1954

Concerning Borel sets and analytic sets

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CONCERNING BOREL SETS AND ANALYTIC SETS

by

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B. A., Westmar College, 1950

Presented in partial fulfillment

of the requirements for the degree of

Master of Arts

MONTANA STATE UNIVERSITY

1954

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July 23 1954

Date

ACKNOWLEDGMENTS

It was Dr. William Myers who taught me to appreciate and to pursue the study of mathematics. It would indeed be an understatement to say that without his help this work could not have been completed. Not only am I indebted to Dr. Myers for his guidance throughout the writing of this paper, but also for his untiring efforts in helping me with my basic mathematical preparation.

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CHAPTER I

INTRODUCTION

In this paper we shall discuss the general properties of the Borel sets and the analytic sets, and shall show several important relationships between these two classes of sets.

The family of Borel sets is defined to be the collection of all the Hausdorff sets P^α and Q^α, where α is an ordinal number of the first or second class. The sets P^α and Q^α are defined by transfinite induction, and are discussed in general in Chapter II. The sets F_α and G_α are then defined in a manner very similar to the Hausdorff sets, and the relationships between the sets of these two families are shown. It is shown also that an equivalent definition of the Borel sets is that they are the smallest family of sets which contain the closed sets, and are closed under countable sums and intersections.

Through the development of the Borel sets in this manner, many properties of the classes of sets P^α and Q^α and of sets F_α and G_α are discussed. The principal problem solved concerning these classes of sets is that of showing in one-dimension Euclidean space that there exists, for each ordinal number α of the first and second classes, sets F_α and sets G_α which are not sets F_β or G_β for each ordinal number β less than α. In Chapter IV, this is established with the aid of sets G_α of the plane which are universal to the linear sets G_α, for each ordinal number α. The proof is completed by applying the "diagonal line" theorem of Sierpinski.

The analytic sets are defined and discussed in general in Chapter V. The principal theorem concerning these sets is that of showing that an analytic operation carried out on a class of analytic sets yields a set of the original class. This leads to the proof that the analytic sets relative to the class of closed sets contains the family of Borel sets.

In the final chapter, it is shown that in one-dimension Euclidean space the family of linear Borel sets is contained properly in the family of linear analytic sets. To show this, a set G_δ of three-dimension Euclidean space universal to all plane sets G_δ is projected onto the plane, the resulting plane set being an analytic set universal to all linear analytic sets. The "diagonal line" theorem of Sierpinski is again employed to complete the proof.

It is assumed that the reader is familiar with the basic topological concepts and with the fundamental properties of continued fractions, cardinal numbers, and ordinal numbers. To avoid ambiguities in the use of terms, we shall define those terms which are used frequently in the text.

A set is any collection of objects which we shall call elements. If x is an element of the set E, then we write $x \in E$. If A is a set such that $x \in A$ implies that $x \in E$, then A is said to be a subset of E, written $A \subset E$.

The sum of two sets A and B is a set, $A+B$, such that $x \in A+B$ if and only if $x \in A$ or $x \in B$ or both. Given a sequence of sets $E_1, E_2, E_3, \ldots$, written $\{E_n\}$, we say that the sum of this sequence of sets is a set $E_1 + E_2 + E_3 + \ldots$ or $\sum_{n=1}^{\infty} E_n$, such that $x \in \sum_{n=1}^{\infty} E_n$, if and only if $x \in E_i$ for at least one integer i. In a like manner, we may define the sum of a non-countable collection of sets.

The product (intersection) of two sets A and B is a set, $A \cdot B$, such that $x \in A \cdot B$ if and only if $x \in A$ and $x \in B$. Given a sequence of sets, $\{E_n\}$, we say that the product of this sequence of sets is a set $E_1 \cdot E_2 \cdot E_3 \cdot \ldots$ or $\prod_{n=1}^{\infty} E_n$, such that $x \in \prod_{n=1}^{\infty} E_n$, if and only if $x \in E_i$ for every integer $i = 1, 2, 3, \ldots$. In a like manner, we may define the product of a non-countable collection of sets.

A set of elements S is said to be a metric space if there is associated with each pair of elements a and b of S a non-negative real number, called the distance between these elements and denoted by $\rho(a,b)$, such that the three following axioms are satisfied.

1) $\rho(a,b)=\rho(b,a)$.

2) $\rho(a,b)=0$, if and only if $a=b$.

3) $\rho(a,c)\leq\rho(a,b)+\rho(b,c)$.

If E is a subset of a metric space S, then E will also be a metric space with proper metrization.

In a metric space S, the complement of a set $E \subset S$ is the set of all elements contained in S but not contained in E. If E and F are two subsets of the space S, then the complement of E relative to F, written as $F \cdot \complement E$ or $F - E$, is the set of all elements of F which are not elements of E.

The least upper bound of the distances between all pairs of elements a and b of a set E is called the diameter of E, and is denoted by $\delta(E)$.

If $x \in S$, and if ϵ is an arbitrary positive real number, then an ϵ-neighborhood of the element x is the set of all elements y of S such that $\rho(x,y)<\epsilon$, and this neighborhood shall be denoted by $N(x,\epsilon)$. A set E will be called an open set if for every element x of E there exists for some $\epsilon>0$, depending on x, an ϵ-neighborhood of x contained entirely in E. A set F will be called a closed set if and only if it is the complement of an open set.

An element x is called a cluster point of a set E if for every $\epsilon>0$, $N(x,\epsilon)$ contains at least one point of E different from x. It can be shown that a set E is closed if and only if it contains all of its cluster points. [6, p. 33] If a set E is such that every element of E is a cluster point, then E is said to be <u>dense-in-itself</u>. The closure of a set E, denoted by $\overline{E}$, is the set of all elements x such that for every $\epsilon > 0$, $N(x,\epsilon)$ contains at least one element of E.

An element which is such that every neighborhood of it contains a non-countable number of elements of a set E is said to be an <u>element of condensation</u> of E.

If $x \in E$, and if $E \subset N(x,\epsilon)$ for some real number $\epsilon > 0$, then E is said to be a <u>bounded set</u>.

An infinite sequence of elements, $a_1, a_2, a_3, \ldots$, denoted by $\{a_n\}$, is said to <u>converge</u> to a limit b if for every positive real number ϵ there exists an integer N such that if $n > N$, then $\rho(a_n, b) < \epsilon$. An infinite sequence of elements $\{a_n\}$ is said to be a <u>Cauchy sequence</u> if for every $\epsilon > 0$ there exists an integer N such that if $n > N$ and $m > N$, then $\rho(a_n, a_m) < \epsilon$. Metric spaces in which Cauchy sequences are always convergent sequences are called <u>complete spaces</u>.

A set $E \subset S$ is said to be <u>dense</u> on S if $\overline{E} = S$. If a space S has a countable dense subset, then S is said to be a <u>separable space</u>.

A space S is said to have a <u>countable open basis</u> if there exists a countable sequence of open sets, $\{U_n\}$, such that any open set of S can be written as a sum of sets belonging to $\{U_n\}$. If S is a metric space, then the conditions of separablity and of having a countable open basis are equivalent. [7, p. 116]

An open covering of E is any aggregate of open sets whose sum contains E. A set E is said to be compact if from every open covering of E a finite subcovering can be selected. A set E is compact if and only if every infinite subset of E has a cluster point in E. In any metric space, a compact set is bounded and closed, and in any n-dimension Euclidean space, a bounded and closed set is compact and vica versa. [5, pp. 41f.]

If E and T are two sets of a metric space S, and if for each element x of E, there corresponds an element f(x) of T, then we say that f is a mapping of E into T. If every element of T is the image of at least one element of E by the mapping f, then f is said to be a mapping of E onto T. A mapping f of E into T is said to be continuous at x_0 of E if for every positive real number ϵ, there exists a positive real number δ such that if $\rho(x,x_0) < \delta$, $x \in E$, then $\rho(f(x),f(x_0)) < \epsilon$. If f is continuous at every point of E, then we say that f is a continuous mapping on E.

If f is a mapping of E into T, and if $y \in T$, then $f^{-1}(y)$, (f-inverse of y), is the set of all points $x \in E$ such that $f(x) = y$. If f is a continuous mapping of E into T, and if f^{-1} is a continuous mapping of T into E, then f is said to be a topological or homeomorphic mapping.

A property of a set E is said to be a <u>topologically invariant property</u> if it is a property possessed by every set which is a homeomorphic image of E. A family of sets F is <u>topologically invariant</u> if every homeomorphic image of a set of the family F also belongs to F.

CHAPTER II

HAUSDORFF SETS P^α AND Q^α

In this chapter we shall define the Hausdorff sets P^α and Q^α, and shall prove several important properties of these sets. Throughout our discussion we shall assume that we are working within a complete metric space M.

<u>Definition</u>: A set E is a set F_σ if $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a closed set.

<u>Definition</u>: A set E is a set G_δ if $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is an open set.

Since $\mathcal{C}(\sum_{n=1}^{\infty} E_n) = \prod_{n=1}^{\infty}(\mathcal{C} E_n)$, a set will be a set F_σ if and only if its complement is a set G_δ.

<u>Theorem 2:1</u> : <u>Every closed set is a set G_δ</u>.

<u>Proof</u>: Suppose that F is a closed set. Let $F_n = \sum_{x\in F} N(x, 1/n)$. Thus each set F_n is open, and $F = \prod_{n=1}^{\infty} F_n$, for if $x \in F$, then for each n, $x \in F_n$ and hence $x \in \prod_{n=1}^{\infty} F_n$. On the other hand if $x \in \prod_{n=1}^{\infty} F_n$, then for each n, $x \in F_n$. Thus for each n there exists a $q_n \in F$ such that $\rho(x, q_n) < 1/n$. Therefore $x \in \overline{F}$, which means that $x \in F$ since F is closed.

Since the complement of a closed set is an open set, and the complement of a set G_δ is a set F_σ, we have the following theorem:

<u>Theorem 2:2</u> : <u>Every open set is a set F_σ</u>.

It can be shown that the homeomorphic image of a set G_δ is again a set G_δ. This is not necessarily true of a set F_σ however. If we assume a stronger condition on our metric space M, namely that "every closed, bounded set is compact", then a continuous image of a set F_σ will be a set F_σ. [8, pp. 121-127]

Hausdorff sets P^α and Q^α are defined in this manner. A set E is a set P^1 if and only if it is an open set, and is a set Q^1 if and only if it is a closed set. For any ordinal number α, $1 < \alpha < \Omega$, where Ω is the first ordinal of the third class, we define sets P^α and Q^α by transfinite induction as follows:

Sets P^α: E is a set P^α if $E = \sum_{n=1}^{\infty} E_n$, where for each n, the set E_n is a set Q^{α_n}, where $\alpha_n < \alpha$.

Sets Q^α: E is a set Q^α if $E = \prod_{n=1}^{\infty} E_n$, where for each n, the set E_n is a set P^{α_n}, where $\alpha_n < \alpha$.

A set P^2, being a countable sum of sets Q^1 (closed sets), is merely a set F_σ, and a set Q^2, being a countable product of sets P^1 (open sets), is a set G_δ.

Theorem 2:3 : Every set P^α is also a set P^β for $\alpha < \beta < \Omega$, and every set Q^α is also a set Q^β for $\alpha < \beta < \Omega$.

Proof: For $\alpha = 1$, we have noted that each set P^1 is a set $P^2(F_\sigma)$ by theorem 2:2, and that each set Q^1 is a set $Q^2(G_\delta)$ by theorem 2:1. If E is a set P^α, $1 < \alpha < \Omega$, then $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set Q^{α_n}, $\alpha_n < \alpha$, hence $\alpha_n < \alpha < \beta$. Thus the definition of a set P^β is satisfied. Likewise if E is a set Q^α, $1 < \alpha < \Omega$, then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set P^{α_n}, $\alpha_n < \alpha$, hence $\alpha_n < \beta$. The set E is therefore a set Q^β, $\alpha < \beta < \Omega$.

<u>Theorem 2:4 : The sum of a finite or countable collection of sets P^α is a set P^α, and the product of a finite or countable collection of sets Q^α is a set Q^α.</u>

<u>Proof:</u> If $\alpha = 1$, the theorem is satisfied by elementary properties of open and closed sets. Suppose $\alpha > 1$, and $E = \sum_{k=1}^{\infty} E_k$, where for each k, E_k is a set P^α. Then $E_k = \sum_{n=1}^{\infty} F_{k,n}$, where $F_{k,n}$ is a set $Q^{\alpha_{k,n}}$, $\alpha_{k,n} < \alpha$. Therefore $E = \sum_{k=1}^{\infty} E_k = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} F_{k,n}$, and hence is a set P^α.

If $E = \prod_{k=1}^{\infty} E_k$, where for each k, E_k is a set Q^α, then $E_k = \prod_{n=1}^{\infty} E_{k,n}$, where $E_{k,n}$ is a set $P^{\alpha_{k,n}}$, $\alpha_{k,n} < \alpha$. Thus $E = \prod_{k=1}^{\infty} \prod_{n=1}^{\infty} E_{k,n}$, and hence is a set Q^α.

<u>Theorem 2:5 : The complement of a set $P^\alpha(Q^\alpha)$ is a set $Q^\alpha(P^\alpha)$.</u>

<u>Proof:</u> The theorem is true for $\alpha = 1$ by the properties of open and closed sets. Proceding by transfinite induction, suppose that α is an ordinal number such that $1 < \alpha < \Omega$, and suppose that the theorem is true for all ordinal numbers $\beta < \alpha$. If E is a set P^α, then $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set Q^{α_n}, $\alpha_n < \alpha$. Thus the set $\complement E_n$ is a set P^{α_n} for each n by our induction assumption, and since $\complement E = \complement \sum_{n=1}^{\infty} E_n = \prod_{n=1}^{\infty} \complement E_n$, $\complement E$ will be a set Q^α.

If E is a set Q^α, then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set P^{α_n}, $\alpha_n < \alpha$. $\complement E = \complement \prod_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \complement E_n$, where $\complement E_n$

is a set Q^{α_n} for each n, and hence $\complement E$ is a set P^{α}.

<u>Theorem 2:6</u> : <u>The sum of a finite number of sets Q^{α} is a set Q^{α}, and the product of a finite number of sets P^{α} is a set P^{α}.</u>

<u>Proof</u>: For $\alpha = 1$ the theorem follows from the properties of open and closed sets. Suppose that α is any ordinal number such that $1 < \alpha < \Omega$. If E and T are both sets P^{α}, then $E = \sum_{n=1}^{\infty} E_n$, and $T = \sum_{k=1}^{\infty} T_k$, where for each n, E is a set Q^{α_n}, $\alpha_n < \alpha$, and where for each k, T_k is a set Q^{β_k}, $\beta_k < \alpha$. Then $E \cdot T = \sum_{n=1}^{\infty} E_n \cdot \sum_{k=1}^{\infty} T_k = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} E_n \cdot T_k$. Denote by $\xi_{n,k}$ the largest of the two ordinals α_n and β_k (or their common value if they are equal) for each pair of sets E_n and T_k. By theorem 2:3, both E_n and T_k are sets $Q^{\xi_{n,k}}$, and by theorem 2:4, the set $E_n \cdot T_k$ is a set $Q^{\xi_{n,k}}$, $\xi_{n,k} < \alpha$. Thus $E \cdot T$ is a set P^{α}.

If F and S are sets Q^{α}, then the set $F + S$ can be written as $\complement \complement (F + S) = \complement (\complement F \cdot \complement S)$. But the sets $\complement F$ and $\complement S$ are sets P^{α}, and so their product is a set P^{α} from the above proof. The set $F + S$ is therefore the complement of a set P^{α}, which is a set Q^{α} by theorem 2:5.

Having proved the theorem in the case of two sets, the proof may be extended to the case of any finite number of sets by ordinary induction methods.

<u>Theorem 2:7</u> : <u>Every set $P^{\alpha}(Q^{\alpha})$ is a set $Q^{\alpha+1}(P^{\alpha+1})$</u>.

