewritten as

$$x_3 = x_2 - y_2 \frac{(x_2-x_1)}{(y_2-y_1)} \qquad (1.1)$$

This is, of course, linear inverse interpolation. Continuing this process we obtain a sequence of points $x_1, x_2, x_3, \ldots$ where

$$x_{i+1} = x_i - \frac{x_i-x_1}{y_i-y_1} y_i \quad (i = 2, 3, \ldots) \qquad (1.2)$$

and x_1, x_2 are our initial approximations. Does the sequence converge?

A sufficient set of conditions to ensure the convergence of the sequence defined by (1.2) are the fourier conditions:

1) $f(x_1)f(x_2) < 0$, 2) $f(x_1)f''(x_1) > 0$, 3) $f''(x) \neq 0$ $(x_1<x<x_2)$

Fig. 1 illustrates the Fourier conditions.

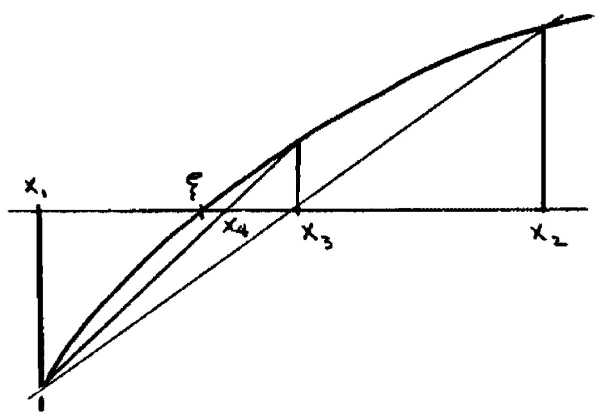


Fig. 1

We note that we are restricted to convex functions by the Fourier conditions.

If the situation is as pictured in Fig. 1 then the sequence (1.2) indeed converges. For x_2, x_3, ... lie on the concave side of the arc and cannot go beyond ξ ; thus we have a monotone decreasing sequence bounded below by ξ . The sequence therefore converges to a limit ξ_o. We now show that ξ_o is the root ξ of $f(x) = 0$ in (x_1, x_2).

We subtract ξ_o from both sides of equation (1.2) and take limits as $i \to \infty$ to obtain

$$0 = \xi_o - \frac{\xi_o - x_1}{f(\xi_o)-f(x_1)} f(\xi_o) - \xi_o$$

$$= \frac{(x_1 - \xi_o)}{f(\xi_o)-f(x_1)} f(\xi_o)$$

Now $x_1 \neq \xi_0$. Thus $f(\xi_0) = 0$ and ξ_0 is a root of $f(x) = 0$ in (x_1, x_2) and hence $\xi_0 = \xi$.

Let us determine the order of the method of false position. By using (0.1) the error is

$$\epsilon_{i+1} = \xi - x_{i+1} = \frac{g''(\eta)}{2} y_1 y_i = -\frac{f''(\bar{\xi})}{2[f'(\bar{\xi})]^3} y_1 y_i$$

since $g''(y) = -\dfrac{f''(x)}{[f'(x)]^3}$.

Using the mean value theorem we have

$y_1 = f(x_1) = f(x_1) - f(\xi) = (x_1 - \xi) f'(\xi_1) = \epsilon_1 f'(\xi_1)$,

$y_i = (x_i - \xi) f'(\xi_i) = \epsilon_i f'(\xi_i)$, ξ_1, ξ_i in appropriate intervals.

$$\text{Therefore } \epsilon_{i+1} = -\frac{f''(\bar{\xi}) f'(\xi_1) f'(\xi_i)}{2[f'(\bar{\xi})]^3} \epsilon_i \epsilon_1 \qquad (1.3)$$

$$\text{Then } \lim_{i \to \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|} = \left| -\frac{f''(\xi^*) f'(\xi_1) f'(\xi)}{2[f'(\xi^*)]^3} \right| |\epsilon_1|$$

since $\xi_i \to \xi$ and $\bar{\xi}$ approaches some limiting value ξ^* as $i \to \infty$. Clearly $f'(x)$ is bounded away from zero in a neighborhood of ξ. Therefore the method of false position has order 1.

The method of "regula falsi" may be modified to increase the rate of convergence. Suppose we do not insist that $f(x_1)f(x_2) < 0$ and that we always use the previous two iterates, x_i and x_{i-1}, to generate x_{i+1}, i.e., we have

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{y_i - y_{i-1}} y_i$$

This modified method is called the secant method. However, the sequence of iterates obtained may not converge (Figure 2 is an example of nonconvergence.

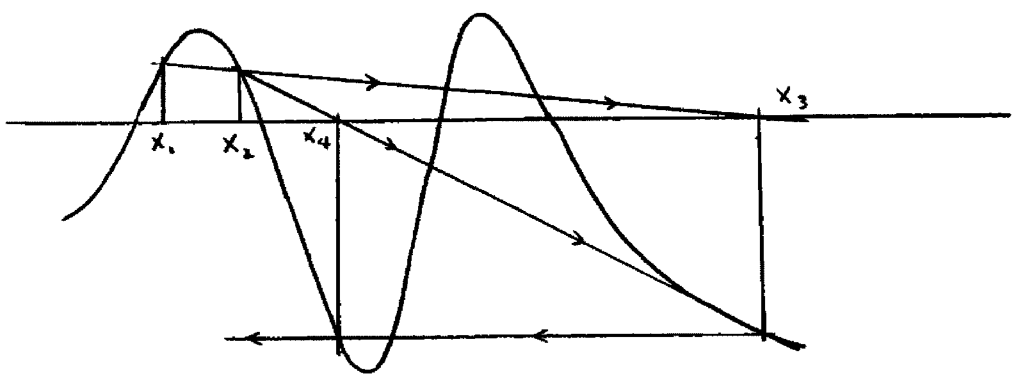


Figure 2

We now ask what is the order of this method assuming that it converges? By reasoning analogous to that used previously the error in the secant method is

$$\varepsilon_{i+1} = -\frac{f''(\bar{\xi})\, f'(\xi_i)\, f'(\xi_{i-1})}{2[f'(\bar{\xi})]^3}\,\varepsilon_i \varepsilon_{i-1} \qquad (1.4)$$

It can be shown that the order of the secant method is $(1 + \sqrt{5})/2$. Ralston [6 , pages 326-327] outlines an argument and Ostrowski [5 , pages 80-81] has a complete proof. Thus the order of convergence of the secant method is substantially greater than the order of the false position method.

Another method for finding the roots of $f(x) = 0$ is the bisection method. If $f(x)$ is continuous on (x_1, x_2) and $f(x_1)$ and $f(x_2)$ have opposite signs then we consider the sequence of points which lie halfway between the previous two points of opposite sign. The bisection method is certainly convergent having once found x_1 and x_2.

A minor variant in the bisection method is the dividing interval method. Given the points x_1 and x_2 such that $f(x_1)f(x_2) < 0$, we subdivide the interval $[x_1, x_2]$ into, say m, subintervals, knowing that we have at least one real root of f(x) in (x_1, x_2). Then we search for a pair of adjacent points $\bar{x}_i$, $\bar{x}_{i+1}$ such that $f(\bar{x}_i)f(\bar{x}_{i+1}) < 0$, $\bar{x}_0 = x_1$, $\bar{x}_i = x_1 + i\,(\frac{x_2 - x_1}{m})$ $(i = 1, 2, \ldots, m)$. Using these two points as endpoints of our next interval we continue the subdividing process until we achieve desired accuracy.

Since the latter two methods are not based on interpolation formulae we do not discuss their order of convergence. These two methods are very useful when a priori information on the location of roots is poor. If such is the case we can start at the origin, say, and test consecutive intervals of an arbitrarily fixed length until we find an interval on which the functional values at the endpoints differ in sign. Having located this fundamental interval we then apply one of the two methods above. If we desire other real roots we can continue along the x-axis in exactly the same manner. Of course it may happen that our test intervals were of too great a length in which case we might miss some roots as shown in Figure 3.

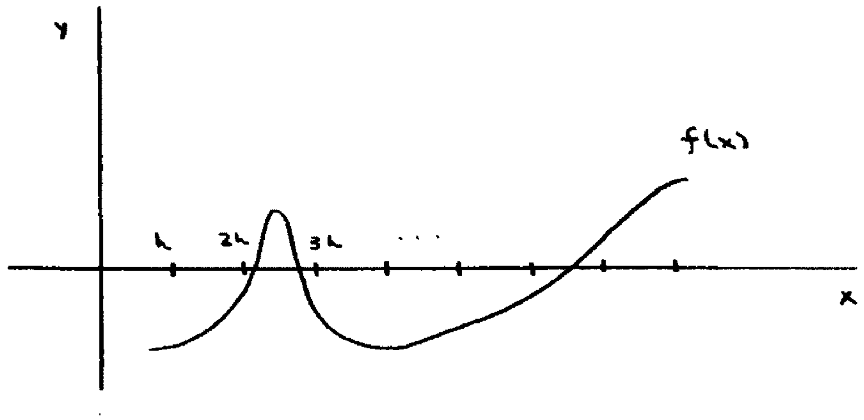


Figure 3

The iteration methods considered thus far have been two-point iteration methods. Next we will consider a class of one-point functional iteration methods of the general form

$$x_{i+1} = F(x_i)$$

We assume that ξ is a simple root of $f(x) = 0$, and that $f(x)$ has an inverse $g(y)$ in a neighborhood of ξ.

We expand $g(y)$ in a Taylor-series about y_i to obtain

$$x = g(y) = \sum_{j=0}^{m+1} \frac{(y-y_i)^j}{j!} g^{(j)}(y_i) + \frac{(y-y_i)^{m+2}}{(m+2)!} g^{(m+2)}(\eta)$$

$$= x_i + \sum_{j=1}^{m+1} \frac{(y-y_i)^j}{j!} g^{(j)}(y_i) + \frac{(y-y_i)^{m+2}}{(m+2)!} g^{(m+2)}(\eta)$$

where η is between y and y_i.

Since $\xi = g(0)$ we have

$$\xi = x_i + \sum_{j=1}^{m+1} \frac{(-1)^j}{j!} y_i^j g^{(j)}(y_i) + \frac{(-1)^{m+2} y_i^{m+2}}{(m+2)!} g^{(m+2)}(\eta)$$

$$= x_i + \sum_{j=1}^{m+1} \frac{(-1)^j}{j!} f_i^j g_i^{(j)} + \frac{(-1)^{m+2}}{(m+2)!} f_i^{m+2} g^{(m+2)}(\eta) \quad (1.5)$$

where $y_i = f(x_i) = f_i$ and $g^{(j)}(y_i) = g_i^{(j)}$.

We define

$$Y_j(x_i) = Y_j = \frac{(-1)^j}{(j+1)!} (f_i')^{j+1} g_i^{(j+1)} \text{ and}$$

$$u_i = \frac{f_i}{f_i'} \qquad j = 0, 1, 2, \ldots \qquad (1.6)$$

Now (1.5) becomes

$$\xi = x_i - \frac{f_i}{f_i'} \sum_{j=0}^{m} \frac{(-1)^j}{(j+1)!} f_i' f_i^j g_i^{(j+1)}$$

$$+ \frac{(-1)^{m+2}}{(m+2)!} f_i^{m+2} g^{(m+2)}(\eta)$$

$$= x_i - \frac{f_i}{f_i'} \sum_{j=0}^{m} \frac{f_i^j}{(f_i')^j} \frac{(-1)^j}{(j+1)!} (f_i')^{j+1} g_i^{(j+1)}$$

$$+ \frac{(-1)^{m+2}}{(m+2)!} f_i^{m+2} g^{(m+2)}(\eta)$$

$$= x_i - u_i \sum_{j=0}^{m} u_i^j Y_j + \frac{(-1)^{m+2}}{(m+2)!} f_i^{m+2} g^{(m+2)}(\eta) \quad (1.7)$$

Now consider an iteration formula of the form

$$x_{i+1} = x_i - u_i \sum_{j=0}^{m} u_i^j Y_j \qquad (1.8)$$

(1.8) will be useful only if the Y_j's are easily calculated. We have $Y_0 = 1$ by (1.6) and by differentiating $Y(x)$ we obtain

$$Y_j = \frac{1}{j+1} (j D_2 Y_{j-1} - Y_{j-1}'), \quad Y_j' = \frac{d}{dx} Y_j(x) \Big| x=x_i \quad (1.10)$$

where $D_j(x_i) = D_j = \dfrac{f_i^{(j)}}{f_i'}$ (1.11)

Also by differentiating (1.11) with $D_1 = 1$,

$$D_j = D_2 D_{j-1} + D_{j-1}' \qquad D_j' = \frac{d}{dx} D_j(x) \Big| x = x_i \quad (1.12)$$

Now $Y_1 = \frac{1}{2} D_2$

$$Y_2 = \frac{1}{3} [D_2^2 - \frac{1}{2} (\frac{d}{dx} D_2)] = \frac{1}{3} (D_2^2 - \frac{1}{2} (D_3 - D_2^2)]$$

and by looking at (1.10) and rewriting (1.12) as

$$D_{j-1}' = D_j - D_2 D_{j-1}$$

we see that Y_j is a polynomial in $D_2, D_3, \ldots, D_{j+1}$.

Thus, the evaluation of (1.8) reduces to the evaluation of u_i and the D_j's.

Subtract (1.8) from (1.7) to obtain the error,

$$\epsilon_{i+1} = \xi - x_{i+1} = \frac{(-1)^{m+2}}{(m+2)!} f_i^{m+2} g^{(m+2)} (\eta).$$

As before $f_i = f(x_i) = f(x_i) - f(\xi) = (x_i - \xi) f'(\xi_i)$, since ξ is a zero of $f(x)$, where ξ_i is between ξ and x_i. Then

$$\epsilon_{i+1} = \frac{1}{(m+2)!} \left\{ [f'(\xi_i)]^{m+2} g^{(m+2)} (\eta) \right\} \epsilon_i^{m+2}$$

Since ξ is a simple root of $f(x) = 0$ the term in braces is bounded in some neighborhood of ξ. The order of (1.8) then is (m+2) provided the method converges.

Let us consider the special case when $m = 0$; hence the order is two. Then

$$x_{i+1} = x_i - u_i = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1.3)$$

which is the Newton-Raphson method of iteration. Geometrically, x_{i+1} is the intersection of the tangent line $f'(x_i)$ with the x-axis.

As m increases then so does the order, but in each case we must evaluate higher and higher order derivatives. Thus the usefulness of this class of methods is dependent on the complexity of $f(x)$, i.e., how hard is it to evaluate higher order derivatives.

Another one-point iterational method is that called the "first-order" iteration method. The principle of the method is to express the equation $f(x) = 0$ in the form

$$x = g(x) \tag{1.14}$$

so that any solution of (1.14) is a solution of $f(x) = 0$. Geometrically a root of (1.14) is a number $x = \xi$ for which the line $y = x$ intersects the curve $y = g(x)$. The iteration formula then has the form

$$x_{i+1} = g(x_i)$$

and it can be shown that if the form of (1.14) is chosen correctly and we have an initial approximation which is "close enough" then the method will converge with order one. In other words equation (1.14) may be written a variety of ways, depending on $f(x)$, but each way does not necessarily lead to convergence.

For example, consider $f(x) = x^2 - x - 6 = 0$, which has as roots 3 and -2. Then (1.14) may assume any of the following forms:

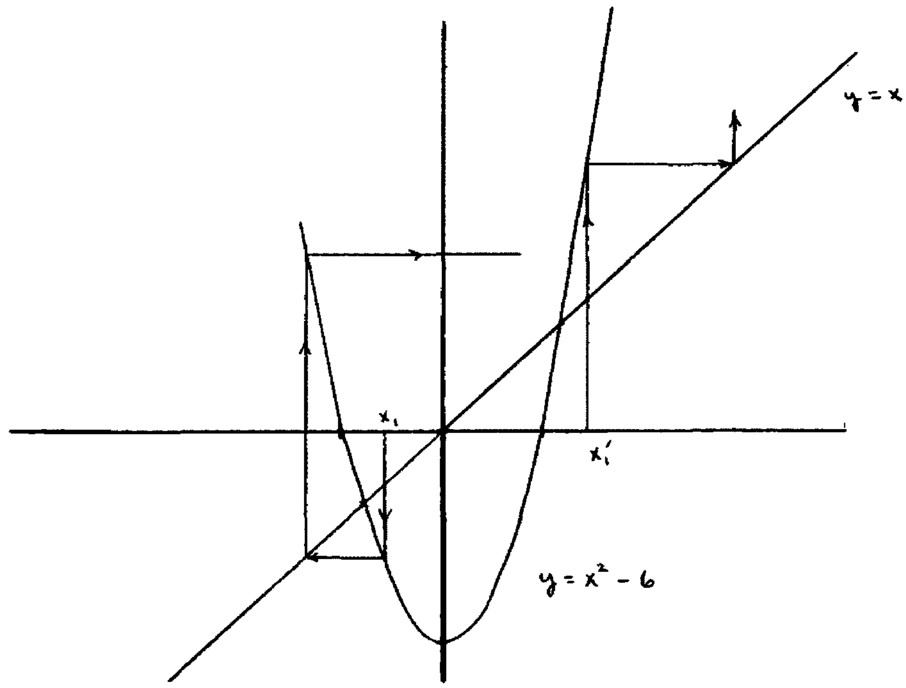
1) $x = x^2 - 6$

2) $x = 1 + \frac{6}{x}$

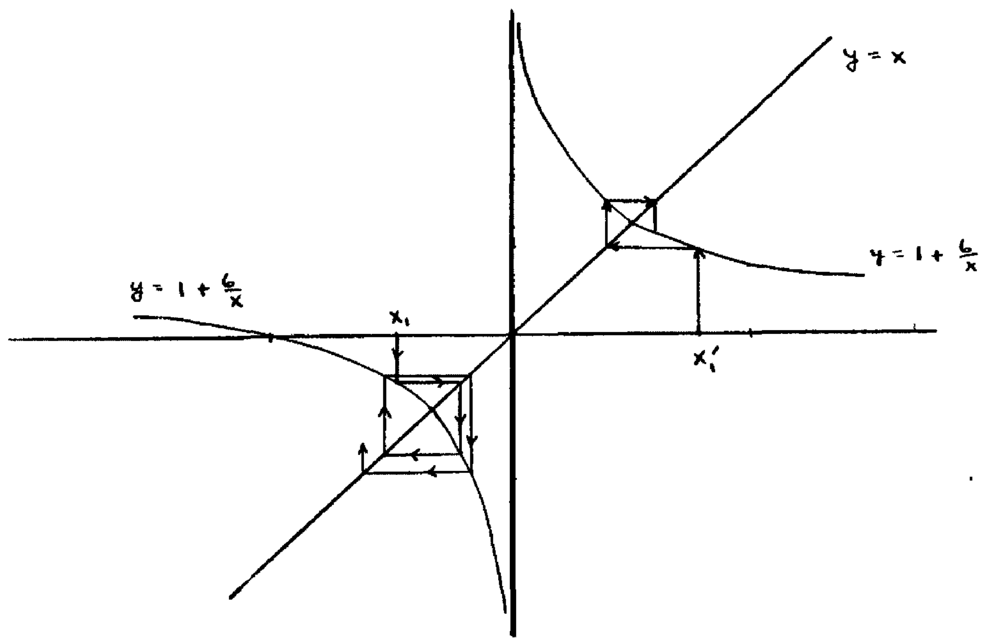
3) $x = \pm\sqrt{x + 6}$

If form 1) is used neither root is found, form 2) will give us the root 3, while form 3) will yield both roots.

