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Spectral theorem for self-adjoint compact operators

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THE SPECTRAL THEOREM FOR SELF-ADJOINT COMPACT OPERATORS

By

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CHAPTER 1

Linear Spaces, Metric Spaces, and Normed Linear Spaces

We begin with some basic definitions.

Definition 1.1. A linear space (or vector space) is an abelian group X with a function $\cdot: F \times X \to X$, where F is either the reals or the complexes, satisfying the following conditions for all α, β in F and all x, x' in X:

(1) $\alpha\cdot(x + x') = \alpha\cdot x + \alpha\cdot x'$.

(2) $\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x$.

(3) $1\cdot x = x$.

(4) $(\alpha + \beta)\cdot x = \alpha\cdot x + \beta\cdot x$.

The elements of F are called scalars; those of X are called vectors.

Definition 1.2. A nonempty subset M of a linear space X is called a linear manifold in X if $\alpha x + \beta y$ is in M whenever x and y are in M and α, β are arbitrary elements of F.

Definition 1.3. Let X and Y be linear spaces over the same field F. If A is a function with domain X and range contained in Y, then A is called a linear operator if the following conditions are satisfied for arbitrary α in F and arbitrary x_1, x_2, x in X:

(1) $A(x_1 + x_2) = Ax_1 + Ax_2$.

(2) $A(\alpha x) = \alpha Ax$.

Definition 1.4. Let A be a linear operator on X into Y, where X and Y are linear spaces over the same field. The null manifold of A is the set of all x in X such that $Ax = 0$. We denote this set by N(A).

Definition 1.5. A linear operator P with domain X and range in X is called a projection (of X) if $P^2 = P$. If M is the range of P, then P is called a projection of X onto M.

Definition 1.6. Let $M_1, \ldots, M_n$ $(n \geq 2)$ be linear manifolds in a linear space X. We say that this set of linear manifolds is linearly independent if x_i in M_i and $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ implies that $x_k = 0$ if $\alpha_k \neq 0$. The linear manifold generated by the elements of $M_1 \cup \ldots \cup M_n$ is denoted by $M_1 \oplus \ldots \oplus M_n$ and is called the direct sum of $M_1, \ldots, M_n$. Elements of the direct sum are representable uniquely in the form $x = x_1 + \ldots + x_n$, with x_i in M_i.

Definition 1.7. Two linear spaces X and Y (over the same field) are said to be isomorphic if there is a linear operator T whose domain is X, whose range is all of Y, and whose inverse T^{-1} exists.

Definition 1.8. A metric space is a set X with a function $d : X \times X \to R$, where R is the real number field, satisfying the following conditons for all x_1, x_2, x_3 in X:

(1) $d(x_1, x_2) = d(x_2, x_1)$.

(2) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$.

(3) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

(4) $d(x_1, x_2) \geq 0$.

The function d is called a distance function (or metric) on X.

Definition 1.9. A sequence $\{x_n\}$ in a metric space X is called a Cauchy sequence if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Definition 1.10. A sequence $\{x_n\}$ in a metric space X is called convergent if there is a point x in X such that $d(x_n, x) \to 0$ as $n \to \infty$. We then write $x_n \to x$ and call x the limit of the sequence.

Definition 1.11. A metric space X is said to be complete if every Cauchy sequence in X has a limit in X.

Definition 1.12. A metric space X is said to be totally bounded if for each $\epsilon > 0$ there is a finite subset $x_1, x_2, \ldots, x_n$ of X such that, if x is an arbitrary element of X, then $d(x, x_i) < \epsilon$ for some $i = 1, 2, \ldots, n$.

Definition 1.13. A subset S of a metric space X is called open if, given x_o in S, there is an $\epsilon > 0$ such that x is in S whenever $d(x, x_o) < \epsilon$.

Definition 1.14. Let S be a subspace of a metric space X. If F is a family of subsets of X such that each point of S belongs to at least one member of F, then F is said to cover S. If all the sets in F are open, F is called an <u>open covering</u> of S.

<u>Definition</u> 1.15. A subset S of a metric space X is called <u>compact</u> if every open covering of S includes a finite subfamily which covers S.

The following well-known results are stated without proof:

<u>Theorem</u> 1.16. A metric space X is compact if and only if it is both totally bounded and complete. ([4], p. 142).

<u>Theorem</u> 1.17. In a metric space, a set S is compact if and only if every sequence in S contains a convergent subsequence with limit in S. ([5], p. 72).

<u>Definition</u> 1.18. Let X and Y be metric spaces, and let f be a function with domain X and range Y. f is said to be <u>continuous</u> at the point x_o in X if to each neighborhood V of $f(x_o)$ in Y there corresponds a neighborhood U of x_o in X such that $f(U) \subset V$.

<u>Theorem</u> 1.19. Suppose f is a function with domain X and range Y, where X and Y are metric spaces. Then f is continuous on X if and only if $f^{-1}(V)$ is an open set in X whenever V is an open set in Y.

Proof: Suppose $f^{-1}(V)$ is open whenever V is open. Let x_o be any point in X and V any neighborhood of $f(x_o)$. Then $f^{-1}(V)$ is open and contains x_o, so there is a neighborhood U of x such that $f(U) \subset V$. Hence f is continuous at x_o. Conversely, suppose f is continuous on X, and let

V be an open set in Y. Then, if x is in $f^{-1}(V)$, V is a neighborhood of f(x), and hence $f^{-1}(V)$ must include a neighborhood of x. This implies that $f^{-1}(V)$ is open.

<u>Theorem</u> 1.20. The continuous image of a compact set is compact.

Proof: Let f be a continuous function which maps the compact set K onto a metric space Y. If F is an open cover for Y, then the collection of sets $f^{-1}(O)$ for all O in F is an open covering of K. By the compactness of K, there is a finite number $O_1, \ldots, O_n$ of sets of F such that the sets $f^{-1}(O_i)$ cover K. Since f is onto Y, the sets $O_1, \ldots, O_n$ cover Y, whence Y is compact.

<u>Theorem</u> 1.21. A closed subset of a compact space is compact.

Proof: Let X be compact, S a closed subset of X, and F an open covering for S. Then $F \cup \{\setminus S\}$ is an open covering for X and so must have a finite subcovering $\{\setminus S, O_1, \ldots, O_n\}$. Then the sets $O_1, \ldots, O_n$ cover S, so S is compact. ([4], p. 137).

<u>Definition</u> 1.22. Two metric spaces X and Y are said to be <u>isometric</u> if there is a function f with domain X and range Y such that $d(x_1, x_2) = d(f(x_1), f(x_2))$ for every pair of points x_1, x_2 in X.

Definitions 1.23, 1.24. A <u>norm</u> on a linear space X is a real-valued function $\|\cdot\|$ with the properties:

(1) $\| x_1 + x_2 \| \leq \| x_1 \| + \| x_2 \|$.

(2) $\| \alpha x \| = |\alpha| \; \| x \|$.

(3) $\| x \| \geq 0$.

(4) $\| x \| \neq 0$ if $x \neq 0$.

A linear space on which a norm is defined becomes a metric space if we define $d(x_1, x_2) = \| x_1 - x_2 \|$. A linear space which is a metric space in this way is called a <u>normed linear space</u>.

We have

$$\| x_1 \| = \| x_1 - x_2 + x_2 \| \leq \| x_1 - x_2 \| + \| x_2 \|$$

$$\| x_1 \| - \| x_2 \| \leq \| x_1 - x_2 \|$$

Similarly, $\| x_2 \| - \| x_1 \| \leq \| x_2 - x_1 \|$. Therefore $| \; \| x_1 \| - \| x_2 \| \; | \leq \| x_1 - x_2 \|$, and it follows that $\| x \|$ is a continuous function of x.

<u>Definition</u> 1.25. Two normed linear spaces are said to be <u>isometrically isomorphic</u>, or congruent, if there is a one-to-one correspondence between the elements of X and Y which makes the spaces both isomorphic as vector spaces and isometric as metric spaces.

Two normed linear spaces X and Y are congruent if and only if there is a linear operator T with domain X and range Y such that T^{-1} exists and such that $\|Tx\| = \|x\|$ for every x in X. For, $\|Tx\| = \|x\|$ for every x in X implies that $\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| = \|x_1 - x_2\|$ for arbitrary x_1, x_2 in X. Conversely, if $\|Tx_1 - Tx_2\| = \|x_1 - x_2\|$ for x_1, x_2 in X, then

$\|Tx\| = \|T(x+x'-x')\| = \|T(x+x')-T(x')\| = \|(x+x')-x'\| = \|x\|$ for every x in X. x' is arbitrary in X.

Definition 1.26. Two normed linear spaces X and Y are said to be topologically isomorphic if there is a linear operator T which establishes an isomorphism of X and Y and which has the property that T and T^{-1} are continuous.

We now derive some results which will be used later.

Theorem 1.27. Let X and Y be normed linear spaces and T a linear operator on X into Y. Then T is continuous either at every point of X or at no point of X. T is continuous on X if and only if there is a constant M such that $\|Tx\| \leq M\|x\|$ for every x in X.

Proof: Let x_o and x_1 be any points of X, and suppose T is continuous at x_o. Then to each $\varepsilon > 0$ corresponds a $\delta > 0$ such that $\|Tx - Tx_o\| < \varepsilon$ if $\|x - x_o\| < \delta$. Now suppose $\|x - x_1\| < \delta$. Then $\|(x + x_o - x_1) - x_o\| < \delta$, so $\|T(x + x_o - x_1) - Tx_o\| < \varepsilon$. But by the linearity of T, $T(x + x_o - x_1) - Tx_o = Tx - Tx_1$, so $\|Tx - Tx_1\| < \varepsilon$. Thus T is continuous at x_1, and the first assertion is proved.

If $\|Tx\| \leq M\|x\|$ for all x, then $\|Tx - T(0)\| = \|Tx\| < \varepsilon$ for arbitrary $\varepsilon > 0$ whenever $\|x\| < \frac{\varepsilon}{M}$, so T is continuous at 0. By the first part of the proof it follows that T is continuous on X. Conversely, if T is continuous at 0, there is a $\delta > 0$ such that $\|Tx\| < 1$ if $\|x\| < \delta$. Now, if $x \neq 0$, let $x_o = \delta x/2\|x\|$, so that $\|x_o\| = \delta/2 < \delta$. Then $\|Tx_o\| < 1$. But $Tx_o = T(\delta x/2\|x\|) = (\delta/2\|x\|)Tx$, so $\|Tx\| = \frac{2\|x\|}{\delta}\|Tx_o\| < 2/\delta\|x\|$. Thus, taking $M = 2/\delta$, we have $\|Tx\| \leq M\|x\|$ if $x \neq 0$, and this inequality is true as well when $x = 0$. The proof is thus complete.

Definition 1.28. Let X and Y be normed linear spaces and T a continuous linear operator on X into Y. Then the smallest admissible value of M in the inequality $\|Tx\| \leq M\|x\|$ is called the norm of T and is denoted by $\|T\|$.

It follows from Definition 1.28 that

$$(1.28\text{-A}) \qquad \|T\| = \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|}$$

$$= \sup_{\|x\|\neq 0} \left\|T\left(\frac{x}{\|x\|}\right)\right\|$$

$$(1.28\text{-B}) \qquad = \sup_{\|x\|=1} \|Tx\|$$

$$(1.28\text{-C}) \qquad = \sup_{\|x\|\leq 1} \|Tx\|.$$

The step from (B) to (C) is justified since if $\|x\| < 1$, then $\left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{1}{\|x\|}\|Tx\| > \|Tx\|$. Also, it is clear from (A) that $\|Tx\| \leq \|T\|\ \|x\|$.

If X and Y are linear spaces over the same field, the set of all linear operators on X into Y is a linear space if we define addition of operators and multiplication of operators by scalars in the following way:

$$(A+B)x = Ax + Bx, \quad (\alpha A)x = \alpha(Ax).$$

If X and Y are normed linear spaces, those linear operators on X into Y which are continuous on X form a subspace of the space of all linear operators on X into Y. We denote this space by [X, Y].

Now let X be a normed linear space and let F be the associated scalar field. A linear operator on X into F is called a linear functional. We call the linear space of all linear functionals on X the algebraic conjugate of X, and denote it by X^f. A continuous linear functional on X is an element of the space [X, F], which we denote by X'. X' is a subspace of X^f.

Definition 1.29. Let X and Y be normed linear spaces, and suppose A is in [X, Y]. If y' is in Y', the linear functional x' defined on X by $x'(x) = y'(Ax)$ is in X'. We write $x' = A'y'$. The operator A' maps Y' into X' and is called the conjugate of A. The definition of A' is expressed by the formula

(1.29-A) $(A'y')(x) = y'(Ax)$, x in X, y' in Y'.

Theorem 1.30. Let T be a linear operator on X to Y, where X and Y are normed linear spaces. Then the inverse T^{-1} exists and is continuous if and only if there is a constant $m > 0$ such that

(1.30-A) $m\|x\| \leq \|Tx\|$

for every x in X.

Proof: If (1.30-A) holds and $Tx = 0$, it follows that $x = 0$. This implies that T^{-1} exists. Now $y = Tx$

is equivalent to $x = T^{-1}y$. Hence (1.30-A) is equivalent to $m\|T^{-1}y\| \leq \|TT^{-1}y\|$, or $\|T^{-1}y\| \leq \frac{1}{m}\|y\|$ for all y in the range of T, which is the domain of T^{-1}. Hence T^{-1} is continuous by Theorem 1.27.

