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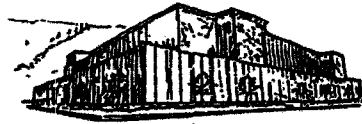
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**THE USE OF COMPUTER MANIPULATIVES IN BUILDING  
INTEGRATED CONCRETE UNDERSTANDINGS  
IN  
SECONDARY MATHEMATICS EDUCATION**

By

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B.S. Washington State University, Pullman Washington 1989

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Presented in partial fulfillment of the requirements  
for the degree of

Doctor of Philosophy

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**The Use of Computer Manipulatives in Building Integrated Concrete Understandings in Secondary Mathematics Education**

Chair: Libby Knott, Ph.D.



This study explored the impact of using virtual computer manipulatives as an aid in developing secondary mathematical abilities. In particular, a comparison was made between the effects of using a concrete or virtual computer manipulative on student semantic processes. Topics included solving two-step linear equations with a concrete or virtual balance beam and multiplying and factoring polynomial expressions using an alternative form of concrete or virtual Algebra Lab Gear. This study provides mathematics educators with insight into learning outcomes that surface as a result of the computer manipulative replacing the concrete manipulative.

The primary research question was “Do learning outcomes differ when activity-based instruction includes the use of a virtual computer or a concrete manipulative?” The objective was to determine potential differences in accuracy measures and adopted solution strategies in problem solving. A secondary question was “Does the sequencing of mathematics instruction from manipulative to symbolic or symbolic to manipulative alter learning outcomes at the secondary level?”

Data were collected from a total of 14 classes that were given one of four treatments: symbolic-virtual computer, symbolic-concrete object, virtual computer-symbolic, and concrete object-symbolic. A total of 304 students participated. Each student participant successfully completed a pretest and similar posttest.

Three randomly selected students from each of 14 classes were chosen for a task-based interview from a list of student volunteers. Interview questions prompted students to voice their thoughts and solve problems with and without the use of a concrete or virtual computer manipulative. Five of a total of 42 student interviews were selected according to student performance and analyzed in terms of semantic processes.

All treatments were effective in improving overall achievement from pretest to posttest. Results favored the concrete manipulative as the manipulative that provided the greatest probability for student posttest improvement. One specific inaccurate strategy was linked to instruction with the concrete manipulative. No differences were found when instruction was sequenced from manipulative to symbolic or symbolic to manipulative.

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# CHAPTER 1

## INTRODUCTION

Computers have become commonplace in academia. Each new generation of students has greater comfort and familiarity with computer programs and usage. The mathematics classroom is enhanced by taking advantage of the new platform that computer-aided instruction can provide. A desire to establish a theoretical foundation for the utility of computer software in the mathematics classroom has led to an interest in exploring the potential of computer-aided instruction as a means by which concrete models can be linked to abstract-symbolic representations of mathematical content.

Suydam (1986) found that students obtain greater mathematical achievement when mathematical concepts are initially introduced with the use of a manipulative. In the opinion of the author, this is a challenge for the secondary level instruction because classrooms are typically filled with students who possess a blend of background experiences in mathematical content. That is, within the same secondary classroom, some students may have received a great deal of symbolic instruction while others may have had little or no exposure to content. Others may have been exposed to content solely with the use of manipulatives. Therefore, the problem requires consideration of what topic specific background experience each student brings to the classroom prior to receiving instruction that utilizes a virtual computer or physical manipulative.

### **Need for study**

Moyer, Bolyard, and Spikell (2002b) found that some educators see virtual manipulatives taking the place of physical manipulatives in some classrooms. Before teachers can replace a proven instructional tool such as physical manipulatives (Olkum, 2003), data will need to be gathered about potential student learning outcomes that may result from the use of this new instructional tool.

Educators should not accept a new instructional tool simply because it is new. Technology is inviting and motivating for both teachers and students. The benefits and challenges of using virtual manipulatives need to be identified before virtual manipulatives are standard in every classroom. In this study, information will be gathered with regard to student learning outcomes that result after exposure to instruction with a virtual manipulative. In particular, we seek to compare the virtual manipulative to an equivalent form of physical manipulative. This information can be used to determine if the virtual manipulative is comparable to the physical manipulative in terms of student learning outcomes.

### **The research questions**

The focus of this study is to determine if the virtual computer manipulative can replace the sensory-concrete manipulative in terms of class achievement and learning outcomes at the secondary level. In particular, research will investigate inaccurate problem solving strategies students adopt when students receive activity-based instruction with a virtual computer or sensory-concrete manipulative and whether or not

these strategies differ among students with various topic specific background experiences. The following research questions are investigated.

1. Does the sequencing of mathematics instruction from manipulative to symbolic or symbolic to manipulative alter student learning outcomes at the secondary level?
2. Do student learning outcomes differ when activity-based instruction includes the use of a virtual computer or a sensory-concrete manipulative?
  - a) Do students complete problems with greater or less accuracy when given instruction using either the virtual computer or sensory-concrete manipulative?
  - b) What inaccurate problem solving strategies might students adopt after receiving instruction with a virtual computer or sensory-concrete manipulative?

### **Limitations**

The hypotheses are based on a small qualitative and quantitative study conducted by the researcher with four secondary teachers and several school administrators. The concerns and opinions of these teachers and administrators formed constructs that guided the study. Conclusions drawn cannot be generalized, but will hopefully prompt more research.

The sample is not large enough to provide adequate statistical power. Broad generalizations are not possible due to a number of limitations. First, the secondary

classrooms sampled were not a random sample from any population. Second, teacher and student participation were completely voluntary. The researcher did not include teachers or students who were not open to the use of manipulatives in classroom instruction.

Each sequence of instruction within a particular manipulative type is viewed as a treatment. The regular classroom teacher is another factor to consider. Each group of students had already been exposed to what might be considered a different treatment with regard to instruction in mathematics. To minimize this potential confounding effect, the researcher conducted the experiment as early in the academic year as possible.

Topics may or may not have been taught to students in previous academic years. Participating teachers agreed to teach the topic immediately prior to or immediately after each treatment depending on the instruction sequence. Teachers also agreed that the topic would not be taught at any other time during the current academic year. The depth and extent of content of which the topic was taught was at the discretion of the participating teacher.

Each treatment included the same lesson plan performed by the researcher for both the virtual and concrete manipulative type. There were, however, differences in what was taught due to the mechanics of computer operation. Simple computer-specific instructions were conveyed within treatments that utilized the virtual manipulative while these mechanics were not needed for treatments that involve the concrete manipulative.

Student participants who had more educational background in the subject area were more likely to receive treatments that assumed initial symbolic instruction. That is, student participants who were in more advanced mathematics classes were given

treatments that assumed some topic specific instruction and practice had occurred in the previous academic years. The educational background of each class was determined by class level. The selective nature of which group receives which treatment restricts the results of the study to this specific group of students. It is therefore inappropriate to use the results for extensions to general secondary mathematics classrooms.

### **Definitions**

A *learning outcome* is the specification of what a student should learn as the result of a period of specified and supported study (Harvey, 2004). Learning outcomes are concerned with the achievements of the learner rather than the intentions of the teacher. They can take many forms and can be broad or narrow in nature (Adam, 2004).

*Semantic analysis* involves the connecting and developing processes, as defined by Wearne and Heibert (1988, p. 375). Students' *connect* mathematical meanings when they link symbols with tangible referents. They *develop* mathematical meanings when actions that parallel those in the referent world are applied in the symbol world. Five interviews were analyzed in terms of the semantic processes of connecting and developing.

*Physical or concrete manipulatives* are tangible objects that can be handled and arranged by students in an effort to stimulate their understanding of abstract mathematical ideas by allowing them to model or represent their ideas concretely. The concrete manipulatives used in this research are colored chips and a home-made form of wooden Algebra Lab Gear.

*Integrated concrete understandings* are formed when a person uses a physical or concrete manipulative to develop current understanding of a mathematical concept to



multiple forms of representation. This understanding of the relationships between tangible objects and a connected mathematical concept enables the use of the physical referent as a guide to associated mathematical symbolic procedures.

*Virtual manipulatives* are “interactive computer-based visual representations of dynamic objects that present opportunities for constructing mathematical knowledge” (Moyer, Bolyard, & Spikell, 2002b, p. 372). Virtual manipulatives are seen as “dynamic visual representations of concrete manipulatives” (Moyer, Bolyard, & Spikell, 2002a, p. 133). Currently, virtual manipulatives are modeled after concrete manipulatives such as base-ten blocks, coins, pattern blocks, tangrams, spinners, rulers, fraction bars, balance scales, algebra tiles, geoboards, and geometric plane and solid figures.

The virtual manipulatives utilized in this research consist of interactive concept tutorials in the form of Java applets. These applets provide dynamic visual representations of balance scales or algebra tiles on a computer monitor. The applet used for the lesson on multiplying and factoring polynomials is named *Algebra Tiles*. This applet is, in fact, a multi-colored form of Algebra Lab Gear. Just as a physical object can be flipped, turned, moved and rotated, so can the visual representation on the computer, by using a mouse and keyboard.

*Sensory-concrete* knowledge is used when students use sensory material to make sense of an idea. For example, at early stages, children cannot count, add, or subtract meaningfully unless they have actual objects to touch (Clements, 1999). “Mathematics cannot be engineered into sensory-concrete materials since ideas such as numbers are not out there” (Clements, 1999, p. 48).

*Integrated-concrete* knowledge is built through learning. It is knowledge that is connected in special ways. When children have this type of interconnected knowledge, the physical objects, the actions they perform on the objects, and the abstractions they make are all interrelated in a strong mental structure (Clements, 1999).

*Dragging* is the process of using a mouse to move visual representations of objects on a computer monitor. Dragging is accomplished by placing the mouse pointer over the object, pressing down with the forefinger on the mouse, and sliding the mouse in the direction of desired object movement.

*Mathematical visualization* is “the process of producing or using geometric or graphic representations of mathematical concepts, principles, or problems, whether hand drawn or computer generated” (Zimmermann & Cunningham, 1991, p. 1).

## CHAPTER 2

### LITERATURE REVIEW

The purpose of this chapter is to present the existing research with regard to student learning outcomes resulting from instruction that incorporates the use of a virtual or physical manipulative. This exploration begins with the historical outline of cognitive development. Next the author will establish the theoretical basis for the research of this dissertation. Developmental stages will be examined to determine what student learning outcomes might be expected from students at the secondary level of mathematics. Definitions of mathematical processes performed by computers will be described to clearly explain distinctions between different types of mathematical education computer software. Finally, studies that utilize dynamic, virtual, or concrete manipulatives within instruction will be described. In this way, the literature review will outline what is known or might be expected with regard to student learning outcomes that result from instruction with a virtual or concrete manipulative.

#### **Theoretical background**

Shaffer and Kaput (1999) outlined a theory of cognitive development developed by Donald (1991). They explain that human cognition has evolved through four distinct stages; episodic, mimetic, mythic, and theoretical. Thinking was based on literal recall of events in the episodic stage, representational acts in the mimetic stage, narrative transmission in the mythic stage, and written symbols and paradigmatic thought in the

theoretical stage.

Although these cognitive abilities developed during different time periods of human development, these ways of thinking currently exist in our minds simultaneously and we move among them and use them in a fluid way. Shaffer and Kaput (1999) theorize that evolutionary development of a cognitive ability and individual development of the same ability might differ in their pattern of acquisition. They believe that the evolutionary development of a new form of representation might have profound developmental consequences. In fact, it is suggested that a new cognitive stage is emerging (Shaffer & Kaput, 1999).

Nelson (1996) studied the work of Donald (1991) from a developmental, rather than evolutionary standpoint. Nelson argues that new cognitive processes affect the way older modes of thought emerge in individual development. The presence of various modes of representation within a culture changes the way we learn to understand our worlds as individuals.

Computers have the capability to store and retrieve information. This reflects the theoretical stage of cognitive development, which, according to Shaffer and Kaput (1999) has been going on for the past 300,000 years. Current scientific culture developed from and depends on the existence of external notations for thinking and on external records for ideas.

In the cognitive stage of the future, computers or other forms of external devices actually perform some of the functions that a mind might take on in a similar circumstance. Shaffer and Kaput describe the capability as autonomous symbolic processing (Shaffer & Kaput, 1999). An example of this type of processing is the student

who uses a calculator to add two fractions. Assuming the student knows how to operate the calculator, the student can produce accurate results, with or without knowing or even considering the meaning of the fractions or the mathematically appropriate algorithm used in finding the sum.

With developing technologies, such as the calculator and computer, the pedagogy of mathematics education will shift towards fluency in representing problem situations in a variety of systems and towards students' ability to coordinate among representations. Kaput (1986) suggests that one of the important features of computational media in mathematics learning is their ability to help students see the relationship among different representations of the same mathematical situation. Similarly, from a virtual perspective, mathematics is not exclusively about calculations. That is the role of the external symbolic processing system. Mathematics is about understanding a problem, representing it in an external processing system, and being able to use the information produced by the external calculations in a meaningful way (Shaffer & Kaput, 1999). Mathematics education in a virtual culture will move us from computational fluency towards representational models. Mathematical experience in a virtual culture will thus be more intimately connected with students' wider worlds of experiences (Kaput, 1986).

### **Theoretical framework**

The underlying foundations of this dissertation rely on exogenous constructivist views on the teaching and learning of mathematics. According to these views, within the individual domain of knowledge there may be a number of individually constructed knowledge representations that are equally valid. The focus of teaching then becomes

one of guiding the learner as they build and modify their existing mental models. It is a focus on knowledge construction, rather than knowledge transmission (Slavin, 1994).

Constructivists generally agree that mathematical knowledge is constructed, at least in part, through a process of reflective abstraction (Noddings, 1990). Reflective abstraction is different from classical abstraction in that it does not proceed from a series of observations of events or objects. Rather, it is a process of interiorizing our physical operations on objects. According to Noddings, “as we move sets of objects about, we interiorize properties of mathematical operations rather than objects; we acquire implicit understanding of commutativity, associativity, and reversibility” (p. 9). This implies an essential connection between purposive activity with concrete manipulatives and the development of mathematical cognitive structure.

A notable example of reflective abstraction is characterized by Papert (1993). As a child, Papert had an erector set from which he actively assembled gear systems. Papert believes that working with the differentials of these gear systems did more for his mathematical development than anything he was taught in elementary school. He explained how his first brush with equations in two variables brought to mind an analogy between the equations and how many teeth each gear needed. In this way, the equations became “a comfortable friend” (p. xix). It was his ability to assimilate mathematical content to his collection of gear models that made the process of learning mathematics easier for him. Papert further hypothesized that what an individual can learn and how they learn depends on what concrete models are available.

According to Piagetian theory, 7-year old and 8-year-old children must have action experiences before they can add new ideas to their cognitive structure. Bruner

(1966) also reported on the importance of actions in learning. However, according to Bruner, the need for actions depends on the stage of maturity and experiential background. The individual is supposedly able to learn with symbols, if appropriate actions have been experienced previously. Hence, the issue of whether it is better to teach mathematical content symbolically or with manipulatives is dependent on individual background experiences.

It is difficult at best to determine what background experiences an individual does or does not possess. Nevertheless, work has been done to establish what is believed to be the optimal sequence of learning mathematical processes. Wearne and Hiebert (1988) identified four critical processes that, when sequenced appropriately, yield mathematical competence. The four major processes, in favorable order of occurrence are *connecting*, *developing*, *elaborating and routinizing*, and *abstracting*. Students' *connect* mathematical meanings when they link symbols with tangible referents. They *develop* mathematical meanings when actions that parallel those in the referent world are applied in the symbol world. The connecting and developing processes can be thought of as semantic analyses. Next, students *elaborate* rules to harder problems. Then students practice or *routinize* rules until these rules require little cognitive effort. Finally, students *abstract* mathematical meanings when the symbols and rules become the referents for building more abstract systems. According to Wearne and Hiebert, "An alternate sequence of acquisition is difficult cognitively and may prevent the development of mathematical competence" (p. 372).

To affirm their claim of *sequence-of-acquisition*, Wearne and Hiebert (1988) conducted a study with 4th, 5th, and 6th graders that focused on the use of written

symbols of the decimal fraction system. Dienes base-10 blocks were used as referents. Nine lessons, covered in seven to nine 25-minute sessions, made up the instructional unit. Instruction began with a unit block and demonstration of how to write the symbols for quantities shown with blocks. Students were asked to use block referents and the combining and separating actions on blocks to decide how to combine the symbols in addition and subtraction problems. All assessments were administered in individual interviews. The computation problems were 'ragged' decimal problems because previous work showed that such problems discriminate most clearly between students who use semantic analyses and those who recall and execute syntactic rules (Wearne & Hiebert, 1988). Results indicate that students who have already routinized syntactic rules without establishing connections between symbols and referents were less likely to engage in the semantic processes than students who are encountering decimal symbols, in the form of Dienes base-10 blocks, for the first time (Wearne & Hiebert, 1988). This result is consistent with the claim that it is preferable to develop meanings for symbols before practicing syntactic routines (Resnick & Omanson, 1987).

Research has also been conducted at the elementary level that compares learning with concrete manipulatives to learning with computer-generated manipulatives. Thompson (1992) performed a study of fourth-grade students who used sensory-concrete manipulatives versus another group that used a computer program written by Thompson and called *Blocks MicroWorld*. The concept was using decimals in addition and subtraction.

Thompson (1992) used wooden Dienes base-10 blocks as the concrete manipulatives of his study. The *Blocks MicroWorld* program was the complementing



computer program. It was designed to support students' continual development of meaning for their notational actions and interpretation of notation. Notation changed automatically when blocks were moved around. For example, a touch of the mouse would alter a ten block to a block with ten single units. This helped students to visually see what it means to regroup in a subtraction problem (Thompson, 1992).

“*Blocks Microworld* was designed so that students could combine collections of blocks in a number of ways. One way would be to treat them as wooden blocks, dragging a collection in one region into the other region, one block at a time, several at a time, or all at once” (Thompson, 1992, p. 128). Figure 2.1 illustrates the image on a *Blocks Microworld* workspace.

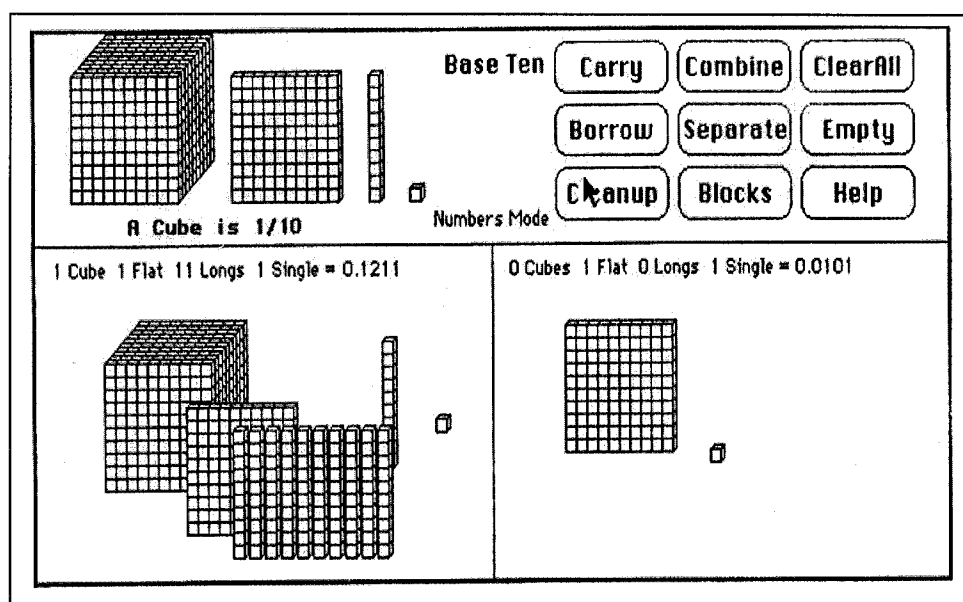


Figure 2.1. Screen display of *Blocks Microworld* after a student has selected *A cube is 1/10* in the *Unit* menu. (Thompson, 1992)

Results of the Thompson (1992) study indicate no significant differences between groups that used the wooden Dienes base-10 blocks and groups that used the *Blocks*

*Microworld* program. Thompson also found no significant differences in improvement from pretest to posttest for the group using computer-generated manipulatives to those using concrete manipulatives. Thompson speculated that this result was due to previous instruction that the students in both groups had received. That is, each student was initially taught a standard procedural method of addition and subtraction of decimals. This instruction was given to classes of students. In subsequent testing, students were attempting to repeat this procedure without reference to work performed on the computer or sensory-concrete manipulatives.

Apparently, it is difficult for semantic processes to alter the tendency to follow routinized procedures (Wearne & Hiebert, 1988). Thompson (1992) reached a similar conclusion stating that, "If students memorize a procedure meaninglessly, it is extremely difficult to get them to change it, even with extended, meaningful remediation" (p. 144). In particular, Thompson pointed out that even though the older children understood more about the task of solving decimal problems (line up decimal points and proceed as with whole numbers), this understanding made the task more complex for them.

The theory of a concept image, developed by Vinner (1991), may help to explain why students ignore semantic processes in favor of procedures. According to Vinner, to understand means to have a concept image. A concept image is more than simply knowing a definition. In fact, knowing the definition of a concept does not imply understanding. Algorithms and procedures associated with a given concept are a part of the concept image. Pictures and individual experiences working with concrete manipulatives also play a role in building the concept image. Students make connections between various aspects of the concept image when they explore different strategies for

solving a problem. It is the ability to make these connections that establishes conceptual understanding.

A concept image is unique to each individual and this same individual may react differently to a concept in different situations. Vinner (1991) use the term “evoked concept image” to describe the part of the concept image evoked in a given context (p. 73). It is not meant to imply that this is all a certain individual knows about a concept.

Vinner (1991) found that people will often ignore other aspects of the concept image, including visual and concrete representation, in favor of using concept associated procedures. Surprisingly, preference for symbolic procedures occurs even when prior problems verify that the student has achieved a visual and concrete understanding of basic underlying notions (Ferrini-Mundy, 1984). Possible explanations for why students tend to avoid visual aspects of a problem include: “a cognitive one (visual is more difficult), a sociological one (visual is harder to teach), and one related to beliefs about mathematics (visual is not mathematical)” (Dreyfus & Eisenberg, 1991, p. 30).

According to Dreyfus and Eisenberg, “While this is also true for many teachers, it does not seem to hold for professional mathematicians. For them, the choice of representation in which to solve a problem seems to depend as much on the problem itself as on personal preferences” (p. 26). Thus, establishing connections between different aspects of ones’ concept image, namely integrated concrete knowledge, may be a determinant of mathematical achievement.

It is necessary to clearly define concrete thinking in order to explore the cognitive needs of typical secondary mathematics students. The notions of concrete and formal thinking stem from the work of Piaget. The cognitive theory of Piaget (1972)

deals with three levels of development; pre-operational, concrete, and formal operational. The pre-operational cognitive level is a low level of thinking. A person in the pre-operational cognitive level “can use symbols from visual and body sensation to represent objects, but has problems mentally reversing actions” (Biehler & Snowman, 1986, p. 62). An example of the pre-operational cognitive level occurs when a person states that a tall container has more water than a squat container, even though the person views the water in the squat container being poured into the tall container. The next cognitive level is concrete operational. A person at this level can understand conservation of matter, classification, and generalization. For example, this person can conclude that all dogs are animals and not all animals are dogs. However, such a person is unable to comprehend mathematical ratios (Barker & Unger, 1983). Formal operational level is the highest cognitive development level defined by Piaget. It is “the ability to deal with abstractions, form hypotheses, solve problems systematically, and engage in mental manipulations” (Biehler & Snowman, 1986, p. 63).

Piaget's theory indicates that formal operational thinking abilities normally develop around age 12 (Chiapetta, 1976). It is at this age that some students begin to move from concrete thinking to formal thinking. However, formal operations, such as thinking in abstractions and logically, can develop at different ages or not at all (Griffiths, 1973; Schwebel, 1975; Pallrand, 1979; Epstein, 1980). Many high school students and adults fail to attain full formal operational thinking (Renner, Grand, & Sutherland, 1978). Several studies have shown that a majority of adults, including college students and professionals, fail at many formal operational tasks (Griffiths, 1973; Schwebel, 1975; Schwebel, 1972). To assume a level of formal operations thinking in the secondary

mathematics classroom is to fail to meet the educational needs of a majority of the students.

If the use of concrete manipulatives leads young children to an ability to use symbolic reasoning at a meaningful level and many adolescent youths have not reached levels of formal operations thinking, then it is not unreasonable to assert that some secondary students will also benefit from the use of concrete manipulatives. Research at the elementary level suggests that the form of the manipulative, concrete or virtual computer, is not a factor in producing the ability to use symbolic reasoning at a meaningful level (Clements & McMillen, 1996). “Mathematical ideas are ultimately made integrated-concrete not by their physical or real-world characteristics, but rather by how meaningfully connected to other ideas and situations - they are” (Clements & McMillen, 1996, p. 273).

Sharp (1995) affirmed this belief with students at the secondary level. Her qualitative study of the use of algebra tiles showed some students found it easy to think about algebraic manipulations when they visualized the tiles. Sharp believed that meanings might be achieved or at least enhanced when individuals construct translations between algebra symbolic systems and physical systems that represent one another. Sharp concluded that “students who successfully make connections between physical representations and mathematical representations have created meaning of mathematical ideas” (p. 4).

The crux of the matter is that it is an entirely different process to construct a rule from the basis of understanding than it is to memorize a rule or procedure that simplifies what might otherwise be a complex conceptual task. Moreover, if we tie this conclusion

to the observation of Thompson (1992) with older students, namely that memorized rules and procedures may hinder the ability to learn underlying principles, literature provides reason to speculate that students at the secondary level are failing to make integrated-concrete connections when learning mathematics begins and ends with syntactic procedural techniques.

Allowing students to use a tool, concrete or virtual computer manipulative, does not guarantee that all students will develop the same meanings for them (Hiebert, 1997). Students who use manipulatives as aids for calculating answers are likely to develop different meanings than students who use them to explore alternative solution methods or reflect on the reasons the methods work. Meanings developed for manipulatives and meanings developed with manipulatives both result from the active use of manipulatives. When students are using a manipulative, they are working on two fronts simultaneously: what the manipulative means and how it can be used effectively to understand something else.

The use of a manipulative in a classroom activity does not guarantee that the manipulative will be used for thoughtful reflection. Of interest is whether the virtual computer manipulative activity influences students to develop different understandings than students who engage in the same activity with a concrete manipulative. One kind of understanding is not necessarily better than another. Rather, it is important to understand potential differences between using a virtual computer or concrete manipulative at the secondary level of mathematics.

## **Historical Background**

The use of dynamic computer manipulatives in teaching secondary mathematics began with software programs such as *Logo* and *Geometers Sketchpad*. Student learning outcomes have the potential to be similar to the learning outcomes already discovered from previous research with these programs. For this reason, a brief historical background of these programs is provided as a means of identifying possible strengths of using a dynamic computer manipulative.

## **Logo Research**

The computer programming language *Logo* was originally developed in 1968 as a part of a National Science Foundation sponsored research project conducted at Bolt, Beranek, and Newman, Inc. in Cambridge, MA (Feurzeig et. al., 1969). *Logo* began to emerge in its present form under the direction of Papert at Massachusetts Institute of Technology (MIT) from 1970 to 1981. Other *Logo* research leaders include Harold Abelson, Andrea di Sessa, Marvin Minsky, and Wallace Feurzeig from Bolt, Beranek, and Neman Inc. (Fiske, 1983). The publication of *Mindstorms: Children, Computers, and Powerful Ideas* in 1980 (Papert), coupled with the increased availability of microcomputers in the schools stimulated more independent research on this topic.

One of the first research studies on *Logo* was the Brookline Project (Papert, et. al., 1979). This project began in 1977 and ended in 1978. The goal of the project was to examine how fourth, fifth, and sixth grade students learned to program the *Logo* turtle. In particular, there was interest in which programming experiences would help students master the mathematical concepts and the degree to which the *Logo* programming

experience would help to develop problem solving skills using debugging strategies. No significant differences were found.

The second Brookline Project focused on the development of a curriculum supporting classroom use of *Logo* (Papert et. al., 1979). Results of the student's involvement with Logo were presented as a breakdown of the mathematical skills and concepts to which the students were exposed during the project. The students using *Logo* in the Brookline Project did better on angle and line estimation than other students with no computer experience. Several studies that included primary school students succeeded the Brookline Projects (Statz, 1973; Howe, O'Shea, & Lane, 1980; Gorman & Bourne, 1983; Clements, 1987). This paper will focus on research at the secondary mathematics level of instruction.