Form 1)



Form 2)



Form 3)

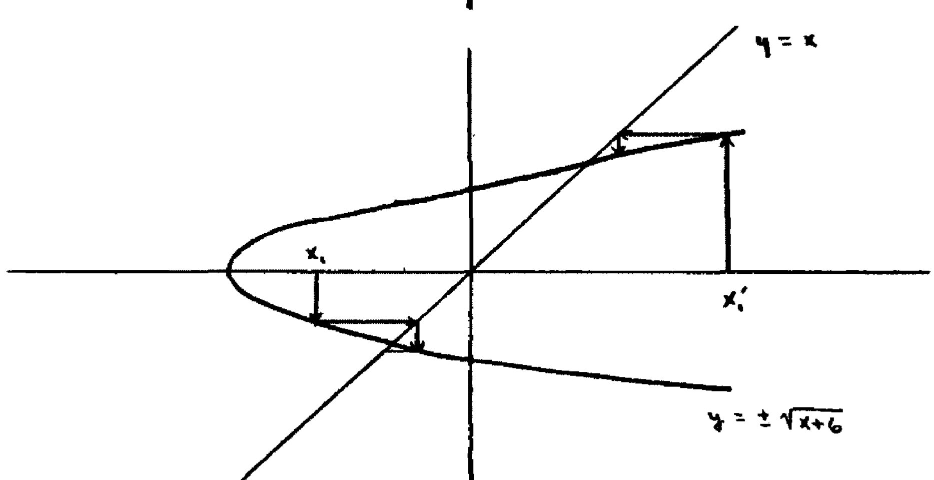


Figure 4

The three forms are illustrated in Figure 4.

As a guide, Newton's method should be used whenever $f'(x)$ is easily calculated. If this is not possible the secant method should be used. If neither of these methods is readily applicable, then try a method with convergence of order one.

In the methods reviewed thus far ξ has been assumed to be a root of multiplicity one of $f(x) = 0$. Suppose now that ξ is a root of multiplicity $r > 1$ of $f(x) = 0$ and that we desire an iteration method whose order of convergence is independent of the multiplicity of the root. Consider $u(x) = \frac{f(x)}{f'(x)}$. No matter what the multiplicity of ξ of $f(x)$, $u(x)$ has ξ as a root of multiplicity one. The roots of $u(x) = 0$ are then identical with the roots of $f(x) = 0$ except they all are simple. Therefore we replace $F(x)$ by $u(x)$ in any method developed thus far and we retain the order of convergence. Newton's method, for instance, becomes

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$= x_i - \frac{f(x_i)\ f'(x_i)}{[f'(x_i)]^2 - f(x_i) f''(x_i)}$$

The order again is two but note the necessity of the evaluation of the second derivative of $f(x)$.

In programming these methods it is necessary to "tell" the computer when to stop the iteration. The criterion adopted was to stop the iteration when $|x_{i+1} - x_i| < \epsilon$, where ϵ is small. As a further check on the convergence the value $f(x_{i+1})$ is punched out and should also be negligible. This latter condition is not a satisfactory criterion for stopping the iteration since for $|f(x_{i+1})| < \bar{\epsilon}$ it may be necessary that $|x_{i+1} - x_i| < \epsilon$ where ϵ is less than the smallest significant number carried in the arithmetic and hence the computer would never stop iterating.

Now we examine the following

Problem 1. Find a real root of the equation

$$f(x) = \sin x - x/2 = 0$$

From the graph given below, Figure 5, we see that $f(\frac{\pi}{2})\ f(\pi) < 0$. Therefore $f(x) = \sin x - x/2$ has a real zero between $\frac{\pi}{2}$ and π. This problem was run using the Newton-Raphson, secant, "first-order" iteration, and dividing interval methods. The programs and complete numerical results appear in the appendix. In each run ϵ was chosen as $.1 \times 10^{-5}$. The real root sought was 1.89549. As initial guesses $\frac{\pi}{2}$ and π were used, and as expected, the Newton-Raphson method converged the fastest, requiring only five iterates. Clearly the derivative $f'(x)$ is easily calculated. This problem was run by Ralston using the false-position method with the same ϵ and same initial guesses but here eleven iterates were required.

Problem 2. Find a positive real zero of the function $f(x) = x^{20} - 1$ using the Newton-Raphson method.

From formula (1.13) it is evident that the larger

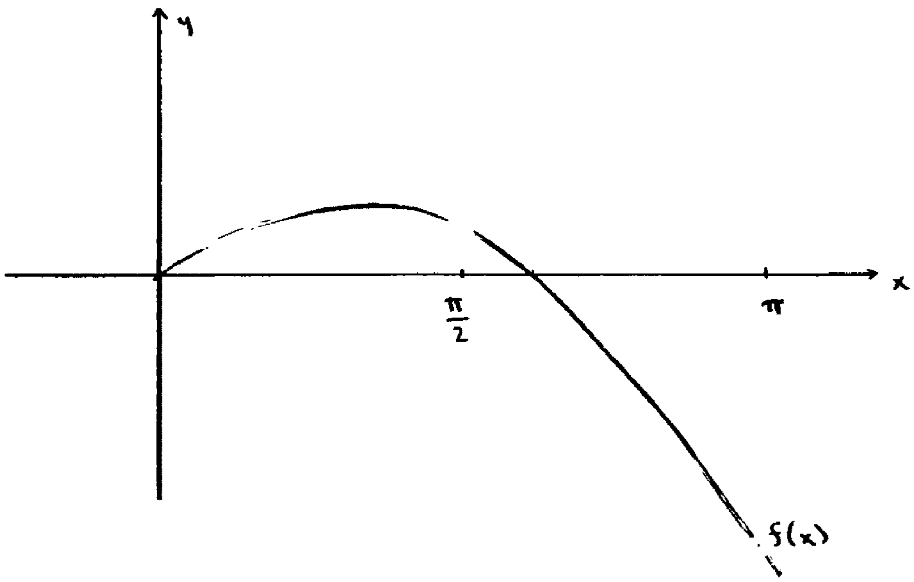


Figure 5

$f(x) = \sin x - x / 2$

the value of f'(x) the smaller is the correction needed to obtain the correct value of the root. This implies that the larger the value of f'(x) in a neighborhood of the root the faster the convergence, and in fact if f'(x) is small in this neighborhood the method would converge very slowly or fail altogether. We see by looking at Figure 6, that if the initial guess x_1 is greater than 1 the method should converge, but for $0 < x_1 < 1$ the most we could hope for is a very slow convergence. In fact with $x_1 = 0.5$ the method has still not converged after 50 iterates and $x_{51} = 2.123 \times 10^3$, whereas with $x_1 = 1.5$ or $x_1 = 5.0$ the method did indeed converge in twelve and thirty-six iterates respectively. Again $\epsilon = .1 \times 10^{-5}$.

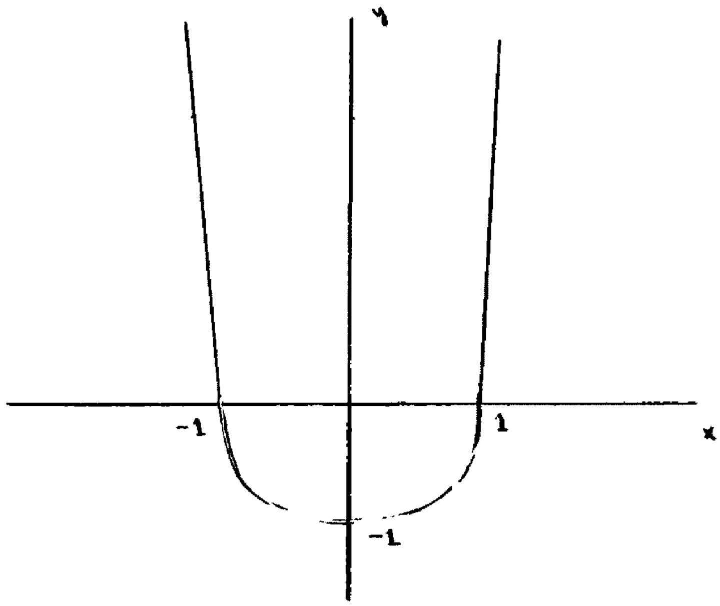


Figure 6

$f(x) = x^{20} - 1$

Problem 3. Find the real root of the equation $f(x) = \frac{1}{x} - 3 = 0$, i.e., find the reciprocal of 3, using the Newton-Raphson method and the "first-order" iteration method. This problem illustrates the concept of order.

The (i+1)st iterate using the Newton-Raphson method is given as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i + \left(\frac{1}{x_i} - 3\right) x_i^2$$

$$= x_i (2 - 3x_i)$$

Let our initial approximation be $x_1 = 0.3$. Then

$$x_2 = 0.3(1.1) = 0.33$$

$$x_3 = 0.33(1.01) = 0.3333$$

$$x_4 = 0.3333(1.0001) = 0.33333333$$
$\vdots$

Each iterate then doubles the number of significant figures. The order of the Newton-Raphson method is two.

To solve this problem using the "first-order" iteration method we rewrite the equation $\frac{1}{x} - 3 = 0$ in the form $x = \frac{1}{2}(-x + 1)$. Thus

$$x_{i+1} = \frac{1}{2}(-x_i + 1)$$

Let $x_1 = 0.3$ once again, and we obtain the sequence of iterates,

$$\{ 0.3, 0.35, 0.325, 0.3375, 0.33125, 0.334375, \ldots \}$$

In this case the sequence oscillates about the root but the sequence is converging to the root. The order of the "first-order" method is one.

Chapter 2

The methods of Chapt. 1 for finding the real roots of transcendental equations are used for finding the real roots of polynomial equations. With some modifications certain of these methods may be applied to the location of complex roots. However, the problem of finding the zeros of polynomials, both real and complex, arises so frequently that special methods have been developed to find them.

We consider the general polynomial equation of the $n^{\underline{th}}$ degree

$$P_n(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \qquad (2.1)$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real numbers, $a_o \neq 0$, and x is a complex variable.

The Newton-Raphson method of Chapt. 1 can be modified so that it may be used to find the complex zeros of polynomials. We have $f(x) = P(x)$ so that the Newton-Raphson method has the form

$$x_{i+1} = x_i - \frac{P(x_i)}{P'(x_i)} \quad , i = 1, 2, \ldots$$

where the initial approximation x_1 is complex, $x_1 = \alpha_1 + i\beta_1$, $\beta_1 \neq 0$.

If $x_n = \alpha_n + i\beta_n$, $P(x_n) = A_n + iB_n$, $P'(x_n) = C_n + i\,D_n$, then we can show that

$$\alpha_{n+1} = \alpha_n - \frac{A_n C_n + B_n D_n}{C_n^2 + D_n^2}$$

$$\beta_{n+1} = \beta_n + \frac{A_n D_n - B_n C_n}{C_n^2 + D_n^2}$$

For $x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$ and by substitution we have

$$\alpha_{n+1} + i\beta_{n+1} = \alpha_n + i\beta_n - \frac{A_n + iB_n}{C_n + iD_n}$$

Rationalizing the denominator yields the desired result.

When using this method to find complex roots we must evaluate quantities such as $(\alpha + i\beta)^k$. This evaluation can certainly be accomplished using the binomial theorem. However it may be accomplished more readily by introducing polar coordinates and using the relation

$$(\alpha + i\beta)^k = r^k(\cos k\,\Theta + i \sin k\,\Theta)$$

where $\alpha = r \cos \Theta$ and $\beta = r \sin \Theta$.

We now discuss a method, which under certain conditions, allows us to find both real and complex roots of a polynomial equation, without any a priori information about the roots. This method is called Graeffe's root-squaring method. The development given here parallels that presented by Scarborough [7 , pages 223-243].

Upon investigation we note that the method is most successful when the roots of the polynomial are all real and unequal. In addition, the method easily handles up to two pairs of complex roots and gives some valuable information if the roots are real and of equal magnitude. In practice, we would first find all of the real roots of the original equation by the root-squaring process of Graeffe. If we were to remove these roots by synthetic division and the order of the remaining polynomial were two or four, then the complex root pairs could be found by examining the quadratic factors given by the root-squaring technique.

If the order of the remaining polynomial was greater than four we could obtain the roots by applying another technique, e.g., the Lin-Bairstow method which is explained later. This technique would be applied either to the original equation or to the reduced polynomial equation.

The principle of the root-squaring method is to transform the equation into an equation which has as its roots higher powers of the roots of the original equation. The roots of the transformed equation are said to be separated if the ratio of the magnitude of any root to the next larger is negligible in comparison with unity. The root-squaring process is continued until this separation of roots is obtained. When the process is programmed for a digital computer it is necessary to "tell" the computer how to recognize this separation.

Consider the general polynomial equation

$$P_n(x) = a_o x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n = 0 \qquad (2.2)$$

If $x_1, x_2, \dots, x_n$ are the roots of equation (2.2) we can rewrite it in the form

$$P_n(x) = a_o(x-x_1)(x-x_2) \dots (x-x_n) = 0 \qquad (2.3)$$

Multiply equation (2.3) by the function $(-1)^n P_n(-x)$,

$$(-1)^n P_n(-x) = (-1)^n a_o(-x-x_1)(-x-x_2) \dots (-x-x_n)$$

$$= a_o(x+x_1)(x+x_2) \dots (x+x_n)$$

to obtain

$$(-1)^n P_n(-x)P_n(x) = a_o^2 (x^2-x_1^2)(x^2-x_2^2) \dots (x^2-x_n^2) = 0 \quad (2.$$

Letting $y = x^2$ in equation (2.4) we have

$$\emptyset(x) = a_o^2 (y-x_1^2)(y-x_2^2) \dots (y-x_n^2) = 0$$

Clearly the roots of the above equation are the squares of the roots of the original equation (2.2). Thus, to form an equation whose roots are the squares of the original equation $P_n(x) = 0$, we multiply the original equation by $(-1)^n P_n(-x)$.

It is instructive to consider as an example the fourth degree equation

$$P_4(x) = a_o x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$$

Now

$$(-1)^4 P_4(-x) = a_o x^4 - a_1 x^3 + a_2 x^2 - a_3 x + a_4$$

Multiplying we have

$$(-1)^4 P_4(-x)P_4(x) = a_o^2 x^8 \begin{array}{r|r|r|l} -a_1^2 & x^6 + a_2^2 & x^4 - a_3^2 & x^2 + a_4^2 = 0 \\ +2a_o a_2 & -2a_1 a_3 & +2a_2 a_4 & \\ & +2a_o a_4 & & \end{array}$$

By considering other examples we would note that the coefficients of the transformed equations are generated in the same manner whether the degree of the polynomial is even or odd. In both cases the odd powers of x vanish. The procedure can be performed schematically. We carry out the multiplication as follows:

$$\begin{array}{cccccc}
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
a_0 & -a_1 & a_2 & -a_3 & a_4 & \cdots \\
\hline
a_0^2 & -a_1^2 & +a_2^2 & -a_3^2 & +a_4^2 & \cdots \\
 & +2a_0a_2 & -2a_1a_3 & +2a_2a_4 & -2a_3a_5 & \\
 & & +2a_0a_4 & -2a_1a_5 & +2a_2a_6 & \\
 & & & +2a_0a_6 & -2a_1a_7 & \cdots \\
 & & & & \ddots & \\
\hline
b_0 & b_1 & b_2 & b_3 & b_4 & \cdots
\end{array} \qquad (2.5)$$

The coefficients of the transformed equation are the sums b_0, b_1, ..., b_n of the several columns shown above. This process is repeated k times to obtain an equation whose roots are the $2k^{\underline{th}}$ power of the roots of the original equation.

First let's consider the case when the roots of equation (2.2) are all real and unequal. Let the order of the magnitude of the roots be

$$|x_1| > |x_2| > \ldots > |x_n|$$

and let the final transformed equation, i.e., the equation in which the roots are separated, be

$$Q(x) = b_o(x^m)^n + b_1(x^m)^{n-1} + \ldots + b_{n-1}(x^m) + b_n = 0 \quad (2.6)$$

The roots $x_1^m, x_2^m, \ldots, x_n^m$ and the coefficients $b_o, b_1, \ldots, b_n$ of equation (2.6) are related as follows:

$$\frac{b_1}{b_o} = -(x_1^m + x_2^m + \ldots x_n^m)$$

$$= -x_1^m(1 + \frac{x_2^m}{x_1^m} + \ldots + - \frac{x_n^m}{x_1^m})$$

$$\frac{b_2}{b_o} = x_1^m x_2^m + x_1^m x_3^m + \ldots + x_1^m x_n^m + x_2^m x_3^m + \ldots + x_{n-1}^m\ x_n^m$$

$$= x_1^m x_2^m(1 + \frac{x_3^m}{x_2^m} + \frac{x_4^m}{x_2^m} + \ldots + \frac{x_n^m}{x_2^m} + \frac{x_3^m}{x_1^m} + \ldots + \frac{x_{n-1}^m\ x_n^m}{x_1^m\ x_2^m})$$

$$\frac{b_3}{b_o} = -(x_1^m x_2^m x_3^m + x_1^m x_2^m x_4^m + \ldots + x_1^m x_2^m x_n^m + x_1^m x_3^m x_4^m + \ldots + x_{n-2}^m x_{n-1}^m x_n^m)$$

$$= -x_1^m x_2^m x_3^m(1 + \frac{x_4^m}{x_3^m} + \frac{x_5^m}{x_3^m} + \ldots + \frac{x_n^m}{x_3^m} + \frac{x_4^m}{x_2^m} + \ldots + \frac{x_n^m}{x_2^m} + \ldots + \frac{x_{n-2}^m\ x_{n-1}^m\ x_n^m}{x_1^m\ x_2^m\ x_3^m})$$

$\vdots$

$$\frac{b_n}{b_o} = (-1)^n\ x_1^m\ x_2^m \ldots x_n^m$$

Since the roots are separated the ratios $\frac{x_2^m}{x_1^m}, \frac{x_3^m}{x_1^m}, \ldots$

are negligible and we have the new relations

$$\frac{b_1}{b_o} \approx - x_1^m \quad \frac{b_2}{b_o} \approx x_1^m x_2^m \ldots \frac{b_k}{b_o} \approx (-1)^k x_1^m x_2^m \ldots x_k^m$$

$$\ldots \frac{b_n}{b_o} \approx (-1)^n x_1^m x_2^m \ldots x_n^m$$

By treating the above approximations as equations we can divide each of these by the preceeding equation to obtain

$$\frac{b_2}{b_1} \approx -x_2^m \quad \frac{b_3}{b_2} \approx -x_3^m \ldots \frac{b_k}{b_{k-1}} \approx -x_k^m \ldots \frac{b_n}{b_{n-1}} \approx -x_n^m \quad (2.7)$$

Using equations (2.7) and the equation $\frac{b_1}{b_o} \approx -x_1^m$ we have the linear factors

$$b_o x_1^m + b_1 \approx 0 \quad b_1 x_2^m + b_2 \approx 0 \ldots \quad b_{n-1} x_n^m + b_n \approx 0$$

We see, therefore, that the root-squaring process has broken up the original equation into n linear factors from which the approximate roots can be found with relative ease. We have in fact

$$|x_k|^m \approx \frac{|b_k|}{|b_{k-1}|}$$

Take the logarithm of both sides and multiply by $\frac{1}{m}$ to get

$$\log |x_k| \approx \frac{1}{m} (\log |b_k| - \log |b_{k-1}|)$$

or

$$|x_k| \approx e^{\frac{1}{m}(\log |b_k| - \log |b_{k-1}|)}$$

To determine the sign of x_k we substitute into the original equation (2.2).