Conversely, if T^{-1} exists and is continuous, by Theorem 1.27 there is a constant M such that $\|T^{-1}y\| \leq M\|y\|$ for all y in the domain of T^{-1}. Since T^{-1} exists, $T^{-1}y = x$ is equivalent to $y = Tx$. Also, the range of T^{-1} is all of X. This implies that $\|x\| \leq M\|Tx\|$ for all x in X, which is equivalent to (1.30-A).

Corollary 1.31. Two normed linear spaces X and Y are topologically isomorphic if and only if there is a linear operator T with domain X and range Y, and positive constants m, M such that

(1.31-A) $m\|x\| \leq \|Tx\| \leq M\|x\|$

for every x in X.

We are ready now to prove that two normed linear spaces of the same finite dimension n over the same field are topologically isomorphic. We begin by stating without proof the Bolzano-Weierstrass theorem for $\ell^2(n)$, the normed linear space of all n-tuples $(\alpha_1, \ldots, \alpha_n)$ of real numbers (or all n-tuples of complex numbers). If $x = (\alpha_1, \ldots, \alpha_n)$ is in $\ell^2(n)$, then the norm of x is defined to be $\|x\| = (\sum_{i=1}^{n} |\alpha_i|^2)^{1/2}$. (If the norm of x is defined to be $\|x\| = \sum_{i=1}^{n} |\alpha_i|$, then this space is called $\ell'(n)$.)

Theorem 1.32. (Bolzano-Weierstrass). Every bounded sequence in $\ell^2(n)$ has at least one cluster point. ([1], p. 49).

Our next step is to establish the following lemma:

Lemma 1.33. The surface of the unit sphere in $\ell'(n)$ (the set S for which $|\alpha_1| + \ldots + |\alpha_n| = 1$) is compact.

Proof: By Theorem 1.32 every sequence in the unit sphere of $\ell^2(n)$ (the set S_2, for which $(\sum_{i=1}^{n}|\alpha_i|^2)^{1/2} \leq 1$) has a convergent subsequence, and it follows from Theorem 1.17 that S_2 is compact. Consider the mapping $f(x) = x$ from S into $\ell^2(n)$. Then f^{-1} exists and has range S, and $f^{-1}(x) = x$. Since $\sum_{i=1}^{n}|\alpha_i| = 1$ implies $|\alpha_i| \leq 1$ for $i = 1, \ldots, n$, it follows that $\sum_{i=1}^{n}|\alpha_i|^2 \leq 1$. Therefore the domain of f^{-1} is a subset of S_2, which we denote by K.

Suppose now that $x = (\alpha_1, \ldots, \alpha_n)$ and $x' = (\alpha_1', \ldots, \alpha_n')$ are elements of K such that $\|x - x'\|_{\ell^2} = (\sum_{i=1}^{n}|\alpha_i - \alpha'_i|^2)^{1/2} < \epsilon/n$ for arbitrary $\epsilon > 0$. Then

$$\sum_{i=1}^{n}|\alpha_i - \alpha'_i|^2 < \epsilon^2/n^2, \text{ so } |\alpha_i - \alpha'_i|^2 < \epsilon^2/n^2$$

for $i = 1, \ldots, n$. Therefore $\|x - x'\|_{\ell'} = \sum_{i=1}^{n}|\alpha_i - \alpha'_i| < \epsilon$, and it follows that f^{-1} is a continuous map from S_2 onto S. Since f^{-1} is continuous, K is a closed subset of S_2, so K is compact by Theorem 1.21. The desired result now follows from Theorem 1.20.

Theorem 1.34. Let X_1 and X_2 be two normed linear spaces of the same finite dimension n over the same field. Then X_1 and X_2 are topologically isomorphic.

Proof: The case $n = 0$ is trivial, and we assume $n \geq 1$. Since the relation of topological isomorphism is transitive, it will suffice to prove that if X is an n-dimensional normed linear space, then X is topologically isomorphic to $\ell'(n)$. Suppose that $x_1, \ldots, x_n$ is a basis for X. If $x = \alpha_1 x_1 + \ldots + \alpha_n x_n$ is the representation of an arbitrary x in X, the correspondence $x \longleftrightarrow (\alpha_1, \ldots, \alpha_n)$ defines an isomorphism of X and $\ell'(n)$. By Corollary 1.31, all we have to prove is that there are positive constants m and M such that

$$\text{(1.34-A)} \quad \|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \leq M\,(|\alpha_1| + \ldots + |\alpha_n|)$$

and

$$\text{(1.34-B)} \quad m(|\alpha_1| + \ldots + |\alpha_n|) \leq \|\alpha_1 x_1 + \ldots + \alpha_n x_n\|$$

for all possible sets of values $(\alpha_1, \ldots, \alpha_n)$. Now (1.34-A) is true if we choose $M = \max \{\|x_1\|, \ldots, \|x_n\|\}$, since

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \leq \|\alpha_1 x_1\| + \ldots + \|\alpha_n x_n\|$$
$$= |\alpha_1|\ \|x_1\| + \ldots + |\alpha_n|\ \|x_n\|.$$

To prove (1.34-B) it suffices to show there is a constant m such that $m \leq \|\alpha_1 x_1 + \ldots + \alpha_n x_n\|$ if $|\alpha_1| + \ldots + |\alpha_n| = 1$, for (1.34-B) holds if $\alpha_1 = \ldots = \alpha_n = 0$, and if $C = |\alpha_1| + \ldots + |\alpha_n| > 0$, we can define $\eta_i = C^{-1}\alpha_i$. Then $|\eta_1| + \ldots + |\eta_n| = 1$ and $\|\eta_1 x_1 + \ldots + \eta_n x_n\| = C^{-1}\|\alpha_1 x_1 + \ldots + \alpha_n x_n\|$, so (1.34-B) holds if $m \leq \|\eta_1 x_1 + \ldots + \eta_n x_n\|$.

Now let $f(\alpha_1, \ldots, \alpha_n) = \|\alpha_1 x_1 + \ldots + \alpha_n x_n\|$. Since $|\ \|x\| - \|y\|\ | \leq \|x-y\|$, we have

$$|f(\alpha_1, \ldots, \alpha_n) - f(\eta_1, \ldots, \eta_n)| \leq \|(\alpha_1-\eta_1)x_1 + \ldots + (\alpha_n-\eta_n)x_n\|$$

$$\leq M(|\alpha_1-\eta_1| + \ldots + |\alpha_n-\eta_n|),$$

and it follows that f is continuous on $\ell'(n)$. By Lemma 1.33, S is compact. Hence f, being continuous on S, attains a minimum value $m \geq 0$ on S. But $m > 0$, for $m = 0$ would imply that $x_1, \ldots, x_n$ are linearly dependent, contrary to the fact that they form a basis for X. The proof of (1.34-B) and of Theorem 1.34 is now complete.

If X and Y are topologically isomorphic normed linear spaces under the mapping T and $\{x_n\}$ is a Cauchy sequence in X, then by Corollary 1.31 there is a positive constant M such that $\|T(x_n - x_m)\| = \|Tx_n - Tx_m\| \leq M\|x_n - x_m\|$, whence $\{Tx_n\}$ is a Cauchy sequence in Y. Conversely, if $\{y_n\}$ is a Cauchy sequence in Y, there is a positive constant m such that

$$m\|T^{-1}y_n - T^{-1}y_m\| \leq \|T(T^{-1}y_n) - T(T^{-1}y_m)\| = \|y_n - y_m\|,$$

whence $\{T^{-1}y_n\}$ is a Cauchy sequence in X. It follows from the continuity of T and T^{-1} that if X is complete, then Y is complete, and conversely. Now $\mathcal{L}'(n)$ is complete as a consequence of the completeness of the real numbers. Thus we have:

Theorem 1.35. A finite-dimensional normed linear space is complete.

Corollary 1.36. Any finite-dimensional subspace of a normed linear space is closed.

Proof: This result follows from the well-known fact that a complete subset of a metric space is closed. ([2], p. 51).

Theorem 1.37. If X is a finite-dimensional normed linear space, then each closed and bounded set in X is compact.

Proof: This result is true by classical analysis for $\mathcal{L}'(n)$. It then follows from Theorem 1.34 that the theorem is true for any finite-dimensional space X, for the properties of being bounded and closed are transferred from a set S to its image S_1 in $\mathcal{L}'(n)$ by the topological isomorphism, and the compactness is then carried back from S_1 to S.

CHAPTER 2

Inner-Product Spaces

Definition 2.1. A complex linear space X is called an inner-product space if there is defined on X x X a complex-valued function (x_1, x_2) (called the inner product of x_1 and x_2) with the following properties for arbitrary x, x_1, x_2, x_3 in X:

(1) $(x_1 + x_2, x_3) = (x_1, x_3) + (x_2, x_3)$.

(2) $(x_1, x_2) = (\overline{x_2, x_1})$.

(3) $(\alpha x_1, x_2) = \alpha(x_1, x_2)$.

(4) $(x, x) \geq 0$ and $(x, x) \neq 0$ if $x \neq 0$ ((x,x) is real by (2)).

A real linear space X is called an inner-product space if there is defined on X x X a real-valued function with the properties (1)-(4), except that (2) is written without the bar over (x_2, x_1).

It follows from (2) and (3) that $(x_1, \alpha x_2) = (\overline{\alpha x_2, x_1}) = \bar{\alpha}\ (\overline{x_2, x_1}) = \bar{\alpha}\ (x_1, x_2)$, where the bar over α is omitted for a real space. Also, from (1) and (2) we obtain

$$(x_1, x_2 + x_3) = (\overline{x_2 + x_3, x_1}) = (\overline{x_2, x_1}) + (\overline{x_3, x_1}) = (x_1, x_2) + (x_1, x_3).$$

Theorem 2.2. If X is an inner-product space, then

(2.2-A) $|(x_1, x_2)| \leq \sqrt{(x_1, x_1)}\sqrt{(x_2, x_2)}$.

Proof: For any α and β we have

$$(\alpha x + \beta y, \alpha x + \beta y) = \alpha\bar{\alpha}(x,x) + \alpha\bar{\beta}(x,y) + \beta\bar{\alpha}(y,x) + \beta\bar{\beta}(y,y) \geq 0.$$

We choose $\alpha = t$, where t is real, and define

$$\beta = \frac{(x,y)}{|(x,y)|} \text{ if } (x,y) \neq 0,$$

$$\beta = 1 \text{ otherwise.}$$

Then $\bar{\beta}(x,y) = \frac{\overline{(x,y)}}{|(x,y)|}(x,y) = |(x,y)|$ if $(x,y) \neq 0$, and this equality also holds if $(x,y) = 0$. Also,

$\beta\bar{\beta} = \frac{(x,y)}{|(x,y)|}\frac{\overline{(x,y)}}{|(x,y)|} = 1$ if $(x,y) \neq 0$, and again this equality holds if $(x,y) = 0$. Therefore

$$t^2(x,x) + t|(x,y)| + [\frac{(x,y)}{|(x,y)|}]t\,\overline{(x,y)} + (y,y)$$
$$= t^2(x,x) + 2t|(x,y)| + (y,y) \geq 0 \text{ for all real } t.$$

If $(x,x) = 0$, (2.2-A) is trivially true. If $(x,x) \neq 0$,

$$t^2(x,x)^2 - 2t(x,x)|(x,y)| + (x,x)(y,y) \geq 0,$$

and letting $t = -\frac{|(x,y)|}{(x,x)}$ we obtain

$$|(x,y)|^2 - 2|(x,y)|^2 + (x,x)(y,y) \geq 0,$$

or

$$(x,x)(y,y) \geq |(x,y)|^2,$$

which is equivalent to (2.2-A).

<u>Theorem</u> 2.3. If X is an inner-product space, then $\sqrt{(x,x)}$ has the properties of a norm.

Proof: We write $\|x\| = \sqrt{(x,x)}$. Then by the definition of inner product we have immediately that $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$, and

$\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{\alpha^2(x,x)} = |\alpha|\ \|x\|$. Also we have

$$\|x + y\|^2 = (x + y,\ x + y) = (x,x) + (x,y) + (y,x) + (y,y).$$

For complex A we write Re A $= \frac{1}{2}(A+\overline{A})$ = real part of A. Since $|\text{Re } A| \leq |A|$, we have

$$\|x + y\|^2 = \|x\|^2 + 2 \text{ Re } (x,y) + \|y\|^2$$
$$\leq \|x\|^2 + 2|(x,y)| + \|y\|^2.$$

With (2.2-A) this gives

$$\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\ \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \text{ so}$$
$$\|x + y\| \leq \|x\| + \|y\|.$$

This completes the proof.

In the following theorem the norm on X x X is defined to be $\|(x_1,\ x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2}$.

<u>Theorem</u> 2.4. The inner product $(x_1,\ x_2)$ is a continuous function on X x X.