In a study by Horton and Ryba (1986), sixteen junior high school students were randomly assigned to treatments with or without the use of *Logo* software. The control group received no treatments apart from the regular school program. In addition to the regular school program, the *Logo* students were given two one-hour *Logo* sessions each week over a seven-week period of instruction. All students were assessed before and after treatments on six tasks; (a) exploration, (b) analysis and planning, (c) creativity, (d) debugging, (e) coding, and (f) prediction.

Within the Horton and Ryba (1986) study, secondary students who worked with *Logo* progressed individually through levels including basic turtle commands, repeat commands, defining procedures, editing and system operating, and sub-procedures. Students advanced according to their abilities to master the thinking skills and programming operations. Instruction was incremental in that no student was allowed to



progress to the next level until all the thinking skills at a previous level were acquired. Progress records for each student were kept using a checklist for: (a) assessing the development of each learner's thinking skills, (b) assessing the progress of a group of learners, and (c) deciding upon the content and organization of activities to be included in each subsequent *Logo* session. After skills were assessed at an appropriate level, students worked on individual *Logo* projects which required them to create a drawing of their own choice by planning and analyzing the steps to completion and then programming the drawing into the Logo language.

Findings of the Horton and Ryba (1986) study indicate that the students in the Logo treatment group tended to outperform their control group counterparts on all tasks. The results suggest that the focus on development of specific thinking skills using Logo can enhance cognitive development.

Yusuf (1991) described a pretest and posttest experimental design study to determine the effects of *Logo*-based instruction compared to instruction by teacher lecture and pencil-and-paper activities. Sixty-seven students in the seventh and eighth grades of a Cincinnati middle school made up an experimental group that received instruction with *Logo*, and a control group that received traditional teacher lecture instruction. Students in the experimental group were taught the basic turtle commands of *Logo*. They then were taught the concepts of points, rays, lines, and segments using *Logo* tutorial modules. The control group was taught the same concepts using the lecture format and paper and pencil activities. An analysis of covariance indicated that students taught within the *Logo* treatments scored significantly higher on the posttest than the control group, and moreover showed significant differences in students' positive attitude

toward mathematics and geometry. Yusuf (1991) concluded that *Logo*-based instruction was an effective tool for teaching geometry at the secondary level.

In summary, the review of literature on the use of Logo in mathematics suggests that students may benefit from using Logo in terms of mathematics achievement, problem solving skills, and the ability to articulate mathematical concepts. At the secondary level, student cognitive development may be enhanced when Logo instruction is added to regular classroom instruction.

### ***Geometric Supposer and Geometers Sketchpad Research***

The *Geometric Supposer* software series (Schwartz & Yerushalmy, 1986) began prior to the introduction of *The Geometers Sketchpad* (Jackiw, 1991). The *Geometric Supposer* programs allow students to choose primitive shapes and perform measurement operations on these shapes. The *Geometers Sketchpad* program allows students to construct a shape, change it, and then maintain any constructions that were created. Both the *Geometric Supposer* and *Geometers Sketchpad* record constructions performed on shapes and can repeat the action on other shapes. The following paragraphs summarize some of the research performed with these programs at the secondary mathematics level.

Yerushalmy, Chazan, and Gordon (1987) presented a yearlong project on the implementation of a guided inquiry approach using *Geometric Supposer* to teach high school geometry in three Boston area suburbs during the 1985-1986 academic school year. The study design included a pretest and posttest to compare the difference of mean scores between the experimental and the control group of five major variables: level, originality, accuracy, the change in accuracy, and the number of arguments. In the beginning of the school year, few students in the experimental or the control classes had

any background in geometry. Results indicate that students who used *Geometers Sketchpad* were significantly more able to develop general conjectures. No other differences between groups were statistically significant.

Chazan (1988) reported in a study that high school students have difficulties in understanding the topic of similarity. A unit addressing these concerns was designed for use with *Geometer Supposer*. Students were observed as they learned similarity with this unit and were given a pretest and posttest on fractions, ratio and proportion, and similarity. Achievement results on the posttest were found to be significantly in favor of the experimental group. In additional qualitative analysis, Chazan (1989) observed that unlike textbook theorems, which students can assume as true because they are in the book, students using *Geometer Supposer* believed that theorems generated with *Supposer* software needed to be proved before they could be accepted as true.

McCoy (1991) studied the geometry achievement of a single class of students that used *Geometers Sketchpad* regularly compared to another class which implemented the traditional path of teaching geometry. This research provided evidence in support of the effectiveness of *Geometers Sketchpad*. Results indicated the integration of *Geometer Sketchpad* activities provided students with a better understanding of mathematical content and improved performance by the high school geometry students.

While positive contributions of the *Geometer Supposer* and *Geometers Sketchpad* software are well documented, the use of technology is not a panacea. For instance, it has been pointed out that *Geometers Sketchpad* does not seem to improve students' abilities to visualize in three dimensions (Dixon, 1997). This is possibly a shortcoming of modeling three-dimensional objects in only two dimensions. Nevertheless, these

programs have been shown to be successful in improving student performance and achievement in a pretest and posttest design.

### **Examples that illustrate computer software definitions**

Additional computer software vocabulary will be introduced as an instrument for distinguishing among the various kinds of educational mathematical programs. These distinctions will clarify how the computer virtual manipulative used in this study is or is not like its concrete counterpart. They will also establish a foundation by which the researcher can compare research results within the literature review.

Cunningham (1991) described three kinds of computer visualization. These are: (a) *post-processing* occurs when the student knowledge is complete and the student is creating a display of the finished product, (b) *tracking* occurs when the knowledge is being developed and the user is watching it being displayed to see its nature, and (c) *steering* occurs when the student is in the processing loop and can interact with and manipulate a simulation (p. 70). This section of the literature review provides examples of post-processing, tracking, and steering as methods for defining what is and what is not a virtual manipulative.

Cunningham provided an example of post-processing with an image of output including a function, its graph, and the computed local minimum and maximum. This image was generated with the use software developed in 1964 by Dartmouth College Professors John G. Kemeny and Thomas E. Kurtz called *True Basic*. Figure 2.2 is a similar image showing the definition of the same function, its graph, and a computed local minimum using *Waterloo Maple Software*. The student who uses this mathematical software knows the computer software specific language of *Maple Software*, and

specifically the commands *restart* and *plot*. She or he also knows how to find the derivative within the *Maple* environment. Finally, the student creates a plot of the derivative to display its graphical interpretation. This is an example of post-processing if we assume the student knows how to take the derivative and is using Maple software to quickly access the graphical representation. In this way, the student has the acquired knowledge and is creating a display of a finished product, namely, the graphical representation of the derivative.

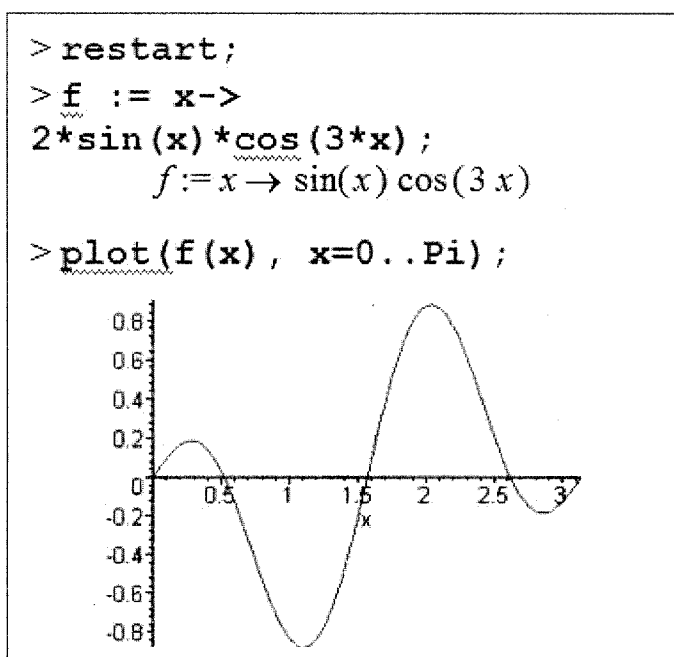


Figure 2.2. Post-processing. A plot of a function with local minimum computed, from the *Maple Version 7* computer algebra software developed by *Waterloo Maple Software*©.

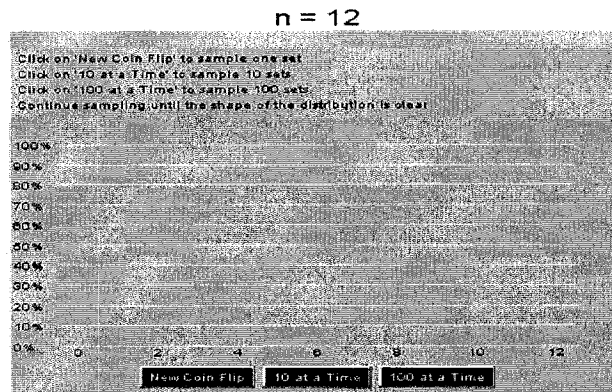
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Tracking involves showing a display as it is computed so that the order of development illustrates the mathematical processes shown. For example, a statistical experiment can show graphical representations of sampling as samples are virtually generated. Figure 2.3 illustrates three of several rapidly changing displays that are meant

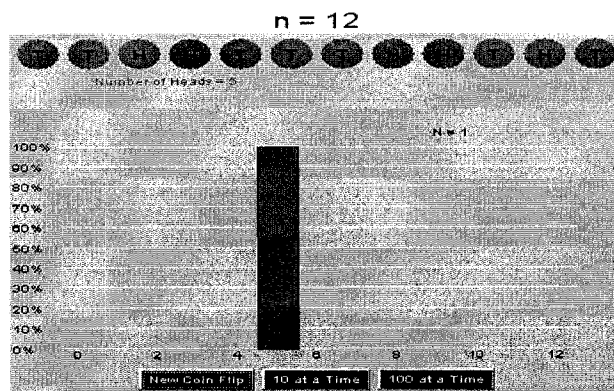
to develop student understanding of the central limit theorem. This display provides user-friendly directions for flipping coins, with an outcome of heads or tails, in samples of size twelve. The images of Figure 2.3 show the virtual workspace: (a) the leftmost image is the initial workspace, (b) the middle image displays the workspace after the user selects the *New Coin Flip* icon, and (c) the rightmost image is the workspace after the user selects *New Coin Flip*, *10 at a Time*, and *100 at a Time*, icons repeatedly until  $N = 3054$  trials of sample size 12 are obtained.

Steering techniques actually get the user involved in the development of the simulation. One of the earliest developed mathematical software programs, *Logo*, provides an example of steering. To produce a triangle similar to a given triangle, the user need only change the scale factor in the procedure. As the simulation is in process, the user can stop the turtle by clicking on its back with the mouse or by backspacing over a dot that appears below the display on the command line. This allows the user to make appropriate modifications in the construction of the defining procedure.

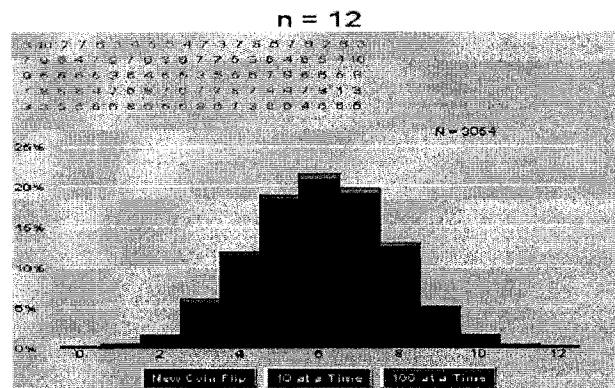
The construction of the thirty-sixty-ninety procedure may be viewed as a prerequisite to developing an understanding of constant ratios associated with similar triangles. In the following diagram, Figure 2.4, the user writes a procedure, named “thirty\_sixty\_ninety” that produces a 30-60-90 degree triangle of a specified scale factor. The turtle on the right is in the process of completing the procedure named thirty-sixty-ninety as indicated by a black dot at the end of the command line. Procedures are written by the researcher using pre-defined *Logo* commands specific to this version of *Logo* software.



(a)



(b)



(c)

*Figure 2.3. Tracking.* These applets were developed by Gary McClelland and published by Duxbury Press© 1999 and are freely available at the website: <http://www.seeingstatistics.com/seeing1999/resources/opening.html>

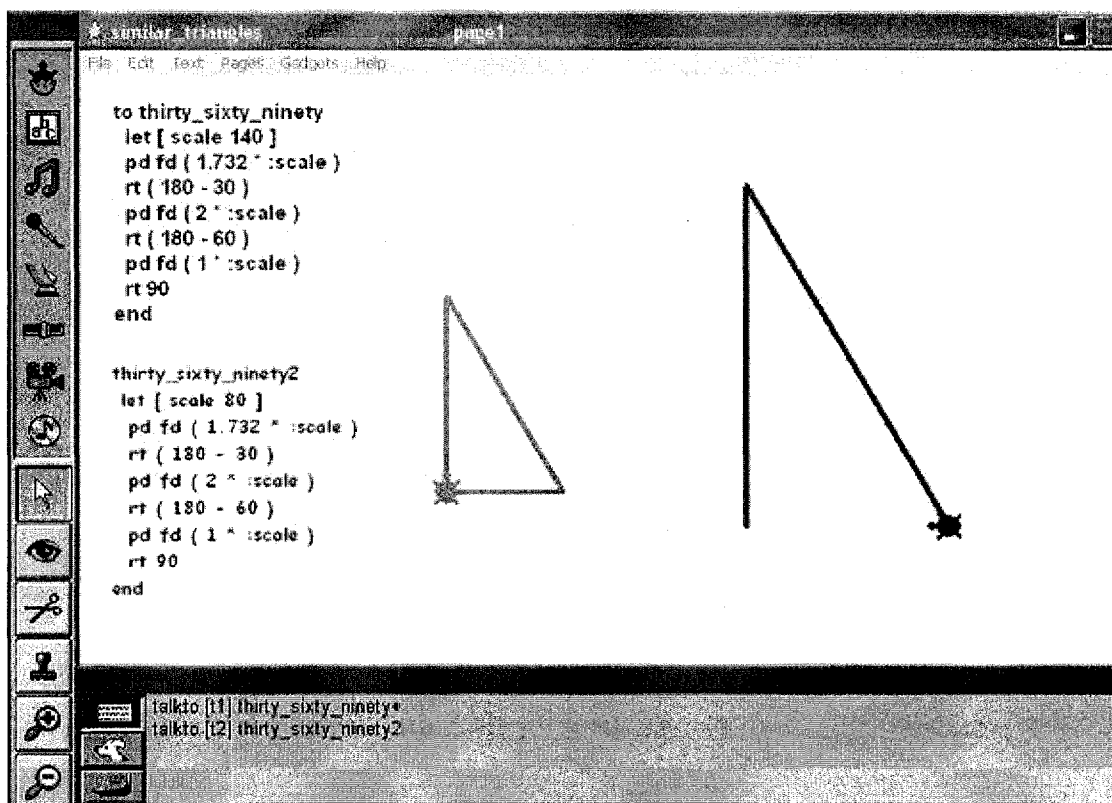


Figure 2.4. Steering. Image is developed with the use of *MicroWorlds™ Version 2.05* © Logo Computer Systems Inc., 2003.

Another example of steering involves the use of *Geometers Sketchpad Version 4.04*. Here, an analogous lesson on similar triangles begins with instruction on how to build two similar triangles within a *Geometers Sketchpad* workspace. The triangle on the right is similar to the triangle on the left in Figure 2.5. Measurements of corresponding line segments and an angle are in the upper left and lower right hand corners.

To produce this image with *Geometers Sketchpad*, the user must (a) construct point A and triangle BCD, (b) double-click point A, (c) draw a marquis around triangle BCD, (d) select dilate from the Transform menu, (e) select a scale factor of 2 by typing



this value into an alternate window and pressing the *Okay* icon, and (f) select Measure Corresponding Sides and Angles to view ratios.

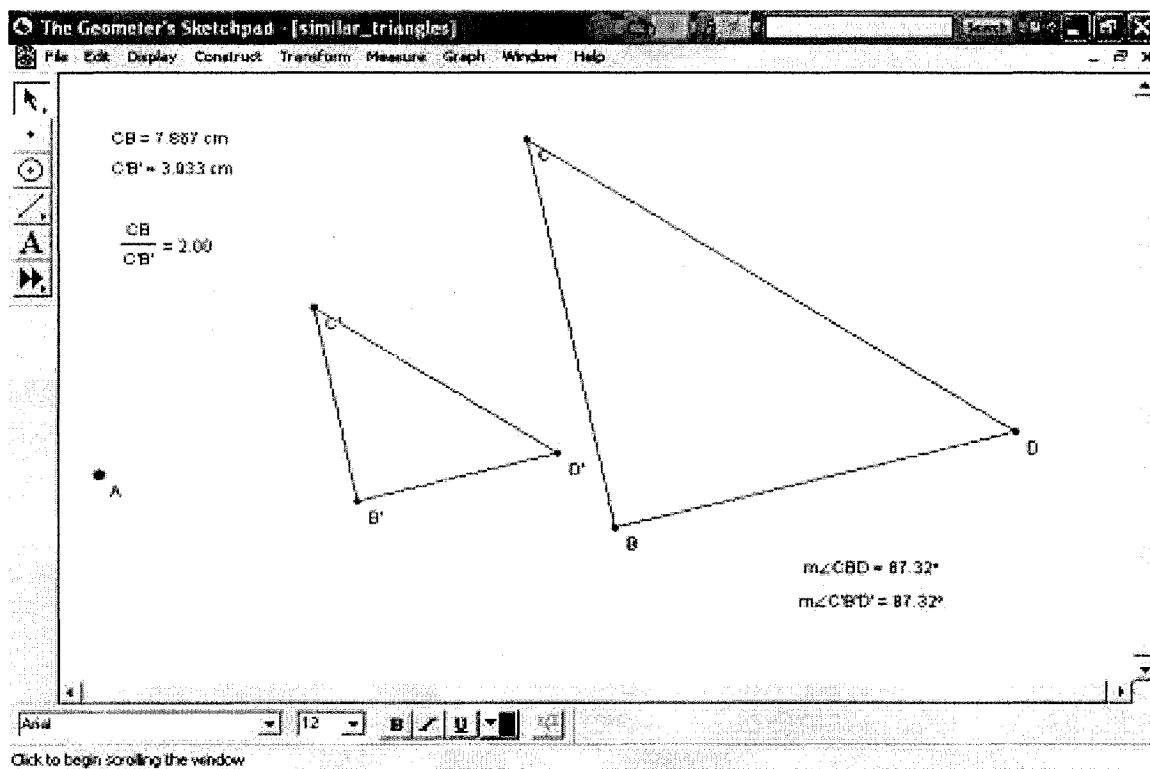


Figure 2.5. Steering. Image is developed with the use of *Geometer's Sketchpad Version 4.04* © Key Curriculum Press, 2003.

Students can alter the formation of any one of the two triangles by placing the mouse pointer over a single vertex and dragging it in any desired direction. This movement automatically causes the second triangle to move in the same direction, thus maintaining the properties of similar triangles. That is, measurements between corresponding lengths and angles change as the image is moved around. What does not change is the ratio between corresponding line segments and the equality in measure of

corresponding interior angles. In this way, the student is able to see that the ratios of side lengths of similar triangles are constant.

The student can also impose a grid over the workspace. The whole triangle can then be freely moved around the workspace by placing the mouse over a red dot that appears on the grid and dragging the grid up, down, left or right. This allows the student to actively discover relevant properties of similar triangles. In this way, the dynamic computer program instantaneously links the triangles to their relevant symbolic mathematical interpretations.

“Informal studies with students across the sciences indicate that students respond much more strongly to dynamic images than to static ones” (Cunningham, 1991, p. 71). These are defined by Cunningham to be precisely the images that are obtained via tracking or steering. In fact, the interactive environment provided by *Geometer's Sketchpad* has the potential to foster students' movement from concrete experiences with mathematics to more formal levels of abstraction, to nurture students' conjecturing spirit, and to improve their mathematical thinking (Manouchehri, Enderson, & Pagnucco, 1998).

The *Logo* and *Geometer's Sketchpad* computer environments illustrate tracking and steering properties in that the user can translate, rotate, dilate, or measure aspects of geometrical shapes by using program specific commands. Many of these commands are built into the mathematical software and must be acquired vocabulary of the user prior to working within either environment. With the necessary vocabulary and knowledge of operating commands, the user can interact with and manipulate any number of simulations as well as explore and develop mathematical properties.

The *Logo* and *Geometer's Sketchpad* computer environments also illustrate dynamic properties. For example, recent versions of *Logo* allow the user to rotate the direction of the turtle by pointing the mouse arrow over the turtle, holding down on the mouse, and moving the mouse in the direction of the desired rotation. In the provided *Geometer's Sketchpad* example, the user was able to change the shape of the original triangle by holding the mouse pointer over one vertex point and dragging the mouse (and subsequently the point) to another location on the workspace. It was also possible to transform the triangle by dragging a grid over the workspace. These capabilities of current versions of *Logo* and *Geometer's Sketchpad* provide a glimpse of virtual manipulatives in that their dynamic nature allows the user to freely move aspects of the visual representation without having to incorporate program specific vocabulary or commands.

Virtual manipulatives are dynamic (computer) visual representations of concrete manipulatives (Moyer, Bolyard, & Spikell, 2002a, p. 133). These manipulatives have tracking and steering capabilities. However, virtual manipulatives incorporate visual cues as a replacement for specific language of computer software commands. Moving objects with a virtual manipulative involves strictly placing the mouse pointer on an object and dragging the mouse in the desired direction, clicking on instructional icons that maximize the use of visual cues, or using the backspace button to delete prior icon selections. Virtual manipulatives make every effort to simplify the movement of computer images. In this way, virtual manipulatives are not unlike any number of computer games.

According to Spicer (2000), there are two types of web-based manipulatives: static and dynamic. Both are referred to as virtual. The virtual manipulatives of this research are described by Moyer, Bolyard, and Spikell (2002b) as dynamic visual representations that are essentially objects in that they can be manipulated in the same way a concrete manipulative is manipulated. For example, one can move a concrete algebra tile by applying pressure on the surface with one finger. Similarly we can move a virtual algebra tile by applying pressure with one finger on the mouse. The action is strikingly similar between the concrete and virtual manipulative.

Static manipulatives are visual images ordinarily associated with pictures in books, drawings on an overhead projector, and sketches on a chalkboard (Moyer, Bolyard, & Spikell, 2002b). They may resemble concrete manipulatives, but they cannot be used in the same way as concrete manipulatives are. In this case, a computer might move the images in response to a command from the user, but the user does not directly manipulate the object.

*Logo* and *Geometers Sketchpad* have been used in the development of their web-based counterparts at Utah State University's National Library of Virtual Manipulatives (NLVM): *Turtle Geometry and Transformations - Dilations*. These manipulatives are freely available on The World Wide Web: <http://www.matti.usu/nlvm>. The applets, *Turtle Geometry and Transformations - Dilations*, were developed as a National Science Foundation (NSF) supported project that began in 1999 to create a library of uniquely interactive, web-based virtual manipulatives or concept tutorials, mostly in the form of Java applets, for mathematics instruction with emphasis in grades kindergarten through high school.

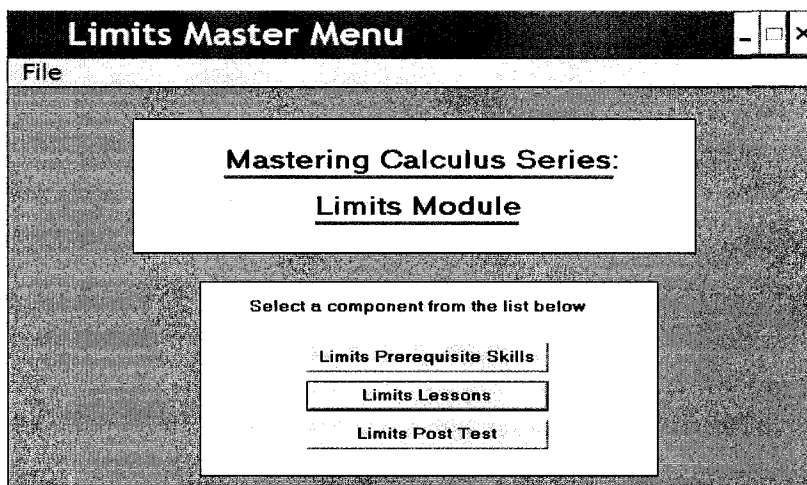
### Other mathematical software programs

To clarify the definition of a virtual manipulative, a comparison to other types of mathematical software will be considered. Some mathematical program designers have advocated using computer technology to create more learner-centered, open-ended learning environments (OELE) in which the learner is provided with varying amounts of help and support to decide what is needed to learn and what resources are required (Cognition & Technology Group at Vanderbilt [CTGV], 1992). Proponents of OELEs assume that by identifying goals and constructing meanings, learners become active managers, rather than passive receptacles, of information.

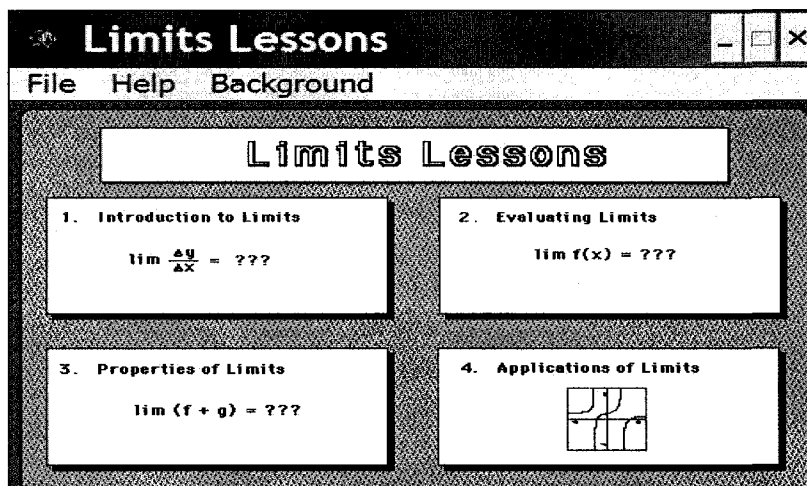
According to Land and Hannafin (1996), one characteristic of an OELE is that it provides learners with opportunities to engage the environment in ways that support their unique needs and intentions for making sense of the world. OELEs generate learning sequences based on the computer's assessment of student prior achievement as interpreted by the teacher. The teacher can assess and develop lessons according to student performance by logging into their teacher account and clicking on the students' name. When this is done, the teacher can view the current lesson and performance as well as choose future lessons.

New Era Classroom, Technology, and Research Foundation developed an example of an OELE with a program called *Math Trek Calculus* (2004). Figure 2.6 shows some of the images a student would see as they complete a lesson on limits. Image (a) is the first window in the lesson. Notice that the student has the option of taking the posttest at any time. Image (b) presents the student with a list of lessons within the lesson. Figure 2.7 is an illustration of Image (c) and Image (d). In Image (c), the

student has selected Evaluating Limits. In Image (d), the student has selected 10 as the value of 'a' and is prompted for an answer. The computer automatically records student responses to questions. This eliminates the need for paper grading and lecture while allowing the student to progress through lessons at their own pace.

