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DIGRAPHS AND HOMOMORPHISMS: CORES, COLORINGS, AND CONSTRUCTIONS

by

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Digraphs and Homomorphisms: cores, colorings, and constructions

Chairperson: Mark Kayll, Ph.D.

   A natural digraph analogue of the graph-theoretic concept of an 'independent set' is that of an acyclic set, namely a set of vertices not spanning a directed cycle. Hence a digraph analogue of a graph coloring is a decomposition of the vertex set into acyclic sets. In the spirit of a famous theorem of P. Erdős [Graph theory and probability, Canad. J. Math. **11**:34–38, (1959)], it was shown probabilistically in [D. Bokal, G. Fijavž, M. Juvan, P. M. Kayll, and B. Mohar, The circular chromatic number of a digraph, J. Graph Theory **46**(3): 227–240, (2004)] that there exist digraphs with arbitrarily large digirth and chromatic number. Here we give a construction of such digraphs and define a new product of these highly chromatic digraphs with the directed analogue of the complete graph. This product gives a construction of uniquely n-colorable digraphs without short cycles.

   The graph-theoretic notion of 'homomorphism' also gives rise to a digraph analogue. An acyclic homomorphism from a digraph D to a digraph H is a mapping $\varphi : V(D) \to V(H)$ such that $uv \in A(D)$ implies that either $\varphi(u)\varphi(v) \in A(H)$ or $\varphi(u) = \varphi(v)$, and all the 'fibers' $\varphi^{-1}(v)$, for $v \in V(H)$, of φ are acyclic. In this language, a core is a digraph D for which there does not exist an acyclic homomorphism from D to a proper subdigraph of itself. Here we prove some basic results about digraph cores and construct new classes of cores. We also define a digraph-theoretic analogue to the graph-theoretic 'fractional chromatic number' and prove results relating it to other well-known digraph invariants. We see that it behaves similarly to the graph-theoretic analogue.

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Chapter 1

Introduction

One of the oldest areas within graph theory is graph coloring. It dates back to the nineteenth century and the Four-Color Conjecture which famously eluded proof for nearly a century. Over the course of the last century the idea of assigning colors to the vertices of a graph such that adjacent vertices are colored differently has blossomed into a beautiful field of mathematics with countless applications in the modern digital era. Colorings naturally give rise to the study of graph homomorphisms: vertex mappings which preserve adjacency. The concept of graph homomorphisms was introduced over fifty years ago by G. Sabidussi [26] and Z. Hedrlín and A. Pultr [17]. For those readers not familiar with colorings and graph homomorphisms, Chapter 2 explores the needed definitions, concepts, and results. For a more thorough treatment of the subject, the reader is encouraged to consult [11], [19], or [13]. For basic notation and terminology concerning graphs and directed graphs (digraphs) we mainly follow [6] and [2] respectively.

This dissertation is mainly concerned with building upon the footings of [3, 10, 15, 16, 35] in order to uncover digraph analogues of some of the rich existing collection of results on graph homomorphisms. Chapter 2 introduces precisely the digraph analogue of graph ho-

momorphism, namely 'acyclic homomorphism', that the authors of the aforementioned works have used. This definition provides the skeleton upon which the rest of the theory hangs. Chapter 2 also surveys selected previous results, both graph- and digraph-theoretic, relating to colorings and homomorphisms. In Chapter 3 we prove a number of basic lemmas for our tool box needed to build up our theory. Here we see such homomorphism nuts and bolts as 'cores', 'retracts', and 'homomorphic equivalence'.

The theory of graph homomorphisms is intimately tied to that of graph products, and in Chapter 4 we prove new results which build the theory of digraph products. For an excellent resource covering graph products, the reader is referred to [14]. If the reader pursues this resource, it will be clear that there is much left to investigate regarding digraph products.

The 'chromatic number', 'circular chromatic number', and 'fractional chromatic number' are three of many standard graph invariants (see, e.g., [11,19]) related to colorings and homomorphisms. Chapter 5 introduces an analogue of the fractional chromatic number for digraphs and establishes several results which actually generalize the analogous graph results. In what sense they are generalized is covered in Chapter 2, where we introduce most of our terminology. Our versions of the circular chromatic number and chromatic number of a digraph were first studied in [3], and Chapter 4 also investigates these parameters.

Chapter 6 presents the main results of this dissertation. These results are digraph analogues of theorems stemming from questions answered by P. Unger and B. Descartes (a well-known pseudonym for W.T. Tutte) [33] in 1954 and A.A. Zykov [41]. The basic question asks if it is possible to have a graph which one would need many colors for proper coloring while remaining sparse in the sense of 'edge density'. The question was answered in the affirmative, going against common intuition, and resulted in numerous theorems from some of the most influential mathematicians of the past century. I am honored and humbled to add my results as a small piece of a puzzle which has been put together incrementally for over a century.

Finally, in Chapter 7, we point to a few future directions possibly continuing from this work. Alas, though I am proud of my work here, there is much to be done. The reader should not want for enticing problems to solve upon finishing this dissertation. As mathematicians, it is clear to us that the questions, not the solutions, are of utmost importance. Before diving into this body of work, the reader is left with a quote from *Anna Karenina* by Leo Tolstoy [32]:

> Some mathematician, I believe, has said that true pleasure lies not in the discovery of truth, but in the search for it.

Chapter 2

Background: notation, definitions, and history

This chapter introduces the definitions, concepts, and results needed to proceed. We survey several results relating homomorphisms and colorings of both graphs and digraphs. We aim both to introduce the reader to precise definitions and to put the original results of this dissertation into their proper historical context.