Proof: Let $x_1 - u_1 = v_1$, $x_2 - u_2 = v_2$ where x_1, x_2, u_1, u_2 are arbitrary and u_1, u_2 are fixed. Then

$$(x_1,\ x_2) - (u_1,\ u_2) = (u_1 + v_1,\ u_2 + v_2) - (u_1,\ u_2)$$
$$= (u_1,\ v_2) + (v_1,\ u_2) + (v_1,\ v_2).$$

Hence by (2.2-A) we have

$$|(x_1,\ x_2) - (u_1,\ u_2)| \leq |(u_1,\ v_2)| + |(v_1,\ u_2)| + |(v_1, v_2)|$$
$$\leq \|u_1\|\ \|x_2 - u_2\| + \|x_1 - u_1\|\ \|u_2\|$$
$$+ \|x_1 - u_1\|\ \|x_2 - u_2\|.$$

If $\delta = \max\left\{\|u_1\|,\ \|u_2\|\right\}$ and $3\delta > \epsilon > 0$, then $\|(x_1,\ x_2) - (u_1,\ u_2)\| < \frac{\epsilon}{3\delta}$ implies that $\|x_1 - u_1\| < \frac{\epsilon}{3\delta}$ and

$\|x_2 - u_2\| < \frac{\epsilon}{3\delta}$. Therefore

$$|(x_1, x_2) - (u_1, u_2)| \leq \|u_1\| \cdot \frac{\epsilon}{3\delta} + \frac{\epsilon}{3\delta} \cdot \|u_2\| + \frac{\epsilon}{3\delta} \cdot \frac{\epsilon}{3\delta}$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

and the theorem is proved.

It follows from Theorem 2.4 that if $\sum_{i=1}^{\infty} x_i = x'$, then $(x,x') = \lim_{n \to \infty} (x, \sum_{i=1}^{n} x_i) = \lim_{n \to \infty} \sum_{i=1}^{n} (x,x_i)$.

Definition 2.5. Two vectors x and y are orthogonal if $(x, y) = 0$. If $(x, y) = 0$ we write $x \perp y$.

Definition 2.6. A set S of vectors is called an orthogonal set if $(x, y) = 0$ for every pair x, y in S such that $x \neq y$. If in addition $\|x\| = 1$ for every x in S, the set is called an orthonormal set.

Definition 2.7. Two linear manifolds M, N in the inner-product space X are said to be orthogonal if $(x, y) = 0$ whenever x is in M and y is in N. A family of linear manifolds is called an orthogonal family if each pair of distinct manifolds from the family is orthogonal.

Definition 2.8. If S is a nonempty subset of X, the set of all x such that $(x, y) = 0$ if y is in S is called the orthogonal complement of S, and is denoted by $S^{\perp}$. If x is in $S^{\perp}$, we write $x \perp S$.

In the proof of the next theorem we require the following well-known inequality, a proof of which is given in the book Inequalities, by Hardy, Littlewood, and Polya:

<u>Cauchy's Inequality</u>. Let $\{a_i\}$ and $\{b_i\}$ be non-negative convergent sequences of real numbers. Then

$$\sum_{i=1}^{\infty} a_i b_i \leq \left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2} \left(\sum_{i=1}^{\infty} b_i^2\right)^{1/2}.$$

<u>Theorem</u> 2.9. Let S be an orthonormal set in X. If $u_1, \ldots, u_n$ is any finite collection of distinct elements of S, then

$$(2.9\text{-A}) \quad \sum_{i=1}^{n} |(x,u_i)|^2 \leq \|x\|^2 \text{ for all } x \text{ in } X.$$

The set of those u in S such that $(x, u) \neq 0$, where x is any fixed element of X, is either finite or countably infinite. If x, y are in X, then

$$(2.9\text{-B}) \quad \sum_{u \in S} |(x, u)(\overline{y, u})| \leq \|x\| \, \|y\|,$$

where the sum on the left includes all u in S for which $(x, y)(\overline{y, u}) \neq 0$.

Proof: We write $\alpha_i = (x, u_i)$. Then

$$0 \leq (x - \sum_{i=1}^{n} \alpha_i u_i, \; x - \sum_{i=1}^{n} \alpha_i u_i)$$

$$= (x,x) - (\sum_{i=1}^{n} \alpha_i u_i, \; x) - (x, \sum_{i=1}^{n} \alpha_i u_i) + (\sum_{i=1}^{n} \alpha_i u_i, \sum_{i=1}^{n} \alpha_i u_i)$$

$$= \|x\|^2 - \sum_{i=1}^{n} \alpha_i (u_i, \; x_i) - \sum_{i=1}^{n} \bar{\alpha}_i (x_i u_i) + \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_i \bar{\alpha}k(u_i, \; u_k)$$

$$= \|x\|^2 - \sum_{i=1}^{n} \alpha_i \bar{\alpha}_i - \sum_{i=1}^{n} \bar{\alpha}_i \alpha_i + \sum_{i=1}^{n} \alpha_i \bar{\alpha}_i$$

$$= \|x\|^2 - \sum_{i=1}^{n} |\alpha_i|^2.$$

Therefore $\|x\|^2 \geq \sum_{i=1}^{n} |\alpha_i|^2 = \sum_{i=1}^{n} |(x, u_i)|^2$.

It follows from (2.9-A) that, if x is in X and n is a positive integer, then the number of elements u of S such that $|(x, u)| \geq \frac{1}{n}$ cannot exceed $n^2\|x\|^2$. Since $|(x, u)| > \frac{1}{n}$ for some n if $(x, u) \neq 0$, the set of those u in S such that $(x, u) \neq 0$ is a countable union of finite sets and is, therefore, either finite or countably infinite. If $u_1, \ldots, u_n$ is a finite collection of distinct elements of S, then by (2.9-A) and Cauchy's inequality we have

$$\sum_{i=1}^{n} |(x,u_i)(\overline{y,u_i})| \leq \left(\sum_{i=1}^{n} |(x, u_i)|^2\right)^{1/2} \left(\sum_{i=1}^{n} |(y,u_i)|^2\right)^{1/2}$$

$$\leq \|x\| \, \|y\|.$$

This completes the proof.

<u>Theorem</u> 2.10. Let $u_1, \ldots, u_n$ be a finite orthonormal set in the space X, and let M be the subspace of X generated by $u_1, \ldots, u_n$. Then $u_1, \ldots, u_n$ is a basis for M, and the coefficients in a representation $x = \alpha_1 u_1 + \ldots + \alpha_n u_n$ of an element of M are related to x by the formulas $\alpha_i = (x, u_i)$.

Proof: If $x = \alpha_1 u_1 + \ldots + \alpha_n u_n$, then $(x, u_i) = \alpha_1(u_1, u_i) + \ldots + \alpha_n (u_n, u_i) = \alpha_i$, by the orthonormality relations. If $x = 0$ it follows that $\alpha_1 = \ldots = \alpha_n = 0$. Hence the u_i's form a linearly independent set, and therefore form a basis for the subspace M which they generate.

<u>Theorem</u> 2.11. Suppose X is complete, and let $\{u_n\}$

be a countably infinite orthonormal set in X. Then a series of the form $\sum_{n=1}^{\infty} \alpha_n u_n$ is convergent if and only if $\sum_{i=1}^{\infty} |\alpha_n|^2 < \infty$, and in that case we have the relations

$$\alpha_n = (x,\ u_n),\quad x = \sum_{n=1}^{\infty} \alpha_n u_n,\quad \|x\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2$$

between the coefficients α_n and the element x defined by the series.

Proof: Let $s_n = \alpha_1 u_1 + \ldots + \alpha_n u_n$. Then by the orthonormality relations we have (if $m < n$)

$$\|s_n - s_m\|^2 = \|\sum_{i=m+1}^{n} \alpha_i u_i\|^2 = (\sum_{i=m+1}^{n} \alpha_i u_i, \sum_{i=m+1}^{n} \alpha_i u_i)$$

$$= \sum_{j=m+1}^{n} \sum_{i=m+1}^{n} \alpha_j \bar{\alpha}_i (u_j, u_i)$$

$$= \sum_{i=m+1}^{n} |\alpha_i|^2.$$

Since X is complete, it is now clear that the sequence $\{s_n\}$ is convergent if and only if $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$. If this latter condition is satisfied, and if $x = \lim_{x \to \infty} s_n$, we prove that $\alpha_i = (x,\ u_i)$ as follows: By Theorem 2.10 we know that $\alpha_i = (s_n,\ u_i)$ if $1 \leq i \leq n$. But $s_n \to x$, so $(s_n,\ u_i) \to (x,\ u_i)$ by the continuity of the inner product. Therefore $\alpha_i = (x,\ u_i)$. Finally,

$$\|x\|^2 = \|\sum_{i=1}^{\infty} \alpha_i u_i\|^2 = \lim_{m \to \infty} (\sum_{i=1}^{m} \alpha_i u_i,\ \sum_{j=1}^{m} \alpha_j u_j)$$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \bar{\alpha}_j (u_i,\ u_j)$$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} |\alpha_i|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2.$$

This completes the proof.

<u>Theorem</u> 2.12. Let S be an orthonormal set in the

space X. If S is infinite, assume further that X is complete. For each x in X there is an element of X unambiguously defined by $x_s = \sum_{u \in S} (x, u)u$. Let M be the closed manifold generated by S. Then x is in M if and only if $x = x_S$. In any case, x_S is in M and $x - x_S$ is in $M^{\perp}$.

Proof: If x is in X, we know by Theorem 2.9 that $(x, u) \neq 0$ for at most a countably infinite number of elements u in S. If we index the u's for which $(x, u) \neq 0$ in some arbitrary order, say $u_1, u_2, \ldots$, then the series $\sum_{n=1}^{\infty}(x, u_n)u_n$, if infinite, is convergent by Theorems 2.9 and 2.11 since $\sum_{n=1}^{\infty}|(x, u_n)|^2 < \|x\|^2$. Moreover, the series remains convergent, no matter how its terms are rearranged, as may be seen by the first part of the proof of Theorem 2.11, using the fact that the series $\sum_{n=1}^{\infty}|(x, u_n)|^2$ is absolutely convergent. We may then show that the series $\sum_{n=1}^{\infty}(x, u_n)u_n$ converges to the same element, no matter how the terms are rearranged. For, if $\{v_n\}$ is a rearrangement of $\{u_n\}$, and $x_1 = \sum_{n=1}^{\infty}(x, u_n)u_n$, $x_2 = \sum_{n=1}^{\infty}(x, v_n)v_n$, we have $(x_1, u_n) = (x, u_n)$ and $(x_2, v_n) = (x, v_n)$ by Theorem 2.11. Thus, if $u_n = v_{m(n)}$, we have

$$(x_1 - x_2, u_n) = (x_1, u_n) - (x_2, u_n) = (x, u_n) - (x_2, v_{m(n)})$$
$$= (x, u_n) - (x, v_{m(n)}) = 0.$$

Similarly, $(x_1 - x_2, v_n) = 0$, and setting $\alpha_n = (x, u_n)$,

$\beta_n = (x, v_n)$, we obtain

$$\|x_1 - x_2\|^2 = (x_1 - x_2, \sum_{n=1}^{\infty}\alpha_n u_n - \sum_{n=1}^{\infty}\beta_n v_n)$$
$$= \sum_{n=1}^{\infty}\bar{\alpha}_n(x_1 - x_2, u_n) - \sum_{n=1}^{\infty}\bar{\beta}_n(x_1 - x_2, v_n) = 0,$$

whence $x_1 = x_2$. These considerations show that the notation $\sum_{u\epsilon S}(x, u)u$ has an unambiguous meaning and defines an element which we denote by x_S. Evidently x_S is in M. To prove $(x - x_S)\perp M$ it suffices to prove $(x - x_S)\perp S$. Let v be an arbitrary element of S. Then

$$(x - x_S, v) = (x, v) - \sum_{u\epsilon S}(x, u)(u, v)$$
$$= (x, v) - (x, v) = 0,$$

and hence $(x - x_S)\perp S$.

Since x_S is in M, it remains only to prove that $x_S = x$ if x is in M. Now, if x is in M, then $x - x_S$ is in M and since $(x - x_S)\perp M$, we have

$$\|x - x_S\|^2 = (x - x_S, x - x_S)$$
$$= (x, x - x_S) - (x_S, x - x_S) = 0,$$

whence $x = x_S$. It is clear from an examination of the proof that we need not assume the completeness of X if S is a finite set, for in that case no convergence questions arise, and the linear manifold generated by S, being finite dimensional, is closed by Corollary 1.36.

<u>Definition</u> 2.13. An orthonormal set S in the space X is called <u>complete</u> if there exists no orthonormal set of which S is a proper subset.

<u>Theorem</u> 2.14. Every inner-product space X having

a nonzero element contains a complete orthonormal set. Moreover, if S is any orthonormal set in X, there is a complete orthonormal set containing S as a subset.

Proof: Let S be an orthonormal set in X. Such sets exist; for instance, if $x \neq 0$, then the set $\frac{x}{\|x\|}$ is orthonormal. Let P be the class of all orthonormal sets having S as a subset. The set inclusion relation partially orders P. Suppose $N = \{N_\alpha\}_{\alpha \in \Lambda}$ is a completely ordered subset of P. Then $\bigcup_{\alpha \in \Lambda} N_\alpha$ is an upper bound of N, and contains S since S is a subset of N_α for every α in Λ. If x and y are vectors in $\bigcup_{\alpha \in \Lambda} N_\alpha$ such that $x \neq y$, then x is in N_α and y is in $N_{\alpha'}$, where α, α' are in Λ. Since N is completely ordered, either $N_\alpha \subseteq N_{\alpha'}$ or $N_{\alpha'} \subseteq N_\alpha$. Suppose for definiteness that $N_\alpha \subseteq N_{\alpha'}$. Then x, y are in $N_{\alpha'}$, whence $(x, y) = 0$. Hence $\bigcup_{\alpha \in \Lambda} N_\alpha$ is an orthonormal set, and it follows that $\bigcup_{\alpha \in \Lambda} N_\alpha$ is a member of P. Therefore by Zorn's lemma P contains a maximal element S'. Since S' is maximal, there exists no orthonormal set of which S' is a proper subset. Hence S' is a complete orthonormal set containing S as a subset.