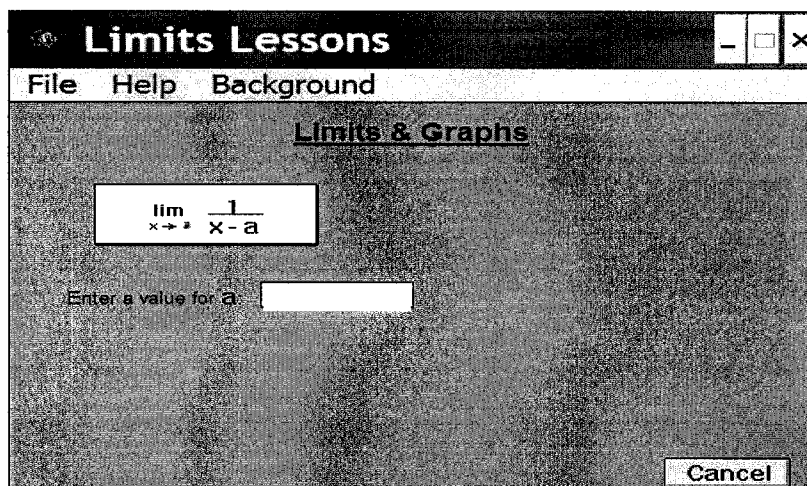


(a)

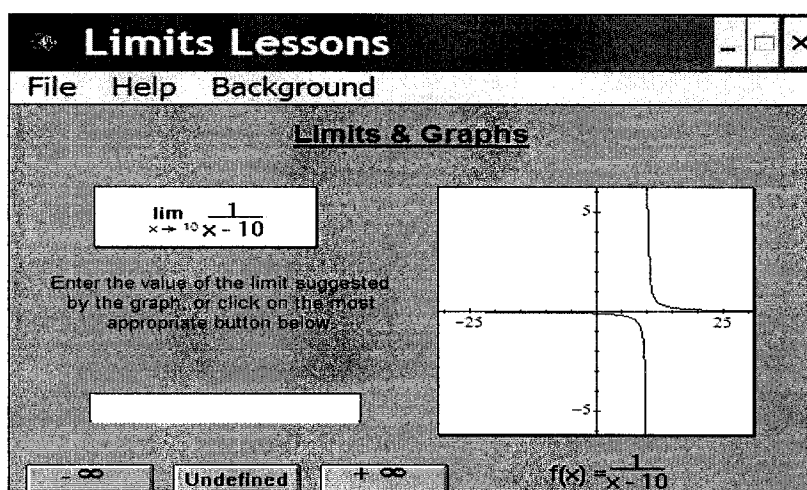


(b)

Figure 2.6. *Math Trek Calculus*©, New Era Classroom, Technology, and Research Foundation, 2004.



(c)



(d)

Figure 2.7. *Math Trek Calculus*©, New Era Classroom, Technology, and Research Foundation, 2004.

Notice that the student who uses an OELE must not only keep track of computer navigations, but must also attempt to understand new problems in isolation of fellow students or the teacher. It is static in that visual images are those that are ordinarily associated with pictures in books or drawings on an overhead projector. Although

student cognitive development may occur, the design of this computer software favors ease and simplicity of instruction rather than supporting the students' cognitive ability to focus on higher levels of abstraction.

In contrast, the *Blocks Microworld* program, presented earlier in this chapter, is an example of a mathematical microworld that employs multiple, linked mathematical systems (Kaput, 1986). The term *microworld* was first coined by Papert (1993) who described the *Logo* microworld for exploring and constructing within a geometrical concept space. A microworld is defined as a model of a concept space, which may be a very simplified version of a real world environment, or it may be a completely abstract environment. Normally, a user can create constructions within a microworld which will behave in a way consistent with the concepts being modeled (Papert, 1993, Rieber, 1992). A popular example of a microworld includes a mechanical problem solving environment called *The Incredible Machine* (1992).

Extensive review of the literature failed to provide a clear distinction between a virtual manipulative and a microworld. The adopted definition of a virtual manipulative refers to a virtual manipulative as being "web-based" (Moyer, Bolyard, & Spikell, 2002b, p. 372). This suggests that virtual manipulatives are web-based computer programs. A microworld is a computer program. This suggests that a computer program written in the format of a virtual manipulative is a form of a microworld presented via access to the internet. No research to date has been performed on the virtual manipulatives offered at the National Library of Virtual Manipulatives website (Jim Dorward, personal communication, March 7, 2005). Therefore, the remaining review of research will focus



on student learning outcomes that result from the use of concrete manipulatives, microworlds, or a comparison between the two.

### **Concrete manipulatives**

The use of concrete manipulatives has a long history of importance in teaching and learning mathematics. Early people used mechanical devices such as fingers, counters, and the abacus to assist with calculations (Toney, 1968). In the 1600s, John Amos Comenius was among the first educational theorists to advocate the use of real and useful things that can make an impression on the senses and on the imagination (Baker, 1977). In the late 1800s, Johann Heinrich Pestalozzi also emphasized the value of using concrete objects for instruction. He believed classroom experiences should be based upon actual experiences of the child, proceeding from the concrete to the abstract, from the particular to the universal (Sobol, 1998).

Many journal articles and research reports have been published over the past 30 years on the use of manipulatives in mathematics. Rather than exploring individual studies, summaries of results of many studies will be addressed to determine overall themes in the literature. After critical analysis, four reviews (Fennema, 1972; Parham, 1983; Sowell, 1989; Suydam & Higgins, 1977) were identified as being exemplary. The remainder of this section will be a summary of the results of these reviews.

Fennema (1972) compared 16 studies on the effectiveness of learning mathematical ideas through the use of concrete manipulatives for students in grades 1 to 8. Results of this review suggest concrete manipulative materials should be included in mathematics instruction. The inclusion and the use of manipulatives were justified as they: (a) help make the abstract world of mathematics meaningful, (b) help provide a

variety of situations that assist the transfer of knowledge from learned to unlearned situations, (c) improve motivation, and (d) help teachers gain insight into children's thinking (p. 637).

Suydam and Higgins (1977) published a comprehensive review and synthesis of research conducted in Grades K-8 on the use of physical manipulatives. They found that students using manipulative materials produced greater achievement scores than those not using them, at all grade and age levels in elementary school. Suydam and Higgins (1977) stated, "We believe that lessons involving manipulative materials will produce greater mathematical achievement than will lessons in which manipulative materials are not used if the manipulative materials are used well" (p. 92).

In particular, the use of counters and base-10 blocks aid the learning of four arithmetic operations (addition, subtraction, multiplication, and division) as well as increasing the understanding of place value and number sense (Kennedy, 1986). Moreover, students at the elementary level develop better proportional reasoning skills when instructed with concrete materials (Hiebert, 1991).

Similar results were reported by Parham (1983) in an analysis of 64 research studies conducted from 1965 to 1979 on the effects of manipulative use on achievement for elementary school students in Grades 1-6. Parham (1983) reported a decided difference in achievement scores, with students who had used concrete manipulatives scoring on average at approximately the 85th percentile on the California Achievement Test, as opposed to similar students not using physical manipulatives scoring at the 50th percentile. Results supported earlier research findings (Suydam & Higgins, 1977) and

avored the use of concrete manipulatives for their positive effect on student achievement.

Parallel conclusions were reported in a comprehensive analysis by Sowell (1989) who examined 60 studies (38 journal reports, three unpublished reports, and 19 dissertations) conducted from 1954 to 1987 on the effectiveness of using concrete manipulatives for students in grades kindergarten through college. According to Sowell (1989), "Results showed that mathematics achievement is increased through the long-term use of concrete instructional materials and that students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use" (p. 498). Length of treatment using concrete manipulatives was linked to achievement. As explained by Sowell (1989), "When treatments lasted a year or longer, the result was significant in favor of the concrete instructional condition" (p. 502). Treatments of shorter duration did not produce statistically significant results.

Kaput (1989) established that "meanings are developed within or relative to particular representations or ensembles of [particular representations]" (p. 38). Sharp (1995) interpreted this statement to mean, "students who successfully make connections between physical representations and mathematical representations have created meaning of mathematical ideas" (p.4). In particular, Sharp hypothesized that meanings might be achieved or at least enhanced when individuals construct translations between algebra symbol systems and physical systems that represent one another.

A qualitative study by Sharp (1995) on the use of algebra tiles in five high school classes showed some students found it easy to think about algebraic manipulations when

they visualized the tiles. Sharp observed that students would often try to apply procedures for factoring to everything whether these procedures were appropriate or not. The algebra tiles gave many of these students enough conceptual understanding for these students to experience some success. However, results indicate no significant differences between groups divided according to: treatment (algebra tiles used only once during a factoring lesson), control (year of manipulatives), and control (algebra tiles as the only manipulative used in instruction). This result may be due in part to small group sizes that range from 10 to 13 students between the ages of 9 to 18 years.

A more noteworthy reason for the lack of significant differences in the Sharp (1995) study is that the participants were chosen from a school district list of students identified as gifted. Some students will learn mathematics regardless of the quality of instruction. It is also difficult to measure success of the manipulative when students can accurately perform procedures before and after treatments.

While much has been written about the perceived benefits of using manipulative aids in the learning of mathematics, concrete manipulative materials are not widely used in secondary mathematics instruction (Char, 1991). The lack of use is due to several difficulties including classroom management; structuring, monitoring, assessing the use of manipulatives, relating manipulatives to mathematical symbols and procedures, lack of financial resources, and lack of professional development (Kim, 1993).

Manipulatives are concrete in one sense because students can experience them in a sensory way. However, concrete manipulatives by themselves are devoid of mathematical meaning (Clements & McMillen, 1996). The connection to mathematics and to abstract symbolism depends on the actions students take on the objects and on

their subsequent reflections about those actions. These actions and reflections depend on the nature of instruction, which includes the activities the students are presented, and the constraints and supports provided by the teacher or computer software.

In summary, one can expect greater mathematical achievement measures when concrete manipulatives are used. This is especially true when there is long-term use of manipulatives. This use of concrete manipulatives can make mathematics more meaningful to students. They also have the potential to improve student attitudes toward mathematical content.

### **Studies that compare virtual to concrete manipulatives**

In a study by Kim (1993), 35 kindergarten children were assigned to hands-on or on-screen teaching groups. Students were taught classification, geometric, and arithmetic concepts using the software, *Hands on Math* (1982), published by Ventura Educational Systems. This study did not consider the use of both types of manipulatives, concrete or virtual computer, in combination. Results were strictly based on pretest and posttest accuracy scores that indicated no statistically significant differences between kindergarten students who used concrete manipulatives and those using virtual manipulatives on measures of addition, geometric classification, and counting skills.

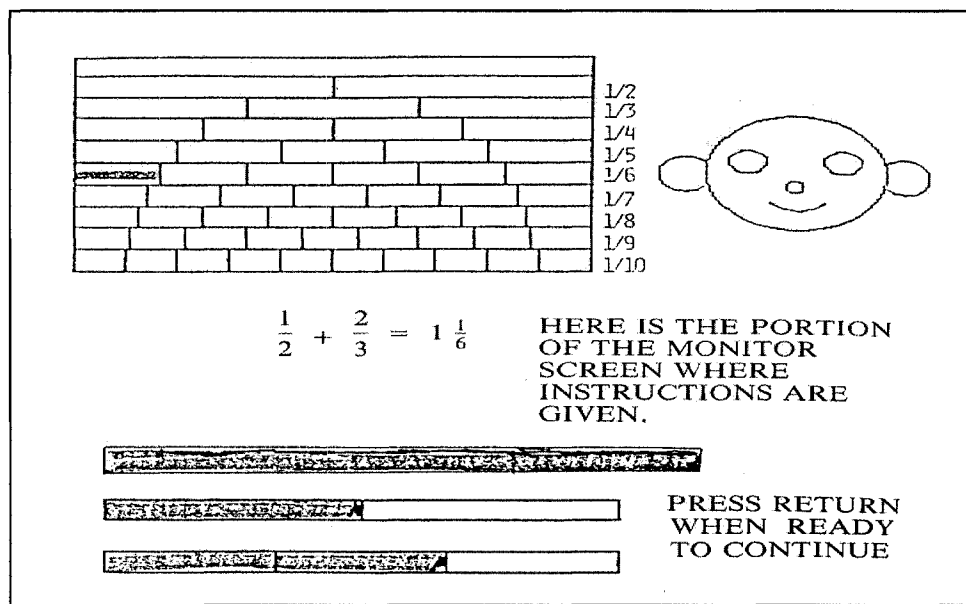
Char (1991) studied 63 kindergarten and first grade students making “computer bean sticks” to develop basic addition concepts. No control group was included in this study. The software allowed students to move images of Popsicle sticks and beans with the computer mouse. They could also add text to the screen. Thirty-one students were observed working in pairs with the virtual computer images of Popsicle sticks and beans. The other 32 children were observed working with physical Popsicle sticks and beans in

two classrooms. All students worked in thirty minute sessions three mornings per week for three weeks. Researchers interviewed all student participants.

Char (1991) found that the virtual computer bean sticks were easier for the children to manage than the physical sticks that involved gluing actual beans and were easier for teachers' housekeeping. She observed that the close resemblance of the virtual computer bean sticks to physical bean sticks contributed to this ease of computer use. She found that children independently integrated numbers with the manipulatives when using the software, which prompted richer exchanges in the classroom.

Ball (1988) used a fraction program that models concrete manipulatives with five fourth grade classes. As shown in Figure 2.7, the following figures are present when the graphic first appears: (a) the fraction strip table in the upper left hand corner, (b) the face, (c) two strips in the lower left-hand corner, and (d) the directions below the face. As the fraction addition problem appears below the strip chart, portions of the two strips at the bottom are shaded representing the two fractions to be added together. Then the shaded strip just above the two bottom strips appears as a result of taping the two strips together end-to-end from the two original strips  $\frac{1}{2}$  and  $\frac{2}{3}$ . The face smiles when a correct response is given. The computer simulated measuring lengths, cutting, and taping which were originally done on the concrete level.

Figure 2.8 displays the graphic image on the monitor screen at the end of a fraction addition problem. The student has selected a row in the table. The strip was colored in a lower row to simulate the actual placing of a strip there. The change in color made it possible for the student to decide if the selection was correct.



*Figure 2.8.* Fraction Strips. Image is taken from *Computers, Concrete Materials and Teaching Fractions* by Stanley Ball found in *School of Science and Mathematics, Volume 88 (6) October 1988*.

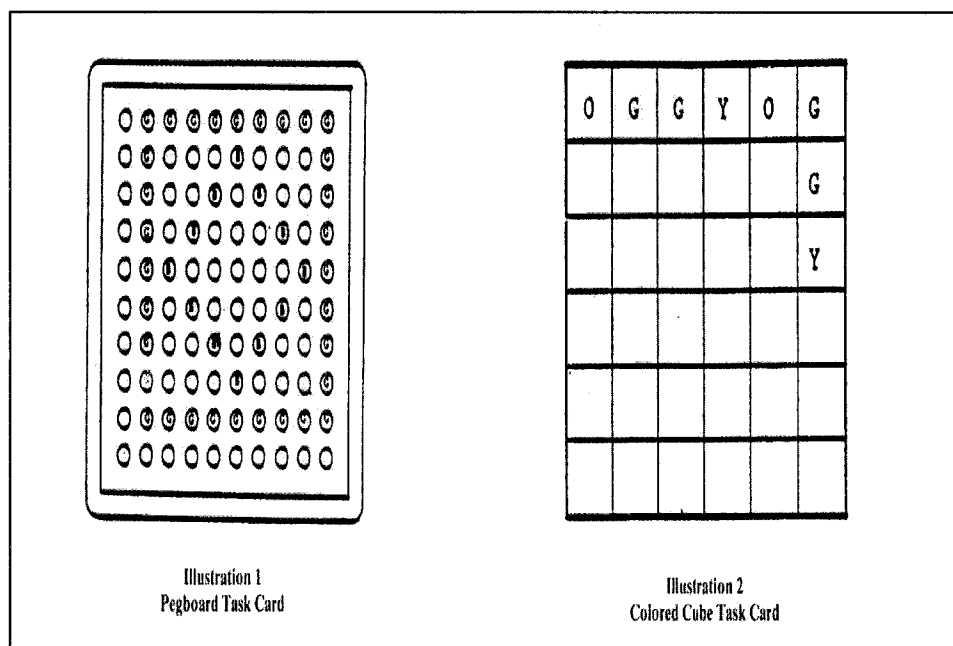
Three classes used concrete and virtual fraction strips and two were instructed using traditional methods with paper fraction strips, but no computers. Teachers of the experimental classes were trained in a summer workshop. The program was used to help students to concretize abstract concepts of adding fractions. The students worked with fraction strips manipulatives where they taped paper strips representing different fractional amounts together and then tried to figure out the new length by comparing the composite strip with other fraction strips. After working with these, the students moved to a computer manipulative with similar fraction strips. A t-test of posttest means revealed a significant difference between the achievement of the experimental group and the control group. Ball concluded that the treatment was effective in improving students' abilities to solve fractional addition problems.

Ball (1988) found that fourth-grade students using both the virtual and physical manipulatives scored significantly higher on conceptual understanding of fractions than students that used no manipulatives. Ball (1988) did not attempt to separate the effect of the physical manipulative from that of the virtual manipulative. Comparisons between the three groups of students using both the concrete and virtual computer strips and the two groups of students using the concrete paper fraction strips were not included in the results. It could be that only one type of manipulative is needed for significantly higher test scores.

Berlin and White (1986) studied 113 second and third grade students' spatial ability while using concrete and virtual manipulatives. The goal of the study was to investigate the effects of combining interactive microcomputer simulations and concrete activities on the development of abstract thinking in elementary school mathematics. Treatments included concrete-only activities, virtual computer only activities, and concrete and virtual computer combined activities.

During a three-week treatment period, instruction consisted of students' completion of concrete and virtual task cards. Task cards required students to duplicate and extend patterns of colored cubes and pegs. The concrete manipulative was in the form of a pegboard task card. The virtual manipulative simulated the pegboard using cubed activities. In the computer simulations, students used the keyboard to select color and location of the pegs and colored cubes on the screen. Figure 2.9 illustrates the appearance of the two types of manipulatives. On the left is a sketch of the wooden pegboards used in the Berlin and White (1986) study. On the right is the general appearance of the virtual computer manipulative.





*Figure 2.9. Pegboard TaskCard and MicroWorld. Images are taken from Computer Simulations and the Transition from Concrete Manipulation of Objects to Abstract Thinking in Elementary School Mathematics by Donna Berlin and Arthur White found in School of Science and Mathematics, Volume 86 (6) October 1986.*

A six-question paper and pencil instrument was used to assess student achievement. Berlin and White (1986) found no statistically significant differences between second- and third- grade students using concrete manipulatives, virtual manipulatives, and both treatments on measures of spatial sense and patterning. However, treatment activities had different effects for different genders and socio-cultural backgrounds. Some students did better with concrete manipulatives while others did better with virtual manipulatives. For example, rural-white boys using virtual manipulatives performed better than suburban-black girls using concrete manipulatives. The authors suggest using all three types of instruction to recognize different processing modes.

Nute (1997) studied 241 fourth, fifth and sixth graders' learning about shapes and transformations by way of quilt making exercises. Children were separated into six experimental groups and one control group. Children created different size quilts using their own four-square designs. This involved transformations on physical and/or virtual manipulatives. After four twenty-minute lessons, Nute used an ANOVA with an achievement score as the response variable and found no effects. This study was limited by the small number of sessions and the narrow curricular goals of the manipulative tasks and thus, generalizations cannot be made about other manipulatives. In this study, teachers provided assistance when asked. Nute found no statistically significant differences between fourth-, fifth-, and sixth-grade students who viewed or used concrete manipulatives, virtual manipulatives, or both on measures of patterning and geometric transformations. However, all groups scored higher than those students with no manipulative exposure.

Drickey (2000) investigated the effectiveness of physical and virtual manipulatives on middle school students' visualization and spatial reasoning skills. Students in two treatment groups, physical manipulatives and virtual manipulatives, were compared to students in a traditional instruction control group using a teacher-guided discussion format without the use of manipulatives. Also of interest in this investigation was the effect of manipulative use on visualization and spatial reasoning skills for students of differing mathematics abilities and attitudes. Comparisons were based on student scores on pretest and posttest measures of visualization and spatial reasoning and attitude about mathematics.

Results of the Drickey (2000) study indicated no statistically significant differences in mathematics posttest mean scores for students in the three treatment groups, as well as for students of differing mathematics abilities and attitudes about mathematics. Students in the concrete and virtual manipulative groups reported a preference for using manipulatives during instruction. Students in the virtual manipulative group had higher rates of on-task behavior than students in the physical and no manipulative groups. Posttest mathematics scores in all treatment groups were associated with the teacher, student gender, amount of homework completed during the unit, and the students' current mathematics grade.

Terry (1996) found that students in the second, third, fourth, and fifth grades using both the physical and virtual manipulatives scored significantly higher on tests of addition, multiplication, and spatial sense than students using either of the treatments alone. Terry also used a combination of physical and virtual manipulatives for instruction and found that students in grades two through five made significantly higher gains from pretest to posttest than students who were instructed using either type of manipulative alone. She studied 102 students at each grade level using base-10 blocks or attribute blocks. Pleet (1990) who also looked at differences between pretest and posttest scores, found no difference between a combination of manipulatives and either type alone. In this case, 56 eighth graders were studied over three weeks as they learned transformation geometry.

Smith (1995) developed a microworld program that was used in the experimental group and called The Cy-Bee Chips. The study took place at two urban middle schools and involved three sixth grade and three eighth grade classes. A total of 128 students

were involved in the study. The treatments of computer microworld, concrete manipulatives, or both computer microworld and concrete manipulatives were randomly assigned to the sixth and eighth grade classes. Seven 45-minute work sessions were used for each group. The sessions were on consecutive days with testing on the first and seventh day and treatments on the second through sixth days.

Prior to treatments in the Smith study, students' understanding and use of integers were assessed by a pretest which was used as a covariate for analyses. The posttest paralleled the pretest in computational exercises. The primary explanatory variables were treatment group membership and level of previous formal instruction. Statistical analysis for this study was divided into parts: (a) factorial analysis of covariance for the entire sample and (b) analysis of variance for subsets of the sample.

The microworld treatment group used only the computer microworld shown in Figure 2.10 to explore adding and subtracting integers. The student can select a positive white or negative black chip by clicking on the icons with the respective images. When this is done, the chip appears in the workspace. In the case shown in Figure 2.10, the student has selected three positive and two negative chips.

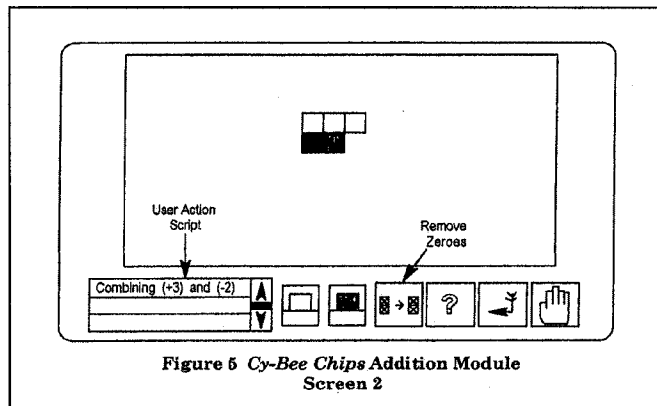


Figure 2.10. Cy-Bee Chips Addition Module.

Smith (1995) found that sixth- and eighth-grade students using the virtual manipulatives scored significantly higher on tests of integer addition and subtraction than both those students who worked with concrete manipulatives and those who used both treatments. The concrete manipulatives group used two-color counters to explore adding and subtracting integers. The objective of the research was to determine the effects of a computer microworld on middle school students' use and understandings of integers.

Results of the Smith study (1995) indicate statistically significant differences between three treatment groups in terms of mean posttest scores of students with and without previous instruction. Data indicate that previous instruction is also statistically significant in analyzing posttest scores. Students without previous instruction actually had higher adjusted mean posttest scores than students with previous instruction. Smith concluded that "previous instruction may interfere with students' acquisition of knowledge under new methods" (p. 130).

In summary, several studies found no difference in student achievement using concrete versus virtual manipulatives. However, when instruction is sequenced with the manipulative instruction preceding symbolic instruction, there is some evidence to suggest that students who use virtual manipulatives experience higher achievement in mathematics than those using only the associated concrete manipulative (Smith, 1995; Thompson, 1992). Two studies suggest that students who use both virtual and physical manipulatives show an increase in conceptual understanding in mathematics (Ball, 1988; Terry, 1996) and one study indicated a decrease (Smith, 1995). Other studies found no statistically significant difference in achievement of students using physical

manipulatives, virtual manipulatives, a combination of both concrete and virtual manipulatives, and no manipulatives (Pleet, 1990).

Comparisons made between physical and virtual manipulatives may be affected by design and sampling characteristics. It is unclear whether increases in academic achievement were partially due to the specific manipulative chosen for each study. For example, computer-simulated base-10 blocks, two-color counters, and fraction strips produced positive results, whereas studies using pegboards and color cubes (Berlin & White, 1986), and geometric shapes (Nute, 1997) realized no noticeable increase in student achievement.

### **Conclusions**

The exploration of virtual manipulatives and/or microworlds has outstanding potential and support. This potential lies in bringing mathematics instruction to a level that balances computational methods with visual and concrete methods that focus on student understanding of concepts. If students are provided with rich environments for learning, they will come to understand mathematical content from numerous perspectives.

## Chapter 3

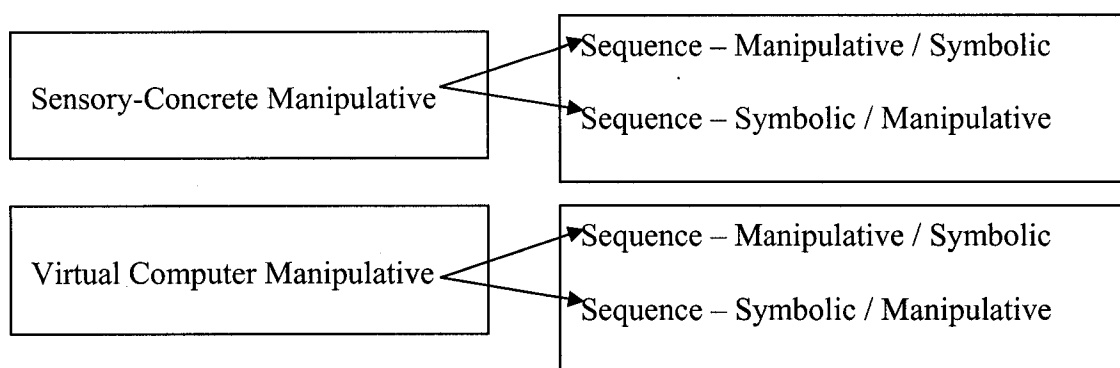
### Methodology

This chapter describes methodology used for answering the research questions outlined in Chapter 1. It includes a description of the participants and selection tools used for participation in the study. It also includes a description of the processes involved in research implementation.

#### Overview and purpose of the study

For over a decade, the importance of implementing technology in the mathematics classroom has been underscored. From *The Agenda for Action* (National Council of Teachers of Mathematics, 1980) to *Measuring Up* (Mathematical Sciences Education Board, 1993), every major document which paints a vision of the future of mathematics education included a description of technologically-enhanced instruction. And while significant transitions are taking place in secondary learning environments, computer software continues to evolve as well, augmenting the teaching tools that promote active construction of mathematical knowledge.

This study was designed to allow the researcher to compare two types of manipulative, concrete and virtual computer, within sequencing of instruction that moves from symbolic to manipulative or vice versa. To do this, the researcher explored various sequences of manipulative types shown in Figure 3.1. Thus, there were four treatments that allowed the researcher to compare the two types of manipulatives within sequencing of instruction.



*Figure 3.1.* Treatments. Manipulative type within sequence of instruction.

Many secondary mathematics classrooms have a mixture of students who have or have not received symbolic instruction in the past. This problem was addressed in the implementation of methods with multiple strategies. First, teachers agreed to not teach either topic until treatment implementation during the current academic year. Second, students in higher level mathematics classes, namely, Integrated Three or Integrated Two Honors were typically placed in sequences that began with symbolic procedures. It is reasonable to assume that students in these math classes had received some symbolic instruction in the years prior to the academic year of this research. It would therefore be unreasonable to place these students in categories that assumed no previous symbolic instruction. Students with more background knowledge of the mathematical content being predominately placed in treatment groups that assume initial symbolic instruction could add bias in favor of sequences with symbolic instruction first. Third, the pretest included questions that asked students about their own perceived background experience in the topic as well as whether or not they had experienced instruction in the topic within the framework of a manipulative. Finally, the pretest score was used as an indicator of student's background knowledge in the topic area.



**The participants**

Several school officials collaborated with the researcher to make this study successful. Approval began with the school district superintendent. This paved the way for the researcher to obtain a computer account that would allow internet access for a class of no more than thirty students. Principals at each school were contacted for approval and informed of presentation dates. Librarians assisted in scheduling the computer lab.

**Teacher selection**

Four teachers at one local high school were selected to be in this study. Each teacher agreed to the conditions of the study (see Appendix A). There were multiple reasons why these four teachers were selected for the study. First, other schools in the district had different time schedules. Classes at another local high school met for fifty-minute class sessions. Selecting all participating teachers at a single high school provided reasonable assurance that all presentations would have equal instruction and activity time periods of ninety minutes. Second, using one location eliminated the additional factor of having multiple schools. Teachers who volunteered outside of this high school were selected for pilot studies.

**Student selection**

Each student received the Parental Permission Form from the classroom teacher (see Appendix B). Students who did not return this form to the classroom teacher by the date of the manipulative presentation were not allowed to participate. Parental Permission Forms were given to the student early in the academic year. As a result, less

than five percent of students asked to participate did not have the forms needed to participate or opted to not participate.

