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The Weierstrass approximation theorem

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THE WEIERSTRASS APPROXIMATION THEOREM

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CHAPTER I

INTRODUCTION

The Weierstrass Approximation Theorem states, "Let f be a continuous function on a compact interval of R and with values in R. Then f can be uniformly approximated by polynomials." M. H. Stone generalized this theorem by placing less restriction on the range and domain of f, and also by showing f can be approximated by a family of functions which are not necessarily polynomials. Stone formulated his generalization, as he states, by answering the following question, "What functions can be built from the functions of a prescribed family by the application of the algebraic operations (addition, multiplication, and multiplication by real numbers) and uniform passages to the limit?"

Stone answers this question by beginning with a compact space X, and the family $\mathfrak{X}$ of all continuous functions on X. He selects a prescribed subfamily $\mathfrak{X}_0$ of $\mathfrak{X}$, and determines what properties of the functions in $\mathfrak{X}_0$ are transmitted by the algebraic operations and uniform passages to the limit. For example, he showed that if all functions in $\mathfrak{X}_0$ had a value of zero at a point x in X, then every function generated from $\mathfrak{X}_0$ by the algebraic operations and uniform passages to the limit also had a value of zero at the point x. By doing this he could determine what prescribed subfamily is needed to generate all continuous functions over a compact domain.

If we apply Stone's generalized theorem to the prescribed subfamily $\mathfrak{X}_0$, where $\mathfrak{X}_0$ contains the two functions $f_1(x) = 1$ and $f_2(x) = x$ for all x in a compact interval, then all the polynomials will be generated by the algebraic operations and all continuous functions will be generated when we apply uniform passage to the limit to the family of polynomials. From this, we can see that we can obtain the Weierstrass Theorem from Stone's generalized theorem. We can also see that from a very small prescribed subfamily a much more inclusive family can be generated.

The second chapter of this paper will present the theorems and their proofs, that M. H. Stone used in generalizing the Weierstrass Theorem. There will be two sections in this chapter. The first section will generalize the Weierstrass Theorem by using the algebraic operations, uniform passages to the limit, and the lattice operations of taking the maximum and minimum of two or more functions.

The second section will treat all continuous functions on a compact set as a linear algebra with addition and multiplication as the ring operations and multiplication by real numbers as the scalar multiplication. This section will also deal with the characterization of closed ideals.

Throughout the second chapter the following notation and definitions will be used.

<u>Definition</u>: Algebraic operations: addition, multiplication, and multiplication by real numbers.

<u>Definition</u>: Uniform passage to the limit: A sequence (f_n) of functions on a subset D of R' to R' is said to converge uniformly on D to a function f if for every $\varepsilon > 0$ there is a natural number N such that if $n \geq N$ then $|f_n(x) - f(x)| < \varepsilon$ for all x in D.

<u>Definition</u>: Lattice operations: Taking the maximum and minimum of two or more functions. This will be denoted by,

$$f \cup g = \max(f, g) \text{ and } f \cap g = \min(f, g).$$

$f \cup g$ and $f \cap g$ will be the functions h and k respectively where

$$h(x) = \max(f(x), g(x)) \text{ and } k(x) = \min(f(x), g(x)) \text{ for all } x.$$

<u>Definition</u>: Bounded function: A function f with domain D is said to be bounded if there exists a natural number M such that $|f(x)| \leq M$ for all x in D.

<u>Definition</u>: Compact Space: A set X is said to be compact if, whenever it is contained in the union of a collection $\mathcal{G} = \{G_\alpha\}$ of open sets, then it is also contained in the union of some finite number of the sets in $\mathcal{G}$.

Notation:

X	A compact space.
$\mathfrak{F}$	Family of all continuous functions on X
$\mathfrak{F}_0$	Prescribed subfamily of $\mathfrak{F}$.
$\mathfrak{U}_1(\mathfrak{F}_0)$	The family of functions generated from $\mathfrak{F}_0$ by the algebraic operations.
$\mathfrak{L}_1(\mathfrak{F}_0)$	The family of functions generated from $\mathfrak{F}_0$ by the lattice operations.
$\mathfrak{L}_2(\mathfrak{F}_0)$	The family of functions generated from $\mathfrak{L}_1(\mathfrak{F}_0)$ by uniform passages to the limit.
$\mathfrak{U}_2(\mathfrak{F}_0)$	The family of all functions generated from $\mathfrak{U}_1(\mathfrak{F}_0)$ by uniform passages to the limit.
$\mathfrak{L}(\mathfrak{F}_0)$	The family of all functions generated from $\mathfrak{F}_0$ by the algebraic operations, uniform passages to the limit, and the lattice operations.
$\mathfrak{F}_0(x,y)$	The family of functions obtained by restricting every function in $\mathfrak{F}_0$ by suppresing all points in X except the two points x and y. That is $\mathfrak{F}_0(x, y) = \{(f(x), f(y)) \mid f \in \mathfrak{F}_0\}$.
$\mathfrak{F}_0(x,y)*$	The closure of $\mathfrak{F}_0(x, y)$.

The following theorems must hold throughout the second chapter and so they will be stated and proved here.

Theorem 1: If f is a continuous function on a compact space X, then f is bounded on X.

Proof: Suppose f is not bounded on X, then for each natural number n there exists a point x_n in X with $|f(x_n)| \geq n$. Since X is bounded, the sequence (x_n) is bounded, hence it follows from the Bolzano-Weierstrass Theorem that there is a subsequence of (x_n) which converges to an element x. Since $x_n \in X$ for $n \in N$, the point x belongs to the set X. Hence, f is continuous at x, so f is bounded by $|f(x)| + 1$ on a neighborhood of x. This contradicts the assumption that $|f(x_n)| \geq n$; therefore f is bounded.

Theorem 2: If f is a continuous function at a point a, then $|f|$ is also continuous at a.

Proof: f is continuous at a implies that for $\varepsilon > 0$ there exist a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. By the triangular inequality we have

$$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)| < \varepsilon.$$

Therefore $|f|$ is continuous at a.

Theorem 3: Let X be a compact space. Let $\mathfrak{F}$ be the family of all continuous real functions defined on X. Let $\mathfrak{F}_0$ be a prescribed subfamily of $\mathfrak{F}$. Let $\mathcal{U}(\mathfrak{F}_0)$ be the family of functions generated from $\mathfrak{F}_0$ by the algebraic operations and by uniform passages to the limit. Then $\mathcal{U}(\mathfrak{F}_0) \subset \mathfrak{F}$ and $\mathcal{U}(\mathcal{U}(\mathfrak{F}_0)) = \mathcal{U}(\mathfrak{F}_0)$.

Proof: First we will show $\mathcal{U}(\mathfrak{F}_0) \subset \mathfrak{F}$. Since the elements of $\mathcal{U}_1(\mathfrak{F}_0)$ are algebraic combinations of functions in $\mathfrak{F}_0$, then must show that if f, $g \in \mathfrak{F}_0$ and $\alpha \in R$, then

$f + g$, fg, and αf are continuous. Since f and g belong to $\mathfrak{X}$ then f and g are continuous. This implies that for $\varepsilon > 0$ there exists a δ_1 and a δ_2 such that if $|a - x| < \delta = \min(\delta_1, \delta_2)$, then $|f(x) - f(a)| < \varepsilon$ and $|g(x) - g(a)| < \varepsilon$. Therefore:

$$|f(x) + g(x) - f(a) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < 2\varepsilon$$ and $f + g$ is continuous.

Also, $$|f(x)g(x) - f(a)g(a)| = |(f(x)g(x) - f(x)g(a)) + f(x)g(a) - f(a)g(a)| \leq$$
$$|f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| = |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)|$$

Since f is continuous on a compact space then f is bounded. Therefore, there exists a natural number M such that $|f(x)| \leq M$.

Therefore, if $|x - a| < \delta$ then $|f(x)g(x) - f(a)g(a)| < M\varepsilon + |g(a)|\varepsilon$ and fg is continuous.