2.1 The chromatic number and homomorphisms

One of the most basic objects we will deal with is that of a graph. A *graph* G is a set $V(G)$ of *vertices* together with a set $E(G)$ of *edges*, each of which is a two-element subset of $V(G)$. Throughout this dissertation we consider only finite simple graphs; i.e., $V(G)$ is finite, $E(G)$ contains only pairs of distinct vertices (no loops), and a pair $x, y \in V(G)$ may appear at most once in $E(G)$ (no multiple edges). We restrict ourselves to such graphs because we are dealing with colorings and homomorphisms. We note here that one diversion from [6] is that we do

not reserve n for $|V(G)|$ or m for $|E(G)|$. We say that two vertices x, y are *adjacent* in G if $\{x, y\} \in E(G)$. As is standard, we will denote an edge $\{x, y\}$ as xy for the remainder of this work. An *independent set* of vertices is a subset S of $V(G)$ such that no two elements of S are adjacent. An assignment $\sigma : V(G) \to C$ of 'colors' C to $V(G)$ is a *proper coloring* if no two adjacent vertices are assigned the same color, i.e. if the set of fibers $\sigma^{-1}(c)$, for $c \in C$, forms a partition of the vertex set into independent sets. Finally we define the *chromatic number* $\chi = \chi(G)$ of G to be the minimum number of sets needed to partition $V(G)$ into independent sets.

As is standard (see, e.g., [11, 19]), we define a *graph homomorphism* from a graph G to a graph H to be a mapping $f : V(G) \to V(H)$ such that $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$, i.e., a mapping from one graph to another which preserves adjacency. If there exists a homomorphism from G to H we say that G is *homomorphic* to H and write $G \to H$. The *fibers* of f are the inverse images of the vertices of H. Notice that the fibers of a homomorphism are independent sets because we are considering exclusively graphs without loops. We now start to see a potential relationship between graph homomorphisms and graph colorings. In order to complete this thought we need an extremely important family of graphs. For $n \geq 1$, the *complete graph* K_n has n vertices and all $\binom{n}{2}$ edges between them. It becomes easy to see that an equivalent definition of the chromatic number is

$$\chi(G) \coloneqq \min\{n \mid G \to K_n\}.$$

In this way, homomorphisms generalize colorings, and we say that G is *H-colorable* if G is homomorphic to H.

For the original results presented in this dissertation, the most important object is a 'digraph'. As defined in [19], a *digraph D* is a finite set $V(D)$ of *vertices*, together with a binary relation $A(D)$, the *arc set*, on D. The elements (u, v) of $A(D)$ are called *arcs* of D and notated as uv for the remainder of the dissertation. As with graphs, we will consider only di-

graphs which have an irreflexive relation, i.e., no loops. Notice that finiteness and not allowing repeated arcs in the same direction are built in to the definition. However it is acceptable to have two arcs in opposite directions between two vertices. We say that a digraph is *symmetric* if its binary relation is symmetric. In fact symmetric digraphs are really graphs in disguise. We define a *complete biorentation* of a graph G, denoted $\overleftrightarrow{G}$, to be the digraph with vertex set $V(G)$ and an arc from u to v and v to u whenever u is adjacent to v in G; i.e., we replace each edge of G by two oppositely directed arcs in $\overleftrightarrow{G}$. It is easy to see that a digraph is symmetric if and only if it is the complete biorientation of some graph. Hence we may think of the set of graphs as a subset of the set of digraphs. This is the sense in which some of our results generalize graph-theoretic results.

The following definitions are modeled from their graph analogues in [19]. A *directed walk* in a digraph D is a sequence of vertices $v_0, v_1, \ldots, v_k$ of D such that $v_{i-1}v_i \in A(D)$, for each $i = 1, 2, \ldots, k$. A directed walk is *closed* if $v_0 = v_k$. The integer k is called the *length* of the walk. A *directed path* in D is a directed walk in which all the vertices are distinct. A *directed cycle* in a digraph is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ of D such that $v_{i-1}v_i \in A(D)$, for $i = 2, 3, \ldots k$, and $v_k v_1 \in A(D)$. Notice that a directed cycle is a directed closed walk, and hence the definition of length is still applicable. We refer to a directed cycle of length k as a *k-cycle*. In general when we talk about a directed cycle in D we are talking about a subdigraph of D with vertex set $\{v_1, v_2, \ldots, v_k\}$ and arc set $\{v_{i-1}v_i \mid i = 2, 3, \ldots k\} \cup \{v_k v_1\}$. We denote this digraph as $\vec{C}_k$. The *digirth* $\vec{g}(D)$ of a digraph is the length of a shortest directed cycle in D. We define the digirth of an acyclic digraph to be infinity.

An *acyclic* digraph is one which does not contain a cycle. The subdigraph of D *induced* by a subset S of $V(D)$, denoted $D[S]$, is the digraph with vertex set S and an arc from u to v when $uv \in A(D)$. An *acyclic* set A in a digraph D is a subset of $V(D)$ such that the subdigraph of D induced by A is acyclic. Notice that an acyclic set in a digraph is a natural generalization of an independent set in a graph as a set is independent in a graph G if and only if the same set is acyclic in $\overleftrightarrow{G}$. We now are prepared to define an analogue of the graph-theoretic concept

of coloring. Here we follow $[3, 10, 25]$ rather than $[2, 6, 19]$. We define a *coloring* of a digraph D to be a partition of $V(D)$ into acyclic sets and the *chromatic number* χ of D to be the minimum number of sets needed for such a partition. Notice that this definition generalizes from graphs to digraphs since $\chi(G) = \chi(\overleftrightarrow{G})$.

