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Study of the acceleration of Coriolis

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A STUDY OF THE ACCELERATION OF CORIOLIS

by

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PART I

INTRODUCTION

In any problem involving precise measurement of acceleration on the earth it is necessary to keep in mind that the only acceleration we can measure directly is the acceleration relative to the earth. Thus, in applying Newton's second law of motion exactly to bodies moving with respect to the earth, we must determine not only the acceleration relative to the earth but also the acceleration produced by the rotation of the earth and add these vectorally to the relative acceleration.

This paper will show that the true acceleration, that is the acceleration relative to a fixed inertial system, is composed of a number of accelerations as measured in a moving system. These are the acceleration of the origin of any moving set of axes, the acceleration due to the angular motion of the moving axes, the acceleration relative to the moving axes, and an acceleration due to the change in the magnitude and direction of the velocity of a point relative to the moving axes because of the rotation of these axes. This last mentioned acceleration is known as complimentary acceleration, compound centripital acceleration, central acceleration, or the acceleration of Coriolis, so named to honor its discoverer, Gaspard Coriolis.

This paper is primarily concerned with giving a modern treatment of the acceleration of Coriolis. Modern physics owes a great debt to Coriolis for his important formulation of Newton's second law of motion when applied to precise measurements of accelerations here on earth.

The following is a short biography of this man, taken from Larousse's Grand Dictionnaire Universel:

> Coriolis (Gaspard-Gustave de), distinguished mathematician, born in Paris in 1792, died in 1843. In 1808 he entered the Polytechnic School, from which he went on to the Civil Engineering School, but soon gave up an engineering career to become an instructor in mathematical analysis and mechanics in the Polytechnic School, where in 1838 he succeeded Dulong in the important position of Director of Studies. Two years previously he had been made a member of the Academy of Sciences. His principal works are: The Calculation of Mechanical Action (Calcul de l'Effet des Machines, Paris, 1829, quarto), reprinted under the title of Treatise on the Mechanics of Solid Bodies, etc. (Traite de la Mecanique des Corps Solides, etc., 1844); and Mathematical Theory of Spin in the Game of Billiards (Theorie Mathematique des Effets du Jeu de Billard, 1835, octavo). He also published numerous articles in the Dictionary of Industry (Dictionnaire de l'Industrie).

PART II

ANALYTICAL TREATMENT OF THE ACCELERATION OF CORIOLIS

To begin the study of the acceleration of Coriolis analytically, let us set up two sets of coordinate axes, one of oxyz coordinates comprising a fixed inertial system to be chosen so that at the particular instant under consideration these axes are parallel to the moving axes; and the other OXYZ, a moving set of rigid coordinate axes. Now let us turn our attention to the behavior of a point P that is in motion with respect to the XYZ system, the moving coordinate axes. In this discussion both the xyz and XYZ systems shall be right handed systems. To clarify the relationships between these two sets or coordinates, the following table may be used:

	X	Y	Z
x	a_1	a_2	a_3
y	b_1	b_2	b_3
z	c_1	c_2	c_3

In this table a_1 is the direction cosine of the angle between x and X axes. a_2 is the direction cosine of the angle between the x and Y axes. a_3 is the direction cosine of the angle between the x and Z axes. b_1 is the cosine of