Also $|\alpha f(x) - \alpha f(a)| = |\alpha||f(x) - f(a)| < |\alpha|\varepsilon$. Therefore, αf is continuous. We must now show that if (f_n) is a sequence of continuous functions which converge uniformly to a function f, then f is continuous. Since (f_n) converges uniformly on X to f, then for $\varepsilon > 0$ there exists a natural number N such that if $n \leq N$ then $|f_n(x) - f(x)| < \varepsilon$ for all x in X. Since f_N is continuous then for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then $|f_N(x) - f_N(a)| < \varepsilon$. Since a in X then $|f_N(a) - f(a)| < \varepsilon$.

Therefore:

$$|f(x) - f(a)| \leq |f_N(x) - f(x)| + |f_N(x) - f_N(a)| +$$

$|f_N(a) - f(a)| < 3\varepsilon$ and f is continuous.

Therefore, $\mathcal{U}(\mathfrak{F}) \subset \mathfrak{F}$.

To show $\mathcal{U}(\mathcal{U}(\mathfrak{F}_0)) = \mathcal{U}(\mathfrak{F}_0)$ is to show $\mathcal{U}(\mathfrak{F}_0)$ is closed under the algebraic operations and uniform passages to the limit.

Let f and g belong to $\mathcal{U}(\mathfrak{F}_0)$; then there exist sequences (f_n) and (g_n) in $\mathcal{U}_1(\mathfrak{F}_0)$ such that $(f_n) \longrightarrow f$ and $(g_n) \longrightarrow g$ uniformly. This implies that for $\varepsilon > 0$ there exists a ntural number N such that if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ and $|g_n(x) - g(x)| < \varepsilon$.

Therefore

$$|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| +$$

$|g_n(x) - g(x)| < 2\varepsilon$ and $(f_n + g_n) \longrightarrow f + g$ which implies $f + g$ is in $\mathcal{U}(\mathfrak{F}_0)$.

Also $|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \leq$

$$|f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| < 2M\varepsilon$$

since f_n and g as continuous functions on the compact space X are abounded by some natural number M. Therefore, $(f_n g_n) \longrightarrow fg$ which implies fg is in $\mathcal{U}(\mathfrak{F}_0)$.

Also $|\alpha f_n(x) - \alpha f(x)| = |\alpha||f_n(x) - f(x)| < |\alpha|\varepsilon$. Therefore, $\alpha(f_n) \longrightarrow \alpha f$ which implies αf is in $\mathcal{U}(\mathfrak{F}_0)$.

Let f be a function such that there exist a sequence (f_n)

in $\mathcal{U}(\mathfrak{X}_0)$ such that $(f_n) \longrightarrow f$ uniformly. This implies that for $\varepsilon > 0$ there exist a natural number N such that if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ for all x in X. Let (f_{n_k}) be a subsequence of (f_n) such that $|f_{n_k}(x) - f(x)| < \frac{1}{2^{k+1}}$ for each k. Since f_{n_k} is in $\mathcal{U}(\mathfrak{X}_0)$ for each k, there exist a sequence (g_{km}) in $\mathcal{U}_1(\mathfrak{X}_0)$ such that $(g_{km}) \longrightarrow f_{n_k}$ uniformly. Let (g_{km_j}) be a subsequence of (g_{km}) such that $|g_{km_j}(x) - f_{n_k}(x)| < \frac{1}{2^{j+1}}$ for each j. Select a subsequence (g_{km_k}). Then

$$|g_{km_k}(x) - f(x)| \leq |g_{km_k}(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)| < \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$$

and by choosing k large enough, $\frac{1}{2^k}$ is less than any given $\varepsilon > 0$.

Therefore f is in $\mathcal{U}(\mathfrak{X}_0)$.

This completes the proof of the theorem.

and $f \cap g$ are continuous. Therefore $\mathcal{L}(\mathfrak{F}_0) \subset \mathfrak{F}$.
Let f and g belong to $\mathcal{L}(\mathfrak{F}_0)$. We need to show that $f \cup g$ and $f \cap g$ belong to $\mathcal{L}(\mathfrak{F}_0)$. Since f and g belong to $\mathcal{L}(\mathfrak{F}_0)$ there exist sequences of functions (f_n) and (g_n) in $\mathcal{L}(\mathfrak{F}_0)$ such that $(f_n) \longrightarrow f$ and $(g_n) \longrightarrow g$ uniformly. This implies that for $\varepsilon > 0$ there exists a natural number N such that if $n \geq N$, then $|f_n - f| < \varepsilon$ and $|g_n - g| < \varepsilon$ for all x in X. Therefore

$$|f \cup g - f_n \cup g_n| = |1/2(f + g + |f - g|) - 1/2(f_n + g_n + |f_n - g_n|)|$$
$$\leq 1/2(|f - f_n| + |(g - g_n| + |f - g| - |f_n - g_n||)$$
$$\leq 1/2(|f - f_n| + |g - g_n| + |f - f_n| + |g - g_n|)$$
$$= |f - f_n| + |g - g_n| < 2\varepsilon$$

and $(f_n \cup g_n) \longrightarrow f \cup g$ which implies $f \cup g$ is in $\mathcal{L}(\mathfrak{F}_0)$. By a similar method $f \cap g$ is in $\mathcal{L}(\mathfrak{F}_0)$. By similar method we can show $\mathcal{L}(\mathfrak{F}_0)$ is closed under the algebraic operations. Therefore $\mathcal{L}(\mathcal{L}(\mathfrak{F}_0)) = \mathcal{L}(\mathfrak{F}_0)$.

The following theorem determines what functions belong to $\mathcal{L}(\mathfrak{F}_0)$.

<u>Theorem 5</u>: Let X be a compact space, $\mathfrak{F}$ the family of all continuous (necessarily bounded) real functions on X, $\mathfrak{F}_0$ an arbitrary subfamily of $\mathfrak{F}$, and $\mathcal{L}(\mathfrak{F}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{F}_0$ by the lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a

function f in $\mathfrak{F}$ to be in $\mathfrak{L}(\mathfrak{F}_0)$ is that, whatever the points x, y in X and whatever the positive number ε, there exists a function f_{xy} obtained by applying the lattice operations alone to $\mathfrak{F}_0$ and such that

$$|f(x) - f_{xy}(x)| < \varepsilon \text{ and } |f(y) - f_{xy}(y)| < \varepsilon.$$

<u>Proof</u>: Let f belong to $\mathfrak{L}(\mathfrak{F}_0)$, then there exists a sequence of functions (f_n) in $\mathfrak{L}_1(\mathfrak{F})$ such that $(f_n) \longrightarrow f$ uniformly. This implies that for x, y in X, there exist a natural number n such that $|f(x) - f_n(x)| < \varepsilon$ and $|f(y) - f_n(y)| < \varepsilon$.

Conversely, let $G_y = \{z | f(z) - f_{xy}(z) < \varepsilon\}$, where x is fixed. G_y is open since f and f_{xy} are continuous which implies $f - f_{xy}$ is continuous, also $-\infty < (f - f_{xy})[G_y] < \varepsilon$, which implies $(f - f_{xy})^{-1}(-\infty, \varepsilon)$ is open but $(f - f_{xy})^{-1}(-\infty, \varepsilon) = G_y$ therefore, G_y is open. By hypothesis x and y are in G_y, so that the union of all the sets G_y is the entire space X. Since X is compact, there exists a finite number of points, $y_1, \ldots, y_n$, such that the union of the sets $G_{y_1}, \ldots, G_{y_n}$ is still the entire space X. Let $g_x = f_{xy_1} \cup f_{xy_2} \cup \ldots \cup f_{xy_n}$. This implies that for any z in X there exists a natural number k such that z is in G_{y_k} which implies

$$g_x(z) \geq f_{xy_k}(z) > f(z) - \varepsilon.$$

We also have, since x is fixed, that

$$f_{xy}(x) < f(x) + \varepsilon$$

which implies that $g_x(x) < f(x) + \varepsilon$ since all the $f_{xy_k}(x) < f(x) + \varepsilon$ by the way the f_{xy_k}'s were chosen. We now work in a similar manner with the functions g_x. Let $H_x = \{z | g_x(z) | g_x(z) < f(z) + \varepsilon\}$. x is in H_x, so that the union of all the sets H_x is the entire space X. The sets H_x are open by using the same reasoning as was used on the sets G_y. Since X is compact, there exists points $x_1, \ldots, x_n$ such that the union of the sets $H_{x_1}, \ldots, H_{x_n}$ is still the entire space X. Let

$$h = g_{x_1} \cap \ldots \cap g_{x_n}.$$

Then we see that for any z in X we have z in H_{x_k} for a suitable choice of k and hence

$$h(z) \leq g_{x_k}(z) < f(z) + \varepsilon.$$

Now since we have that

$$g_x(z) > f(z) - \varepsilon$$

for all x and all z, we then have

$$h(z) > f(z) - \varepsilon$$

for all z. Therefore, we have $|f(z) - h(z)| < \varepsilon$ for all z in X. Now since only the lattice operations have been used in constructing the functions g_x and h from the functions f_{xy}, these functions are all in $\mathcal{L}_1(\mathfrak{X}_0)$. Therefore f is in $\mathcal{L}(\mathfrak{X}_0)$.