<u>Proof</u>: If E is a set P^{α}, then we may write $E = E \cdot E \cdot E \cdots$, thus satisfying the definition of a set $Q^{\alpha+1}$.

Likewise, if E is a set Q^{α}, then $E = E + E + E + \cdots$, and is therefore a set $P^{\alpha+1}$.

<u>Theorem 2:8</u> : <u>The sum of a countable collection of sets P^{α} is a set $Q^{\alpha+1}$. The product of a countable collection of sets Q^{α} is a set $P^{\alpha+1}$.</u>

<u>Proof</u>: Suppose that $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set P^{α}. By theorem 2:4 the set E is a set P^{α}, and is therefore a set $Q^{\alpha+1}$ by theorem 2:7.

Suppose that $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set Q^{α}. By theorem 2:4 the set E is a set Q^{α}, and is therefore a set $P^{\alpha+1}$ by theorem 2:7.

<u>Theorem 2:9</u> : <u>The difference of two sets P^{α}, or of two sets Q^{α} is both a set $Q^{\alpha+1}$ and a set $P^{\alpha+1}$.</u>

<u>Proof</u>: Let $T = E_1 - E_2$, where E_1 and E_2 are sets P^{α}. Thus $T = E_1 \cdot \complement E_2$. But E_1 is a set $P^{\alpha+1}$ and a set $Q^{\alpha+1}$ by theorem 2:3 and theorem 2:7 respectively. In a like manner, E_2 is a set $P^{\alpha+1}$ and a set $Q^{\alpha+1}$, and so $\complement E_2$ is also. By theorem 2:6 and theorem 2:4, T is a set $P^{\alpha+1}$ and a set $Q^{\alpha+1}$. By taking complements, the second part of the theorem follows directly.

<u>Theorem 2:10</u> : <u>For $3 \leq \alpha < \Omega$, every set P^{α} is the sum of a countable collection of disjoint sets $E_1, E_2, E_3, \cdots$, where for each n, E_n is a set Q^{ε_n}, $\varepsilon_n < \alpha$.</u>

<u>Proof</u>: Suppose E is a set P^{α}, where $3 \leq \alpha < \Omega$.

Then $E = \sum_{n=1}^{\infty} T_n$, where for each n, T_n is a set Q^{β_n}, $2 \leq \beta_n < \alpha$. (For if T_n were a set Q^1, then it would also be a set Q^2 by theorem 2:3)

Let $S_K = \sum_{n=1}^{K} T_n$, and let ε_K be the maximum of the ordinals $\beta_1, \beta_2, \beta_3, \cdots, \beta_K$. Thus $2 \leq \varepsilon_K < \alpha$ for each k. S_K is a set Q^{ε_K} for each k by theorem 2:3 and theorem 2:6. We note $S_1 \subset S_2 \subset S_3 \subset \cdots$.

Let $R_1 = S_1$ and $R_{K+1} = S_{K+1} \cdot \complement S_K$ for each k. But $\complement S_K$ is a set P^{ε_K} by theorem 2:5, so that $\complement S_K = T_{K,1} + T_{K,2} + T_{K,3} + \cdots + T_{K,\ell} + \cdots$, where for each ℓ, $T_{K,\ell}$ is a set $Q^{\varepsilon_{K,\ell}}$, $\varepsilon_{K,\ell} < \varepsilon_K$. Let $\delta_{K,\ell}$ be the maximum of the ordinals $\varepsilon_{K,1}, \varepsilon_{K,2}, \varepsilon_{K,3}, \cdots, \varepsilon_{K,\ell}$ for each ℓ. Thus $\delta_{K,\ell} < \varepsilon_K$ for each ℓ.

Let $S_{K,\ell} = T_{K,1} + T_{K,2} + \cdots + T_{K,\ell}$ for each ℓ. By theorem 2:6, $S_{K,\ell}$ is a set $Q^{\delta_{K,\ell}}$ for each ℓ, and we note $S_{K,1} \subset S_{K,2} \subset S_{K,3} \subset \cdots$.

Let $R_{K,1} = S_{K,1}$ and $R_{K,\ell} = S_{K,\ell} \cdot \complement S_{K,\ell-1}$ for $\ell = 2, 3, 4, \cdots$. Since $\delta_{K,\ell-1} \leq \delta_{K,\ell}$ for $\ell \geq 2$, by theorem 2:9, $R_{K,\ell}$ is a set $Q^{\delta_{K,\ell}+1}$ for each ℓ; and since $\delta_{K,\ell} < \varepsilon_K$ for each ℓ, $\delta_{K,\ell}+1 \leq \varepsilon_K$, $R_{K,\ell}$ is a set Q^{ε_K}. But $\varepsilon_K \leq \varepsilon_{K+1}$, so $S_{K+1} \cdot R_{K,\ell}$ is a set $Q^{\varepsilon_{K+1}}$ for each ℓ by theorem 2:3 and theorem 2:4.

Let $F = S_1 + \sum_{K=1}^{\infty} \sum_{\ell=1}^{\infty} S_{K+1} \cdot R_{K,\ell}$. The set F is the sum of a countable collection of sets $Q^{\varepsilon_{K+1}}$ for $\varepsilon_{K+1} < \alpha$. The set F is also the sum of a countable collection of disjoint sets, for we note that $R_i \cdot R_K = 0$ if $i \neq k$ since $R_K = S_K \cdot \complement S_{K-1}$ and the sets S_K form an increasing sequence of sets. Following the same line of reasoning, we note that the sets $R_{k,\ell}$ are disjoint for a fixed k and where $\ell = 1,2,3,\cdots$.

Thus for a fixed k, we have the sets $R_1 + \sum_{\ell=1}^{\infty} S_{k+1} \cdot R_{k,\ell}$ where the sets $S_{k+1} \cdot R_{k,\ell}$ are disjoint since sets $R_{k,\ell}$ are disjoint for a fixed k and $\ell = 1,2,3,\cdots$. Since for any fixed k, $\sum_{\ell=1}^{\infty} R_{k,\ell} = \sum_{\ell=1}^{\infty} S_{k,\ell} = \complement S_k \subset R_{k+1}$, R_1 is disjoint from the other sets.

For a fixed number ℓ, we have the sets $R_1 + \sum_{k=1}^{\infty} S_{k+1} \cdot R_{k,\ell}$. For each k, $R_{k,\ell} \subset R_{k+1}$, and hence the sets are disjoint.

It remains to be shown that $E = F$. Since it is evident that $\sum_{\ell=1}^{\infty} R_{k,\ell} = \sum_{\ell=1}^{\infty} S_{k,\ell} = \complement S_k$, $F = R_1 + \sum_{k=1}^{\infty} S_{k+1} \cdot \complement S_k$. But $S_{k+1} \cdot \complement S_k = R_{k+1}$, so $F = \sum_{k=1}^{\infty} R_k = \sum_{k=1}^{\infty} S_k = \sum_{n=1}^{\infty} T_n = E$. The proof is complete.

<u>Theorem 2:11</u> : <u>For $3 \leq \alpha < \Omega$, sets P^{α} are topologically invariant, and for $2 \leq \alpha < \Omega$, sets Q^{α} are topologically invariant.</u>

<u>Proof</u>: Sets $Q^2(G_\delta)$ are topologically invariant. (See Chapter I, page 7) Preceding by transfinite induction, suppose that the theorem is true for every ordinal β where $2 \leq \beta < \alpha$, and let E be a set P^{α}. Then $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set Q^{α_n}, $\alpha_n < \alpha$. By theorem 2:3, we may assume that $\alpha_n \geq 2$ for each n.

Let T be a set which is homeomorphic to E by a mapping f. Let $T_n = f(E_n)$ for each integer n. Then the set $T = \sum_{n=1}^{\infty} T_n$, where for each n, T_n is a set Q^{α_n} by our induction assumption. T is therefore a set P^{α}.

Suppose that H is a set Q^{α}, where $\alpha \geq 3$, and let $\complement H = E$. Suppose that T is the homeomorphic image of H by a function f. There exists by Lavrentieff's theorem sets M and N, each a set $G_\delta(Q^2)$, such that $H \subset M$, $T \subset N$, and M is homeomorphic to N by a function ϕ such that $\phi(p) = f(p)$ if $p \in H$. [8, p. 126] Since $H \subset M$ and $H = \complement E$, $H = M \cdot \complement E = M - E$ $M - M \cdot E$. $T = f(H) = \phi(H) = \phi(M - M \cdot E) = \phi(M) - \phi(M \cdot E) = N - \phi(M \cdot E)$. The set M is a set $G_\delta(Q^2)$, and E is a set $P^{\alpha}, \alpha \geq 3$; so $M \cdot E$ is a set P^{α}, $\alpha \geq 3$, and $\phi(M \cdot E) = S$ is a set P^{α}. But this gives $T = N - \phi(M \cdot E) = N - S = N \cdot \complement S$. Since N is a set $G_\delta(Q^2)$, and $\complement S$ is a set Q^{α}, T is a set Q^{α}. The proof is complete.

<u>Definition</u>: A set E is said to be a Borel set if for some ordinal α, where $1 \leq \alpha < \Omega$, E is a set P^{α} or a set Q^{α}.

Thus the family of Borel sets (B) is merely the collection of all sets P^{α} and Q^{α} for all ordinals α of the first or second classes. The Borel sets satisfy the following conditions:

1) Every closed set belongs to B.

2) The sum of a countable aggregate of sets belonging to B belongs to B.

3) The product of a countable aggregate of sets belonging to B belongs to B.

Condition 1) follows directly from the definition of sets Q^1. Suppose that $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set belonging to B. By theorem 2:7, we may assume that each set E is a set Q^{ξ_n}, $\xi_n < \Omega$. For this infinite sequence of ordinals $\{\xi_n\}$, there exists an ordinal β such that $\xi_n < \beta < \Omega$ for each n. [3, p. 91] Thus E is a set P^{β}, and so it belongs to B. Condition 2) is therefore satisfied. In a very similar manner, it may be shown that condition 3) is satisfied.

Having shown that the family of Borel sets satisfies conditions 1), 2), and 3), it will now be shown that the family of Borel sets is the smallest family of sets which does satisfy these conditions. With this fact proved, we will have established an equivalent definition for the family of Borel sets.

Suppose that W is any family of sets satisfying conditions 1), 2), and 3). Sets Q^1 belong to W by their definition. Sets $F_{\sigma}(P^2)$ then belong to W as a countable sum of sets Q^1. Since sets P^1 are sets P^2, they also belong to W.

Proceding by transfinite induction, suppose that α is any ordinal such that $1 < \alpha < \Omega$, and assume that all sets P^{β} and Q^{β} belong to W for $\beta < \alpha$. If E is a set P^{α}, then $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set Q^{α_n}, $\alpha_n < \alpha$. Thus E_n belongs to W for each n, and by condition 2), E belongs to W.

Similarily, if E is a set Q^{α}, then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set P^{α_n}, $\alpha_n < \alpha$. Thus E_n belongs to W for each n, and by condition 3), E belongs to W. The family of Borel sets is therefore included in the family W.

Other properties of the family of Borel sets which follow from the theorems already established are as follows:

4) The complement of a set belonging to B belongs to B.

5) The difference of two sets belonging to B belongs to B.

6) A set which is homeomorphic to a set belonging to B belongs to B.

The family of Borel sets is also the smallest family which satisfies conditions 7), 8), and 9) as follows:

7) Every open set belongs to B.

8) The sum of a countable collection of disjoint sets belonging to B belongs to B.

9) The product of a countable collection of sets belonging to B belongs to B.

Suppose that W is any family of sets satisfying conditions 7), 8), and 9). By condition 7), sets P^1 belong to W, and so sets Q^2 belong to W by condition 9). Since sets Q^1 are sets Q^2, sets Q^1 belong to W. Each set P^3 is a countable sum of disjoint sets Q^1 and Q^2 by theorem 2:10, and so they belong to W by condition 8). Sets P^2, being sets P^3, also belong to W. Now let α be an ordinal such that $3 \leq \alpha < \Omega$, and suppose that all sets P^β and Q^β belong to W for $\beta < \alpha$. If E is a set P^α, then by theorem 2:10 the set E may be expressed as the sum of a countable collection of disjoint sets Q^{α_n}, $\alpha_n < \alpha$. Thus E is a set belonging to W by condition 8). If E is a set Q^α, then it belongs to W by condition 9).

CHAPTER III

BOREL SETS F_α AND G_α

In this chapter we shall express the Borel sets in yet a different manner, namely in terms of sets F_α and G_α. We shall also establish several important properties of these sets F_α and G_α.

In the definition of these sets F_α and G_α, it will be necessary to consider any ordinal $\alpha < \Omega$ as being even or odd. If α is a finite ordinal, then α will be considered even or odd in the usual manner. If α is a limit ordinal, that is, a transfinite ordinal with no immediate predecessor, then α is considered to be even. Other ordinals will be defined to be even or odd by transfinite induction as follows. Suppose that α is a given transfinite ordinal with an immediate predecessor, and suppose that we have determined each ordinal β to be even or odd if $\beta < \alpha$. Then if the immediate predecessor of α is even, α will be odd; if the immediate predecessor of α is odd, α will be even.

A set E is a set F_0 if and only if it is a closed set. For any ordinal $\alpha > 0$, α odd, E is a set F_α if and only if $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$. If $\alpha > 0$, α even, E is a set F_α if and only if $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$.

In a corresponding manner, let E be a set G_0 if and only if it is an open set. For an ordinal $\alpha > 0$, α odd, E is a set G_α if and only if $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set G_{α_n}, $\alpha_n < \alpha$. For $\alpha > 0$, α even, E is a set G_α if and only if $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set G_{α_n}, $\alpha_n < \alpha$.

<u>Theorem 3:1</u> : <u>The complement of a set F_α is a set G_α, and the complement of a set G_α is a set F_α, for $\alpha < \Omega$</u>.

<u>Proof</u>: The theorem is true for sets G_0(open) and sets F_0(closed) by the properties of open and closed sets, and is true also for sets $G_1(G_\delta)$ and sets $F_1(F_\sigma)$ as shown in Chapter II. Proceding by transfinite induction, suppose that α is an ordinal such that $1 < \alpha < \Omega$, and assume that the theorem is true for all sets G_β and F_β, where $\beta < \alpha$.

If α is even, and if E is a set F_α, then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$. For each n, $\complement E_n$ is a set G_{α_n}, $\alpha_n < \alpha$, by our induction assumption. The set $\complement E$ is then a set G_α since $\complement E = \complement \prod_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \complement E_n$. The proofs for the other possible cases are very similar.

<u>Theorem 3:2</u> : <u>If $\alpha < \Omega$ is odd, the sum of a countable collection of sets F_α is a set F_α, and the product of a countable collection of sets G_α is a set G_α. If $\alpha < \Omega$ is even, the product of a countable collection of sets F_α is a set F_α, and the sum of a countable collection of sets G_α is a set G_α</u>.

Proof: Suppose $\alpha < \Omega$, α is odd, and $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_α. Then for each n, $E_n = \sum_{k=1}^{\infty} H_{n,k}$, where for each k, $H_{n,k}$ is a set $F_{\alpha_{n,k}}$, $\alpha_{n,k} < \alpha$. Thus $E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} H_{n,k}$, and is by definition a set F_α.

If $\alpha < \Omega$, α odd, and $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set G_α, then $E = \complement(\complement \prod_{n=1}^{\infty} E_n) = \complement(\sum_{n=1}^{\infty} \complement E_n)$. Since $\complement E_n$ is a set F_α for each n, α is odd, $\sum_{n=1}^{\infty} \complement E_n$ is a set F_α. Then E is a set G_α as the complement of a set F_α by the previous theorem. The proofs for the other possible cases are very similar.

Theorem 3:3 : A set F_α is a set F_β for $\beta > \alpha$, and a set G_α is a set G_β for $\beta > \alpha$.

Proof: Suppose E is a set F_α, $\alpha < \beta$. If β is even, then since $E = E \cdot E \cdot E \cdots$, E is a set F_β. If β is odd, then since $E = E + E + E + \cdots$, E is a set F_β.