Presentations included an explanation of what was required of each student participant. The researcher read the Student Informed Assent aloud in front of the class of students (see Appendix C). Next, the researcher answered questions and addressed concerns of the students. If the student agreed to participate, she/he signed the assent form and immediately returned it to the researcher. Students who did not immediately return this form to the researcher were not allowed to participate.

A total of 304 students with both signed forms were escorted to a computer lab by the researcher after Parental Consent and Student Assent forms were collected from 14 classes. The computer lab was located in the library of the high school. Students without both signed forms remained in the regular classroom with their assigned classroom teacher.

Students received a pretest upon entering the computer lab (see Appendix D). Depending on the treatment, either the concrete or virtual computer manipulative was on every desk and available for student use. Each student was given fifteen minutes to complete the pretest. Students also completed a posttest the same week of one of the four instruction sequences (see Appendix E). The posttest included a final question that asked the student participant if they would be willing to participate in a task-based personal interview at a later date. Students selected for interviewing were chosen randomly from a list of students who agreed to be interviewed by indicating so on this final posttest question.

Three students were selected to be interviewed from each of the 14 class presentations. Four of these students were absent on the selected interview date. Twenty-three interviews were performed on the topic of multiplying and factoring polynomial expressions. Fifteen interviews were performed on the topic of solving two-step linear equations.

Interviews ranged from 20 to 45 minutes in length and covered basic topic foundational knowledge, problems presented in the treatment activity, and extension problems. For example, an interviewee on the topic of multiplying and factoring polynomial expressions was asked about which pieces of Algebra Lab Gear represented the variable  $x$ . This represents basic topic foundational knowledge. Next the student was given a problem that required the student to multiply two binomials. This is the type of problem the student practiced during the treatment activity. Finally, the student was asked to factor polynomials with more than four terms. This is an extension of the problems the student practiced during the treatment activity.

### **Class levels**

Students at the high school are tracked according to their academic records, age, grade level, and perceived abilities in mathematics. This means that upon entry into the high school, the student's counselor determines math placement using the student academic record. Class levels included in this study were Pre-Algebra, Integrated One, Integrated Two, Integrated Two-Honors, and Integrated Three. Students were in ninth grade, tenth grade, eleventh grade, or twelfth grade.

**The researcher**

I earned a M.S. in mathematics in 1995. I have had a variety of courses including differential equations, engineering physics, analysis, probability, statistics, and learning theories in mathematics education. My teaching experience includes four years in the public schools, one year at a tribal community college, and several years as a graduate student at two different universities. During this time, I have taught a number of classes ranging from basic math to multi-variable calculus.

As a mathematics educator, I believe that the people who understand mathematics best are precisely those people who have multiple ways of expressing mathematical content. What a person knows about any mathematical concept is inseparably linked to the number of ways in which that person can express the concept. Indeed, this research assumes a person understands a concept when that person can demonstrate the concept in a manner that a fellow mathematician can understand, but can also demonstrate the concept in a manner that a non-mathematician can understand. In order to do this, one must understand the concept from multiple perspectives. It is the number of mental connections that a person makes with respect to a given concept that establishes how much that person truly understands the mathematical concept.

The researcher also believes that students' perceptions and understandings of different aspects of a mathematical concept are often incomplete. The typical secondary student will tend to use taught procedural techniques without referring to the relevant definitions or meanings behind the procedures. When a procedural technique fails to provide a student with a straight answer, the student will sometimes attempt to solve the problem with self-defined logical operations.

Through exploration of student errors, one can attempt to gain insight into pre-conceived student misconceptions. The researcher favors the belief that some misconceptions can be linked to instruction that focused on procedures prior to providing meaningful representation of the symbolic structure. There is bias in favor of using an alternate form of mathematical representation. This representation could be sensory-concrete, pictorial, or virtual computer.

Since many secondary students are not at a formal operations stage of thinking, it is, in the opinion of this researcher, better to begin any new secondary mathematical concept with activity-based instruction that provides a sensory-concrete or virtual computer form of symbolic representation. In doing so, the instructor is providing the student with a frame of reference from which the student can build meanings behind mathematical content.

Results of all relevant research conclusions in this dissertation will be reported without favoritism towards admitted bias. Above personal preferences is the desire to investigate student learning outcomes that result from the use of a concrete or virtual manipulative in secondary mathematics instruction.

### **The research tools and their implementation**

Many educational mathematical software programs have the steering feature. That is, the user can move objects by providing instructions. These instructions are typically defined within the software program. For example, the Logo program allows the user to instruct a turtle to move forward a number of units, to rotate a degree angle, and/or follow prewritten procedures. To operate this type of steering manipulative, the user must become familiar with the command set of the particular program.

The virtual applets chosen in this study move beyond this steering capability in that they do not require the user to learn predefined routines to move or manipulate objects. The movement, rather, is caused by simply placing the mouse pointer over the object on the computer monitor, holding the forefinger down on the mouse, and moving the mouse left, right, up, or down. The mouse movement directly causes the objects to move on the computer screen in the same direction that the hand is moving with the mouse. Thus, mental processes one must undergo when using predefined commands are avoided. In this way, the virtual manipulatives utilized within this study more closely model actual physical movements.

### **Topics**

The research consisted of two topics: solving two-step linear equations and multiplying and factoring polynomial expressions. These topics were chosen because of the features of the virtual computer manipulative that complemented each topic, the frequency with which the topic was taught within the chosen high school, and the potential to duplicate the images on the virtual applet into a concrete form.

### **Virtual solving of two-step linear equations**

The virtual computer manipulative that was used for instruction in solving two-step linear equations is an applet located at the National Library of Virtual Manipulatives website: <http://www.matti.usu.edu/>. This applet can be reached by selecting the *Virtual Library* icon available under the topic of algebra grades 6-9, and choosing the option titled, *Algebra Balance Scales*. Each student manipulated this applet with the use of the computer keyboard and mouse. Movements were controlled with the computer mouse within an area on the computer screen called the workspace.

The *Algebra Balance Scales* virtual manipulative allows the student to solve simple linear equations through the use of a balance beam. Unit blocks (representing 1s) and X-blocks (for the unknown variable), were placed on the pans of a virtual balance beam. The beam balances to represent equilibrium of the given linear equation. As long as the student keeps the scale in balance by performing equivalent operations on both sides of the equation, the student can choose to perform any arithmetic operation. The goal is to get a single X-block on one side with any number of unit blocks needed for balance, thus revealing the appropriate value of X that makes the original equation a true statement.

To place blocks on the balance scale, the student would have to click on an object and drag it toward the side of the beam she/he wanted to place it on. When the student releases the object, it snaps into place on the scale. When the student first places an object on a pan the scale swings down on that side (no longer balances), but when the given equation is fully represented, the balance is restored. Note that a student cannot click the *continue* button until they have successfully represented the equation, whether or not the scale balances. Blocks and boxes may be placed on either pan and in any order.

To remove blocks from the scale, the student would click and drag any object (even from the middle of a stack) to the *trash can* in the lower right corner of the workspace. This removal would cause the balance scale to tilt in one direction or another if a similar object had not been removed from both sides of the balance scale.

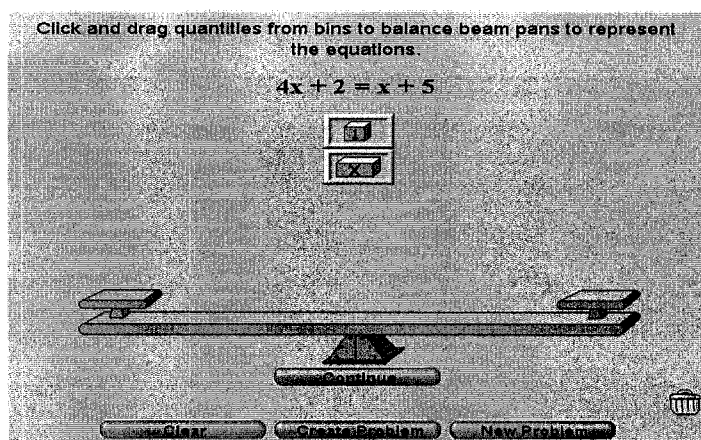
When the student believes she or he has correctly represented the equation, they can click the *continue* button. A message appears if the equation is not set up correctly. The message informs the student that the two sides do not match the equation. If the

equation is set up correctly, the initial display shows the options to add, subtract, multiply, or divide. The only allowable operations require the student to perform the same operations on both sides of the balance scale (and thus to both sides of the equation). The equation is updated with each operation. When the student has a single X-block on one pan and any number of unit blocks on the other pan, the virtual manipulative displays the solution in the form of an equation where X is equal to the number of unit blocks on the right pan of the balance beam.

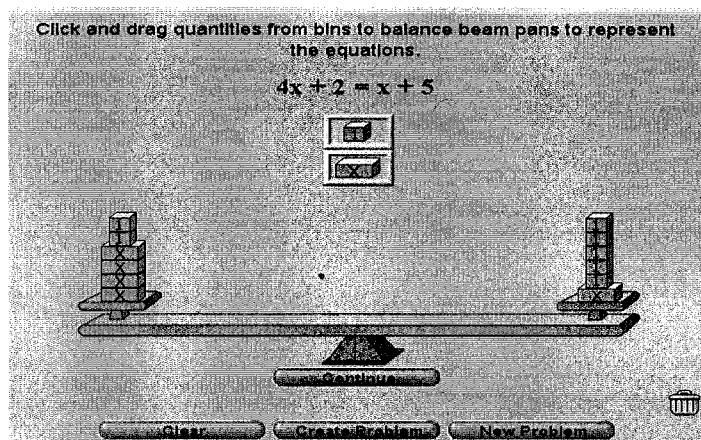
The following is an example of what the successful student would do to solve the two-step linear equation,  $4x + 2 = x + 5$ , using this virtual computer applet. Figure 3.2 displays the process of setting up the problem on the virtual applet. The student begins with the workspace on the computer screen. Next the student places 4 X-blocks and 2 unit blocks on the left pan of the balance scale. Then the student places one X-block and 5 unit blocks on the right pan of the balance scale. At this point the problem is correctly set up and the student can click the *continue* button. When this happens, a third virtual image appears giving the student the option of adding, subtracting, multiplying, or dividing from both sides of the equation.

The problem is now ready for solving. Suppose the student decides to subtract 2 unit blocks. The student would click the icon with a subtraction symbol. This becomes highlighted when selected. Then the student would type in the number 2 in the white box that appears just after the words, subtract from both sides. At this point, the student would click on the *go* button. Two unit blocks would instantly disappear from both sides of the balance beam and a new equivalent equation appears:  $4x = x + 3$ . Next, the

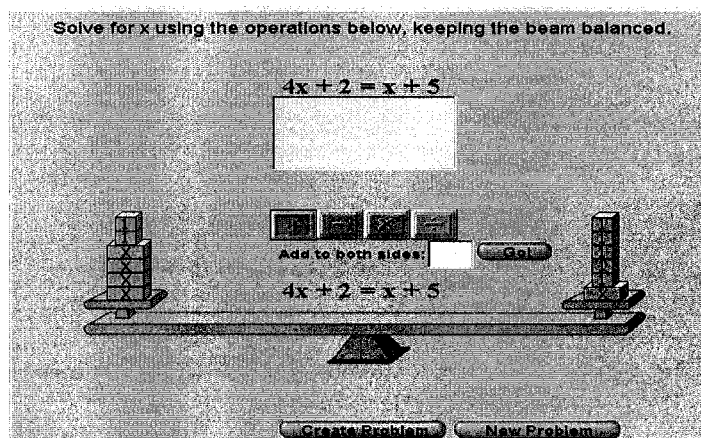




(a)



(b)



(c)

Figure 3.2. The process of setting up an equation for solving on the virtual balance scale. *Algebra Balance Scales*, National Library of Virtual Manipulatives, © 1999-2005 Utah State University.


student would subtract one X-block from both sides of the balance scale. This is done by selecting the subtraction icon and typing the letter X or the number and letter 1X into the white box. Again, the student clicks on the *go* button. Finally, the student divides both sides by three using a similar process. Figure 3.3 displays the steps involved in this process.

This virtual manipulative has no predetermined sequence of operations that must be discovered. The student chooses the operation to be performed and after each operation the displayed equation is updated so that both the original equation and the latest equivalent form are seen together. Note that the student can choose to represent either side of the equation on either balance pan, and after pressing the *continue* button, the student works with the form of the equation thus selected. The only operations that are allowed are those that leave positive whole numbers as coefficients. Thus, for example, it is not possible to divide by 2 unless the numbers of unit-blocks and X-blocks on each side is even. The student must decide when the equation is solved; there are no whistles or bells when there is just one X-block appearing on one pan or the other and the student can continue on through another loop of operations if desired.

#### **Concrete solving of two-step linear equations.**

The concrete manipulative that was used for instruction in solving two-step linear equations was similar to the virtual computer manipulative in that it included a balance beam for instructional demonstration of solving equations. Materials for instruction included laminated sheets of paper and adhesive square plastic chips that represented an X-block and a unit block respectively. The square plastic chips were approximately one-tenth of a centimeter in thickness. For this reason, they are referred to as chips instead of

Solve for  $x$  using the operations below, keeping the beam balanced.

$$4x + 2 = x + 5$$



Subtract from both sides:

$$4x + 2 = x + 5$$

(a)

Solve for  $x$  using the operations below, keeping the beam balanced.

$$4x + 2 = x + 5$$

$$4x = x + 3$$


Subtract from both sides:

$$4x = x + 3$$

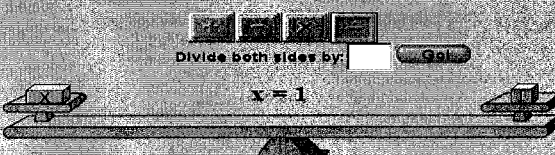
(b)

Solve for  $x$  using the operations below, keeping the beam balanced.

$$4x + 2 = x + 5$$

$$4x = x + 3$$

$$3x = 3$$

$$x = 1$$


Divide both sides by:

$$x = 1$$

Correct

(c)

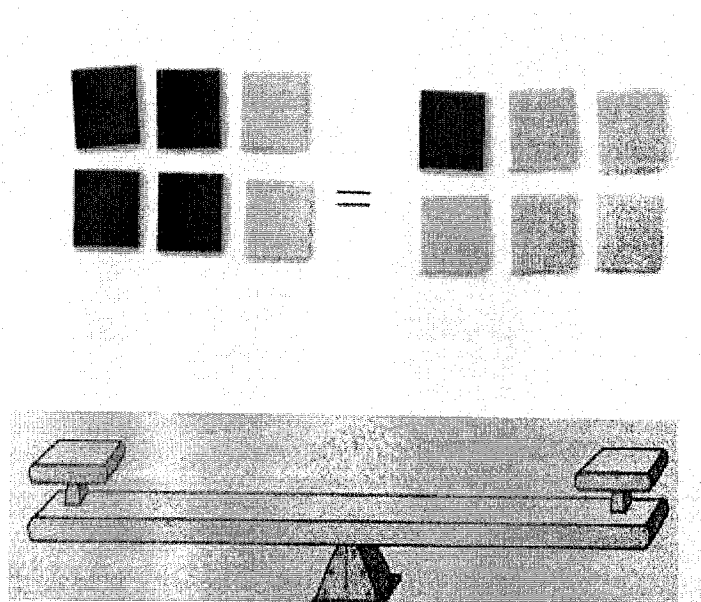
Figure 3.3. The completion of the process for solving a two-step linear equation on the balance scale applet. *Algebra Balance Scales*, National Library of Virtual Manipulatives, © 1999-2005 Utah State University.

blocks. All plastic chips were the same size. However, the X-chip was blue and the unit chip was yellow.

A poster board was set up at the front of the classroom to remind students that the blue chip was the X-chip and the yellow chip was the unit chip. Each student was given a sheet of laminated paper, an activity sheet, and twenty-five plastic square chips in each color. The laminated sheet of paper had the image of a balance scale at the bottom of the page.

The operation of subtraction was similar to what was performed on the virtual computer in that equal amounts of X-chips or unit chips were removed from both sides of the balance beam. However, the operation of division involved grouping chips according to the number of remaining X-chips and identifying how many unit chips were in a group with a single X-chip.

To set up a problem, students placed the chips on both sides of a large equal sign in the center of the laminated page. For example, if asked to solve the two-step linear equation,  $4x + 2 = x + 5$ , the student would begin by placing 4 blue X-chips and 2 yellow unit chips on the leftmost side of their equal sign. Next the student would place 1 blue X-chip and 5 yellow unit chips on the rightmost side of their equal sign. At this point, the problem would be correctly represented with the chips. Now, the student could begin the process of solving the equation. Figure 3.4 displays the proper set up of the equation  $4x + 2 = x + 5$  with the concrete manipulative.



*Figure 3.4.* The equation  $4x + 2 = x + 5$  set up on the concrete balance scale. Homemade manipulative created by the researcher with laminated sheet of paper and plastic detachable squares.

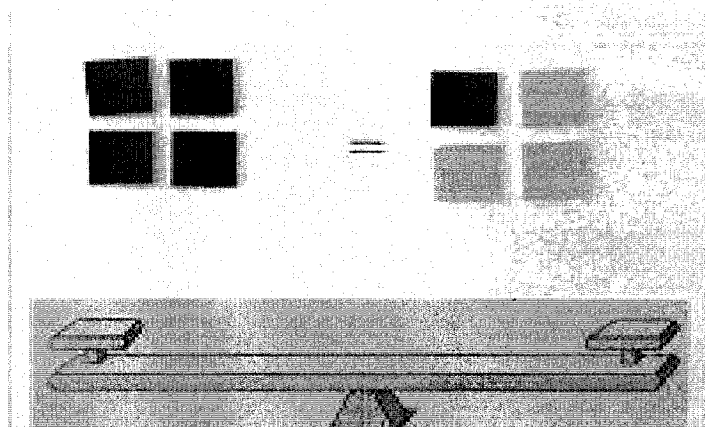
To solve the two-step linear equation, the student was asked to think of the problem in terms of the balance scale. First, the student would remove equal amounts of the same color chip from both sides. In our example, the student would remove two yellow unit chips from both sides of the equal sign. Next, the student would remove one X-chip from each side of the equal sign. The remaining chips include 3 blue X-chips on the leftmost side and 3 yellow unit chips on the rightmost side of the equal sign. At this point, the student can no longer remove chips of the same color from both sides. One side, the leftmost, has only X-chips and the other side, the rightmost, has only yellow unit chips. The student must recognize that the process of removing equal amounts is

complete and that for each blue X-chip there is exactly one yellow unit chip. Therefore, the final solution is:  $x = 1$ . Figure 3.5 illustrates these final steps.

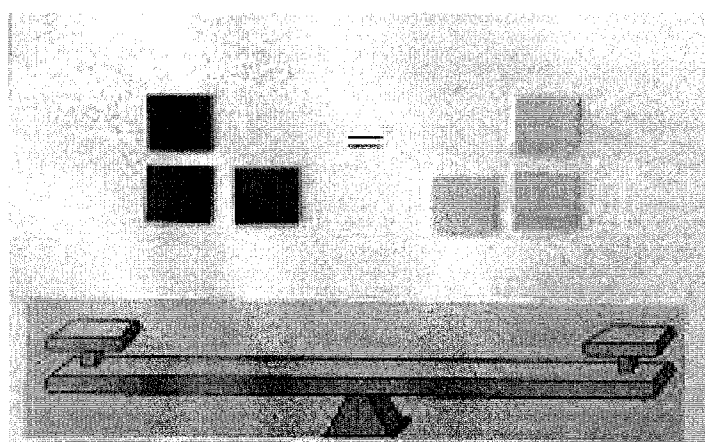
Although necessary with the concrete manipulative, the physical process for performing division was not clearly illustrated by the virtual manipulative. For example, the problem  $2x = 6$  requires division by 2. Students who used the concrete manipulative divided all the chips into two groups of equal color distribution. The answer was the number of unit chips in a group that included one X-chip. The same problem on the virtual manipulative required the student to select the division icon with the mouse and type the value 2 on the keyboard.

As with the virtual manipulative, the concrete manipulative had no predetermined sequence of operations. However, unlike the virtual manipulative, the student read the problem from their activity sheet and was asked to represent the initial setup with pencil and paper by drawing an X in appropriate provided squares. This allowed the researcher the opportunity to check that all student participants were correctly setting up each equation during the activity portion of the treatment. The virtual manipulative accomplished this by not allowing the student to continue unless the blocks were properly placed on the pads of the balance scale.

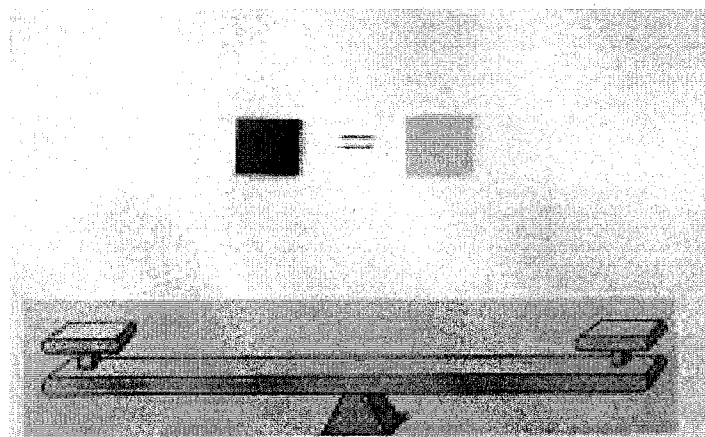
Every effort was made to make content of both the concrete and virtual manipulative as similar as possible. Some differences, such as the advantage of the virtual manipulative to place the scale in equilibrium without regard to the weight of the X-block or unit block, were impossible to overcome. However, content was mathematically the same. The activity sheet for classes using the concrete manipulative



(a)



(b)



(c)

*Figure 3.5.* The process of solving the equation  $4x + 2 = x + 5$  on the concrete balance scale. Homemade manipulative created by the researcher with laminated sheet of paper and plastic detachable squares.

was created from problems that were generated on the virtual applet. This would insure that the activity problems both groups encountered would be similar, if not identical.

### **Virtual algebra lab gear**

The virtual computer manipulative that was used for instruction in multiplying and factoring polynomial expressions is an applet located at the National Library of Virtual Manipulatives website: <http://www.matti.usu.edu/>. This applet can be reached by selecting the *Virtual Library* icon available under the topic of algebra grades 6-9, and choosing the option titled, *Algebra Tiles*. The manipulative created by the applet is very useful as an introduction to the physical representation of multiplication and division. However, the applet is limited to binomial operations, positive integers and variables, and factors with no integer exponents greater than one.

Using the *Algebra Tiles* virtual manipulative, the student can add tiles to a workspace, rotate tiles, change the length of the X or Y tiles, and delete tiles by putting them in a trash bin. To add tiles to the workspace the student simply clicks on any of the buttons below the workspace. The respective algebra tile instantly appears in the center of the workspace. Once the tile is added to the workspace, the student can click and drag the algebra tile to the desired location in the workspace. To rotate an algebra tile, the student must move the mouse over the corner of the tile. A round black dot appears. This dot can be used to rotate the tile. When the student places the mouse over the location of the dot and drags the mouse in a circular motion, the tile will rotate. When the student releases the mouse, the tile will snap into a vertical or horizontal orientation. The student can also change the length of the X- and Y-tiles. Two scroll bars located at the bottom of the workspace allow the student to adjust the size of the tiles. The scroll



bars are colored in a manner that matches the tile they resize. Finally, students can delete tiles by dragging the tile over a trash bin located in the lower right hand corner of the workspace. It is also possible to clear the screen by clicking on the *Clear* icon. Figure 3.6 illustrates the *Algebra Tiles* workspace.

To form a product of two polynomial expressions, the student must place one factor in the space just above the variable icons and the other factor in the space on the far left. When the student selects the icon of choice, the item appears in the center of the workspace. Then the student drags the item to the provided space. When both factors are in place, two lines outline a rectangle that forms the space where the product solution should be placed.

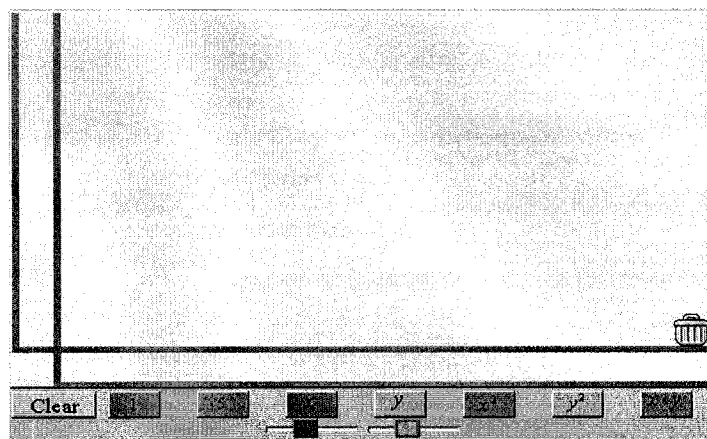


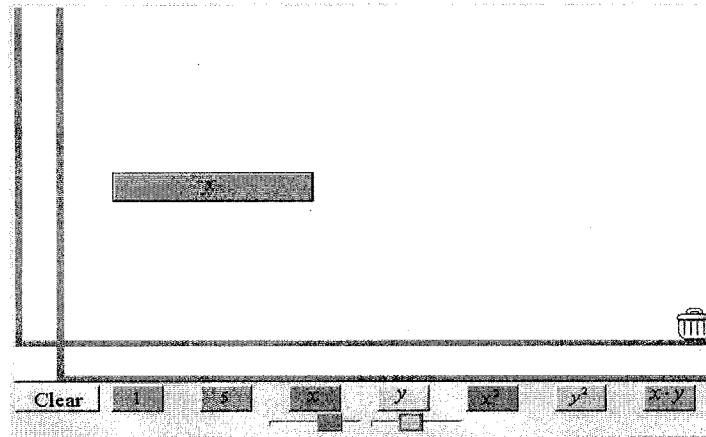
Figure 3.6. The workspace. *Algebra Tiles*, National Library of Virtual Manipulatives, © 1999-2005 Utah State University.

Figure 3.7 illustrates how a student would set up the problem:  $(2x + 1)(y + 3)$ . Here, the student has selected the  $x$  icon in the topmost image. In the middle image, the student has placed the factor  $(2x + 1)$  in the provided space above the variable icons. The student has placed the factor  $(y + 3)$  in the far left space in the bottom image.

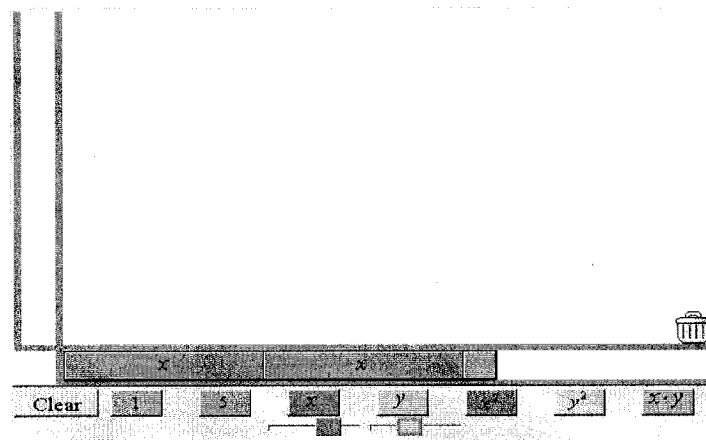
To find the product of  $(2x + 1)(y + 3)$ , the student must fill in the rectangular space with variables that correspond to the cross product of what is below and to the left of the rectangle. Depending on the particular problem, this may or may not require the student to rotate the variables after they are introduced into the workspace. Nevertheless, the solution will be the sum of all the tiles in the rectangular box.

This virtual manipulative also provided a method by which students could check that their solutions were correct. If students had already filled in the rectangle, they could check their work by adjusting the size of the tiles. The students knew that their solution were not correct if the tiles no longer fit into the outlining rectangle when the tiles were resized. Figure 3.8 illustrates how students, who incorrectly filled in the rectangle, may discover that their answer is not correct. In the image on the left, a student has filled in the rectangle with algebra tiles that do not correspond to the factors. In the image on the right, a student has used the scroll bars located on the bottom of the workspace, to resize the tiles. Notice that the tiles that once seemed to fit into the rectangular space no longer fit.