From theorem 5 we have the following corollaries:

Corollary 1: If $\mathfrak{X}_0$ has the property that, whatever the points x, y with $x \neq y$, in X and whatever the real numbers α and β, there exists a function f_0 in $\mathfrak{F}_0$ for which $f_0(x) = \alpha$ and $f_0(y) = \beta$, then $\mathcal{L}(\mathfrak{F}_0) = \mathfrak{F}$. In other words, any continuous function on X can be uniformly approximated by lattice polynomials in functions belonging to the prescribed family $\mathfrak{F}_0$.

Proof: Let f belong to $\mathfrak{F}$. Let x, y be any two points in X such that $x \neq y$. $f(x)$ and $f(y)$ are real numbers, therefore there exists a function f_0 in $\mathfrak{F}_0$ such that $f_0(x) = f(x)$ and $f_0(y) = f(y)$ which implies that $|f_0(x) - f(x)| < \varepsilon$ and $|f_0(y) - f(y)| < \varepsilon$ which, by theorem 5, implies f is in $\mathcal{L}(\mathfrak{F}_0)$.

Corollary 2: If a continuous real function f on a compact space X is the limit of a monotonic sequence (f_n) of continuous functions, then the sequence converges uniformly to f.

Proof: Let $\mathfrak{F}_0$ be the totality of functions occurring in the sequence (f_n). Then $\mathcal{L}_1(\mathfrak{F}_0) = \mathfrak{F}_0$ since monotonicity implies that $f_n \cup f_m$ coincides with one of the two functions f_n and f_m while $f_n \cap f_m$ coincides with the other. From the hypothesis we know that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

for every x, which shows that the condition of theorem 5 is satisfied. Hence f is in $\mathcal{L}(\mathfrak{F}_0)$.

Theorem 5 tells us that whether a given function f can be approximated in terms of the prescribed family $\mathfrak{F}_0$ is dependent on the way in which f and $\mathfrak{F}_0$ behave on pairs of points in X. By using the notation $\mathfrak{F}_0(x, y)$ and $\mathfrak{F}_0(x, y)*$ as defined in the introduction, we can restate theorem 5 in the following form: If f is in $\mathfrak{F}$, then f is in $\mathfrak{L}(\mathfrak{F}_0)$ if and only if

$$(f(x), f(y)) \text{ is in } \mathfrak{F}_0(x, y)*$$

for every pair of distinct points x, y in X.

<u>Theorem 6</u>: Let X be a compact space, $\mathfrak{F}$ the family of continuous functions on X, and $\mathfrak{F}_0$ a subfamily of $\mathfrak{F}$ which is closed under the lattice operations and uniform passage to the limit. Then $\mathfrak{F}_0$ is completely characterized by the system of planar sets $\mathfrak{F}_0(x, y)* = \mathfrak{F}_0(x, y)$.

<u>Proof</u>: Our hypothesis that $\mathfrak{F}_0 = \mathfrak{L}(\mathfrak{F}_0)$ shows that $\mathfrak{F}_0(x, y)$ has $\mathfrak{F}_0(x, y)*$ as its closure. Let us suppose that

$$\mathfrak{G}_0 = \mathfrak{L}(\mathfrak{G}_0) \subset \mathfrak{F}$$

and that $\mathfrak{F}_0(x, y)* = \mathfrak{G}_0(x, y)*$ for all pairs of points x, y in X. Then the conditions for f in $\mathfrak{F}$ to belong to $\mathfrak{F}_0$ are identical with those for it to belong $\mathfrak{G}_0$, hence $\mathfrak{F}_0$ and $\mathfrak{G}_0$ coincide.

In view of the equations for $f \cup g$ and $f \cap g$, which express the lattice operations in terms of the linear operations and the single operation of forming the absolute value, we may take the specified algebraic operations to be simply addition, multiplication by real numbers, and formation of absolute values. The family $\mathfrak{L}(\mathfrak{F}_0)$ of all functions which can be constructed from $\mathfrak{F}_0$ in $\mathfrak{F}$ by application of the lattice operations and uniform passage to the limit is still obtainable in two steps, the first being algebraic and the second by uniform passage to the limit. As shown in theorems 3 and 4, $\mathfrak{L}(\mathfrak{F}_0)$ is closed under the operations used to generate it.

The following theorem is an analog of the results contained in theorems 5 and 6.

Theorem 7: Let X be a compact space, $\mathfrak{F}$ the family of all continuous real (necessarily bounded) functions on X, $\mathfrak{F}_0$ an arbitrary subfamily of $\mathfrak{F}$, and $\mathfrak{L}(\mathfrak{F}_0)$ the family of all functions (necessarily continuous) generated from by the linear lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{F}$ to be in $\mathfrak{L}(\mathfrak{F}_0)$ is that f satisfy every linear relation of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta \geq 0$, which is satisfied by all functions in $\mathfrak{F}_0$. If $\mathfrak{F}_0$ is a closed linear sublattice of $\mathfrak{F}$ – that is, if $\mathfrak{F}_0 = \mathfrak{L}(\mathfrak{F}_0)$ – then $\mathfrak{F}_0$ is characterized by the system of all the linear relations of this form which are satisfied by every function belonging to it. The linear relations associated with an arbitrary pair of points x, y in X must be equivalent to one of the following distinct types:

(1) $g(x) = 0$ and $g(y) = 0$;

(2) $g(x) = 0$ and $g(y)$ unrestricted, or vice versa;

(3) $g(x) = g(y)$ without restriction on the common value;

(4) $g(x) = \lambda g(y)$ or $g(y) = \lambda g(x)$ for a unique value λ, $0 < \lambda < 1$.

Proof: Suppose the functions in $\mathfrak{F}_0$ satisfy the relation $\alpha f(x) = \beta f(y)$ for all f in $\mathfrak{F}_0$. Then we will show that the functions in $\mathcal{L}_1(\mathfrak{F}_0)$ satisfy the same relation.

Let f and g belong to $\mathfrak{F}_0$, then $\alpha f(x) = \beta f(y)$ and $\alpha g(x) = \beta g(y)$ where $\alpha\beta \geq 0$. From this it is obvious that $\alpha(f(x) + g(x)) = \beta(f(y) + g(y))$, $\alpha(\delta f(x)) = \beta(\delta f(y))$ where δ is real, and $\alpha|f(x) + g(x)| = \beta|f(y) + g(y)|$. From this it is seen that if f is in $\mathcal{L}_1(\mathfrak{F}_0)$, then $\alpha f(x) = \beta f(y)$ for $\alpha\beta \geq 0$. Let f belong to $\mathcal{L}(\mathfrak{F}_0)$ then there exists a sequence of functions (f_n) in $\mathcal{L}_1(\mathfrak{F}_0)$ such that $(f_n) \rightarrow f$ uniformly. This implies that $(f_n(x)) \rightarrow f(x)$ and $(f_n(y)) \rightarrow f(y)$. Since f_n is in $\mathcal{L}_1(\mathfrak{F}_0)$ for each n, then $\alpha f_n(x) = \beta f_n(y)$, and since $\alpha(f_n(x)) \rightarrow \alpha f(x)$ and $\beta(f_n(y)) \rightarrow \beta f(y)$, then for $\varepsilon > 0$ there exists a natural number N such that if $n \geq N$ then $|\alpha f_n(x) - \alpha f(x)| < \varepsilon$ and $|\beta f_n(y) - \beta f(y)| < \varepsilon$. Therefore,

$$\begin{aligned}|\alpha f(x) - \beta f(y)| &= |\alpha f(x) - \alpha f_n(x) + \beta f_n(y) - \beta f(y)| \\ &\leq |\alpha f(x) - \alpha f_n(x)| + |\beta f_n(y) - \beta f(y)| \\ &< 2\varepsilon.\end{aligned}$$

Therefore, $\alpha f(x) = \beta f(y)$.