We now ask the question how many root-squarings are necessary in order to insure that the eqs. (2.7) are indeed valid. Suppose an additional root-squaring is performed on Q(x) to obtain the equation

$$\overline{Q}(x) = \overline{b}_o(x^{2m})^n + \overline{b}_1(x^{2m})^{n-1} + \ldots + \overline{b}_{n-1}(x^{2m}) + \overline{b}_n = 0$$

whose roots are x_1^{2m}, x_2^{2m}, ..., x_n^{2m}. With the additional root-squaring we have separated the roots even further than before.

Now

$$\overline{b}_k \approx (-1)^k x_1^{2m} \ldots x_k^{2m} \overline{b}_o$$

from our known relations between the coefficients and the roots of a polynomial equation. We have $\overline{b}_o = b_o^2$ directly from the root-squaring process. Therefore

$$\overline{b}_k \approx (-1)^k (x_1^m)^2 \ldots (x_k^m)^2 b_o^2 \approx (-1)^k b_k^2$$

By examining the form of (2.5) it is evident that $\overline{b}_1 \approx - b_1^2$, $\overline{b}_2 \approx b_2^2$, ..., and $\overline{b}_n \approx (-1)^n b_n^2$ if the cross product terms in the root-squaring process are negligible in comparison to the squared terms. In this case further root-squaring is useless. It is possible that the coefficients will become "too large" for the computer before separation occurs. The programmer must provide a means for recognizing and allowing for such cases.

Graeffe's method was applied to several polynomial equations, all of whose roots were real and unequal. Complete numerical results are given in the appendix. This program and any further programs use eight-place arithmetic unless stated otherwise. For the benefit of the reader we list the polynomial equations to be solved, their actual roots, the approximate roots given by the root-squaring method, the number of root-squarings performed (RSP), and the functional values of the approximate roots. In each case the cross product terms became negligible which indicated that the criterion for separation was satisfied.

EXAMPLE 1. $P_3(x) = x^3 - 2x^2 - 5x + 6 = 0$

Actual roots: $x_1 = 3$, $x_2 = -2$, $x_3 = 1$

Approximate roots: $x_1 = 3.0000000$, $x_2 = -1.9999998$, $x_3 = 1.0000000$

RSP: 5

$f(x_1) = 0$, $f(x_2) = .000003$, $f(x_3) = 0$

EXAMPLE 2.

$$P_5(x) = 1.23x^5 - 2.52x^4 - 1.61x^3 + 1.73x^2 + 2.94x - 1.34 = 0$$

Actual roots: unknown

Approximate roots: $x_1 = 4.0657071$, $x_2 = -2.9916832$, $x_3 = 1.9587274$, $x_4 = -1.0284223$, $x_5 = .044463368$

RSP: 5

$f(x_1) = .0024924$, $f(x_2) = .001363$, $f(x_3) = .0000202$

$f(x_4) = -.000008$, $f(x_5) = 0.00000$

The sum of these roots is 2.04879 whereas it should be $2.52 / 1.23 = 2.04878$

EXAMPLE 3.

$$P_4(x) = x^4 - 5x^3 + 9.35x^2 - 7.750x + 2.4024 = 0$$

Actual roots: $x_1 = 1.4$, $x_2 = 1.3$, $x_3 = 1.2$, $x_4 = 1.1$

Approximate roots: $x_1 = 1.4000016$, $x_2 = 1.2999978$

$x_3 = 1.2000007$, $x_4 = 1.0999998$

RSP: 7

$f(x_1) = 0$, $f(x_2) = .0000001$, $f(x_3) = 0$, $f(x_4) = 0$

EXAMPLE 4. $P_3(x) = x^3 - 3.06x^2 + 3.1211x - 1.061106 = 0$

Actual roots: $x_1 = 1.03$, $x_2 = 1.02$, $x_3 = 1.01$

Approximate roots: $x_1 = 1.0299843$, $x_2 = 1.0200309$,

$x_3 = 1.0099847$

RSP: 10

$f(x_1) = 0$, $f(x_2) = 0$, $f(x_3) = 0$

EXAMPLE 5. $P_3(x) = x^3 - 3.006x^2 + 3.012011x - 1.00601106 = 0$

Actual roots: $x_1 = 1.003$, $x_2 = 1.002$, $x_3 = 1.001$

The polynomial actually examined in example 5 was $\bar{P}_3(x) = x^3 - 3.006x^2 + 3.012011x - 1.0060110 = 0$, because the program was written for eight place arithmetic, i.e., the constant term of $P_3(x)$ was rounded to eight significant figures. The approximate roots listed then are actually approximations to the real roots of $\bar{P}_3(x)$.

Approximate roots: $x_1 = 1.0034118$, $x_2 = 1.0027331$, $x_3 = 0.99985752$

RSP: 11

$f(x_1) = 0$, $f(x_2) = 0$, $f(x_3) = 0$.

Since the functional values are all zero we conclude that the roots so obtained are quite close to the actual roots of $\bar{P}_3(x)$.

We now consider the case when the polynomial equation has some complex roots and the equation cannot then be expressed as a product of linear factors with real coefficients. Instead the factored form of the equation is a product of real linear and real quadratic factors.

Consider for example an equation having two distinct real roots, x_1, x_3, and a pair of complex roots $re^{i\theta}$, $re^{-i\theta}$, such that $|x_1| > r > |x_3|$. Then the equation having these as roots is

$$(x-x_1)\ (x - re^{i\theta})\ (x - re^{-i\theta})\ (x-x_3) = 0$$

An equation whose roots are the mth powers of the roots of this equation is

$$(y-x_1{}^m)\ (y - r^m e^{im\theta})\ (y - r^m e^{-im\theta})\ (y-x_3{}^m) = 0$$

or

$$y^4 - (x_1{}^m + r^m e^{im\theta} + r^m e^{-im\theta} + x_3{}^m)y^3$$

$$+ (x_1{}^m\ r^m\ e^{im\theta} + x_1{}^m\ r^m\ e^{-im\theta} + \ldots)y^2$$

$$- (x_1{}^m\ r^m\ e^{im\theta}\ r^m\ e^{-im\theta} + \ldots)y$$

$$+ (x_1{}^m\ r^m\ e^{im\theta}\ r^m\ e^{-im\theta}\ x_3{}^m) = 0$$

Taking out $x_1{}^m$, $x_1{}^m r^m$, $x_1{}^m r^{2m}$, $x_1{}^m r^{2m} x_3{}^m$ and neglecting the ratios $\frac{r^m}{x_1{}^m}$, $\frac{x_3{}^m}{x_1{}^m}$, $\frac{x_3{}^m}{r^m}$ (the roots being separated) we have

$$y^4 - x_1{}^m y^3 + 2x_1{}^m r^m \cos m\theta\, y^2 - x_1{}^m r^{2m} y + x_1{}^m r^{2m} x_3{}^m = 0 \quad (2.8)$$

We now separate equation (2.8) into quadratic and linear factors from which we can approximate the real and complex roots, i.e.,

$$y - x_1^m \approx 0$$

$$-x_1^m y^2 + 2x_1^m r^m \cos m\theta y + x_1^m r^{2m} \approx 0$$

$$-x_1^m r^{2m} y + x_1^m r^{2m} x_3^m \approx 0$$

Suppose we apply the root squaring once more.

	y^4	y^3	y^2	y^1	y^0
m^{th}	1	$-x_1^m$	$2x_1^m r^m \cos m\theta$	$-x_1^m r^{2m}$	$x_1^m r^{2m} x_3^m$
	1	$-x_1^{2m}$	$4x_1^{2m} r^{2m} \cos^2 m\theta$	$-x_1^{2m} r^{4m}$	$x_1^{2m} r^{4m} x_3^{2m}$
		$2x_1^m r^m \cos m\theta$	$-2x_1^{2m} r^{2m}$	$4x_1^{2m} r^{3m} x_3^m \cos m\theta$	
			$2x_1^m r^{2m} x_3^m$		
$2m^{th}$	1	$-x_1^m$	$4x_1^{2m} r^{2m} \cos^2 m\theta$	$-x_1^{2m} r^{4m}$	$x_1^{2m} r^{4m} x_3^{2m}$
			$-2x_1^{2m} r^{2m}$		

Note that all the doubled products in the first row are not negligible. Furthermore since $2\cos^2 m\theta - 1 = \cos 2m\theta$ we can rewrite the final coefficient of y^2 as $2x_1^{2m} r^{2m} \cos 2m\theta$. Thus the final transformed equation is

$$y^4 - x_1^{2m} y^3 + 2x_1^{2m} r^{2m} \cos 2m\theta \, y^2 - x_1^{2m} r^{4m} y + x_1^{2m} r^{4m} x_3^{2m} = 0$$

Comparing this with the equation for the $m^{\underline{th}}$ roots we see that the root-squaring has doubled the amplitudes of the complex roots. Thus the cosine of the phase angle may change signs frequently and this may be used to indicate complex roots. However the presence of complex roots is probably most easily detected by the fact that the doubled cross-product terms of the first row do not all disappear.

Let us consider a couple of typical examples and use the relationship between the roots and the coefficients of an equation to aid us in the computation of the complex roots. As written, the program gives only real roots and not complex roots. The program does however give the necessary quadratic factors and with additional programming it would carry out all the operations done by hand in the following two examples.

EXAMPLE 6. Find all the roots of the equation $x^3 - 3x^2 + 4x - 5 = 0$. The root-squaring stopped with the 32nd power of the roots, and the original equation has been broken into one linear and one quadratic factor. From the linear factor we have $x_1 = 2.2134112$

In order to obtain the complex roots we recall that the roots $x^2 + bx + c = 0$ may be written as $re^{i\theta}$, $re^{-i\theta}$. Then

$$x^2 + bx + c = (x - re^{i\theta})(x - re^{-i\theta})$$

$$= x^2 - r(e^{i\theta} + e^{-i\theta})x + r^2$$

$$= x^2 - 2r\cos\theta x + r^2$$

i.e., the absolute term in the quadratic is equal to the square of the modulus of the complex roots. Then we may readily evaluate the modulus r.

As the quadratic factor in the above example we have

a) $1.1015091 \times 10^{11} y^2 - 5.8707920 \times 10^{17} y + 2.3283064 \times 10^{22} = 0.$

The modulus of the complex roots of (a) is actually the 32^{nd} power of the modulus of the complex roots of the original equation.

Therefore

$$r^{64} = \frac{2.3283064 \times 10^{11}}{1.1015091},$$

$$\log r = \frac{11 + .36698 - .04218}{64}$$

$$= .1769$$

or $$r = 1.503.$$

Now let the complex pair be denoted by $u \pm iv$. The sum of the roots of the given equation is $-(-3/1) = 3$.

Thus

$$x_1 + 2u = 3$$

or $$u = \frac{3 - 2.2134112}{2}$$

$$= .3933 ,$$

and $$v = \sqrt{r^2 - u^2}$$

$$= \sqrt{2.259 - .155}$$

$$= 1.45$$

The complex roots are then $.3933 \pm 1.45i$.

In the following example we illustrate the application of the root squaring process to an equation with four complex roots.

EXAMPLE 7. Consider the polynomial

$$P_6(x) = x^6 + 3x^5 - x^4 - 7x^3 + 10x^2 + 14x - 20 = 0$$

which has the roots 1, $1 \pm i$, -2, $-2 \pm i$.

We apply Graeffe's method to get approximations to the real roots of the above equation. The process was stopped after the sixth root-squaring since another root-squaring would have produced coefficients that would be too large for the computer. In this problem we obtained the real roots $x_1 = -2.0000084$ and $x_2 = .99999951$, with $f(x_1) = .252 \times 10^{-3}$, $f(x_2) = -.15 \times 10^{-4}$. By synthetic division we reduced the original polynomial to one of order four with only complex roots.

We obtained

$$P_4(x) = x^4 + 1.9999911x^3 - 1.0000014x^2 - 1.9999928x + 10.000001 = 0$$

We performed six root-squarings on this equation and this resulted in the two quadratic factors

$$y^2 - .7950482 \times 10^{22} y + .54204046 \times 10^{45} = 0$$

and $.54204046 \times 10^{45}\, y^2 + .46563726 \times 10^{55}\, y + .10000064 \times 10^{65} = 0$

From the first quadratic factor

$$r_1^{128} = 5.4210086 \times 10^{44},$$

or $\log r_1 = \frac{44.73404}{128} = .349485,$

or $r_1 = 2.236$ $\qquad (r_1^2 = 5)$

Using the second quadratic factor we obtained

$$r_2^{128} = \frac{10^{64}}{5.4210086 \times 10^{44}}$$

or $$\log r_2 = \frac{64-44.73405}{128} = .150515$$

or $$r_2 = 1.414 \qquad (r_2^2 = 2)$$

Let the complex roots be $u_1 \pm iv_1$, $u_2 \pm iv_2$, and since the sum of the roots is approximately -2 we have

$$2u_1 + 2u_2 = -2$$

or $$u_1 + u_2 = -1 \qquad (2.9)$$

The relationship between the coefficients and the reciprocals of the roots may be used to obtain

$$\frac{1}{u_1+iv_1} + \frac{1}{u_1-iv_1} + \frac{1}{u_2+iv_2} + \frac{1}{u_2-iv_2} = \frac{1}{5}$$

Rationalize the denominators of the complex terms and, since $u_1^2 + v_1^2 = r_1^2$, $u_2^2 + v_2^2 = r_2^2$, we have

$$\frac{2u_1}{r_1^2} + \frac{2u_2}{r_2^2} = \frac{1}{5}$$

or $$\frac{2u_1}{5} + u_2 = \frac{1}{5} \qquad (2.10)$$

(2.9) and (2.10) may be solved simultaneously to obtain

$$u_1 = -2, \; u_2 = 1$$

Now $$v_1 = \sqrt{r_1^2-u_1^2} = \sqrt{5-4} = 1,$$

$$v_2 = \sqrt{r_2^2-u_2^2} = \sqrt{2-1} = 1$$

and hence the two pairs of complex roots are

$$-2 \pm i \text{ and } 1 \pm i$$

If more than two pair of complex roots occur the difficulties encountered in using Graeffe's method are nearly insurmountable. Hence, in the case of three or more pairs of complex roots we must turn either to the Newton-Raphson method for complex roots or to the Lin-Bairstow method which is discussed later. We will find that the Lin-Bairstow method does not require the use of complex arithmetic to find the complex root pairs of polynomials.

We now consider the effectiveness of the Graeffe method for the solution of polynomial equations whose roots are multiple real roots. Since such roots are equal in magnitude, no amount of squaring would separate them. The original equation can be broken down into linear equations for the real and unequal roots and quadratic equations for pairs of real roots of equal magnitude. The presence of two real roots of equal magnitude is noted by the nonvanishing of cross-product terms. These cross-product terms, in this case, approach a value equal to half the squared term.

The possible real roots given by our present computer program are arrived at by considering only the linear fragments. This program may not be used to find real roots of equal magnitude, since we must consider quadratic factors. The program does however give the coefficients of the quadratic factors and in the following examples we worked with these factors in determining the roots. The computer program could readily be modified to determine real multiple roots.

We consider the following

EXAMPLE 8. $P_4(x) = x^4 - 4x^3 - .75x^2 + 16.25x - 12.5 = 0$ has the roots 2.5, 2.5, -2., 1.

The process was stopped after six root-squarings since another root-squaring would have made the coefficients too large for the computer to handle. The final equation should be broken into one quadratic factor and two linear factors. The quadratic factor is easily detected by noticing that the second coefficient of this final transformed equation, $.5877 \times 10^{26}$, is just half the square of the corresponding coefficient, $.1085 \times 10^{14}$, of the preceeding equation. Hence the quadratic factor is

$$y^2 + .5877 \times 10^{26} y + .8636 \times 10^{51} = 0$$

and since the roots are known to be equal and since their product is equal to the constant term of the quadratic, we have

$$(x^2)^{64} = x^{128} = .8636 \times 10^{51}$$

Using logarithms we get $x = |2.5|$ and by testing the values 2.5 and -2.5, we see that $x_{1,2} = 2.5$. The approximate root +1, with a functional value of zero, is given us by one linear fragment, while the other linear fragment gives us $\pm .5657$, neither of which has a negligible functional value. This presents no problem however. We just use the relationship between the coefficients and the roots of the original equation, i.e., $x_1 + x_2 + x_3 + x_4 = -(-4/1) = 4$

or $\quad x_4 = -2$. As a check we have

$$x_1x_2x_3x_4 = -12.5 = -12.5/1$$

EXAMPLE 9. $P_4(x) = x^4 - 4.5x^3 + 5.5x^2 - 2 = 0$ has the roots 2, 2, 1, and $-\frac{1}{2}$.

Seven root-squarings were performed. The quadratic factor is $y^2 + .6806 \times 10^{39}y + .1158 \times 10^{78} = 0$ since $.6806 \times 10^{39}$ is just half the square of $.3689 \times 10^{20}$. As above we have

$$x_{1,2} = 2$$

and -.5, where f(-.5) = 0, is given by a linear fragment. We have $\pm$ .707 as the other approximate root, but again the functional values are not negligible. In this case $x_1 + x_2 + x_3 + x_4 = 4.5$ or $x_3 = 1$.

EXAMPLE 10.

$$P_5(x) = x^5 + 1.5x^4 - 2.5x^3 - 6.5x^2 - 4.5x - 1. = 0$$

has the roots 2., -1, -1, -1, and $-\frac{1}{2}$.

In this example we are examining a polynomial equation with three roots of equal magnitude. No quadratic factors are possible in this case but Graeffe's method is still of great value.

Eight root-squaring were performed. As approximate roots we obtain $x_1 = 1.9999999$, $x_2 = -1$, $x_3 = -.99571775$, with $f(x_1) = -.75 \times 10^{-5}$, $f(x_2) = 0$, $f(x_3) = .1 \times 10^{-6}$.

Now the three approximate roots could be removed from the original equation using synthetic division and the remaining two real roots could be approximated by solving the resulting quadratic equation.

Hence, we can safely say that the Graeffe method gives much valuable information about the roots of polynomial equations regardless of the distribution of these roots.

Carvallo [Resolution Numerique des Equations, page 24] has extended Graeffe's method to the solution of transcendental equations by expanding the equation into a Taylor series, neglecting the remainder term, and then treating the resulting polynomial as an algebraic equation.

A more general method of finding the complex roots of a polynomial equation is the Lin-Bairstow method. The procedure is to find a quadratic factor $x^2 + \alpha x + \beta$ of the polynomial by an iterative process. If we divide $P_n(x)$ by an initial guess at our factor, say $x^2 + rx + s$, we obtain, as a quotient, a polynomial $Q_{n-2}(x)$ of degree n-2 and a remainder $Rx + S$. We therefore write

$$P_n(x) = \sum_{k=0}^{n} a_k x^{n-k} = (x^2 + rx + s) \sum_{k=0}^{n-2} b_k x^{n-k-2} + Rx + S \qquad (2.11)$$

It follows then that

$$\begin{aligned} a_o &= b_o \\ a_1 &= b_1 + rb_o \\ a_2 &= b_2 + rb_1 + sb_o \qquad (2.12) \\ &\vdots \end{aligned}$$

$$a_k = b_k + rb_{k-1} + sb_{k-2}$$
$$\vdots$$
$$a_{n-1} = R + rb_{n-2} + sb_{n-3}$$
$$a_n = S + sb_{n-2}$$

This is easily seen by multiplying out and matching coefficients or by considering the synthetic division scheme for a quadratic factor given below:

$$\begin{array}{cccccccc|l} a_o & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n & \underline{r \quad s} \\ & & sb_o & sb_1 & & sb_{n-4} & sb_{n-3} & sb_{n-2} & \\ & rb_o & rb_1 & rb_2 & & rb_{n-3} & rb_{n-2} & & \\ \hline b_o & b_1 & b_2 & b_3 & \cdots & b_{n-2} & R & S & \end{array}$$

By setting $b_{-1} = b_{-2} = 0$, $b_{n-1} = R$, and $b_n = S - rR$ (2.13)

equations (2.12) can be written as

$$b_k = a_k - rb_{k-1} - sb_{k-2} \qquad k = 0, 1, 2, \ldots, n \quad (2.14)$$

R and S then are functions of r and s and we now try to solve the simultaneous nonlinear equations

$$R(r, s) = 0 \quad \text{and} \quad S(r, s) = 0$$

by an iterative procedure. If $\bar{r}$ and $\bar{s}$ satisfy the system then $x^2 + \bar{r}x + \bar{s}$ is the factor of $P_n(x)$ which we are seeking. To find $\bar{r}$ and $\bar{s}$ we suppose that r and s are such that

$$\bar{r} = r + \Delta r$$
$$\bar{s} = s + \Delta s$$

where Δr and Δs are small. Let us use Taylor's expansion for functions of two variables and neglect second and higher powers of Δr and Δs, to obtain

$$R(r,s) + \frac{\partial R}{\partial r}\Delta r + \frac{\partial R}{\partial s}\Delta s \approx R(\bar{r},\bar{s}) = 0$$

$$S(r,s) + \frac{\partial S}{\partial r}\Delta r + \frac{\partial S}{\partial s}\Delta s \approx S(\bar{r},\bar{s}) = 0 \qquad (2.15)$$

We now find the partial derivatives in equations (2.15) and solve these equations for Δr and Δs.