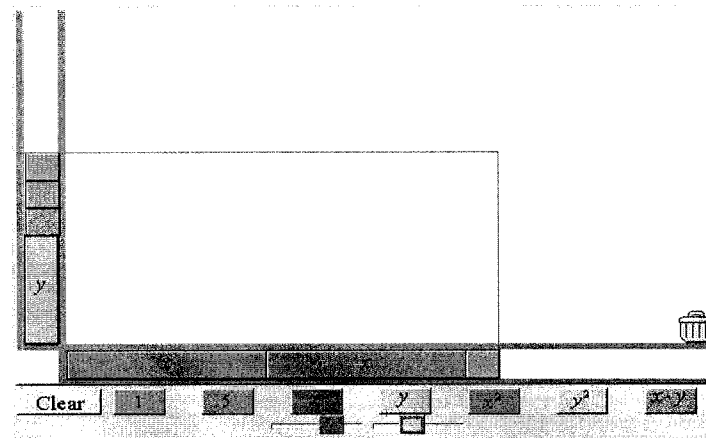
This applet was also used to factor polynomial expressions. Problems involving factoring required the student to form a rectangular shape. Then, based on the rectangular shape, the student had to determine the appropriate factors by filling in the space just above the variable icons and on the far left. The process of factorization is simply the reversal of the process of multiplication.



(a)



(b)

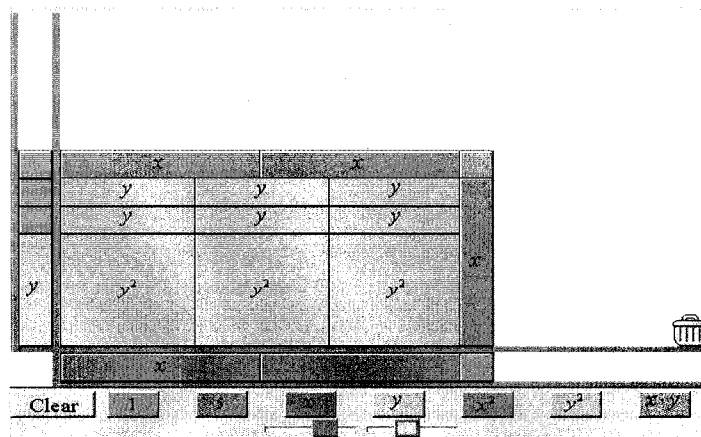


(c)

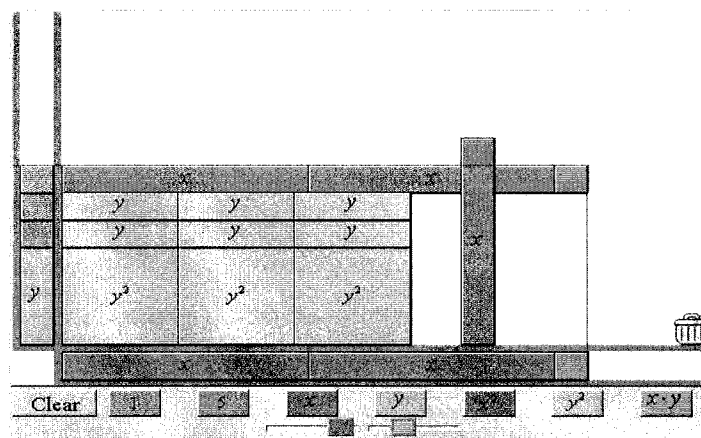
Figure 3.7. Algebra Tiles, National Library of Virtual Manipulatives©, 1999-2005 Utah State University.

### Concrete Algebra Lab Gear

*Algebra Lab Gear* is a manipulative that was developed by Creative Publications and is sold by Wright Group/McGraw-Hill 2004. Tiles in this commercial set are typically light blue and yellow in color. The concrete manipulative used in the instruction of multiplying and factoring polynomials was not this type of *Algebra Lab Gear*. Rather, the researcher used a homemade form of *Algebra Lab Gear* that was made out of wood and colored in a manner similar to the virtual manipulative.



(a)



(b)

Figure 3.8. *Algebra Tiles*, © 1999-2005 Utah State University.

The X-tiles were painted red, the Y-tiles were painted blue, the unit tiles were painted green, and the XY-tiles were painted purple. The homemade set also included a wooden L-shaped frame. It did not, however, include the string of 5 unit tiles. Each student participant was provided with a homemade set of tiles that included six red X-squared-tiles, six blue Y-squared-tiles, eight X-tiles, eight Y-tiles, six XY-tiles, and fifteen unit tiles.

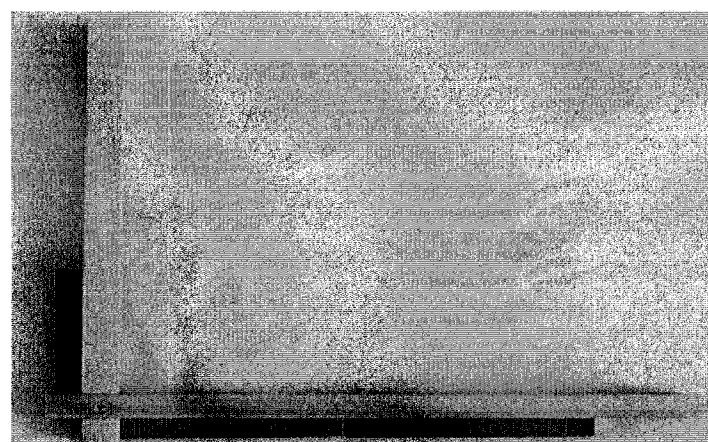
Students who used the concrete manipulative to multiply or factor polynomials did so by moving their tiles, with their hands, into their appropriate positions around the L-shaped wooden frame. For example, a student who wanted to find the product of  $(2x + 1)(y + 3)$  would begin by placing two X-tiles and one unit tile below the horizontal bar of the L-shaped wooden frame. Next, the student would place one Y-tile and three unit tiles on the leftmost side of the L-shaped wooden frame. Finally, the student would indicate multiplication by creating a rectangle that fit into the space outlined by the lower and leftmost factors.

Figure 3.9 illustrates this process with the concrete manipulative. In the image on the top, the student has placed two X-tiles and one unit tile below the axis on the L-shaped wooden frame. In the middle image, the student has placed a single Y-tile and three unit tiles in the leftmost region of the L-shaped wooden frame. The bottom image displays the final step in finding the solution. That is, the tiles that form the interior rectangle represent the expression:  $2xy + 6x + y + 3$ .

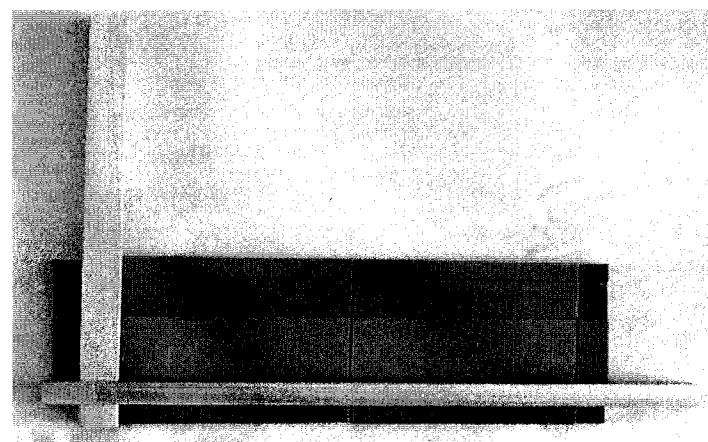
To factor an expression, say,  $2xy + 6x + y + 3$ , the student would begin with 12 tiles. Namely, 2 XY-tiles, 6 X-tiles, 1 Y-tile, and 3 unit tiles. Next, the student would try to form a rectangle in the interior of the L-shaped wooden frame. Once this is done, the



(a)



(b)



(c)

*Figure 3.9.* The process of finding the product of  $(2x + 1)(y + 3)$  with the concrete manipulative. Homemade manipulative created by the researcher with colored wooden pieces.

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student places the appropriate tiles below and to the left of the L-shaped wooden frame to determine the factors of the polynomial expression.

The concrete manipulative was designed to be as similar to the virtual computer manipulative as possible. However, obvious differences exist. First, the concrete manipulative cannot vary in size. Second, a rectangular outline did not appear on the concrete manipulative when the factors were placed below or to the left of the L-shaped wooden frame. Third, the length of the Y-tile was similar to the length of three unit tiles and could therefore cause the confusion that Y is equal to 3. These differences tend to favor the virtual computer manipulative. However, the concrete manipulative offered a tactile experience that could reasonably make the experience more meaningful for some learners.

### **Lesson plans**

One lesson plan was designed for each topic: solving two-step linear equations and multiplying and factoring polynomial expressions (see Appendix F). Regardless of manipulative type, instruction included the use of a computer applet in the computer lab. That is, instruction on the topic of how to solve a two-step linear equation was given with the use of the *Algebra Balance Scales* applet but included the demonstration of one problem on a wooden balance scale. Instruction on the topic of how to multiply or factor polynomial expressions was given strictly with the use of the *Algebra Tiles* applet.

The researcher prepared the lab so that it would be ready for use prior to each presentation. This preparation involved setting out the objects when a concrete manipulative would be used in instruction. In the case of the virtual manipulative, preparation involved turning on the computers and selecting the applet Website.

Students entered the computer lab and sat down behind any one of eight rows of computers. The computer lab has a total of 40 computers. Thus, each student had their own computer and a small desk area about which they could work.

Instruction was aimed at semantic analysis outlined in Chapter 2. Semantic analysis is similar to mapping instruction. Both approaches “aim to help students make sense of symbolic rules or algorithms by connecting the rules with the referents” (Wearne & Hiebert, 1988, p. 381). However, instruction aimed towards semantic analysis begins by establishing the meaning of individual symbols and spends a major portion of the instructional time connecting symbols with referents. Actions on referents are then used to generate procedures with symbols. Some rules, such as add the product of the first, outer, inner, and last terms, are not needed as independent rules in the syntactic system. According to Wearne and Hiebert (1988), the differences between the instructional approaches might be summarized by noting that the goal of semantic analysis instruction is to promote the analysis of symbol expressions in terms of meaningful referents and thereby eliminate the need for many syntactic rules, whereas the goal of mapping instruction is to help students understand syntactic rules and apply them appropriately.

Each lesson was activity based. The lecture portion of instruction lasted no more than 10 minutes. The remaining 80 minutes of class time was dedicated to the treatment activity. This design was chosen in part because of bias in favor of action-based learning experiences. It is felt that some students learn best when they are engaged in mental activities that require them to move objects. Another reason for this design is to eliminate, as much as possible, the influence of instructional abilities.



At the end of the short presentation, an activity sheet was passed out to the students. Once the students began the activity, the researcher walked around the room checking the work of each student. The researcher checked a box on the activity sheet with her initials. The researcher initialed a single problem on the activity sheet when the student presented a visual representation of the solution and had correctly written the answer on her/his activity sheet. This process was used to encourage students to use the manipulative to solve problems.

### **Pretest and posttest questions**

Pretest questions addressed the students' prior experience using manipulatives, recollection of prior instruction in the content area, and content specific problem solving abilities. Student participants were asked about previous instruction with manipulatives to validate the underlying assumption that many of the participants have not previously experienced content from the perspective of the manipulative. Questions on recollection of prior instruction were given to validate student background experience in the topic.

Content specific problems are problems that require the same skills as those that the student is asked to perform in the treatment activity. Pretest problems in the content area were given to provide a measure of how much background experience in the topic the student possessed when entering the computer lab. Upon entry in the computer lab, each student picked a seat. In front of each seat was a fully equipped computer that was previously turned on, logged into the appropriate website, and ready to use. Regardless of treatment type, the concrete manipulative was also placed by the computer. Each student had these resources when taking the pretest. Students were neither encouraged nor discouraged from using either manipulative to solve problems.

Posttest content questions repeated the pretest content questions in that they required the same skills to complete. The process for giving the posttest was similar to the pretest. Students were escorted to the computer lab. Upon entry into the computer lab, each student picked a seat. In front of each seat was a fully equipped computer that was previously turned on, logged into the appropriate website, and ready to use. The concrete manipulative was placed by each computer. Each student had these resources when taking the posttest. The posttest also included questions with regard to student attitudes about the instructional methods incorporated in the administration of the treatment and a final question addressing the students' willingness to participate in the interview process.

Each content specific problem included the image of a square box on both the pretest and posttest. Under the box were words indicating the use of the manipulative. The students were told to place a check mark into the box if they used a manipulative to solve the problem or thought of objects and used these images to solve the problem. It was hoped that this would serve as a measure of how many students were using a manipulative to solve problems on both the pretest and posttest.

### **Solving two-step linear equations**

The pretest and posttest given to students asked to solve two-step linear equations had 7 content specific problems. Problems on the posttest were designed to be similar to the pretest problems. Content specific problems on the pretest were randomly selected by the *Algebra Balance Scales* applet that was used as an instructional aide in all classes that received treatments involving finding the solution to two-step linear equations.

Content problems on the pretest and posttest of students receiving instruction in solving two-step linear equations differed only in that some problems required division to complete while others simply required removing equal quantities from both sides of the equal sign. Thus, the level of difficulty of problems had a small margin of variation within the tests themselves. All problems required a reasonably equal amount of mathematical skill to solve. This design was adopted because students who received the treatments in solving two-step linear equations are typically younger and have less background experience than students who received instruction in multiplying and factoring polynomial expressions.

### **Multiplying and factoring polynomial expressions**

The pretest and posttest given to students in multiplying and factoring polynomial expressions had 10 content specific problems. Problems on the posttest were designed to be similar to the pretest problems. Unlike the pretest and posttest given to students in solving two-step linear equations, the pretest and posttest of this topic required a variety of mathematical skills. For example, both tests had problems involving finding the product of a monomial and a binomial, two binomials, binomials in one variable, and binomials in two variables. They also included factoring of expressions with two, three, four, and five unlike terms.

These tests were designed to cover several concepts because students who received this pretest or posttest were typically students with a wide range of background knowledge and skill in simplifying polynomial expressions. Additionally, the lesson on multiplying and factoring polynomial expressions focused on relationships between the two operations and the similarities of concrete procedural approaches to solving both

types of problems. This relationship between multiplication and factoring is represented by the process of forming a rectangle from two factors in multiplication and the reverse process of forming a rectangle and then determining the factors based on the length and width of the formed rectangle. Both the virtual and concrete manipulatives were available to the student to use during the pretest and posttest. Treatments and instruction provided students with an activity using one type of manipulative to explore content relationships within the content area.

### **Data analysis methods**

Quantitative data were collected from pretest and posttest questions given to all student participants. Of interest is whether or not instruction with the use of a manipulative improved student accuracy or student attitudes on the use of a manipulative, and whether or not students tended to answer questions differently based on the type of manipulative. Accuracy was measured in terms of the mean number of correct answers on the pretest and posttest for each class. Student accuracy was measured in terms of achievement groups. Attitudes were measured in terms of weighted mean Likert scale class values. Finally, student incorrect answers were grouped and compared by placing identical incorrect solutions into researcher defined categories.

### **Background knowledge categories**

Students who had previous knowledge of the correct methods for solving problems were not able to demonstrate improvement. That is, any student participant who scored 100% on a pretest had no room to demonstrate improvement on the posttest. Since the quantitative measurement of treatment effect is largely based on this improvement score, the inclusion of these students makes it more difficult to identify

variables that are possibly essential to determining treatment effects. For this reason, analysis will be conducted on specific categories of pretest score.

Category divisions were chosen based on the approximate percentage equivalence. For example, students who scored 6 or 7 out of 7 problems on the pretest in solving two-step linear equations were thought to have mastered the concept prior to taking the pretest. These students scored 86% or more on the pretest. Similarly, students who scored 8, 9, or 10 out of 10 problems on the pretest in multiplying and factoring polynomial expressions answered approximately 80% or above correctly. These students were grouped together in a category indicating students who had reached topic mastery prior to taking the pretest. This grouping facilitated the evaluation of overall success by providing a measure of background knowledge for each student.

Students who received instruction on solving two-step linear equations were divided into three categories according to a pretest score of 7 possible points;

1. NO KNOWLEDGE - SCORE 0 OR 1

Assume the student participant has no sufficient understanding of how to solve two-step linear equations.

2. SOME KNOWLEDGE - SCORE 2, 3, 4, OR 5

Assume the student participant has some background understanding of how to solve two-step linear equations.

3. MASTERY - SCORE 6,7

Assume the student has mastered the concept of how to solve two-step linear equations with at least one type of strategy.

Students who received instruction on multiplying and factoring polynomial equations were divided into the same three categories according to a pretest score of 10 possible points;

1. NO KNOWLEDGE - SCORE 0, 1, 2

Assume that either the student participant has no sufficient understanding of how to multiply polynomials or the student participant has no sufficient understanding of how to factor polynomials.

2. SOME KNOWLEDGE - SCORE 3, 4, 5, 6 OR 7

Assume that either the student has some background understanding of how to multiply polynomial expressions or some background understanding of how to factor polynomial expressions or both.

3. MASTERY - SCORE 8, 9, OR 10

Assume the student has mastered the concept of how to multiply and factor polynomial expressions with at least one type of strategy.

The sequencing of instruction is inherently bound to the use of manipulatives. Studies indicate the learning outcomes that result from instruction that adopts manipulatives differ among students who have experienced procedural/symbolic instruction and those who have not (Thompson, 1992).

The student background knowledge is related to the issue of sequencing. More than likely, students who demonstrated mastery of content on the pretest have had some instruction in the topic in their prior academic years of education. It is reasonable to assume that this instruction was primarily procedural if the student agrees, in the form of a pretest question, that they have not received instruction that utilized a manipulative. By

default, students who demonstrated mastery on the pretest and indicated not having experienced the topic with the use of a manipulative were receiving instruction that began with a procedural approach prior to the introduction of methods that incorporate the use of a manipulative.

On the other hand, students who score 0, 1, or 2 problems on a pretest may or may not have received prior instruction. These students were also asked if they had received prior instruction with the use of a manipulative. It is not reasonable to assume these students received prior symbolic instruction of content. Thus, within this pool of students are the students who received initial instruction in the form of a manipulative.

### **Statistical methods**

This study uses analysis of covariance to determine if the sequence of instruction involving the two manipulatives, concrete and virtual computer, has any effect on improvement in test score from pretest to posttest. Analysis of covariance (ANCOVA) is a statistical technique that allows examination of the effect of an explanatory variable on a response variable, while removing the effects of other variables. In this case, interest lies in the improvement students made in their test scores after receiving instruction under one of the four manipulative sequences. However, there are other factors that contribute to an improvement in test score. These factors, or covariates, may include familiarity with computers, stage of learning, classroom teacher, class level, and the age of the student. ANCOVA will look at the effect of the manipulative teaching sequence treatment after accounting for the effects of the covariates.

Next, students are divided into background knowledge categories. ANOVA procedures were performed for each background category to determine differences

among students based on their pretest scores. The hypothesis is that students who scored above 80% on the pretest had more background experience or ability with the topic than those who scored below 25% on the pretest. This analysis explores the issue of whether the manipulative type was more or less effective among students of varying background experience and ability in both topics.

Answers to posttest student opinion questions will be categorized according to a Likert scale. Student attitude towards the use of the concrete or virtual manipulative may influence learning outcomes. Chi-square contingency tables compare the concrete and virtual computer manipulative in terms of overall student attitude. This test provides some indication of whether student opinion is a factor in the final overall results.

Close examination of the accuracy of each pretest and corresponding posttest problem is considered for both topics. There are four achievement groups for each test problem. The first group represents a count of students who incorrectly answered a question on the pretest but correctly answered the corresponding posttest question. The second group represents a count of students who correctly answered the pretest but incorrectly answered the corresponding posttest question. The third group represents a count of students who correctly answered both the pretest and corresponding posttest question. The fourth group represents a count of students who incorrectly answered both the pretest and corresponding posttest question. Chi-square analysis is used to determine if these counts were significantly different between the two manipulative types for each type of question.



### **Interview methods**

Thirty-eight task-based interviews were given to two or three student participants in each class. These interviews were 20 to 45 minutes in length and occurred within 10 days of the treatment. Students were asked questions similar to those on the activity sheet they received during the presentation (see Appendix G). They were also asked questions beyond the scope of what was covered in class. The researcher consistently asked students to solve the interview problems using more than one method. Interviewed students were asked to say what they were thinking while solving problems.

Qualitative analysis focused on the semantic processes involved in making connections between the manipulative and the corresponding symbolic representations and procedures. This approach was based on semantic analysis categories developed by Wearne and Hiebert (1988) that are outlined in the literature review of this dissertation. The analysis process involved the researcher making the determination if the student was able to connect the concrete or virtual manipulative to its corresponding symbolic representation. The researcher also determined if the student was able to identify actions on the concrete or virtual manipulative as analogous to mathematical operations. Finally, the researcher observed whether students who had incorrect answers used the manipulative to correct or guide their symbolic procedures.

Interviews with students who received instruction in solving two-step linear equations were omitted from the final analysis and were not included here. This was done because of unforeseen discrepancies between the treatments. Students who received instruction with the virtual manipulative did not directly receive instruction in the physical representation of division. Students who received instruction with the concrete

manipulative had to physically represent division by grouping X-chips with unit chips in such a way that each group included exactly one X-chip. Instruction in this representation of long division with the chips typically was given as one-on-one communication between the researcher and the student or the teacher and the student during the course of the concrete manipulative activity. It was mistakenly assumed that students in all treatment groups would be able to connect the symbolic representation of long division to its analogous concrete representation.

The difference between the instructional activity of the concrete and virtual manipulative in the topic of solving two-step linear equations is related to the connecting process outlined by Wearne and Hiebert (1988). Direct comparisons of the connecting process were inappropriate within these interviews because students who used the concrete manipulative may have received instruction on the physical representation of long division while students who received instruction with the virtual manipulative did not. It was also impossible to compare whether the student was able to represent symbolic division with the virtual manipulative because students could not freely move the blocks on the virtual manipulative into groups. Thus, analysis of student connecting processes for the topic of solving two-step linear equations was omitted from the final qualitative results.

Twenty-three interviews were performed on the topic of multiplying and factoring polynomial expressions. Interviewees selected for final analysis were chosen based on whether or not a single incorrect solution strategy was adopted during the course of the interview. The only exception to this rule was one question that required the student to factor a trinomial of negative terms ( $4x^2 - 2x - 2$ ). This question cannot be answered

using the provided form of concrete or virtual Algebra Lab Gear. It was included strictly as an indicator of student background knowledge in procedural techniques of multiplying and factoring polynomial expressions. If the student was able to factor this problem using a procedural approach, then it is not unreasonable to assume that the student had considerable background instruction in the symbolic procedures of multiplying and factoring polynomial expressions.

In a similar study with fourth and fifth grade students using Dienes base-10 blocks by Wearne and Heibert (1988), interview questions were designed to discriminate between semantic analyzers and syntactic-rule appliers. This was accomplished with the inclusion of problems that would be difficult to solve unless the student engaged in semantic analysis. Students were credited with using the semantic processes if they referred to the values of the numerals (either read or written) in explaining how they decided what to do. In this study, students are credited with using semantic processes if statements refer to the manipulative for symbols, description of quantities, or choice of procedural technique.

Direct and transfer measures were used to model this approach. Direct measures are tasks that had been discussed and practiced during instruction. They assessed students' use of the processes in familiar context. Specifically, these were tasks that involved finding the product of two binomial expressions or factoring a trinomial expression. Such problems discriminate most clearly between students who use semantic analyses and those who recall and execute syntactic rules. Transfer measures involved problems that were not taught in the lesson and would be difficult to solve using syntactic

procedures. For example, interviewed students were asked to factor the polynomial expression  $xy + y^2 + y + x + y + 1$ . A complete list of interview questions is in Appendix H.

Every student selected for an interview was asked to solve problems using two techniques of strategy. The two assumed techniques would be a procedural approach towards problem solution using paper and pencil or solving the problem with the use of the manipulative. The student was provided with the manipulative that was used in the activity of the treatment they received. This manipulative was placed on the desk in front of the student. If the student used the computer manipulative, the computer was turned on prior to the interview logged into the appropriate website.

Each problem was presented to the student on a clean sheet of paper. The student began by solving the problem with their own preference on the method of solution. The student was asked to say their thoughts out loud as they were working on each problem. After the student completed the problem with their chosen method, the student was asked to repeat the problem using the unselected solution strategy.

Interviews took place in the computer lab. This was an isolated room that extended from the library and was reserved for the interview sessions. It is also the room where the student received treatments. Although distractions did occur, they were relatively rare and infrequent.

## Chapter 4

### Quantitative Research Findings

Quantitative results are generated using comparisons among the 14 classes involved in this study. The overall treatment effect is measured with the use of class averages and proportions of students who responded in a specific manner to Likert scale attitude questions. A class level analysis was performed because the teaching method treatments were applied to entire classes instead of individual students and hence classes were the experimental units. Each class had a respective mean pretest score, mean posttest score, class level, manipulative type, topic, and teacher. The mean pretest and mean posttest scores were proportions of correct responses. This minimized the effect of having a different number of questions on tests in the two topics. Unless stated otherwise, inferential analysis is two-tailed at an  $\alpha = 0.05$  significance level.

#### Analysis of Covariance

Analysis of covariance (ANCOVA) was performed to compare manipulative treatments between classes. The response variable is the mean posttest score for each class. Covariates were the mean pretest score, class level, and teacher. The selection of the mean pretest score as a covariate was based on the belief that students who performed well on the pretest will likely perform well on the posttest. The class level was also considered as a covariate because students at higher class levels are more likely to have higher pretest and posttest scores. The teacher is also included as a potential covariate to account for any pre-existing differences in student ability due to teacher differences.

The design employed was a two factor  $4 \times 2$  ANCOVA. The first factor was the manipulative sequence at four levels: symbolic-computer, symbolic-concrete, computer-symbolic, and concrete-symbolic. Each level represents a sequence of instruction for a particular manipulative type. The second factor was the topic at two levels: solving two-step linear equations and multiplying and factoring polynomial expressions. All ANCOVA assumptions were met as indicated by Levene's test of equality of error variance and approximate normality of the errors.

Results of ANCOVA indicate that class mean post-test scores are not significantly different among teachers, topics, class level, and treatments, but are significantly associated with mean pretest scores. ANCOVA results are provided in Table 4.1.

There was a statistically significant improvement from mean pretest scores to mean posttest scores among all treatments ( $p = 0.021$ ). Classes of students who received treatments beginning with symbolic instruction tended to have mean pretest scores that were slightly higher than classes who received treatments that began with the use of a concrete or virtual manipulative. For example, the class mean pretest score for treatments that began with symbolic instruction and ended with instruction including the virtual computer manipulative was 3.823. This means, on average, students in this treatment group correctly answered approximately 4 out of 10 questions on the pretest. On the other hand, the class mean pretest score for treatments that began with instruction using the concrete manipulative was 1.538. This result is expected since higher level math classes were selected for treatments that began with symbolic instruction.

Table 4.1

Analysis of Covariance for Treatment Effect on Mean Posttest Score

Source	<i>df</i>	$\eta$	<i>F</i>	<i>p</i>
Treatment	3	.346	0.625	.624
Teacher	3	.011	0.019	.895
Topics	1	.044	0.080	.787
Mean Pretest Score	1	5.395	9.745	.021
Class Level	1	.563	1.016	.352
Error	6	.554		

Figure 4.1 illustrates the change in mean pretest score to mean posttest score among the four treatments. In each case, the mean score increased by approximately 3 to 4 points. The slope of each line represents the change in class mean test scores from pretest to posttest. The slopes are approximately equal in that they do not differ by an amount that is statistically significant. Hence, all treatments were effective in that they significantly improved the overall class mean scores, but did not differ from one another.

