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ON TWO-FOLD GENERALIZATIONS OF CAUCHY'S LEMMA

by

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B.E.D., Illinois State Normal University, 1932

Presented in partial fulfillment of the require-
ment for the Degree of Master of Arts

Montana State University
1950

APPROVED:

A handwritten signature in cursive, appearing to be "H. C. ...", is written above a horizontal line.
Chairman of the Board of Examiners

A handwritten signature in cursive, appearing to be "W. F. Clark", is written above a horizontal line.
Chairman of Committee on Graduate Study

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INTRODUCTION

We investigate the conditions on a and b under which there exists integral solutions greater than $-K$ of the pair of equations:

$$(1) \quad a = \sum_{i=1}^s c_i x_i^2 \quad ; \quad b = \sum_{i=1}^s c_i x_i \quad ,$$

where K is an integer, while each c_i is a given positive integer. Since each solution is greater than $-K$, each $x_i \geq -K + 1$. For solutions ≥ 0 we take $K = 1$. Denote the sum of the c_i by t .

Necessary conditions for solutions are:

$$(2) \quad a, b \text{ integers, } b \geq 11(1 - K), \quad a \equiv b \pmod{2}.$$

A. L. Cauchy investigated the conditions on a and b when each $c_i = 1$. In his paper, "Two-Fold Generalizations of Cauchy's Lemma", L. E. Dickson [1]¹ treated all cases in which

$\sum_{i=1}^4$, $t = 8$. H. Chatland [2] investigated the cases in which

$\sum_{i=1}^4$, $t = 9$. Emma Bravo [3] investigated the cases in which

$\sum_{i=1}^4$, $t = 10$. In this paper cases in which $\sum_{i=1}^4$, $t = 11$, are

considered.

Theorem: Let $(c_1 \cdot \cdot \cdot c_4)$ be one of the eleven sets below.

Let (2) hold and

$$(3) \quad ta \geq b^2, \quad (t - 1)a < b^2 + 2bK + tK^2.$$

Then there exist integral solutions greater than $-K$ of (1) with $s = 4$, if the following forms are "regular", where regular is

¹ Numbers in brackets refer to the references cited at the end of the paper.

defined as follows:

Definition: Let f denote a certain form. If all the positive integers not represented by f coincide with all the positive integers contained in certain arithmetical progressions, the form f is called regular.

The general $f = ax^2 + by^2 + cz^2$ denoted by (a,b,c) was treated by B. W. Jones [4].

When the greatest common divisor of a,b,c is 1, he found that exactly 102 forms are regular.

The same general method may be applied in all cases.

The problem of finding the conditions on a and b has been reduced by L. E. Dickson [1] to one of finding the representations of a ternary quadratic form. He arrives at the identity:

$$(4) \quad (c_3+c_4)(ta-b^2) = (c_1+c_2)v^2 + c_1(c_3+c_4)(2vd+gd^2) + c_3c_4tw^2$$

$$g = c_2 + c_3 + c_4$$

$$t = c_1 + c_2 + c_3 + c_4$$

$$w = x_3 - x_4$$

$$d = x_1 - x_2$$

$$v = (c_3 + c_4)x_2 - c_3x_3 - c_4x_4$$

By transforming this identity into one involving only squares, we get our ternary quadratic forms.

We shall then show the x_1 's to be integers. It then happens that the eleven ternary quadratic forms of this paper are all shown to be irregular by comparison with the table of B. W. Jones [5].

Case I:

$$s = 4, \quad t = 11, \quad c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad c_4 = 8.$$

We have by (4)

$$9(11a - b^2) = 2v^2 + 18vd + 90d^2 + 88w^2$$

$$\begin{aligned} 18(11a - b^2) &= 4v^2 + 36vd + 180d^2 + 176w^2 \\ &= (2v + 9d)^2 + 99d^2 + 176w^2 \end{aligned}$$

$$\text{Set } \xi = (2v + 9d) \qquad \text{Set } \theta = 11a - b^2$$

$$\text{Then } f = 18\theta = \xi^2 + 99d^2 + 176w^2.$$

We have

$$\begin{aligned} b &= x_1 + x_2 + x_3 + 8x_4 \\ v &= 9x_2 - x_3 - 8x_4 \\ d &= x_1 - x_2 \\ w &= x_3 - x_4 \end{aligned}$$

Solving these we find that

$$\begin{aligned} 11x_1 &= b + v + 10d \\ 11x_2 &= b + v - d \\ 99x_3 &= 9b - 9d - 2v + 88w \\ 99x_4 &= 9b - 9d - 2v - 11w \end{aligned}$$

Our problem here is to show that the x 's are integers.