Differentiate equation (2.14) to get

$$\frac{\partial b_k}{\partial r} = -b_{k-1} - r\frac{\partial b_{k-1}}{\partial r} - s\frac{\partial b_{k-2}}{\partial r}$$

$$\frac{\partial b_k}{\partial s} = -b_{k-2} - r\frac{\partial b_{k-1}}{\partial s} - s\frac{\partial b_{k-2}}{\partial s} \qquad (2.16)$$

We now have the

THEOREM 2.1 $\frac{\partial b_k}{\partial r} = \frac{\partial b_{k+1}}{\partial s}$ for $k = 0, 1, \ldots, n-1$

PROOF. Since $b_o = a_o$, it is a constant function of r and s; hence from equations (2.16) we have

$$\frac{\partial b_o}{\partial r} = 0 \qquad \frac{\partial b_1}{s} = 0$$

$$\frac{\partial b_1}{\partial r} = -b_o \qquad \frac{\partial b_2}{\partial s} = -b_o - r\frac{\partial b_1}{\partial s} = -b_o$$

$$\frac{\partial b_2}{\partial r} = -b_1 - r\frac{\partial b_1}{\partial r} \qquad \frac{\partial b_3}{\partial s} = -b_1 - r\frac{\partial b_2}{\partial s} - s\frac{\partial b_1}{\partial s}$$

$$= -b_1 + rb_o \qquad = -b_1 + rb_o$$

Thus the theorem is true for k = 0, 1, 2.

Suppose that the theorem holds for all k up to m-1. Then by equations (2.16)

$$\frac{\partial b_m}{\partial r} = -b_{m-1} - r\frac{\partial b_{m-1}}{\partial r} - s\frac{\partial b_{m-2}}{\partial s}$$

$$= -b_{m-1} - r\frac{\partial b_m}{\partial s} - \frac{\partial b_{m-1}}{\partial s} = \frac{\partial b_{m+1}}{\partial s}$$

and thus it holds for m.

DEFINITION 2.1 $\quad -c_{k-1} = \frac{\partial b_k}{\partial r} = \frac{\partial b_{k+1}}{\partial s}$ $(k= 0, 1, \ldots, n-1)$

One may now make use of Definition 2.1 to write a single recurrence relation in place of equations (2.16), i.e.,

$$c_k = b_k - rc_{k-1} - sc_{k-2} \tag{2.17}$$

and in particular $c_{-1} = 0$ anc $c_o = b_o$. Thus we note that the c's are obtained from the b's in exactly the same way as the b's were obtained from the a's.

Using equations (2.13) and Theorem 2.1 we have

$$R = b_{n-1}$$

$$\frac{\partial R}{\partial r} = \frac{\partial b_{n-1}}{\partial r} = -c_{n-2}$$

$$\frac{\partial R}{\partial s} = \frac{\partial b_{n-1}}{\partial s} = -c_{n-3}$$

and

$$S = b_n + rb_{n-1}$$

$$\frac{\partial S}{\partial r} = \frac{\partial b_n}{\partial r} + b_{n-1} + r\frac{\partial b_{n-1}}{\partial r} = -c_{n-1} - r\,c_{n-2} + b_{n-1}$$

$$\frac{\partial S}{\partial s} = \frac{\partial b_n}{\partial s} + r\frac{\partial b_{n-1}}{\partial s} = -c_{n-2} - r\,c_{n-3}$$

We can now solve for Δr and Δs, and in fact,

$$c_{n-2}\,\Delta r + c_{n-3}\,\Delta s = b_{n-1}$$

$$(c_{n-1} - b_{n-1})\,\Delta r + c_{n-2}\,\Delta s = b_n$$

Having solved for Δr and Δs we add these values to r and s to improve the estimates for $\bar{r}$ and $\bar{s}$. The procedure is repeated until a quadratic factor $x^2 + \bar{r}x + \bar{s}$ is found with sufficient accuracy; then two roots of the given equation are determined by setting $x^2 + \bar{r}x + \bar{s}$ equal to zero.

The development given here can be found in Kunz [3, pages 34-37].

The computer program as written below finds all of the roots, both real and complex, of a polynomial equation. The procedure is to find a quadratic factor of the original equation, remove this factor, and then search for a quadratic factor of the remaining polynomial of reduced degree. This process is repeated until the remaining polynomial is of degree one or two. In either case the roots of this final polynomial are easily extracted.

Again, some examples were run using this program. The usual choices for r and s were both zero, and in only one case did the procedure fail to converge with x^2 as our trial factor. In the case of nonconvergence, $x^2 + 2x + 2$ was used as the initial guess, and the method then converged. When it converges, Bairstow's method has the characteristic rapid convergence of the Newton-Raphson method.

In the search for each quadratic factor the iterative procedure was continued until $|r_{i+1} - r_i| < \epsilon$ and also $|s_{i+1} - s_i| < \epsilon$, where again ϵ is chosen to insure a prescribed accuracy in the approximate roots. The ϵ used in each example is given in parenthesis following the statement of the problem.

EXAMPLE 11. $P_3(x) = x^3 - x - 1 = 0$ $\qquad (.1 \times 10^{-4})$

In this example we chose x^2 as the trial quadratic factor, i.e., $r = s = 0$. The matrix of coefficients for Δr and Δs was singular. Therefore we used $x^2 + 2x + 2$ as the trial factor and then arrived at the necessary quadratic factor. The approximate roots are $x_{1,2} = -.66235900 \pm .56227950i$ and $x_3 = 1.3247180$, $f(x_3) = 0$. $x_1 + x_2 + x_3 = 0$ as it should be.

EXAMPLE 12.

$P_5(x) = x^5 - 17x^4 + 124x^3 - 508x^2 + 1035x - 875 = 0$

$(.1 \times 10^{-3})$

Actual roots: $x_{1,2} = 2 \pm i$, $x_{3,4} = 3 \pm 4i$, $x_5 = 7$.

Approximate roots: $x_{1,2} = 2.0000004 \pm .99999945i$

$x_{3,4} = 2.9999872 \pm 4.0000034i$, $x_5 = 7.0000260$

$f(x_5) = .02425$

EXAMPLE 13 (a)

$P_6(x) = 3.26x^6 + 4.2x^4 + 3.08x^3 - 7.16x^2 + 1.92x - 7.76 = 0$ $(.1 \times 10^{-4})$

This problem is taken from Scarborough [, page 257]. He gives as answers $x_{1,2} = -.051040 \pm .94212i$, $x_3 = 1.06393$ $x_4 = -1.31327$, $x_{5,6} = .17571 \pm 1.37214i$

Approximate roots: $x_{1,2} = -.056091180 \pm .94183490i$,

$x_3 = 1.0639999$, $x_4 = -1.3182197$, $x_{5,6} = .18320110 \pm 1.3685389i$, $f(x_3) = .0000427$, $f(x_4) = -.0000066$

The agreement in the above example is not too good, yet the sum of the approximate roots is $.4 \times 10^{-7} \approx -(0/1)$

EXAMPLE 13 (b) $(.1 \times 10^{-7})$

The same problem was run with a smaller epsilon. In this case the approximate roots are

$x_{1,2} = -.056091140 \pm .94183495i,\ x_3 = 1.0639989$

$x_4 = -1.3182197,\ x_{5,6} = .18320156 \pm 1.36853861$

$f(x_3) = -.0000017,\ f(x_4) = -.0000066$

EXAMPLE 13 (c) $(.1 \times 10^{-4})$

We now used sixteen place arithmetic and the original epsilon. The approximate roots, truncated to eight figures, are $x_{1,2} = -.056091172 \pm .94183493i,\ x_3 = 1.0639998$

$x_4 = -1.3182197,\ x_{5,6} = .18320110 \pm 1.3685389i$

$f(x_3) = 0,\ f(x_4) = -.0000066$

EXAMPLE 14.

$P_7(x) = x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0 \quad (.1 \times 10^{-4})$

This is an example in Scarborough and his answers rounded to three or four decimal places are

$x_{1,2} = .3028 \pm 1.018i,\ x_3 = 1.1080,\ x_4 = -1.9625$

$x_{5,6} = -.6445 \pm 1.118i,\ x_7 = 1.5379$

The approximate roots rounded to the same number of significant figures are $x_{1,2} = .3046 \pm .9919i,\ x_3 = 1.1080$

$x_4 = -1.9625,\ x_{5,6} = -.6463 \pm 1.117i,\ x_7 = 1.5379$

$f(x_3) = .0000072,\ f(x_4) = .0000111,\ f(x_7) = .0000115$

Note the exact agreement of the real roots.

EXAMPLE 15.

$$P_8(x) = x^8 + 20.4x^7 + 151.3x^6 + 490x^5 + 687x^4 + 719x^3 + 150x^2 + 109x + 6.87 = 0 \quad (.1 \times 10^{-4})$$

This also is an example in Scarborough and as answers he gives

$x_{1,2} = .002818 \pm .413i$, $x_3 = -.0674$, $x_4 = -7.78$
$x_{5,6} = -.6678 \pm 1.322i$, $x_{7,8} = -5.604 \pm 1.891i$

The approximate roots given by the Bairstow method are

$x_{1,2} = .002829 \pm .413i$, $x_3 = -.0674$, $x_4 = -7.79$
$x_{5,6} = -.6678 \pm 1.322i$, $x_{7,8} = -5.608 \pm 1.875i$ where $f(x_3) = .0002568$ and $f(x_4) = -.0520949$. The agreement in this example is quite good, both for the real and complex roots.

The last example has three pair of complex roots. Yet no difficulties were encountered in finding approximations to the roots. This same problem is unmanageable with Graeffe's method.

A few examples run with Graeffe's method were rerun using the Lin-Bairstow method, in order that a comparison could be made. In particular, examples 6, 7, and 4 were rerun. The final results are given below with computer time in seconds (TIS) included.

EXAMPLE 6 (b) $P_3(x) = x^3 - 3x^2 + 4x - 5 = 0$ $(.1 \times 10^{-4})$

Approximate roots (Graeffe): $x_1 = 2.2134112$, $x_{2,3} = .3933 \pm 1.451i$

TIS (Graeffe): 33.1

Approximate roots (Bairstow): $x_1 = 2.2134125$, $x_{2,3} = .3933 \pm 1.451i$

$f(x_1) = .0000045$

TIS (Bairstow): 14.5

EXAMPLE 7 (c)

$P_6(x) = x^6 + 3x^5 - x^4 - 7x^3 + 10x^2 + 14x - 20 = 0$ $(.1 \times 10^{-4})$

Actual roots: 1, 1 $\pm$ i, -2, -2 $\pm$ i

Approximate roots (Graeffe): .99999951, 1 $\pm$ i, -2.0000084, -2 $\pm$ i

TIS (Graeffe): 96.1

Approximate roots (Bairstow): 1, 1 $\pm$ i, -1.9999998, -2 $\pm$ i

TIS (Bairstow): 37.5

EXAMPLE 4 (b)

$P_3(x) = x^3 -3.06x^2 + 3.1211x -1.061106 = 0$ $(.1 \times 10^{-4})$

Actual roots: 1.01, 1.02, 1.03

The approximate roots in this example are rounded to five significant figures.

Approximate roots (Graeffe): 1.0100, 1.0200, 1.0300

TIS (Graeffe): 44.0

Approximate roots (Bairstow): 1.0098, 1.0204, 1.0298

$f(1.0098) = -.0000001$, $f(1.0204) = 0$, $f(1.0298) = 0$

TIS (Bairstow): 24.3

When seeking the complex roots of polynomials, it is frequently of interest to determine the sign of the real part of the complex roots. The Lin-Bairstow method then is certainly very useful as it not only gives the signs of the real and imaginary parts but also approximates the magnitudes with favorable accuracy.

A polynomial in which a small change in a coefficient may cause a significant change in one or more zeros is called ill-conditioned. By significant we mean either a change from a real to a complex root or a change such that the magnitude of a root increases appreciably. As a simple example the equation

$x^2 - 8x + 16 = 0$ has a double root $x = 4$,

$x^2 - 8x + 16.01 = 0$ has complex roots $x_{1,2} = 4 \pm \frac{i}{10}$.

The problem of determining the roots of ill-conditioned polynomials arises quite frequently in numerical work. The coefficients of these polynomials may arise from empirical data, in which case we do not know the exact value of the coefficients or we may know the exact value of the coefficients but may find it necessary to round them when inserting them into the computer.

One coefficient of the polynomial in example 5 was rounded and we noted the presence of the unfavorable approximations to the actual roots. In this case the change was not too extreme.

Ralston [6 , page 379] considered a more sophistacated example. The polynomial equation $P_{20}(z) = (z+1)(z+2) \ldots (z+20) = 0$ has as roots -1, -2, ..., -20. We then consider $P_{20}(z) + 2^{-23} z^{19} = 0$ and the roots are now -1, -2, -3, -4, -4.999999928, -6.000006944, -8.007267603, -8.917250249, -20.84690810, -10.095266145 ± 0.643500904i, -11.793633881 ± 1.652329728i, -13.992358137 ± 2.518830070i, -16.730737466 ± 2.812624894i, and -19.502439400 ± 1.940330347i. In this example not only are the changes substantial but half of the roots become complex.

By using as many places of accuracy in the computer as possible the error from ill-conditioned polynomials is reduced. Example 5 was rerun using sixteen place arithmetic instead of eight. The approximate roots were truncated to eight significant figures. The results are given in

EXAMPLE 5 (b).

$P_3(x) = x^3 - 3.006x^2 + 3.012011x - 1.006011006 = 0 \quad (.1 \times 10^{-4})$

Actual roots: 1.003, 1.002, 1.001

Approximate roots (Graeffe): $x_1 = 1.0030000$, $x_2 = 1.0019999$,

$x_3 = 1.0009999$

RSP: 13

$f(x_1) = .69 \times 10^{-13}$, $f(x_2) = 0$, $f(x_3) = -.68 \times 10^{-12}$

TIS (Graeffe): 72.7

Approximate roots (Bairstow): 1.0030082, 1.0019908, 1.0010008

TIS (Bairstow): 44.0

The approximate roots are now satisfactory.

Chapter 3

The real roots of n simultaneous nonlinear equations in n unknowns can be found by several methods. Two such methods will be outlined in this chapter. One is a direct extension of the Newton-Raphson method for a single equation in a single unknown and the other is based on the numerical solution of a properly chosen initial value problem. Each method is described only for the case of two equations in two unknowns, however, each method may be generalized to the case of n equations in n unknowns.

Let the given nonlinear equations be

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned} \tag{3.1}$$

where (ξ, η) is the solution.

If (x_1, y_1) is an approximation to the solution and h, k are the corrections such that

$$\xi = x_1 + h$$

$$\eta = y_1 + k$$

then

$$\begin{aligned} f(x_1+h, y_1+k) &= 0 \\ g(x_1+h, y_1+k) &= 0 \end{aligned} \tag{3.2}$$

Assuming that f and g are sufficiently differentiable, we expand equations (3.2) about (x_1, y_1) using Taylor's series for functions of two variables. We have

$$f(x_1+h,\ y_1+k) = f(x_1,y_1) + hf_x(x_1,y_1) + kf_y(x_1,y_1) + \ldots$$

(3.3)

$$g(x_1+h,\ y_1+k) = g(x_1,y_1) + hg_x(x_1,y_1) + kg_y(x_1,y_1) + \ldots$$

If $(x_1,\ y_1)$ is "sufficiently close" to the solution (ξ,η), i.e., if h and k are sufficiently small, we can neglect higher order terms so that equations (3.3) become simply

$$f(x_1,y_1) + h_1 f_x(x_1,y_1) + k_1 f_y(x_1,y_1) = 0$$

(3.4)

$$g(x_1,y_1) + h_1 g_x(x_1,y_1) + k_1 g_y(x_1,y_1) = 0$$

Using Cramer's rule to solve (3.4) for the approximations h_1, k_1 of h and k we obtain

$$h_1 = \frac{-f(x_1,y_1)g_y(x_1,y_1) + g(x_1,y_1)f_y(x_1,y_1)}{J(f_1,g_1)}$$

$$k_1 = \frac{-g(x_1,y_1)f_x(x_1,y_1) + f(x_1,y_1)g_x(x_1,y_1)}{J(f_1,g_1)}$$

provided $J(f,\ g) \neq 0$ where

$$J(f_i,g_i) = f_x(x_i,y_i)\ g_y(x_i,y_i) - f_y(x_i,y_i)\ g_x(x_i,y_i)$$

Then $x_2 = x_1 + h_1$, $y_2 = y_1 + k_1$ and $(x_2,\ y_2)$ is the new approximation to the solution (ξ,η). We expect $(x_2,\ y_2)$ to be closer to the solution (ξ,η) than $(x_1,\ y_1)$. The iteration formula for the approximations to the roots then has the form

$$x_{i+1} = x_i + h_i$$

$$= x_i - \left[\frac{fg_y - gf_y}{J(f,g)}\right]_i$$

$$y_{i+1} = y_i + k_i$$

$$= y_i - \left[\frac{gf_x - fg_x}{J(f,g)}\right]_i$$

where all functions involved are evaluated at (x_i, y_i).

Ralston [6 , pages 348-350] extended this method to n equations in n unknowns.

Consider the following

EXAMPLE 1. Compute by the Newton-Raphson method two real solutions of the equations

$$f(x, y) = x + 3 \log_{10}x - y^2 = 0$$

$$g(x, y) = 2x^2 - xy - 5x + 1 = 0$$

(This example is taken from Scarborough [ ].)

The FORTRAN program and the corresponding computer results are given in the appendix. The iteration was continued until $|x_{i+1} - x_i| < \varepsilon$ and $|y_{i+1} - y_i| < \varepsilon$, where ε is chosen to insure a prescribed accuracy in the approximate roots. For this problem we let $\varepsilon = .1 \times 10^{-5}$. As an initial approximation to the roots we used (3.4, 2.2). The method converged in four iterates and gave as approximate roots, $x_1 = 3.4874404$, $y_1 = 2.2616242$ where $f(x_1, y_1) = .0000128$, $g(x_1, y_1) = .0000006$. Another initial approximation (1.4, -1.5) was employed and again the method converged in four iterates, but this time to a different solution, $x_2 = 1.4588911$, $y_2 = -1.3967658$ where

$f(x_2, y_2) = .000013$, $g(x_2, y_2) = .0000002$.

We recall that this method was applied effectively in the Lin-Bairstow method to find Δr and Δs.