Let E be a set G_α, $\alpha < \beta$. If β is even, E is a set G_β since $E = E + E + E + \cdots$, and if β is odd, E is a set G_β since $E = E \cdot E \cdot E \cdots$.

Theorem 3:4 : For every ordinal $\alpha < \Omega$, the sum and product of any finite number of sets $F_\alpha(G_\alpha)$ is a set $F_\alpha(G_\alpha)$.

Proof: It is noted that in several cases, this theorem is established by theorem 3:2.

Suppose $\alpha < \Omega$, α odd. Let E and H be sets F_α, and let $S = E \cdot H$. $E = \sum_{m=1}^{\infty} E_m$, where for each m, E_m is a set F_{α_m}, $\alpha_m < \alpha$, and $H = \sum_{n=1}^{\infty} H_n$, where for each n, H_n is a set F_{β_n},

$\beta_n < \alpha$. Then $S = E \cdot H = \sum_{m=1}^{\infty} E_m \cdot \sum_{n=1}^{\infty} H_n = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_m \cdot H_n$. Let $\lambda_{m,n}$ be an even ordinal such that $\alpha_m \leqslant \lambda_{m,n}$, $\beta_n < \lambda_{m,n}$, and $\lambda_{m,n} < \alpha$ for each pair of indices m and n. Then each set $E_m \cdot H_n$ is a set $F_{\lambda_{m,n}}$, $\lambda_{m,n} < \alpha$. S is therefore a set F_α.

Suppose $\alpha < \Omega$, α odd. Let E and H be sets G_α, and let $S = E + H$. Then $S = \complement(\complement(E+H)) = \complement(\complement E \cdot \complement H)$. But $\complement E$ and $\complement H$ are sets F_α, hence their product is a set F_α by the above proof. The complement of their product, the set S, is then a set G_α. The proofs for other cases are similar to the above. Having proved the theorem in the case of two sets, the proof may be extended to any finite number of sets by ordinary induction.

<u>Theorem 3:5</u> : <u>For every ordinal $\alpha < \Omega$, every set F_α is also a set $G_{\alpha+1}$, and every set G_α is also a set $F_{\alpha+1}$.</u>

<u>Proof</u>: By theorem 2:1 and theorem 2:2 it is known that a set G_0(open) is a set F_1(F_σ), and that a set F_0 (closed) is a set G_1(G_δ). Given an ordinal α, $1 \leqslant \alpha < \Omega$, assume that for every ordinal $\beta < \alpha$, a set G_β is a set $F_{\beta+1}$, and a set F_β is a set $G_{\beta+1}$. Let E be a set G_α, and suppose that α is odd. Then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set G_{α_n}, $\alpha_n < \alpha$. Therefore each set E_n is a set F_{α_n+1}, where $\alpha_n + 1 < \alpha + 1$. Since $\alpha + 1$ is even, E is a set $F_{\alpha+1}$. If we suppose that α is even, the proof is very similar.

Let E be a set F_α. Then $\complement E$, as a set G_α, is a set $F_{\alpha+1}$ from the above proof. Thus $E = \complement(\complement E)$ is the complement of a set $F_{\alpha+1}$, and hence is a set $G_{\alpha+1}$.

It will now be shown that if R is the family of all sets F_α and G_α, $0 \leq \alpha < \Omega$, then the family R is identical to the family B, the Borel sets. We have noted that the family B is the smallest family of sets to satisfy the following conditions.

1) Every closed set belongs to B.

2) The sum of a countable aggregate of sets belonging to B belongs to B.

3) The product of a countable aggregate of sets belonging to B belongs to B.

Directly from the definitions of the sets of the family R, it can be concluded that the family R satisfies conditions 1), 2), and 3). If we can show that the family R is included in the family B, then the family R must be identical to the family B.

Sets F_0(closed) belong to the family B. Proceding by transfinite induction, suppose that α is an ordinal such that $\alpha < \Omega$, and assume that sets F_β belong to the family B if $\beta < \alpha$. If α is even, let E be a set F_α. Then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$. Thus for each n, E_n is a set of the family B, and by condition 3), E is a set of the family B. If α is odd, then $E = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$. Thus for each n, E_n is a set of the family B, and by condition 2), E is a set of the family B.

Since sets G_α are sets $F_{\alpha+1}$, sets G_α are included in the family of sets B. The family R is therefore included in and identical to the family B.

Having established that the two families of sets R and B are identical, we shall now show the relationships between the sets F_α, G_α and the sets P^α, Q^α of these two families. If ω is the least limit ordinal, we have:

<u>Theorem 3:6</u> : <u>For $\alpha<\omega$, if α is even, sets F_α are identical to the sets $Q^{\alpha+1}$, and sets G_α are identical to the sets $P^{\alpha+1}$. For $\alpha<\omega$, if α is odd, sets F_α are identical to the sets $P^{\alpha+1}$, and sets G_α are identical to the sets $Q^{\alpha+1}$.</u>

<u>Proof</u>: The sets F_0 are identical to the sets Q^1 by their definitions. Given an ordinal α, $0<\alpha<\omega$, assume that the theorem is true for all ordinals β if $\beta<\alpha$. Suppose that α is even, and let E be a set F_α. Then $E=\prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n<\alpha$. If α_n is even, E_n is a set F_{α_n+1}, and hence a set P^{α_n+2}, where $\alpha_n+2\leq\alpha<\alpha+1$. If α_n is odd, E_n, as a set F_{α_n}, is a set P^{α_n+1}, $\alpha_n+1\leq\alpha<\alpha+1$. E is therefore a set $Q^{\alpha+1}$.

If α is even, and if H is a set $Q^{\alpha+1}$, then $H=\prod_{n=1}^{\infty} H_n$, where for each n, H_n is a set P^{α_n}, $\alpha_n<\alpha+1$. If α_n is odd, P^{α_n}, as a set P^{α_n+1}, is a set F_{α_n}, $\alpha_n<\alpha$. If α_n is even, P^{α_n} is a set F_{α_n-1}, $\alpha_n-1<\alpha$. H is therefore a set

F_α, and thus the sets F_α are identical to the sets $Q^{\alpha+1}$.

By taking complements, sets G_α may be shown to be identical to the sets $P^{\alpha+1}$. In the case where α is odd, the proof is similar to the above.

<u>Theorem 3:7 : If $\alpha \geq \omega$, α is even, then sets F_α are identical to the sets Q^α, and sets G_α are identical to the sets P^α. If $\alpha > \omega$, α is odd, then sets F_α are identical to the sets P^α, and sets G_α are identical to the sets Q^α.</u>

<u>Proof</u>: If $\alpha = \omega$, and E is a set F_α, then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \omega$. E is therefore a set Q^{α_n+1} for α_n even, and hence a set P^{α_n+2}, where $\alpha_n+2<\alpha$. If α_n is odd, then E_n is a set P^{α_n+1}, $\alpha_n+1<\alpha$. E is then a set Q^α.

If H is a set Q^α, then $H = \prod_{n=1}^{\infty} H_n$, where for each n, H_n is a set P^{α_n}, $\alpha_n < \omega$. If α_n is odd, then H_n is a set G_{α_n-1}, and hence a set F_{α_n}, $\alpha_n < \omega$. If α_n is even, then H_n is a set F_{α_n-1}, $\alpha_n-1 < \alpha$. H is then a set F_α. By taking complements, it follows that sets G_α are identical to the sets P^α for $\alpha = \omega$.

Now suppose that $\alpha > \omega$, and assume that the theorem is true for all ordinals β where $\omega \leq \beta < \alpha$. There are three possible cases to consider. The first is where α is a limit ordinal, the second is where α is even and not a limit ordinal, and the third is where α is odd.

First, suppose that α is a limit ordinal, and let E be a set F_α. Then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\omega \leq \alpha_n < \alpha$. Then E_n will be a set Q^{α_n} if α_n is even, and thus a set P^{α_n+1}, $\alpha_n + 1 < \alpha$. If α_n is odd, E_n will be a set P^{α_n}, $\alpha_n < \alpha$. E is then a set Q^α.

If H is a set Q^α, $H = \prod_{n=1}^{\infty} H_n$, where for each n, H_n is a set P^{α_n}, $\alpha_n < \alpha$. If α_n is even, then H_n is a set G_{α_n}, and thus a set F_{α_n+1}, $\alpha_n + 1 < \alpha$. If α_n is odd, then H_n is a set F_{α_n}, $\alpha_n < \alpha$. H is then a set F_α.

Suppose that α is an even ordinal, and is not a limit ordinal. Let E be a set F_α. Then $E = \prod_{n=1}^{\infty} E_n$, where for each n, E_n is a set F_{α_n}, $\alpha_n < \alpha$. If α_n is even, E_n is a set Q^{α_n}, and hence a set P^{α_n+1}, $\alpha_n + 1 < \alpha$. If α_n is odd, E_n is a set P^{α_n}, $\alpha_n < \alpha$. E is then a set Q^α.

If H is a set Q^α, then $H = \prod_{n=1}^{\infty} H_n$, where for each n, H_n is a set P^{α_n}, $\alpha_n < \alpha$. If α_n is even, H_n is a set G_{α_n}, and hence a set F_{α_n+1}, $\alpha_n + 1 < \alpha$. If α_n is odd, H_n is a set F_{α_n}, $\alpha_n < \alpha$. H is then a set F_α.

If α is an odd ordinal, then the proof is very similar to the case where α is an even ordinal, and is not a limit ordinal. By taking complements, the remaining parts of the theorem can be shown.

We have shown that for any ordinal α, $0 < \alpha < \Omega$, sets F_α include all sets F_β, $\beta < \alpha$, and all sets G_β, $\beta < \alpha$. Likewise sets G_α include all sets G_β, $\beta < \alpha$, and all sets F_β,

$\beta<\alpha$. The question might arise as to whether there exists for each ordinal α, $0<\alpha<\Omega$, sets F_α which are not sets F_β, for each ordinal $\beta<\alpha$, or sets G_α which are not sets G_β, for each ordinal $\beta<\alpha$. This would follow if it can be shown that there exist sets F_α which are not sets G_α for each ordinal α, $0\leqslant\alpha<\Omega$.

In the case where $\alpha=0$, there exist sets which are sets F_0(closed), but are not sets G_0(open) by the properties of open and closed sets, and by taking complements it follows that there exist sets G_0 which are not sets F_0. We shall show next that in R_1, one-dimension Euclidean space, there exist sets F_1 (F_σ) which are not sets G_1 (G_δ), and vice versa. Several preliminary theorems will now be established.

<u>Definition</u>: A set E is nowhere dense in R_1, the set of all real numbers, if for every open interval (a,b) there is an open interval (c,d) such that $(c,d)\subset(a,b)$, and $(c,d)\cdot E=0$.

It can be shown that a set E is nowhere dense if and only if $\complement(\overline{E})$ is dense. [6, p. 35]

<u>Definition</u>: A set E is a set of the first category if and only if $E=\sum_{n=1}^{\infty}E_n$, where for each n, E_n is nowhere dense. A set E is a set of the second category if it is not of the first category. A set E is a residual set if it is of the second category, and $\complement E$ is of the first category.

The first category shall be denoted as category I, and the second category as category II.

Theorem 3:8 : If a set S is a complete metric space, then S is of category II.

Proof: Suppose that a set S is a complete metric space, and suppose that S is of category I. Then $S = \sum_{n=1}^{\infty} E_n$, where for each n, E_n is a nowhere sense set. There exists an $x_1 \in (\complement \overline{E_1})$ and an $\epsilon_1 > 0$ such that $N(x_1, 2\epsilon_1) \cdot E_1 = 0$. Thus $\overline{N(x_1, \epsilon_1)} \cdot E_1 = 0$. Likewise for each integer n, there exists an $x_n \in N(x_{n-1}, \epsilon_{n-1}) \cdot \complement(\overline{E_n})$ such that for some $\epsilon_n > 0$, $\epsilon_n < \frac{\epsilon_{n-1}}{2}$, $\overline{N(x_n, \epsilon_n)} \cdot E_n = 0$, and such that $\overline{N(x_n, \epsilon_n)} \subset N(x_{n-1}, \epsilon_{n-1})$. We obtain a sequence of points x_n corresponding to a decreasing sequence of closed sets whose diameters approach zero. Since S is a complete space, there exists an element x_0 common to all the intervals, by Cantor's theorem. [8, p. 30] But $x_0 \in \complement E_n$ for each n, hence $x_0 \notin S$, which leads to a contradiction. Thus the theorem is established.

Theorem 3:9 : If a set S is a complete metric space, and if H is a set G_δ which is dense in S, then H is a residual set, that is, H is a set of category II and $\complement H$ is a set of category I.

Proof: Suppose that H is a set G_δ which is dense in S, a complete metric space. The set $\complement H$ is a set F_σ, thus $\complement H = \sum_{n=1}^{\infty} H_n$, where for each n, H_n is a closed set. Since $H_n \subset \complement H$, $\complement H_n \supset H$, and H being dense in S implies that $\complement H_n$ is dense in S for each n. $H_n = \complement(\complement H_n)$ is therefore nowhere dense in S for each n. The set $\complement H$ is then of category I, and $\complement H + H = S$, where S is of category II by theorem 3:8. Thus H is of category II, for if H were of category I, then $H = \sum_{m=1}^{\infty} K_m$, where for each m, K_m is a set nowhere dense. Then $S = \complement H + H = \sum_{n=1}^{\infty} H_n + \sum_{m=1}^{\infty} K_m$, and would be of category I, but this is a contradiction.

<u>Theorem 3:10</u> : <u>The set of all rational numbers, N, is a set F_σ, but is not a set G_δ.</u>

<u>Proof</u>: The set N, all rational numbers, is a set F_σ since $N = \sum_{k=1}^{\infty} N_k$, where for each k, N_k is a rational number. Suppose that N is also a set G_δ. Since R_1, the set of all real numbers, is a complete metric space, and since N is dense on R_1, N will be a residual set by theorem 3:9. That means that N is a set of category II, and $\complement N$ is a set of category I. But $N = \sum_{k=1}^{\infty} N_k$, where for each k, N_k is a rational number which is a nowhere dense set on R_1. Thus N is a set of category I, which leads to a contradiction.

From this theorem, we may further conclude that the set of all irrational numbers is a set G_δ, but is not a set F_σ. In Chapter IV, we shall show further that there exists for each ordinal α, $0 < \alpha < \Omega$, sets F_α which are not sets G_α, and vice versa.

CHAPTER IV

SETS UNIVERSAL TO SETS G_α

1) Borel Sets Relative to their Containing Space.

From the construction of the Borel sets, it is apparent that if E is a Borel set, say an F_α, in a space A, it is not necessarily a set F_α in a different space B. For example, an open interval is a set G_o in a space consisting of itself only, but is not a set G_o in the plane.

Suppose that we have a given metric space M, and suppose that E is a subset of M, and is a metric space itself. Then for any ordinal α, $0 \leq \alpha < \Omega$, a set $F_\alpha(G_\alpha)$ relative to the metric space E is denoted as $(F_\alpha)_E \left((G_\alpha)_E\right)$.

Theorem 4:1 : Given a metric space M, and $E \subset M$, then a set $H \subset E$ is a set $F_\alpha(G_\alpha)$ relative to E if and only if it is the intersection of E and a set $F_\alpha(G_\alpha)$ relative to the space M.