Theorem 2.15. Let S be an orthonormal set in X, and let M be the closed linear manifold generated by S. If M = X, it follows that S is complete. If the space X is complete and the set S is complete (maximal), then M = X.

Proof: If S is not complete, then there is an $x \neq 0$

such that $x \perp S$, and hence also $x \perp M$. Now, if $M = X$, we have $x \perp x$, which implies $x = 0$. This proves the first part of the theorem. Assume now that X is complete. Then, if $M \neq X$, suppose x is in $X \setminus M$ and construct x_S as in Theorem 2.12. Then let $y = \frac{x - x_S}{\|x - x_S\|}$. By Theorem 2.12 $y \perp M$, so the set consisting of S and y is orthonormal. But y is not in S, whence S is not complete. This completes the proof.

Definition 2.16. An inner-product space which is infinite dimensional and complete is called a Hilbert space.

Theorem 2.17. Let X be an inner-product space and M a complete vector subspace of X. For any x in X, there is one and only one point $y = P_M(x)$ in M such that $\|x-y\| = d(x,M) = \inf_{m \in M} \|x-m\|$. The point $y = P_M(x)$ is also the only point z in M such that $x - z$ is orthogonal to M. The mapping $x \to P_M(x)$ of X onto M is linear; its kernel $M' = P_M^{-1}(0)$ is the subspace orthogonal to M, and X is the direct sum of M and M'. Finally, M is the subspace orthogonal to M'.

Proof: Let $\alpha = d(x, M)$. By definition, there is a sequence $\{y_n\}$ of points of M such that $\lim_{n \to \infty} \|x-y_n\| = \alpha$. For any two points u, v of X we have

$$\text{(2.17-A)} \quad \|u + v\|^2 + \|u - v\|^2 = (u + v, u + v) + (u-v, u-v)$$
$$= (u, u) + (u, v) + (v,u)+(v,v) +(u,u)-(u,v)-(v,u)+(v,v)$$

$$= 2(u,u) + 2(v,v)$$

$$= 2(\|u\|^2 + \|v\|^2);\ \text{hence}$$

$$\|y_m - y_n\|^2 = 2(\|x-y_m\|^2 + \|x-y_n\|^2) - \|2x-y_m+y_n\|^2$$

$$= 2(\|x-y_m\|^2 + \|x-y_n\|^2) - 4\|x - \tfrac{1}{2}(y_m+y_n)\|^2.$$

But $1/2(y_m + y_n)$ is in M, so $\|x - \frac{1}{2}(y_m+y_n)\|^2 \geq \alpha^2$. Since $\|x - y_n\| \to \alpha$, given $\epsilon > 0$ there is an n_o such that $n \geq n_o$ implies $\|x - y_n\|^2 \leq \alpha^2 + \epsilon/4$. Then if $m \geq n_o$ and $n \geq n_o$ we have

$$\|y_m - y_n\|^2 \leq 2(\alpha^2 + \epsilon/4 + \alpha^2 + \epsilon/4) - 4\alpha^2 = \epsilon,\ \text{so}$$

$\{y_n\}$ is a Cauchy sequence. Since M is complete, the sequence $\{y_n\}$ has a limit y in M, for which $\|x - y\| = d(x, M)$. Suppose y' in M also satisfies $\|x - y'\| = d(x, M)$. Then using (2.17-A) again, we obtain

$$\|y-y'\|^2 = 2(\|x-y\|^2 + \|x-y'\|^2) - 4\|x - \tfrac{1}{2}(y+y')\|^2$$

$$= 4\alpha^2 - 4\|x - \tfrac{1}{2}(y, y')\|^2.$$

But $\frac{1}{2}(y + y')$ is in M, so $\|x - \frac{1}{2}(y + y')\| \geq \alpha$, and it follows that $\|y - y'\|^2 \leq 4\alpha^2 - 4\alpha^2 = 0$. Therefore $y' = y$ and $y = P_M(x)$ is uniquely defined.

Now let $z \neq 0$ be any point of M. Then $\|x - (y + \lambda z)\|^2 > \alpha^2$ for any real scalar $\lambda \neq 0$; this gives

$$\|x - (y + \lambda z)\|^2 = (x - y + \lambda z,\ x - y + \lambda z)$$

$$= (x-y, x-y) + (x-y, \lambda z) + (\lambda z, x-y) + (\lambda z, \lambda z)$$

$$= \|x-y\|^2 + 2\lambda \operatorname{Re}(x-y, z) + \lambda^2 \|z\|^2$$

$$= \alpha^2 + 2\lambda \operatorname{Re}(x-y, z) + \lambda^2 \|z\|^2 > \alpha^2,$$

This implies that $2\lambda \mathrm{Re}(x-y, z) + \lambda^2 \|z\|^2$ $= \lambda[2 \mathrm{Re}(x-y, z) + \lambda \|z\|^2] > 0$. But if $\lambda = \frac{2 \mathrm{Re}(x-y,z)}{\|z\|^2}$, then $\lambda[2 \mathrm{Re}(x-y, z) + \lambda\|z\|^2] = 0$, so the above inequality is true only if $\mathrm{Re}(x-y, z) = 0$. Replacing z by iz (if X is a complex inner-product space) shows that $\mathrm{Im}(x-y, z) = 0$; hence $(x-y, z) = 0$ in every case, so x-y is orthogonal to M. Let y' in M be such that x-y' is orthogonal to M. Then, for any $z \neq 0$ in M, we have (since $(x-y', y'-z) = 0$)

$$\|x-z\|^2 = \|(x-y') + (y'-z)\|^2 = \|x-y'\|^2 + \|y'-z\|^2,$$

so $\|x-y'\|^2 \leq \|x-z\|^2$ for all z in M. This implies that $\|x-y'\| = d(x, M)$, and it follows that $y' = y$ by the uniqueness of $P_M(x)$. Hence the point $y = P_M(x)$ is the only point z in M such that $(x-z) \perp M$.

Now if $x - y$ and $x' - y'$ are orthogonal to M, then $\lambda x - \lambda y \perp M$, as is $(x+x') - (y+y') = (x-y) + (x'-y')$. Since $y + y'$ and λy are in M, this shows that $P_M(x+x') = y + y' = P_M(x) + P_M(x')$ and $P_M(\lambda x) = \lambda y = \lambda P_M(x)$. Hence P_M is linear.

If $P_M(x) = 0$, then $x - 0 = x$ is orthogonal to M. Hence $M' = P_M^{-1}(0)$ consists of the vectors x orthogonal to M. Thus $x - P_M(x)$ is in M', and since $x = P_M(x) + (x - P_M(x))$ for any x in X, we have $X = M + M'$. Further, if x is in $M \cap M'$, then $(x, x) = 0$ which implies that $x = 0$. Hence $X = M \oplus M'$.

Finally, if x is in X and x is orthogonal to M', we have in particular that $(x, x - P_M(x)) = 0$. But $(P_M(x), x - P_M(x)) = 0$ since $M' \perp M$; hence

$$0 = (x, x-P_M(x)) - (P_M(x), x-P_M(x))$$
$$= (x-P_M(x), x-P_M(x)) = \|x-P_M(x)\|^2,$$

so $x = P_M(x)$ and x is in M. Therefore M is the subspace orthogonal to M'. The proof is now complete. ([2], p. 115).

Theorem 2.18. Let H be a Hilbert space. Then H is isometrically (conjugate) isomorphic to the linear space H' of all bounded linear functionals of H under the mapping $\sigma: H \rightarrow H'$ defined by $[\sigma(x)](y) = (y, x)$, x, y in H.

Proof: Let $x_1 = x_2$, where x_1 and x_2 are in H. Then

$$0 = (y,0) = (y, x_1-x_2) = (y, x_1) - (y, x_2)$$
$$= [\sigma(x_1)](y) - [\sigma(x_2)](y)$$

for all y in H. Hence $\sigma(x_1) = \sigma(x_2)$ and σ is well-defined.

Conversely, suppose $\sigma(x_1) = \sigma(x_2)$, so that $[\sigma(x_1)](y) = [\sigma(x_2)](y)$ for all y in H. Then $(y, x_1) = (y, x_2)$ and $(y, x_1 - x_2) = 0$ for all y in H. In particular, $(x_1 - x_2, x_1 - x_2) = 0$, whence $x_1 = x_2$. Hence σ is one-to-one and σ^{-1} exists.

We have

$$[\sigma(\alpha_1 x_1 + \alpha_2 x_2)](y) = (y, \alpha_1 x_1 + \alpha_2 x_2)$$
$$= (y, \alpha_1 x_1) + (y, \alpha_2 x_2)$$

$$= \bar{\alpha}_1(y, x_1) + \bar{\alpha}_2(y, x_2)$$
$$= \bar{\alpha}_1[\sigma(x_1)](y) + \bar{\alpha}_2[\sigma(x_2)](y).$$

Hence σ is (conjugate) linear.

Also, $|(y, x)| \le \|y\| \, \|x\|$ by (2.2-A), so

$$\|\sigma(x)\| = \sup_{\|y\|\le 1} |[\sigma(x)](y)| = \sup_{\|y\|\le 1} |(y, x)|$$
$$\le \sup_{\|y\|\le 1} \|y\| \, \|x\| \le \|x\|.$$

But $[\sigma(x)](x) = (x, x) = \|x\|^2$, so

$$\|\sigma(x)\| = \sup_{\|y\|\le 1} |[\sigma(x)](y)| \ge |[\sigma(x)] (\tfrac{x}{\|x\|})|$$
$$= \frac{1}{\|x\|} |[\sigma(x)](x)| = \|x\|.$$

Hence $\|\sigma(x)\| = \|x\|$.

It remains to be shown that σ is an onto mapping. Let x' be arbitrary in H'. $M = N(x') = \{x \in H \mid x'(x) = 0\}$ is a closed linear subspace since x' is continuous. Then by Theorem 2.17, $H = M \oplus M$. Let x_1 be in $M^{\perp}$, $x_1 \ne 0$, and let $P : H \to M^{\perp}$ be a projection. For any y in H,

$$x'(y - \frac{x'(y)}{x'(x_1)} x_1) = x'(y) - \frac{x'(y)}{x'(x_1)} x'(x_1) = 0,$$

so $y - \frac{x'(y)}{x'(x_1)} x_1$ is in M. By Theorem 2.17,

$H = (I-P)H + P(H) = M \oplus M^{\perp}$. Thus

$y = y - \frac{x'(y)}{x'(x_1)} x_1 + \frac{x'(y)}{x'(x_1)} x_1$ implies that

$P(y) = \frac{x'(y)}{x'(x_1)} x_1$. Set $x = \overline{x'(x_1)} \, \|x_1\|^{-2} x_1$. Then x is in $M^{\perp}$ and

$$(y, x) = (y, \overline{x'(x_1)}\ \|x_1\|^{-2}\ x_1)$$
$$= x'(x_1)\|x_1\|^{-2}\ (y, x_1)$$
$$= x'(x_1)\|x_1\|^{-2}\ (P(y) + y - P(y), x_1)$$
$$= x'(x_1)\|x_1\|^{-2}\ [(P(y), x_1) - (y - P(y), x_1)]$$
$$= x'(x_1)\|x_1\|^{-2}\ (P(y), x_1)$$
$$= x'(x_1)\|x_1\|^{-2}\ (\tfrac{x'(y)}{x'(x_1)}, x_1)$$
$$= x'(x_1)\|x_1\|^{-2}\ \frac{x'(y)}{x'(x_1)}\ (x_1, x_1) = x'(y).$$

But y was arbitrary, so $[\sigma(x)](y) = (y, x) = x'(y)$ for all y in H, whence σ maps H onto H'. This completes the proof.

<u>Definition</u> 2.19. Let X and Y be arbitrary complete inner-product spaces. Suppose y_o is fixed in Y. Let $x'(x) = (Ax, y_o)$, where A is in [X, Y]. Then x' is in X', and hence by Theorem 2.18 there is a unique x_o in X such that $x'(x) = (x, x_o)$. We write $x_o = A^*y_o$, thus defining an operator A* on Y into X. The definition of A* is fully expressed by the equation

(2.19-A) $(Ax, y) = (x, A^*y)$, x in X, y in Y. The operator A* is called the <u>adjoint</u> of A.

If $X \neq (0)$ is a complete inner-product space and if A is in [X, X], then A* is in [X, X] also. If $A^* = A$ we say that A is <u>self-adjoint</u>.

<u>Definition</u> 2.20. Let X be an inner-product space. A scalar-valued function ϕ on X x X is called a <u>bilinear</u>

<u>form</u> if $\phi(x, y)$ is linear in x for each y, and $\overline{\phi(x, y)}$ is linear in y for each x.

With ϕ we associate the function ψ on X defined by $\psi(x) = \phi(x, x)$, and we call ψ the <u>quadratic</u> <u>form</u> corresponding to ϕ.