### **Manipulative use for students of differing mathematical abilities**

This analysis will focus on achievement measures within the predefined pretest categories: *No Knowledge*, *Some Knowledge*, and *Mastery*. This division is justified in that most secondary classrooms contain a mixture of students that bring a variety of

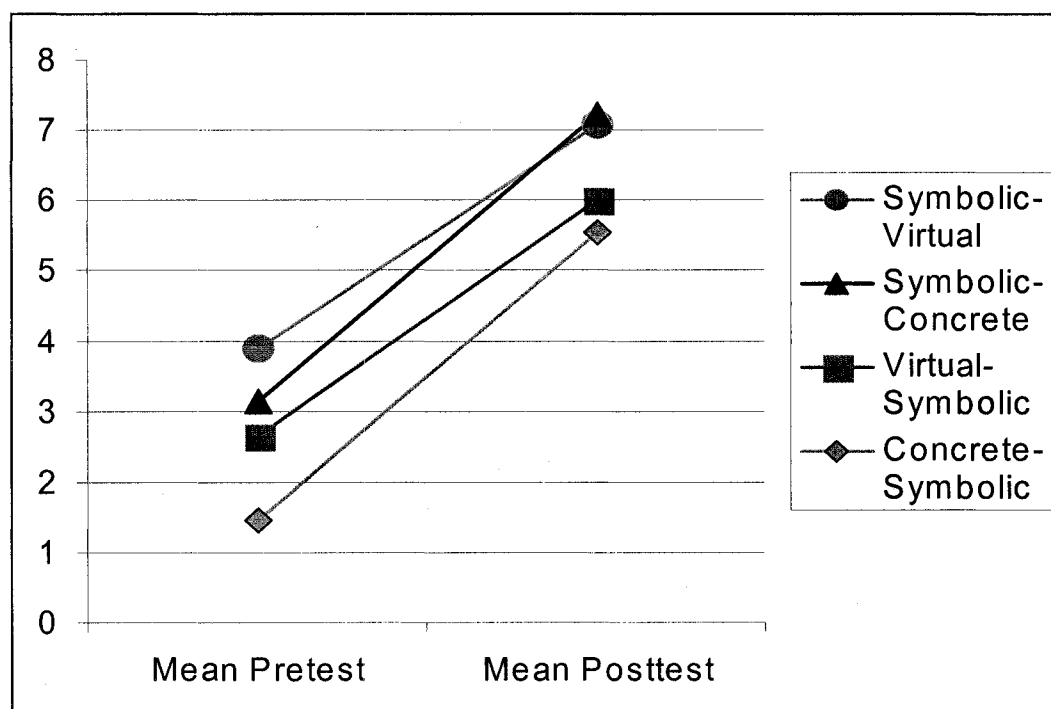


Figure 4.1. Comparison of class mean pretest and mean posttest scores among treatments.

background experiences. Among the two manipulative types, concrete and virtual computer, it is of interest to know if there are any differences in the increased accuracy of pretest to posttest responses for classes of students that have different mathematical abilities.

The mean pretest and posttest scores were recalculated within each category for each class. Classes that contained zero students in a category were omitted from analysis. This omission is because without any students in any one category, the opportunity for mean improvement does not exist. The separation of classes by knowledge category created 35 class groups of students: 13 classes of students in the *No Knowledge* category, 13 in the *Some Knowledge* category, and 9 in the *Mastery* category. Comparisons are



based on the difference in the means of the proportion of correct responses. This difference reflects the mean proportional increase in test score from pretest to posttest and is referred to herein as the *mean improvement proportion*. Table 4.2 depicts the student per class breakdown and mean improvement proportion when classes are separated according to knowledge categories.

Table 4.2  
Number of Students in Each Knowledge Category  
and Mean Improvement Proportion (N=304)

<u>Class Number</u>	<u>Treatment</u>	<u>No Knowledge</u>		<u>Some Knowledge</u>		<u>Mastery</u>	
		<u>N</u>	<u>M</u>	<u>N</u>	<u>M</u>	<u>N</u>	<u>M</u>
1	4	18	.305	(0)	--	(0)	--
2	1	11	.897	3	.286	8	.000
3	2	10	.700	2	.071	2	.143
4	3	13	.423	12	.383	(0)	--
5	2	5	.540	15	.353	1	.100
6	1	13	.462	11	.300	1	-.100
7	4	17	.576	2	.300	(0)	--
8	4	9	.937	5	.571	9	.048
9	3	12	.750	1	-.143	6	.000
10	2	18	.533	8	.462	(0)	--
11	3	12	.726	5	.229	7	-.061
12	2	(0)	--	23	.382	2	.150
13	1	3	.700	11	.236	6	.033
14	4	17	.541	6	.083	(0)	--

*Note.* Values enclosed in parentheses represent groups of zero students that are omitted from analysis. Treatments are: (1) Symbolic-Computer Manipulative, (2) Symbolic-Concrete Manipulative, (3) Computer Manipulative-Symbolic, and (4) Concrete Object Manipulative-Symbolic.

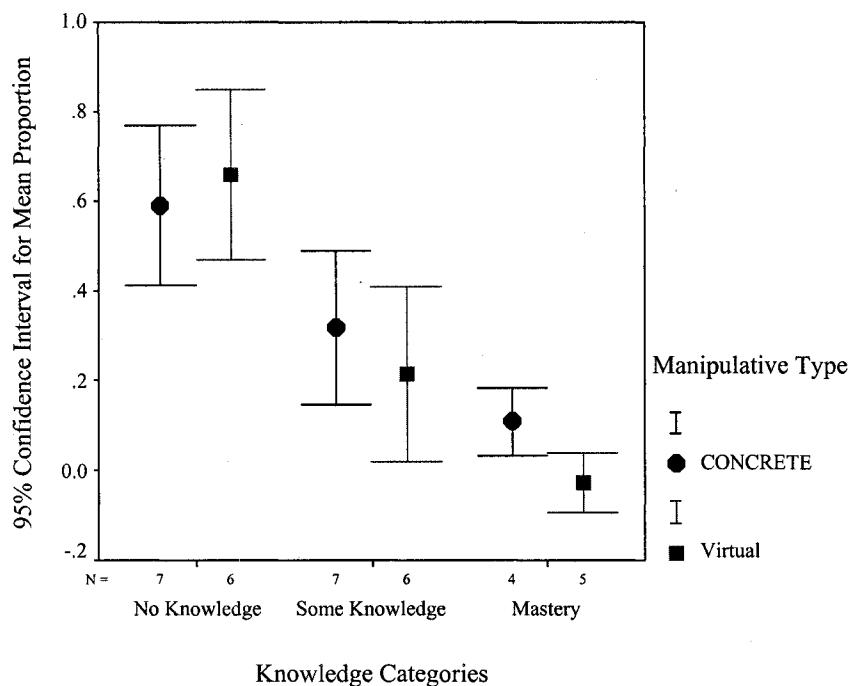
Each of the mean improvement proportions were weighted according to class size.

The design employed was a two factor  $2 \times 3$  ANOVA. The first factor was the

manipulative at two levels: concrete or virtual computer. The second factor was the knowledge category at three levels: *No Knowledge*, *Some Knowledge*, or *Mastery*. Results indicate no significant differences in mean improvement scores between manipulative types in the *No Knowledge*, *Some Knowledge*, and *Mastery* categories ( $F = 0.030, df = 1, p = 0.863$ ).

There were significant differences ( $p < .0001$ ) in mean improvement proportions among knowledge categories. This result is not surprising due to the fixed number of questions on the pretest and posttest, classes of students in the *Mastery* category did not have as much potential for improvement as classes of students in the *No Knowledge* category. For example, a student who scored 8 out of 10 correct problems on the pretest would be placed in a class of students in the *Mastery* category. The highest possible score this student could achieve on the posttest would be 10 out of 10 correct problems for an overall improvement of 2 problems. A student placed in a class in the *No Knowledge* category could improve as many as 9 or 10 problems. Figure 4.2 illustrates the mean improvement proportion of each manipulative within each knowledge category.

Bonferroni pairwise comparisons indicate statistically significant differences between knowledge categories and the weighted mean improvement proportions: *No Knowledge* versus *Mastery* ( $p < 0.001$ ), *Some Knowledge* versus *Mastery* ( $p = 0.003$ ), and *No Knowledge* versus *Some Knowledge* ( $p < 0.001$ ). Classes in the *No Knowledge* category had greater opportunity to improve their scores than classes in the *Some Knowledge* category.



*Figure 4.2.* Ninety-five percent confidence intervals for weighted mean improvement proportions among different knowledge categories.

### Student attitudes and manipulative use

There were 3 posttest opinion questions. Questions were the same for all treatments. Posttest opinion questions asked students to rank their experience on a Likert scale. A Likert scale measures the extent to which a person agrees or disagrees with the question. Responses were ranked on a scale from 1 to 5. Numerical values were assigned to responses as follows; 1=strongly disagree, 2=disagree, 3=neutral, 4=agree, and 5=strongly agree. Students who did not answer were considered to be neutral in their opinion of the lesson. Figure 4.3 shows the three opinion questions that appeared on each posttest.

- 
- A. You have experienced a math lesson that uses a computer or physical manipulative. Did you enjoy learning mathematics this way?
- Strongly Disagree    Disagree    Agree    Strongly Agree
- B. Do you feel it is practical for you to learn mathematical content using this type of instruction?
- Strongly Disagree    Disagree    Agree    Strongly Agree
- C. Do you feel it is practical for others to learn mathematical content using this type of instruction?
- Strongly Disagree    Disagree    Agree    Strongly Agree
- 

*Figure 4.3.* Posttest opinion questions A, B, and C.

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Students were divided according to manipulative type to determine the exact count of students within each response category of opinion questions A, B, and C. Students who did not respond to any of the 3 questions were omitted from analysis. The remaining data were recoded into two categories. This division was necessary for sufficient counts for each category. The first category included students whose response indicated that they *Strongly Disagree* or *Disagree*. The second category included students who selected the *Agree* or *Strongly Agree* options. Students were thus split according to whether they agreed or disagreed with posttest opinion questions A, B, or C. Contingency tables for each posttest question are given in Figure 4.4.

Chi-square tests of independence in the contingency tables indicate no significant relationship between the manipulative type and the student response for posttest opinion questions B ( $\chi^2 = 2.726, df = 1, p = 0.099$ ) and C ( $\chi^2 = 0.267, df = 1, p = 0.605$ ). This

Count

		Manipulative Type		Total
		Concrete	Virtual	
Posttest Opinion	Disagree	33	11	44
Question A	Agree	135	122	257
Total		168	133	301

Count

		Manipulative Type		Total
		Concrete	Virtual	
Posttest Opinion	Disagree	35	18	53
Question B	Agree	133	115	248
Total		168	133	301

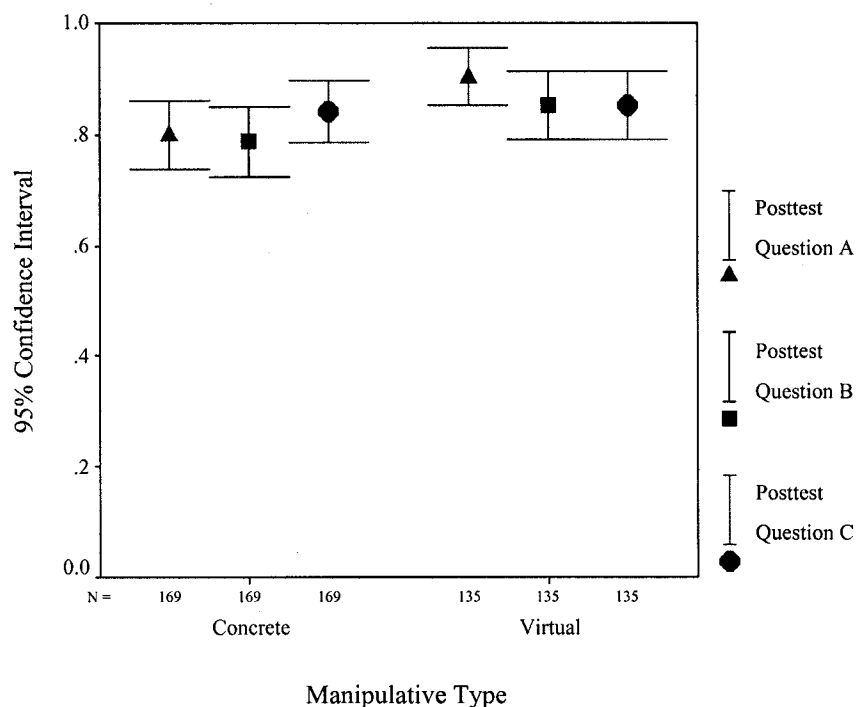
Count

		Manipulative Type		Total
		Concrete	Virtual	
Posttest Opinion	Disagree	25	17	42
Question C	Agree	142	115	257
Total		167	132	299

Figure 4.4. Contingency tables for posttest opinion questions A, B, and C.

implies that student opinion on the issue of whether or not it was practical for them or others to learn mathematical content with a manipulative was not related to the type of manipulative that was used in instruction.

Results of the chi-square analysis were statistically significant for posttest opinion question A ( $\chi^2 = 7.692, df = 1, p = 0.006$ ). In this case, there was a relationship between the manipulative type and student response. In fact, a greater proportion of students agreed with the statement that they enjoyed learning mathematics this way when the manipulative was presented in the form of a virtual computer Java applet instead of a concrete manipulative (92% versus 83% respectively). Figure 4.5 illustrates the 95%



*Figure 4.5.* Ninety-five percent confidence intervals for favorable responses to posttest opinion questions A, B, and C among manipulative types.

confidence intervals for the proportion of favorable responses within each manipulative type.

### **Relationship between achievement measures and manipulative type**

This portion of the analysis will provide investigation of achievement measures for specific problem solving skills. In terms of accuracy, it may be that one type of manipulative was more beneficial than another for a specific problem solving skill. Examination of pairs of problems (pretest and posttest) indicate that some students correctly answered both the pretest and posttest question, some students incorrectly answered both the pretest and posttest question, some students correctly answered the

pretest question but failed to correctly answer the posttest question, and some students incorrectly answered the pretest question but correctly answered the posttest question. These groups of students determine the overall accuracy of response and will henceforth be called achievement groups. The contingency table in Figure 4.6 provides the number of entries in each achievement group by manipulative. There were 179 students included in treatments in the topic of multiplying and factoring polynomial expressions. Each student answered 10 questions.

**Multiplying and Factoring Polynomial Expressions**

			Achievement Groups for Ten Problems				Total
			Correct Pretest Incorrect Posttest	Incorrect Pretest Correct Posttest	Correct Pretest Correct Posttest	Incorrect Pretest Incorrect Posttest	
Manipulative	Concrete N=109	Count %	22 2.02%	483 44.31%	263 24.13%	322 29.54%	1090 100.00%
	Virtual N=70	Count %	31 4.43%	265 37.86%	229 32.71%	175 25.00%	700 100.00%
Total		Count	53	748	492	497	1790

*Figure 4.6.* Contingency table of achievement groups for the topic of multiplying and factoring polynomials.

The proportion of questions given to students who correctly answered a pretest question but incorrectly answered the same posttest question is considerably higher among students who received instruction with the virtual computer manipulative (4.43% versus 2.02% respectively). In addition, the proportion of questions given to students for which the response went from incorrect on the pretest to correct on the posttest is considerably higher among students who received instruction with the concrete manipulative (44.31% versus 37.86% respectively).

A similar analysis of achievement groups was performed in the topic of solving two-step linear equations. There were 125 students included in treatments in the topic of solving two-step linear equations. Each student answered 7 questions. The contingency table in Figure 4.7 provides the number of entries in each achievement group by manipulative type.

**Solving Two-Step Linear Equations**

			Achievement Groups for Seven Problems				Total
			Correct Pretest Incorrect Posttest	Incorrect Pretest Correct Posttest	Correct Pretest Correct Posttest	Incorrect Pretest Incorrect Posttest	
Manipulative	Concrete N=60	Count %	5 1.19%	243 57.86%	121 28.81%	51 12.14%	420 100.00%
	Virtual N=65	Count %	15 3.30%	215 47.25%	165 36.26%	60 13.19%	455 100.00%
Total		Count	20	450	286	111	875

*Figure 4.7.* Contingency table of achievement groups for the topic of solving two-step linear equations.

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Among questions given to students who received instruction with the virtual manipulative, 3.30% of the questions are in the achievement group that correctly answered the pretest question but did not correctly answer the same posttest question. The percentage of entries in this category is only 1.19% among questions given to students who received instruction with the concrete manipulative. The proportion of questions given to students whose response went from incorrect on the pretest to correct on the posttest is considerably higher among students who received instruction with the concrete manipulative (57.86% versus 47.25% respectively).



In both topics, the percentage of questions given to students who went from an incorrect response to a correct response on the posttest is somewhat higher among treatments that included the concrete manipulative. Similarly, the percentage of questions given to students who went from a correct response to an incorrect response is higher among treatments that included the virtual manipulative.

Within the topic of solving two-step linear equations, the proportion of students that incorrectly answered both the pretest and posttest questions is lower than the topic of multiplying and factoring polynomial expressions. This result makes sense because instruction and testing on the topic of solving two-step linear equations focused on one mathematical concept while the topic of multiplying and factoring polynomial expressions had a wider variety of problem levels.

Separating counts into achievement groups for the topic of solving two-step linear equations is inappropriate because all problems were at the same difficulty level. Problem solving skills necessary to solve any one problem were identical to those required to solve another. However, the varying difficulty levels of problems in the topic of multiplying and factoring polynomials prompted investigation into whether or not the concrete and virtual manipulative differ in their problem solving utility for the specific problem solving skills.

Each of these problems will be examined in pairs to analyze the same problem solving skills on the pretest and posttest. For example, analysis will explore whether students who factored a trinomial (problem seven) or multiplied two binomials (problem three) were more likely to answer correctly after receiving treatments that used either a concrete or virtual computer manipulative. Table 4.3 presents the counts of students

within each achievement group for each of the ten problems for the topic of multiplying and factoring polynomial expressions.

Table 4.3  
Achievement Groups for each Problem Within Manipulative Types (N=179)

<u>Problem</u>	<u>Correct Pretest</u>		<u>Incorrect Pretest</u>		<u>Correct Pretest</u>		<u>Incorrect Pretest</u>	
	<u>Incorrect Posttest</u>		<u>Correct Posttest</u>		<u>Correct Posttest</u>		<u>Incorrect Posttest</u>	
	<u>Concrete</u>	<u>Virtual</u>	<u>Concrete</u>	<u>Virtual</u>	<u>Concrete</u>	<u>Virtual</u>	<u>Concrete</u>	<u>Virtual</u>
1	4	5	23	10	74	54	8	1
2	2	5	43	15	42	46	22	4
3	5	2	33	22	43	35	28	11
4	3	8	51	29	25	22	30	11
5	5	6	41	27	32	23	31	14
6	-	2	67	36	3	6	39	26
7	-	-	47	28	28	26	34	15
8	1	1	67	40	8	9	33	20
9	2	2	54	33	8	7	45	28
10	-	-	57	25	-	-	52	45

A chi-square test could not be applied directly to contingency tables of each problem since all problems contain multiple cells with counts of less than five. Instead, the data were regrouped into two categories. The first category included students whose scores changed (either correct on the pretest and incorrect on the posttest or incorrect on the pretest and correct on the posttest). The second category included students whose scores did not change. After applying a Bonferroni correction, counts and proportions shown in Table 4.4, in the changed or unchanged achievement groups are essentially the same without regard to whether the manipulative is concrete or virtual.

Table 4.4  
 Changed or Unchanged Achievement Group Counts and Percentages  
 for each Problem Within Manipulative Types (N=179)

<u>Problem</u>	<u>Score Changed</u>		<u>Score Did Not Change</u>	
	<u>Concrete</u>	<u>Virtual</u>	<u>Concrete</u>	<u>Virtual</u>
	<u>Count/Percentage</u>	<u>Count/Percentage</u>	<u>Count/Percentage</u>	<u>Count/Percentage</u>
1	27 / 15.08%	15 / 8.38%	82 / 45.81%	55 / 30.73%
2	45 / 25.14%	20 / 11.17%	64 / 35.75%	50 / 27.93%
3	38 / 21.23%	24 / 13.41%	71 / 39.66%	46 / 25.70%
4	54 / 30.17%	37 / 20.67%	55 / 30.73%	33 / 18.44%
5	46 / 25.70%	33 / 18.34%	63 / 35.20%	37 / 20.67%
6	67 / 37.43%	38 / 21.23%	42 / 23.46%	32 / 17.88%
7	47 / 26.26%	28 / 15.64%	62 / 34.64%	41 / 22.91%
8	68 / 37.99%	41 / 22.91%	41 / 22.91%	29 / 16.20%
9	56 / 31.28%	35 / 19.55%	53 / 29.61%	35 / 19.55%
10	57 / 31.84%	25 / 13.97%	52 / 29.05%	45 / 25.14%

*Note.* Percentages are rounded to two decimal places.

### Summary

The results suggest that the use of either the virtual computer or the concrete manipulative will significantly increase overall class mean scores. This result held true without regard to the class knowledge categories. That is, when students were grouped into knowledge category classes based on their pretest score, there were no significant differences in class mean improvement between the two manipulative types.

Next, achievement groups were examined to determine if students who received treatments with the virtual computer or concrete manipulative were more likely to change their pretest and posttest response from an incorrect answer to a correct answer. No significant differences between treatments were found. However, the proportion of

students who moved from an incorrect answer on the pretest to a correct response on the posttest were greater among students who received treatments with the concrete manipulative than for those who received treatments with the virtual computer manipulative in both topics.

Finally, a chi-square test was performed on achievement groups for each problem within the topic of multiplying and factoring polynomial expressions. This test was performed to determine if students who received instruction with the virtual computer or concrete manipulative performed differently with respect to a particular type of problem. For example, it may be that the concrete manipulative was more beneficial to students in a problem that involved factoring a trinomial whereas the virtual manipulative is more suited for a problem that involves multiplying two binomials. However, results showed no difference in achievement groups between the concrete and virtual manipulative. That is, regardless of the type of problem given on the pretest or posttest, students who received instruction with the virtual computer manipulative were just as likely to improve test performance from an incorrect pretest to a correct posttest as students who received instruction with the concrete manipulative.

## Chapter 5

### Interviews

Five of the thirty-eight interviewed students gave incorrect answers. Henceforth, these students will be referred to as Ivan, Jan, Kay, Lynn, and Meely. All students attended the same high school. Complete transcripts of interviews are located in Appendices I, J, K, L, and M respectively. Students not chosen for analysis correctly answered all interview questions using procedural methods and the manipulative. The goal of conducting the interviews is to gain insight into semantic processes for students with difference topic backgrounds and identify potential manipulative contributions.

#### Illustrations of theory

Cognitive processes “need to be developed in the context of particular subject matters because specific subject matter knowledge, as well as specific task variables, can have a profound influence on the types of processes brought to bear on a task” (Wearne & Hiebert, 1988, p. 371). Qualitative interviews in this study focus on cognitive processes involved in completing tasks in the topic of multiplying and factoring polynomial expressions. Rather than providing static descriptions of cognitive processes, interview methods are geared toward identifying cognitive change.

Interview analysis was approached from a perspective that views competence as “the cumulative and sequential mastery of four separate cognitive processes in working with written symbols: (a) *connecting*, (b) *developing*, (c) *elaborating*, and (d) *abstracting*” (Wearne & Heibert, 1988, p. 371). The first two processes, connecting

and developing, make up what is referred to as semantic analysis. It is theorized by Wearne and Hiebert (1988) that any alternative sequence of acquisition is “cognitively difficult and may explain deficiencies students exhibit on decimal fraction tasks by suggesting that they have acquired later processes without the foundation provided by the earlier ones” (p. 372). Although these processes were described in Chapter 2, they are restated here within the context of their appearance in analyzed interviews and with respect to the specific subject matter of multiplying and factoring polynomial expressions.

The process of connecting is twofold because it involves both making an association between the referent and its corresponding symbolism and identifying actions on the referent that illustrate mathematical operations. For example, consider the work of Kay who wrote the expression  $x + y$  on paper after placing the  $x$  and  $y$  pieces together under and adjacent to the concrete *Algebra Lab Gear* frame. This student established a one-to-one correspondence between the concrete representation of  $x$  and  $y$  and the symbolic representation of  $x$  and  $y$ . She identified the action of placing the objects together in a group as one that illustrates the operation of addition.

When asked to factor  $x^2 + 6x + 9$  Kay correctly followed the procedure of initially forming the interior rectangle inside the concrete *Algebra Lab Gear* frame. However, it is unclear whether Kay viewed the area of this rectangle as a representation of a specific product. The choice of concrete factors and the corresponding answer of  $(1x + y)(1x + y)$  imply a misunderstanding. Kay seemed to understand the procedure as one of forming a rectangle with the wooden pieces and then finding the pieces that fit

along the edges. In fact, the measure of the three unit pieces that would form a correct solution was less than one-tenth of a centimeter in length different than the  $y$  piece that Kay used. Kay did not seem to recognize that the pieces that fit along the edges are the specific factors of the products held within. Thus, Kay provides an example of a student who is in the initial stages of the connecting process.

The process of developing involves the construction of connections between key ideas of the referent world to their corresponding representations in a symbolic world. This differs from the process of connecting in that in the developing process a student uses the referent as a guide to understanding mathematical procedures. The best example of this stems from the work of Lynn who was asked to find the product of  $x + y + 1$  and  $2y + 1$ . She began by using procedural techniques to find an incorrect solution of  $2xy + x + 6y + 1$ . Then she found the correct solution of  $2xy + x + 2y^2 + 3y + 1$  by placing the pieces in their appropriate positions around the concrete *Algebra Lab Gear* frame. When she realized the solutions were not identical, she insisted on going back over her procedural solution. By doing this, Lynn was able to identify her procedural error. In her words, "I messed up, because I was adding. See right here, I was adding it. I was doing  $y$  plus  $2y$  instead of thinking in my head, to multiply." Thus, by using the referent in parallel with the procedures, Lynn was able to identify and correct her procedural mistakes.

Lynn was able to set up the *Algebra Lab Gear* frame for finding the correct solution of  $2xy + x + 2y^2 + 3y + 1$ . In doing so, she not only was able to identify the blue and red pieces as representations of variables, but also recognized the innermost squares

and rectangles as representations of the operation of multiplication. This illustrates Lynn's mastery of the connecting process. Her work shows that she was engaged in the developing process in that she was using her correct concrete display of the problem to discover her procedural mistake.

The elaborating process involves extending syntactic procedures to other appropriate contexts. Extension problems in this research were problems that involved factoring polynomials with more than three unlike terms. The factorization of a polynomial with more than three unlike terms is an extension problem because students at this particular high school were not familiar with procedures or strategies for solving this type of problem. They had not been taught how to find the solution to this type of problem during any of the treatments. Without considerable mathematical background, the solution of this type of problem is difficult to form.

The elaborating process differs from the developing process in that the student must go beyond making corrections or adaptations to known procedures. Rather, in the elaborating process, the student is presented with unfamiliar content that requires their creative imagination to design new approaches to problem solving.

An example of the elaborating process stems from the work of Jan. Using procedural techniques this student found the product of  $x + y$  and  $2x + 1$  to be  $3xy$ . Jan was unfamiliar with the procedural process for multiplying two binomial expressions. When Jan was asked to verify her solution with the virtual *Algebra Lab Gear*, she found the correct answer. She recognized that her two solutions were different ( $3xy$  is not the



same as  $(2x^2 + x + 2xy + y)$ ). Her next step was not to simply identify procedural mistakes, but to derive the correct procedure for multiplying binomial expressions.

Concurrent with the elaborating process is the process of routinization. Wearne and Hiebert (1988) grouped the processes of elaborating and routinization together because they both tend to produce mathematical efficiency. The routinization process occurs when memorized and practiced rules become automatic to the point that problems require “little cognitive effort” (Wearne & Hiebert, 1988, p. 373). Meely appeared to be using a routinization process when she found the correct product of  $x + y$  and  $2x + 1$ . She voiced some of the routine in words while in the process of performing a procedure as follows: ‘And then outside for FOIL. Which would be the  $x$ , inside,  $2xy$ , and last.’ Meely was finished with the problem in less than a minute and confident of her answer. It was clear that the problem was not cognitively demanding for her. This process of routinization is characterized by the adoption of symbolic procedures apart from and without reference to analogous concrete referents.

The final process is abstracting. Here, the symbols and rules become the referents. In other words, the symbols or rules take on the role of the referent in the connecting process. An example of this process is found in the words of Meely when she was asked to factor the expression  $xy + y^2 + y + x + y + 1$ . In the following excerpt, Meely realizes that one factor may have three terms in it after struggling to find the factorization using reverse FOIL techniques. With this insight, Meely determines that reverse FOIL procedures would not be appropriate for this type of problem. She speculates that the correct procedure would involve a new acronym. In this way, Meely

is using a previously acquired rule to establish that a similar rule must exist for factoring the expression  $xy + y^2 + y + x + y + 1$ .

### **Sequencing of instruction**

Quantitative results indicate that the sequencing of instruction, from manipulative to symbolic or symbolic to manipulative, does not alter learning outcomes. Qualitative interviews, although limited in scope, tend to contradict this finding. When compared to students with considerable background in procedural methods, students who had little or no background knowledge in the subject of multiplying and factoring polynomial expressions were more likely to adopt the manipulative as a resource to guide procedural methods. That is, students who received scores of below two out of ten points on their pretest were more likely to engage in the developing processes.