Finally we would like to relate digraph colorings to homomorphisms. An *acyclic homomorphism* from a digraph D to a digraph H is a mapping $\varphi : V(D) \to V(H)$ such that $uv \in A(D)$ implies that either $\varphi(u)\varphi(v) \in A(H)$ or $\varphi(u) = \varphi(v)$, and all the fibers of φ are acyclic. If φ is an acyclic homomorphism such that all $uv \in A(D)$ satisfy $\varphi(u) \neq \varphi(v)$ then φ is a *non-contracting homomorphism*. As with graphs, if there exists an acyclic homomorphism from a digraph D to a digraph H we say that D is *homomorphic* to H and denote this by $D \to H$. Since we deal almost exclusively with acyclic homomorphisms when considering digraphs, we often write 'homomorphism' when it is clear from the context that we mean 'acyclic homomorphism'. It is an easy exercise to check that acyclic homomorphisms compose. Furthermore, as in the case of graphs, the digraph chromatic number satisfies $\chi(D) = \min\{n \mid D \to \overleftrightarrow{K}_n\}$. Thus we obtain the following result; see $[3]$:

Lemma 2.1. *For any digraphs D and H, if $D \to H$ then $\chi(D) \leq \chi(H)$.*

2.2 The circular chromatic number

A well-studied refinement of the (graph) chromatic number is the circular chromatic number χ_c which was first introduced by Vince $[34]$ in 1988 as the 'star chromatic number'. The original definition, as it appears in $[39]$, is quoted below:

For two integers $1 \leq q \leq p$, a (p, q)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0, 1, 2, \ldots, p - 1\}$ such that:

$$xy \in E(G) \implies q \leq |c(x) - c(y)| \leq p - q.$$

The *circular chromatic number* of G is defined as:

$$\chi_c(G) = \inf\{p/q \mid G \text{ has a } (p,q)\text{-coloring}\}.$$

It was proved in [34] that the infimum in this definition is attained; i.e.,

$$\chi_c(G) = \min\{p/q \mid G \text{ has a } (p,q)\text{-coloring}\}.$$

It was shown by Zhu [36] that an equivalent definition of the circular chromatic number is as follows:

Let C be a circle in $\mathbb{R}^2$ of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of C of length $1/r$. An r-circle-coloring of a graph G is a mapping c from $V(G)$ to $C^{(r)}$ such that $c(x) \cap c(y) = \varnothing$ whenever $xy \in E(G)$. If such an r-coloring exists, we say that G is r−circle-colorable. The circular chromatic number of G is

$$\chi_c(G) = \inf\{r \mid G \text{ is } r\text{-circle-colorable}\}.$$

We now relate the circular chromatic number to homomorphisms. In order to do so we once again need to define an important family of graphs. For $1 \leq q \leq p$, the *rational complete graph* $K_{p/q}$ has vertices $\{0, 1, \ldots, p-1\}$ and edge set $\{ij \mid q \leq |i-j| \leq p-q\}$. An alternate but equivalent definition of the circular chromatic number of a graph G is then

$$\chi_c(G) = \min\{p/q \mid G \to K_{p/q}\}$$

(see [19]). It is worth noting that $K_{p/q}$ is empty unless $p \geq 2q$, that $K_{p/1}$ is isomorphic to K_p, and that $K_{(2k+1)/k}$ is isomorphic to the odd cycle C_{2k+1}. In order to justify calling the circular chromatic number a refinement of the chromatic number we give the following standard result

(see e.g. [19, 39]) valid for all graphs G:

$$\chi(G) - 1 < \chi_c(G) \le \chi(G).$$

One obvious question to ask is for which graphs does $\chi_c(G) = \chi(G)$? Sufficient conditions for graphs under which equality holds were investigated in [1, 5, 30]. The interested reader should consult the survey [39] for an in-depth discussion of the circular chromatic number for graphs; now we move on to the digraph analogue.

The circular chromatic number of a digraph first appeared in [3] and was defined as follows (where, to remain faithful to the quote, we write $E(D)$ for our own $A(D)$ and 'edge' for 'arc'):

For a positive real number p, denote by $S_p \subset \mathbb{R}^2$ the circle with perimeter p (hence with radius $p/2\pi$) centered at the origin of $\mathbb{R}^2$. We can identify S_p with the set $\mathbb{R}/p\mathbb{Z}$ in the obvious way. For $x, y \in S_p$, let us denote by $S_p(x, y)$ the arc on S_p from x to y in the clockwise direction, and let $d(x, y)$ denote the length of this arc. The set $\mathbb{R}/p\mathbb{Z}$ can also be identified with the real interval $[0, p)$, where the "distance" function $d(x, y)$ can be expressed as

$$d(x, y) = \begin{cases} y - x, & \text{if } x \text{ precedes } y \text{ on } [0, p) \\ p + y - x, & \text{otherwise.} \end{cases}$$

A *circular p-coloring* of a digraph D is a function $c : V(D) \to S_p$ such that every edge $uv \in E(D)$ satisfies $d(c(u), c(v)) \ge 1$. If D has at least one edge, then the *circular chromatic number* $\chi_c(D)$ of D is the infimum of all real numbers p for which there exists a circular p-coloring of D. If D has no edges, then we define $\chi_c(D) = 1$.