We have

$$\psi(\tfrac{x+y}{2}) - \psi(\tfrac{x-y}{2}) = \phi(\tfrac{x+y}{2}, \tfrac{x+y}{2}) - \phi(\tfrac{x-y}{2}, \tfrac{x-y}{2})$$

$$= \tfrac{1}{4}\phi(x,x) + \tfrac{1}{4}\phi(x,y) + \tfrac{1}{4}\phi(y,x) + \tfrac{1}{4}\phi(y,y)$$

$$- \tfrac{1}{4}\phi(x,x) + \tfrac{1}{4}\phi(x,y) + \tfrac{1}{4}\phi(y,x) - \tfrac{1}{4}\phi(y,y),$$ or

$$(2.21)\quad \psi(\tfrac{x+y}{2}) - \psi(\tfrac{x-y}{2}) = \tfrac{1}{2}\left\{\phi(x,y) + \phi(y,x)\right\}.$$

<u>Theorem</u> 2.22. Let ϕ be a bilinear form on X x X. Then ϕ is continuous on X x X if and only if it is continuous at (0, 0). ϕ is continuous jointly in its two variables if and only if $\sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |\phi(x,y)|$ is finite.

Proof: First, we note that $\rho((x_1, y_1), (x_2, y_2)) = (\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2)^{1/2}$ is a metric on X x X.

If ϕ is continuous on X x X it is continuous at (0, 0). Conversely, if ϕ is continuous at (0, 0) there is a $\delta > 0$ such that if $\rho((x, y), (0, 0))$

$= (\|x\|^2 + \|y\|^2)^{1/2} < \delta$, then $|\phi(x, y)| < 1$.

Let $(x_o, y_o) = (\frac{\delta}{\sqrt{8}} \frac{x}{\|x\|}, \frac{\delta}{\sqrt{8}} \frac{y}{\|y\|})$. Then

$$\rho((x_o, y_o), (0, 0)) = (\tfrac{\delta^2}{8} + \tfrac{\delta^2}{8})^{1/2} = \delta/2 < \delta,$$

so $\left|\phi\left(\frac{\delta}{\sqrt{8}}\ \frac{x}{\|x\|}, \frac{\delta}{\sqrt{8}}\ \frac{y}{\|y\|}\right)\right| = \frac{\delta^2}{8\|x\|\ \|y\|} |\phi(x, y)| < 1.$

Thus $|\phi(x, y)| < \frac{8\|x\|\ \|y\|}{\delta^2}$, and

$$S = \sup_{\substack{\|x\|\leq 1\\ \|y\|\leq 1}} |\phi(x, y)| < \sup_{\substack{\|x\|\leq 1\\ \|y\|\leq 1}} \frac{8\|x\|\ \|y\|}{\delta^2} \leq \frac{8}{\delta^2}.$$

Also, if $x \neq 0$, $y \neq 0$, then $\left|\phi\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)\right| = \frac{|\phi(x,y)|}{\|x\|\ \|y\|} \leq S$, so $|\phi(x, y)| \leq S\ \|x\|\ \|y\|$.

Consider (x_1, y_1) in $X \times X$. If $x_1 = 0$ and $y_1 = 0$, then ϕ is continuous at (x_1, y_1). Suppose now that $x_1 \neq 0$. Then $\rho((x_1, y_1), (x_2, y_2)) < 1$ implies $(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2)^{1/2} < 1$. It follows that $\|y_2 - y_1\| < 1$, and since $\|y_2\| - \|y_1\| \leq \|y_2 - y_1\|$ we have $\|y_2\| < 1 + \|y_1\|$. Let $\epsilon > 0$ be given, and let $M = \max\left\{S\|x_1\|, S(\|y_1\| + 1)\right\}$, $\delta' = \min\left\{1, \frac{\epsilon}{2M}\right\}$.

Then if $\rho((x_1, y_1), (x_2, y_2)) < \delta'$ it follows that

$$\|x_1-x_2\| < \frac{\epsilon}{2M} \leq \frac{\epsilon}{2S(\|y_1\|+1)} \text{ and } \|y_1-y_2\| < \frac{\epsilon}{2M} \leq \frac{\epsilon}{2S\|x_1\|},$$

whence

$$\begin{aligned}
|\phi(x_1,y_1) - \phi(x_2,y_2)| &= |\phi(x_1,y_1)-\phi(x_1,y_2)+\phi(x_1,y_2) -\phi(x_2,y_2)| \\
&\leq |\phi(x_1,y_1)-\phi(x_1,y_2)|+|\phi(x_1,y_2) -\phi(x_2,y_2)| \\
&= |\phi(x_1,y_1-y_2)|+|\phi(x_1-x_2,y_2)| \\
&\leq S\|x_1\|\ \|y_1-y_2\|+S\|x_1-x_2\|\ \|y_2\| \\
&< S\|x_1\|\ \|y_1-y_2\|+S\|x_1-x_2\|(1+\|y_1\|)
\end{aligned}$$

$$< S\|x_1\| \left(\frac{\epsilon}{2S\|x_1\|}\right)$$

$$+ S\left(\frac{\epsilon}{2S(\|y_1\|+1)}\right) (\|y_1\| + 1)$$

$$= \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore ϕ is continuous on X x X, and the first assertion of the theorem is proved.

If ϕ is continuous on X x X, then it is continuous at (0, 0) and $S = \sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} |\phi(x,y)|$ is bounded by the first part of the proof. Conversely, if there is a constant M such that $\sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} |\phi(x,y)| \leq M$, then $\left|\phi\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)\right| = \frac{|\phi(x,y)|}{\|x\| \; \|y\|} \leq M$, so $|\phi(x,y)| \leq M \|x\| \; \|y\|$. Let $\epsilon > 0$ be given. Then $\rho((x,y), (0, 0)) < \sqrt{\epsilon/M}$ implies that $\|x\| < \sqrt{\epsilon/M}$ and $\|y\| < \sqrt{\epsilon/M}$. Hence

$$|\phi(x,y)| < M \left(\sqrt{\epsilon/M}\right) \left(\sqrt{\epsilon/M}\right) = \epsilon,$$

so ϕ is continuous at (0, 0). It follows from the first part of the proof that ϕ is continuous on X x X.

<u>Definition</u> 2.23. Let ϕ be a continuous bilinear form on X x X, where X is an inner-product space. Then the <u>norm</u> of ϕ, denoted by $\|\phi\|$, is defined as follows:

(2.23-A) $$\|\phi\| = \sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} |\phi(x,y)|$$

(2.23-B) $$= \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\phi(x,y)|$$

$$= \sup_{\substack{\|x\|\neq 0 \\ \|y\|\neq 0}} |\phi(\tfrac{x}{\|x\|}, \tfrac{y}{\|y\|})|$$

$$\text{(2.23-C)} \qquad = \sup_{\substack{\|x\|\neq 0 \\ \|y\|\neq 0}} \frac{|\phi(x,y)|}{\|x\| \; \|y\|} .$$

It follows from (2.23-C) that $|\phi(x,y)| \leq \|\phi\| \; \|x\| \; \|y\|$ for all x and y in X.

For the quadratic form ψ corresponding to the bilinear form ϕ we define

$$\|\psi\| = \sup_{\|x\|=1} |\psi(x)| = \sup_{\|x\|=1} |\phi(x,x)| = \sup_{\|x\|\neq 0} |\phi(\tfrac{x}{\|x\|}, \tfrac{x}{\|x\|})|$$

$$= \sup_{\|x\|\neq 0} \frac{|\phi(x,x)|}{\|x\|^2} = \sup_{\|x\|\neq 0} \frac{|\psi(x)|}{\|x\|^2} .$$

<u>Definition</u> 2.24. A bilinear form ϕ is called <u>symmetric</u> if $\phi(x,y) = \overline{\phi(y,x)}$.

<u>Theorem</u> 2.25. Let X be an inner-product space. If ϕ is a symmetric bilinear form on X x X and ψ is the corresponding quadratic form, then $\|\phi\| = \|\psi\|$.

Proof: Since $\{\phi(x, x)| \; \|x\| = 1\} \subseteq \{\phi(x,y)| \; \|x\| = \|y\| = 1\}$, we have $\|\psi\| \leq \|\phi\|$. Also,

$$|\text{Re } \phi(x,y)| = |\tfrac{1}{2} \{\phi(x, y) + \overline{\phi(x, y)}\} | .$$

Since ϕ is symmetric this gives $|\text{Re } \phi(x, y)| = |\frac{1}{2} \{\phi(x, y) + \phi(y, x)\} |$, and from 2.21 it follows that

$$\begin{aligned} |\text{Re } \phi(x, y)| &= |\psi(\tfrac{x+y}{2}) - \psi(\tfrac{x-y}{2})| \\ &\leq |\psi(\tfrac{x+y}{2})| + |\psi(\tfrac{x-y}{2})| \\ &\leq \|\psi\| \; \|\tfrac{x+y}{2}\|^2 + \|\psi\| \; \|\tfrac{x-y}{2}\|^2 \\ &= \frac{\|\psi\|}{4} (\|x+y\|^2 + \|x-y\|^2). \end{aligned}$$

Applying 2.17-A, we obtain

$$|\text{Re } \phi(x, y)| = \frac{\|\psi\|}{4} (2(\|x\|^2 + \|y\|^2))$$

$= \frac{\|\psi\|}{2} (\|x\|^2 + \|y\|^2)$. (The Re symbol, for the "real part" is superfluous if ϕ is a real-valued function.) For fixed x and y with $\|x\| = \|y\| = 1$ we can choose α so that $|\alpha| = 1$ and $\alpha\phi(x, y) = |\phi(x, y)|$. If ϕ is real-valued we choose $\alpha = -1$ if $\phi(x, y) < 0$, $\alpha = 1$ if $\phi(x, y) \geq 0$. If ϕ is complex-valued we choose $\alpha = (\frac{\overline{\phi(x,y)}}{\phi(x,y)})^{1/2}$. Then

$$|\phi(x,y)| = (\phi(x,y)\overline{\phi(x,y)})^{1/2} = \phi(x,y)(\frac{\overline{\phi(x,y)}}{\phi(x,y)})^{1/2}$$
$$= \alpha\ \phi(x,y).$$

Since $|\phi(x,y)| = |\overline{\phi(x,y)}|$, $|\alpha| = |(\frac{\overline{\phi(x,y)}}{\phi(x,y)})^{1/2}| = (\left|\frac{\overline{\phi(x,y)}}{\phi(x,y)}\right|)=1$. Therefore

$$|\phi(x,y)| = \phi(\alpha x,y) = |\text{Re } \phi(\alpha x,y)| \leq \frac{\|\psi\|}{2}(\|x\|^2+\|y\|^2)=\|\psi\|,$$

so $\|\phi\| = \sup\limits_{\substack{\|x\|=1\\ \|y\|=1}} |\phi(x,y)| \leq \|\psi\|$. Hence $\|\phi\| = \|\psi\|$ and the proof is complete.

<u>Theorem</u> 2.26. Let X be an inner-product space. Suppose $\phi(x, y) = (Ax, y)$, where A is linear on X into X. Then ϕ is continuous if and only if A is continuous, and in that case $\|A\| = \|\phi\|$.

Proof: If A is continuous, then $\|Ax\| \leq \|A\|\ \|x\|$ for all x in X. Hence by (2.2-A) we have

$|(Ax, y)| \leq \|Ax\|\ \|y\| \leq \|A\|\ \|x\|\ \|y\|$, and so

$$\sup_{\substack{\|x\|\leq 1\\ \|y\|\leq 1}} |\phi(x, y)| \leq \sup_{\substack{\|x\|\leq 1\\ \|y\|\leq 1}} \|A\|\ \|x\|\ \|y\| \leq \|A\|.$$

Therefore ϕ is continuous by Theorem 2.22, and $\|\phi\| \leq \|A\|$.

Conversely, if ϕ is continuous, then $|\phi(x, y)| \leq \|\phi\| \, \|x\| \, \|y\|$, and we have

$$\|Ax\|^2 = (Ax, Ax) \leq \|\phi\| \, \|x\| \, \|Ax\|.$$

If $\|Ax\| = 0$, A is the zero operator and is continuous. If $\|Ax\| \neq 0$, then $\|Ax\| \leq \|\phi\| \, \|x\|$, whence A is continuous by Theorem 1.27. Also,

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} \leq \sup_{\|x\| \neq 0} \frac{\|\phi\| \, \|x\|}{\|x\|} = \|\phi\|,$$

and together with the first part of the proof this implies $\|A\| = \|\phi\|$.

<u>Definition</u> 2.27. A linear operator A with domain and range in the inner-product space X is called <u>symmetric</u> if (Ax, y) = (x, Ay) for every x, y in D(A).

We see that a self-adjoint operator is symmetric. Conversely, a symmetric operator whose domain is all of X, with X complete, is self-adjoint. In the next section we consider only operators whose domains are the whole space on which they are defined, so in this case we make no distinction between symmetric operators and self-adjoint operators.

CHAPTER 3

Compact Operators in Hilbert Space

<u>Definition</u> 3.1. Let X and Y be normed linear spaces. Suppose T is a linear operator with domain X and range in Y. We say that T is <u>compact</u> if, for each bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in Y.