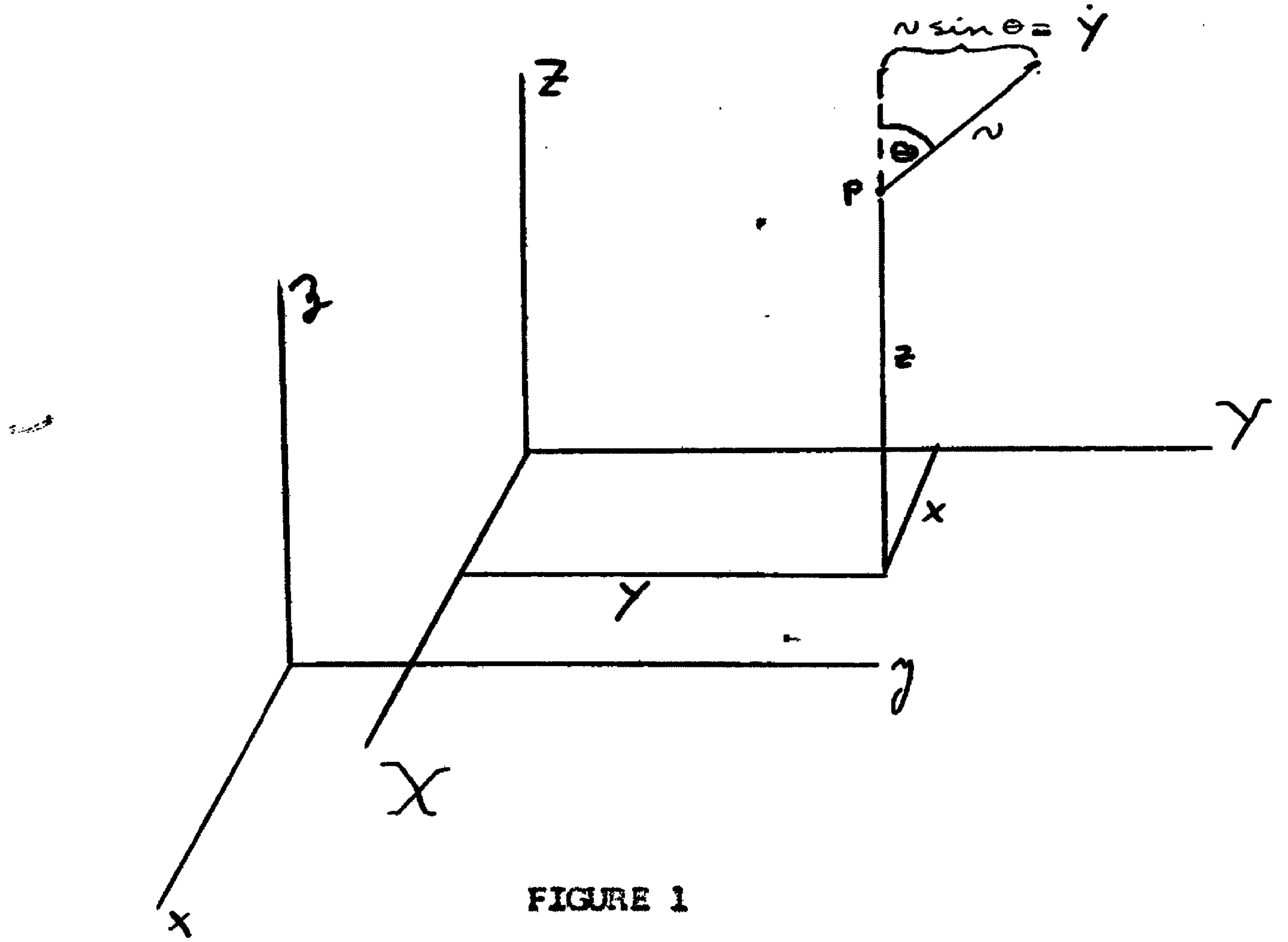


FIGURE 1

POSITION OF THE TWO COORDINATE SYSTEMS AT THE INSTANT UNDER CONSIDERATION

the angle between y and X axes, b_2 is the direction cosine of the angle between y and Y axes, b_3 is the direction cosine of the angle between y and Z axes. c_1 is the direction cosine of the angle between z and X axes, c_2 is the direction cosine of the angle between z and Y axes, c_3 is the direction cosine of the angle between z and Z axes.

From this table the following equations may be set up:

$X = x_0 \quad a_1X \quad a_2Y \quad a_3Z$

$y = y_0 \quad b_1X \quad b_2Y \quad b_3Z$ and $z = z_0 \quad c_1X \quad c_2Y \quad c_3Z$

In these equations $x_0 y_0$ and z_0 represent the position of "O" the origin of the moving ordinates with respect to the fixed system. Now let us differentiate these equations with respect to time. In these equations the derivative of x with respect to time shall be denoted by

$\dot{x}$, etc. $\quad \dot{x} = \dot{x}_0 + a_1\dot{X} + a_2\dot{Y} + a_3\dot{Z} + \dot{a}_1X + \dot{a}_3Z$

$$\dot{y} = \dot{y}_0 + b_1\dot{X} + b_2\dot{Y} + b_3\dot{Z} + \dot{b}_1X + \dot{b}_2Y + \dot{b}_3Z$$