Proof: From the properties of open and closed sets, it is known that a set is an F_o(closed) in E if and only if it is the intersection of E and a set F_o in M, and that a set is a G_o(open) in E if and only if it is the intersection of E and a set G_o in M. [6, p. 50]

Proceding by transfinite induction, suppose that $\alpha \geq 1$ is a given ordinal, and assume that the theorem is true for all ordinals β, $\beta<\alpha$. Let H be a set $(F_\alpha)_E$, $H\subset E$, and suppose that α is even. Then $H=\prod_{n=1}^{\infty}H_n$, where for each n, H_n is a set $(F_{\alpha_n})_E$, $\alpha_n<\alpha$. By our induction assumption, H_n is the intersection of E and a set K_n, where K_n is a set F_{α_n}, $\alpha_n<\alpha$. Thus $H=\prod_{n=1}^{\infty}(E\cdot K_n)=E\cdot\prod_{n=1}^{\infty}K_n$, and hence is the intersection of E and a set F . If α is odd, and H is a set $(F_\alpha)_E$, then $H=\sum_{n=1}^{\infty}H_n$, where for each n, H_n is a set $(F_{\alpha_n})_E$. Then H_n is the intersection of E and a set K_n, where K_n is a set F_{α_n}, $\alpha_n<\alpha$. Thus $H=\sum_{n=1}^{\infty}(E\cdot K_n)=E\cdot\sum_{n=1}^{\infty}K_n$, and is therefore the intersection of E and a set F_α.

Now suppose that $H=K\cdot E$, where K is a set F_α, $0<\alpha<\Omega$, and suppose that α is even. Then $H=K\cdot E=E\cdot\prod_{n=1}^{\infty}K_n$, where for each n, K_n is a set F_{α_n}, $\alpha_n<\alpha$. Thus $H=\prod_{n=1}^{\infty}E\cdot K_n$, where each set $E\cdot K_n$ is a set $(F_{\alpha_n})_E$ by our induction assumption. H is then a set $(F_\alpha)_E$. If α is odd, then $H=K\cdot E=E\cdot\sum_{n=1}^{\infty}K_n=\sum_{n=1}^{\infty}E\cdot K_n$, where for each n, K_n is a set F_{α_n}, $\alpha_n<\alpha$. $E\cdot K_n$ is therefore a set $(F_{\alpha_n})_E$ for each n, and H is a set $(F_\alpha)_E$. Proof for the sets G_α is very similar.

<u>Theorem 4:2</u> : <u>Given a metric space M, and $E\subset M$, then a set $H\subset E$ is a Borel set relative to E if and only if it is the intersection of E and a Borel set relative to the space M.</u>

The proof of this theorem follows from theorem 4:1.

Since the intersection of two sets $F_\alpha(G_\alpha)$ is again a set $F_\alpha(G_\alpha)$, we have the following theorems which follow directly from the above.

Theorem 4:3 : <u>Given a metric space M, and $H \subset E \subset M$, if H is a set $F_\alpha(G_\alpha)$ relative to M, then H is a set $F_\alpha(G_\alpha)$ relative to E.</u>

<u>If $H \subset E \subset M$, and E is a set $F_\alpha(G_\alpha)$ relative to M, then H is a set $F_\alpha(G_\alpha)$ relative to E if and only if it is a set $F_\alpha(G_\alpha)$ relative to M.</u>

Theorem 4:4 : <u>Given a metric space M, and $H \subset E \subset M$, if H is a Borel set relative to M, then H is a Borel set relative to E.</u>

<u>If $H \subset E \subset M$, and E is a Borel set relative to M, then H is a Borel set relative to E if and only if it is a Borel set relative to M.</u>

2) <u>Construction of Sets Universal to Linear Sets G_α.</u>

<u>Definition</u>: A set U of the plane is said to be a set universal to all linear sets of a family R if the intersection of U and any vertical line gives a linear set of R, and if any linear set of R can be obtained by the intersection of U and some vertical line.

We shall now construct plane sets U which are universal to all linear sets G_α. These sets U will be defined in a space S, where S is a subset of the plane which consists of all vertical lines $x=r$, where $0<r<1$, and r is irrational. By observing that the set $\mathcal{C}S$ is a set F_σ, being the sum of a countable collection of closed sets, it is seen that S is a set G_δ, and thus a set G_α for $\alpha \geq 1$ relative to the plane.

Let N_o be the set of all irrational numbers such that if $x \in N_o$, then $0 < x < 1$. If $x \in N_o$, then x can be written uniquely as a continued fraction as

$$x = \frac{1}{\alpha^1} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \cdots + \frac{1}{\alpha^n} + \cdots$$

where for each n, α^n is a positive integer. Thus we may associate with each number $x \in N_o$ a unique infinite sequence of positive integers, $\alpha^1, \alpha^2, \alpha^3, \cdots$, which we shall denote by $x = \{\alpha^n\}$. [2, pp. 273-281]

In turn, each number x gives rise to a countable sequence of irrational numbers $x_1, x_2, x_3, \cdots$, obtained as follows by continued fractions.

$$x_1 = \alpha^1, \alpha^3, \alpha^5, \alpha^7, \cdots$$

$$x_2 = \alpha^2, \alpha^6, \alpha^{10}, \alpha^{14}, \cdots$$

$$x_3 = \alpha^4, \alpha^{12}, \alpha^{20}, \alpha^{28}, \cdots$$

and in general,

$$x_n = \alpha^{2^{n-1}(2\cdot 1-1)}, \alpha^{2^{n-1}(2\cdot 2-1)}, \cdots, \alpha^{2^{n-1}(2\cdot m-1)}, \cdots$$

By the properties of continued fractions, $0 < x_n < 1$, hence $x_n \in N_o$ for each n. Also, if x and y are two numbers such that $x \in N_o$, $y \in N_o$, $x = \{\alpha^n\}$, $y = \{\beta^n\}$, then given any $\epsilon > 0$, and given a fixed integer k, there exists an integer L such that if $\alpha^n = \beta^n$ for $n \leq L$, $\rho(x_k, y_k) < \epsilon$. This gives rise to a $\delta > 0$ such that if $\rho(x, z) < \delta$, and $z = \{\gamma^n\}$, then $\alpha^n = \gamma^n$ for $n \leq L$. Hence we have shown that for any fixed integer k,

x_K is a continuous function of x.

Let $R_1, R_2, R_3, \cdots$ be a countable open base of the real number line P'. Then if $x_0 \in N_0$, and $x_0 = \{\alpha^n\}$, let $H_0(x_0) = \sum_{n=1}^{\infty} R_{\alpha^n}$. Thus $H_0(x_0)$ will be an open linear set. Then let $M_0(x_0) = E_p[p = (x_0, y),\ y \in H_0(x_0)]$, and

$$M_0 = \sum_{x \in N_0} M_0(x) = E_p[p = (x, y),\ y \in H_0(x)],$$

for $x \in N_0$.

M_0 is an open set in S, for if $p \in M_0$, then there exists an x_0 such that $p \in M_0(x_0)$, and $p = (x_0, y_0)$, where $y_0 \in H_0(x_0)$. Thus for some α^K, $y_0 \in R_{\alpha^K}$. There exists a neighborhood of y_0 such that if q is a point in this neighborhood intersected with the space S, then $q = (x_1, y_1)$, where if $x_1 = \{\beta^n\}$, $x_0 = \{\alpha^n\}$ by continued fractions, then $\beta^n = \alpha^n$ for $n \leq k$. Thus $y_1 \in R_{\alpha^K}$, and q is contained in the set M_0.

M_0 is a set universal to all open linear sets, that is, we can obtain any open linear set, and only such a set, by intersecting M_0 with a vertical line $L(x)$, $x \in N_0$. For if Q is a given linear set, then $Q = \sum_{K=1}^{\infty} R_{n_K}$, where R_{n_K} is a set of the countable open base of P' previously selected. Let $x = \alpha^1, \alpha^2, \alpha^3, \cdots$, where $\alpha^K = n_K$ for each k, and x is defined by continued fractions. Then $H_0(x) = \sum_{K=1}^{\infty} R_{\alpha^K} = \sum_{K=1}^{\infty} R_{n_K} = Q$. But $M_0(x) = M_0 \cdot L(x)$, for x defined above, and $M_0(x)$ is identical to $H_0(x)$ except for position. Thus we may obtain any given open linear set by intersecting M_0

with some vertical line of S. On the other hand, the intersection of M_o and any line $L(x)$, $x \in N_o$, gives a set $M_o(x)$ which is by definition the sum of a countable collection of open linear sets, and is therefore an open linear set itself.

If $x_o \in N_o$, then we have shown that x_o determines a sequence $\{x_n\}$ of numbers such that $x_n \in N_o$ for each n. For this given x_o, let $H_1(x_o) = \prod_{n=1}^{\infty} H_o(x_n)$, let

$$M_1(x_o) = E_p[p = (x_o, y),\ y \in H_1(x_o)], \text{ and}$$

$$M_1 = \sum_{x \in N_o} M_1(x) = E_p[p = (x, y),\ y \in H_1(x)],\ x \in N_o.$$

The set M_1 as defined above is a set universal to all linear sets $G_1(G_\delta)$; for let Q be any linear G_δ, then $Q = \prod_{n=1}^{\infty} Q_n$, where for each n, Q_n is an open linear set. For each n, there exists an x_n such that $Q_n = H_o(x_n)$, where $x_n = \alpha_n^1, \alpha_n^2, \ldots, \alpha_n^k, \ldots$ by continued fractions. Define x as follows:

$$x = \alpha_1^1, \alpha_2^1, \alpha_1^2, \alpha_3^1, \ldots, \alpha_n^m, \ldots$$
$$= \alpha^1, \alpha^2, \alpha^3, \alpha^4, \ldots, \alpha^k, \ldots$$

where in general, $\alpha^k = \alpha_n^m$ where $k = 2^{n-1}(2m-1)$. We then have $H_1(x) = \prod_{n=1}^{\infty} H_o(x_n) = \prod_{n=1}^{\infty} Q_n = Q$.

It can be shown directly that the intersection of M_1 and a line $L(x)$, $x \in N_o$, is a linear set G_1; however it would be sufficient to show that M_1 is a set $G_1(G_\delta)$ itself since the line $L(x)$, being a closed set, is a set $G_1(G_\delta)$. The fact that the set M_1 is a set G_1 will be shown later.

In general, we shall define by transfinite induction the sets $H_\alpha(x) = \sum_{n=1}^{\infty} H_{\alpha-1}(x_n)$ for $\alpha < \Omega$, α is even and not a limit ordinal. If $\alpha < \Omega$, α is odd, $H_\alpha(x) = \prod_{n=1}^{\infty} H_{\alpha-1}(x_n)$, and if $\alpha < \Omega$, α is a limit ordinal, then $H_\alpha(x) = \sum_{n=1}^{\infty} H_{\lambda_n}(x_n)$, where $\{\lambda_n\}$ is a sequence of ordinals such that $\lambda_n < \alpha$ for each n, and $\alpha = \lim_{n\to\infty} \lambda_n$. In each case,

$$M_\alpha(x_0) = E_p[p = (x_0, y),\ y \in H_\alpha(x_0)],$$
$$M_\alpha = E_p[p = (x, y),\ y \in H_\alpha(x)],$$

where $x_0 \in N_0$, $x \in N_0$.

Sets M_α are universal to the linear sets G_α, $0 < \alpha < \Omega$, for if Q is any linear G_α, then Q will be shown to be the intersection of a vertical line $L(x)$, $x \in N_0$, with M_α, that is, Q will be a set $M_\alpha(x)$. For suppose that α is even and not a limit ordinal, and assume that the set M_β is universal to all linear sets G_β, $\beta < \alpha$, then $Q = \sum_{n=1}^{\infty} Q_n$, where for each n, Q_n is a set G_{β_n}, $\beta_n < \alpha$. Each set Q_n is then a set $G_{\alpha-1}$, and $Q_n = M_{\alpha-1} \cdot L(x_n) = H_{\alpha-1}(x_n)$ for each n. Now define x as follows:

$$x = \alpha_1^1, \alpha_2^1, \alpha_1^2, \alpha_3^1, \ldots, \alpha_n^m, \ldots$$
$$= \alpha^1, \alpha^2, \alpha^3, \alpha^4, \ldots, \alpha^k, \ldots$$

where for each n, $x_n = \alpha_n^1, \alpha_n^2, \alpha_n^3, \ldots, \alpha_n^j, \ldots$, and in general, $\alpha^k = \alpha_n^m$ where $k = 2^{n-1}(2\cdot m - 1)$. We then have

$$H_\alpha(x) = \sum_{n=1}^{\infty} H_{\alpha-1}(x_n) = \sum_{n=1}^{\infty} Q_n = Q.$$

Suppose that α is even, $\alpha < \Omega$, and α is a limit ordinal. Let Q be a set G_α. Then $Q = \sum_{n=1}^{\infty} Q_n$, where for each n, Q_n is a set G_{β_n}, $\beta_n < \alpha$. There exists a sequence $\{\lambda_n\}$,

$\lambda_n < \alpha$ for each n, such that $\lim_{n\to\infty} \lambda_n = \alpha$. For each set Q_n, which is a set G_{β_n}, there exists a λ_{κ_n} such that $\lambda_{\kappa_n} \geq \beta_n$, and $\lambda_{\kappa_n} > \lambda_{\kappa_{n-1}}$. Thus Q_n is a set $G_{\lambda_{\kappa_n}}$. By our induction assumption, there exists a number x_{κ_n}, $x_{\kappa_n} \in N_o$, such that $H_{\lambda_{\kappa_n}}(x_{\kappa_n}) = Q_n$ for each n. Where $\lambda_i \neq \lambda_{\kappa_n}$ for any n, let $H_{\lambda_i}(x_i) = 0$, the empty set. Thus we have the following: $H_\alpha(x) = \sum_{n=1}^{\infty} H_{\lambda_n}(x_n) = \sum_{n=1}^{\infty} H_{\lambda_{\kappa_n}}(x_{\kappa_n}) = \sum_{n=1}^{\infty} Q_n = Q$.

If α is odd, $\alpha < \Omega$, the proof follows in a manner similar to the case where α is even and not a limit ordinal. Since a line $L(x)$, $x \in N_o$, is a set G_α, $\alpha \geq 1$, the intersection of this line and a set M_α will be a linear set G_α if M is itself a set G_α. That each set M_α is a set G_α relative to S will be shown next.

It has been shown that M_o is a set G_o(open) relative to S, and that for each $x \in N_o$, and for each fixed n, x_n is a continuous function of x.

We shall define F_n to be a mapping of S such that $F_n(p) = F_n(x,y) = (x_n,y)$, and therefore F_n is a continuous mapping of vertical lines into vertical lines. The mapping will be an onto mapping, for if $q = (z,y)$, where z is an element of N_o, then $z = \alpha^1, \alpha^2, \cdots, \alpha^\kappa, \cdots$ by continued fractions. Let $x = \alpha_1^1, \alpha_2^1, \alpha_1^2, \cdots, \alpha_n^m, \cdots$ where for some fixed k, $\alpha_\kappa^m = \alpha^m$. Thus $x \in N_o$, and $x_\kappa = \alpha_\kappa^1, \alpha_\kappa^2, \cdots, \alpha_\kappa^m, \cdots$, that is, $x_\kappa = z$. Hence $F_n(x,y) = (x_n,y) = (z,y)$.

Given an ordinal $\alpha < \Omega$, suppose that if $\beta < \alpha$, then

M_ρ is a set G_ρ in S. Suppose that α is odd. Then we have the following identities:

$$M_\alpha = E_\rho[p=(x,y),\ y \in H_\alpha(x)]$$
$$= E_\rho[p=(x,y),\ y \in \prod_{n=1}^{\infty} H_{\alpha-1}(x_n)]$$
$$= \prod_{n=1}^{\infty} E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n)].$$

To establish the last identity, suppose that $p_o \in E_\rho[p=(x,y),\ y \in \prod_{n=1}^{\infty} H_{\alpha-1}(x_n)]$, and $p_o=(x_o,y_o)$. Then, where x_o gives rise to the sequence $\{x_n^o\}$, $y_o \in \prod_{n=1}^{\infty} H_{\alpha-1}(x_n^o)$; thus $y_o \in H_{\alpha-1}(x_n^o)$ for each n. Thus

$$p_o \in E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n^o)]$$

for each n, which means

$$p_o \in \prod_{n=1}^{\infty} E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n^o)].$$

On the other hand, suppose that

$$p_o \in \prod_{n=1}^{\infty} E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n)],$$

which means that, for each n,

$$p_o \in E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n)],$$

so $y \in H_{\alpha-1}(x_n^o)$ for each n. Therefore

$$p_o \in E_\rho[p=(x,y),\ y \in \prod_{n=1}^{\infty} H_{\alpha-1}(x_n^o)],$$

and the identity is established.