<u>Theorem</u> 3.2. Let H_1 and H_2 be Hilbert spaces, A an operator from H_1 to H_2, and B an operator from H_2 to H_1. If A is compact with $D(A) = H_1$, and if B is defined and bounded everywhere in H_2, then the operators AB and BA are compact.

Proof: Suppose $\{x_n\}$ is a bounded sequence in H_1. Since A is compact, $\{x_n\}$ contains a subsequence $\{x'_n\}$ such that $\{Ax'_n\}$ converges. Suppose $Ax'_n \to x$. Then since B is bounded (and hence continuous), $BA\, x'_n = B(Ax'_n) \to Bx$, whence BA is compact. Further, the sequence $\{Bx_n\}$ is bounded, so the sequence $\{A(Bx_n)\} = \{AB\, x_n\}$ contains a convergent subsequence. Therefore AB is compact. ([3], p. 101).

<u>Theorem</u> 3.3. Let H_1 and H_2 be Hilbert spaces, and A a bounded linear operator from H_1 to H_2. If A is defined everywhere in H_1, then A is compact if and only if A^*A is compact, where A^* is the adjoint of A.

Proof: If A is compact, then A*A is compact from Theorem 3.2 since A* is bounded.

Conversely, suppose the operator A*A is compact and let $\{x_n\}$ be a bounded sequence of elements in H_1 such that $\|x_n\| \leq C$ for some constant C and all n. From $\{x_n\}$ we can select a subsequence $\{x'_n\}$ such that $\{A^*A\, x'_n\}$ converges. But then

$$\begin{aligned}\|Ax'_n - Ax'_m\|^2 &= (A(x'_n - x'_m), A(x'_n - x'_m)) \\ &= (x'_n - x'_m, A^*A\, x'_n - A^*A\, x'_m) \\ &\leq \|x'_n - x'_m\| \, \|A^*Ax'_n - A^*Ax'_m\| \\ &\leq 2C \, \|A^*Ax'_n - A^*Ax'_m\|.\end{aligned}$$

Given an $\epsilon > 0$, there is a positive integer N such that $n, m \geq N$ implies $\|A^*A\, x'_n - A^*A\, x'_m\| < \frac{\epsilon^2}{2C}$. Hence $n, m > N$ implies $\|Ax'_n - Ax'_m\| < \epsilon$, so the sequence $\{Ax'_n\}$ converges. Therefore A is compact. ([3], p. 106).

<u>Theorem</u> 3.4. Let H_1 and H_2 be Hilbert spaces, A a linear operator from H_1 into H_2 defined everywhere in H_1. If A is compact, then A* is also compact.

Proof: Given x, y in H_1, we have

$$(y, Ax) = (\overline{Ax,y}) = (\overline{x,A^*y}) = (A^*y,x) = (y,A^{**}x),$$

where $A^{**} = (A^*)^*$. Hence $0 = (y,Ax) - (y,A^{**}x) = (y,Ax-A^{**}x)$. Since x and y were arbitrary, this implies that $Ax - A^{**}x = 0$ for all x in H_1, or $A = A^{**}$. From Theorem 3.2, if A is compact, then AA* is compact. But $AA^* = (A^*)^*\, A^*$, so A* is compact by Theorem 3.3. This completes the proof. ([3], p. 106).

Theorem 3.5. Let H_1, H_2 be Hilbert spaces, $\{A_n\}$ a sequence of compact linear operators in $[H_1, H_2]$ such that $\|A_n - A\| \to 0$. Then A is compact.

Proof: Let $S = \{x \text{ in } H_1 \mid \|x\| \leq 1\}$. We will show that $\overline{A(S)}$ is compact; first we show that A(S) is totally bounded. Since $\|A_n - A\| \to 0$, given $\epsilon > 0$ there is a positive integer N such that, if $n \geq N$, then $\|A_n x - Ax\| = \|(A_n - A)x\| < \epsilon/3$ for all x in S. Since A_n is compact, every sequence in $A_n(S)$ has a convergent subsequence, whence $\overline{A_n(S)}$ is compact by Theorem 1.17. Then $\overline{A_n(S)}$ is totally bounded by Theorem 1.16, so $A_n(S)$ is totally bounded. (If $\overline{G}$ is totally bounded, then G is totally bounded. For, given $\epsilon > 0$ there is a finite subset $\{x_1, \ldots, x_n\}$ of $\overline{G}$ such that, if x is in G, then $d(x, x_i) < \epsilon/2$ for some i. But there is an x'_i in G such that $d(x_i, x'_i) < \epsilon/2$, so $d(x, x'_i) \leq d(x, x_i) + d(x_i, x'_i) < \epsilon$. Hence $\{x'_1, \ldots, x'_n\} \subseteq G$ is a finite ϵ-net, and G is totally bounded.) Hence given $n \geq N$ there is a finite subset $\{A_n x_1, \ldots, A_n x_m\}$ of $A_n(S)$ such that for any x in S, $\|A_n x - A_n x_i\| < \epsilon/3$ for some i, $1 \leq i \leq m$. Then given $n \geq N$ and x in S, there is an i, $1 \leq i \leq m$ such that

$$\|Ax - Ax_i\| \leq \|Ax - A_n x\| + \|A_n x - A_n x_i\| + \|A_n x_i - Ax_i\|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Hence A(S) is totally bounded, so $\overline{A(S)}$ is totally bounded.

(If G is totally bounded, then $\overline{G}$ is totally bounded. For, let x be in $\overline{G}$ and let $\epsilon > 0$ be given. Then there is an x' in G such that $d(x, x') < \epsilon/2$. Also, there is a subset $\{x_1, \ldots, x_n\}$ of G such that $d(x', x_i) < \epsilon/2$ for some i, $1 \leq i \leq n$. Then $d(x, x_i) \leq d(x, x') + d(x', x_i) < \epsilon$, whence $\{x_1, \ldots, x_n\}$ is an ϵ-net for $\overline{G}$, and $\overline{G}$ is totally bounded.) Since H_2 is complete, $\overline{A(S)}$ is compact.

Now let $\{x_n\}$ be a sequence in H such that $\|x_n\| \leq C$ for all n, where C is a positive constant. Then $\left\{\frac{x_n}{C}\right\}$ is in S, and since $\overline{A(S)}$ is compact, the sequence $\left\{A(\frac{x_n}{C})\right\} = \left\{\frac{1}{C} A x_n\right\}$ has a convergent subsequence $\left\{\frac{1}{C} A x'_n\right\}$. Suppose $\frac{1}{C} A x'_n \rightarrow x$, where x is in $\overline{A(S)}$. Then $Ax'_n \rightarrow Cx$, whence A is compact.

Definitions 3.6, 3.7, 3.8, 3.9, 3.10. Let X be a normed linear space, T a linear operator whose domain D(T) and range R(T) lie in X. Consider the operator $\lambda I-T$, where λ is a scalar and I is the identity operator. We write $\lambda-T$ in place of $\lambda I-T$. If λ is such that $\overline{R(\lambda-T)} = X$ and if $\lambda-T$ has a continuous inverse, we say that λ is in the resolvent set of T. All scalar values of λ not in the resolvent set comprise the set called the spectrum of T.

If λ is a scalar such that $\lambda-T$ has no inverse, then there is at least one nonzero vector x such that

$Tx = \lambda x$. In this case λ is called an <u>eigenvalue</u> of T, and x is a corresponding <u>eigenvector</u>.

The null manifold of $\lambda - T$ is called the <u>eigenmanifold</u> corresponding to λ.

Suppose A is a linear operator with domain and range in the inner-product space X. If we regard D(A) by itself as an inner-product space, then $\phi(x, y) = (Ax, y)$ is a bilinear form on D(A) x D(A), and ϕ is symmetric if and only if A is symmetric. The corresponding quadratic form, defined on D(A), is (Ax, x).

We assume now that A is symmetric and that $D(A) \neq 0$. Then (Ax, x) is real since $(Ax, x) = (x, Ax) = (\overline{Ax, x})$. We define

$$(3.11) \quad m(A) = \inf_{\|x\|=1} (Ax, x), \quad M(A) = \sup_{\|x\|=1} (Ax, x).$$

The possibilities $m(A) = -\infty$, $M(A) = +\infty$ are not excluded.

<u>Theorem</u> 3.12. If A is a symmetric operator on an inner-product space X and λ is an eigenvalue of A, then λ is real and $m(A) \leq \lambda \leq M(A)$. Also, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof: Suppose $Ax = \lambda x$, and assume without loss of generality that $\|x\| = 1$. Then $(Ax, x) = (\lambda x, x) = \lambda(x, x) = \lambda$, so λ is real and $m(A) \leq \lambda \leq M(A)$. If $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ where $\lambda_1 \neq \lambda_2$, then $\lambda_1(x, y) = (\lambda_1 x, y) = (Ax, y) = (x, Ay) = (x, \lambda_2 y) = \lambda_2(x, y)$. Hence $(\lambda_1 - \lambda_2)(x, y) = 0$, and it follows that $(x, y) = 0$.

<u>Theorem</u> 3.13. Let A be a symmetric operator on the inner-product space X with $D(A) = X$. Then A is continuous if and only if $m(A)$ and $M(A)$ are both finite, and in that case $\|A\| = \max \{ |m(A)|, |M(A)| \}$.

Proof: By Theorem 2.26, A is continuous if and only if $\phi(x, y) = (Ax, y)$ is continuous, and then $\|A\| = \|\phi\|$. By Theorem 2.22, $\phi(x, y)$ is continuous if and only if $\|\phi\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |(Ax, y)| < \infty$. Since ϕ is symmetric if A is symmetric, we have from Theorem 2.25 that $\|\phi\| = \|\psi\| = \sup_{\|x\|=1} |\psi(x)| = \sup_{\|x\|=1} |(Ax, x)|$. Hence A is continuous if and only if $\sup_{\|x\|=1} |(Ax, x)|$ is finite. But this implies that $m(A)$ and $M(A)$ are finite. Further, if A is continuous, then $\|A\| = \|\phi\| = \sup_{\|x\|=1} |(Ax, x)|$ $= \max \{ |m(A)|, |M(A)| \}$. The proof is now complete.

For the remaining theorems we assume that A is in $[X, X]$, where X is a real or complex inner-product space.

<u>Theorem</u> 3.14. Let $A \neq 0$ be a compact, symmetric operator. Then either $\|A\|$ or $-\|A\|$ is an eigenvalue of A, and there is a corresponding eigenvector x such that $\|x\| = 1$ and $|(Ax, x)| = \|A\|$.

Proof: Since A is compact it is continuous. For, discontinuity of A would, by Theorem 1.27, imply the existence of a sequence $\{x_n\}$ such that $\|x_n\| = 1$ for all n and $\|Ax_n\| \to \infty$; this cannot occur if A is compact. Since A is continuous, we have by Theorem 3.13 that $|m(A)| < \infty$, $|M(A)| < \infty$, and $\|A\| = \max\{|m(A)|, |M(A)|\}$. This implies that there is a sequence $\{x_n\}$ such that $\|x_n\| = 1$ for all n and $(Ax_n, x_n) \to \lambda$, where λ is real and $|\lambda| = \|A\|$. Now

$$0 \leq \|Ax_n - \lambda x_n\|^2 = (Ax_n - \lambda x_n, Ax_n - \lambda x_n)$$
$$= (Ax_n, Ax_n) - (\lambda x_n, Ax_n) - (Ax_n, \lambda x_n)$$
$$+ (\lambda x_n, \lambda x_n) = \|Ax_n\|^2 - 2\lambda(Ax_n, x_n)$$
$$+ \lambda^2\|x_n\|^2 \leq (\|A\| \, \|x_n\|)^2 - 2\lambda(Ax_n, x_n)$$
$$+ \lambda^2\|x_n\|^2 = \|A\|^2 - 2\lambda(Ax_n, x_n) + \lambda^2.$$

Since $(Ax_n, x_n) \to \lambda$, given $\epsilon > 0$ there is a positive integer N such that $n \geq N$ implies $|(Ax_n, x_n) - \lambda| < \frac{\epsilon}{2|\lambda|}$, or $\lambda - \frac{\epsilon}{2|\lambda|} < (Ax_n, x_n) < \lambda + \frac{\epsilon}{2|\lambda|}$. If $\lambda > 0$ we have

$$\|Ax_n - \lambda x_n\|^2 < \|A\|^2 - 2\lambda(\lambda - \frac{\epsilon}{2|\lambda|}) + \lambda^2$$
$$= \lambda^2 - 2\lambda^2 + \epsilon + \lambda^2 = \epsilon,$$

and if $\lambda < 0$,

$$\|Ax_n - \lambda x_n\|^2 < \|A\|^2 - 2\lambda(\lambda + \frac{\epsilon}{2|\lambda|}) + \lambda^2$$
$$= \lambda^2 - 2\lambda^2 + \epsilon + \lambda^2 = \epsilon.$$

Therefore $\|Ax_n - \lambda x_n\| \to 0$, which implies that $Ax_n - \lambda x_n \to 0$ since $\|x\| = 0$ if and only if $x = 0$. Since A is compact, $\{Ax_n\}$ contains a convergent subsequence which we denote by $\{Ay_k\}$, where $\{y_k\}$ is a subsequence of $\{x_n\}$. Suppose $Ay_k \to x$. Then given

$\epsilon > 0$ there is a positive integer K_1 such that $k \geq K_1$ implies $\|Ay_k - x\| < \epsilon/2$. Also, there is a K_2 such that $k \geq K_2$ implies $\|Ay_k - \lambda y_k\| > \epsilon/2$. Let $K = \max\{K_1, K_2\}$. Then $k \geq K$ implies

$$\|x - \lambda y_k\| \leq \|x - Ay_k\| + \|Ay_k - \lambda y_k\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $\lambda y_k \to x$ and $y_k \to \frac{x}{\lambda}$ since $\lambda \neq 0$. Then $\|\frac{x}{\lambda}\| = 1$ and $(A-\lambda)\, y_k \to (A-\lambda)\, \frac{x}{\lambda} = \frac{1}{\lambda}Ax - x$. Since $Ax_n - \lambda x_n \to 0$, this implies that $\frac{1}{\lambda}Ax - x = 0$, or $Ax - \lambda x = 0$. Thus $\lambda = \|A\|$ or $\lambda = -\|A\|$ is an eigenvalue of A. Also,

$$|(A(\tfrac{x}{\lambda}), \tfrac{x}{\lambda})| = |(\lambda(\tfrac{x}{\lambda}), \tfrac{x}{\lambda})| = |\lambda\, \|\tfrac{x}{\lambda}\|^2| = |\lambda| = \|A\|.$$

This completes the proof.