As we have done repeatedly in this chapter, the authors of [3] recast their new coloring invariant using the language of homomorphisms. For $p \geq q$, we define the *directed complete rational graph* $\vec{K}_{p/q}$ to be the digraph with vertex set $\{0, 1, \ldots, p-1\}$ and an arc from i to j if $j - i \in \{q, q+1, \ldots, p-1\}$ (with arithmetic modulo p). It was shown in [3] that we may define

$$\chi_c(D) = \min\{p/q \mid D \to \vec{K}_{p/q}\}.$$

Thus if $D \to H$ then $\chi_c(D) \leq \chi_c(H)$. Two of the more important elementary results appearing in [3] are the following.

Proposition 2.2. *Every digraph D satisfies $\chi(D) - 1 < \chi_c(D) \leq \chi(D)$.*

Proposition 2.3. *If p and q are positive integers with $p \geq q$, then $\chi_c(\vec{K}_{p/q}) = p/q$.*

In the proof [3] of Proposition 2.3 we see that $K_{p/q}$ is isomorphic to the symmetric part of $\vec{K}_{p/q}$. Hence it is easy to check that all graphs G satisfy $\chi_c(G) = \chi_c(\overleftrightarrow{G})$, and we see that the digraph circular chromatic number generalizes the graph version of this invariant.

2.3   Cores

In the last two sections we have seen two important families of graphs in coloring theory, namely the complete graphs K_n and the complete rational graphs $K_{p/q}$. These two families are examples of 'cores', a class of graphs much studied over the past 20 years. For example, a MathSciNet search on the keywords 'graph' and 'core' produces 61 hits [22]. The modern definition of core first appeared in [18] and was defined in the following manner.

> A subgraph H of a graph G is called a *core of G* if there is a homomorphism of G to H, but no homomorphism of G to any proper subgraph of H. A graph is called a *core* if it is its own core.

One of the more useful definitions of a core, of which there are numerous equivalent ones, involves an important type of homomorphism. A *retraction* of a graph G to a subgraph H is a homomorphism $\varphi : V(G) \to V(H)$ such that the restriction of φ to $V(H)$ is the identity. If there is a retraction from G to H we say that H is a *retract* of G and G *retracts* to H. It turns out that the core of a finite graph is its smallest retract, and is unique up to isomorphism (see, e.g., [11, 19]). If H is the core of G we denote it by $G^{\bullet}$.

Notice that $G \to G^{\bullet}$ and $G^{\bullet} \to G$ which implies that G and $G^{\bullet}$ share some qualities; for example, $\chi(G) = \chi(G^{\bullet})$. In general we say that G and H are *homomorphically equivalent* if $G \to H$ and $H \to G$. The following result appears in [11].

Proposition 2.4. *Two graphs are homomorphically equivalent if and only if their cores are isomorphic.*

It is of interest to catalog cores because of their importance. At the start of this section we have seen a couple of examples of cores. A couple more are odd cycles and 'wheels'. The *wheel* W_k on $k + 1$ vertices is a cycle of length k together with a central vertex that is adjacent to every vertex of the cycle. The authors in [11] prove the following theorem and corollary which together give us a class of cores related to prime numbers. We call a graph *vertex transitive* if its automorphism group acts transitively on its vertex set.

Theorem 2.5. *If G is a vertex transitive graph, then $|V(G^{\bullet})|$ divides $|V(G)|$.*

Corollary 2.6. *If G is a non-empty vertex transitive graph with a prime number of vertices, then G is a core.*

In Chapter 3 we define the core of a digraph similarly to that of a graph. We prove a few basic lemmas in order to establish digraph analogues of various results about graphs, their cores, and homomorphisms. These lemmas serve as a tool box for the rest of the dissertation. We also generalize Theorem 2.5 and Corollary 2.6 to digraphs in Chapter 3.

2.4 Large girth and chromatic number

It is natural to consider if it is possible to have a graph which is both highly chromatic and devoid of short cycles. One might venture to guess that it is not possible, for it is intuitive to think of highly chromatic graphs as having dense edge sets and graphs with large girth as having sparse edge sets. However this is not the case as Tutte [33] constructed highly chromatic graphs with girth at least 6. He did this in answer to the following problem posed by Ungar: Show that for any $n > 1$ there exists a triangle-free graph (girth at least 4) with chromatic number n. Then in 1959 Erdős answered the question for arbitrarily large girth with a ground-breaking probabilistic proof [9]. This result gives us insight into the fact that the chromatic number is a global parameter rather than a local one.

At this point Ungar's original question was more than answered but we still had no idea what a graph with large girth and chromatic number looked like due to the nonconstructive nature of Erdős' proof. Then in 1968 Lovász [21] constructed hypergraphs with arbitrarily large girth and chromatic number. However he noted that he was unable to give a construction in purely graph-theoretic terms, although a simpler hypergraph construction [24] appeared in 1979. Finally in 1989 Kříž [20] produced the first purely graph-theoretic construction of graphs with arbitrarily large girth and chromatic number. This however, we shall see, is far from the end of the story.

To continue, we shall need a new graph coloring concept. We define a graph G to be *uniquely n-colorable* if G is n-chromatic and any two n-colorings of G induce the same partition of $V(G)$. In light of Erdős' theorem, one would like to know if there exist graphs with arbitrarily large girth which are also uniquely n-colorable for any prescribed $n > 1$. In 1973 Nešetřil [23] presented a construction which gives for each $n \geq 3$ an infinite class of uniquely n-colorable graphs of girth 4. Nešetřil noted that his construction for one of the graphs with $n = 3$ produces a graph with over half a billion vertices. To proceed we define the *direct product*

$G \times H$ of two graphs G, H to have vertex set $V(G) \times V(H)$ and an edge between (g_1, h_1) and (g_2, h_2) if and only if $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. Using this product Greenwell and Lovász [12] gave the following results in 1974.