Proof: By hypothesis $a \equiv b \pmod{2}$. Then $\xi^2 + 99d^2 \equiv 0 \pmod{2}$ and $\xi^2 + d^2 \equiv 0 \pmod{2}$ and $\xi \equiv d \pmod{2}$. Therefore v is an integer. From 7θ we have $\xi^2 \equiv 7\theta \pmod{11}$ and $\xi^2 \equiv -7b^2 \pmod{11}$ and $\xi^2 \equiv 4b^2 \pmod{11}$. $\xi \equiv \pm 2b \pmod{11}$. By choice of signs $\xi \equiv -2b \pmod{11}$. Then $2v + 2b + 9d \equiv 0 \pmod{11}$ and $2v + 2b + 20d \equiv 0 \pmod{11}$, and $2(v + b + 10d) \equiv 0 \pmod{11}$; also, $v + b + 10d \equiv 0 \pmod{11}$, and $v + b - d \equiv 0 \pmod{11}$ and x_1 and x_2 are integers.

Let $b + v - d = n$, and $9b - 9d - 2v = m$. Then $9n \equiv m \pmod{11}$, also $-9n + m + 88w = 11(8w - v)$. By 90, $88w^2 + 2v^2 \equiv 0 \pmod{9}$. Therefore $8w - v \equiv 0 \pmod{9}$, or $8w + v \equiv 0 \pmod{9}$. Choose the sign of v so that $8w - v \equiv 0 \pmod{9}$. Then $11(8w-v) \equiv 0 \pmod{99}$, also $9n - m + 44w = 11(4w + v)$. Since $8w - v \equiv 0 \pmod{9}$, $v - 8w \equiv v + w \equiv 0 \pmod{9}$. Therefore $11(v + w) \equiv 0 \pmod{99}$ and the x_1 's are integers.

There are eleven possible combinations of the sum of four integers t , $t = 11$, as follows:

(1,1,1,8), (1,1,2,7), (1,1,3,6), (1,1,4,5), (1,2,2,6), (1,2,3,5),
 (1,2,4,4), (1,3,3,4), (2,2,2,5), (2,2,3,4), (2,3,3,3).

Using the identity (4) we get the following ternary quadratic forms:

Case 1: $t = 11$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = 8$.

$$\begin{aligned} \text{By (4)} \quad 9(11a - b^2) &= 2v^2 + 18vd + 90d^2 + 88w^2 \\ 18(11a - b^2) &= 4v^2 + 36vd + 180d^2 + 176w^2 \\ &= (2v + 9d)^2 + 99d^2 + 176w^2 \end{aligned}$$

Set $\xi^2 = (2v + 9d)^2$

$$\text{Then} \quad 18(11a - b^2) = \xi^2 + 99d^2 + 176w^2$$

Case 2: $t = 11$, $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, $c_4 = 7$.

$$\begin{aligned} \text{By (4)} \quad 9(11a - b^2) &= 2v^2 + 18vd + 90d^2 + 154w^2 \\ 18(11a - b^2) &= 4v^2 + 36vd + 180d^2 + 308w^2 \\ &= (2v + 9d)^2 + 99d^2 + 308w^2 \end{aligned}$$

Set $\xi^2 = (2v + 9d)^2$

$$\text{Then} \quad 18(11a - b^2) = \xi^2 + 99d^2 + 308w^2$$

By (4) the remaining ternary quadratic forms are:

Case 3: $t = 11, c_1 = 1, c_2 = 1, c_3 = 3, c_4 = 6.$

$$18(11a - b^2) = \xi^2 + 99d^2 + 396w^2$$

Case 4: $t = 11, c_1 = 1, c_2 = 1, c_3 = 4, c_4 = 5.$

$$18(11a - b^2) = \xi^2 + 99d^2 + 440w^2$$

Case 5: $t = 11, c_1 = 1, c_2 = 2, c_3 = 2, c_4 = 6.$

$$24(11a - b^2) = \xi^2 + 176d^2 + 396w^2$$

Case 6: $t = 11, c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 5.$

$$24(11a - b^2) = \xi^2 + 176d^2 + 495w^2$$

Case 7: $t = 11, c_1 = 1, c_2 = 2, c_3 = 4, c_4 = 4.$

$$24(11a - b^2) = \xi^2 + 176d^2 + 528w^2$$

Case 8: $t = 11, c_1 = 1, c_2 = 3, c_3 = 3, c_4 = 4.$

$$28(11a - b^2) = \xi^2 + 231d^2 + 528w^2$$

Case 9: $t = 11, c_1 = 2, c_2 = 2, c_3 = 2, c_4 = 5.$

$$7(11a - b^2) = \xi^2 + 77d^2 + 110w^2$$

Case 10: $t = 11, c_1 = 2, c_2 = 2, c_3 = 3, c_4 = 4.$

$$7(11a - b^2) = \xi^2 + 77d^2 + 132w^2$$

Case 11: $t = 11, c_1 = 2, c_2 = 3, c_3 = 3, c_4 = 3.$

$$30(11a - b^2) = \xi^2 + 396d^2 + 495w^2$$

By comparing the eleven ternary quadratic forms above with the table of B. W. Jones [5] we find all of these forms to be irregular.