Differentiating again with respect to time we get the x, y, and z components of the acceleration of P. These components are seen to be:

$$\ddot{x} = \ddot{x}_0 + a_1\ddot{X} + a_2\ddot{Y} + a_3\ddot{Z} + 2\dot{a}_1\dot{X} + 2\dot{a}_2\dot{Y} + 2\dot{a}_3\dot{Z} + \ddot{a}_1X + \ddot{a}_2Y + \ddot{a}_3Z \quad ($$

Similar equations follow for $\ddot{y}$ and $\ddot{z}$. The terms $a_1\ddot{X}$, $a_2\ddot{Y}$ and $a_3\ddot{Z}$, represent the linear acceleration of "P," relative to the moving axes. The terms $\ddot{a}_1X$, $\ddot{a}_2Y$ and $\ddot{a}_3Z$, etc., are the terms giving the angular acceleration of "P" due to the angular acceleration of the moving axes. The terms $2\dot{a}_1\dot{X}$, $2\dot{a}_2\dot{Y}$, $2\dot{a}_3\dot{Z}$, $2\dot{b}_1\dot{X}$, $2\dot{b}_2\dot{Y}$, $2\dot{b}_3\dot{Z}$, $2\dot{c}_1\dot{X}$, $2\dot{c}_2\dot{Y}$, $2\dot{c}_3\dot{Z}$ comprise the acceleration of Coriolis, also known as compound centripital or central acceleration. To arrive at an understanding of this acceleration most easily, let us choose the Z axis as the axis of rotation for the moving set of coordinates and let us choose the Y axis so that YZ plane is parallel to the plane of the instantaneous velocity of "P" and as indicated previously, our fixed axes at this particular instant are parallel to the moving set of axes. We may choose our axes in this manner without destroying the generality of this discussion since the laws of physics hold in any coordinate system. Now observe that $\dot{a}_1$ equals minus the sine of the angle between x and X times the derivative with respect to time of the angle between x and X. $\dot{a}_2$ equals minus the sine of the angle between x and Y times the derivative with respect to time of the angle between x and Y while $\dot{a}_3$ equals minus the sine of the angle between x and Z times the derivative with respect to time of the angle between x and Z. Similar equations hold for $\dot{b}_1$ $\dot{b}_2$, $\dot{b}_3$ and $\dot{c}_1$, $\dot{c}_2$ and $\dot{c}_3$. Now, observe that with the above choice of axes the only term in our Coriolis acceleration that is not equal to zero is the $\dot{a}_2\dot{Y}$ term. There will be no change in the angles between the z and Z axes so that $\dot{c}_1\dot{X}$, $\dot{c}_2\dot{Y}$ and $\dot{c}_3\dot{Z}$ are equal to zero, and $\dot{a}_3$ and $\dot{b}_3$ are equal to zero. Since the sine of the angle between x and X will equal zero and the sine of the angle between y and Y equal zero, $\dot{a}_1$ and $\dot{b}_2$ equal zero. Further, since there is no velocity of our point in the X direction $\dot{b}_1\dot{X}$ equals zero. Because the angle between x and Y is 90° the sine of this angle will be unity, thus our acceleration of Coriolis reduces to a minus 2 omega times $\dot{Y}$, where ω is the angular velocity of the moving set of axes also the derivative with respect to time of the angle between Y and x. Now if θ is the angle between a line parallel to the Z axes in the plane of the velocity of P, and the direction of v, then the change in Y with respect to time of P will be v sin θ , since v sin θ will equal the projection of v on the XY plane, and since we have chosen the Y axis parallel to this projection v sin θ will equal Y, hence our Coriolis acceleration equals $2\omega v \sin\theta$ where as noted, θ is the angle between the direction of the axis of rotation and the direction of velocity of P.

Because of the orientation of axes, the only component of Coriolis acceleration remaining is the x compondnt. Solving equation (L) for an acceleration relative to the XYZ axes, this acceleration being equal to $\ddot{a}_1X$, we find that $\ddot{a}_1X$ equals $\ddot{x} - \ddot{a}_1X - \ddot{a}_2Y - \ddot{a}_3Z - 2\omega v \sin\theta$. Consequently, since instantaneously x and X are in the same direction, our Coriolis acceleration will lie in the X direction, provided that ω is positive. ω will be positive if the moving axes are rotating in a counterclockwise direction, since the angle whose cosine is a_2 will be increasing. If the axes rotate clockwise ω will be negative and our Coriolis acceleration as observed from the moving axes will lie in the negative X direction.