We now apply Theorem 3.14 repeatedly. Denote the eigenvalue and eigenvector of Theorem 3.14 by λ_1 and x_1 respectively. Let $X_1 = X$ and $X_2 = \{x \mid (x, x_1) = 0\}$. Then X_2 is closed. For, if $\{y_n\}$ is a sequence in X_2 such that $y_n \to y$, then given $\epsilon > 0$ there is a positive integer N such that $n \geq N$ implies $\|y - y_n\| < \frac{\epsilon}{\|x_1\|}$. Hence $|(y, x_1)| = |(y + y_n - y_n, x_1)| = |(y - y_n, x_1)$

$$+ (y_n, x_1)| = |(y - y_n, x_1)| \leq \|y - y_n\|\, \|x_1\| < \epsilon.$$

But since ϵ was arbitrary, it follows that $(y, x_1) = 0$ and y is in X_2. Hence X_2 is closed. Also, $A(X_2) \subset X_2$; if x is in X_2, then $(Ax, x_1) = (x, Ax_1) = (x, \lambda_1 x_1) = \lambda_1(x, x_1) = 0$, whence Ax is in X_2. Now let $\{x_k\}$ be a bounded sequence in X_2. Then $\{Ax_k\}$ has a convergent subsequence which converges to a point in X_2

since X_2 is closed. Thus the restriction of A to X_2 is compact and symmetric. If this restriction is not the zero operator, then by Theorem 3.14 we can assert the existence of λ_2 and x_2 such that x_2 is in X_2, $\|x_2\| = 1$, $Ax_2 = \lambda_2 x_2$, and $|\lambda_2|$ is the norm of the restriction of A to X_2. Since $X_2 \subseteq X_1$, $|\lambda_2| \leq |\lambda_1|$. Continuing in this way we obtain the nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $x_1, x_2, \ldots, x_n$ of unit norm. We also obtain $X_1, X_2, \ldots, X_{n+1}$, with X_{k+1} the set of elements of X_k which are orthogonal to $x_1, x_2, \ldots, x_k$, that is, if x is in X_{k+1}, then $(x, x_i) = 0$ for $i = 1, 2, \ldots, k$. At each step x_k is in X_k and $|\lambda_k|$ is the norm of the restriction of A to X_k, so $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. The process stops with λ_n, x_n, and X_{n+1} if and only if the restriction of A to X_{n+1} is the zero operator. In that case the range of A lies in the linear manifold $S = \left\{ \sum_{i=1}^{m} \alpha_i x_i \,|\, m \leq n \right\}$ generated by $x_1, x_2, \ldots, x_n$. For, if x is in X, let

$$(3.15)\quad y_n = x - \sum_{k=1}^{n} (x, x_k)x_k.$$

Then

$$\begin{aligned}(y_n, x_i) &= (x - \sum_{k=1}^{n}(x, x_k)x_k, x_i) \\ &= (x, x_i) - \sum_{k=1}^{n}(x, x_k)(x_k, x_i) \\ &= (x, x_i) - (x, x_i) = 0, \; i = 1, 2, \ldots, n.\end{aligned}$$

Hence y_n is in X_{n+1} and $Ay_n = 0$ since $A|X_{n+1}$ is the zero operator. Thus

$$(3.16)\quad Ax = A(y_n + \sum_{k=1}^{n} (x, x_k)x_k)$$
$$= Ay_n + \sum_{k=1}^{n} (x, x_k)Ax_k$$
$$= \sum_{k=1}^{n} (x, x_k)\lambda_k x_k = \sum_{k=1}^{n} \lambda_k (x, x_k)x_k.$$

This situation may occur even if X is infinite dimensional. It will certainly occur if X is finite dimensional since a linear space with finite dimension m can have at most m linearly independent vectors.

The foregoing considerations lead us to the statement of the following fundamental theorem:

<u>Theorem</u> 3.17. Suppose A is a compact symmetric operator, and $A \neq 0$. The procedure described in the foregoing discussion yields a possibly terminating sequence of nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$ and a corresponding orthonormal set of eigenvectors $x_1, x_2, \ldots$. If the sequences do not terminate, then $|\lambda_n| \to 0$. The expansion

$$(3.17\text{-A})\quad Ax = \Sigma\, (Ax, x_k)x_k = \Sigma\, \lambda_k (x, x_k)x_k$$

is valid for each x in X, the summation being extended over the entire sequence, whether finite or infinite. Each nonzero eigenvalue of A occurs in the sequence $\{\lambda_n\}$. The eigenmanifold corresponding to a particular λ_i is finite dimensional and its dimension is exactly the number of times this particular eigenvalue is repeated in the sequence $\{\lambda_n\}$.

Proof: Since $|\lambda_k| \geq |\lambda_{k+1}|$, we either have $\lambda_n \to 0$ or $|\lambda_n| \geq \epsilon > 0$ for some ϵ and all n. Suppose the latter and suppose that the sequence is infinite. Then $\|\frac{x_n}{\lambda_n}\| \leq \frac{\|x_n\|}{\epsilon} = \frac{1}{\epsilon}$, so $\left\{\frac{x_n}{\lambda_n}\right\}$ is a bounded sequence. But $A(\frac{x_n}{\lambda_n}) = \frac{1}{\lambda_n} Ax_n = \frac{1}{\lambda_n} \lambda_n x_n = x_n$, so $\{x_n\}$ must contain a convergent subsequence since A is compact. This is impossible, for the orthogonality of the x_i's yields

$$\begin{aligned}\|x_n - x_m\|^2 &= (x_n - x_m, x_n - x_m)\\ &= (x_n, x_n) - (x_n, x_m) - (x_m, x_n) + (x_m, x_m)\\ &= \|x_n\|^2 + \|x_m\|^2 = 2 \text{ for all m, n such that } m \neq n.\end{aligned}$$

Therefore $\{x_n\}$ contains no Cauchy subsequence, so $\{x_n\}$ has no convergent subsequence. It follows that $\lambda_n \to 0$ when the sequence $\{\lambda_n\}$ is infinite.

If the sequence of λ_k's terminates with λ_n, (3.17-A) is equivalent to 3.16. In the nonterminating case we define y_n by 3.15 and obtain

$$\begin{aligned}\|y_n\|^2 &= (x - \sum_{k=1}^{n}(x, x_k)\, x_k,\ x - \sum_{k=1}^{n}(x, x_k)x_k)\\ &= (x, x) - \sum_{k=1}^{n}(x, x_k)(x_k, x) - \sum_{k=1}^{n}\overline{(x, x_k)}(x, x_k)\\ &+ \sum_{k=1}^{n}(x, x_k) \sum_{j=1}^{n}\overline{(x, x_j)}(x_k, x_j)\\ &= (x, x) - \sum_{k=1}^{n}(x, x_k)\overline{(x, x_k)} - \sum_{k=1}^{n}\overline{(x, x_k)}(x, x_k)\\ &+ \sum_{k=1}^{n}(x, x_k)\overline{(x, x_k)} = \|x\|^2 - \sum_{k=1}^{n}|(x, x_k)|^2 \leq \|x\|^2.\end{aligned}$$

Since y_n is in X_{n+1} and $|\lambda_{n+1}|$ is the norm of the restriction of A to X_{n+1}, we have

$$\|Ay_n\| \le \|A\| \, \|y_n\| \le |\lambda_{n+1}| \, \|y_n\| \le |\lambda_{n+1}| \, \|x\|.$$

But $|\lambda_n| \to 0$, so given $\varepsilon > 0$ there is a positive integer N such that $n \ge N$ implies $|\lambda_n| < \frac{\varepsilon}{\|x\|}$. Hence $n \ge N$ implies $\|Ay_n\| < \varepsilon$, and $\|Ay_n\| \to 0$. It follows that $Ay_n \to 0$. Also,

$$Ay_n = A(x - \sum_{k=1}^{n}(x, x_k)x_k) = Ax - \sum_{k=1}^{n}(x, x_k)Ax_k,$$

so

$$\begin{aligned} Ax = \lim_{n \to \infty} \sum_{k=1}^{n} (x, x_k)Ax_k &= \Sigma(x, x_k)Ax_k \\ &= \Sigma(x, x_k)\lambda_k x_k \\ &= \Sigma(x, \lambda_k x_k)x_k \\ &= \Sigma(x, Ax_k)x_k \\ &= \Sigma(Ax, x_k)x_k. \end{aligned}$$

This proves 3.17-A.

If λ is a nonzero eigenvalue of A which is not in the sequence $\{\lambda_k\}$, then by Theorem 3.14 there is a corresponding eigenvector x of unit norm, and x must be orthogonal to x_n for every n by Theorem 3.12. Then $Ax = 0$ by 3.17-A. This contradicts $Ax = \lambda x \ne 0$. Hence each nonzero eigenvalue of A occurs in the sequence $\{\lambda_k\}$.

An eigenvalue cannot be repeated infinitely often in the sequence $\{\lambda_k\}$, because $\lambda_k \to 0$. Suppose that λ_k occurs p times. Then the corresponding eigenmanifold contains an orthonormal set of p eigenvectors, and is therefore at least p-dimensional. It cannot be of dimension greater than p, for this would entail the existence of an x such that $Ax = \lambda_k x \neq 0$, $\|x\| = 1$, and $(x, x_n) = 0$ for every n. But this is impossible, for then $Ax = 0$ by 3.17-A. The proof is now complete.

Theorem 3.18. Suppose X is complete. Let $\{x_n\}$ be an orthonormal set, and let $\{\lambda_n\}$ be any sequence of real numbers such that $\lambda_n \to 0$. Let A be defined by $Ax = \sum_{k=1}^{\infty} \lambda_k (x, x_k) x_k$. Then A is self-adjoint and compact.

Proof: First of all we show that Ax is defined for all x in X. By Theorem 2.11, $\sum_{k=1}^{\infty} \lambda_k (x, x_k) x_k$ converges if and only if $\Sigma |\lambda_k (x, x_k)|^2 = \Sigma |\lambda_k|^2 |(x,x_k)|^2 < \infty$. But $\Sigma |(x, x_k)|^2 \leq \|x\|^2$ by Theorem 2.9, and since $\lambda_n \to 0$ there is a constant $M > 0$ such that $|\lambda_n| \leq M$ for all n. Therefore $\Sigma |\lambda_k|^2 |(x,x_k)|^2 \leq M^2 \|x\|^2$, so $\Sigma \lambda_k (x, x_k) x_k$ converges for all x in X, whence Ax is defined for all x in X.

To prove A self-adjoint we must show that $(Ax, y) = (x, Ay)$ for all x, y in X. We have, by the continuity of the inner product,

$$\begin{aligned}(Ax, y) &= \lim_{n \to \infty} \left(\sum_{k=1}^{n} \lambda_k (x, x_k) x_k, y\right) \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k (x, x_k)(x_k, y) \\ &= \lim_{n \to \infty} \left(x, \sum_{k=1}^{n} \lambda_k (y, x_k)\, x_k\right) = (x, Ay)\end{aligned}$$

Hence A is self-adjoint.

Now let $A_n x = \sum_{k=1}^{n} \lambda_k (x, x_k) x_k$. Then A_n is compact by Theorem 1.37 since $R(A_n)$ is finite-dimensional ($R(A_n)$

is closed by Corollary 1.36.). We have

$$\|(A_n - A)(x)\|^2 = \|A_n x - A_x\|^2 = \|\sum_{k=n+1}^{\infty} \lambda_k (x, x_k) x_k\|^2$$

$$= \sum_{k=n+1}^{\infty} |\lambda_k (x, x_k)|^2.$$

Since $\lambda_k \to 0$, given a integer $n > 0$ there is a real number $\lambda(n)$ such that $\lambda(n) = \sup \{|\lambda_{n+1}|, |\lambda_{n+2}|, \ldots\}$. Hence

$$\|(A_n - A)(x)\|^2 \leq |\lambda(n)|^2 \sum_{k=n+1}^{\infty} |(x, x_k)|^2$$

$$\leq |\lambda(n)|^2 \|x\|^2.$$

But $\lambda(n) \to 0$, so given $\epsilon > 0$ there is a positive integer N such that $n \geq N$ implies $|\lambda(n)| < \epsilon$. Hence if $n \geq N$ we have

$$\|A_n - A\| = \sup_{\|x\|=1} \|(A_n - A)(x)\| \leq \sup_{\|x\|=1} |\lambda(n)| \ \|x\|$$

$$= |\lambda(n)| < \epsilon.$$

Hence $\|A_n - A\| \to 0$, and it follows from Theorem 3.5 that A is compact.