Theorem 2.7. *If $\chi(G) > n$ and G is connected, then $K_n \times G$ is uniquely n-colorable (and the shortest odd cycle in $K_n \times G$ is at least as large as the shortest odd cycle in G).*

Corollary 2.8. *For all $n \geq 3$, there is a uniquely n-colorable graph without odd cycles shorter than any prescribed integer $s \geq 3$.*

Notice that Corollary 2.8 was for odd girth exclusively and was nonconstructive until Kříž's work. Before that the graph used for Corollary 2.8 was known to exist only due to Erdős' probabilistic proof. Two years later Bollobás and Sauer [4] proved probabilistically that there exist graphs with arbitrarily large girth which are uniquely n-colorable for arbitrarily large integers n.

Next came a generalization from n-coloring to H-coloring. For graphs G and H we say that G is *uniquely H-colorable* if it is surjectively H-colorable, and for any two H-colorings ϕ, ψ of G, the functions ϕ and ψ differ by an automorphism of H. Notice that a graph is uniquely n-colorable if and only if it is uniquely K_n-colorable. Also, if there exists a uniquely H-colorable graph, then H must be a core (see [19]). In 1996 Zhu [37] established probabilistically the existence of uniquely H-colorable graphs with arbitrarily large girth for any given core H. A few years later he was able to construct [38] uniquely H-colorable graphs with arbitrarily large *odd girth* using the graph direct product that Greenwell and Lovász had exploited.

What came next were a couple of analogous results for digraphs. Bokal et al. proved probabilistically in [3] that there exist digraphs with arbitrarily large digirth and chromatic number. In Chapter 6 we construct digraphs with digirth k and chromatic number n for any given pair of integers k and n exceeding one. This is analogous to Kříž's result [20]; however notice that it is slightly more precise because it is a construction for every pair of

integers n and k whereas Kříž did not pin down a specific chromatic number or girth. A later paper by Harutyunyan et al. [15] proved probabilistically the existence of digraphs with arbitrarily large digirth which are uniquely H-colorable (defined similarly to graphs) for any given core H. In Chapter 6 we construct uniquely $\overleftrightarrow{K}_n$-colorable digraphs with girth k for any pair of integers k and n exceeding one. Thus our main results are constructive versions of key theorems from [3] and [15], that were themselves sometimes intricate applications of the (nonconstructive) probabilistic method.

Chapter 3

Preliminary results

In this chapter we prove some basic original results that serve as convenient tools throughout this dissertation. We start with a few results that confirm digraph-theoretic cores share many of the same properties as the analogous graph-theoretic cores. We then show that acyclic homomorphisms act similarly to graph homomorphisms on cycles. For instance we show that $D \to H$ implies that the order of a minimum-length directed cycle in D is at least as big as a minimum-length directed cycle in H. Finally we identify a few classes of cores in Section 3.3, the last of which is a result establishing a connection between cores of 'vertex transitive' digraphs and prime numbers.

3.1 Basic results about cores

In [15] the authors defined a few basic terms necessary for the study of cores of digraphs. A digraph D is *uniquely H-colorable* if it is surjectively H-colorable, and for any two H-colorings ϕ, ψ of D, the functions ϕ and ψ differ by an automorphism of H, and a digraph D is a *core* if it is uniquely D-colorable. The authors also proved the following useful lemma in [15].

Lemma 3.1. *A digraph D is a core if and only if every acyclic homomorphism $V(D) \to V(D)$ is a bijection.*

The condition that every acyclic homomorphism $V(D) \to V(D)$ is a bijection is equivalent to saying that D is not homomorphic to a proper subdigraph of itself. An *(acyclic) retraction* of a digraph D is an acyclic homomorphism ϕ from D to a subdigraph H of D such that the restriction $\phi|_H$ is the identity map on H. Now we can state an equivalent definition of a core.

Lemma 3.2. *A digraph D is a core if and only if it does not retract to a proper subdigraph of itself.*

Proof. The necessity is clear since an acyclic retraction is an acyclic homomorphism. Now suppose that D is not a core, and let H be a proper subdigraph of D such that ϕ is an acyclic homomorphism from D to H and D is not homomorphic to any proper subdigraph of H. The existence of such an H is ensured by Lemma 3.1. We claim that H is a core, for suppose it is not and let f be an acyclic homomorphism from H to a proper subdigraph of H. Then $f \circ \phi$ is an acyclic homomorphism from D to a proper subdigraph of H, contradicting our choice of H. Because of the claim, any homomorphism from H to itself is an automorphism of H. Let $\varphi : V(D) \to V(H)$ be an acyclic homomorphism. Since the restriction of a homomorphism is a homomorphism, $\psi := \varphi|_H$ is an automorphism of H. Hence ψ^{-1} exists and $\psi^{-1} \circ \varphi : V(D) \to V(H)$ is an acyclic retraction to a proper subdigraph of D. $\square$

We now define a subdigraph H of a digraph D to be a *core in D* if there exists an acyclic retraction from D to H and H is a core.