To gain a clearer understanding of the terms $\ddot{a}_1X$, $\ddot{a}_2Y$, etc., note that

$\dot{a}_1 = -\omega \sin(l)$ where l is the angle between x and X

$\dot{a}_2 = -\omega \sin(m)$ where m is the angle between x and Y

and

$$\ddot{a}_1 = \omega^2 \cos(l) - \dot{\omega}\sin(l)$$
$$\ddot{a}_2 = \omega^2 \cos(m) - \dot{\omega}\sin(m)$$
$$\ddot{a}_3 = 0$$

due to our choice of axes $\cos(l) = 1$

$\sin(l) = 0$

$\cos(m) = 0$

$\sin(m) = 1$

consequently $\ddot{x} = -\omega^2 X - \dot{\omega} Y - 2\omega v \sin\theta + a_1\ddot{X} + a_2\ddot{Y} + a_3\ddot{Z} + \ddot{X}_0$

or we can rewrite this equation

$$\ddot{x} = -\omega^2 X - \dot{\omega} Y - 2\omega v \sin\theta + \ddot{f}_x + \ddot{x}.$$

where $f_x = a\ddot{X} + a_2\ddot{Y} + a_3\ddot{Z}$ and is the linear acceleration of P with respect to the moving origin.

$\omega_2^2 X$ represents the x component of acceleration of p due to the angular acceleration of the moving system.

Thus we see that the true acceleration of p is composed of five accelerations. Similarly $\ddot{y}$ and $\ddot{z}$ have these components.

VECTOR TREATMENT OF THE ACCELERATION OF CORIOLIS*

As before, let a set of rectangular axes $X_0Y_0Z_0$ with origin O_0 be fixed in an inertial system. A second set of axes X Y Z with origin at O are free to move in the inertial system. Defining the unit vectors i_0, j_0, and k_0 and the unit vectors i, j, and k in the usual manner, the position of a point, P, free to move with respect to either set of axes, will be given by the equation

$$r_0 = \underline{\rho} + r \qquad (1)$$

where: $r_0 = i_0 x_0 + j_0 y_0 + k_0 z_0$

x_0, y_0, z_0 being the coordinates of point, P, in the fixed coordinate system, and

*This treatment by vector methods follows the standard type of development as exemplified by the presentation in Page's "Introduction to Theoretical Physics."

$$\underline{r} = \underline{i}x + \underline{j}y + \underline{k}z$$

where x, y and z are the coordinates of the point, P, in the moving system and ρ the position vector of O with respect to O_o

Denoting the time derivative or $\underline{r}_o$ as $\underline{v}$

$$\underline{v} = \dot{\underline{r}}_o = \dot{\rho} + \dot{\underline{r}} \qquad (2)$$

where: $\dot{\underline{r}} = \underline{i}\dot{x} + \underline{j}\dot{y} + \underline{k}\dot{z} + \dot{\underline{i}}x + \dot{\underline{j}}y + \dot{\underline{k}}z \qquad (3)$

let us say that the first three terms of equation (3) represent the apparent velocity, $\underline{V}$ of P relative to the moving axes. The remaining three terms of equation (3) represent the angular velocity of P due to the rotation of the XYZ axes, or equation (3) may be written

$$\dot{\underline{r}} = \underline{V} + \dot{\underline{i}}x + \dot{\underline{j}}y + \dot{\underline{k}}z \qquad (3')$$

differentiating $\underline{V}$ with respect to time we get

$$\dot{\underline{V}} = \dot{\underline{i}}\dot{x} + \dot{\underline{j}}\dot{y} + \dot{\underline{k}}\dot{z} + \underline{i}\ddot{x} + \underline{j}\ddot{y} + \underline{k}\ddot{z} \qquad (4a)$$

differentiating the remaining three terms of (3') with respect to time we get

$$\frac{d}{dt}(\dot{\underline{i}}x + \dot{\underline{j}}y + \dot{\underline{k}}z) = \ddot{\underline{i}}x + \ddot{\underline{j}}y + \ddot{\underline{k}}z + \dot{\underline{i}}\dot{x} + \dot{\underline{j}}\dot{y} + \dot{\underline{k}}\dot{z} \qquad (4b)$$

we note that in $\ddot{\underline{r}}$ there are two sets of terms $\dot{\underline{i}}\dot{x}$, $\dot{\underline{j}}\dot{y}$, and $\dot{\underline{k}}\dot{z}$. We can say that the first set of these terms arises from the change in direction of $\underline{V}$ with respect to the moving axes, while the second set is due to the change in magnitude of the velocity due to the rotation of the moving axes.