The "first-order" iteration method is also easily extended to simultaneous nonlinear equations. For a complete account of this extension the reader is referred to Scarborough [7 , pages 217-221].

We now consider a second method which is based on the numerical solution of initial value problems, which are solved quite easily on a computer. Suppose we are given the equation

$$f(x, y) = 0 \tag{3.5}$$

To find a differential equation which has $f(x, y) = 0$ as its solution we proceed as follows.

We differentiate $f(x, y)$ with respect to x, set this derivative equal to zero, and solve for $\frac{dy}{dx}$, i.e.,

$$f_x + f_y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{f_x}{f_y} \tag{3.6}$$

The general solution of equation (3.6) is $f(x, y) = c$, where c is an arbitrary constant. We impose an initial condition $y(x_1) = y_1$. That is, we chose a value x_1 and substitute this value into the equation $f(x, y) = 0$. Equation (3.5) is then reduced to an equation in one unknown, $g(y_1) = 0$. If the reduced equation is linear we can easily find y_1, and if the reduced equation is nonlinear we use the methods reviewed in this thesis for the solution of y_1.

Therefore, if we have a system of nonlinear equations

$$
\begin{aligned}
f_1(x, y) &= 0 \\
f_2(x, y) &= 0
\end{aligned}
\tag{3.7}
$$

we can find the differential equations which have $f_1(x,y) = 0$ and $f_2(x,y) = 0$ as solutions. To find an approximate real solution of (3.7), we produce the solutions of the derived differential equations by numerical methods, and see where they intersect (approximately). This gives us an initial approximation to the solution which can now be improved upon by using the Newton-Raphson method for two nonlinear equations.

EXAMPLE 2. Consider the set of simultaneous nonlinear equations

$$f_1(x,y) = xy - 6 = 0$$
$$f_2(x,y) = x^3 - y^4 - 11 = 0$$

with a real solution (3, 2).

We form the appropriate differential equations

$$\frac{dy_1}{dx} = -\frac{(f_1)_x}{(f_1)_y} = -\frac{y}{x}$$

$$\frac{dy_2}{dx} = -\frac{(f_2)_x}{(f_2)_y} = \frac{3x^2}{4y^3}$$

with imposed initial conditions $y_1(x_1) = y_1$, $y_2(x_1) = y_2$.

Let the initial approximation for x be $x_1 = 2.5$.

Then $y_1(2.5) = 2.4$, $y_2(2.5) = (4.625)^{\frac{1}{4}}$ and we then produce the numeric solutions, say by Euler's method or the Runge Kutta method. This procedure is illustrated in Figure 1.

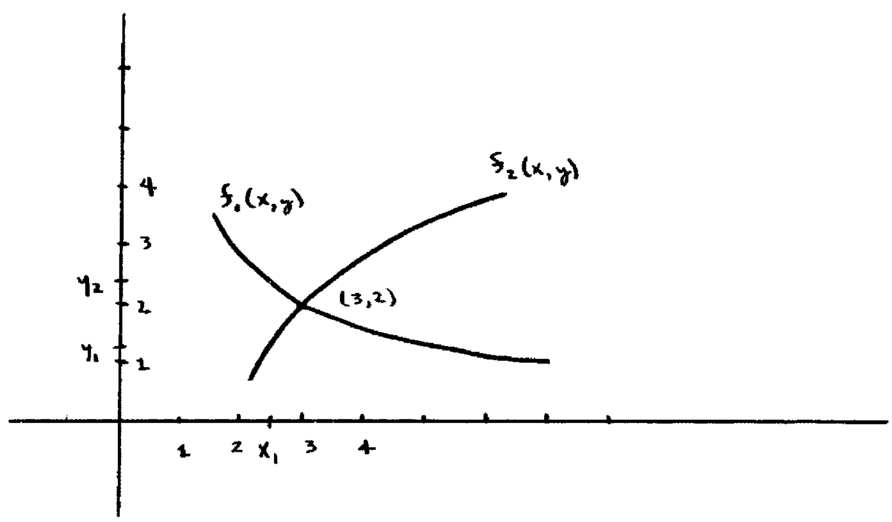


Figure 1

When we extend this method to three nonlinear equations in three unknowns the problem becomes increasingly difficult. In this case we must find where three surfaces intersect.

REFERENCES

The text, Iterative Methods for the Solution of Equations by J. F. Traub, Prentice-Hall Inc., 1964, contains a very extensive bibliography. Consequently, the following list of references will include only those papers dated after 1963. In addition, some texts which were found helpful are also listed.

TEXTS

1. S. D. Conte, Elementary Numerical Analysis, McGraw-Hill, 1965
2. A. S. Householder, Principles of Numerical Analysis, McGraw-Hill, 1953
3. K. S. Kunz, Numerical Analysis, McGraw-Hill, 1957
4. S. S. Kuo, Numerical Methods and Computers, Addison-Wesley, 1965
5. A. M. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, 1960
6. A. Ralston, A First Course in Numerical Analysis, McGraw-Hill, 1965
7. J. B. Scarborough, Numerical Mathematical Analysis, John Hopkins Press, 1962
8. J. Todd (Editor), Survey of Numerical Analysis, McGraw-Hill, 1962 (Also contains an extensive bibliography)

PAPERS

1. C. A. Barlow, Jr. and E. L. Jones, A Method for the Solution of Roots of a Nonlinear Equation and for the Solution of the General Eigenvalue Problem, Jour. of the Assoc. for Comp. Mach., Vol 13 (Jan 66), 135-143

2. G. L. Cain, Jr., A Method for Locating Zeros of Complex Functions, Communications of the ACM, Vol 9 (Apr 66), 305

3. P. Henrici and B. O. Watkins, Finding Zeros of a Polynomial by the Q-D Algorithm, Communications of the ACM, Vol 8 (Sept 1965), 570

4. D. G. Moursund, Examination of Multiple Roots and Root Clusters of a Polynomial Using the Bernoulli Principle, Jour. of the Assoc. for Comp. Mach., Vol 12 (Apr 65), 169-175

5. R. S. Pinkham, Random Root Location, Jour. Soc. Indust. and Appl. Math., Vol 12 (64), 855-864

6. O. L. Rasmussen, Solution of Polynomial Equations by the Method of D. H. Lehmer, Nordisk Tidshr. Informations, Vol 4 (64), 250-260

7. H. C. Thacher, Solution of Transcendental Equations by Series Revision, Communications of the ACM, Vol 9 (Jan 66), 10-11

```
C                    THE NEWTON-RAPHSON METHOD
C
C       WHEN THE DERIVATIVE OF THE NUMERICAL EXPRESSION F(X) = 0 CAN BE
C       FOUND THE REAL ROOTS OF THE EQUATION CAN BE COMPUTED BY THE
C       NEWTON-RAPHSON METHOD
C
C       MUST HAVE A SUBROUTINE FOR F AND DXF, THE DERIVATIVE OF F(X)
C
C       X0 IS THE APPROXIMATE VALUE OF THE DESIRED ROOT
C       X0 IS PREDETERMINED AND IS READ IN
C
C       RT IS THE EXACT VALUE OF THE ROOT
C
C       AN EPSILON CRITERION MUST BE SATISFIED AND EPS IS READ IN
C
C       THE LARGER THE VALUE OF DXF(X) IN THE NBHD. OF THE ROOT THE
C       FASTER THE CONVERGENCE
C       THE NEWTON-RAPHSON METHOD WILL FAIL IF DXF(X) = 0 IN THE NHBHD OF
C       THE ROOT
C
C                                            JANUARY 1966    CARD
C
      DIMENSION ID(15)
    1 READ 101,ID
      PUNCH 102,ID
      READ 103,X0,EPS
      PUNCH 104,X0,EPS
      PUNCH 105
      ITER = 1
    2 CALL DO(X0,F,DXF)
      RT = X0-F/DXF
      PUNCH 106,ITER,RT
      IF(ABSF(RT-X0)-EPS)3,3,4
    4 X0 = RT
      ITER = ITER+1
      IF(ITER-50)2,2,5
    3 CALL DO(RT,F,DXF)
      PUNCH 107,RT,F
      GO TO 1
    5 PUNCH 108
      GO TO 1
  101 FORMAT(15A2)
  102 FORMAT(41HEVALUATION OF A REAL ROOT OF THE FUNCTION/2X,7HF(X) = 15
     1A2/6X,28HBY THE NEWTON-RAPHSON METHOD/)
  103 FORMAT(2E14.8)
  104 FORMAT(2X,37HINITIAL APPROXIMATION TO THE ROOT IS E14.8/29X,10HEPS
     1ILON = E14.8)
  105 FORMAT(3X,13HITERATION NO.,5X,16HAPPROXIMATE ROOT)
  106 FORMAT(8X,I2,11X,E14.8)
  107 FORMAT(2X,21HTHE REAL ROOT IS X = E14.8/10X,7HF(X) = E14.8)
  108 FORMAT(63HTHE EPSILON CRITERIA HAS NOT BEEN SATISFIED AFTER 50 ITE
     1RATIONS)
      END
```

```
C                    THE SFCANT METHOD
C      A TWO POINT ITERATION METHOD FOR FINDING REAL ROOTS
C
C      RT IS THE EXACT VALUE OF THE ROOT
C
C      XO AND X1 ARE THE APPROXIMATE VALUES OF THE DESIRED ROOT
C      XO AND X1 ARE PREDETERMINED AND ARE READ IN
C
C      AN EPSILON CRITERION MUST BE SATISFIED AND EPS IS READ IN
C      MUST HAVE A SUBROUTINE FOR F
C
C      REFERENCE  SCARBOROUGH
C
C                                        JANUARY  1966,   CARD
C
C
      DIMENSION ID(15)
    1 READ 101,ID
      PUNCH 102,ID
      READ 103,XO,X1,EPS
      PUNCH 104,XO,X1,EPS
      PUNCH 105
      ITER = 1
    2 RT = X1-(X1-XO)/(F(X1)-F(XO))*F(X1)
      PUNCH 106,ITER,RT
      IF(ABSF(RT-X1)-EPS)3,3,4
    4 XO = X1
      X1 = RT
      ITER = ITER+1
      IF(ITER-50)2,2,5
    3 FRT = F(RT)
      PUNCH 107,RT,FRT
      GO TO 1
    5 PUNCH 108
      GO TO 1
  101 FORMAT(15A2)
  102 FORMAT(41HEVALUATION OF A REAL ROOT OF THE FUNCTION/2X,7HF(X) = 15
     1A2/10X,20HBY THE SECANT METHOD/)
  103 FORMAT(3E14.8)
  104 FORMAT(8X,28HTHE FIRST APPROXIMATIONS ARE/5HXO = E14.8,7H  AND  ,5
     1HX1 = E14.8/18X,10HEPSILON = E14.8/)
  105 FORMAT(3X,13HITERATION NO.,5X,16HAPPROXIMATE ROOT)
  106 FORMAT(8X,I2,11X,E14.8)
  107 FORMAT(2X,21HTHE REAL ROOT IS X = E14.8/10X,7HF(X) = E14.8)
  108 FORMAT(63HTHE EPSILON CRITERIA HAS NOT BEEN SATISFIED AFTER 50 ITE
     1RATIONS)
      END
```

```
C             THF METHOD OF ITERATION
C
C      WHEN A NUMERICAL FQUATION, F(X) = 0, CAN BE EXPRESSED IN THE FORM
C      X = PHI(X), AND A CONVERGENCE CRITERION IS SATISFIED, THEN THE
C      REAL ROOTS CAN BE FOUND BY THE PROCESS OF ITERATION
C
C      MUST HAVE A FUNCTION SUBPROGRAM FOR PHI(X)
C
C      THE CONVERGENCE CRITERION IS AS FOLLOWS,
C      THE ABSOLUTF VALUF OF THE DERIVATIVE OF PHI(X) MUST BE LESS THAN 1
C      IN THE NEIGHBORHOOD OF THE APPROXIMATE ROOT
C      SENSE SWITCH 1 IS ON IF THIS CRITERION IS TO BE TESTED
C
C      MUST HAVE A FUNCTION SUBPROGRAM FOR DXPHI(X)
C
C      APRT IS THE APPROXIMATE VALUE OF THE DESIRED ROOT, APRT IS PRE-
C      DETERMINED AND IS READ IN
C
C      RT IS THE EXACT VALUE OF THF ROOT
C
C      AN EPSILON CRITERION MUST BE SATISFIED AND EPS IS READ IN
C
C                                              MARCH 1966,  CARD
C
C
      DIMENSION ID(15)
    1 READ 101,ID
      PUNCH 102,ID
      READ 10,APRT
      READ 10,EPS
      PUNCH 11,APRT,EPS
      IF(SENSE SWITCH 1)4,2
    4 ABSDX = ABSF(DXPHI(APRT))
      PUNCH 17, ABSDX
      IF (ABSDX-1.)2,25,25
   25 PUNCH 16
    2 ITER = 1
    3 RT = PHI(APRT)
      PUNCH 12,ITER,RT
      IF(ABSF(APRT-RT)-FPS)15,15,5
    5 ITER = ITER+1
      APRT = RT
      IF(ITER-50)3,3,20
   15 PUNCH 13,RT
      GO TO 1
   20 PUNCH 14
      GO TO 1
```

```
 10 FORMAT(E14.8)
 11 FORMAT(38HTHE PREDETERMINED APPROXIMATE ROOT IS E14.8//11HEPSILON
   1IS E14.8//)
 12 FORMAT(14HITERATION NO. I3,5X,15HAPPROX. ROOT = E14.8)
 13 FORMAT(2X,21HTHE REAL ROOT IS X = E14.8)
 14 FORMAT(64HTHE EPSILON CRITERION HAS NOT BEEN SATISFIED AFTER 50 IT
   1ERATIONS)
 16 FORMAT(42HPROCESS WILL CONVERGE SLOWLY OR NOT AT ALL/)
 17 FORMAT(50HTHE ABSOLUTE VALUE OF THE DERIVATIVE OF PHI(X) IS E14.8/
   1)
101 FORMAT(15A2)
102 FORMAT(41HEVALUATION OF A REAL ROOT OF THE FUNCTION/2X,7HF(X) = 15
   1A2/7X,26HBY THE METHOD OF ITERATION/)
    END
```

```
C     PROGRAM TO COMPUTE REAL ROOTS OF A NUMERICAL EQUATION F(X) = 0
C
C              DIVIDING INTERVAL METHOD
C     SOLVE ALGEBRAIC AND TRANSCENDENTAL EQUATIONS OF ONE UNKNOWN
C
C     THE METHOD OF COMPUTATION IS BASED ON THE FOLLOWING FUNDAMENTAL
C     THEOREM. IF F(X) IS CONTINUOUS FROM X=A TO X=B AND IF F(A) AND
C     F(B) HAVE OPPOSITE SIGNS , THEN THERE IS AT LEAST ONE REAL ROOT
C     BETWEEN A AND B
C
C     THE STARTING POINT A IS READ IN
C     A IS USUALLY TAKEN TO BE ZERO UNLESS AN OBVIOUS VALUE FOR A CAN
C     BE OBTAINED BY LOOKING AT A GRAPH OF F(X)
C
C     D IS THE INCREMENT
C
C     N IS THE UPPER LIMIT OF THE INCREMENTS
C
C     AN EPSILON CRITERION MUST BE SATISFIED
C
C     MUST HAVE A FUNCTION SUBPROGRAM FOR F(X)
C
C                                  JANUARY  1966,   CARD
C
C
      DIMENSION ID(15)
    1 READ 101,ID
      READ 100,A
      READ 100,D
      READ 100, EPS
      READ 300,N
      PRINT 700,A,D,EPS,N
      PUNCH 102,ID
      PUNCH 900,A
      J = 1
      PN = N
   50 PI = 0.
      C1 = F(A)
      A1 = A
      IF(C1)5,10,5
   10 PUNCH 200,A1
      GO TO 1
    5 PI = 1.
      B = A+PI*D
      C2 = F(B)
   35 IF(C1*C2)20,25,30
   25 PUNCH 200,B
      GO TO 1
```

```
 30 A1 = B
    C1 = C2
    PI = PI+1.
    IF(PI-PN)40,40,45
 40 B = A+PI*D
    C2 = F(B)
    GO TO 35
 45 PUNCH 400
    GO TO 1
 20 GO TO(55,60),J
 55 PUNCH 500
    J = 2
 60 PUNCH 600,A1,B
    IF(ABSF(A1-B)-EPS)110,110,105
105 D = D/10.
    A = A1
    GO TO 50
110 PUNCH 800,A1,B
    GO TO 1
101 FORMAT(15A2)
102 FORMAT(3X,25H DIVIDING INTERVAL METHOD/33H FOR A REAL ROOT OF THE
   1 FUNCTION/15A2//)
100 FORMAT(E14.8)
200 FORMAT(19HA REAL ROOT IS A = E14.8)
300 FORMAT(I3)
400 FORMAT(53HTHE FUNCTION HAS NOT CHANGED SIGNS AFTER N INCREMENTS/33
   1HCHOOSE A DIFFERENT STARTING POINT/)
500 FORMAT(10X,20HSUCCESSIVE INTERVALS/)
600 FORMAT(4HA = E14.8,5X,4HB = E14.8/)
700 FORMAT(   4HA = E14.8,5X,4HD = E14.8/6HEPS = E14.8,5X,4HN = I3)
800 FORMAT(45HTHE REAL ROOT LIES IN THE OPEN INTERVAL (A,B)//6HWHERE ,
   14HA = E14.8,8HAND B = E14.8/)
900 FORMAT(7X,17H INITIAL GUESS IS E14.8/)
    END
```

PROBLEM 1
CHAPTER 1

```
EVALUATION OF A REAL ROOT OF THE FUNCTION
  F(X) =          SINF(X)-X/2.
      BY THE NEWTON-RAPHSON METHOD

   INITIAL APPROXIMATION TO THE ROOT IS  .15708000E+01
                                EPSILON =  .10000000E-05
    ITERATION NO.     APPROXIMATE ROOT
          1              .19999968E+01
          2              .19009953E+01
          3              .18955117E+01
          4              .18954943E+01
          5              .18954943E+01
   THE REAL ROOT IS X =  .18954943E+01
           F(X) = -.30000000E-07
```

```
EVALUATION OF A REAL ROOT OF THE FUNCTION
  F(X) =          SINF(X)-X/2.
      BY THE NEWTON-RAPHSON METHOD

   INITIAL APPROXIMATION TO THE ROOT IS  .31416000E+01
                                EPSILON =  .10000000E-05
    ITERATION NO.     APPROXIMATE ROOT
          1              .20943952E+01
          2              .19132229E+01
          3              .18956718E+01
          4              .18954943E+01
          5              .18954943E+01
   THE REAL ROOT IS X =  .18954943E+01
           F(X) = -.30000000E-07
```

```
EVALUATION OF A REAL ROOT OF THE FUNCTION
  F(X) =            SINF(X)-X/2.
           BY THE SECANT METHOD

         THE FIRST APPROXIMATIONS ARE
XO =  .31415900E+01  AND  X1 =  .15707963E+01
                    EPSILON =  .10000000E-05

   ITERATION NO.      APPROXIMATE ROOT
         1             .17596035E+01
         2             .19320037E+01
         3             .18924157E+01
         4             .18954307E+01
         5             .18954943E+01
         6             .18954943E+01
  THE REAL ROOT IS X =  .18954943E+01
          F(X) = -.30000000E-07
```

```
EVALUATION OF A REAL ROOT OF THE FUNCTION
  F(X) =            SINF(X)-X/2.
           BY THE SECANT METHOD

         THE FIRST APPROXIMATIONS ARE
XO =  .31415900E+01  AND  X1 =  .25000000E+01
                    EPSILON =  .10000000E-05

   ITERATION NO.      APPROXIMATE ROOT
         1             .20452737E+01
         2             .19285226E+01
         3             .18980283E+01
         4             .18955416E+01
         5             .18954944E+01
         6             .18954943E+01
  THE REAL ROOT IS X =  .18954943E+01
          F(X) = -.30000000E-07
```