Four of the five students selected for interview analysis had pretest scores of less than three out of ten points. Meely was the single student selected for analysis who demonstrated mastery of content on the pretest with a score of 9 out of 10, but did not find a correct answer using the virtual computer manipulative during her interview. Meely indicated, on pretest Question C, that she had never received any previous instruction in how to multiply or factor polynomial expressions using a computer or physical object. However, her response to Questions A and B indicate that she had some background instruction in the topic. The following is a summary of the work performed by Meely followed by a contrast to Ivan, Jan, Kay and Lynn.

### **Interview with Meely**

Meely received instruction and was interviewed with the virtual computer *Algebra Lab Gear*. She is unique in that she freely uses words like coefficient, binomial,

and acronym. She is, in the view of the researcher, a syntactic rule applier, reluctant to engage in activities with the virtual computer manipulative. She says very little while verifying her previously obtained procedural solutions.

Meely was chosen for analysis because of her incorrect answer to the question on factoring the expression  $2x^2 + 4xy + 2x - xy + y^2 + y$ . She is given this problem after she has successfully solved the three previous problems using procedures and then verifying on the virtual computer manipulative. As she began to work on the problem, Meely quickly identified the like terms in the expression and wrote the equivalent expression. Unlike previous attempts, she decided to begin with the use of the virtual computer manipulative.

- JW: ... Try this one. Let's see you have to factor some huge thing. 2  $x$ -squared plus  $4xy$  plus  $2x$  minus  $xy$  plus  $y$ -squared plus  $y$ . You're going after it with pencil and paper?
- MEELY: Yeah. Just because then I can get rid of this and simplify them.
- JW: Oh. Okay. You're simplifying them.
- MEELY: Now I have to go to the computer.
- JW: Oh. Okay. It's what I want to do. So, you've got 2  $x$ -squares, right. Then you got  $3xy$ 's. I see those.
- MEELY: I have 2  $x$ -squares,  $3xy$ 's, 2  $x$ 's, a  $y$ -squared, and a  $y$ .
- JW: Okay.
- JW: So you turned that  $y$ .
- MEELY: I was thinking I would need a  $y$  over here if that would be an  $x$ -squared, but that would have to be a  $y$  as well.
- JW: Okay.
- MEELY: The  $y$  doesn't fit.

At this point in the interview, Meely is attempting to form a rectangle within the frame of the virtual computer workspace. She tries several different arrangements, but does not form a rectangle in the provided virtual workspace.

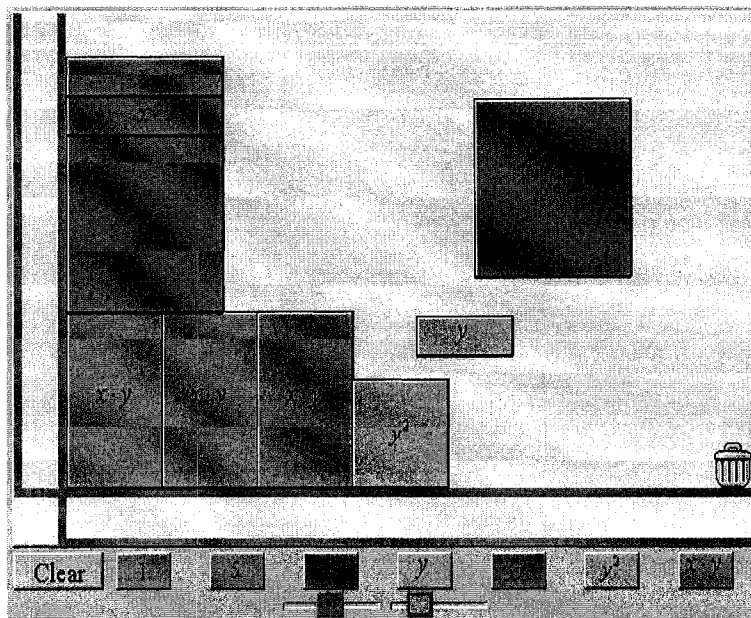
JW: You said it doesn't fit?  
 MEELY: No, the  $y$  doesn't fit.  
 JW: Any ideas?  
 MEELY: No, cause I'm thinking I would need. Maybe if I get this out of the way, but I don't think that would work. This is really hard.  
 JW: The day you had this in class, do you remember of them being really hard?  
 MEELY: Um, I remember having to play with the blocks a lot.  
 JW: The blocks?  
 MEELY: With the squares a lot.  
 JW: To get it to work?  
 MEELY: Yeah, but I don't remember it being that hard.  
 JW: Okay.  
 MEELY: Like today, we're learning the coefficients of both of the  $a$ 's more than one.  
 MEELY: I have just no idea how to do with the computer.  
 JW: Okay, let me ask you this. Do you know any other way of doing this problem?  
 MEELY: No.  
 JW: No.  
 MEELY: I wouldn't know how to, well.  
 JW: You wouldn't want to use your FOIL?  
 MEELY: I could try my FOIL.

Meely has agreed to try the FOIL method to solve the problem. Using this method, she acquires factors of  $2x + y$  and  $x + 1$  but does not attempt to represent this product on the computer. She does not use the manipulative as a resource for solving. There appears to be a no connected link between what is done on the computer and what is performed symbolically. In the following excerpt, Meely sticks to procedural methods without reflecting on her visual computer display. The final computer image and work of Meely on this problem appears in Figure 5.1.

JW: Okay, tell me what you're thinking while you're doing it.  
 MEELY: Um, I could start out with the brackets, then I would need, um, for one reason or another, I'm guessing everything there is going to be positive.  
 JW: Okay.

MEELY: Um.  
 JW: Any reason for that positive guess?  
 MEELY: Oh, just because everything's positive.  
 JW: Okay.  
 MEELY: So, there's a  $2x$ ,  $2x$ -squared, the first. (Pause) Maybe a  $y$ .  
 Skipping the outside and the in.  $xy$  yeah. Three of them.  
 JW: Did you say you have 3 of those?  
 MEELY: Well, I need 3 of the  $xy$ 's, so, I don't know how I get those in.  
 JW: Then you kinda wrote a  $2x$  in the middle of it all.  
 MEELY: Yeah,  $2x$ , just cause I, da, guessed there's a 1 here.  
 JW: Oh, okay.  
 MEELY: And turned that into  $2x$ . It took care of that one.  
 JW: Okay, you took care of that one.  
 MEELY: That's the inside, and then the last part,  $y$ .  
 JW: Cool.  
 MEELY: Then I need a  $y$ -squared and 2 more  $xy$ 's. Maybe if I switched  
 that. Don't want erasing?  
 JW: No. I appreciate that.  
 MEELY:  $2x$ -squared. I'm going to leave this open for my outside.  
 JW: Okay.  
 MEELY: Oh, I am stuck.  
 JW: Want to go on?  
 MEELY: I think so.

Meely's performance in another problem is very similar. This time Meely is asked to factor  $xy + y^2 + y + x + y + 1$ . Here again, the problem is one of transfer in that it is a difficult problem to perform without semantic processes. The problem is given in an effort to move Meely towards verbally demonstrating the *connecting* process. That is, Meely is being prompted to utilize the manipulative as a model for her symbolic solution. Meely begins the problem by simplifying the expression. Then she draws brackets indicating that she would begin with procedural methods. When asked if she had lost faith in the computer, Meely acknowledges that she finds the computer to be '*confusing*'.



(a)

Factor  $2x^2 + 4xy + 2x - xy + y^2 + y$   
 $2x^2 + 3xy + 2x + y^2 + y$

$$(2x + y)(x + 1)$$

$$2x^2 + 2x + xy + y$$

$$(2x + y)(x + 1)$$

$$2x^2 + \quad + xy$$

(b)

Figure 5.1. Meely's final solution of factor  $2x^2 + 4xy + 2x - xy + y^2 + y$  in (a) virtual and (b) written formats. National Library of Virtual Manipulatives, © 1999-2005 Utah State University.

She forms the product  $x + y$  times  $y + 1$  while uttering words such as '*outside*' and '*inside*'. It is apparent that she is attempting reverse FOIL methods.

- JW: Okay, It's alright. Let's try this one.  
 MEELY: I'm going to simplify it.  
 JW: Have you lost faith in the computer?  
 MEELY: Yeah.  
 JW: Okay. It's second best.  
 MEELY: I think it's pretty confusing.  
 JW: Okay.  
 MEELY: Cause you have to find, get them to fit right, and then, I just, I don't know. I'm not much of a hands-on. As my second grade teacher, I prefer meat and potatoes, the basics.  
 JW: Okay. You called it hands-on.  
 MEELY: Yeah.  
 JW: But your hands are only on the mouse.  
 MEELY: Yeah, but you're, like using manipulatives. Where, in my mind, and I find them a little more confusing.  
 JW: Okay.  
 MEELY: If I didn't have to deal with the shapes and get everything to fit right.  
 JW: So, it's like creating confusion where none doesn't have to be.  
 MEELY: Yeah, in my mind it doesn't, struggling with this a little bit as it is, but... (Pause) Outside. My outside plus  $x$ . The inside was a  $y$ -squared, not a  $2y$ .  
 JW: Okay.  
 MEELY: Oh, you don't want me to erase it.

At this point, Meely verifies her solution of  $(x + y)(y + 1)$  is not correct by finding the product and writing  $xy + x + y^2$ . Then she asks the question about whether the solution is 'just binomials'. It is clear she has had considerable instruction in this subject.

- JW: It's okay, your thinking while you're doing this, it's difficult.  
 MEELY: All these problems, it's just, um, with like just binomials?  
 JW: Oh, it could be.  
 MEELY: Or it could be more?  
 JW: Uh-huh.

- MEELY: Oh, bummer. I've never been taught, like what to, like how to expand if there were like 3 things in, um, like in the bracket.
- JW: Uh-huh.
- MEELY: Cause all I've ever worked with is like FOIL.
- JW: Do you think it would be the same kind of technique or you think it would be something completely different?
- MEELY: It might, but it might be more confusing with the inside.
- JW: Outside, middle, stuff? (Laughs)
- MEELY: Yeah.
- JW: It's not inside, outside. It's like inside, last. I don't know. I can't remember. You know it better than I do.
- MEELY: FOIL, the first, outside, inside, last. But then with the 3, what would be inside.
- JW: Well if it were 3, wouldn't you have a first and a last still?
- MEELY: Yeah, then what would you do with the middle? It wouldn't be FOIL anymore.
- JW: It wouldn't be FOIL anymore?
- MEELY: No, because it'd be a new um, acronym.
- JW: Are you feeling frustrated with this one too?
- MEELY: Yes.
- JW: Why don't you try this one on the computer?
- MEELY: Okay.

At this point in the interview, Meely begins her work on the computer. She quickly forms the appropriate rectangle, factors, and writes the correct solution on her paper. It is now apparent that she can connect the concrete depiction of the solution to its corresponding symbolic representation. She does not, however, use the concrete depiction of the solution to develop the acronym that she found to be lacking in her background. Rather, she expresses that she really hopes the solution obtained by the virtual computer manipulative is correct. It is not clear if Meely believes the new answer found with the virtual manipulative is correct because she had previously affirmed that the solution included an expression with more than two terms or whether she believes the manipulative provides accurate results. Her final written solution is shown in Figure 5.2.



Factor  $xy + y^2 + y + x + y + 1$   
 $xy + y^2 + 2y + x + 1$   
 $(x + y)(xy + 1)$   
 $xy + x - y^2$   
 $(x + y + 1)(y + 1)$

Figure 5.2. Meely's final solution of factor  $xy + y^2 + y + x + y + 1$  in written format.

- MEELY:  $xy$   
 JW: Okay, you have an  $xy$ .  
 MEELY: A  $y$ -squared.  
 JW: A  $y$ -squared.  
 MEELY:  $2y$ 's, and  $x$ .  
 MEELY: I really haven't given up hope yet, on the computer.  
 JW: Okay. You haven't?  
 MEELY: No, just like, if I were to get really get super confused, that's why this would be nice. It doesn't seem very, uh, like I would want to have to rely on it for class work.  
 JW: Uh-huh. Not trustworthy?  
 MEELY: It's not convenient, as much, like in class, I wouldn't have it, like you know, sitting on my desk.  
 JW: Okay. So you wrote  $x + y + 1$  times  $y + 1$ .  
 MEELY: Yeah  
 JW: How confident are you of that?

- MEELY: I really hope it's the right answer, because that's what I got up there, but I'm not confident about how you write it, but I'm pretty sure that's it.
- JW: You sure?
- MEELY: Yeah.

Meely was able to connect the virtual solution to the symbolic representation. However, she was not a semantic analyzer in that she did not use the manipulative as a resource for correcting procedural methods. Despite her exceptional background and evidence that the manipulative led her to the correct solution, Meely did not seem to view the manipulative as related to the procedural technique of solution. This is evident in that after performing a procedural technique she did not use the resource of the *Algebra Lab Gear* to check her work even when she suspected her solution was incorrect. This interview suggests a theory that students who receive initial instruction with procedural methods are less likely to use a referent for developing procedural methods.

### **Contrast of Meely to Ivan**

Ivan, Lynn, Kay, and Jan were not familiar with procedural techniques for factoring polynomial expressions with more than four terms. Nevertheless, Ivan and Lynn quickly factored  $2x^2 + 4xy + 2x - xy + y^2 + y$  using the concrete manipulative. Jan factored  $xy + y^2 + y + x + y + 1$  using the computer manipulative. Ivan checked his work by writing down the answer on paper. Then, with his pencil, he pointed to each term in the first expression and identified the term in the second expression that gave each product of terms in the solution. Lynn and Jan did not check their solutions with procedural techniques. This is reasonable since Lynn and Jan's work indicated little or

no background education in the topic of how to multiply or factor polynomial expressions using symbolic procedural methods.

The fact that Ivan, Jan, Kay, and Lynn were able to obtain the correct answer to problems involving the elaboration process is similar to the work of Meely. While unable to find the correct answer to factoring  $2x^2 + 4xy + 2x - xy + y^2 + y$ , Meely was able to obtain the correct answer to factoring  $xy + y^2 + y + x + y + 1$ . What is strikingly different is the response to the solutions obtained with the manipulative. When Meely found the correct factorization of  $xy + y^2 + y + x + y + 1$ , she was faced with two separate solutions that did not agree. It did not seem to occur to Meely that one method could be used to invalidate or validate the other or that the symbolic process could be guided by the actions on the manipulative. This represents a clear difference in attitude and approach when compared with the other analyzed interviewees. The interview with Ivan best illustrates this difference.

Ivan's pretest score was one out of ten problems. He indicated that he had no instruction in factoring on the pretest question C, and he could not factor the problem designed to detect symbolic procedural understanding ( $4x^2 - 2x - 2$ ). This provides evidence that Ivan, unlike Meely, had relatively little background in procedural methods for factoring polynomial expressions.

Ivan's approach to problem solving began with the manipulative. Ivan was relentless in his efforts to use the manipulative to make sense of the procedural approach. His interview is unique in that it was conducted with concrete *Algebra Lab Gear* even though Ivan received instruction with the virtual computer *Algebra Lab Gear*. This

occurred because the computers were unexpectedly down during the scheduled interview time.

Unlike Meely, Ivan did not verbally express any confusion between the two types of manipulatives or preference for one over the other. Rather, Ivan used semantic processes to come up with the correct answers and develop his underlying understanding of syntactical procedures. For example, Ivan began the problem of factoring  $2xy + 4x + y + 2$  procedurally by writing  $(xy - 1)(xy + 1)$ . When Ivan became unsure of this work, he quickly switched back to using the *Algebra Lab Gear*. In doing so, Ivan realized that  $2xy$  is the product of factors 2,  $x$ , and  $y$ . His work can best be characterized as a back and forth process between the symbols on the paper and the actions with the concrete manipulative. Ivan tended to view the work with the manipulative as connected to procedural approaches while the existence of this connection between the manipulative and procedures was confusing to Meely.

Ivan conveyed mental connections of the symbols to their referents as well as actions on the symbols to the operation of multiplication. He explained variables by pointing to the concrete piece of *Algebra Lab Gear* and pointing to the corresponding written expression. Ivan continually used semantic processes of connecting while explaining the solutions obtained via syntactic procedures with the concrete manipulative. Throughout the interview, it was not necessary to ask Ivan to use a pencil and paper procedural approach or to use the manipulative to solve a problem. Ivan always worked problems with a combination of both strategies. His level of understanding is conveyed in the sporadic but detailed explanations of procedurally obtained solutions with the manipulative.

The problem that Ivan missed during his interview illustrates his willingness to understand the problem using the manipulative as a resource. The following is an example of Ivan's approach to contradicting solutions. Ivan was asked to factor  $2xy + 4x + y + 2$  and began with an attempt to form a rectangle with the concrete *Algebra Lab Gear*.

- IVAN:  $2xy$ , so we're going to have two of the purple ones.  $4x$ .  
 Yep. Plus 2. I think I've assembled these before. There, modified, modifications. We must have one more  $y$ , don't we. That's not going to work. Why. We turn it this way. If we turn it that way, then these are going to be longer, so that's not right. Put the two up there. We have one  $y$  and two twos. Maybe these don't go that same way. Never know till you try. It's still plus 2, not plus 4. So if we put that there. Nah, it's still going to be plus 4. No, that's still going to be plus 4. Ah no, how do I. I can't remember how I normally factor it.
- JW: Oh. Okay.
- IVAN: It's like, I know I tried that.
- JW: You mean, what occurred to you then, what you're saying is, you thought maybe you'd just try to factor it, the normal, you know, some other way?
- IVAN: Yeah.
- JW: Okay.
- IVAN: Visualize.

The struggle to form a rectangle tended to take considerable effort when the ratio of the length to width of the rectangle was large. In other words, rectangles that were almost square were typically easier for students to quickly form and identify. This struggle was not unlike the struggle Meely underwent when trying to form a rectangle with the pieces  $2x^2$ ,  $3xy$ ,  $2x$ ,  $y^2$ , and  $y$ . When Ivan was unsuccessful forming the appropriate rectangle, he began to compare his visual display to his procedural solution. He verified that the procedural solution was incorrect by finding an incorrect product and attempting to form the corresponding rectangle with the *Algebra Lab Gear*. In other

words, he tried to verify his solution with procedural and concrete methods. Procedural work is shown in Figure 5.3.

Factor  $2xy + 4x + y + 2$

$(x) (y)$

$2xy + 1xy + 2xy$

$3xy$

Figure 5.3. Ivan written solution of factor  $2xy + 4x + y + 2$ .

Ivan found the product of  $xy$  and  $xy$  to be  $2xy$  while attempting procedural methods. This is evident by his statement implying that  $2xy$  must have something to do with an  $xy$  and  $xy$ . His subsequent corresponding concrete display of  $x$  and  $y$  as factors on each side of the workspace frame is shown in Figure 5.4.

At this point, Ivan determines that  $xy$  and  $xy$  as front terms cannot be correct because it cannot be represented with the concrete *Algebra Lab Gear*. The researcher probes Ivan for an explanation in the following discussion.

IVAN: What about if we do. No, no, maybe they don't go the way I'm thinking. No, that doesn't work. No. Fiddle faddle. No, that ones not all the way. That might explain that.

JW: Okay.

IVAN: So it will be  $2xy$ , you're gonna have to have  $xy$  and  $xy$ . See, I got to do these last year.  $4x$  is in the middle, first, then outer. Uh-oh. Somewhere you have to get  $2xy$  though. No, that doesn't work.

IVAN: You can't have  $xy$  first cause then it doesn't fit over here though. So that can't be it.

JW: So the first thing you wrote down is  $xy$ ,  $xy$ ?

IVAN: Yeah.

JW: Kind of like, uh, the front terms in FOIL?

IVAN: Yeah.

JW: I don't know if that's what you were thinking?

IVAN: That's what I'm thinking, but it doesn't work.

JW: Why doesn't it work?

IVAN: Cause it won't fit on here.

IVAN: Yeah.

JW: Kind of like, uh, the front terms in FOIL?

IVAN: Yeah.

JW: I don't know if that's what you were thinking?

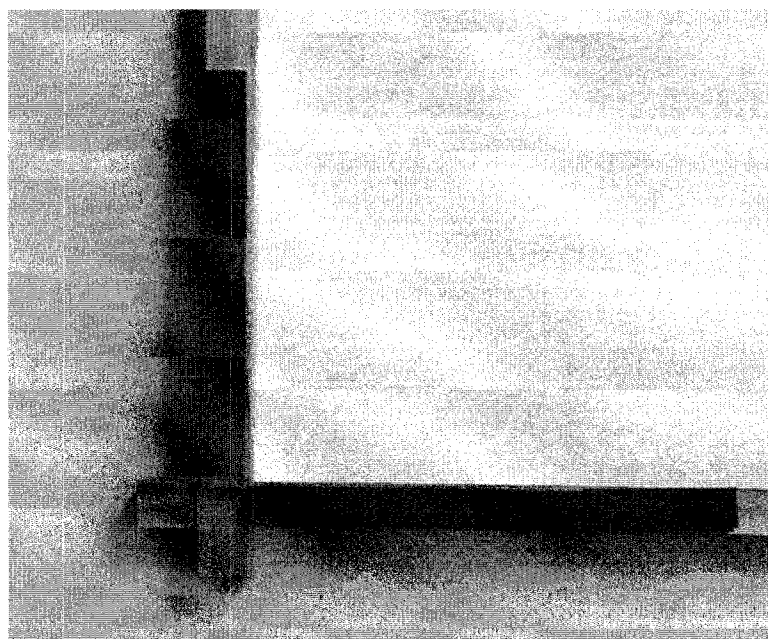
IVAN: That's what I'm thinking, but it doesn't work.

JW: Why doesn't it work?

IVAN: Cause it won't fit on here.

JW: Okay.

IVAN: Remember the program wouldn't let us do that. Cause I tried that one.



*Figure 5.4.* Ivan's concrete representation of the product of  $xy$  and  $xy$ .

- IVAN: What about if we do. No, no, maybe they don't go the way I'm thinking. No, that doesn't work. No. Fiddle faddle. No, that ones not all the way. That might explain that.
- JW: Okay.
- IVAN: So it will be  $2xy$ , you're gonna have to have  $xy$  and  $xy$ . See, I got to do these last year.  $4x$  is in the middle, first, then outer. Uh-oh. Somewhere you have to get  $2xy$  though. No, that doesn't work.
- IVAN: You can't have  $xy$  first cause then it doesn't fit over here though. So that can't be it.
- JW: So the first thing you wrote down is  $xy$ ,  $xy$ ?
- IVAN: Yeah.
- JW: Kind of like, uh, the front terms in FOIL?
- IVAN: Yeah.
- JW: I don't know if that's what you were thinking?
- IVAN: That's what I'm thinking, but it doesn't work.
- JW: Why doesn't it work?
- IVAN: Cause it won't fit on here.
- JW: Okay.
- IVAN: Remember the program wouldn't let us do that. Cause I tried that one.

Ivan is referring to work completed during the activities of the lesson. This highlights a difference between the concrete and virtual computer manipulative. The virtual computer manipulative will not allow the user to place the  $xy$  piece outside the workspace frame. If or when a student attempted to place the  $xy$  piece outside the virtual workspace frame, they would not be able to do so by dragging the  $xy$  piece with the mouse. The  $xy$  piece would simply not follow the mouse movement. On the other had, this is physically possible with the concrete *Algebra Lab Gear*. Figure 5.5 illustrates this difference between the two manipulatives.

Realizing the previous strategy did not match what was allowed on the computer program, Ivan repeated his work with the concrete manipulative. That is, Ivan attempted to form a rectangle on the inside of the lab gear frame. Using the referent as a guide to symbolic procedures, Ivan is able to identify the incorrect procedural strategy and modify

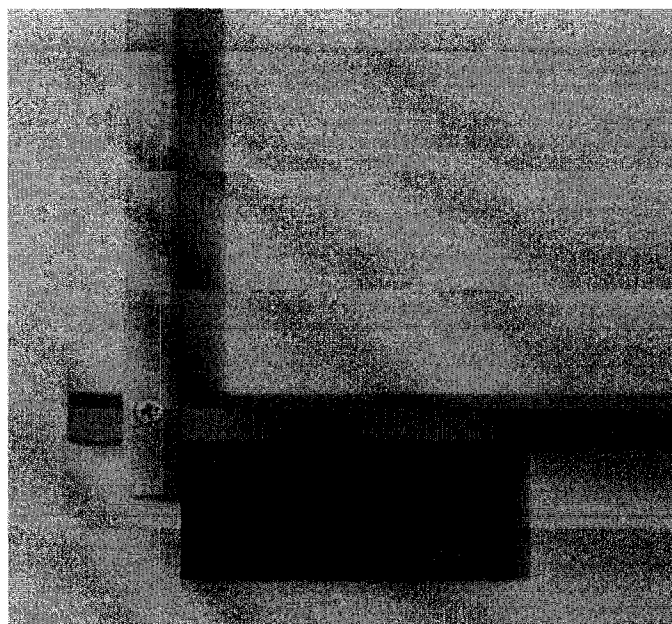


his result. In this way, Ivan is using the developing process. His actions in the referent world paralleled the symbolic world when he changed his solution to model the work performed with the *Algebra Lab Gear*. In the following excerpt, Ivan has successfully formed the appropriate rectangle with *Algebra Lab Gear* and begins to place the factors on the edges of the frame.

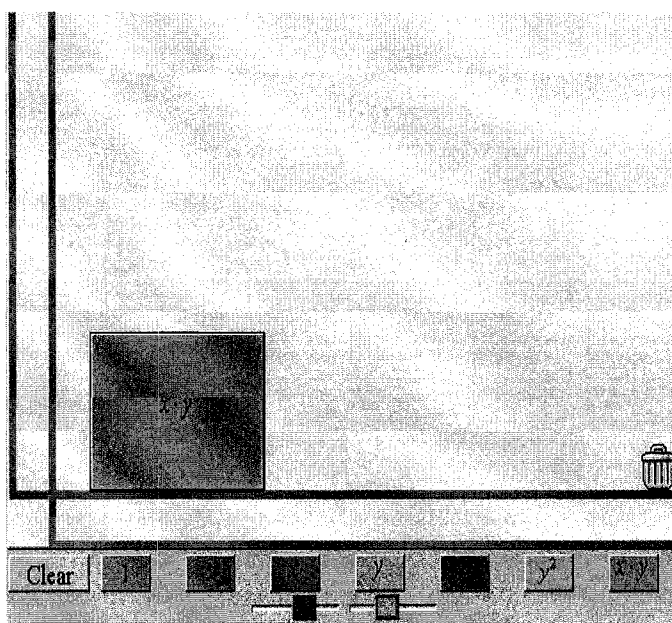
- IVAN:           There we go. I think I got it. I think I. Oh yeah. Look at that. Oh yes. There we go.
- JW:             (Laughs) It's the ah-ha syndrome.
- IVAN:           Finally. The only way to get what's gonna to be, so we're gonna have  $x$ . Either way it's going to work. I want this one. There we go. That will give us  $x$  times  $y$ . Then were going to have to alternate it. Wait a sec. No, I'm thinking wrong. That times that and that times that, so we need another red one. Then, this one would have to be one. That's gonna make that. Yep. And those are going to be two more greens. So lets see how far off I truly was now. So now we got, we got  $2x + 1y + 2$ .

It is apparent that Ivan had some previous experience with multiplying binomial expressions and factoring trinomials. Ivan's final solution appears in Figure 5.6.

However, Ivan was less familiar with syntactical procedures than Meely. Not having a great deal of prior instruction in syntactical procedures may explain Ivan's willingness to engage in semantic analysis while having a great deal of prior instruction may explain Meely's unwillingness. This implies that the sequence of instruction from manipulative to symbolic or vice versa may influence student learning outcomes in that students who have initially learned symbolic approaches tend to adhere to these techniques in isolation of other potentially helpful methods.



(a)



(b)

*Figure 5.5.* Illustration of why it is not possible to place  $xy$  piece outside the virtual workspace frame. National Library of Virtual Manipulatives, © 1999-2005 Utah State University.

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⓪ Factor  $2xy + 4x + y + 2$   
 $(x, 1)(xy, 1)$   
 $2xy + 1xy + 2xy$   
 $3xy$   
 $1x$        $2xy$   
 $1xy + 1xy$   
 $(2x + 1)(y + 2)$

Figure 5.6. Ivan's corrected written solution to factoring  $2xy + 4x + y + 2$ .