Thus we see that

$$\ddot{\underline{r}}_o = \ddot{\rho} + \underline{i}\ddot{x} + \underline{j}\ddot{y} + \underline{k}\ddot{z} + 2(\dot{\underline{i}}\dot{x} + \dot{\underline{j}}\dot{y} + \dot{\underline{k}}\dot{z}) + \ddot{\underline{i}}x + \ddot{\underline{j}}y + \ddot{\underline{k}}z \qquad (4)$$

Since $\underline{i}$, $\underline{j}$, and $\underline{k}$ are defined as unit vectors, they can only change in direction and hence the time derivative of each must be perpendicular to the vector itself.

Or
$$\dot{i} = cj - gk$$
$$\dot{j} = ak - fi$$
$$\dot{k} = ei - dj \qquad (5)$$

however:

$$i = j \times k$$
$$j = k \times i$$
$$k = i \times j$$

and it follows that:

$$\dot{i} = \dot{j} \times k + j \times \dot{k}$$
$$\dot{j} = \dot{k} \times i + k \times \dot{i}$$
$$\dot{k} = \dot{i} \times j + i \times \dot{j} \qquad (6)$$

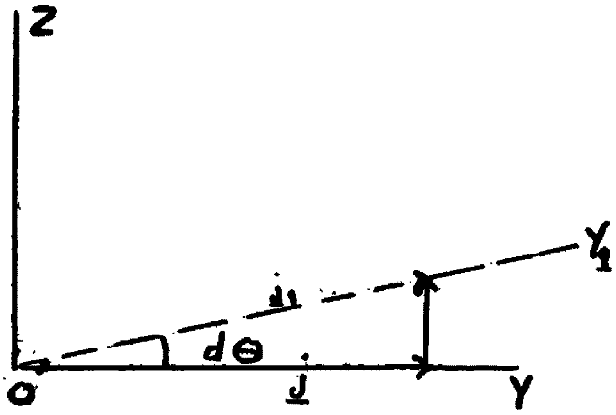
Combining equations (5) and equations (6) we find that $c = f$, $b = c$ and $a = d$.

It then follows that:

$$\dot{i} = cj - bk$$
$$\dot{j} = ak - ci$$
$$\dot{k} = bi - aj \qquad (7)$$

To examine the physical significance of the coefficients, a, b, c, consider the coefficient a which appears as the z component of $\dot{j}$. It represents the projection of $\dot{j}$ on the YZ plane. Therefore adt is the projection on this plane of the vector increment in j which has been produced by a rotation of the axes in a time dt.

Let j_{11} be the projection of j on the YZ plane at the end of a time dt. Then tangent $d\theta = adt = adt = d\theta$



Further, $d\theta$ is the angle about the X axis through which the Y and Z axes have turned in a time dt. Therefore, a is the angular velocity of the moving system about the X axis.

Similarly, b and c are the angular velocities of the system respectively about the Y and Z axes.

Defining the vector ω as the angular velocity of the system; $\omega_y = a,\ \omega_y = b,\ \omega_z = qc.$ (8)

Returning to equation (4) and considering those terms having the coeficient two we find that these terms,

$$2(\dot{i}\dot{x} + \dot{j}\dot{y} + \dot{k}\dot{z}), \text{ become}$$

$2\dot{x}\,(cj - qk) + 2\dot{y}\,(ak - ci) + 2\dot{z}\,(bi - aj)$ or upon substitution of equations (8)

$2\dot{x}\,(\omega_z j - \omega_y k) + 2\dot{y}\,(\omega_x k - \omega_z i) + 2\dot{z}\,(\omega_y i - \omega_x j)$ upon collecting we get

$$2\left[i\,(\omega_y \dot{z} - \omega_z \dot{y}) + j\,(\omega_z \dot{x} - \omega_x \dot{z}) + k\,(\omega_x \dot{y} - \omega_y \dot{x})\right] \quad (9)$$