Lemma 3.3. *An acyclic retract of a digraph D is an induced subdigraph of D.*

Proof. Let ϕ be an acyclic retraction from D to a subdigraph H of D. Suppose that $x, y \in V(H)$ and $xy \in A(D)$. Since H is a retract, both $\phi(x) = x$ and $\phi(y) = y$ and, as ϕ is a homomorphism, xy is an arc in H. $\square$

We say that two digraphs D and H are *homomorphically equivalent* if H is homomorphic to D and D is homomorphic to H.

Lemma 3.4. *If H and K are cores then they are homomorphically equivalent if and only if they are isomorphic.*

Proof. Let $\phi : H \rightarrow K$ and $\psi : K \rightarrow H$ be acyclic homomorphisms. This implies that $\psi \circ \phi$ and $\phi \circ \psi$ are bijections since H and K are cores. Thus ϕ and ψ are both bijective and hence $H \cong K$ since, e.g., ϕ is a bijective homomorphism. $\qquad\square$

Lemma 3.5. *Every finite digraph D has a core, which is an induced subdigraph and is unique up to isomorphism.*

Proof. Since D is finite and the identity mapping is an acyclic retraction, the family of subdigraphs of D to which D has an acyclic retraction is finite and nonempty and thus has a minimal element $D^\bullet$ with respect to inclusion. From the definition of 'core in D' and Lemma 3.2, we see that $D^\bullet$ is a core in D. Since $D^\bullet$ is an acyclic retract, it is an induced subdigraph by Lemma 3.3. Now let H_1 and H_2 be cores of D, and, for $i = 1, 2$, let ϕ_i be an acyclic retraction from D to H_i. Then $\phi_1|_{H_2}$ is an acyclic homomorphism from H_2 to H_1 and similarly there exists an acyclic homomorphism from H_1 to H_2. Therefore, by the preceding lemma, $H_1 \cong H_2$. $\qquad\square$

In the remainder of this dissertation we will always use $D^\bullet$ for 'core of D' as is done for the graph-theoretic analogue in [11].

Lemma 3.6. *Cores of connected digraphs are connected.*

Proof. Let D be a connected digraph and φ a retraction to $D^\bullet$. Suppose that $x, y \in V(D^\bullet)$. Then x, y are vertices of D because φ is a retraction. Since D is connected there exists a sequence of vertices $x = u_1, u_2, \ldots, u_n = y$ in D such that for all $i \in [n-1]$ we have $u_i u_{i+1} \in A(D)$

or $u_{i+1}u_i \in A(D)$ (possibly both). For $i \in [n]$, define $v_i := \varphi(u_i)$. The fact that φ is a retraction implies that $v_1 = x$, $v_n = y$, and the sequence of vertices $v_1, v_2, \ldots, v_n$ has the property that all $i \in [n-1]$ satisfy $v_iv_{i+1} \in A(D)$, $v_{i+1}v_i \in A(D)$, or $v_i = v_{i+1}$. Therefore $D^\bullet$ is connected. $\qquad\square$

The following result displays one use of cores for testing homomorphic equivalence.

Lemma 3.7. *Two digraphs are homomorphically equivalent if and only if their cores are isomorphic.*

Proof. Clearly, a digraph and its core are homomorphically equivalent. The sufficiency of the condition follows. For necessity, let $D^\bullet$ and $H^\bullet$ be cores of the digraphs D and H respectively. Assuming D and H are homomorphically equivalent, we have that $D^\bullet$ is homomorphic to D, D is homomorphic to H, and H is homomorphic to $H^\bullet$. Thus $D^\bullet$ is homorphic to $H^\bullet$ using the fact that the composition of acyclic homomorphisms is an acyclic homomorphism. Similarly $H^\bullet$ is homomorphic to $D^\bullet$. Hence by Lemma 3.4, $H^\bullet$ and $D^\bullet$ are isomorphic. $\qquad\square$

Earlier we defined a digraph H to be a core if it is uniquely H-colorable. In fact we will see that there is a looser condition governing whether H is a core. The next result shows that if we find any digraph which is uniquely H-colorable, then H is a core.

Lemma 3.8. *If there exists a uniquely H-colorable digraph, then H must be a core.*

Proof. Let D be uniquely H-colorable and $\phi : V(D) \to V(H)$ a surjective acyclic homomorphism. Now suppose that $\psi : V(H) \to V(H^\bullet)$ is an acyclic retraction and hence $\psi \circ \phi : V(D) \to V(H)$ is an acyclic homomorphism. Thus $\psi \circ \phi = \pi \circ \phi$ for some $\pi \in \text{Aut}(H)$, since D is uniquely H-colorable. Now since ϕ is surjective, $\text{Im}(\pi \circ \phi) = V(H)$. This implies that $\text{Im}(\psi \circ \phi) = V(H)$. But $\text{Im}(\psi) = V(H^\bullet)$, so we've shown that $V(H) \subseteq V(H^\bullet)$. Since the reverse containment is always true, we conclude that $H = H^\bullet$. $\qquad\square$

3.2 Homomorphisms and directed cycles

We now explore how acyclic homomorphisms interact with directed cycles. We will see it is similar to the interaction between odd cycles and homomorphisms on the domain of graphs. Corollary 3.10 is an indispensable tool throughout this dissertation, especially for the main results in Chapter 6.