But we then have $E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n)] = F_n^{-1}(E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x)])$, for if

$$p_o \in E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x_n)],$$

where $p_o=(x_o,y_o)$, $y_o \in H_{\alpha-1}(x_n^o)$. If $q_o=(x_n^o,y_o)$, then

$$q_o \in E_\rho[p=(x,y),\ y \in H_{\alpha-1}(x)],$$

and $F_n(p_o)=F_n(x_o,y_o)=(x_n^o,y_o)=q$. Thus $p_o=F_n^{-1}(q_o)$, so

$$p_o \in F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]).$$

On the other hand, let

$$p_o \in F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]).$$

Then $(x_n^o, y_o) = F_n(p_o) = E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]$; thus $y_o \in H_{\alpha-1}(x_n^o)$, and $p_o \in E_p[p=(x,y),\ y \in H_{\alpha-1}(x_n)]$. The identity is established. Thus

$$M_\alpha = \prod_{n=1}^{\infty} F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]),$$

where $\alpha < \Omega$, α odd.

If α is even, not a limit ordinal, then it can be shown in a similar manner that

$$M_\alpha = \sum_{n=1}^{\infty} F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)].$$

If α is even, and α is a limit ordinal, then

$$\begin{aligned} M_\alpha &= E_p[p=(x,y),\ y \in H_\alpha(x)] \\ &= E_p[p=(x,y),\ y \in \sum_{n=1}^{\infty} H_{\lambda_n}(x_n)] \\ &= \sum_{n=1}^{\infty} E_p[p=(x,y),\ y \in H_{\lambda_n}(x_n)], \end{aligned}$$

where $\{\lambda_n\}$ is a sequence of ordinals such that $\lim_{n\to\infty} \lambda_n = \alpha$. For suppose $p_o \in E_p[p=(x,y),\ y \in \sum_{n=1}^{\infty} H_{\lambda_n}(x_n)]$, $p_o = (x_o, y_o)$, and x_o gives rise to the sequence $\{x_n^o\}$. Then

$$p_o \in E_p[p=(x,y),\ y \in \sum_{n=1}^{\infty} H_{\lambda_n}(x_n^o)].$$

Thus for some n, $y \in H_{\lambda_n}(x_n^o)$, and so

$$p_o \in \sum_{n=1}^{\infty} E_p[p=(x,y),\ y \in H_{\lambda_n}(x_n)].$$

If $p_o \in \sum_{n=1}^{\infty} E_p[p=(x,y),\ y \in H_{\lambda_n}(x_n)]$, then for some index n, $p_o \in E_p[p=(x,y),\ y \in H_{\lambda_n}(x_n)]$. Hence

$$p_o \in E_p[p=(x,y),\ y \in \sum_{n=1}^{\infty} H_{\lambda_n}(x_n)].$$

But $E_p[p=(x,y),\ y \in H_{\lambda_n}(x_n)] = F_n^{-1}(E_p[p=(x,y),\ y \in H_{\lambda_n}(x)]$, for each n, as shown previously. Thus

$$M_\alpha = \sum_{n=1}^{\infty} F_n(E_p[p=(x,y),\ y \in H_{\lambda_n}(x)],$$

where α is a limit ordinal.

Thus we have shown that for each n, the function F_n maps the space S continuously onto S. Relative to the space S, $F_n^{-1}(Q)$, where Q is a set G_0(open), is a set G_0. [8, p. 27] Proceeding by transfinite induction, suppose that $\alpha < \Omega$, and assume that $F_n^{-1}(Q)$, where Q is a set G_β, $\beta<\alpha$, is a set G_β relative to the space S. Let T be a set G_α, and suppose that α is odd. Then $T = \prod_{m=1}^{\infty} T_m$, where for each m, T_m is a set G_{β_m}, $\beta_m < \alpha$. Thus $F_n^{-1}(T) = F_n^{-1}(\prod_{m=1}^{\infty} T_m) = \prod_{m=1}^{\infty} F_n^{-1}(T_m)$, where each set $F_n^{-1}(T_m)$ is a set G_{β_m} in S. Hence $F_n^{-1}(T)$ is a set G_α in S. If α is even, then $T = \sum_{m=1}^{\infty} T_m$, where for each m, T_m is a set G_{β_m}, $\beta_m < \alpha$. Then $F_n^{-1}(T) = F_n^{-1}(\sum_{m=1}^{\infty} T_m) = \sum_{m=1}^{\infty} F_n^{-1}(T_m)$, where each set $F_n^{-1}(T_m)$ is a set G_{β_m} in S, $\beta_m < \alpha$. Hence $F_n^{-1}(T)$ is a set G_α in S.

By the identities that we have established, namely

$$M_\alpha = \prod_{n=1}^{\infty} F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]) = \prod_{n=1}^{\infty} F_n^{-1}(M_{\alpha-1}),$$

where α is odd,

$$M_\alpha = \sum_{n=1}^{\infty} F_n^{-1}(E_p[p=(x,y),\ y \in H_{\alpha-1}(x)]) = \sum_{n=1}^{\infty} F_n^{-1}(M_{\alpha-1}),$$

where α is even, not a limit ordinal, and

$$M_\alpha = \sum_{n=1}^{\infty} F_n^{-1}(E_p[p=(x,y),\ y \in H_{\lambda_n}(x)]) = \sum_{n=1}^{\infty} F_n^{-1}(M_{\lambda_n}),$$

where α is a limit ordinal, we may conclude that each set M_α is a set G_α relative to the space S. Since the space S is a set G_1 in the plane, and hence a set G_α, $\alpha \geqslant 1$, each set M_α

is a set G_α in the plane for $\alpha \geqslant 1$, by theorem 4:3.

As has been previously stated, any line $L(x)$, $x \in N_0$, is a set G_α, $\alpha \geqslant 1$. Thus the intersection of such a line and a set M_α is a set G_α. The sets universal to all linear sets G_α, $0 \leqslant \alpha < \Omega$, are defined.

3) Sets $F_\alpha(G_\alpha)$ Not Sets $F_\beta(G_\beta)$ for $\beta < \alpha$.

Theorem 4:5 : If R is any family of all linear and plane sets possessing the following properties, (A) the intersection of a plane set of R with a line is a set of R, and (B) any linear set of R projected onto the y-axis is a set of R, then if D is the set of all points on a line $y = x$, and U is a set of R in the plane universal to all linear sets of R which are subsets of the y-axis, then $D \cdot U \in R$, and $(D - U) \notin R$.

Proof: 1) $D \cdot U \in R$ from our hypothesis.

2) Deny our conclusion supposing that $(D - U) \in R$. The projection H of $D - U$ on the y-axis is a set of R by property (B) of the hypothesis. Since U is a set universal to all linear sets of R, there exists a real number α such that the intersection of $x = \alpha$ and U gives a set E whose projection on the y-axis is H. Let Q be the projection of $D \cdot U$ on the y-axis. Thus H is the complement of Q relative to the y-axis. Suppose that $p = (\alpha, \alpha)$. Either $p \in (D \cdot U)$, or $p \in (D - U)$. If $p \in D \cdot U$, then $\alpha \in Q$, $\alpha \notin H$.

Hence $p \notin E$. But E contains all points in which $x=\alpha$ meets the set U, hence $p \notin U$, which gives a contradiction.

On the other hand, if we suppose that $p \in (D-U)$, then $\alpha \in H$, so $(\alpha,\alpha) \in E$. But $E \subset U$, hence $p \in U$, which is again a contradiction. The theorem is established.

The class of plane sets $G_\alpha, \alpha \geqslant 1$, satisfies the conditions for the family R of the above theorem. The intersection of a set G_α and a line (a set G_1) is again a set $G_\alpha, \alpha \geqslant 1$, thus satisfying condition (A). As for condition (B), that the projection of a linear G_α onto the y-axis is a set G_α, two cases are to be considered. If the linear set G_α is perpendicular to the y-axis, then its projection is merely a point. But a point, as a closed set, is a set G_1, and hence a set $G_\alpha, \alpha \geqslant 1$. If the linear set G_α is not perpendicular to the y-axis, then its projection is merely a homeomorphic image, and thus is a set $G_\alpha, \alpha \geqslant 1$.

Since sets universal to all linear sets G_α, $0 \leqslant \alpha < \Omega$, have been defined, and since the class of plane sets G_α satisfies the conditions for the family R, it can be concluded that there exists sets G_α which are not sets F_α for each $\alpha \geqslant 1$. This in turn implies that there exists sets F_α which are not sets $F_\beta, \beta < \alpha$, and sets G_α which are not sets $G_\beta, \beta < \alpha$, for each ordinal α, $0 < \alpha < \Omega$.

CHAPTER V

ANALYTIC SETS

Suppose that we have a given space M, and a family of sets F contained in this space. For every finite sequence of positive integers $n_1, n_2, n_3, \cdots, n_k$, suppose that we have a set of the family F assigned, and denote this set by $E_{n_1, n_2, \cdots, n_k}$. Thus we have a given defining system of sets which we shall designate by $[E_{n_1, n_2, \cdots, n_k}]$.

If a set $E = \sum_{\{n_k\}} E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \cdots$, where the summation extends over all possible infinite sequences of positive integers $\{n_k\}$, then we say that E is the nucleus of the defining system $[E_{n_1, n_2, \cdots, n_k}]$ of sets of the family F. Also we say that E is the result of operation A on the given family of sets F, or that E is analytic relative to the family F. The class of sets analytic relative to a family of sets F will be designated as A(F).

For economy of notation, a finite sequence of integers $n_1, n_2, \cdots, n_k$, will be designated as $n_{(k)}$. The nucleus E of a defining system $[E_{n_{(k)}}]$ will then be designated as $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n_{(k)}}$, where the summation extends over all possible infinite sequences $\{n_k\}$.

Any set E of the family of sets F is included in the family of sets A(F), for if the set E itself is assigned to each finite sequence of positive integers $n_{(k)}$, that is, $E_{n(k)}=E$, then the condition is satisfied. Several of the fundamental theorems concerning analytic sets will now be shown.

Theorem 5:1 : The sum of a countable number of sets of the family of sets F is analytic relative to the family F. $(S(F)\subset A(F))$

Proof: Suppose $H=\sum_{n=1}^{\infty} H_n$, where for each n, $H_n \in F$. For each finite sequence of indices $n_{(k)}$, let $H_{n_1}=E_{n(k)}$, for k=1, 2, 3,$\cdots$. Thus $H_n=\prod_{k=1}^{\infty} E_{n(k)}$ for all possible sequences of integers $\{n_k\}$ where $n_1 = n$. The set H is then analytic since $H=\sum_{n=1}^{\infty} H_n = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$.

Theorem 5:2 : The intersection of a countable number of sets of the family of sets F is analytic relative to the family F. $(P(F)\subset A(F))$

Proof: Suppose $H=\prod_{k=1}^{\infty} H_k$, where for each k, $H_k \in F$. Let $H_k=E_{n(k)}$ for k=1, 2, 3,$\cdots$, and for every infinite sequence of positive integers $\{n_k\}$. $H=\prod_{k=1}^{\infty} H_k=\prod_{k=1}^{\infty} E_{n(k)}$ for all possible sequences $\{n_k\}$, hence $H=\sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$.

Theorem 5:3 : If each set $E^{r(s)}$ is analytic relative to the family of sets F, then the nucleus of the defining system $[E^{r(s)}]$ is also analytic relative to the family of sets F. $[A(A(F))\subset A(F)]$

Proof: A (1,1) correspondence may be established between the sequence of all positive integers $\{k\}$ and a sequence of all pairs of positive integers $\{p_k, q_k\}$ by letting k correspond to the pair of integers (p_k, q_k), where the equation $k = 2^{p_k - 1}(2q_k - 1)$ is satisfied. Now let $p_k = \phi(k)$ and $q_k = \psi(k)$, and for every pair of integers (p, q), let $\nu(p,q) = 2^{p-1}(2q-1)$. Then the following relationships are valid:

$$\nu\big(\phi(k), \psi(k)\big) = k, \text{ for each } k,$$

$$\psi(k) \leq k, \text{ for each } k,$$

$$\nu\big(n, \psi(k)\big) \leq k, \text{ for each } k,\ n = 1, 2, \cdots, \phi(k),$$

$$\phi\big(\nu(p,q)\big) = p,\ \psi\big(\nu(p,q)\big) = q, \text{ for each } p, \text{ and}$$

for each q.

Each set $E^{r(s)}$ is analytic relative to the family of sets F, so for each combination of positive integers r , $E^{r(s)} = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E^{r(s)}_{n(k)}$, where $[E^{r(s)}_{n(k)}]$ is a defining system of sets of the family of sets F, and where the summation is extended over all infinite sequences of positive integers $\{n_k\}$. Define $E_{n(k)}$ to be a set of the family F such that

$$E_{n(k)} = E^{\phi(n_1), \phi(n_2), \cdots, \phi(n_{\psi(k)})}_{\psi(n_{\nu(1,\psi(k))}),\ \psi(n_{\nu(2,\psi(k))}),\ \cdots,\ \psi(n_{\nu(\phi(k),\psi(k))})}$$

It must be shown that $\sum_{\{r_s\}} \prod_{s=1}^{\infty} E^{r(s)} = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$. Let $x \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$. There exists an infinite sequence $\{n_k\}$ such that $x \in \prod_{k=1}^{\infty} E_{n(k)}$. Let $r_s = \phi(n_s)$ for $s = 1, 2, 3, \cdots$; for a fixed integer s, let $j_h = \psi(n_{\nu(h,s)})$ for each h. Then we obtain $E_{n_1, n_2, n_3, \cdots, n_{\nu(h,s)}} = E^{r(s)}_{j(h)}$ for each h. Therefore $x \in \prod_{h=1}^{\infty} E^{r(s)}_{j(h)} \subset \sum_{\{j_h\}} \prod_{h=1}^{\infty} E^{r(s)}_{j(h)}$, for each fixed s, which means that

$x \in E^{r(s)}$ for each s. Therefore $x \in \sum_{\{r_s\}} \prod_{s=1}^{\infty} E^{r(s)}$.

On the other hand, suppose that $x \in \sum_{\{r_s\}} \prod_{s=1}^{\infty} E^{r(s)}$. Then there exists an infinite sequence of positive integers $\{r_s\}$ such that $x \in \prod_{s=1}^{\infty} E^{r(s)}$. By the charactor of the sets $E^{r(s)}$, there exists an infinite sequence of indices $\{m_k^s\}$ for each s such that $x \in E^{r(s)}_{m^s_{(k)}}$ for $s=1, 2, 3, \cdots$, and each $k=1, 2, 3, \cdots$, where each set $E^{r(s)}_{m^s_{(k)}}$ is of the family of sets F.

Put $n_h = \nu\left(r_h, m^{(\psi(h))}_{\phi(h)}\right)$ for each $h=1, 2, 3, \cdots$. This means $\phi(n_h) = r_h$, and $\psi(n_h) = m^{(\psi(h))}_{(\phi(h))}$ for each integer h. Also $h=\nu(i, \psi(k))$ implies $\psi\left(n_{\nu(i,\psi(k))}\right) = m_i^{(\psi(k))}$ for $i=1, 2, 3, \cdots$, $k=1, 2, 3, \cdots$.

Since we have

$$E_{n(k)} = E^{\phi(n_1),\ \phi(n_2),\ \cdots,\ \phi(n_{\psi(k)})}_{\psi\left(n_{\nu(1,\psi(k))}\right),\ \psi\left(n_{\nu(2,\psi(k))}\right),\ \cdots,\ \psi\left(n_{\nu(\phi(k),\psi(k))}\right)}$$

we get $E_{n(k)} = E^{r_1,\ r_2,\ \cdots,\ r_{\psi(k)}}_{m_1^{\psi(k)},\ m_2^{\psi(k)},\ \cdots,\ m^{\psi(k)}_{\phi(k)}}$ by substitution. Thus $x \in \prod_{k=1}^{\infty} E_{n(k)} \subset \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$. Hence the systems $[E_{n(k)}]$ and $[E^{r(s)}]$ have the same nucleus. Since each set $E_{n(k)}$ is a set of the family of sets F, the nucleus of the system $[E^{r(s)}]$ is analytic relative to the family F.