Definition: Given a form f . If there does not exist an arithmetical progression containing the positive integer K , not

represented by f , all of whose positive terms are not represented by f , f is called irregular.

Six Lemmas useful in obtaining integral solutions for irregular forms [5].

Lemma 1: If p is an odd prime dividing neither a nor b and if K is any integer, $ax^2 + by^2 \equiv K \pmod{p}$ is solvable.

Proof: x^2 takes $1 + \frac{1}{2}(p - 1)$ values incongruent modulo p . The same is therefore true of $ax^2 - K$ and of $-by^2$. One of the values of the former is congruent to one of the values of the latter, since otherwise there would be $p + 1$ integers incongruent modulo p , which is absurd.

Write $f = ax^2 + by^2 + cz^2$

Lemma 2: If abc is not divisible by the odd prime p , and if K is any integer, $f \equiv K \pmod{p}$ has solutions with x and y not both divisible by p .

Proof: According as K is not or is divisible by p , take $z \equiv 0$, or $z \equiv 1 \pmod{p}$. Then $\ell = K - cz^2$ is not divisible by p . By Lemma 1, $ax^2 + by^2 \equiv \ell \pmod{p}$ has integral solutions x, y , which are evidently not both divisible by p .

Lemma 3: If abc is not divisible by the odd prime p , and if K and n are arbitrary integers, $n \geq 1$, $f \equiv K \pmod{p^n}$ has solutions with x and y not both divisible by p .

Proof: This is true when $n = 1$ by Lemma 2. To proceed by induction from $n = m \geq 1$ to $n = m + 1$, let $f \equiv K \pmod{p^m}$ have solutions ξ, η, \mathcal{J} , such that ξ and η are not both divisible by p . Then $a\xi^2 + b\eta^2 + c\mathcal{J}^2 = K + p^m q$, where q is an integer. Take $x = \xi + p^m X$, $y = \eta + p^m Y$, and $z = \mathcal{J} + p^m Z$.

Then $f \equiv K + p^m q + 2p^m L \pmod{p^{m+1}}$
 $L = a\xi X + b\eta Y + c\zeta Z.$

We can choose X, Y, Z as follows so that $q + 2L \equiv 0 \pmod{p}$.
 If ξ is not divisible by p , take $Y = Z = 0$, $2a\xi X \equiv -q \pmod{p}$.
 If η is not divisible by p , take $X = Z = 0$. In either case,
 $f \equiv K \pmod{p^{m+1}}$ and x and y are not both divisible by p . The
 induction is complete.

Lemma 4: If an odd prime p divides c , but not abK , $f \equiv K \pmod{p^n}$
 has solutions with x and y not both divisible by p .

Proof: This is true when $n = 1$ by Lemma 1. It follows for any n
 by induction as in the proof of Lemma 3.

Lemma 5: If an odd prime p divides c and K , but not ab , and if $-ab$
 is a quadratic residue of p , $f \equiv K \pmod{p^n}$ has solutions with x
 and y both prime to p .

Proof: There exist integers P and R , prime to p , such that $-ab \equiv P^2$,
 $P \equiv aR$, whence $aR^2 + b \equiv 0 \pmod{p}$. Hence $f \equiv K \pmod{p}$ has the
 solutions $x = R, y = 1, z$ arbitrary. The proof for any n by in-
 duction is like that of Lemma 3 with $Y = Z = 0$.

Lemma 6: If K is odd and if $f \equiv K \pmod{8}$ is solvable, then
 $f \equiv K \pmod{2^n}$ is solvable when n is arbitrary.

Proof: To proceed by induction, let $f \equiv K \pmod{2^n}$ have solutions
 ξ, η, ζ , for $m = 3$. Then $a\xi^2 + b\eta^2 + c\zeta^2 = K + 2^m q$.
 Take $x = \xi + 2^{m-1} X$, $y = \eta + 2^{m-1} Y$, and $z = \zeta + 2^{m-1} Z$.
 Then $f \equiv K + 2^m q + 2^m L \pmod{2^{m+1}}$,
 $L = a\xi X + b\eta Y + c\zeta Z.$

Since K is odd, $a\xi, b\eta, c\zeta$, are not all even. Hence there exist

integral solutions X, Y, Z , of $q + L \equiv 0 \pmod{2}$. Hence $f \equiv K \pmod{2^{m+1}}$ and the induction is complete.

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