Let $\mathfrak{G} = \mathfrak{L}(\mathfrak{X})$. Since $\mathfrak{L}(\mathfrak{X})$ is closed under the lattice operations and uniform passage to the limit, then $\mathfrak{G}$ is. Therefore theorem 6 will be used to prove this theorem. The planar set $\mathfrak{G}(x, y)$, where x and y are arbitrary points in X, must be the entire plane, a straight line through the origin, or the one-point set consisting of the origin alone. This appears at once when we observe that if

$$(\alpha, \beta) \in \mathfrak{G}(x, y),$$

then there exists a function f in $\mathfrak{G}$ such that $f(x) = \alpha$ and $f(y) = \beta$. But since f is in $\mathfrak{G}$, then λf is in $\mathfrak{G}$ and $\lambda f(x) = \lambda\alpha$ and $\lambda f(y) = \lambda\beta$ which implies that $(\lambda\alpha, \lambda\beta)$ is in $\mathfrak{G}(x, y)$ for every λ. We also have that if (α, β) and (γ, δ) are in $\mathfrak{G}(x, y)$, then there exist two functions f and g such that $f(x) = \alpha$, $f(y) = \beta$, $g(x) = \gamma$, and $g(y) = \delta$. Since f and g are in $\mathfrak{G}$, then $f + g$ is in $\mathfrak{G}$ and $(f + g)(x) = \alpha + \gamma$ and $(f + g)(y) = \beta + \delta$ which implies that $(\alpha + \gamma, \beta + \delta)$ is in $\mathfrak{G}(x, y)$. From this we see that $\mathfrak{G}(x, y)$ is a closed subset of the plane, and we have $\mathfrak{G}(x, y) = \mathfrak{G}(x, y)^*$. When $\mathfrak{G}(x, y)$ is a straight line through the origin, we write its equation as $\alpha\xi = \beta\kappa$ and observe that

$$(\beta, \alpha) \in \mathfrak{G}(x, y).$$

Since $\mathfrak{G}$ is closed under the operations of forming absolute values, we see that

$$(|\beta|, |\alpha|) \in \mathfrak{Y}_0(x, y).$$

Therefore by using the equation for $\mathfrak{Y}_0(x, y)$ we get $\alpha\xi = \beta\eta$ which implies $\alpha|\beta| = \beta|\alpha|$ which implies $\alpha\beta|\beta| = \beta^2|\alpha| \geq 0$ which implies that $\alpha\beta \geq 0$. When $\mathfrak{Y}_0(x, y)$ consists of the origin alone, we have the case enumerated as (1) in the statement of the theorem. When $\mathfrak{Y}_0(x, y)$ is a straight line through the origin we have case (2) if it coincides with one of the coordinate axes, case (3) if it coincides with the bisector of the angle between the positive coordinate axes, and case (4) otherwise. When $\mathfrak{Y}_0(x, y)$ is the entire plane, there is no corresponding linear relation, of course. Theorem 6 shows that $\mathfrak{Y}_0$ is characterized by the sets

$$\mathfrak{Y}_0(x, y) = \mathfrak{Y}_0(x, y)*$$

In other words that f in $\mathfrak{F}$ belongs to $\mathfrak{Y}_0 = \mathfrak{L}(\mathfrak{X}_0)$ if and only if

$$(f(x), f(y)) \in \mathfrak{Y}_0(x, y) \text{ for all } x, y.$$

Since $\mathfrak{X}_0 \subset \mathfrak{Y}_0$, it is clear that the conditions thus imposed on the functions in $\mathfrak{L}(\mathfrak{X}_0)$ are satisfied by the functions in $\mathfrak{X}_0$.

From this theorem we obtain the following corollaries.

Corollary 1: In order that $\mathfrak{L}(\mathfrak{X}_0)$ contain a nonvanishing constant function, it is necessary and sufficient that the only linear relations of the form

$\alpha g(x) = \beta g(y)$, $\alpha\beta > 0$, satisfied by every function in $\mathfrak{F}_0$ be those reducible to the form $g(x) = g(y)$.

<u>Proof</u>: It is obvious that of conditions (1)-(4) in Theorem 7, only condition (3) can be satisfied by a non-vanishing constant function.

<u>Corollary 2</u>: In order that $\mathcal{L}(\mathfrak{F}_0) = \mathfrak{F}$, it is sufficient that the functions in $\mathfrak{F}_0$ satisfy no linear relation of the form (1)-(4) of Theorem 7.

<u>Proof</u>: If every function in $\mathfrak{F}_0$ satisfied any of the relations (1)-(4), then by Theorem 7 every function in $\mathcal{L}(\mathfrak{F}_0)$ would satisfy the same relation which would imply every function in $\mathfrak{F}$ would satisfy the same relation which is a contradiction. Therefore the corollary holds.

In order to state a further corollary, we first introduce a convenient definition.

<u>Definition</u>: A family of arbitrary functions on a domain X is said to be a separating family (for that domain) if, whenever x and y are distinct points in X, there is some function f in the family with distinct values f(x), f(y) at these points.

<u>Corollary 3</u>: If X is compact and if $\mathfrak{F}_0$ is a separating family for X and contains a nonvanishing constant function, then $\mathcal{L}(\mathfrak{F}_0) = \mathfrak{F}$.

<u>Proof</u>: Since $\mathfrak{F}_0$ contains a nonvanishing constant function, the only conditions of (1)-(4) satisfied by

every function in $\mathfrak{F}_0$ are those of the form (3) because of Corollary 1. But since $\mathfrak{F}_0$ is a separating family, no linear relation of the form $g(x) = g(y)$, where $x \neq y$, is satisfied by every function in $\mathfrak{F}_0$. Therefore, from Corollary 2 we obtain the desired results.

Corollary 4: If $\mathfrak{F}_0$ is a separating family, then so is $\mathfrak{F}$. If $\mathfrak{F}$ is a separating family and $\mathcal{L}(\mathfrak{F}_0) = \mathfrak{F}$, then $\mathfrak{F}_0$ is also a separating family.

Proof: $\mathfrak{F}_0$ is a separating family implies $\mathcal{L}(\mathfrak{F}_0)$ is a separating family and $\mathcal{L}(\mathfrak{F}_0) \subset \mathfrak{F}$. Therefore, $\mathfrak{F}$ is a separating family. $\mathcal{L}(\mathfrak{F}_0) = \mathfrak{F}$ implies that $\mathfrak{F}_0$ is not subject to conditions (1)-(4) which implies $\mathfrak{F}_0$ is not subject to condition (3) which implies $\mathfrak{F}_0$ is a separating family.

SECTION 2

LINEAR RING FORMULATIONS OF THE GENERALIZED THEOREM AND THE CHARACTERIZATION OF CLOSED IDEALS

As stated in the introduction, in this section we will treat the family of all continuous functions on a compact domain X, as a linear ring. By definition a linear ring is an algebra. The ring operations for $\mathfrak{F}$ will be the addition and multiplication of functions and the scalar multiplication will be multiplication by real numbers. By use of Theorem 3 it is an easy proof to show that $\mathfrak{F}$ is a linear ring. Since by Theorem 1 we know that all functions in $\mathfrak{F}$ are bounded, the material in the introduction applies to this section.

In order to prove the principal theorem of this section, we will introduce and prove the following theorem.

Theorem 8: If ε is any positive number and $[\alpha, \beta]$ any real interval, then there exists a polynomial $p(\xi)$ in the real variable ξ with $p(0) = 0$ such that $||\xi| - p(\xi)| < \varepsilon$ for $\alpha \leq \xi \leq \beta$.