It is apparent that Ivan had some previous experience with multiplying binomial expressions and factoring trinomials. However, Ivan was less familiar with syntactical procedures than Meely. Not having a great deal of prior instruction in syntactical procedures may explain Ivan's willingness to engage in semantic analysis while having a great deal of prior instruction may explain Meely's unwillingness. This implies that the sequence of instruction from manipulative to symbolic or vice versa may influence student learning outcomes in that students who have initially learned symbolic

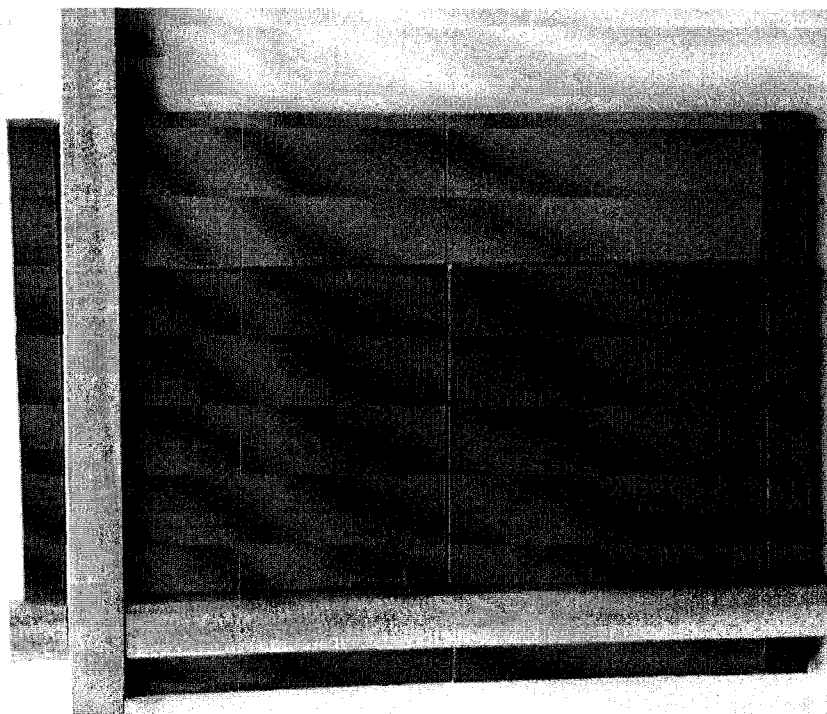
approaches tend to adhere to these techniques instead of using other potentially helpful methods.

### **Differences in learning outcomes for the concrete and virtual manipulative**

Qualitative results confirm differences between accuracy measures within the subject of multiplying and factoring polynomial expressions. The researcher observed that many students included a new variable in their posttest answer to a problem that involved factoring an expression in a single variable. The posttest question was problem seven and required the student to factor  $x^2 + 6x + 9$ . This error did not occur on any of the pretest answers to problem seven where the student was prompted to factor  $x^2 + 4x + 4$ . The answer given to factoring  $x^2 + 6x + 9$  was  $(x + y)(x + y)$  for 10 out of 109 students who received instruction with the concrete manipulative. One out of 70 students who received instruction with the virtual computer manipulative gave an identical response. This result favors the virtual computer manipulative in that a smaller percentage of students who used the virtual manipulative gave an incorrect response that included an additional variable. Fortunately, the approach to finding the incorrect solution was witnessed during the interview with Kay.

Kay received instruction and was interviewed with the concrete manipulative. Although prompted to do so, she did not successfully solve any problems with procedural methods. During the first problem of the interview, Kay was asked to multiply  $x + y$  times  $2x + 1$ . She proceeded to place the appropriate factors along the edges of the frame, fill in a rectangular region, and wrote the correct answer on her paper. What is

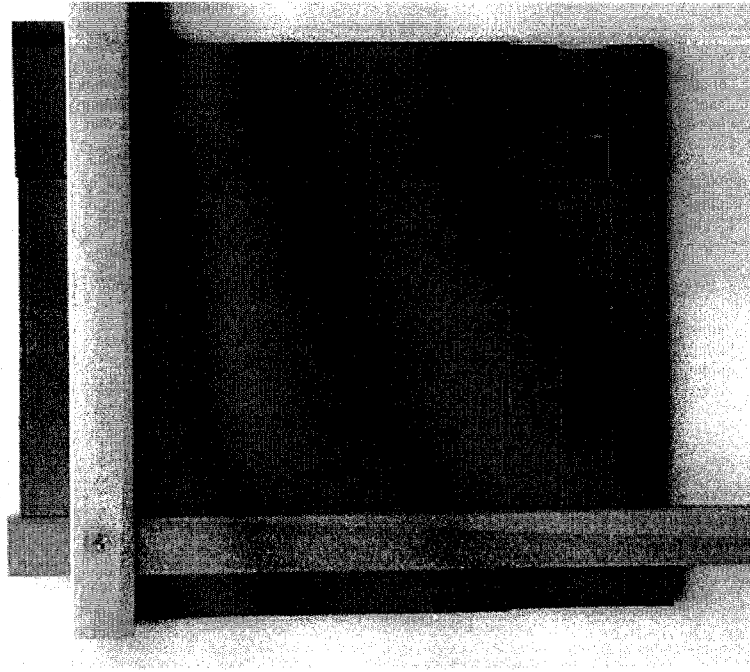
unusual about her concrete display of the solution is that the pieces don't line up, as shown in Figure 5.7.



*Figure 5.7.* Kay's concrete representation of multiplying  $(x + y)(2x + 1)$ .

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This inconsistency would not be such a big deal if it did not reappear in the next problem when Kay was asked to factor  $x^2 + 6x + 6 + 3$ . This time Kay began with the appropriate rectangle, placed inappropriate factors along the edges, and wrote the corresponding solution as shown in Figure 5.8. Figure 5.8 illustrates that the  $y$  piece was approximately equal in length to three unit pieces.



(a)

Factor  $x^2 + 6x + 6 + 3$   
 $(x+4)(x+4)$

(b)

*Figure 5.8.* Kay's concrete representation and solution to factor  $x^2 + 6x + 6 + 3$ .

The researcher queried Kay as to whether a solution can have a  $y$  term in it while beginning with only  $x$  variable terms. When Kay did not see an issue with the inconsistent variables, the researcher asked Kay to repeat the problem using the blue piece as the  $x$  variable. The intent of the question was to determine if Kay could find the

correct answer when the procedure of creating a rectangle and subsequent factors would not lead to inaccurate solutions based on the size of the *Algebra Lab Gear* pieces.

- JW: Times  $x$ , plus  $y$ . Um, is that your answer? Confident?
- KAY: Yeah.
- JW: Can I ask you, um, the question, it says factor  $x$ -squared, plus  $6x$ , plus 6, plus 3, um, doesn't have any  $y$ 's in it. Is it possible to get an answer with  $y$ 's in it without any  $y$ 's in the original? I'm just asking.
- KAY: Um, sure.
- JW: Sure. Okay, in this case, um, you know, if you ask a little child what does  $x$  mean, they'd say, "It's the first letter in x-ray." Well, what does  $x$  mean in this case?
- KAY: Um, it's like a block or zero.
- JW: It means zero?
- KAY: It's, your try, you're trying to find, like, the number.
- JW: The number. The number, but  $x$  isn't a number. It's an  $x$ . It's a letter of the alphabet. So what I'm asking you to do is to try to make sense of why in the world a math teacher would give you something like, factor  $x$ -squared, plus  $6x$ , plus, plus 6, plus 3? And what in the world does that mean? Does it mean anything to you or just?
- KAY: It means like, trying to find the value of it or.
- JW: The value of what?
- KAY:  $x$
- JW: The value of  $x$ . Okay.
- KAY: Yeah.
- JW: Okay, so they want you to try to find the value of  $x$ .
- KAY: Yeah.
- JW: But, ah, here, ah, you have a value of  $x$ . It was that red thing. This is what you were calling  $x$ . Is that right? This thing?
- KAY: Yeah.
- JW: But before you were calling  $x$  this one? Is that okay? To call  $x$  that one?
- KAY: Uh. No, because this is only equal to one.
- JW: Okay, it's equal to one.
- KAY: Yeah.
- JW: So you can't call  $x$  that?
- KAY: No.
- JW: How bout the blue one? What if I wanted to say the blue one is  $x$ ? Is that okay?
- KAY: The blue one would, well the blue one's  $y$  though.
- JW: I know, but if I just, one time I just want to call them. Let's say I want to do the problem calling the blue one  $x$ .

- KAY: Yeah, you could do that.  
 JW: Okay, go ahead. Will you do that for me? Pretend the blue one's  $x$ .  
 KAY: So that's like.  
 JW: You know. Just, just do the problem all over again.

At this point, Kay moves 6  $x$  and 9 unit pieces away from the original red rectangle and begins to move blue pieces around the red square. She seems to have overlooked the fact that the red square would no longer represent  $x$ -squared. The researcher attempts to point this out in the following statements.

- JW: If this one, if the blue one is  $x$ , then which one would be  $x$ -squared?  
 KAY: The red.

Kay points to the red  $x$ -squared piece inside the concrete *Algebra Lab Gear* frame. The  $x$  and  $y$  factors on the edges of the frame and the  $x$ -square piece in the center are never moved.

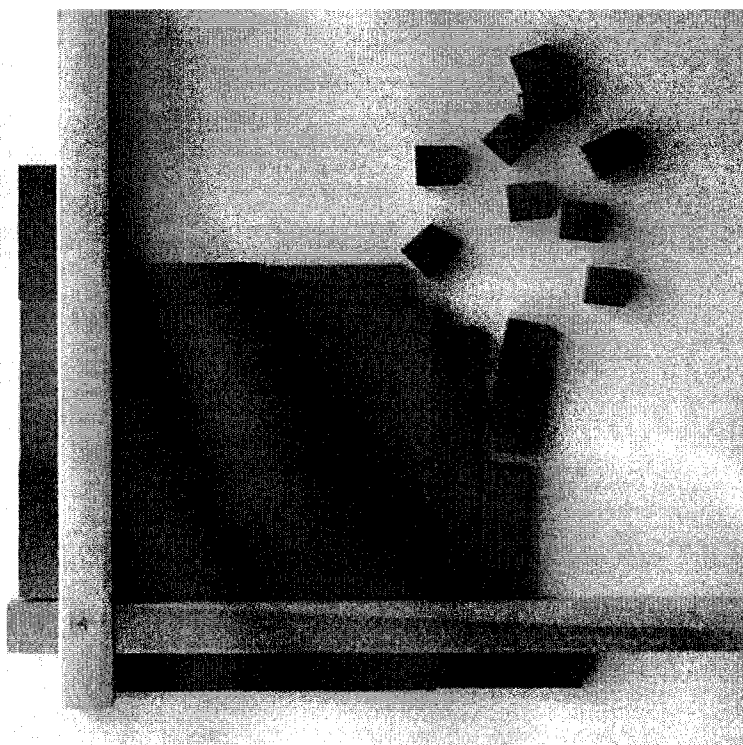
- JW: The red ones still  $x$ -squared?  
 KAY: Yeah.

Kay was unable to make connections between the pieces despite their similarities in color and shape. When asked to perform the problem with  $y$  as a replacement for  $x$ , she was unable to see how this change would alter which pieces would subsequently represent  $x$ -squared and  $y$ -squared. Consequently, she was unable to form the appropriate rectangle and find a solution. A copy of the rectangle she was attempting to form when she decided it was not working is shown in Figure 5.9.

The next example indicates that Kay does not understand the concrete representation of a product. Here, Kay is asked to factor the expression  $2xy + 4x + y + 2$ . She begins by forming a shape that is somewhat less than a rectangle,



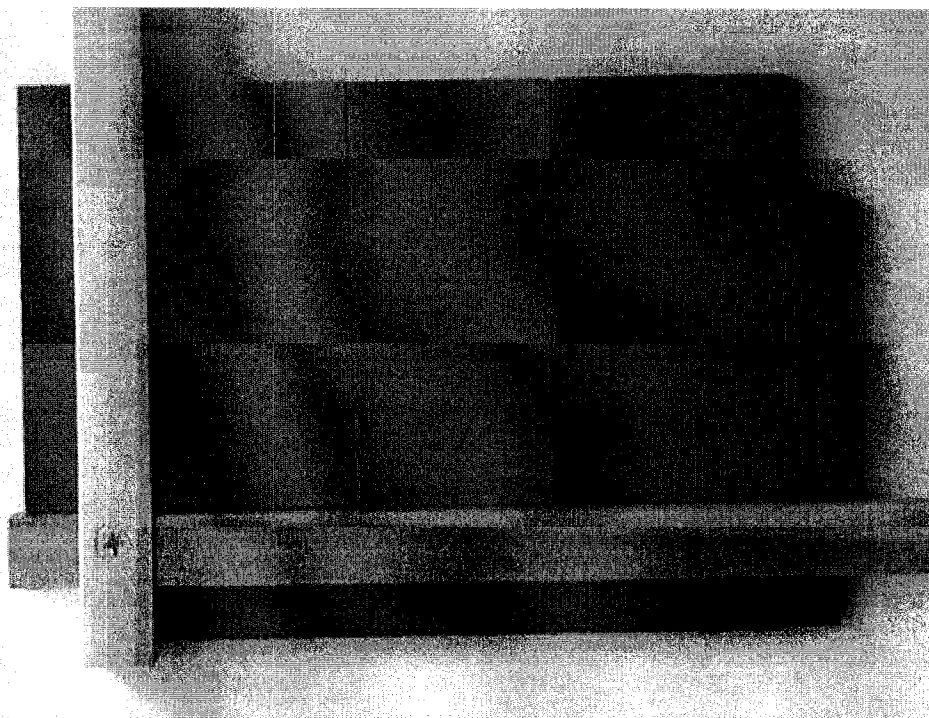
places the  $x$ ,  $y$ , and unit pieces below the frame, the  $x$  piece adjacent and left of the frame, and writes the solution of  $1x + 1xy + 1$ .



*Figure 5.9.* Kay's concrete representation of factor  $x^2 + 6x + 6 + 3$  using the blue pieces to represent  $x$ .

- KAY: I think I got it.  
 JW: Huh?  
 KAY: I think I got it.  
 JW: What? You don't sound very sure in your voice. Why are you unsure in your voice?  
 KAY: Um.  
 JW: I sense that.  
 KAY: Two pieces missing.  
 JW: Okay, two pieces missing, so it would look more like a square? I don't know, if I put two pieces there, it might look more like a rectangle.  
 KAY: Yeah.

The work of Kay and her final solution to this problem are shown in Figure 5.10.



(a)

⊙ Factor  $2xy + 4x + y + 2$   
 $x + x + 1$

(b)

*Figure 5.10.* Kay's concrete representation and written solution of factor  $2xy + 4x + y + 2$ .

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According to Wearne and Hiebert (1988), the connecting process involves a combination of establishing connections for both symbols and operations. Kay was unable to make the appropriate connections to the operation of multiplication even

though she could connect symbols to their corresponding referent. Without this essential connection, she is unable to refine symbolic manipulations based on the referent and fails to enter the developing process.

The interview with Kay highlights a distinct difference in the two types of manipulatives. The virtual computer manipulative allows users to adjust the size of the variables. This is not possible with the concrete manipulative. It may be that the one student who made this mistake with the virtual computer manipulative on posttest problem seven, did so when the size adjustments of the  $x$  and  $y$  variables just happened to be similar to the size differences of the concrete manipulative. That is, it may be that virtual computer *Algebra Lab Gear* size adjustments were such that 3 unit pieces were approximately equal in length to one  $y$  piece.

This example was provided as an illustration of a difference in student learning outcomes between the two types of manipulatives. Students who received instruction with the concrete manipulative were more likely to include a new variable in posttest problem seven. The lab gear pieces that represent products were highlighted during the lesson presentation. That is, the researcher began each lesson with a verbal explanation of the  $x^2$ ,  $y^2$ , and  $xy$  pieces as being representative of products of  $x$  and  $x$ ,  $y$  and  $y$ , and  $x$  and  $y$  respectively.

This area of an *Algebra Lab Gear* piece is a physical representation of a product of the length and width. The connection of area as a physical representation of a product was viewed as somewhat elementary. Therefore, although addressed in the initial presentation, it was not directly addressed in any of the student activities of any treatment. The observation of this error in posttest problem seven for about 10% of the

population of students receiving instruction with the concrete manipulative indicates the topic was not so elementary.

If this oversight had not occurred, quantitative results for differences between the manipulatives would possibly produce results in favor of the concrete manipulative. The error of including an additional variable in the solution to a factoring problem may have been avoided among students using the concrete manipulative with some activities that had focused on the relationship between multiplication and the physical representation of a product. Assuming these activities were successful, the proportion of students who missed problems on the pretest but answered correctly on the posttest would increase among students using the concrete manipulative.

Nevertheless, the mistake of including an additional variable is an identified difference in the adopted solution strategies for the two types of manipulatives. It appears that students who use the concrete manipulative and do not have an understanding of the physical representation of a product are more likely to include an additional variable in their answer than students who received the same instruction with the virtual manipulative.

### **Improvement in accuracy with the concrete and virtual manipulative**

Qualitative interviews affirm student ability to provide accurate solutions to problems that cannot easily be solved with procedural methods. For example, Lynn and Jan both struggled to find the product of  $x + y$  and  $2x + 1$ . Nevertheless, they both could find the correct factorizations of  $xy + y^2 + y + x + y + 1$  and  $2x^2 + 4xy + 2x - xy + y^2 + y$ . Jan received instruction with the computer manipulative. Lynn received instruction with the concrete manipulative. Clearly, both manipulatives gave these students the

opportunity to be successful at solving problems that they would not otherwise be able to solve. Here, qualitative results affirm quantitative results in that both manipulatives pave the way to correct answers even though one leads to occasional incorrect answers.

## Chapter 6

### Conclusions

This study focused on secondary mathematics student learning outcomes when topics are sequenced using a concrete manipulative or a similar virtual computer manipulative. Topics included solving two-step linear equations with a concrete or virtual balance beam and multiplying and factoring polynomial expressions using an alternative form of concrete or virtual *Algebra Lab Gear*.

Data was collected from a total of 14 classes that were given one of four treatments: (a) symbolic, virtual computer, (b) symbolic, concrete object, (c) virtual computer, symbolic, and (d) concrete object, symbolic. A total of 304 students participated. Students were recruited from the classrooms of 4 teachers. All participation was completely voluntary. All students were tested before and after treatments. Three randomly selected students were chosen from each class for a task-based interview for a total of 42 students. Interview questions from 5 students were analyzed in terms of semantic processes undergone when students were prompted to voice their thoughts and solve problems with and without the use of the concrete or virtual computer manipulative.

The first question was “Does the sequencing of mathematics instruction from manipulative to symbolic or symbolic to manipulative alter student learning outcomes at the secondary level?” The premise of this question relies on a theory of learning mathematics that suggests student understanding of mathematical content relies on having previously experienced semantic processes involving sensory-concrete referents.

The second research question was “Do student learning outcomes differ when activity based instruction includes the use of a virtual computer or a concrete manipulative?” The objective was to identify potential differences in achievement measures and adopted solution strategies for problem solving.

### **Results**

The sequencing of instruction did not influence the final class mean posttest score. This is indicated through analysis of covariance. To examine sequencing in detail, classes were split according to background knowledge of groups of students. That is, students who scored below 3 correct responses on the pretest were assumed to have little or no background knowledge in the topic of multiplying and factoring polynomial expressions. Students who got 3 or more problems right were placed into classes of students with some background knowledge. Finally, students who correctly answered at or over 80% of the pretest questions were assumed to have mastery of the topic. Analysis of covariance was repeated with these knowledge categories. There was no indication that background knowledge was an influential determinant of class mean posttest score. This implies that the sequence of instruction does not influence learning outcomes in terms of getting the correct answer.

Interview analysis indicates differences in student learning outcomes based on student background knowledge of procedural approaches. Students whose first exposure to multiplying and factoring polynomial expressions was in the form of a concrete or virtual manipulative tended to use the manipulative to develop, build, and correct symbolic procedures. Correcting of symbolic procedures was not observed with the single student who had demonstrated mastery of the topic on a pretest. This student did

not work back and forth between the manipulative and the procedure. This seems to imply that among the students who were taught the topic of multiplying and factoring polynomials initially with procedural approaches, while being able to understand the manipulative, did not engage in the process of combining the two methods of strategy to monitor their own problem solving techniques.

All treatments were significant in improving overall achievement from pretest to posttest. Thus, classes of students who received instruction with either a concrete or virtual manipulative showed significant mean improvement in test scores. Classes of students who received symbolic instruction before or after the treatment also showed significant mean improvement in test scores.

Analyzed interviews tend to confirm the conclusion that instruction with either the concrete or virtual manipulative may improve student ability to obtain the correct answer. Students were observed being unable to symbolically multiply  $x + y$  times  $2x + 1$  while demonstrating the ability to factor  $xy + y^2 + y + x + y + 1$  with the use of the concrete or virtual manipulative. Both manipulatives allowed students to obtain correct answers that they could not get using procedural methods. Further study is needed to determine whether the ability to factor polynomial expressions will help students to learn symbolic procedures.

Counts were taken of the number of times a student was able to change an incorrect answer on a pretest to a correct answer on a similar posttest question. A greater proportion of students who received instruction with the concrete manipulative were able to change their incorrect answers to correct answers. Likewise, similar counts indicated a smaller than expected proportion of students who received instruction with the concrete



manipulative correctly answered the pretest question while incorrectly answering the posttest question. This result occurred in both topics and favors the concrete manipulative as the manipulative that provides the greatest probability for student posttest improvement.

There was also a clear incorrect solution strategy caused by one treatment. This strategy involved forming a rectangle for factoring  $x^2 + 6x + 9$ . The solution given was  $(x + y)(x + y)$ . The concrete *Algebra Lab Gear* pieces were fixed in length. The length of the  $y$  piece was approximately equal to the length of 3 unit pieces. With this in mind,  $x^2 + 6x + 9$  looks like  $x^2 + (2y)x + y^2 = (x + y)(x + y)$ . Hence, this mistake makes sense and is likely due to the length of the  $y$  *Algebra Lab Gear* piece.

Ten out of 109 students gave this answer after receiving instruction with the concrete manipulative. One student out of 70 made the same mistake with the virtual manipulative. No students made this error on a similar pretest problem. It is assumed that the students who made this mistake were using the  $y$  *Algebra Lab Gear* piece as a replacement for 3 unit pieces.

### **Limitations Revisited**

This study began with several limitations. There was a lack of precision in measuring student background knowledge, teacher effect, and control of treatments. It was not possible to precisely measure student educational background in each topic. Teacher participation or lack of participation was not carefully measured. Treatments were not controlled in that all students received instruction with the virtual computer manipulative. These problems added to the limitations of the study and will be discussed in greater detail in the following paragraphs.

Student participants came from a wide span of educational backgrounds. Comparisons of results for the sequencing of instruction for classes and individual students are extremely broad. Nevertheless, these comparisons were made to contribute to a wide range of mathematical understanding of the complexities surrounding the issue of whether secondary instruction should begin with the use of manipulatives or with symbolic procedures.

The influence of the current classroom teacher is a confounding variable to student learning outcomes. Teachers were all willing volunteers who had no objections to instruction with the use of a concrete or virtual manipulative. Teacher bias tended to favor the use of manipulatives in mathematics instruction. Predictions that these teachers were more likely to have already used alternative forms of instruction cannot be disputed.

Teacher participation in activities varied among treatments. Some teachers were actively walking around the room helping students during the planned activities, others graded papers during the activity, and still others were completely absent. This lack of treatment control contributed to potentially undetected differences in student learning outcomes.

Students volunteered to participate both as a part of treatments and for individual interviews. Students who were interviewed were typically those who also enjoyed learning with the use of a concrete or virtual manipulative. Therefore, the selection of students for analysis was not random. In this respect, student interview results cannot be applied to a general population.

All treatments were taught with the virtual computer manipulative in a single computer lab. Differences in the mechanics of computer operations, however simple,

were emphasized within treatments that utilized the virtual manipulative while these mechanics were not emphasized during treatments that involve the concrete manipulative. These differences were ignored in qualitative analysis of students who received treatments in multiplying and factoring polynomial expressions. These differences clearly hindered qualitative analysis in both topics. It is not unreasonable to assume that instruction methods altered final results.

Another setback of this study was the lack of a control group. It would have been beneficial to have classrooms that received a traditional procedural approach to content at the same school. This additional data could possibly have provided a comparison group. The comparison group would have afforded the researcher the opportunity to distinguish group learning outcomes as unique from those experienced when another form of instruction was administered.

The results of this study cannot be applied to all mathematics classrooms. This study was performed in a single location and is limited in scope. The sample size is not large enough to apply results to similar populations. The validity of conclusions drawn should not be used to make generalizations, but will hopefully prompt more research.

### **Final Remarks**

Some secondary mathematics classrooms begin topics with procedural techniques that assume the student has obtained a formal level of thought necessary for understanding syntactic approaches. If we adopt the theory that understanding mathematics begins with the use of a referent, then there may be a substantial percentage of students in secondary classrooms who would benefit from the use of a manipulative prior to introduction of concepts in symbolic form.

Students at the secondary level do not have trouble associating symbols with their concrete representations. For example, a group of 7 chips is easily represented by the number 7. Seven chips that are marked with an  $x$  can be expressed as  $7x$ . However, secondary students will struggle with the physical actions that represent operations in mathematics. Many secondary students do not typically approach mathematical procedures from a kinesthetic perspective of physical actions on a manipulative. This may be due to underlying beliefs about mathematics as being isolated or separate from other actions or things a person does in life. Students who receive instruction with a concrete or virtual manipulative are given an alternative means for understanding mathematical procedures as an integral part of active play.

It is easy for a secondary mathematics instructor to assume students have the necessary background knowledge that eliminates the need for referents. Indeed, the researcher in this dissertation is guilty of assuming background knowledge that did not exist among students who received treatments involving the topic of solving two-step linear equations. In this case, the virtual manipulative differed from the concrete manipulative in a form that was too abstract to be intuitive. Students who used the virtual manipulative had to click on a button within the applet to divide each side of the virtual balance scale. For example, when solving  $3x = 3$ , a student using the virtual manipulative would divide three  $x$  blocks on the left side and three unit blocks on the right side of the balance scale by clicking on the divide symbol icon and typing in 3. Instantly, the solution would be visually displayed both as an equation and as the balance scale appearing as one  $x$  block on the left and a single unit block on the right.

The process of solving the same equation,  $3x = 3$ , was somewhat different with the concrete manipulative. The answer to division by three would be the number of unit chips that matched up with exactly one  $x$  chip. In this way, students with the concrete manipulative were learning to connect the action of separating pieces into a given number of groups to perform the operation of division.

During the treatment activities, the researcher was asked to explain division to students who used the concrete manipulative, but did not have to do so when the same instruction was given with the virtual manipulative. Subsequent interviews indicated a considerable difference in student ability to connect the grouping process to the operation of division. Almost unanimously, interviewed students who received instruction with the concrete manipulative knew the physical representation of division while students who received instruction with the virtual manipulative did not.

These interviews were not considered for final analysis since the process of connecting the action of grouping was taught during the activities of treatments with the concrete manipulative, but not taught with treatments that utilized the virtual manipulative. Thus, when trying to determine if a virtual manipulative is comparable to a concrete manipulative in instruction, one would be advised to consider how the virtual manipulative models the actual physical movement of objects.

Virtual manipulatives are capable of performing actions that cannot be physically duplicated. This aspect seems advantageous in that it eliminates the burden on the instructor during activities that seem intuitively simple. However, this abstract ability can be detrimental to student understanding of underlying mathematical processes. This was observed when students who received instruction in solving two-step linear

equations did not develop an understanding of what it meant to divide any number of concrete objects. On the other hand, the *Algebra Lab Gear* virtual manipulative drew a red rectangular boundary in the virtual workspace that could be adjusted for a student to check that they had correctly formed the appropriate solution. There is no indication that this additional feature prevented students from forming a connection between multiplication and the rectangular image of a product of length and width.

Current mathematics education theory indicates that the use of referents in building connections to mathematical operations and procedures is essential to meaningful understanding of mathematical content. In the opinion of this researcher, the virtual manipulatives that are most likely to produce mathematical understandings similar to those produced by concrete manipulatives are precisely those virtual manipulatives that most closely model the actual concrete manipulatives.

The question of whether or not the computer is of assistance in mathematics education can be extended to the college level. Here, the comparison between concrete and virtual manipulative becomes increasingly difficult to administer as the objects increase in complexity. For example, three dimensional functions generated on a mathematical computer program are tough to duplicate with actual objects. However, the availability of quick complex three-dimensional images in college level mathematics may support student understanding of content. Thus, at this level, it makes sense to compare student learning outcomes with the virtual computer manipulative to student learning outcomes from a traditional lecture based class. This is the direction of research I would like to pursue in the future.

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