Now from equations (3) and (3') we see that the linear velocity V relative to the moving axes is given by

$$V = i\dot{x} + j\dot{y} + k\dot{z}$$

and letting the vector $\omega = i\omega_x - j\omega_y - k\omega_z$ we observe that expression (9) is twice vector product of $\omega \times V$ or $2\omega \times V$

As a final equation we have:

$$x_o = \ddot{\rho}\ \ (\ddot{i}x + \ddot{j}y + \ddot{k}z) + 2(\dot{i}\dot{x} + \dot{j}\dot{y} + \dot{k}\dot{z}) + (i\ddot{x} + j\ddot{y} + k\ddot{z})$$

consider the quantity $(i\ddot{x} + j\ddot{y} + k\ddot{z})$, this is the linear acceleration of the point, P, relative to the moving axes.

As has been shown before, the quantity $2(\dot{i}\dot{x} + \dot{j}\dot{y} + \dot{k}\dot{z})$ may be reduced to the vector product $2(\underline{\omega} \times \underline{V})$ which is the Coriolis acceleration.

The quantity $\underline{\ddot{\rho}}$ is simply the acceleration of the moving origin with respect to the fixed origin.

Consider the relationship $(\ddot{i}x + \ddot{j}y + \ddot{k}z)$

$$\ddot{i} = \dot{c}j + c\dot{j} - \dot{b}k - b\dot{k}$$
$$\ddot{j} = \dot{a}k + a\dot{k} - \dot{c}i - c\dot{i}$$
$$\ddot{k} = \dot{b}i + b\dot{i} - \dot{a}j - a\dot{j}$$

This relationship becomes

$$(\ddot{i}x + \ddot{j}y + \ddot{k}z) = i(\dot{b}z - \dot{c}y) + j(\dot{c}x - \dot{a}z) + k(\dot{a}y - \dot{b}x) +$$
$$\dot{i}(bz - cy) + \dot{j}(cx - az) + \dot{k}(ay - bx)$$

the last three terms, those involving $\dot{i}$, $\dot{j}$, and $\dot{k}$, may be set equal to

$$(cj - bk)(bz - cy) + (ak - ci)(cx - az) + (bi - aj)(ay - bx)$$

collecting these terms on i, j and k we get

$$i[b(ay - bx) - c(cx - az)] + j[c(bz - cy) - a(ay - bz)]$$
$$+ k[a(cx - az) - b(bz - cy)]$$

Now expanding $\underline{\omega} \times (\omega \times \underline{r})$ we get

$$\underline{\omega} \times \underline{r} = i(bz - cy) + j(cx - az) + k(ay - bx) \text{ and}$$
$$\underline{\omega} \times \underline{\omega} \times \underline{r} = i\{b(ay - bx) - c(cx - az)\} + j[c(bz - cy)$$
$$- a(ay - bx)\} + k[a(cx - az) - (bz - cy)]$$

These three terms, those involving $\dot{i}$, $\dot{j}$ and $\dot{k}$, in the expansion of $[\ddot{i}x + \ddot{j}y + \ddot{k}z]$ are the expansion of $\underline{\omega} \times \underline{\omega} \times \underline{r}$.

Let us now expand the vector product $\underline{\dot{\omega}} \times \underline{r}$

$$\underline{\dot{w}} = \dot{i}a + j\dot{b} + k\dot{c} + \dot{i}a + \dot{j}b + \dot{k}c$$

Comparing the last three terms of $\underline{\dot{\omega}}$, those involving $\dot{\mathbf{i}}$, $\dot{\mathbf{j}}$ and $\dot{\mathbf{k}}$ with the expansion of $\omega \times r$ which may be reduced to

$$\dot{\mathbf{i}}x + \dot{\mathbf{j}}y + \dot{\mathbf{k}}z$$

by analogy the three terms will be the expansion of $\underline{\omega} \times \underline{\omega}$ which is identically zero.