Proof: If $\xi = 0$ is not in the given interval $[\alpha, \beta]$, then we can take $p(\xi) = \xi$ if $\xi \geq 0$ and $p(\xi) = -\xi$ if $\xi < 0$ and then we are done. If 0 is in the interval $[\alpha, \beta]$ then there is no loss of generality in confining our attention to intervals of the form $[-\gamma, \gamma]$ where $\gamma > 0$, since the given interval $[\alpha, \beta]$ can be included in an interval of this form by letting $\gamma \geq$ max. It is also sufficient to study only the case of the interval $[-1, 1]$ since, if $q(\eta)$, $q(0) = 0$, is a polynomial such that

$$||\eta| - q(\eta)| < \varepsilon/\gamma$$

for $-1 \leq \eta < 1$, then

$$p(\xi) = \gamma q(\varepsilon/\gamma),\ p(0) = 0,$$

is a polynomial such that $||\xi| - p(\xi)|\ |\ \varepsilon$ for $-\gamma \leq \xi \leq \gamma$. To see this, note that each ξ in $[-\gamma, \gamma]$ can be expressed in the form $\gamma\eta$, for $-1 \leq \eta < 1$, which implies $\eta = \xi/\gamma$. Therefore $||\eta| - q(\eta)| < \varepsilon/\gamma$ implies that $||\xi/\gamma| - q(\xi/\gamma$ which implies $\gamma||\xi/\gamma| - q(\xi/\gamma)| < \varepsilon$ which implies, since $\gamma > 0$, that $||\xi| - \gamma q(\xi/\gamma)| < \varepsilon$ and since $p(\xi) = \gamma q(\xi/\gamma)$, then for each ξ in $[\gamma, \gamma]$, $||\xi| - p(\xi)| < \varepsilon$ and $p(0) = 0$.

We shall obtain the desired polynomial q for the interval $-1 \leq \eta \leq 1$ as a partial sum of the power series development for $\sqrt{1 - \xi}$ where $\xi = 1 - \eta^2$.

We will begin by defining a sequence of constants α_k recursively from the relations

$$\alpha_1 = \frac{1}{2}$$
$$\alpha_k = \frac{1}{2} \sum_{m+n=k} \alpha_m\alpha_n = \frac{1}{2}(\alpha_1\alpha_{k-1} + \alpha_2\alpha_{k-2} + \dots + \alpha_{k-1}\alpha_1)$$

It is obvious that $\alpha_k > 0$. Putting

$$\sigma_n = \sum_1^n \alpha_k,$$

we can show inductively that $\sigma_n < 1$. We have $\sigma_1 = \alpha_1 = \frac{1}{2} < 1$. We assume $\sigma_n < 1$, which implies

$$\sigma_{n+1} = \alpha_1 + \sum_{k=2}^{n+1} \alpha_k = \frac{1}{2} + \frac{1}{2}\sum_{k=2}^{n+1} \sum_{i,j\geq 1}^{i+j=k} \alpha_i\alpha_j \leq \frac{1}{2} + \frac{1}{2}\sum_{i,j=1}^{n} \alpha_i\alpha_j \leq \frac{1}{2}(1 + \sigma_n^2) < 1.$$

Accordingly the positive term series $\sum_{k=1}^{\infty}\alpha_k$ converges to a sum σ satisfying the inequality $\sigma \leq 1$; and the power series

$$\sum_{k=1}^{\infty}\alpha_k x^k$$

converges uniformly for $|x| \leq 1$ to a continuous function $\sigma(x)$. To identify the function $\sigma(x)$ with the function $1 - \sqrt{1 - x}$, we prove that $\sigma(x)$, like $1 - \sqrt{1 - x}$, is a solution of the equation $f(x)(2 - f(x)) = x$. Looking at the partial sums of the power series for $\sigma(x)$, we observe that

$$\left(\sum_{i=1}^{n}\alpha_i x^i\right)\left(2 - \sum_{j=1}^{n}\alpha_i x^j\right) = 2\sum_{k=1}^{n} \alpha_k x^k - \sum_{i,j=1}^{n} \alpha_i\alpha_j x^{i+j} = 2\sum_{k=1}^{n} \alpha_k x^k - 2\sum_{k=2}^{n}\alpha_k x^k - \sum_{k\leq i,j\leq n} \alpha_i\alpha_j x^{i+j}$$

$$= x - \sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i\alpha_j x^{i+j}$$

in accordance with the definition of the coefficients α_k. The final term here can now be estimated as follows:

$$\left|\sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i\alpha_j x^{i+j}\right| \leq \sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i\alpha_j \leq \sum_{k=n+1}^{\infty} \sum_{i,j\geq 1}^{i+j=k} \alpha_i\alpha_j \leq 2\sum_{k=n+1}^{\infty} \alpha_k.$$

As n tends towards infinity, this term tends towards zero, and passage to the limit in the identity above accordingly yields the relation $\sigma(x)(2 - \sigma(x)) = x$. Therefore, for each x such that $-1 \leq x \leq 1$, we have $\sigma(x) = 1 \pm \sqrt{1 - x}$. Here we decide upon the choice of sign by showing that $\sigma(x) \leq 1$ is an inequality incompatible with the positive sign. It is evident that $\sigma(1) = 1$, independently of the choice of sign, and hence that $\sum_{k=1}^{\infty}\alpha_k = \sigma(1) = 1$. Inasmuch as α_k is positive, it follows that $\sigma(x) \leq \sigma(|x|) \leq \sigma(1) = 1$, as we intended to show. It is now clear that the power series for $\sqrt{1 - x}$ is given by

$$\sqrt{1 - x} = 1 - \sigma(x) = 1 - \sum_{k=1}^{\infty}\alpha_k x^k = \sum_{k=1}^{\infty}\alpha_k(1 - x^k).$$

Taking η so that $-1 \leq \eta \leq 1$, we have $0 \leq 1 - \eta \leq 1$ and hence

$$|\eta| = \sqrt{\eta^2} = 1 - \sigma(1 - \eta^2) = \sum_{k=1}^{\infty}\alpha_k(1 - (1 - \eta^2)^k),$$

the series being uniformly convergent. The general term of this series is a polynomial in η which vanishes for $\eta = 0$. Hence we can take a suitable one of its partial sums as the required polynomial $q(\eta)$, completing the proof.

The next theorem is the principal result concerning the generalization of the Weierstrass theorem for the linear ring operations.

Theorem 9: Let X be a compact space, $\mathfrak{X}$ the family of all continuous real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear-ring operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathcal{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form g(x) = 0 or g(x) = g(y) which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear subring of $\mathfrak{X}$ – that is, if $\mathfrak{X}_0 = \mathcal{U}(\mathfrak{X}_0)$ – then $\mathfrak{X}_0$ is characterized by the system of all linear relations of this kind which are satisfied by every function belonging to it. In other words, $\mathfrak{X}_0$ is characterized by the partition of X into mutually disjoint closed subsets on each of which every function in $\mathfrak{X}_0$ is constant and by the specification of that one, if any, of these subsets on which every function in $\mathfrak{X}_0$ vanishes.

Proof: Since X is compact, the functions in $\mathcal{U}(\mathfrak{X}_0)$ are bounded. Let f belong to $\mathcal{U}(\mathfrak{X}_0)$; then f is bounded. This implies there exists numbers α and β such that $\alpha \leq f(x) \leq \beta$ for all x in X. By the preceding theorem, we can find a polynomial $p_n(\xi)$ such that $||\xi| - p_n(\xi)| < \frac{1}{n}$ for $\alpha \leq \xi \leq \beta$ where p(0) = 0. It is clear that $p_n(f)$ is in $\mathcal{U}(\mathfrak{X}_0)$ since $p_n(f)$ is linear combinations of f and $\mathcal{U}(\mathfrak{X}_0)$ is closed under the algebraic operations. Since $\alpha \leq f(x) \leq \beta$ and $||\xi| - p_n(\xi)| < \frac{1}{n}$ for $\alpha \leq \xi \leq \beta$, then $||f(x)| - p_n(f(x))| < \frac{1}{n}$ for all x in X. Hence $|f|$ is the uniform limit of the functions $p_n(f)$. Referring to the formulas $\max(\xi, \eta) = 1/2(\xi + \eta + |\xi - \eta|)$ and $\min(\xi, \eta) = 1/2(\xi + \eta - |\xi - \eta|)$, connecting the operations $\cup$ and $\cap$ with the operations of forming the absolute value, we now see that whenever f and g are in $\mathfrak{U}(\mathfrak{F}_0)$, then so also are $f \cup g$ and $f \cap g$. But this implies $\mathfrak{U}(\mathfrak{F}_0)$ is closed under the linear lattice operations as well as under the ring operations and uniform passage to the limit. Therefore the characterization of closed linear sublattices of $\mathfrak{F}$, as stated in Theorem 7, applies to $\mathfrak{U}(\mathfrak{F}_0)$. It is easy to see that none of the characteristic linear relations can be of type (4) described there, since, if every function in $\mathfrak{U}(\mathfrak{F}_0)$ were to satisfy a linear relation of the form $g(x) = \gamma g(y)$, we would find for every f in $\mathfrak{U}(\mathfrak{F}_0)$ that, f^2 being also in $\mathfrak{U}(\mathfrak{F}_0)$, the relations

$f(x) = \lambda f(y)$ and $f^2(x) = \lambda f^2(y)$ hold. These imply $\lambda^2 f^2(y) = \lambda f^2(y)$ and we conclude that $f(y) = 0$ for every f in $\mathfrak{U}(\mathfrak{F}_0)$ or that $\lambda = 0, 1$. Thus we conclude that f is in $\mathfrak{U}(\mathfrak{F}_0)$ if and only if it satisfies all the linear relations $g(x) = 0$ or $g(x) = g(y)$ satisfied by every function in $\mathfrak{F}_0$.