This theorem may be expressed as $A(A(F)) \subset A(F)$. Since the inclusion in the other way is apparent, we can conclude that $A(A(F)) = A(F)$. With this fact, and with the aid of theorem 5:1 and theorem 5:2, we conclude that the sum of a countable collection of sets analytic relative to a family of sets F is analytic relative to the family of sets F since $S(A(F)) \subset A(A(F)) \subset A(F)$. Since $P(A(F)) \subset A(A(F)) \subset A(F)$, the intersection of a countable collection of sets analytic relative to a family of sets F is analytic to the family of sets F.

<u>Theorem 5:4</u> : <u>The family of sets A(F) is topologically invariant if the family of sets F is itself topologically invariant, and if the intersection of a set of the family F with a set G_δ is a set of the family F.</u>

<u>Proof</u>: Let $H = \sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} E_{n(\kappa)}$, where each set $E_{n(\kappa)}$ is of the family F. Let T be the homeomorphic image of H by a function f. By Lavrentieff's theorem, [8, p. 126], there exists sets M and N such that $H \subset M$, $T \subset N$, M and N are sets G_δ, and M is homeomorphic to N by a function ϕ, where $\phi(p) = f(p)$ if $p \in H$.

Let $Y_{n(\kappa)} = M \cdot E_{n(\kappa)}$, which by our hypothesis is a set of the family of sets F. Thus $H = M \cdot \sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} E_{n(\kappa)} = \sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} M \cdot E_{n(\kappa)}$. Hence $H = \sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} Y_{n(\kappa)}$. Then

$$T = \phi(H) = \phi\left(\sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} Y_{n(\kappa)}\right) = \sum_{\{n_\kappa\}} \prod_{\kappa=1}^{\infty} \phi(Y_{n(\kappa)}).$$

But each set $\phi(Y_{n(\kappa)})$ belongs to the family of sets F, so T is a set of the family A(F), and the proof is complete.

In the discussion of analytic sets thus far, the sets of the family F have not been specifically defined. Throughout the remainder of this discussion, however, the family of sets F will be considered to be the class of all closed sets (C). The general results already established will be true for the class of analytic sets relative to the class of closed sets, and in particular theorem 5:4 will be valid.

It can be concluded that every Borel set is an analytic set since the class of sets A(C) satisfy the following conditions:

1) Every closed set is a set A(C)

2) $S(A(C)) \subset A(C)$

3) $P(A(C)) \subset A(C)$

It is evident from the definition of the analytic sets that the property of being an analytic set will be dependent on the space in which the set is contained. Relative to this fact, we have the following theorems:

<u>Theorem 5:5</u> : <u>If S is a subset of a given space M, then a set E is an analytic set in the space S if and only if E is the intersection of S and an analytic set of the space M.</u>

<u>Proof</u>: Suppose that $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$, where for each integer k, $E_{n(k)}$ is a closed set in S. Then $E_{n(k)} = H_{n(k)} \cdot S$, where for each k, $H_{n(k)}$ is closed in M. [6, p. 50] Thus $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)} = \sum_{\{n_k\}} \prod_{k=1}^{\infty} S \cdot H_{n(k)} = S \cdot \sum_{\{n_k\}} \prod_{k=1}^{\infty} H_{n(k)}$, where the set $\sum_{\{n_k\}} \prod_{k=1}^{\infty} H_{n(k)}$ is an analytic set in M.

On the other hand, if E is an analytic set in M, then $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$, where for each k, $E_{n(k)}$ is closed in M. The set $S \cdot E = S \cdot \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)} = \sum_{\{n_k\}} \prod_{k=1}^{\infty} S \cdot E_{n(k)}$, where for each k, $S \cdot E_{n(k)}$ is closed in S. S·E is therefore analytic in S.

From this theorem we conclude the following:

<u>Theorem 5:6</u> : <u>If S is a subset of a given space M, and if S is an analytic set in the space M, then a set $E \subset S$ is an analytic set in S if and only if it is an analytic set in the space M.</u>

<u>Definition</u>: A defining system $[E_{n(k)}]$ is regular if the closed sets $E_{n(k)}$ satisfy the following conditions for $k = 1, 2, 3, \cdots$.

$$\delta(E_{n(k)}) < \frac{1}{k}$$

$$E_{n(k+1)} \subset E_{n(k)}$$

$$E_{n(k)} \neq 0$$

<u>Theorem 5:7</u> : <u>If E is a non-empty analytic set in a complete separable space M, then $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} Y_{n(k)}$, where $[Y_{n(k)}]$ is a regular defining system.</u>

<u>Proof</u>: Given that E is an analytic set in the space M, then $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} F_{n(k)}$, where each set $F_{n(k)}$ is a closed set in the space M. Since M is a separable space, (see introduction), and M is a metric space, M possesses the Lindelof property. Thus $M = \sum N_n^{(k)}$, $k = 1, 2, 3, \cdots$, where for each n, $N_n^{(k)}$ is an open set such that $\delta(N_n^{(k)}) < \frac{1}{k}$. [7, p. 116] Let $M_n^{(k)} = \overline{N_n^{(k)}}$, then $M \subset \sum_{n=1}^{\infty} M_n^{(k)}$, $k = 1, 2, 3, \cdots$, where for each n, $M_n^{(k)}$ is closed, and $\delta(M_n^{(k)}) < \frac{1}{k}$. [6, p. 27]

Let $E_{n_1} = M^{(2)}_{n_1}$ for $n_1 = 1, 2, 3, \cdots$, and let $E_{n_1,n_2} = E_{n_1}$ for all n_1 and n_2. Thus E_{n_1} and E_{n_1,n_2} are closed, and $\delta(E_{n_1}) = \delta(E_{n_1,n_2}) < \frac{1}{2}$. For $k > 1$, let $E_{n_1,n_2,\cdots,n_{2k-1}} = E_{n_1,n_2,\cdots,n_{2k}} = F_{n_2,n_4,\cdots,n_{2k-2}} \cdot M^{(2k)}_{n_{2k-1}}$, for each finite sequence of 2k positive integers, denoted as $n_1, n_2, n_3, \cdots, n_{2k}$. The sets $E_{n(k)}$ are closed, and $\delta(E_{n_1,n_2,\cdots,n_{2k-1}}) = \delta(E_{n_1,n_2,\cdots,n_{2k}}) \leq \delta(M^{(2k)}_{n_{2k-1}}) < \frac{1}{2k}$, for each k. It will now be shown that $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$. If $x \in E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} F_{n(k)}$, there exists an infinite sequence of indices $\{m_k\}$ such that $x \in \prod_{k=1}^{\infty} F_{m(k)}$. Since $x \in M$, $x \in \sum_{n=1}^{\infty} M^{(2k)}_n$, for $k = 1, 2, 3, \cdots$. There exists an integer i_k such that $x \in \prod_{k=1}^{\infty} M^{(2k)}_{i_k}$. Let $n_1, n_2, n_3, \cdots$ be the terms of the sequence $i_1, m_1, i_2, m_2, \cdots$. Then for each integer k, $x \in F_{m_1,m_2,\cdots,m_{k-1}}$, $k > 1$, so $x \in F_{n_2,n_4,\cdots,n_{2k-2}}$. Also $x \in M^{(2k)}_{i_{k-1}}$, so $x \in M^{(2k)}_{n_{2k-1}}$. Therefore, for each integer k, $x \in E_{n_1,n_2,\cdots,n_{2k-1}} = E_{n_1,n_2,\cdots,n_{2k}}$, and $x \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$.

On the other hand, if $x \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$, then there exists an infinite sequence of indices $\{n_k\}$ such that $x \in \prod_{k=1}^{\infty} E_{n(k)}$. Thus $x \in F_{n_2,n_4,\cdots,n_{2k-2}}$, so $x \in F_{m_1,m_2,\cdots,m_{k-1}}$, for $k > 1$. Thus $x \in \prod_{k=1}^{\infty} F_{m(k)} \subset E$, and the identity is established.

Let $X_{n(k)} = \prod_{i=1}^{k} E_{n(i)}$. Each set $X_{n(k)}$ will be closed, and $\delta(X_{n(k)}) \leq \delta(E_{n(k)}) < \frac{1}{k}$. Also $X_{n(k+1)} \subset X_{n(k)}$ by the properties of intersections of sets. The identity $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} X_{n(k)}$ will now be established.

If $x \in \sum_{\{n_K\}} \prod_{K=1}^{\infty} X_{n(K)}$, then since $X_{n(K)} \subset E_{n(K)}$ for each integer k, and for each infinite sequence of indices $\{n_K\}$, $x \in \sum_{\{n_K\}} \prod_{K=1}^{\infty} E_{n(K)} = E$. If $x \in E$, then there exists an infinite sequence of indices $\{n_K\}$ such that $x \in \prod_{K=1}^{\infty} E_{n(K)}$. Thus $x \in \prod_{K=1}^{\infty} \prod_{i=1}^{K} E_{n(i)}$, hence $x \in \prod_{K=1}^{\infty} X_{n(K)} \subset \sum_{\{n_K\}} \prod_{K=1}^{\infty} X_{n(K)}$.

The set E is the nucleus of the defining system $[X_{n(K)}]$ which has all of the properties of a regular system except the assurance that each set is non-empty. Let $X^{r(s)} = \sum X_{r(s),n_1} \cdot X_{r(s),n_1,n_2} \cdot X_{r(s),n_1,n_2,n_3} \cdots$ for each infinite sequence of indices $\{n_K\}$. If the set $X^{r(s)}$ is not empty, let $\{x_{r(s)}\}$ be one of its elements. Since $X_{n(K+1)} \subset X_{n(K)}$, $X^{r(s)} \subset \sum_{\{r_s\}} \prod_{S=1}^{\infty} X_{r(s)}$, so $x_{r(s)} \in E$. There will be at least one element $\{x_0\}$ of E since E is not empty. The sets $Y_{r(s)}$ are defined as follows:

$$Y_{r(s)} = X_{r(s)} \text{ if } X^{r(s)} \neq 0,$$
$$Y_{r(s)} = \{x_0\} \text{ if } X^{r(s)} = 0,\ X^{r(1)} = 0,$$
$$Y_{r(s)} = \{x_{r(q)}\} \text{ if } X^{r(s)} = 0,\ X^{r(1)} \neq 0,$$ and where $q+1$ is the smallest index such that $X^{r(q+1)} = 0$, and where $\{x_{r(q)}\} \in X^{r(q)}$.

The defining system $[Y_{n(K)}]$ is regular. That the sets $Y_{n(K)}$ are each closed follows from the fact that $Y_{n(K)} = X_{n(K)}$, or else $Y_{n(K)}$ is a single point. The condition concerning the diameters of the sets is satisfied since $\delta(Y_{n(K)}) = \delta(X_{n(K)}) < \frac{1}{K}$. All sets $Y_{n(K)}$ are non-empty since by their definition they contain at least one point.

It remains to be shown that $Y_{n(k+1)} \subset Y_{n(k)}$ for each integer k. If $X^{n(k)}$ and $X^{n(k+1)}$ are not empty, $Y_{n(k+1)} \subset Y_{n(k)}$ since $X_{n(k+1)} \subset X_{n(k)}$. If $X^{n(1)} = 0$, then $X^{n(k)} = 0$, and $X^{n(k+1)} = 0$. Then $Y_{n(k+1)} = \{x_0\} = Y_{n(k)}$. If $X^{n(1)} \neq 0$, and $X^{n(k)} = 0$, then $X^{n(k+1)} = 0$. Then $Y_{n(k+1)} = Y_{n(k)} = \{x_{n(q)}\}$. If $X^{n(k)} \neq 0$, $X^{n(k+1)} = 0$, then $Y_{n(k+1)} = \{x_{n(k)}\} \in X_{n(k)} = Y_{n(k)}$. Thus in all cases, $Y_{n(k+1)} \subset Y_{n(k)}$, for each k.

To complete the theorem, it must be established that $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} Y_{n(k)}$. If $p \in E$, then $p \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} X_{n(k)}$. There exists an infinite sequence of indices $\{n_k\}$ such that $p \in \prod_{k=1}^{\infty} X_{n(k)}$. For $s = 1, 2, 3, \cdots$, and $j = 1, 2, 3, \cdots$, let $m_j = n_{s+j}$. Let $r_i = n_i$ for $i \leq s$. Then $p \in X^{r(1)} \cdot X^{r(2)} \cdot \cdot \cdot$ for the given sequence, hence $p \in \sum_{\{r_s\}} \prod_{s=1}^{\infty} X^{r(s)}$. This means that $Y_{n(k)} = X_{n(k)}$ for each k, and $p \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} Y_{n(k)}$.

If $p \in \sum_{\{n_k\}} \prod_{k=1}^{\infty} Y_{n(k)}$, then there exists an infinite sequence of indices $\{n_k\}$ such that $p \in \prod_{k=1}^{\infty} Y_{n(k)}$. Several cases may arise. If $X^{n(k)} \neq 0$ for each k, then $p \in E$ directly. If $X^{n(k)} = 0$ for each k, then $p = \{x_0\} \in E$. If $X^{n(k)} = 0$ for all integers $k > q$, $X^{n(k)} \neq 0$ for $k \leq q$, then $Y^{n(k)} = \{x_{n(q)}\}$ for $k > q$. Thus $p = \{x_{n(q)}\}$. But $\{x_{n(q)}\} \in X^{n(q)}$ which means that there exists an infinite sequence of indices $\{m_k\}$ such that $p \in X_{n(q)} \cdot X_{n(q), m_1} \cdot X_{n(q), m(2)} \cdots$. Since $X_{n(k)}$ is a descending sequence of sets,

$$p \in X_{n(1)} \cdot X_{n(2)} \cdot X_{n(3)} \cdots X_{n(q)} \cdot X_{n(q), m(1)} \cdots$$

and so $p \in E$. The theorem is established.

An important application of theorem 5:7 is its use in establishing a condition for a set to be analytic, as is done in the following theorem.

Theorem 5:8 : A necessary and sufficient condition for a non-empty set E contained in a complete separable space M to be analytic is that it be the continuous image of the set N of all irrational numbers.

Proof: If E is a non-empty analytic set in a complete separable space M, then $E=\sum_{\{n_K\}}\prod_{K=1}^{\infty}E_{n(K)}$, where $[E_{n(K)}]$ is a regular defining system of closed sets. If $x\in N$, then $x=[x]+\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\cdots$, where $[x]$ is the largest positive integer less than x, and $n_1, n_2, n_3, \cdots$ is the infinite sequence of positive integers obtained from the continued fraction development of x. (See Chapter IV)

Let $F(x)=\prod_{K=1}^{\infty}E_{n(K)}$. F(x) will be a single point since the sets $E_{n(K)}$ form a descending sequence of non-empty closed sets whose diameters tend towards zero, and since the space M is complete. [7, p. 189] Let this point be called f(x). Thus for each $x\in N$, f(x) is defined. Also, f is a mapping from N onto E, for suppose that $q\in E$, then there exists a sequence of indices $\{n_K\}$ such that $q\in\prod_{K=1}^{\infty}E_{n(K)}$. Thus $q=f(x)$ where $x=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\cdots$, that is, $x=\{n_K\}$ by continued fractions.

The function f(x) is a continuous mapping of N onto E. To show this, suppose that $x_o=\{n_k^o\}$ by continued fractions, and suppose a number $\epsilon>0$ is given. Then there exists a number $\delta>0$, and an integer k such that $\frac{1}{k}<\epsilon$, and $\rho(x,x_o)<\delta$ implies $n_i=n_i^o$ for $i\leq k$, where $x=\{n_k\}$. Thus $f(x_o)\in E_{n^o_{(k)}}=E_{n_{(k)}}$, and $f(x)$ is contained in $E_{n_{(k)}}$ for this given integer k. Hence $\rho\big(f(x),\ f(x_o)\big)\leq \delta(E_{n_{(k)}})<\frac{1}{k}<\epsilon$. The continuity of the function, as well as the necessary condition of the theorem, is established.