PROBLEM 1
CHAPTER 1

EVALUATION OF A REAL ROOT OF THE FUNCTION
F(X) = SINF(X)-X/2.
BY THE METHOD OF ITERATION

THE PREDETERMINED APPROXIMATE ROOT IS .15708000E+01

EPSILON IS .10000000E-05

ITERATION NO.	1	APPROX. ROOT =	.20000000E+01
ITERATION NO.	2	APPROX. ROOT =	.18185948E+01
ITERATION NO.	3	APPROX. ROOT =	.19389094E+01
ITERATION NO.	4	APPROX. ROOT =	.18660160E+01
ITERATION NO.	5	APPROX. ROOT =	.19134765E+01
ITERATION NO.	6	APPROX. ROOT =	.18837149E+01
ITERATION NO.	7	APPROX. ROOT =	.19028783E+01
ITERATION NO.	8	APPROX. ROOT =	.18907312E+01
ITERATION NO.	9	APPROX. ROOT =	.18985118E+01
ITERATION NO.	10	APPROX. ROOT =	.18935603E+01
ITERATION NO.	11	APPROX. ROOT =	.18967246E+01
ITERATION NO.	12	APPROX. ROOT =	.18947078E+01
ITERATION NO.	13	APPROX. ROOT =	.18959954E+01
ITERATION NO.	14	APPROX. ROOT =	.18951742E+01
ITERATION NO.	15	APPROX. ROOT =	.18956983E+01
ITERATION NO.	16	APPROX. ROOT =	.18953640E+01
ITERATION NO.	17	APPROX. ROOT =	.18955773E+01
ITERATION NO.	18	APPROX. ROOT =	.18954412E+01
ITERATION NO.	19	APPROX. ROOT =	.18955281E+01
ITERATION NO.	20	APPROX. ROOT =	.18954726E+01
ITERATION NO.	21	APPROX. ROOT =	.18955080E+01
ITERATION NO.	22	APPROX. ROOT =	.18954855E+01
ITERATION NO.	23	APPROX. ROOT =	.18954998E+01
ITERATION NO.	24	APPROX. ROOT =	.18954907E+01
ITERATION NO.	25	APPROX. ROOT =	.18954965E+01
ITERATION NO.	26	APPROX. ROOT =	.18954928E+01
ITERATION NO.	27	APPROX. ROOT =	.18954952E+01
ITERATION NO.	28	APPROX. ROOT =	.18954936E+01
ITERATION NO.	29	APPROX. ROOT =	.18954946E+01

THE REAL ROOT IS X = .18954946E+01

PROBLEM 1
CHAPTER 1

EVALUATION OF A REAL ROOT OF THE FUNCTION
F(X) = SINF(X)-X/2.
BY THE METHOD OF ITERATION

THF PREDFTERMINED APPROXIMATE ROOT IS .31416000E+01

EPSILON IS .10000000E-05

ITERATION NO.	1	APPROX. ROOT = -.14680000E-04
ITERATION NO.	2	APPROX. ROOT = -.29360000E-04
ITERATION NO.	3	APPROX. ROOT = -.58720000E-04
ITERATION NO.	4	APPROX. ROOT = -.11744000E-03
ITERATION NO.	5	APPROX. ROOT = -.23486000E-03
ITERATION NO.	6	APPROX. ROOT = -.46970000E-03
ITERATION NO.	7	APPROX. ROOT = -.93938000E-03
ITERATION NO.	8	APPROX. ROOT = -.18787400E-02
ITERATION NO.	9	APPROX. ROOT = -.37574600E-02
ITERATION NO.	10	APPROX. ROOT = -.75149000E-02
ITERATION NO.	11	APPROX. ROOT = -.15029640E-01
ITERATION NO.	12	APPROX. ROOT = -.30058140E-01
ITERATION NO.	13	APPROX. ROOT = -.60107220E-01
ITERATION NO.	14	APPROX. ROOT = -.12014206E+00
ITERATION NO.	15	APPROX. ROOT = -.23970648E+00
ITERATION NO.	16	APPROX. ROOT = -.47483500E+00
ITERATION NO.	17	APPROX. ROOT = -.91438340E+00
ITERATION NO.	18	APPROX. ROOT = -.15843728E+01
ITERATION NO.	19	APPROX. ROOT = -.19998156E+01
ITERATION NO.	20	APPROX. ROOT = -.18187483E+01
ITERATION NO.	21	APPROX. ROOT = -.19388341E+01
ITERATION NO.	22	APPROX. ROOT = -.18660702E+01
ITERATION NO.	23	APPROX. ROOT = -.19134449E+01
ITERATION NO.	24	APPROX. ROOT = -.18837361E+01
ITERATION NO.	25	APPROX. ROOT = -.19028653E+01
ITERATION NO.	26	APPROX. ROOT = -.18907397E+01
ITERATION NO.	27	APPROX. ROOT = -.18985064E+01
ITERATION NO.	28	APPROX. ROOT = -.18935637E+01
ITERATION NO.	29	APPROX. ROOT = -.18967225E+01
ITERATION NO.	30	APPROX. ROOT = -.18947091E+01
ITERATION NO.	31	APPROX. ROOT = -.18959946E+01
ITERATION NO.	32	APPROX. ROOT = -.18951747E+01
ITERATION NO.	33	APPROX. ROOT = -.18956980E+01
ITERATION NO.	34	APPROX. ROOT = -.18953642E+01
ITERATION NO.	35	APPROX. ROOT = -.18955772E+01

```
ITERATION NO.  36     APPROX. ROOT = -.18954413E+01
ITERATION NO.  37     APPROX. ROOT = -.18955280E+01
ITERATION NO.  38     APPROX. ROOT = -.18954727E+01
ITERATION NO.  39     APPROX. ROOT = -.18955080E+01
ITERATION NO.  40     APPROX. ROOT = -.18954855E+01
ITERATION NO.  41     APPROX. ROOT = -.18954998E+01
ITERATION NO.  42     APPROX. ROOT = -.18954907E+01
ITERATION NO.  43     APPROX. ROOT = -.18954965E+01
ITERATION NO.  44     APPROX. ROOT = -.18954928E+01
ITERATION NO.  45     APPROX. ROOT = -.18954952E+01
ITERATION NO.  46     APPROX. ROOT = -.18954936E+01
ITERATION NO.  47     APPROX. ROOT = -.18954946E+01
  THE REAL ROOT IS X = -.18954946E+01
```

DIVIDING INTERVAL METHOD
FOR A REAL ROOT OF THE FUNCTION
SINF(X)-X/2.

INITIAL GUESS IS .15708000E+01

SUCCESSIVE INTERVALS

A = .18708000E+01 B = .19708000E+01

A = .18908000E+01 B = .19008000E+01

A = .18948000E+01 B = .18958000E+01

A = .18954000E+01 B = .18955000E+01

A = .18954900E+01 B = .18955000E+01

A = .18954940E+01 B = .18954950E+01

THE REAL ROOT LIES IN THE OPEN INTERVAL (A,B)

WHERE A = .18954940E+01AND B = .18954950E+01

DIVIDING INTERVAL METHOD
FOR A REAL ROOT OF THE FUNCTION
SINF(X)-X/2.

INITIAL GUESS IS .31416000E+01

SUCCESSIVE INTERVALS

A = .19416000E+01 B = .18416000E+01

A = .19016000E+01 B = .18916000E+01

A = .18956000E+01 B = .18946000E+01

A = .18955000E+01 B = .18954000E+01

A = .18955000E+01 B = .18954900E+01

A = .18954950E+01 B = .18954940E+01

THE REAL ROOT LIES IN THE OPEN INTERVAL (A,B)

WHERE A = .18954950E+01AND B = .18954940E+01

PROBLEM 2
CHAPTER 1

EVALUATION OF A REAL ROOT OF THE FUNCTION
F(X) = X**20-1.
BY THE NEWTON-RAPHSON METHOD

INITIAL APPROXIMATION TO THE ROOT IS .50000000E+00
EPSILON = .10000000E-05

ITERATION NO.	APPROXIMATE ROOT
1	.26214876E+05
2	.24904133E+05
3	.23658927E+05
4	.22475981E+05
5	.21352182E+05
6	.20284573E+05
7	.19270345E+05
8	.18306828E+05
9	.17391487E+05
10	.16521913E+05
11	.15695818E+05
12	.14911028E+05
13	.14165477E+05
14	.13457204E+05
15	.12784344E+05
16	.12145127E+05
17	.11537871E+05
18	.10960978E+05
19	.10412930E+05
20	.98922840E+04
21	.93976698E+04
22	.89277864E+04
23	.84813971E+04
24	.80573273E+04
25	.76544610E+04
26	.72717380E+04
27	.69081511E+04
28	.65627436E+04
29	.62346065E+04
30	.59228762E+04
31	.56267324E+04
32	.53453958E+04
33	.50781261E+04
34	.48242198E+04
35	.45830089E+04
36	.43538585E+04
37	.41361656E+04
38	.39293574E+04
39	.37328896E+04
40	.35462452E+04

41	.33689330E+04
42	.32004864E+04
43	.30404621E+04
44	.28884390E+04
45	.27440171E+04
46	.26068163E+04
47	.24764755E+04
48	.23526518E+04
49	.22350193E+04
50	.21232684E+04

THE EPSILON CRITERIA HAS NOT BEEN SATISFIED AFTER 50 ITERATIONS

PROBLEM 2
CHAPTER 1

EVALUATION OF A REAL ROOT OF THE FUNCTION
F(X) = X**20-1.
BY THE NEWTON-RAPHSON METHOD

INITIAL APPROXIMATION TO THE ROOT IS .15000000E+01
EPSILON = .10000000E-05

ITERATION NO.	APPROXIMATE ROOT
1	.14250226E+01
2	.13538313E+01
3	.12862980E+01
4	.12224014E+01
5	.11623827E+01
6	.11071300E+01
7	.10590045E+01
8	.10228776E+01
9	.10042665E+01
10	.10001679E+01
11	.10000003E+01
12	.10000001E+01

THE REAL ROOT IS X = .10000001E+01
F(X) = .20000000E-05

```
C           **  GRAEFFE'S ROOT SQUARING METHOD  **
C
C      THE UNDERLYING PRINCIPLE OF GRAEFFE'S METHOD IS THIS-THE GIVEN
C      EQUATION IS TRANSFORMED INTO ANOTHER WHOSE ROOTS ARE HIGH POWERS
C      OF THOSE OF THE ORIGINAL EQUATION. THE ROOTS OF THE TRANSFORMED
C      EQUATION ARE WIDELY SEPARATED, AND BECAUSE OF THIS FACT ARE EASILY
C      FOUND. THE ROOTS OF THE TRANSFORMED EQUATION ARE SAID TO BE
C      SEPARATED WHEN THE RATIO OF ANY ROOT TO THE NEXT LARGER IS NEGLI-
C      GIBLE IN COMPARISON WITH UNITY.
C
C      REFERENCE    NUMERICAL MATHEMATICAL ANALYSIS - SCARBOROUGH
C
C                                PHILLIP CARD    MARCH 1966
C
C      SEPARATED WHEN THE RATIO OF ANY ROOT TO THE NEXT LARGER IS NEGLIGI
C
    1 READ 100,EPS
      READ 101,N
      DIMENSION A(10),B(10),C(10,10),AVB(10),X(10),XN(10),SAVE(10)
C
C      READ IN THE ORDER AND COEFFICIENTS OF THE ORIGINAL EQUATION
C            THE ORDER N IS LESS THAN 30
C
      M = N+1
      READ 102,(A(I),I=1,M)
      PUNCH 103,N,(A(I),I=1,M)
      PUNCH 114,EPS
      P = 1
      DO 55 I=1,M
   55 SAVE(I) = A(I)
C
C      COMPUTE THE ELEMENTS OF THE MATRIX C
C
      M2 = (M+1)/2
   77 DO 10 I=1,M2
      DO 10 J=1,M
   10 C(I,J) = 0
      DO 20 I=1,M
   20 C(1,I) = A(I)**2
      MM1 = M-1
      DO 30 I=2,MM1
   30 C(2,I) = -2.*A(I-1)*A(I+1)
      GO TO 3
   19 DO 90 I=1,M
   90 AVB(I) = ABSF(B(I))
C
C      THE PREVIOUS B(I)'S (OR THE PRESENT A(I)'S) ARE THE COEFFICIENTS
C      OF OUR FINAL TRANSFORMED EQUATION
C
    7 PUNCH 104
      PUNCH 105
```

```
C
C      CALCULATE REAL ROOTS ACCORDING TO THE SIMPLE EQUATIONS
C      CALL SYNTHETIC DIVISION SUBROUTINE TO CHECK FOR SIGNS OF ROOTS
C      PUNCH X(I),F(X(I)),-X(I),F(-X(I))
C
       DO 110 I=1,N
  110  X(I) = EXPF((1./P)*(LOGF(AVB(I+1))-LOGF(AVB(I))))
       DO 120 I=1,N
  120  XN(I) = -X(I)
       DO 130 I=1,N
       CALL SYND(M,SAVE,X(I),F)
       FP = F
       CALL SYND(M,SAVE,XN(I),FN)
       PUNCH 108,I,X(I),FP,XN(I),FN
  130  CONTINUE
       PRINT 106
       PAUSE
       GO TO 1
C
C      PROCESS NOT COMPLETE, COMPUTE REMAINING ELEMENTS OF THE MATRIX C
C
    3  IF(M2-3)17,13,13
   13  DO 50 I=3,M2
       MM = M-I+1
       IF(MM-I)17,27,27
   27  JJ = 0
       DO 50 J=I,MM
       JJ = JJ+1
       K = 2*(I-1)+JJ
       C(I,J) = 2.*A(JJ)*A(K)*(-1.)**(I-1)
   50  CONTINUE
C
C      COMPUTE COEFFICIENTS OF THE TRANSFORMED EQUATION
C
   17  P = P*2.
       DO 60 I=1,M
       B(I) = 0
       DO 60 J=1,M2
   60  B(I) = B(I)+C(J,I)
       IP = P
       PUNCH 109,IP,(B(I),I=1,M)
       IF(IP-4)18,18,28
   28  DO 88 I=2,N
       IF(ABSF(B(I)/C(1,I))-EPS)18,88,88
   88  CONTINUE
       PUNCH 1001
       GO TO 19
   18  DO 70 I=1,M
       AVB(I) = ABSF(B(I))
       IF(AVB(I)-.99999999E49)70,70,7
   70  CONTINUE
       DO 80 I=1,M
   80  A(I) = B(I)
```

```
      GO TO 77
  100 FORMAT(F6.4)
  101 FORMAT(I3)
  102 FORMAT(5E14.8)
  103 FORMAT(8X,29H   ROOTS OF THE POLYNOMIAL   /46HP(X) = A(1)*X**N+A(2
     1)*X**N-1+...+A(N)*X+A(N+1)/1X,39HTHE DEGREE N OF THE POLYNOMIAL P(
     1X) IS I5/ 46HTHE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS/
     2(16X,E14.8)/)
  104 FORMAT(/59HTHE COFFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICI
     1ENTS/18X,24HOF THF TERMINAL EQUATION//)
  105 FORMAT(18X,31HTHE POSSIBLE REAL ROOTS OF P(X)/
     12H I,8X,4HX(I),12X,7HF(X(I)),10X,5H-X(I),11X,8HF(-X(I))/)
  106 FORMAT(16HPROCESS COMPLETE)
  108 FORMAT(I2,4(3X,E14.8)/)
  109 FORMAT(/4HP = I3/
     144HTHE COEFFICIENTS OF THE TRANSFORMED EQUATION/
     24(3X,E14.8))
  114 FORMAT(/11HEPSILON IS F6.4)
 1001 FORMAT(/34HCROSS PRODUCT TERMS ARE NEGLIGIBLE)
C
      END
```

SUBROUTINE
SYNTHETIC DIVISION

```
      SUBROUTINE SYND(M,A,XO,F)
      DIMENSION A(30),B(30)
      B(1) = A(1)
      DO 5 I=2,M
    5 B(I) = B(I-1)*XO+A(I)
      F = B(M)
      RFTURN
      END
```

```
            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     3
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                  .10000000E+01

                 -.20000000E+01

                 -.50000000E+01

                  .60000000E+01


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .14000000E+02    .49000000E+02    .36000000E+02

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .98000000E+02    .13930000E+04    .12960000E+04

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .68180000E+04    .16864330E+07    .16796160E+07

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .43112258E+08    .28211530E+13    .28211099E+13

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .18530244E+16    .79586610E+25    .79586610E+25

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
                    OF THE TERMINAL EQUATION


                    THE POSSIBLE REAL ROOTS OF P(X)
 I        X(I)            F(X(I))          -X(I)            F(-X(I))

 1    .30000000E+01    .00000000E-99   -.30000000E+01   -.24000000E+02

 2    .19999998E+01   -.39999995E+01   -.19999998E+01    .30000000E-05

 3    .10000000E+01    .00000000E-99   -.10000000E+01    .80000000E+01
```

```
            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS      5
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                   .12300000E+01

                  -.25200000E+01

                  -.16100000E+02

                   .17300000E+02

                   .29400000E+02

                  -.13400000E+01


EPSILON IS  .9500

P =    2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .15129000E+01    .45956400E+02    .41872600E+03    .12527236E+04
    .91072400E+03    .17956000E+01

P =    4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .22888664E+01    .84500960E+03    .62945798E+05    .80679383E+06
    .82491942E+06    .32241793E+01

P =    8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .52389093E+01    .42589218E+06    .26024526E+10    .54706587E+12
    .68048684E+12    .10395332E+02

P =   16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .27446170E+02    .15411612E+12    .63067845E+19    .29573920E+24
    .46306233E+24    .10806292E+03

P =   32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .75329224E+03    .23405584E+23    .39684374E+38    .87455834E+47
    .21442672E+48    .11677594E+05

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
                  OF THE TERMINAL EQUATION
```

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.40657071E+01	.24924000E-02	-.40657071E+01	-.80787521E+03
2	.29916832E+01	-.96737312E+02	-.29916832E+01	.13630000E-02
3	.19587274F+01	.20200000F-04	-.19587274F+01	.55880009E+02
4	.10284223E+01	.28276895E+02	-.10284223E+01	-.80000000E-05
5	.44463368E-01	.00000000E-99	-.44463368E-01	-.26116159E+01

```
             ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     4
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                    .10000000E+01

                   -.50000000E+01

                    .93500000E+01

                   -.77500000E+01

                    .24024000E+01


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .63000000E+01    .14727300E+02    .15137620E+02
    .57715257E+01

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .10235400E+02    .37702401E+02    .59149550E+02
    .33310508E+02

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .29358610E+02    .27725341E+03    .98689700E+03
    .11095899E+04

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .30742116E+03    .21140784E+05    .35869052E+06
    .12311897E+07

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .52226201E+05    .22885700E+09    .76602250E+11
    .15158280E+13

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .22698620E+10    .44377269E+17    .51740891E+22
    .22977345E+25
```

P = 128
THE COEFFICIFNTS OF THE TRANSFORMED EQUATION
.10000000E+01 .50635189E+19 .19458531E+34 .26567264E+44
.52795838E+49