Lemma 3.9. *Given any integers k, ℓ exceeding one, $\vec{C}_k \to \vec{C}_\ell$ if and only if $\ell \le k$.*

Proof. Let $V(\vec{C}_k) \coloneqq \{v_1, v_2, \ldots, v_k\}$ and $A(\vec{C}_k) \coloneqq \{v_i v_{i+1} \mid i = 1, 2, \ldots, k-1\} \cup \{v_k v_1\}$. Similarly, we define $V(\vec{C}_\ell) \coloneqq \{w_1, w_2, \ldots, w_\ell\}$ and $A(\vec{C}_\ell) \coloneqq \{w_i w_{i+1} \mid i = 1, 2, \ldots, \ell - 1\} \cup \{w_\ell w_1\}$. We first suppose that $\ell \le k$. We may then define the mapping $\varphi : V(\vec{C}_k) \to V(\vec{C}_\ell)$ by:

$$\varphi(v_i) = \begin{cases} w_i & \text{if } i \in \{1, 2, \ldots, \ell\} \\ w_\ell & \text{otherwise.} \end{cases}$$

This mapping is well-defined because $\ell \le k$ and it is easy to check that φ is a homomorphism provided $\ell > 1$. In order to prove the 'only if' direction we assume that $\sigma : V(\vec{C}_k) \to V(\vec{C}_\ell)$ is a homomorphism. This implies that for $i = 1, 2, \ldots, k - 1$ we have either $\sigma(v_i)\sigma(v_{i+1}) \in A(\vec{C}_\ell)$ or $\sigma(v_i) = \sigma(v_{i+1})$ and $\sigma(v_k)\sigma(v_1) \in A(\vec{C}_\ell)$ or $\sigma(v_k) = \sigma(v_1)$. Thus the sequence of vertices $\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_k), \sigma(v_1)$ is a closed directed walk possibly with some repeated entries. Let r be the length of the walk. It is clear that $r \le k$ and $r > 0$ because $\{v_1, v_2, \ldots, v_k\}$ is not an acyclic set. The shortest closed walk in $\vec{C}_\ell$ that is not a single vertex is $w_1, w_2, \ldots, w_\ell$. Therefore $\ell \le k$. $\square$

Corollary 3.10. *$D \to H$ implies that the order of a minimum-length directed cycle in D is at least as big as a minimum-length directed cycle in H.*

Proof. Suppose that the minimum-length of a directed cycle in D is k, so that $\vec{C}_k$ is a subdi-

graph of D. If $\varphi : V(D) \to V(H)$ is a homomorphism from D to H, then $\varphi|_{\overrightarrow{C}_k} : V(\overrightarrow{C}_k) \to V(H)$ is a homomorphism. It is easy to see that that the image of $\varphi|_{\overrightarrow{C}_k}$ is a cycle and thus Lemma 3.9 implies that this cycle must have length at most k. The assertion follows. $\qquad\square$

Corollary 3.11. *If $\phi : V(D) \to V(H)$ is a homomorphism then for every acyclic subset $B \subseteq V(H)$, the union $\bigcup\limits_{v \in \beta} \phi^{-1}(v)$ is acyclic in $V(D)$.*

3.3 A few classes of cores

Guided by analogous classes of graph-theoretic cores we explore a few classes of digraph-theoretic cores in this section. We first notice that for any integer $k \geq 2$ the cycle $\overrightarrow{C}_k$ is a core because the only induced subdigraphs of $\overrightarrow{C}_k$ are acyclic (hence $\overrightarrow{C}_k$ does not retract to any of its induced subdigraphs).

Lemma 3.12. *For each integer $k \geq 2$ the directed wheel $\overrightarrow{W}_k$ with all spokes directed in (or out) is a core.*

Proof. Let $v_0, v_1, ..., v_{k-1}$ be the vertices of $\overrightarrow{W}_k$ such that the subdigraph induced by $\{v_0, v_1, ..., v_{k-1}\}$ is $\overrightarrow{C}_k$. Notice that $\overrightarrow{C}_k$ is the only directed cycle of $\overrightarrow{W}_k$ and hence any other subset of $V(\overrightarrow{W}_k)$ with size at most k is acyclic. Thus any acyclic homomorphism ϕ from $\overrightarrow{W}_k$ to itself has the property that $\phi(V(\overrightarrow{C}_k)) = \overrightarrow{V}(C_k)$ using Corollary 3.11. Hence the restriction of ϕ to $V(\overrightarrow{C}_k)$ is a bijection. Thus we may assume, without loss of generality, that for each $i = 0, 1, ..., k - 1$ (and subscripts taken mod k), we have $\phi(v_i)\phi(v_{i+1})$ is an arc of $\overrightarrow{W}_k$ and $\phi(v_i)\phi(v_{i-1})$ is not. Let c be the center vertex of $\overrightarrow{W}_k$, and assume, for a contradiction, that $\phi(c) = \phi(v_0)$ ($\neq \phi(v_1)$). This would imply that ϕ is not an acyclic homomorphism, for the arc $v_1 c$ being in $\overrightarrow{W}_k$ would force the arc $\phi(v_1)\phi(c) = \phi(v_1)\phi(v_0)$ to be in the image (knowing as we do that $\phi(v_0) \neq \phi(v_1)$), while in fact it is not. Similarly, we see that $\phi(c) \neq \phi(v_i)$ for each $i = 0, 1, ..., k - 1$, so that $\phi(c) = c$, and ϕ is a bijection. Lemma 3.1 now implies that $\overrightarrow{W}_k$ is a

core. □

Recall from Section 2.1 that the *digirth* of a digraph D is the length of a shortest directed cycle in D and infinity if D is acyclic. A *2-arc* of a digraph D is a sequence of three vertices xyz such that there is an arc in D between x and y (in either direction) and an arc in D between y and z.