To show that the condition of the theorem is sufficient, let $f(x)$ be a function defined and continuous on N which assumes values in a complete separable space M. Since the sum of a countable collection of analytic sets is again an analytic set, it will be sufficient to consider the function $f(x)$ only on the set N_o, the set of all irrational numbers x, $0<x<1$. Let $f(N_o)=E$.

For each finite sequence of positive integers, $n_1, n_2, n_3, \cdots, n_k$, let $X_{n_{(k)}}$ be a set such that $x\in X_{n_{(k)}}$ if $x\in N_o$, and if $x=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\cdots+\frac{1}{n_k}+\cdots$ by continued fractions. Let $E_{n_{(k)}}=\overline{f(X_{n_{(k)}})}$. Thus $E_{n_{(k)}}$ will be a closed set. We shall now show that $E=\sum_{\{n_k\}}\prod_{k=1}^{\infty}E_{n_{(k)}}$.

Suppose $q\in E$. There exists an $x\in N_o$ such that $f(x)=q$. But $x=\{n_k\}$ by continued fractions, hence $x\in X_{n_{(k)}}$ for each k. Thus $x\in\prod_{k=1}^{\infty}X_{n_{(k)}}$. But $E_{n_{(k)}}=\overline{f(X_{n_{(k)}})}$, so $f(x)\in f\Big(\prod_{k=1}^{\infty}(X_{n_{(k)}})\Big)\subset\prod_{k=1}^{\infty}f(X_{n_{(k)}})=\prod_{k=1}^{\infty}E_{n_{(k)}}\subset\sum_{\{n_k\}}\prod_{k=1}^{\infty}E_{n_{(k)}}$.

Suppose $q\in\sum_{\{n_k\}}\prod_{k=1}^{\infty}E_{n_{(k)}}$. Then there exists an infinite sequence of indices $\{n_k^\circ\}$ such that $q \in \prod_{k=1}^{\infty} E_{n(k)}$. Let $x_0 = \{n_k^\circ\}$ by continued fractions. Then $x_0 \in N_0$. It will now be shown that $f(x_0) = q$ by showing that they are arbitrarily close to each other. Given any number $\epsilon > 0$, there exists a number $\delta > 0$ such that $\rho(x, x_0) < \delta$ implies that $\rho(f(x), f(x_0)) < \frac{\epsilon}{2}$, for $x \in N_0$, by the definition of continuity. By the properties of continued fractions, there exists an integer L such that if $n_i = n_i^\circ$ for $i \leq L$, and $x = \{n_k\}$ by continued fractions, then $\rho(x, x_0) < \delta$, and thus $\rho(f(x), f(x_0)) < \frac{\epsilon}{2}$. It follows that $\delta(f(X^\circ_{n(L)})) \leq \frac{\epsilon}{2}$, hence $\delta(E^\circ_{n(L)}) < \epsilon$. The point $q \in E^\circ_{n(L)}$, and $f(x_0) \in E^\circ_{n(L)}$, thus $\rho(q, f(x_0)) < \epsilon$.

The identity $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n(k)}$, where each set $E_{n(k)}$ is closed, is established, and therefore E is an analytic set.

Theorem 5:9 : The continuous image of an analytic set in a complete separable space is an analytic set.

Proof: Let E be an analytic set in a complete separable space M. Let f be a continuous function on E, and let $f(E) = T$. Then there exists a function ψ on N, the set of all irrational numbers, such that $\psi(N) = E$. Let $\phi(x) = f(\psi(x))$. Then $\phi(N) = T$. Thus T is the continuous image of N, and is an analytic set.

Since a Borel set is also an analytic set, its continuous image in a complete separable space is an analytic set. Also it follows from the last theorem that in complete separable spaces analytic sets are topologically invariant. It can be shown that both the analytic sets and the complements of analytic sets are topologically invariant in any complete space, not necessarily separable. [7, p. 220]

We shall now show that the power (cardinal number) of a non-countable analytic set contained in a separable metric space is equal to C, the power of the continuum. First we shall prove this preliminary theorem.

<u>Theorem 5:10</u> : <u>If E is a set contained in a separable space, and if S is a neighborhood such that E·S is non-countable, then there exists neighborhoods S_0 and S_1 whose diameters are as small as we choose, and such that $\overline{S_0}\cdot\overline{S_1}=0$, $S_0 \subset S$, $S_1 \subset S$, and the sets $E\cdot S_0$ and $E\cdot S_1$ are non-countable</u>.

<u>Proof</u>: Suppose that E is a set in a separable metric space, and that S is a neighborhood such that E·S is non-countable. Then there exists a non-countable set $E_1 \subset E\cdot S$ such that $x \in E_1$ if and only if x is an element of condensation of E·S. [8, p. 43] Let p and q be two points of E_1. Since S is an open set, there exists numbers r_0 and r_1 sufficiently small so that $N(p,r_0)$ and $N(q,r_1)$ each are contained in S, and such that $\overline{N(p,r_0)}\cdot\overline{N(q,r_1)}=0$. [6, p. 21] Let $S_0 = N(p,r_0)$, and $S_1 = N(q,r_1)$. By the definition of an element of condensation, $E\cdot S_0$ and $E\cdot S_1$ are both non-count-

able sets.

<u>Theorem 5:11</u> : <u>Every non-countable analytic set which is contained in a complete separable space contains a subset which is non-empty and perfect</u>.

<u>Proof</u>: Suppose that E is non-countable and is contained in a complete separable space M. By theorem 5:7, $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n_{(k)}}$, where the defining system $[E_{n_{(k)}}]$ is regular.

For each finite combination of positive integers $r_1, r_2, r_3, \cdots, r_s$, let

$$E^{r_{(s)}} = \sum_{\{n_k\}} E_{r_{(1)}} \cdot E_{r_{(2)}} \cdots E_{r_{(s)}} \cdot E_{r_{(s)}, n_1} \cdot E_{r_{(s)}, n_{(2)}} \cdots$$

where the summation extends over all possible sequences of integers $\{n_k\}$. It follows that $E = E^1 + E^2 + E^3 + \cdots$, and that $E^{r_{(s)}} = E^{r_{(s)},1} + E^{r_{(s)},2} + E^{r_{(s)},3} + \cdots$ for every finite combination of indices $r_{(s)}$.

Let p be an element of condensation of E, and let $S = N(p,1)$. E S is non-countable by the definition of an element of condensation, thus we can apply theorem 5:10 directly. There exist two neighborhoods S_0 and S_1 which are contained in S such that $\bar{S}_0 \cdot \bar{S}_1 = 0$, $E \cdot S_0$ and $E \cdot S_1$ are non-countable, and $\delta(S_0) < 1$, $\delta(S_1) < 1$. From above we have $E \cdot S_0 = E^1 \cdot S_0 + E^2 \cdot S_0 + E^3 \cdot S_0 + \cdots$. Since $E \cdot S_0$ is non-countable, there exists at least one index m_0 such that $E^{m_0} \cdot S_0$ is non-countable. In a like manner, there exists an index m_1 such that $E^{m_1} \cdot S_1$ is non-countable.

Proceding by induction, suppose that we have defined for a given integer k the neighborhoods $S_{a_{(k)}}$ and the integers $m_{a_{(k)}}$, where $a_{(k)}$ is a finite sequence of numbers which are either 0 or 1, such that

$$\delta(S_{a_{(k)}}) < \frac{1}{k},$$
$$S_{a_{(k)}} \subset S_{a_{(k-1)}}, \text{ if } k>1,$$
$$\bar{S}_{a_{(k-1)},0} \cdot \bar{S}_{a_{(k-1)},1} = 0, \text{ and}$$
$$E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}} \cdot S_{a_{(k)}} \text{ is non-countable.}$$

From theorem 5:10, there exist neighborhoods $S_{a_{(k)},0}$ and $S_{a_{(k)},1}$ contained in $S_{a_{(k)}}$ such that

$$\bar{S}_{a_{(k)},0} \; \bar{S}_{a_{(k)},1} = 0$$
$$\delta(S_{a_{(k)},0}) < \frac{1}{k+1}, \; \delta(S_{a_{(k)},1}) < \frac{1}{k+1}, \text{ and sets}$$
$$S_{a_{(k)},0} \cdot E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}} \text{ and}$$
$$S_{a_{(k)},1} \cdot E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}} \text{ are non-countable.}$$

Then since

$$E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}} = \sum_{n=1}^{\infty} E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}, n}$$

there exists an integer $m_{a_{(k)},0}$ such that the set

$$S_{a_{(k)},0} \cdot E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}, m_{a_{(k)},0}}$$

is non-countable. Likewise there exists an integer $m_{a_{(k)},1}$ such that the set $S_{a_{(k)},1} \cdot E^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}, m_{a_{(k)},1}}$ is non-countable. Thus by induction the neighborhoods $S_{a_{(k)}}$ and the indices $m_{a_{(k)}}$ have been defined for every finite combination of numbers $a_{(k)}$ which are either 0 or 1, and these neighborhoods $S_{a_{(k)}}$ and indices $m_{a_{(k)}}$ are such that the preceding conditions are satisfied.

Let $H_k = \sum_{a_{(k)}} \bar{E}^{m_{a_{(1)}}, m_{a_{(2)}}, \dots, m_{a_{(k)}}} \cdot \bar{S}_{a_{(k)}}$, where the summation extends over all possible sequences of k numbers which are either 0 or 1. Since the summation is of a finite number of closed and bounded sets, each set H_k will be closed and bounded. It follows that $H_{k+1} \subset H_k$ for each k, and that H_k is not empty.

Let $H = \prod_{k=1}^{\infty} H_k$. Since H is the intersection of a descending sequence of closed sets in a complete space, H is non-empty. [6, p. 52] To show that H is perfect, that is, H is closed and dense-in-itself, we must show that $p \in H$ if and only if it is a cluster point of H.

Suppose that $p \in H$. Then $p \in H_1 = \overline{E}^{m_0} \cdot \overline{S}_0 + \overline{E}^{m_1} \cdot \overline{S}_1$. The element p belongs either to $\overline{E}^{m_0} \cdot \overline{S}_0$ or to $\overline{E}^{m_1} \cdot \overline{S}_1$. (It cannot belong to both since they are disjoint) Let $\alpha_1 = 0$ or $\alpha_1 = 1$ so that $p \in \overline{E}^{m_{\alpha_1}} \cdot \overline{S}_{\alpha_1}$. In a like manner, $p \in H_2 = \overline{E}^{m_0, m_{0,0}} \cdot \overline{S}_{0,0} + \overline{E}^{m_0, m_{0,1}} \cdot \overline{S}_{0,1} + \overline{E}^{m_1, m_{1,0}} \cdot \overline{S}_{1,0} + \overline{E}^{m_1, m_{1,1}} \overline{S}_{1,1}$. But from the construction of these sets, p can belong only to the set $\overline{E}^{m_{\alpha_1}, m_{\alpha_1,0}} \cdot \overline{S}_{\alpha_1,0}$ or to $\overline{E}^{m_{\alpha_1}, m_{\alpha_1,1}} \cdot \overline{S}_{\alpha_1,1}$. Let $\alpha_2 = 0$ or $\alpha_2 = 1$ accordingly such that $p \in \overline{E}^{m_{\alpha_1}, m_{\alpha_1,\alpha_2}} \cdot \overline{S}_{\alpha_1,\alpha_2}$. Since for each k, $p \in H_k$, we continue to obtain the elements of the infinite sequence $\{\alpha_n\}$ in a like manner, where each term of the sequence is either 0 or 1, and such that $p \in \overline{E}^{m_{\alpha_1}, m_{\alpha(2)}, \cdots, m_{\alpha(k)}} \cdot S_{\alpha(k)}$ for $k = 1, 2, 3, \cdots$.

Given a number $\epsilon > 0$, let s be an integer such that $\frac{1}{s} < \epsilon$. Let $\{\beta_n\}$ be an infinite sequence of numbers either 0 or 1 as follows. If $k \leq s$, let $\beta_k = \alpha_k$. If $k = s + 1$, let $\beta_k = 1-\alpha_k$. If $k > s+1$, let $\beta_k = 0$. Consider the set $Q = \prod_{k=1}^{\infty} \overline{E}^{m_{\beta(1)}, m_{\beta(2)}, \cdots, m_{\beta(k)}} \cdot \overline{S}_{\beta(k)}$ defined by the sequence $\{\beta_n\}$. Since Q is the intersection of a descending sequence of closed sets whose diameters tend towards zero, and since these sets are in a complete space, Q will be a single element which will be denoted by q. [6, p. 52] The element q will be an element of each set H_k, $k = 1, 2, 3, \cdots$, by the definition of those sets, so $q \in H$. Since $\beta_k = \alpha_k$ for $k \leq s$ and $p \in \overline{S}_{\alpha(s)}$, $q \in \overline{S}_{\beta(s)}$, q is an element of the set $\overline{S}_{\alpha(s)}$. But $\delta(\overline{S}_{\alpha(s)}) < \frac{1}{s} < \epsilon$, thus $\rho(p,q) < \epsilon$.

The element p is different from q since $p \in \overline{S}_{\alpha(s+1)}$, $q \in \overline{S}_{\beta(s+1)}$ where these two sets are disjoint since $\beta_{s+1} = 1-\alpha_{s+1}$. Therefore p is a cluster point of H. On the other hand, if p is a cluster point of H, then $p \in H$ since H, as the intersection of a countable collection of closed sets, is closed. H is therefore perfect.

It remains to be shown that $H \subset E$. Suppose that $p \in H$. As previously defined, there exists a specific sequence $\{\alpha_n\}$ of numbers either 0 or 1 such that the element $p \in \prod_{k=1}^{\infty} \overline{E}^{m_{\alpha(1)}, m_{\alpha(2)}, \cdots, m_{\alpha(k)}} \cdot \overline{S}_{\alpha(k)}$. Thus $p \in \prod_{k=1}^{\infty} \overline{E}^{m_{\alpha(1)}, m_{\alpha(2)}, \cdots, m_{\alpha(k)}}$. From the construction of the sets $E^{r(s)}$, it is noted that for each finite combination of indices $r_{(s)}$, $E^{r(s)} \subset E_{r(s)}$. Since $E_{r(s)}$ is closed, $\overline{E}^{r(s)} \subset \overline{E}_{r(s)} \subset E_{r(s)}$. Therefore the element $p \in \prod_{k=1}^{\infty} E_{m_{\alpha(1)}, m_{\alpha(2)}, \cdots, m_{\alpha(k)}} \subset E$, and the theorem is complete.

Noting that for each infinite sequence $\{\alpha_n\}$ of numbers 0 or 1 there is a distinct point of the set E which is defined by $\prod_{K=1}^{\infty} \bar{E}^{m_{\alpha(1)}, m_{\alpha(2)}, \ldots, m_{\alpha(K)}} \cdot \bar{S}_{\alpha(K)}$, it can be said that the cardinal of E is greater than or equal to C, the cardinal of the continuum. [7, p. 263] Since E is contained in a separable metric space, and therefore has a countable basis, the cardinal of E is less than or equal to C. Hence the cardinal of E is C. From this we may conclude that the cardinal of a non-countable Borel set contained in a separable complete space is C.

<u>Definition</u>: Two sets P and Q are said to be exclusive B if there exists two disjoint Borel sets M and N such that $P \subset M$, $Q \subset N$.