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
OF THE TERMINAL EQUATION

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.14000016E+01	.00000000E-99	-.14000016F+01	.49140117E+02
2	.12999978E+01	.10000000E-06	-.12999978E+01	.42119853E+02
3	.12000007F+01	.00000000E-99	-.12000007F+01	.35880039E+02
4	.10999998E+01	.00000000E-99	-.10999998E+01	.30359988E+02

```
              ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     3
THF COFFFICIFNTS A(1) TO A(N+1) ARE AS FOLLOWS
                        .10000000E+01

                       -.30600000F+01

                        .31211000E+01

                       -.10611060E+01


EPSILON IS  .9500

P =   2
THE COEFFICIFNTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .31214000F+01    .32472965E+01    .11259459E+01

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .32485449E+01    .35158790E+01    .12677541E+01

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .35212850E+01    .41246930E+01    .16072004E+01

P =  16
THF COEFFICIFNTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .41500620E+01    .56942710E+01    .25830931E+01

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .58344720E+01    .10984729E+02    .66723699E+01

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .12071605E+02    .42804760E+02    .44520520E+02

P = 128
THF COEFFICIFNTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .60114120F+02    .75737920E+03    .19820767E+04

P = 256
THF COEFFICIENTS OF THF TRANSFORMED EQUATION
    .10000000F+01    .20989490E+04    .33532166E+06    .39286280E+07
```

P = 512
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.37349436E+07	.95948640E+11	.15434117E+14

P = 1024
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.13757906E+14	.90908504E+22	.23821196E+27

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS OF THE TERMINAL EQUATION

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.10299843E+01	.00000000E-99	-.10299843E+01	-.86147220E+01
2	.10200309E+01	.00000000E-99	-.10200309E+01	-.84898456E+01
3	.10099847E+01	.00000000E-99	-.10099847E+01	-.83650347E+01

```
             ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS      3
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                     .10000000E+01

                    -.30060000E+01

                     .30120110E+01

                    -.10060110E+01


FPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .30120140E+01    .30240721E+01    .10120581E+01

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .30240841E+01    .30483457E+01    .10242615E+01

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .30483932E+01    .30975057E+01    .10491116E+01

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .30976897E+01    .31983322E+01    .11006351E+01

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .31990170E+01    .34104760E+01    .12113976E+01

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .34127570E+01    .38807830E+01    .14674841E+01

P = 128
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .38853440E+01    .50441430E+01    .21535095E+01

P = 256
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .50076110E+01    .87091280E+01    .46376031E+01
```

P = 512
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.76579110E+01	.29402286E+02	.21507362E+02

P = 1024
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	-.16097200E+00	.53509150E+03	.46256662E+03

P = 2048
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	-.10701571E+04	.28647183E+06	.21396787E+06

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS OF THE TERMINAL EQUATION

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.10034118E+01	.00000000E-99	-.10034118E+01	-.80651154E+01
2	.10027331E+01	.00000000E-99	-.10027331E+01	-.80569296E+01
3	.99985752E+00	.00000000E-99	-.99985752E+00	-.80223088E+01

```
            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     3
THF COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                    .10000000E+01

                   -.30000000E+01

                    .40000000E+01

                   -.50000000E+01


FPSILON IS  .9500

P =   2
THF COEFFICIENTS OF THE TRANSFORMED FQUATION
    .10000000F+01    .10000000F+01   -.14000000F+02    .25000000F+02

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .29000000F+02    .14600000E+03    .62500000E+03

P =   8
THF COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .54900000E+03   -.14934000E+05    .39062500E+06

P =  16
THF COEFFICIENTS OF THE TRANSFORMFD EQUATION
    .10000000F+01    .33126900E+06   -.20588190F+09    .15258789E+12

P =  32
THF COEFFICIENTS OF THE TRANSFORMFD EQUATION
    .10000000E+01    .11015091E+12   -.58707920E+17    .23283064E+23

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THF COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COFFFICIENTS
                     OF THE TERMINAL EQUATION


                     THF POSSIBLE REAL ROOTS OF P(X)
 I        X(I)             F(X(I))           -X(I)            F(-X(I))

 1    .22134112E+01    -.24000000E-05    -.22134112E+01    -.39395130E+02

 2    .15099398E+01    -.23574562E+01    -.15099398E+01    -.21322052E+02

 3    .14960572E+01    -.23818766E+01    -.14960572F+01    -.21047245E+02
```

```
             ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     6
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                    .10000000E+01

                    .30000000E+01

                   -.10000000E+01

                   -.70000000E+01

                    .10000000E+02

                    .14000000E+02

                   -.20000000E+02


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01   .11000000E+02   .63000000E+02   .19300000E+03
   .33600000E+03   .59600000E+03   .40000000E+03

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01  -.50000000E+01   .39500000E+03   .72250000E+04
  -.66760000E+05   .86416000E+05   .16000000E+06

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01  -.76500000E+03   .94755000E+05   .10375686E+09
   .33345864E+10   .28830925E+11   .25600000E+11

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01   .39571500E+06   .17439567E+12   .10089386E+17
   .51415054E+19   .66049141E+21   .65536000E+21

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01  -.19220098E+12   .22439090E+23   .10000291E+33
   .13107400E+38   .42950983E+42   .42949672E+42

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01  -.79369640E+22   .54195409E+45   .99999940E+64
   .85899470E+74   .18446744E+84   .18446743E+84
```

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
OF THE TERMINAL EQUATION

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.21987816E+01	.22852832E+03	-.21987816E+01	.74243100E+01
2	.22739770E+01	.27517328E+03	-.22739770E+01	.11300856E+02
3	.20000084E+01	.13600308E+03	-.20000084E+01	.25200000E-03
4	.38015546E+00	-.13611266E+02	-.38015546E+00	-.23534105E+02
5	.11763972E+01	.64068700E+01	-.11763972E+01	-.17258128E+02
6	.99999951E+00	-.15000000E-04	-.99999951E+00	-.20000006E+02

```
              ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREF N OF THE POLYNOMIAL P(X) IS     4
THF COEFFICIFNTS A(1) TO A(N+1) ARE AS FOLLOWS
                    .10000000E+01

                    .19999911E+01

                   -.10000014E+01

                   -.19999928E+01

                    .10000001E+02


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .59999672E+01    .28999940E+02    .24000001E+02
    .10000002E+03

P =   4
THF COEFFICIFNTS OF THF TRANSFORMED EQUATION
    .10000000F+01   -.22000274F+02    .75299814E+03   -.52239891E+04
    .10000004F+05

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01   -.10219842E+04    .35714781E+06    .12230094E+08
    .10000008E+09

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000F+01    .33015610E+06    .15275247E+12    .78145580E+14
    .10000016F+17

P =  32
THE COEFFICIFNTS OF THE TRANSFORMFD EQUATION
    .10000000F+01   -.19650189E+12    .23281737E+23    .30516774E+28
    .10000032E+33

P =  64
THE COEFFICIFNTS OF THE TRANSFORMED EQUATION
    .10000000F+01   -.79504820F+22    .54204046E+45    .46563726E+55
    .10000064E+65

THF COEFFICIENTS LISTED DIRECTLY ABOVF ARE THE COEFFICIENTS
                    OF THF TERMINAL EQUATION
```

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.21988400E+01	.45405902E+02	-.21988400E+01	.11676756E+02
2	.22739223E+01	.50533354E+02	-.22739223E+01	.12597933E+02
3	.14296147E+01	.15117753E+02	-.14296147E+01	.91488690E+01
4	.44239656E+00	.91309670E+01	-.44239656E+00	.10554213E+02

```
                                                          EXAMPLE  8
                                                          CHAPTER  2


            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     4
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                  .10000000E+01

                 -.40000000E+01

                 -.75000000E+00

                  .16250000E+02

                 -.12500000E+02


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .17500000E+02     .10556250E+03     .24531250E+03
    .15625000E+03

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .95125000E+02     .28700040E+04     .27189941E+05
    .24414062E+05

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .33087576E+04     .31128648E+07     .59915598E+09
    .59604642E+09

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .47221470E+07     .57261954E+13     .35527706E+18
    .35527133E+18

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .10846282E+14     .29433972E+26     .12621772E+36
    .12621771E+36

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .58773890E+26     .86362072E+51     .15930912E+71
    .15930910E+71

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
                   OF THE TERMINAL EQUATION
```

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.25272226E+01	.51240000E-02	-.25272226E+01	.46998401E+02
2	.24730708E+01	.47800000E-02	-.24730708E+01	.40633948E+02
3	.56568540E+00	-.41692900E+01	-.56568540E+00	-.21105910E+02
4	.10000000E+01	.00000000E-99	-.10000000E+01	-.24500000E+02

EXAMPLE 9
CHAPTER 2

```
            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     4
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                 .10000000E+01

                -.45000000E+01

                 .55000000E+01

                 .00000000E-99

                -.20000000E+01


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .92500000E+01    .26250000E+02    .22000000E+02
   .40000000E+01

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .33062500E+02    .29006250E+03    .27400000E+03
   .16000000E+02

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .51300390E+03    .66050003E+05    .65794000E+05
   .25600000E+03

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .13107300E+06    .42950982E+10    .42950328E+10
   .65536000E+05

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .85899350E+10    .18446743E+20    .18446744E+20
   .42949672E+10

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .36893497E+20    .34028232E+39    .34028236E+39
   .18446743E+20

P = 128
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
   .10000000E+01    .68056550E+39    .11579205E+78    .11579208E+78
   .34028232E+39

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
                 OF THE TERMINAL EQUATION
```

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.20108597E+01	.29930000E-03	-.20108597E+01	.73179524E+02
2	.19891988E+01	.28730000E-03	-.19891988E+01	.70840044E+02
3	.50000000E+00	-.11250000E+01	-.50000000E+00	.00000000E-99
4	.70710678E+00	-.59099040E+00	-.70710678E+00	.25909901E+01

```
            ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
 THE DEGREE N OF THE POLYNOMIAL P(X) IS     5
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                   .10000000E+01

                   .15000000E+01

                  -.25000000E+01

                  -.65000000E+01

                  -.45000000E+01

                  -.10000000E+01


EPSILON IS  .9500

P =   2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .72500000E+01    .16750000E+02    .16750000E+02
    .72500000E+01    .10000000E+01

P =   4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .19062500E+02    .52187500E+02    .52187500E+02
    .19062500E+02    .10000000E+01

P =   8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .25900390E+03    .77201170E+03    .77201170E+03
    .25900390E+03    .10000000E+01

P =  16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .65538997E+05    .19661198E+06    .19661198E+06
    .65538997E+05    .10000000E+01

P =  32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .42949669E+10    .12884898E+11    .12884898E+11
    .42949669E+10    .10000000E+01

P =  64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01    .18446740E+20    .55340170E+20    .55340170E+20
    .18446740E+20    .10000000E+01
```

```
P = 128
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .34028221E+39     .10208430E+40     .10208430E+40
    .34028221E+39     .10000000E+01

P = 256
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
    .10000000E+01     .11579198E+78     .34737100E+78     .34737100E+78
    .11579198E+78     .10000000E+01

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS
                  OF THE TERMINAL EQUATION
```

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.19999999E+01	-.75000000E-05	-.19999999E+01	-.59999973E+01
2	.71014781E+00	-.78069124E+01	-.71014781E+00	.13869000E-01
3	.10000000E+01	-.12000000E+02	-.10000000E+01	.00000000E-99
4	.99571775E+00	-.11939956E+02	-.99571775E+00	.10000000E-06
5	.70710678E+00	-.77640872E+01	-.70710678E+00	.14087200E-01

```
C      **  LIN-BAIRSTOW METHOD FOR COMPLEX ROOTS   **
C
C      A GENERAL METHOD FOR DETERMINING THE COMPLEX ROOTS OF A POLYNOMIAL
C      EQUATION
C         P(X) = AO*X**N+A1*X**N-1+...+AN-1*X+AN = 0
C      INVOLVES FINDING A QUADRATIC FACTOR X**2+ALP*X+BETA OF THE POLY-
C      NOMIAL BY AN ITERATIVE PROCEDURE.
C
C      REFFRENCE      NUMERICAL ANALYSIS-KUNZ
C
C                                    PHILLIP CARD    MARCH 1966
C
    1 READ 100,EPS
      READ 101,N
      DIMENSION A(100),B(100),C(100)
C
C      READ THE ORDER AND COEFFICIENTS OF THE ORIGINAL EQUATION
C                 THE ORDER N IS LESS THAN 100
C                      AND GREATER THAN 3
C
      J = N
      M = N+1
      L1 = 1
      L2 = 2
      READ 102,(A(I),I = 1,M)
      PUNCH 103,N,(A(I),I = 1,M)
      PUNCH 114,EPS
C
C      R AND S INITIALLY ARE GUESSES AT THE QUADRATIC COEFFICIENTS
C
      JP1 = J+1
    2 READ 104,R,S
      PUNCH 115
      PUNCH 105,R,S
      K = 0
C
C      CALCULATE THE COEFFICIENTS B(I) AND C(I)
C
    3 K = K+1
      B(1) = A(1)
      B(2) = A(2)-R*B(1)
      DO 10 I = 3,JP1
   10 B(I) = A(I)-R*B(I-1)-S*B(I-2)
      C(1) = B(1)
      C(2) = B(2)-R*C(1)
      DO 20 I = 3,J
   20 C(I) = B(I)-R*C(I-1)-S*C(I-2)
C
C      CALCULATE DELR AND DELS
C
```

```
      DEN = C(J-1)**2-C(J-2)*(C(J)-B(J))
      IF(DEN)21,22,21
   22 PRINT 116
      GO TO 2
   21 DELS = (C(J-1)*B(J+1)-B(J)*(C(J)-B(J)))/DEN
      DELR = (B(J)*C(J-1)-C(J-2)*B(J+1))/DEN
      RS = R+DELR
      SS = S+DELS
      PUNCH 106,K,RS,SS
      IF(ABSF(R-RS)-EPS)5,5,15
    5 IF(ABSF(S-SS)-EPS)25,25,15
   15 IF(K-50)35,45,45
   35 R = RS
      S = SS
C
C     REPEAT THE PROCESS WITH NEW R AND S
C
      GO TO 3
C
C     METHOD HAS CONVERGED ,COMPUTE ROOTS USING QUADRATIC FORMULA
C
   25 T = 1
      CALL QES(T,R,S,RR1,RI1,RR2,RI2)
      PUNCH 108
      PUNCH 109,L1,RR1,RI1,L2,RR2,RI2
      L1 = L1 + 2
      L2 = L2+2
      PRINT 117
      PAUSE
      GO TO 4
   45 PUNCH 107
      PRINT 107
      PAUSE
C
C     HIT START TO READ IN NEW VALUES FOR R AND S
C
      GO TO 2
C
    4 J = J-2
      IF(J-2)65,75,85
   85 JP1 = J+1
      DO 50 I=1,JP1
   50 A(I) = B(I)
      GO TO 2
   75 CALL QES(B(1),B(2),B(3),RR1,RI1,RR2,RI2)
      PUNCH 118
      PUNCH 109,L1,RR1,RI1,L2,RR2,RI2
      PRINT 121
      PAUSE
      GO TO 1
   65 RR = -B(2)/B(1)
      RI = 0.
      PUNCH 118
      PUNCH 109,L1,RR,RI
      PRINT 121
      GO TO 1
```

```
100 FORMAT(E14.8)
101 FORMAT(I3)
102 FORMAT(5E14.8)
103 FORMAT(    62HBAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF P
   1OLYNOMIALS/8X,46HP(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)/
   2 8X,39HTHE DEGREE N OF THE POLYNOMIAL P(X) IS I5/8X,46HTHE COEFFIC
   3IENTS A(1) TO A(N+1) ARE AS FOLLOWS/(24X,E14.8))
104 FORMAT(2E14.8)
105 FORMAT(21X,E14.8,5X,E14.8)
106 FORMAT(12X,I2,7X,E14.8,5X,E14.8)
107 FORMAT(/39HMETHOD HAS NOT CONVERGED IN 50 ITERATES/)
108 FORMAT(7H  ROOTS,8X,4HREAL,10X,9HIMAGINARY)
109 FORMAT(3X,I2,5X,E14.8,3X,E14.8)
114 FORMAT(/12X,11HEPSILON IS E14.8)
115 FORMAT(/9X,7HITERATE,11X,1HR,18X,1HS/)
116 FORMAT(18HCHOOSE NEW R AND S)
117 FORMAT(11HCONVERGENCE)
118 FORMAT(/7H  ROOTS,8X,4HREAL,10X,9HIMAGINARY)
121 FORMAT(10HFINAL HALT)
    END
```

SUBROUTINE
QUADRATIC EQUATION SOLVER

```
      SUBROUTINE QES(A3,A2,A1,RR1,RI1,RR2,RI2)
      D = A2**2-4.*A3*A1
      IF(D)5,15,15
   15 RR1 = (-A2+SQRTF(D))/(2.*A3)
      RR2 = (-A2-SQRTF(D))/(2.*A3)
      RI1 = 0
      RI2 = 0
      RETURN
    5 RR1 = -A2/(2.*A3)
      RR2 = RR1
      RI1 = SQRTF(-D)/(2.*A3)
      RI2 = -RI1
      RETURN
      END
```

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
        P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
        THE DEGREE N OF THE POLYNOMIAL P(X) IS     3
        THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                         .10000000E+01
                         .00000000E-99
                        -.10000000E+01
                        -.10000000E+01

            EPSILON IS  .00001

         ITERATE              R                      S

                       .10000000E+01          .10000000E+01
              1        .13333333E+01          .66666670E+00
              2        .13245615E+01          .75438592E+00
              3        .13247180E+01          .75487770E+00
              4        .13247180E+01          .75487770E+00
   ROOTS         REAL              IMAGINARY
     1      -.66235900E+00      .56227950E+00
     2      -.66235900E+00     -.56227950E+00

   ROOTS         REAL              IMAGINARY
     3       .13247180E+01      .00000000E-99
```

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
       P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
       THE DEGREE N OF THE POLYNOMIAL P(X) IS     5
       THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                        .10000000E+01
                       -.17000000E+02
                        .12400000E+03
                       -.50800000E+03
                        .10350000E+04
                       -.87500000E+03

          EPSILON IS  .00010

        ITERATE             R                    S

                     .00000000E-99        .00000000E-99
             1      -.16169632E+01        .17224409E+01
             2      -.28679906E+01        .33046610E+01
             3      -.36781091E+01        .44873180E+01
             4      -.39708941E+01        .49519633E+01
             5      -.39997555E+01        .49995860E+01
             6      -.40000009E+01        .50000008E+01
             7      -.40000009E+01        .50000008E+01
   ROOTS        REAL          IMAGINARY
     1      .20000004E+01    .99999945E+00
     2      .20000004E+01   -.99999945E+00

        ITERATE             R                    S

                     .00000000E-99        .00000000E-99
             1      -.41183426E+01        .13461536E+02
             2      -.58254301E+01        .22290883E+02
             3      -.60079523E+01        .24958785E+02
             4      -.59999744E+01        .24999951E+02
             5      -.59999982E+01        .24999990E+02
   ROOTS        REAL          IMAGINARY
     3      .29999872E+01    .40000034E+01
     4      .29999872E+01   -.40000034E+01

   ROOTS        REAL          IMAGINARY
     5      .70000260E+01    .00000000E-99
```

EXAMPLE 13 (A)
CHAPTER 2

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
        P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
        THE DEGREE N OF THE POLYNOMIAL P(X) IS     6
        THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                                .32600000E+01
                                .00000000E-99
                                .42000000E+01
                                .30800000E+01
                               -.71600000E+01
                                .19200000E+01
                               -.77600000E+01

            EPSILON IS   .10000000E-04

         ITERATE              R                     S

                          .00000000E-99        .00000000E-99
              1           .19805873E+00        .10837988E+01
              2           .95997580E-01        .90173020E+00
              3           .11244911E+00        .88997786E+00
              4           .11218236E+00        .89019927E+00
              5           .11218228E+00        .89019935E+00
   ROOTS          REAL            IMAGINARY
     1       -.56091180E-01     .94183490E+00
     2       -.56091180E-01    -.94183490E+00