The first characterization of the closed linear subrings of $\mathfrak{F}$ given in the statement of the theorem follows immediately. As to the second characterization, we remark first that the relation $\equiv$ defined by putting $x \equiv y$ if and only if $f(x) = f(y)$ for all f in $\mathfrak{F}_0$ is obviously an equivalence relation in X: $x = y$ implies $x \equiv y$; $x \equiv y$ implies $y \equiv x$; $x \equiv y$ and $y \equiv z$ implies $x \equiv z$. Consequently, X is partitioned by this equivalence relation into mutually disjoint subsets, each a maximal set of mutually equivalent elements. The set of all points y such that $x \equiv y$ is just that member of the partition which contains x. Since this set is the intersection or common part of all the sets

$$X_f = (y;\ f(x) = f(y))$$

for the various functions f in $\mathfrak{X}_o$ and since each set X_f is closed by virtue of the continuity of f, we see that the equivalence class containing x is closed. If x and y are in distinct partition classes, then there exists a function f in $\mathfrak{X}_o$ such that $f(x) \neq f(y)$, since otherwise we would have $x \equiv y$ and the two given partition classes could not be distinct. If a partition class contains a single point x such that $f(x) = 0$ for every f in $\mathfrak{X}_o$, then all its points obviously have this property. On the other hand, at most one partition class can contain such a point since, if x and y are points such that $f(x) = 0$, $f(y) = 0$ for every f in $\mathfrak{X}_o$, then $f(x) = f(y)$ for every f in $\mathfrak{X}_o$, $x \equiv y$, and x and y are in the same partition class.

We cannot expect that an arbitrary partition of X into mutually disjoint closed subsets can be derived in the manner just described from some closed linear subring $\mathfrak{X}_o$ of $\mathfrak{X}$. However, partitions obtained from distinct closed linear subrings are necessarily distinct-except in the case where one subring consists of all the functions in $\mathfrak{X}$ which are constant on each partition class and the other consists of all those functions which are in the first subring and in addition vanish on one specified partition class. Thus we see that a closed linear subring is specified by the partition of X into the closed subsets on each of which all its members are constant and the specification of that particular partition class, if any, on which all its members vanish.

From this theorem we have the following useful corollaries.

Corollary 1: In order that $\mathcal{U}(\mathfrak{F}_0)$ contain a non-vanishing constant function, it is necessary and sufficient that for every x in X there exists some f in $\mathfrak{F}_0$ such that $f(x) \neq 0$.

Proof: The corollary is an immediate consequence of theorem 9.

Corollary 2: If $\mathfrak{F}_0$ is a separating family for X, then $\mathcal{U}(\mathfrak{F}_0)$ either coincides with $\mathfrak{F}$ or is, for a uniquely determined point x_0, the family of all functions f in $\mathfrak{F}$ such that $f(x_0) = 0$. Conversely, if $\mathfrak{F}$ is a separating family for X, and $\mathcal{U}(\mathfrak{F}_0)$ either coincides with $\mathfrak{F}$ or is the family of all those f in $\mathfrak{F}$ which vanish at some fixed point x_0 in X, then $\mathfrak{F}_0$ is a separating family.

Proof: If $\mathfrak{F}_0$ is a separating family, so also are $\mathcal{U}(\mathfrak{F}_0)$ and $\mathfrak{F}$. Hence the partition classes associated with $\mathcal{U}(\mathfrak{F}_0)$ must each consist of a single point. It follows that $\mathcal{U}(\mathfrak{F}_0)$

must be as indicated. Conversely, when $\mathfrak{F}$ is a separating family and $\mathfrak{U}(\mathfrak{F}_0)$ is as stated, then $\mathfrak{U}(\mathfrak{F}_0)$ is a separating family. If it were not, every f in $\mathfrak{U}(\mathfrak{F}_0)$ would vanish at some point x_0; and there would exist distinct points x and y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{U}(\mathfrak{F}_0)$. Consider now an arbitrary function f in $\mathfrak{F}$. Clearly, the function f_0 defined by putting

$$f_0(z) = f(z) - f(x_0)$$

is continuous and vanishes at x_0. Thus f_0 is in $\mathfrak{U}(\mathfrak{F}_0)$, the equation $f_0(x) = f_0(y)$ is verified, and in consequence $f(x) = f(y)$. Thus we find that $f(x) = f(y)$ for every f in $\mathfrak{F}$, against hypothesis. Since $\mathfrak{U}(\mathfrak{F}_0)$ is a separating family, $\mathfrak{F}_0$ must be also. Otherwise, of course, there would exist distinct points x, y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{F}_0$; and then the equation $f(x) = f(y)$ would hold for every f in $\mathfrak{U}(\mathfrak{F}_0)$ contrary to what was just established.

For the next two theorems, we will think of $\mathfrak{F}$ as a lattice, that is the only operations we will take into consideration will be the operations $\cup$ and $\cap$. We will define an ideal as follows.

Definition: $\mathfrak{F}_0$ is an ideal of $\mathfrak{F}$, if $\mathfrak{F}_0$ is a non-void subclass of $\mathfrak{F}$ and if f, g are in $\mathfrak{F}_0$ and h is in $\mathfrak{F}$, then $f \cup g$ and $f \cap h$ are in $\mathfrak{F}_0$.

The second condition of this definition is equivalent to the requirement that $\mathfrak{L}_0$ should contain h whenever it contains f and $f(x) \geq h(x)$ for every x. This is true since on the one hand if $f(x) \geq h(x)$ for all x, then $f \cap h = h$, and on the other hand, $f \geq f \cap h$ for all h. Therefore, h is in $\mathfrak{L}_0$.

We now have the first theorem stating the characterizations of the closed ideals in $\mathfrak{F}$.

<u>Theorem 10</u>: Let $\mathfrak{F}$ be the lattice of all continuous real functions on a compact space X, $\mathfrak{L}_0$ an arbitrary subfamily of $\mathfrak{F}$, F_0 the extended-real function defined on X through the equation $F_0(x) = \sup_{f \in \mathfrak{L}_0} f(x)$, and $\mathfrak{J}_0$ the family of all those functions f in $\mathfrak{F}$ such that $f(x) \leq F_0(x)$ for every x in X. When $\mathfrak{L}_0$ is void, $F_0(x) = -\infty$ for every x and $\mathfrak{J}_0$ is void. Otherwise, $\mathfrak{J}_0$ is the smallest closed ideal containing $\mathfrak{L}_0$; further, $\mathfrak{L}_0$ is a closed ideal if and only if $\mathfrak{L}_0 = \mathfrak{J}_0$. A closed ideal $\mathfrak{L}_0$ is characterized by the associated function F_0.