<u>Theorem 5:12</u> : <u>If $P = \sum_{j=1}^{\infty} P_j$, and $Q = \sum_{K=1}^{\infty} Q_K$, and if P and Q are not exclusive B, then for some indices j and k the sets P_j and Q_K are not exclusive B.</u>

<u>Proof</u>: Suppose that $P = \sum_{j=1}^{\infty} P_j$, and $Q = \sum_{K=1}^{\infty} Q_K$, and that P and Q are not exclusive B. Then suppose that for every pair of indices j and k, sets P_j and Q_K are exclusive B. Then there would exist disjoint Borel sets $M_{j,K}$ and $N_{j,K}$ for every pair of indices j and k such that $P_j \subset M_{j,K}$ and $Q_K \subset N_{j,K}$. Let $M = \sum_{j=1}^{\infty} \prod_{K=1}^{\infty} M_{j,K}$ and $N = \sum_{K=1}^{\infty} \prod_{j=1}^{\infty} N_{j,K}$. The sets M and N are Borel sets by theorem 2:4 and theorem 2:8. For each j, $P_j \subset \prod_{K=1}^{\infty} M_{j,K}$, so $\sum_{j=1}^{\infty} P_j \subset \sum_{j=1}^{\infty} \prod_{K=1}^{\infty} M_{j,K}$, that is, $P \subset M$. For each k, $Q_K \subset \prod_{j=1}^{\infty} N_{j,K}$, so $\sum_{K=1}^{\infty} Q_K \subset \sum_{K=1}^{\infty} \prod_{j=1}^{\infty} N_{j,K}$, that

is, $Q \subset N$. Sets M and N are disjoint, for if $x \in M$, there exists an index j such that $x \in \prod_{k=1}^{\infty} M_{j,k}$. Since $M_{j,k} \cdot N_{j,k} = 0$ for $j = 1, 2, 3, \cdots$, $k = 1, 2, 3, \cdots$, $x \notin N_{j,k}$ for the given index j, and for all indices k. Thus $x \notin \sum_{k=1}^{\infty} \prod_{j=1}^{\infty} N_{j,k}$. Thus the sets P and Q are exclusive B which contradicts the hypothesis of the theorem.

Theorem 5:13 : If E and T are two analytic sets contained in a complete separable space, and if $E \cdot T = 0$, then E and T are exclusive B.

Proof: The sets E and T may each be written as the nucleus of a regular defining system by theorem 5:7. Thus $E = \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{n_{(k)}}$, and $T = \sum_{\{n_k\}} \prod_{k=1}^{\infty} T_{n_{(k)}}$, where $[E_{n_{(k)}}]$ and $[T_{n_{(k)}}]$ are regular defining systems.

For every finite combination of positive integers $r_{(s)}$, let $E^{r_{(s)}} = \sum_{\{n_k\}} E_{r_{(1)}} \cdot E_{r_{(2)}} \cdots E_{r_{(s)}} \cdot E_{r_{(s)}, n_{(1)}} \cdot E_{r_{(s)}, n_{(2)}} \cdots$, where the summation extends over all infinite sequences of positive integers $\{n_k\}$. Likewise, for every finite combination of indices $r_{(s)}$, let

$$T^{r_{(s)}} = \sum_{\{n_k\}} T_{r_{(1)}} \cdot T_{r_{(2)}} \cdots T_{r_{(s)}} \cdot T_{r_{(s)}, n_{(1)}} \cdots.$$

From theorem 5:11, we note that

$$E^{r_{(s)}} = E^{r_{(s)},1} + E^{r_{(s)},2} + \cdots,$$

$$T^{r_{(s)}} = T^{r_{(s)},1} + T^{r_{(s)},2} + \cdots,$$

$$E = E^1 + E^2 + E^3 + \cdots,$$

$$T = T^1 + T^2 + T^3 + \cdots.$$

Now let us suppose that E and T are not exclusive B.

By theorem 5:12, there exist indices p_1 and q_1 such that the sets E^{p_1} and T^{q_1} are not exclusive B. But since

$$E^{p_1} = E^{p_1,1} + E^{p_1,2} + E^{p_1,3} + \cdots,$$
$$T^{q_1} = T^{q_1,1} + T^{q_1,2} + T^{q_1,3} + \cdots,$$

there exist indices p_2 and q_2 such that E^{p_1,p_2} and T^{q_1,q_2} are not exclusive B. Continuing in a similar manner, we can obtain two infinite sequences of integers $\{p_n\}$ and $\{q_n\}$ such that $E^{p(k)}$ and $T^{q(k)}$ are not exclusive B for each k.

From theorem 5:11 it is noted that the sets $E^{p(k)} \subset E_{p(k)}$, and $T^{q(k)} \subset T_{q(k)}$ for $k = 1, 2, 3, \cdots$. Since each set $E_{p(k)}$ and $T_{q(k)}$ is a set of a regular defining system, it is closed and therefore a Borel set. Thus if $E_{p(k)} \cdot T_{q(k)} = 0$ for any integer k, the sets $E^{p(k)}$ and $T^{q(k)}$ would be exclusive B. Therefore $E_{p(k)} \cdot T_{q(k)} \neq 0$ for $k = 1, 2, 3, \cdots$. Let $R_k = E_{p(k)} \cdot T_{q(k)}$. The sets R_k form a descending sequence of non-empty closed sets since $\delta(R_k) \leq \delta(E_{p(k)}) < \frac{1}{k}$, $\delta(T_{q(k)}) < \frac{1}{k}$, $E_{p(k+1)} \subset E_{p(k)}$, $T_{q(k+1)} \subset T_{q(k)}$, and the sets $E_{p(k)}$ and $T_{q(k)}$ are closed for each k. Let $R = \prod_{k=1}^{\infty} R_k$. The set R will be non-empty since the containing space is complete. Suppose that y is an element of R. Then

$$y \in \prod_{k=1}^{\infty} E_{p(k)} \subset \sum_{\{n_k\}} \prod_{k=1}^{\infty} E_{p(k)} = E, \text{ and}$$
$$y \in \prod_{k=1}^{\infty} T_{q(k)} \subset \sum_{\{n_k\}} \prod_{k=1}^{\infty} T_{q(k)} = T.$$

Hence $E \cdot T \neq 0$ which contradicts the hypothesis. Therefore the theorem is established.

With the aid of theorem 5:13, a criterion for an analytic set to be a Borel set can be established.

<u>Theorem 5:14</u> : <u>An analytic set E contained in a complete separable space is a Borel set if and only if its complement is an analytic set</u>.

<u>Proof</u>: Suppose that E is a Borel set. Then its complement is also a Borel set and therefore an analytic set.

On the other hand, suppose that E is an analytic set in a complete separable space, and suppose that $\mathcal{C}E$ is an analytic set. Then since $E \cdot \mathcal{C}E = 0$, there exist two Borel sets M and N such that $E \subset M$, $\mathcal{C}E \subset N$, and $M \cdot N = 0$. (Theorem 5:13) Since $\mathcal{C}E \subset N$, $\mathcal{C}(\mathcal{C}(E)) \supset \mathcal{C}N$, that is, $M \subset \mathcal{C}N \subset E$. Hence $E = M$; thus E is a Borel set.

In a similar manner, the following theorem could be established.

<u>Theorem 5:15</u> : <u>A set E in a complete separable space S is a Borel set if and only if E and $\mathcal{C}E$ are analytic sets</u>.

CHAPTER VI

A UNIVERSAL ANALYTIC SET

In the concluding chapter, we shall show that there exist sets which are not analytic sets relative to their containing space, and that there exist sets which are analytic sets but not Borel sets. In showing this, we shall discuss projection and projective sets, and shall establish a plane analytic set W which is universal to all linear analytic sets.

Definition: The projection of a point $x = (x_1, x_2, \cdots, x_{m+1})$ of the space R_{m+1} (m+1-dimension Euclidean space) is the point $y = (x_1, x_2, \cdots, x_m)$ of the space R_m, and we write $P(x) = y$. The projection of a set $E \subset R_{m+1}$ is the set $P(E) \subset R_m$ which consists of the projections of all of the elements of E.

Since the distance between two elements of a set E in R_{m+1} is greater than or equal to the distance between the images of these two points in P(E) in R_m, a projection is a continuous mapping of E onto P(E). Therefore the projection of a sum of sets is equal to the sum of the projections, $P(\sum_{E \in \mathcal{B}} E) = \sum_{E \in \mathcal{B}} (P(E))$; and the projection of a product of sets is included in the product of the projections, $P(\prod_{E \in \mathcal{B}} E) \subset \prod_{E \in \mathcal{B}} (P(E))$.

Theorem 6:1 : ***A set T is a set F_σ in R_m if and only if it is the projection of a set E which is a closed set in R_{m+1}.***

Proof: We shall prove the theorem in the case where $m = 2$ which is analogous to the proof for any dimension m.

Suppose that E is a closed set in R_3, three-dimension Euclidean space. For each positive integer k, let $E_k = E \cdot \overline{N(0,k)}$. Then $E = \sum_{k=1}^{\infty} E_k$, where for each k, the set E_k is closed and bounded, and is therefore compact. Then $T = P(E) = P(\sum_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} P(E_k)$, where each set $P(E_k)$ is a compact and closed set. The set T is then an F_σ.
[6, p. 68]

Now suppose that T is a set F_σ in the plane. Then $T = \sum_{k=1}^{\infty} T_k$, where for each k, T_k is a closed set. For each positive integer k, let $E_k = E_{(x,y,z)}[(x,y) \in T_k,\ z = k]$. For integers $i \neq j$, the sets E_i and E_j will be disjoint, having their nearest points a distance of at least 1 from each other. Since E_k is congruent geometrically to T_k, $k = 1, 2, 3, \cdots$, each set E_k will be a closed plane set.

Let $E = \sum_{k=1}^{\infty} E_k$. If $p = (x,y,z)$ is a cluster point of E, then there exists an infinite sequence $\{p_n\} = \{(x_n, y_n, z_n)\}$ such that $\lim_{n\to\infty} (x_n, y_n, z_n) = (x,y,z) = p$. Thus $\lim_{n\to\infty} z_n = z$. Given a number $\epsilon = \frac{1}{2}$, there exists an integer K such that if $n > K$, $\rho(z_n, z) < \frac{1}{2}$. Since the sets E_i and E_j, $i \neq j$, are a distance apart of at least 1, there exists an integer k such that $p_n \in E_K$, $n > K$. The subsequence $p_{K+1}, p_{K+2}, \dots$ is contained in E_K, and will converge to p. The element p is therefore a cluster point of E_K. Since E_K is closed, $p \in E_K \subset E$. Thus E is closed. It then follows that

$$P(E) = P(\sum_{K=1}^{\infty} E_K) = \sum_{K=1}^{\infty} P(E_K) = \sum_{K=1}^{\infty} T_K = T,$$

and the theorem is established.

<u>Theorem 6:2</u> : <u>A set E is an analytic set in R_m if and only if it is the projection of a set H which is a set G_δ in R_{m+1}.</u>

<u>Proof</u>: Suppose that H is a set G_δ in R_{m+1}. Since a projection is a continuous mapping, $P(H) = T$, as the image of a Borel set, is an analytic set in R_m by theorem 5:9.

We shall show that if E is an analytic set in R_2, then it is the projection of a set H which is a set G_δ in R_3. The proof for the more general case is very similar. Suppose that E is an analytic set in R_2. By theorem 5:8, E is the continuous image of N, the set of all irrational numbers, by a mapping f.

Let $H = E_{(x,y,z)}[z \in N,\ (x,y) = f(z)]$. Then

$P(H) = E_{(x,y)}[z \in N,\ (x,y) = f(z)] = f(N) = E.$

It remains to be shown that H is a set G_δ. Let T be the set of all planes in R_3 with rational z-coordinates. Thus T is a set F_σ as the sum of a countable collection of closed sets. We shall now establish the identity, $H = \bar{H} - T$.

Since $H \cdot T = 0$, $H \subset \complement T$. Then $H \subset \complement T \cdot \overline{H} = \overline{H} - T$. If $(x_o, y_o, z_o) \in \overline{H} - T$, then $z_o \notin T$, $z_o \in N$. Since $(x_o, y_o, z_o) \in \overline{H}$, there exists an infinite sequence $\{(x_n, y_n, z_n)\}$ of the set H such that $\lim_{n\to\infty} (x_n, y_n, z_n) = (x_o, y_o, z_o)$. In turn

$$\lim_{n\to\infty} x_n = x_o,$$
$$\lim_{n\to\infty} y_n = y_o,$$
$$\lim_{n\to\infty} z_n = z_o.$$

Since $(x_n, y_n, z_n) \in H$ for each n, $z_n \in N$ for each n, and $(x_n, y_n) = f(z_n)$, $\lim_{n\to\infty} (x_n, y_n) = \lim f(z_n) = (x_o, y_o)$. Also, since f is continuous, $\lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(z_o) = (x_o, y_o)$. Therefore $(x_o, y_o, z_o) \in H$, and $H = \overline{H} - T$.

Since the closed set $\overline{H}$ is a set G_δ, and $\complement T$ as the complement of a set F_σ is a set G_δ, their intersection, $\overline{H} \cdot \complement T = H$, is also a set G_δ. The proof is therefore established.

Following the method used in Chapter IV, we shall construct a set M_1 in R_3 which is a set G_δ, and which is universal to all plane sets G_δ. Then we shall show that the projection of this set M_1 is an analytic set in R_2 which is universal to all linear analytic sets.

Let S be a subset of R_3 consisting of all planes S_{x_o}, where $p \in S_{x_o}$ if and only if $p = (x_o, y, z)$, $x_o \in N_o$, and y, z have any real values. (The set N_o is the set of all irrational numbers x, $0 < x < 1$) Thus the planes S_{x_o} will be perpendicular to the x-axis. Let $\{K_n\}$ be a sequence of open plane sets which form a countable open basis for the (y,z) plane.

If $x_0 \in N_0$, and if $x_0 = \{\alpha^n\}$ by continued fractions, then let $H_0(x_0) = \sum_{n=1}^{\infty} K_{\alpha^n}$, and let

$$M_0(x_0) = E_{(x,y,z)}[x = x_0,\ (y,z) \in H_0(x_0)].$$

Then let $M = \sum_{x \in N_0} M_0(x)$

Following the method described in Chapter IV, let each number x_0, $x_0 \in N_0$, determine an infinite sequence of numbers $\{x_n^0\}$, where for each n, $x_n^0 \in N_0$. Let

$$H_1(x_0) = \prod_{n=1}^{\infty} H_0(x_n^0),$$

$$M_1(x_0) = E_{(x,y,z)}[x = x_0,\ (y,z) \in H_1(x_0)],$$

$$M_1 = \sum_{x \in N_0} M_1(x).$$

In a manner entirely analogous to that used in Chapter IV, it can be shown that the set M_1 is a set G_δ in R_3 which is universal to all plane sets G_δ. These plane sets G_δ are obtained by intersecting M_1 with planes S_x, $x \in N_0$, and S_x perpendicular to the x-axis.

Consider the projection of M_1, $P(M_1) = W$. The set W is an analytic set in R_2 (the (x,y) plane) by theorem 6:2. Next we shall show that W is universal to all linear analytic sets by means of intersections with lines L(x), $x \in N_0$.

If E is a linear analytic set, then there exists a set H of the plane which is a set G_δ such that $P(H) = E$. Since M_1 is universal to all plane sets G_δ, H is the intersection of a plane S_{x_0}, $x_0 \in N_0$, and the set M_1. Then E is the intersection of the line L(x) with W, where W is the projection of M_1.

On the other hand, if W is intersected with a line L(x), then the intersection is a linear analytic set since the set W is itself an analytic set, and the line, as a Borel set, is an analytic set.

The class of all sets which are analytic sets relative to the plane satisfy the hypothesis of theorem 4:5. First, the intersection of a line (Borel set) and an analytic set is an analytic set. Second, if E is an analytic set on a line x, then f(E) is an analytic set on the y-axis where f is a horizontal projection. If the line x is not perpendicular to the y-axis, this will be true since f is a topological mapping. If the line x is perpendicular to the y-axis, then f(E) will be a single point, and hence an analytic set.

Thus, by applying theorem 4:5 directly, the set D·W is shown to be an analytic set, and the set D·𝔈W is not an analytic set. By theorem 5:14 (since the line D is a complete separable space) we can conclude that the set D·W is not a Borel set; for if it were, then its complement, D·𝔈W, would be an analytic set.

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