         ITERATE              R                     S

                          .00000000E-99        .00000000E-99
              1           .65306290E+00       -.65103051E+01
              2           .35247705E+00       -.34335817E+01
              3           .24574896E+00       -.19838148E+01
              4           .24185552E+00       -.14808962E+01
              5           .25340898E+00       -.14044602E+01
              6           .25421987E+00       -.14025857E+01
              7           .25422081E+00       -.14025844E+01
   ROOTS          REAL            IMAGINARY
     3        .10639999E+01     .00000000E-99
     4       -.13182197E+01     .00000000E-99

   ROOTS          REAL            IMAGINARY
     5        .18320110E+00     .13685389E+01
     6        .18320110E+00    -.13685389E+01
```

BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS

```
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
THE DEGREE N OF THE POLYNOMIAL P(X) IS     6
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                 .32600000E+01
                 .00000000E-99
                 .42000000E+01
                 .30800000E+01
                -.71600000E+01
                 .19200000E+01
                -.77600000E+01

    EPSILON IS  .10000000E-07
```

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	.19805873E+00	.10837988E+01
2	.95997580E-01	.90173020E+00
3	.11244911E+00	.88997786E+00
4	.11218236E+00	.89019927E+00
5	.11218228E+00	.89019935E+00
6	.11218228E+00	.89019935E+00

ROOTS	REAL	IMAGINARY
1	-.56091140E-01	.94183495E+00
2	-.56091140E-01	-.94183495E+00

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	.65306408E+00	-.65103057E+01
2	.35247765E+00	-.34335823E+01
3	.24574921E+00	-.19838151E+01
4	.24185558E+00	-.14808962E+01
5	.25340902E+00	-.14044601E+01
6	.25421991E+00	-.14025857E+01
7	.25422085E+00	-.14025844E+01
8	.25422085E+00	-.14025844E+01

ROOTS	REAL	IMAGINARY
3	.10639989E+01	.00000000E-99
4	-.13182197E+01	.00000000E-99

ROOTS	REAL	IMAGINARY
5	.18320156E+00	.13685386E+01
6	.18320156E+00	-.13685386E+01

EXAMPLE 13 (C)
CHAPTER 2

BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS

```
        P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
        THE DEGREE N OF THE POLYNOMIAL P(X) IS      6
        THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                                  .32600000E+01
                                  .00000000E-99
                                  .42000000E+01
                                  .30800000E+01
                                 -.71600000E+01
                                  .19200000E+01
                                 -.77600000E+01

            EPSILON IS  .10000000E-04

         ITERATE             R                    S

                        .00000000E-99         .00000000E-99
            1           .19805873E-00         .10837988E+01
            2           .95997548E-01         .90173012E-00
            3           .11244910E-00         .88997783E-00
            4           .11218234E-00         .89019925E-00
            5           .11218227E-00         .89019933E-00
   ROOTS          REAL            IMAGINARY
     1       -.56091172E-01     .94183493E-00
     2       -.56091172E-01    -.94183493E-00

         ITERATE             R                    S

                        .00000000E-99         .00000000E-99
            1           .65306317E-00        -.65103047E+01
            2           .35247717E-00        -.34335812E+01
            3           .24574899E-00        -.19838145E+01
            4           .24185551E-00        -.14808960E+01
            5           .25340897E-00        -.14044600E+01
            6           .25421986E-00        -.14025856E+01
            7           .25422081E-00        -.14025843E+01
   ROOTS          REAL            IMAGINARY
     3        .10639998E+01     .00000000E-99
     4       -.13182197E+01     .00000000E-99

   ROOTS          REAL            IMAGINARY
     5        .18320110E-00     .13685389E+01
     6        .18320110E-00    -.13685389E+01
```

EXAMPLE 14
CHAPTER 2

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
        P(X)  = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
        THE DEGREE N OF THE POLYNOMIAL P(X) IS       7
        THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                                .10000000E+01
                                .00000000E-99
                               -.20000000E+01
                                .00000000E-99
                               -.30000000E+01
                                .40000000E+01
                               -.50000000E+01
                                .60000000E+01

            EPSILON IS  .10000000E-04

         ITERATE              R                    S

                         .00000000E-99       .00000000E-99
              1         -.12500000E+00       .15000000E+01
              2         -.33026133E+00       .88502620E+00
              3         -.69440005E+00       .10784368E+01
              4         -.60706881E+00       .10696905E+01
              5         -.60921879E+00       .10767151E+01
              6         -.60921328F+00       .10766801E+01
              7         -.60921328E+00       .10766801E+01
  ROOTS        REAL             IMAGINARY
    1       .30460664E+00     .99191475E+00
    2       .30460664E+00    -.99191475E+00

         ITERATE              R                    S

                         .00000000F-99       .00000000E-99
              1          .34867614E+01      -.24185137E+01
              2          .26015274E+01      -.19842264E+01
              3          .19315073E+01      -.17834572E+01
              4          .14045295E+01      -.18387089E+01
              5          .10336157E+01      -.20264880E+01
              6          .88039870E+00      -.21434174E+01
              7          .85524766E+00      -.21732008E+01
              8          .85447464E+00      -.21744699E+01
              9          .85447380E+00      -.21744715E+01
  ROOTS        REAL             IMAGINARY
    3       .11080156E+01     .00000000E-99
    4      -.19624902E+01     .00000000F-99
```

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	.43915078E+02	.10449172E+02
2	.21959686E+02	.52530510E+01
3	.10987535E+02	.27110865E+01
4	.55172830E+01	.15484181E+01
5	.28305900E+01	.11666539E+01
6	.16291744E+01	.12888837E+01
7	.13034803E+01	.15911775E+01
8	.12925243E+01	.16660050E+01
9	.12926297E+01	.16664238E+01
10	.12926297E+01	.16664238E+01

ROOTS	REAL	IMAGINARY
5	-.64631485E+00	.11174528E+01
6	-.64631485E+00	-.11174528E+01

ROOTS	REAL	IMAGINARY
7	.15378910E+01	.00000000E-99

EXAMPLE 15
CHAPTER 2

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
       P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
       THE DEGREE N OF THE POLYNOMIAL P(X) IS      8
       THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                           .10000000E+01
                           .20400000E+02
                           .15130000E+03
                           .49000000E+03
                           .68700000E+03
                           .71900000E+03
                           .15000000E+03
                           .10900000E+03
                           .68700000E+01

          EPSILON IS  .00001

        ITERATE            R                      S

                      .00000000E-99          .00000000E-99
           1          .50713200E+00          .45800000E-01
           2         -.50994650E-01          .12056259E-01
           3          .29452665E+00          .58640522E-01
           4         -.16727224E+00          .78684068E-01
           5          .35399700E-01          .10122710E+00
           6          .20501505E-01          .17233900E+00
           7         -.56442910E-02          .17012366E+00
           8         -.56575771E-02          .17079725E+00
           9         -.56604909E-02          .17079728E+00
  ROOTS          REAL             IMAGINARY
    1        .28287885E-02      .41326656E+00
    2        .28287885E-02     -.41326656E+00

        ITERATE            R                      S

                      .00000000E-99          .00000000E-99
           1          .91876182E+00          .60581391E-01
           2          .42300950E+01          .28963854E+00
           3          .76635625E+01          .50986716E+00
           4          .78875067E+01          .52713425E+00
           5          .78540419E+01          .52466013E+00
           6          .78531312E+01          .52459383E+00
           7          .78531280E+01          .52459362E+00
  ROOTS          REAL             IMAGINARY
    3       -.67378750E-01      .00000000E-99
    4       -.77857520E+01      .00000000E-99
```

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	.10133398E+01	.14704039E+01
2	.13146751E+01	.21107625E+01
3	.13354893E+01	.21917771E+01
4	.13355030E+01	.21924679E+01
5	.13355030E+01	.21924679E+01

ROOTS	REAL	IMAGINARY
5	-.66775150E+00	.13215808E+01
6	-.66775150E+00	-.13215808E+01

ROOTS	REAL	IMAGINARY
7	-.56085115E+01	.18748846E+01
8	-.56085115E+01	-.18748846E+01

EXAMPLE 6 (B)
CHAPTER 2

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
        P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
        THE DEGREE N OF THE POLYNOMIAL P(X) IS      3
        THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                            .10000000E+01
                           -.30000000E+01
                            .40000000E+01
                           -.50000000E+01

            FPSILON IS  .10000000F-04

          ITERATE            R                   S

                         .00000000E-99       .00000000E-99
              1         -.77777777E+00       .16666666E+01
              2         -.78762305E+00       .22573839E+01
              3         -.78658759E+00       .22589561E+01
              4         -.78658832E+00       .22589561E+01
    ROOTS       REAL           IMAGINARY
      1     .39329379E+00     .14506123E+01
      2     .39329379E+00    -.14506123E+01

    ROOTS       REAL           IMAGINARY
      3     .22134125E+01     .00000000E-99
```

EXAMPLE 7 (C)
CHAPTER 2

```
BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS
       P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
       THE DEGREE N OF THE POLYNOMIAL P(X) IS      6
       THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS
                         .10000000E+01
                         .30000000E+01
                        -.10000000E+01
                        -.70000000E+01
                         .10000000E+02
                         .14000000E+02
                        -.20000000E+02

           EPSILON IS  .10000000E-04

        ITERATE              R                    S

                      .00000000E-99        .00000000E-99
            1         .00000000F-99       -.20000000E+01
            2         .71532846E+00       -.21751824E+01
            3         .98185698E+00       -.20464492E+01
            4         .99971621E+00       -.20005900E+01
            5         .99999988E+00       -.20000000E+01
            6         .10000001E+01       -.20000000E+01
  ROOTS         REAL            IMAGINARY
    1        .10000000E+01     .00000000E-99
    2       -.19999998E+01     .00000000E-99

        ITERATE              R                    S

                      .00000000E-99        .00000000E-99
            1        -.18000006E+02       -.10000001E+02
            2        -.11688143E+02       -.65660120E+01
            3        -.74978080E+01       -.42488138E+01
            4        -.47364012E+01       -.26356980E+01
            5        -.29642537E+01       -.13968442E+01
            6        -.19614340E+01       -.18518910E+00
            7        -.17926335E+01        .13697357E+01
            8        -.20306692E+01        .20601793E+01
            9        -.20005634E+01        .20007869E+01
           10        -.20000001E+01        .20000002E+01
           11        -.20000000E+01        .20000001E+01
  ROOTS         REAL            IMAGINARY
    3        .10000000E+01     .10000000E+01
    4        .10000000F+01    -.10000000F+01

  ROOTS         REAL            IMAGINARY
    5       -.20000001E+01     .10000001E+01
    6       -.20000001E+01    -.10000001F+01
```

EXAMPLE 4 (B)
CHAPTER 2

BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS

P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)

THE DEGREE N OF THE POLYNOMIAL P(X) IS 3

THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS

.10000000E+01
-.30600000E+01
.31211000E+01
-.10611060E+01

EPSILON IS .10000000E-04

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	-.90664488E+00	.34676666E+00
2	-.13868472E+01	.57009833E+00
3	-.16523766E+01	.72467022E+00
4	-.18054803E+01	.83264872E+00
5	-.18964961E+01	.90623555E+00
6	-.19515692E+01	.95488758E+00
7	-.19851523E+01	.98623586E+00
8	-.20056114E+01	.10059877E+01
9	-.20178974E+01	.10180930E+01
10	-.20249377E+01	.10251138E+01
11	-.20285192E+01	.10287087E+01
12	-.20293595E+01	.10295592E+01
13	-.20302004E+01	.10303998E+01
14	-.20301990E+01	.10303991E+01

ROOTS	REAL	IMAGINARY
1	.10204480E+01	.00000000E-99
2	.10097523E+01	.00000000E-99

ROOTS	REAL	IMAGINARY
3	.10297996E+01	.00000000E-99

EXAMPLE 5 (B)
CHAPTER 2

ROOTS OF THE POLYNOMIAL
P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)
THE DEGREE N OF THE POLYNOMIAL P(X) IS 3
THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS

.10000000E+01

-.30060000E+01

.30120110E+01

-.10060110E+01

EPSILON IS .9500

P = 2
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .30120140E+01 .30240720E+01 .10120581E+01

P = 4
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .30240841E+01 .30483454E+01 .10242616E+01

P = 8
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .30483940E+01 .30975028E+01 .10491120E+01

P = 16
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .30977003E+01 .31983106E+01 .11006359E+01

P = 32
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .31991261E+01 .34103098E+01 .12113995E+01

P = 64
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .34137883E+01 .38793731E+01 .14674889E+01

P = 128
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .38952045E+01 .50301422E+01 .21535239E+01

P = 256
THE COEFFICIENTS OF THE TRANSFORMED EQUATION
.10000000E+01 .51123339E+01 .85254989E+01 .46376652E+01

P = 512
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.90849606E+01	.25265545E+02	.21507939E+02

P = 1024
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.32005418E+02	.24755023E+03	.46259144E+03

P = 2048
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.52924637E+03	.31670250E+05	.21399084E+06

P = 4096
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.21676122E+06	.77649701E+09	.45792082E+11

P = 8192
THE COEFFICIENTS OF THE TRANSFORMED EQUATION

.10000000E+01	.45432434E+11	.58309572E+18	.20969148E+22

CROSS PRODUCT TERMS ARE NEGLIGIBLE

THE COEFFICIENTS LISTED DIRECTLY ABOVE ARE THE COEFFICIENTS OF THE TERMINAL EQUATION

THE POSSIBLE REAL ROOTS OF P(X)

I	X(I)	F(X(I))	-X(I)	F(-X(I))
1	.10030000E+01	.69000000E-13	-.10030000E+01	-.80601485E+01
2	.10019999E+01	.00000000E-99	-.10019999E+01	-.80480940E+01
3	.10009999E+01	-.68000000E-13	-.10009999E+01	-.80360516E+01

EXAMPLE 5 (B)
CHAPTER 2

BAIRSTOW'S METHOD FOR FINDING QUADRATIC FACTORS OF POLYNOMIALS

P(X) = A(1)*X**N+A(2)*X**N-1+...+A(N)*X+A(N+1)

THE DEGREE N OF THE POLYNOMIAL P(X) IS 3

THE COEFFICIENTS A(1) TO A(N+1) ARE AS FOLLOWS

.10000000E+01
-.30060000E+01
.30120110E+01
-.10060110E+01

EPSILON IS .10000000E-04

ITERATE	R	S
	.00000000E-99	.00000000E-99
1	-.89066644E-00	.33466766E-00
2	-.13624176E+01	.55021623E-00
3	-.16232926E+01	.69941653E-00
4	-.17737425E+01	.80366835E-00
5	-.18632289E+01	.87475906E-00
6	-.19174554E+01	.92183480E-00
7	-.19506445E+01	.95228612E-00
8	-.19710588E+01	.97166430E-00
9	-.19836440E+01	.98386234E-00
10	-.19914085E+01	.99148464E-00
11	-.19961961E+01	.99622151E-00
12	-.19991401E+01	.99914840E-00
13	-.20009366E+01	.10009396E+01
14	-.20020101E+01	.10020119E+01
15	-.20026156E+01	.10026174E+01
16	-.20029051E+01	.10029070E+01
17	-.20029917E+01	.10029937E+01
18	-.20029999E+01	.10030019E+01

ROOTS	REAL	IMAGINARY
1	.10019908E+01	.00000000E-99
2	.10010008E+01	.00000000E-99

ROOTS	REAL	IMAGINARY
3	.10030082E+01	.00000000E-99

```
C      NEWTON-RAPHSON METHOD FOR SIMULTANEOUS EQUATIONS
C
C      METHOD OF SOLUTION FOR FINDING THE REAL ROOTS OF TWO EQUATIONS IN
C      TWO UNKNOWNS, F1(X,Y) = 0, F2(X,Y) = 0
C
C      MUST HAVE SUBROUTINE FOR F1,F2,DXF1,DYF1,DXF2,DYF2
C
C      XO AND YO ARE THE APPROXIMATE VALUES FOR A PAIR OF ROOTS
C      XO AND YO ARE PREDETERMINED AND ARE READ IN
C
C      X AND Y ARE THE EXACT VALUES OF THE PAIR OF ROOTS
C      AN EPSILON CRITERION MUST BE SATISFIED, EPSILON IS READ IN
C
C      A CONVERGENCE CRITERION EXISTS
C
C                                  JANUARY 1966,  CARD
C
     1 READ 10,XO
       READ 10,YO
       READ 10,EPS
       PUNCH 11,XO,YO,EPS
       ITER = 1
     2 CALL DO(XO,YO,F1,F2,DXF1,DYF1,DXF2,DYF2)
       D = DXF1*DYF2-DXF2*DYF1
       H = (-F1*DYF2+DYF1*F2)
       G = (-F2*DXF1+F1*DXF2)
       X = XO+H/D
       Y = YO+G/D
       PUNCH 12,ITER,X,Y
       IF(ABSF(XO-X)-EPS)3,3,4
     3 IF(ABSF(YO-Y)-EPS)5,5,4
     4 ITER = ITER+1
       XO = X
       YO = Y
       IF(ITER-50)2,2,6
     5 PUNCH 13,X,Y
       GO TO 1
     6 PUNCH 14
       GO TO 1
    10 FORMAT(E14.8)
    11 FORMAT(41HTHE PREDETERMINED APPROXIMATE ROOT XO IS E14.8//41HTHE P
      1REDETERMINED APPROXIMATE ROOT YO IS E14.8//11HEPSILON IS E14.8//)
    12 FORMAT(14HITERATION NO. I3,5X,9HROOT X = E14.8//22X,9HROOT Y = E14
      1.8//)
    13 FORMAT(40HTHE EPSILON CRITERION HAS BEEN SATISFIED//5X,14HAND ROOT
      1 X IS E14.8,7X,10HROOT Y IS E14.8)
    14 FORMAT(64HTHE EPSILON CRITERION HAS NOT BEEN SATISFIED AFTER 50 IT
      1ERATIONS)
       END
```

EXAMPLE 1
CHAPTER 3

```
THE PREDETERMINED APPROXIMATE ROOT XO IS  .34000000E+01

THE PREDETERMINED APPROXIMATE ROOT YO IS  .22000000E+01

EPSILON IS  .10000000E-05

ITERATION NO.   1     ROOT X =  .34899099E+01

                      ROOT Y =  .22633598E+01

ITERATION NO.   2     ROOT X =  .34874422E+01

                      ROOT Y =  .22616255E+01

ITERATION NO.   3     ROOT X =  .34874405E+01

                      ROOT Y =  .22616242E+01

ITERATION NO.   4     ROOT X =  .34874404E+01

                      ROOT Y =  .22616242E+01

THE EPSILON CRITERION HAS BEEN SATISFIED

     AND ROOT X IS  .34874404E+01       ROOT Y IS  .22616242E+01
```

EXAMPLE 1
CHAPTER 3

```
THE PREDETERMINED APPROXIMATE ROOT XO IS  .14000000E+01
THE PREDETERMINED APPROXIMATE ROOT YO IS -.15000000E+01
EPSILON IS  .10000000E-05

ITERATION NO.   1      ROOT X =  .14573449E+01
                       ROOT Y = -.13996970E+01

ITERATION NO.   2      ROOT X =  .14588896E+01
                       ROOT Y = -.13967682E+01

ITERATION NO.   3      ROOT X =  .14588911E+01
                       ROOT Y = -.13967658E+01

ITERATION NO.   4      ROOT X =  .14588911E+01
                       ROOT Y = -.13967658E+01

THE EPSILON CRITERION HAS BEEN SATISFIED
     AND ROOT X IS  .14588911E+01       ROOT Y IS -.13967658E+01
```