<u>Proof</u>: As indicated in the theorem, we permit $+\infty$ and $-\infty$ to appear as values of F_0, when necessary. Suppose $\mathfrak{L}_0$ is void, this implies that $F_0(x) = -\infty$ since if $F(x) > -\infty$, then there would exist a function f such that $F_0(x) \geq f(x)$ which would imply that f(x) belongs to $\mathfrak{L}_0$, which is a contradiction since $\mathfrak{L}_0$ is void. $F_0(x) = -\infty$ implies $\mathfrak{J}_0$ is void since if $\mathfrak{J}_0$ is not void, there exists a function f in $\mathfrak{F}$ such that $f(x) \leq F_0(x)$ which implies $f(x) \leq -\infty$ which is a

contradiction. Suppose $\mathfrak{L}$ is not void; then there exists an f in $\mathfrak{L}$ such that $f(x) > -\infty$ which implies there exists a g in $\mathfrak{F}$ such that $g(x) \leq f(x)$ which implies $g(x) \leq F_0(x)$ which implies g is in $\mathfrak{G}$. Suppose $\mathfrak{L}$ is a closed ideal. By Theorem 6, $\mathfrak{L}_0$ is characterized by the set $\mathfrak{L}_0(x, y)*$ as a closed sublattice of $\mathfrak{F}$. First of all, it is evident that $\mathfrak{L}_0(x, y)$ and hence also its closure $\mathfrak{L}_0(x, y)*$ must be contained in the set of points (α, β) such that $\alpha \leq F_0(x)$ and $\beta \leq F_0(x)$. Let $\varepsilon > 0$, then there exist functions f and g in $\mathfrak{L}_0$ such that $f(x) > F_0(x) - \varepsilon$ and $g(y) > F_0(y) - \varepsilon$ for any prescribed pair of points x, y in X. The function $h = f \cup g$ is in the ideal $\mathfrak{L}_0$ and satisfies the relations

$$h(x) \geq f(x) > F_0(x) - \varepsilon \text{ and } h(y) \geq g(y) > F_0(y) - \varepsilon.$$

Thus $(h(x), h(y))$ is a point in $\mathfrak{L}_0(x, y)$ and

$$|h(x) - F_0(x)| < \varepsilon \text{ and } |h(y) - F_0(y)| < \varepsilon,$$

so that $(F_0(x), F_0(y))$ is in $\mathfrak{L}_0(x, y)*$ which implies $(F_0(x), F_0(y))$ is a limit point of $\mathfrak{L}_0(x, y)$.

Now we establish the fact that f is in $\mathfrak{L}_0$ when $f(x) \leq F_0(x)$ for every x. Let f_ε be the function in $\mathfrak{F}$ defined by putting

$$f_\varepsilon(x) = f(x) - \varepsilon, \ \varepsilon > 0.$$

If x, y are arbitrary points in X, and h in $\mathfrak{L}_0$ such that

$$h(x) > f_\varepsilon(x) = f(x) - \varepsilon, \ h(y) > f_\varepsilon(y) - \varepsilon,$$

then the function $f_{xy} = h \cap f_\varepsilon$ belongs to the ideal $\mathfrak{L}_0$ and has the property that

$$f(x) - f_{xy}(x) = \varepsilon \text{ and } f(y) - f_{xy}(y) = \varepsilon.$$

By Theorem 5 we conclude that f is the uniform limit of functions in the closed ideal $\mathfrak{J}_0$ and hence that f itself is in $\mathfrak{J}_0$. We have now shown that $\mathfrak{Y}_0 \subset \mathfrak{J}_0$. Since $\mathfrak{J}_0 \subset \mathfrak{Y}$ by construction, we conclude that $\mathfrak{J}_0 = \mathfrak{Y}_0$ under the present hypothesis.

Returning to the case where $\mathfrak{J}_0$ is an arbitrary nonvoid family, we consider a closed ideal $\mathfrak{J}_1$ containing $\mathfrak{J}_0$. Evidently $\mathfrak{J}_1$ has an associated function F_1 such that $F_1(x) \geq F_0(x)$ for every x. Otherwise, if $F_1(x) < F_0(x)$ for some x, then there would exist a function f in $\mathfrak{J}_0$ such that $F_1(x) < f(x) \leq F_0(x)$ which would imply $\mathfrak{J}_0$ is not in $\mathfrak{J}_1$, which is contrary to hypothesis. From the fact then, that $F_1(x) \geq F_0(x)$ for every x, we can conclude that

$$\mathfrak{J}_1 = \mathfrak{Y}_1 \supset \mathfrak{Y}_0$$

Therefore $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{J}_0$. With this the proof of the theorem is complete.

Next we shall consider the case where $\mathfrak{J}$ is treated as a linear lattice, the algebraic operations allowed including the linear operations as well as the two lattice operation. Here an ideal is to be defined as a nonvoid class closed under the allowed algebraic operations and enjoying the additional property that it contains with f every g such that

$$|g(x)| \leq |f(x)|$$

for all x.

<u>Theorem 11</u>: Let $\mathfrak{F}$ be the linear lattice of all the continuous real functions on a compact space X, let $\mathfrak{F}_0$ be an arbitrary nonvoid subfamily of $\mathfrak{F}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{F}_0$ vanishes, and let $\mathfrak{J}_0$ be the family of those functions f in $\mathfrak{F}$ which vanish at every point of X_0. Then $\mathfrak{J}_0$ is the smallest closed ideal containing $\mathfrak{F}_0$; and $\mathfrak{F}_0$ is a closed ideal if and only if $\mathfrak{F}_0 = \mathfrak{J}_0$. A closed ideal $\mathfrak{F}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{F}_0 = \mathfrak{J}_0 = \mathfrak{F}$ if and only if X_0 is void.

<u>Proof</u>: It is evident that $\mathfrak{J}_0$ is a closed ideal containing $\mathfrak{F}_0$. For example, if f is in $\mathfrak{J}_0$ and $|g(x)| \leq |f(x)|$ for every x, then g vanishes everywhere on X_0 and therefore belongs to $\mathfrak{J}_0$. If $\mathfrak{F}_0$ is a closed ideal we can show that $\mathfrak{F}_0 = \mathfrak{J}_0$. To do so we refer to Theorem 7 and consider what linear relations of the form indicated there can be satisfied by every function in $\mathfrak{F}_0$. Obviously the pairs of points x, y which have one or both members in X_0 are of no further interest, as the corresponding linear conditions are those of types (1) and (2), the effect of which has already been taken into account through the introduction of the closed set X_0. Turning to the case where x and y are distinct points not in X_0, we first remark that if we have $f(x) = f(y)$ for every f in $\mathfrak{F}$, then no effective restriction

is implied by the linear relation corresponding to the pair of points in question. Assuming therefore that g is a function in $\mathfrak{F}$ with $g(x) \neq g(y)$, we may suppose without loss of generality that $g(x) = 1$, $g(y) = 0$, for we may replace g if necessary by the function h defined through the equation

$$h(z) = \frac{g(z) - g(y)}{g(x) - g(y)}$$

for all z in X. Since x is not in X_0 there is a function f in $\mathfrak{F}_0$ such that $f(x) \neq 0$. We may suppose without loss of generality that $f(x) > 1$ for we may replace f is necessary by the function $h = \alpha f$ with a suitable value of α. The function $h = |f| \cap |g|$ is now seen to be in the ideal $\mathfrak{F}_0$ and to satisfy the equations $h(x) = 1$, $h(y) = 0$. Accordingly, no linear relation of the type (3) or type (4) is satisfied by h. Hence the linear relations which characterize $\mathfrak{F}_0$ as a closed linear sublattice of $\mathfrak{F}$ reduce effectively to those implicit in the statement that every function in $\mathfrak{F}_0$ vanishes throughout X_0. It follows that $\mathfrak{F}_0 = \mathfrak{G}_0$. Obviously if $\mathfrak{F}_0$ is an arbitrary nonvoid family and $\mathfrak{F}_1$ is a closed ideal containing $\mathfrak{F}_0$, then the associated closed set X_1 is part of X_0; and $\mathfrak{F}_1 = \mathfrak{G}_1 \supset \mathfrak{G}_0$. This completes the proof of the theorem.

CHAPTER III

TWO CLASSICAL PROOFS OF THE WEIERSTRASS APPROXIMATION THEOREM

<u>Definition</u>: Let f be a function with domain $I = [0, 1]$ and range in R. The n^{th} Bernstein polynomial for f is defined to be

$$(1) \quad B_n(x) = B_n(x;\ f) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1 - x)^{n-k}$$

<u>Bernstein Approximation Theorem</u>: Let f be continuous on I with values in R. Then the sequence of Bernstein polynomials for f, as defined above, converges uniformly on I to f.