<u>Theorem</u> 3.19. Let $A, \{\lambda_n\}$, $\{x_n\}$ be as in Theorem 3.17. Then, if $\lambda \neq 0$ and $\lambda \neq \lambda_k$ for each k, $\lambda - A$ has a continuous inverse defined on all of X and given by $x = (\lambda - A)^{-1} y$, where

$$\text{(3.19-A)} \quad x = \frac{1}{\lambda} y + \frac{1}{\lambda} \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k.$$

Proof: Suppose x and y are given such that $\lambda x - Ax = y$. Then $Ax = \lambda x - y$, and so from 3.17-A we have

$$\lambda x - y = \Sigma \lambda_k (x, x_k) x_k.$$

We form the inner product with x_i and obtain

$$(\lambda x, x_i) - (y, x_i) = (\lambda x - y, x_i)$$
$$= (\Sigma \lambda_k (x, x_k) x_k, x_i)$$
$$= \Sigma \lambda_k (x, x_k)(x_k, x_i) = \lambda_i (x, x_i).$$

Thus

$$(x, x_i) = \frac{(y, x_i)}{\lambda - \lambda_i},$$

and so

$$x = y + \Sigma \lambda_k (x, x_k) x_k = y + \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k,$$

which gives 3.19-A. This shows that the solution of $(\lambda - A)x = y$ is unique, if it exists. On the other hand, if the series in 3.19-A is convergent, the element x defined by 3.19-A satisfies $(\lambda - A)\,x = y$, for then

$$(\lambda - A)\,x = (\lambda - A)\left(\frac{1}{\lambda}\,y + \frac{1}{\lambda}\,\Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k\right)$$
$$= y + \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k - \frac{1}{\lambda} Ay$$
$$- \frac{1}{\lambda} \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} Ax_k.$$

We put $Ax_k = \lambda_k x_k$ in the last sum and use 3.17-A with y in place of x; then

$$(\lambda - A)x = y + \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k - \frac{1}{\lambda} \Sigma \lambda_k (y, x_k) x_k$$
$$- \frac{1}{\lambda} \Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} \lambda_k x_k$$
$$= y + \Sigma\left(1 - \frac{\lambda - \lambda_k}{\lambda} - \frac{\lambda_k}{\lambda}\right) \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k = y.$$

We now show that the series in 3.19-A converges, no matter how y is chosen. Let $\alpha = \sup_k \left| \frac{\lambda_k}{\lambda - \lambda_k} \right|$,

$\beta = \sup_k \frac{1}{|\lambda - \lambda_k|}$, $u_n = \sum_{k=1}^{n} \lambda_k \frac{(y,x_k)}{\lambda - \lambda_k} x_k$,

$v_n = \sum_{k=1}^{n} \frac{(y,x_k)}{\lambda - \lambda_k} x_k$. α and β are finite. Now, if $m < n$,

$$\begin{aligned} \|u_n - u_m\|^2 &= \left(\sum_{k=m+1}^{n} \lambda_k \frac{(y,x_k)}{\lambda - \lambda_k} x_k,\ \sum_{k=m+1}^{n} \lambda_k \frac{(y,x_k)}{\lambda - \lambda_k} x_k\right) \\ &= \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \lambda_j \frac{(y,x_j)}{\lambda - \lambda_j} \lambda_k \frac{\overline{(y,x_k)}}{\lambda - \lambda_k} (x_j, x_k) \\ &= \sum_{k=m+1}^{n} \lambda_k^2 \frac{|(y,x_k)|^2}{(\lambda - \lambda_k)^2} \\ &= \sum_{k=m+1}^{n} \left|\frac{\lambda_k}{\lambda - \lambda_k}\right|^2 |(y, x_k)|^2 \\ &\leq \alpha^2 \sum_{k=m+1}^{n} |(y, x_k)|^2. \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence, because $\Sigma|(y, x_k)|^2$ is convergent by Theorem 2.9. If X is complete, this implies that $u_n \to u$, where u is in X. If X is not complete we continue the argument as follows:

$$\|v_n\|^2 = \sum_{k=1}^{n} \frac{|(y,x_k)|^2}{|\lambda - \lambda_k|^2} \leq \beta^2 \sum_{k=1}^{n} |(y, x_k)|^2 \leq \beta^2 \|y\|^2,$$

so $\{v_n\}$ is bounded. Now
$$\begin{aligned} Av_n &= \sum_{k=1}^{n} \frac{(y,x_k)}{\lambda - \lambda_k} Ax_k \\ &= \sum_{k=1}^{n} \frac{(y,x_k)}{\lambda - \lambda_k} \lambda_k x_k = u_n. \end{aligned}$$

Hence, the compactness of A implies that $\{u_n\}$ contains a convergent subsequence. Since $\{u_n\}$ is Cauchy, it must then be convergent to the same limit as the subsequence. Hence the series in 3.19-A converges. We see from 3.19-A that

$$\|x\| = \| \frac{1}{\lambda} y + \frac{1}{\lambda} \quad \Sigma \lambda_k \frac{(y,x_k)}{\lambda - \lambda_k} x_k \|$$

$$\leq \| \frac{1}{\lambda} y \| + \| \frac{1}{\lambda} \Sigma \lambda_k \frac{(y,x_k)}{\lambda - \lambda_k} x_k \|$$

$$\leq \frac{1}{|\lambda|} \|y\| + \frac{1}{|\lambda|} \alpha \; \| \Sigma \; (y, \; x_k) \; x_k \|.$$

But $\|\Sigma(y,x_k)x_k\|^2 = \|y\|$ by Theorems 2.9 and 2.11, so

$$\|x\| \leq \frac{1}{|\lambda|} \|y\| + \frac{1}{|\lambda|} \alpha \; \|y\|.$$

Hence

$$\|(\lambda - A)^{-1}\| = \sup_{\|y\| \neq 0} \frac{\|(\lambda - A)^{-1} y\|}{\|y\|} = \sup_{\|y\| \neq 0} \frac{\|x\|}{\|y\|}$$

$$\leq \frac{1}{|\lambda|} + \frac{1}{|\lambda|} \alpha,$$

so $(\lambda - A)^{-1}$ is continuous and defined on all of X.

<u>Theorem</u> 3.20. Let A, $\{\lambda_n\}$, $\{x_n\}$ be as in Theorem 3.17. If $\lambda = \lambda_j$ for some j, then the range of λ -A consists of all vectors orthogonal to the eigenmanifold E_j corresponding to λ_j. For such a vector y the general solution of $(\lambda - A) \; x = y$ is

$$(3.20\text{-A}) \quad x = \frac{1}{\lambda} y + \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(x,x_k)}{\lambda - \lambda_k} x_k + \omega ,$$

where ω is an arbitrary element of E_j.

Proof: Let x be in X, and let z_j be an arbitrary vector in E_j. Then

$$((\lambda - A)x, \; z_j) = (\lambda x - Ax, \; z_j)$$

$$= (\lambda x, \; z_j) - (Ax, \; z_j)$$

$$= (\lambda x, \; z_j) - (x, \; Az_j)$$

$$= (\lambda x, z_j) - (x, \lambda z_j)$$
$$= \lambda(x, z_j) - \lambda(x, z_j) = 0.$$

Hence if y is in $R(\lambda - A)$, then $y \perp E_j$.

Now suppose y is given, and that $\lambda x - Ax = y$. Then from 3.17-A we have

$$\lambda x - y = Ax = \Sigma \lambda_k (x, x_k) x_k.$$

Forming the inner product with x_i gives

$$(\lambda x, x_i) - (y, x_i) = \lambda_i (x, x_i).$$

Thus, if $\lambda \neq \lambda_i$, we have

$$(x, x_i) = \frac{(y, x_i)}{\lambda - \lambda_i},$$

and so

$$\lambda x = y + \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k + \sum_{\lambda_k = \lambda} \lambda_k (x, x_k) x_k.$$

But by Theorem 2.12, $P(x) = \sum_{\lambda_k = \lambda} (x, x_k) x_k$ is the orthogonal projection of x onto the linear manifold generated by E_j. Hence

$$x = \frac{1}{\lambda} y + \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} + P(x), \text{ or}$$

$$(I-P)x = \frac{1}{\lambda} y + \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k}.$$

If ω is an arbitrary element of E_j, then $(I-P)(x+\omega) = (I-P)x + (I-P)\omega = (I-P)x$. Therefore, given y in $R(\lambda - A)$, $\frac{1}{\lambda} y + \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k$ is a solution of $(\lambda - A)\, x = y$, as is $\frac{1}{\lambda} y + \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k + \omega$. We have shown in Theorem 3.19 that $\Sigma \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k$ converges, no matter how y is chosen. If y is orthogonal to E_j, the element x defined by 3.20-A then satisfies $(\lambda - A)\,x = y$, for

$$\lambda x - Ax = y + \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k + \lambda\omega - \frac{1}{\lambda} Ay$$

$$- \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} Ax_k - A\omega$$

$$= y + \Big[\sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k - \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k (y, x_k) x_k$$

$$- \frac{1}{\lambda} \sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} \lambda_k x_k\Big]$$

$$+ \lambda\omega - A\omega - \frac{1}{\lambda} \sum_{\lambda_k = \lambda} \lambda_k (y, x_k) x_k$$

$$= y + \Big(\sum_{\lambda_k \neq \lambda} \lambda_k \frac{(y, x_k)}{\lambda - \lambda_k} x_k \Big)\Big(1 - \frac{\lambda - \lambda_k}{\lambda} - \frac{\lambda_k}{\lambda}\Big)$$

$$+ (\lambda - A)\omega - \frac{1}{\lambda} \sum_{\lambda_k = \lambda} \lambda_k (y, x_k) x_k = y.$$

Therefore the range of $\lambda - A$ consists of all vectors orthogonal to E_j, and for such a vector y the general solution of $(\lambda - A)\,x = y$ is given by 3.20-A.

We round out the foregoing discussion by considering the null manifold and range of A when X is a Hilbert space.

<u>Theorem</u> 3.21. (a) Let A, $\{\lambda_n\}$, $\{x_n\}$ be as in 3.17, and let M be the closed linear manifold generated by the eigenvectors $x_1, x_2, \ldots$. Then $M^{\perp} = N(A)$. Hence the orthonormal set $\{x_n\}$ is complete if and only if 0 is not an eigenvalue of A. When X is complete we have $X = M \oplus N(A)$.

(b) Suppose X is complete. Then the range of A is composed of those elements in M which are such that the series

$$(3.21\text{-A}) \quad \Sigma \frac{(y, x_k)}{\lambda_k} x_k$$

is convergent.

Proof: (a) Let x be in $M^{\perp}$. Then $(x, x_k) = 0$ for all k, so $Ax = 0$ by 3.17-A. Hence x is in N(A) and $M^{\perp} \subseteq N(A)$. Conversely, if x is in N(A), then for any k, $(x, x_k) = \lambda_k^{-1} (x, \lambda_k x_k) = \lambda_k^{-1} (x, Ax_k)$ $= \lambda_k^{-1} (Ax, x_k) = 0$, so x is in $M^{\perp}$. Hence $M^{\perp} = N(A)$. The orthonormal set $\{x_n\}$ is complete if and only if $M^{\perp} = (0)$, which means that if $x \neq 0$, then $Ax \neq 0$. It follows that if $\{x_n\}$ is complete, then 0 is not an eigenvalue of A. If X is complete, we have $X = M \oplus N(A)$ by Theorem 2.17.

(b) Suppose $Ax = y$. Then from 3.17-A, $y = Ax$ $= \Sigma \lambda_k (x, x_k) x_k$, so y is in M. From the orthonormality and continuity of the inner product we obtain

$$(y, x_i) = (\Sigma \lambda_k (x, x_k) x_k, x_i) = \Sigma \lambda_k (x, x_k)(x_k, x_i)$$
$$= \lambda_i (x, x_i).$$

Since $X = M \oplus N(A)$ we can write $x = u + v$, where u is in M and v is in N(A). Then $(x, x_k) = (u, x_k) + (v, x_k)$ $= (u, x_k)$ since $N(A) = M^{\perp}$, and by Theorem 2.12 we have

$$u = \Sigma (u, x_k) x_k = \Sigma (x, x_k) x_k = \Sigma \frac{(x, \lambda_k x_k)}{\lambda_k} x_k$$

$= \Sigma \frac{(x, Ax_k)}{\lambda_k} x_k = \Sigma \frac{(Ax, x_k)}{\lambda_k} x_k = \Sigma \frac{(y, x_k)}{\lambda_k} x_k$, the series necessarily being convergent if it is infinite. Conversely, suppose y is in M and that the series 3.21-A is convergent, with u as its sum. Then

$$Au = \Sigma \frac{(y, x_k)}{\lambda_k} Ax_k = \Sigma \frac{(y, x_k)}{\lambda_k} \lambda_k x_k = \Sigma(y, x_k)x_k = y,$$

so y is in R(A). This completes the proof.

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