$\therefore \quad \underline{\dot{\omega}} = \mathbf{i}\dot{a} + \mathbf{j}\dot{b} + \mathbf{k}\dot{c}$ and $\underline{\dot{\omega}} \times r = \mathbf{i}(\dot{b}z - \dot{c}y) + \mathbf{j}(\dot{c}x - \dot{a}z) + \mathbf{k}(\dot{a}y - \dot{b}x)$

This expansion of $\underline{\dot{\omega}} \times \underline{r}$ is the same as the three terms in the expansion of $\ddot{\mathbf{i}}x + \ddot{\mathbf{j}}y + \ddot{\mathbf{k}}z$ which involve $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$\therefore$ we can conclude that $(\ddot{\mathbf{i}}x + \ddot{\mathbf{j}}y + \ddot{\mathbf{k}}z) = \underline{\dot{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$

Now: $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is the quantity defined as centripetal acceleration. $\underline{\dot{\omega}} \times \underline{r}$ is defined as the acceleration of a point in the moving coordinate system due to the angular acceleration of the moving axes.

We may now write $\ddot{\underline{r}}_o = \ddot{\underline{\rho}} + \underline{\dot{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + 2(\underline{\omega} \times \underline{v}) + \underline{f}$ where $\underline{f}$ = linear acceleration of P relative to the moving origin. Now $\underline{f}$ is the observed acceleration. Its relationship to the true acceleration in an inertial system is

$$\underline{f} = \ddot{\underline{r}}_o - \ddot{\underline{\rho}} - \underline{\dot{\omega}} \times \underline{r} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) - 2(\underline{\omega} \times \underline{v}).$$

It is apparent that in any problem in mechanics in which the quantity $2(\underline{\omega} \times \underline{v})$ must be applied and in which $\underline{f}$ is measured, the quantity will have a negative sign. Thus the acceleration will have a direction opposite to the vector $2(\underline{\omega} \times \underline{v})$.

EXAMPLE OF A PROBLEM IN CORIOLIS ACCELERATION

Part I: Solution by elementary means

Let us suppose that a projectile is fired from a position of 60° N latitude. Its average velocity, v, parallel to the earth's surface is BOX 10^3 f plus 1 sec. The target is 10^5 feet due south at the time of firing.

It is apparent that the linear velocity, due to the rotation of the earth, of a point 10^5 feet south of a point at 60° N latitude will be greater than that of the point at 60° N latitude since the radius of the earth's small circle parallel to the equatorial plane farther south will be greater than will be the radius of a similar small circle at 60° N latitude.

The radius of the small circle at 60° is given by

$r = r_e \cos 60°$ where r_e = 3,950 miles app

The radius of the southern small circle is given by

$r = r_e \cos (60 - \alpha)$

Where α is the angle at the center of the earth subtended by the line connecting the firing and target points,

$$\alpha = \frac{10^5}{3.95 \times 5.28 \times 10^6} \quad 57.3 \frac{\text{degrees}}{\text{radius}} = 2.74 \times 10^{10}$$

The difference in the velocity between a point at 60° and a point at $(60 - \alpha)°$ will be

$$\Delta N_1 = \left[r_e \cos (60 - \alpha) - r_e \cos 60\right] \omega.$$

It may be seen that this velocity multiplied by the time required for the flight of the projectile will give the projectile a displacement to the westward of the target given by the equation

$$\Delta S = \left[r_e \cos (60 - \alpha) - r_e \cos 60\right] \omega \, t$$

However, this displacement to the west will be only a part of the error due to the effect of Coriolis acceleration. We will also have an acceleration due to the change in the direction of the velocity of the projectile. If we form a vector diagram giving the difference between v_a, the actual velocity and v_o, the original velocity of the projectile with respect to the earth, we note that the tangent of the angle between v_a and v_o is given by the equation

$$\tan \phi = \frac{\Delta V_L t}{R}$$

where ϕ is the angle between $v_o - v_a$ and R is the range (10^5 ft). But $R = vt$ where $v = v_o = 3 \times 10^3$ ft/sec therefore

$$\tan \phi = \frac{\Delta v_L t}{v_o t} = \frac{\Delta V_L}{v_o}$$

Now from the vector difference diagram it is seen that the vector difference between V_a and V_o is:

(5) $\Delta V = V_o \tan \phi$ (since ϕ is a very small angle)

Hence from equations (4) and (5) we see:

(6) $\Delta V = \Delta V_L$

or the apparent change in the velocity of the projectile due to the change in direction of its velocity relative to the earth is equal to the difference in velocity that arises due to the difference in linear velocity of the earth's surface at the points of departure and return of the projectile to the earth.