<u>Proof</u>: The Binomial Theorem asserts that

$$(s + t)^n = \sum_{k=0}^{n}\binom{n}{k}s^k t^{n-k}.$$

If we let $s = x$ and $t = 1 - x$ we obtain

$$1 = \sum_{k=0}^{n}\binom{n}{k}x^k(1 - x)^{n-x}.$$

Multiplying this equation by $f(x)$ we get

$$f(x) = \sum_{k=0}^{n} f(x)\binom{n}{k}x^k(1 - x)^{n-k}.$$

Therefore, we obtain the relation

$$f(x) = B_n(x) = \sum_{k=0}^{n}\{f(x) - f(k/n)\}\binom{n}{k}x^k(1 - x)^{n-k}$$

from which it follows that

$$(2) \quad |f(x) - B_n(x)| \leq \sum_{k=0}^{n}|f(x) - f(k/n)|\binom{n}{k}x^k(1 - x)^{n-k}.$$

Now f is bounded, say by M, and also uniformly continuous. Note that if k is such that k/n is near x, then the corresponding term in the sum (2) is small because of the continuity of f at x; on the other hand, if k/n is far from x, the factor involving f can only be said to be less than 2M and any samllness must arise from the other factors. We are led, therefore, to break (2) into two parts: those values of k where x - k/n is small and those for which x - k/n is large.

Let $\varepsilon > 0$ and let δ be as in the definition of uniform continuity for f. It truns out to be convenient to choose n so large that

$$(3)\quad n \geq \sup\{\delta^{-4}, M^2/\varepsilon^2\},$$

and break (2) into two sums. The sum taken over those k for which $|x - k/n| < n^{-1/4} \leq \delta$ yields the estimate

$$\sum_k \varepsilon \binom{n}{k} x^k(1 - x)^{n-k} \leq \varepsilon \sum_{k=1}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = \varepsilon.$$

The sum taken over those k for which $|x - k/n| \geq n^{-1/4}$, that is, $(x - k/m) \geq n^{-1/2}$, can be estimated by using the formula

$$(1/n)x(1 - x) = \sum_{k=0}^{n} (x - k/n)^2 \binom{n}{k} x^k (1 - x)^{n-k}.$$

For this part of the sum in (2) we obtain the upper bound

$$\sum_k 2M\binom{n}{k} x^k (1 - x)^{n-k} = 2M \sum_k \frac{(x - k/n)^2}{(x - k/n)^2} \binom{n}{k} x^k (1 - x)^{n-k}$$

$$\leq 2M \sqrt{n} \sum_{k=1}^{n} (x - k/n)^2 \binom{n}{k} x^k (1 - x)^{n-k}$$

$$\leq 2M \sqrt{n}\, \{1/nx(1 - x)\} \leq \frac{M}{2\sqrt{n}},$$

since $x(1 - x) \leq 1/4$ on the interval I. Recalling the determination (3) for n, we conclude that each of these two parts of (2) is bounded above by ε. Hence, for n chosen in

(3) we have

$$|f(x) - B_n(x)| < 2\varepsilon,$$

independently of the value of x. This shows that the sequence (B_n) converges uniformly on I to f.

<u>Weierstrass Approximation Theorem</u>: Let f be a continuous function on a compact interval of R and with values in R. Then f can be uniformly approximated by polynomials.

<u>Proof</u>: If f is defined on [a, b], then the function g defined on I = [0, 1] by

$$g(t) = f((b - a)t + a), \text{ t in I},$$

is continuous. Hence g can be uniformly approximated by Berstein polynomials and a simple change of varialbe yields a polynomial approximation to f. We have $|g(t) - B_n(t)| < \varepsilon$ for $0 \leq t \leq 1$. Since $x = (b - a)t + a$, then $t = \frac{x - a}{b - a}$. Therefore

$$f(x) = g(\frac{x - a}{b - a}) \text{ and } 0 \leq \frac{x - a}{b - a} \leq 1.$$

From this we see that

$$|f(x) - B_n(\frac{x - a}{b - a})| = | g(\frac{x - a}{b - a}) - B_n(\frac{x - a}{b - a})| < \varepsilon.$$

The following proof of the Weierstrass Theorem is attributed to Landau.

<u>Proof</u>: To prove this, we assume that the interval $a \leq x \leq b$ lies wholly in the interior of the interval $0 < x < 1$; thus, two numbers α and β may be found with $0 < \alpha < a < b < \beta < 1$. We may suppose that the function f(x), which is by assumption continuous in the interval

$a \leq x \leq b$, has been extended continuously to the entire interval $\alpha \leq x \leq \beta$.

Let us now consider the integral

$$J_n = \int_0^1 (1 - v^2)^n dv.$$

Now, if δ is a fixed number in the interval $0 < \delta < 1$, and if we set

$$J_n^* = \int_\delta^1 (1 - v^2)^n dv,$$

we assert that

$$\lim_{n \to \infty} \frac{J_n^*}{J_n} = 0,$$

which means that for sufficiently large n the integral from 0 to δ forms the dominant part of the whole integral from 0 to 1. In fact, for $n \geq 1$,

$$J_n > \int_0^1 (1 - v)^n \, dv = \frac{1}{n + 1},$$

$$J_n^* = \int_\delta^1 (1 - v^2)^n dv < (1 - \delta^2)^n (1 - \delta) < (1 - \delta^2)^n,$$

$$\frac{J_n^*}{J_n} < (n + 1(1 - \delta^2)^n$$

and hence

$$\lim_{n \to \infty} \frac{J_n^*}{J_n} = 0.$$

We now assume $a \leq x \leq b$ and from the expressions

$$P_n(x) = \frac{\int_\alpha^\beta f(u)[1 - (u - x)^2]^n du}{\int_{-1}^1 (1 - u^2)^n du} \quad (n = 1, 2, \ldots),$$

which are polynomials in x of degree 2n whose coefficients are quotients of definite integrals. We shall show that they afford the desired approximation.

By making the substitution $u = v + x$ we find for the numerator

$$\int_{\alpha}^{1} f(u)[1 - (u - x)^2]^n du = \int_{\alpha-x}^{\beta-x} f(v + x)[1 - v^2]^n dv$$

$$= \int_{\alpha-x}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\beta-x}$$

$$= I_1 + I_2 + I_3,$$

where the positive number δ in the interval $0 < \delta < 1$ will be suitably fixed later. The integral I_2 may be transformed to

$$I_2 = f(x) \int_{-\delta}^{\delta}(1 - v^2)^n dv + \int_{-\delta}^{\delta}[f(v + x) - f(x)](1 - v^2)^n dv$$

$$= 2f(x)(J_n - J_n^*) + \int_{-\delta}^{\delta}[f(v + x) - f(x)](1 - v^2)^n dv.$$

Because of the uniform continuity of $f(x)$ in the interval $\alpha \leq x \leq \beta$ it is possible, for arbitrarily small $\varepsilon > 0$, to choose a $\delta = \delta(\varepsilon)$ in the interval $0 < \delta < 1$, depending only on ε, such that, for $|v| \leq \delta$ and $a \leq x \leq b$, $|f(v - x) - f(x)| \leq \varepsilon$. It then follows that

$$\left|\int_{-\delta}^{\delta}[f(v + x) - f(x)](1 - v^2)^n dv\right| \leq \varepsilon \int_{-\delta}^{\delta}(1 - v^2)^n dv$$

$$\leq \varepsilon \int_{-1}^{1}(1 - v^2)^n dv$$

$$= 2\varepsilon J_n.$$

Furthermore, if M is the maximum of $|f(x)|$ for $\alpha \leq x \leq \beta$ we obtain

$$|I_1| < M \int_{-1}^{-\delta} (1 - v^2)^n dv = MJ_n^*,$$

$$|I_3| < M \int_{\delta}^{1} (1 - v^2)^n dv = MJ_n^*.$$

Therefore, since the denominator in $P_n(x)$ is equal to $2J_n$,

$$|P_n(x) - f(x)| < 2M \frac{J_n^*}{J_n} + \varepsilon.$$

Since lim $(J_n^*/J_n) = 0$, the right side may be made less than 2ε by a suitable choice of n; thus f(x) is indeed approximated by $P_n(x)$ uniformly in the interval $a \leq x \leq b$.

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