The total change in velocity of the projectile relative to the earth during the time of flight t is therefore

$2\,\Delta V_L$. The average acceleration being equal to the change in velocity divided by the time we then must have for the Coriolis acceleration in this case

$$a = \frac{2\,\Delta V_L}{t}$$

Numerical evaluation:

From equation 1

$$\Delta V_L = \left[r \cos(60 - \alpha) - r \cos 60\right]\omega$$
$$= \left[\cos(60 - .274) - \cos 60\right] r\,\omega$$
$$= (.50414 - .5)\ 3950 \times 5280 \times 7.292 \times 10^5$$
$$= 4.14 \times 10^{-3} \times 3.950 \times 10^3 \times 5.28 \times 10^3 \times 7.292 \times 10^{-5}$$

$$\Delta V_L = 6.28 \text{ ft/sec}$$

$$a = \frac{2\,\Delta V_L}{t} = \frac{6.28 \times 2}{\frac{10^5}{3 \times 10^3}} = \frac{6.28 \times 2 \times 3 \times 10^3}{10^5} = .377 \text{ ft/sec}$$

$$s = \frac{1}{2}at^2 = \frac{1}{2}\,.377 \times (33.\ \ )^2 = 209 \text{ ft.}$$

Part II: Evaluation of vector product

$2\underline{\omega} \times \underline{v}$

A. Direction: rotation of $\underline{\omega}$ into $\underline{v}$ gives $\underline{\omega} \times \underline{v}$ a direction into paper which would correspond to an eastward direction

$\therefore\ -2\,\underline{\omega} \times \underline{v}$ is tb the west.

B. Magnitude

$|-2\,\underline{\omega} \times \underline{v}| = 2\,|\omega||v|\ \sin\theta$ where $\theta = 120^\circ$ from diagram

$\sin 120^\circ = \sin 60^\circ = .86603$

$$|-2\omega \times x| = 2 \times 7.292 \times 10^{-5} \times 3 \times 10^{3} \times .866 = .379 \text{ ft/sec}^2$$

$$a = .379 \text{ ft/sec}^2$$

$$S = \frac{1}{2} a t^2 = \frac{1}{2} .379 \times (33.3)^2 = 210.5 \text{ ft.}$$

Part III: Conclusion

In the elementary approach certain approximations were made. These were:

1. In finding the angle
2. In treating the problem as a plane
3. Value of r_e

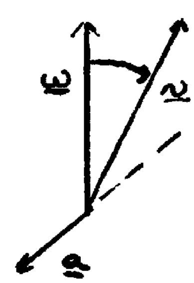
From this we conclude that our results from the two methods agree within reasonable exactness.

PART III

SUMMARY AND CONCLUSIONS

To summarize this work: There are five components of acceleration comprising the true acceleration of a point located in a moving coordinate system. They are: the centripetal acceleration, the acceleration due to the angular acceleration of the moving system, the linear acceleration of the point relative to the moving axis, the acceleration of the moving origin, and the acceleration of Coriolis.

To restate, the acceleration of Coriolis equals in magnitude twice the product of the angular velocity of the moving system and thellinear velocity of the point relative to the moving system multiplied by the sine of the angle between the axis of rotation and the direction of the velocity. The direction of the Coriolis Acceleration is perpendicular to both the axis of rotation and the direction of the velocity of the moving point in such a manner that if we rotate the vector representing the axis of rotation towards the vector representing the velocity (see diagram) the acceleration will have a direction opposite to the direction of advance of a right handed screw thus rotated.



Special attention has been given to the Coriolis acceleration because it is not as immediately apparent as the other components of acceleration. Furthermore, it is an important factor in determining the direction of winds and because of its relation to the problem of the Foucault pendulum we are able to demonstrate beyond a question that the earth does rotate.

For the reader who wishes details on the last items, a detailed discussion of the problem of the Foucault pendulum may be found in Page's Introduction to Theoretical Physics, Article 35. A discussion of the relationship of Coriolis Acceleration to weather may be found in Humphrey's Physics of Air, under the chapter title, "Possible Theory of Origin and Maintenance of the Extra Tropical Cyclone."

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