Proposition 3.13. *If D is a connected digraph with finite digirth such that every two-arc lies in a shortest directed cycle, then $D \cong \vec{C}_k$ for some $k \geq 2$.*

Proof. Let D be a digraph such that every two-arc lies in a shortest directed cycle of length k. This implies that $\vec{C}_k$ is a subdigraph of D. Let $v_0, v_1, ..., v_{k-1}$ be the vertices of $\vec{C}_k$ and $\{v_i v_{i+1}\}$ be the arc set with the subscripts taken modulo k. As $\vec{C}_k$ is a shortest cycle, there are no arcs with ends in $\{v_0, v_1, \ldots, v_{k-1}\}$ besides the arcs of $\vec{C}_k$. If $V(D) \smallsetminus V(\vec{C}_k)$ is nonempty, then, because D is connected, there exists $x \in V(D) \smallsetminus C(\vec{C}_k)$ and $v_i \in V(\vec{C}_k)$ such that xv_i or $v_i x$ is in the arc set of D. If xv_i is an arc, then $xv_i v_{i-1}$ is a two-arc (not necessarily a directed two-arc). However $v_i v_{i-1}$ is not an arc of D which implies that $xv_i v_{i-1}$ is not in a directed cycle. A similar argument holds when $v_i x$ is an arc of D. Thus D contains no vertices or arcs besides those of $\vec{C}_k$, and therefore $D \cong \vec{C}_k$. □

Lemma 3.14. *If D is a vertex-transitive digraph, then its core $D^\bullet$ is vertex transitive.*

Proof. Let x and y be vertices of $D^\bullet$. Since D is vertex transitive there is an automorphism φ of D such that $\varphi(x) = y$. Let ψ be an acyclic retraction from D to $D^\bullet$. Hence the restriction $\psi \circ \varphi|_{D^\bullet}$ is an acyclic homomorphism from $D^\bullet$ to $D^\bullet$. Thus $\psi \circ \varphi|_{D^\bullet}$ is an automorphism of $D^\bullet$ by Lemma 3.1. Also $\psi \circ \varphi(x) = \psi(y) = y$ since ψ is a retraction. □

Theorem 3.15. *If D is a vertex-transitive digraph, then $|V(D^\bullet)|$ divides $|V(D)|$.*

Proof. It is enough to show that the fibers of any acyclic retraction φ from D to $D^\bullet$ have the same size. For any automorphism π of D the restriction $\pi|_{V(D^\bullet)}$ is an acyclic homomorphism. This implies that $\varphi \circ \pi|_{V(D^\bullet)}$ is an acyclic homomorphism and hence $\varphi \circ \pi|_{V(D^\bullet)}$ is surjective by Lemma 3.1 since $D^\bullet$ is a core. Thus $\pi(V(D^\bullet))$ has exactly one vertex in each fiber of φ since the number of vertices in $\pi(V(D^\bullet))$ is the same as the number of vertices in $D^\bullet$. Now let x be a vertex in D and F be the fiber of φ that contains x. Since D is vertex transitive the number N of automorphisms π of D such that $\pi(V(D^\bullet))$ contains x is independent of the choice of x. Since $\pi(V(D^\bullet))$ has exactly one vertex in common with F for all $\pi \in \text{Aut}(D)$, it follows that $\text{Aut}(D) = \bigcup_{y \in F} \{\pi \in \text{Aut}(D) \mid \pi(D^\bullet) \cap F = \{y\}\}$, and as this union is disjoint, we see that $|\text{Aut}(D)| = |F| \cdot N$. But N is independent of $|F|$ and therefore all fibers of φ have the same size. $\qquad\square$

Corollary 3.16. *If D is a vertex-transitive digraph with a prime number of vertices then D is a core.*

In this chapter we have confirmed that digraph cores behave similarly to their graph analogues. We have also confirmed that acyclic homomorphisms interact with directed cycles in the same way homomorphisms interact with odd cycles. We now have the tools needed to prove some results about products of digraphs in Chapter 4; these tools also find extensive use in Chapter 6.

Chapter 4

Products of digraphs

In this chapter we define three products of digraphs and prove a couple of results about each regarding colorings. We define the 'direct product', 'cartesian product', and 'lexicographic product'. The direct product is related to Hedetniemi's Conjecture, a well known open problem in graph theory. The lexicographic product is the most important product for this dissertation and we will see it again in Chapter 5. We also prove a few results about the core of the lexicographic product of two digraphs.

4.1   The direct product

The *direct product* $D \times H$ of two digraphs D, H is the digraph with $V(D) \times V(H)$ as its vertex set and an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2$ is an arc of D and $h_1 h_2$ is an arc of H. It is easy to see that this product is commutative. Important and useful mappings from any direct product are the projections which are defined canonically; e.g., $\pi_D : V(D \times H) \to V(D)$ is defined by $\pi_D((d, h)) = d$ whenever $(d, h) \in V(D \times H)$.

Lemma 4.1. *The projections $\pi_D : V(D \times H) \to V(D)$ and $\pi_H : V(D \times H) \to V(H)$ are acyclic*

homomorphisms.

Proof. Since the direct product is commutative we need only show that π_D is an acyclic homomorphism. If $(d_1, h_1)(d_2, h_2)$ is an arc in $D \times H$ then $d_1 d_2 = \pi_D((d_1, h_1))\pi_D((d_2, h_2))$ must be an arc in D by the definition of $D \times H$. As D has no loops, the fibers of π_D are acyclic and thus π_D is an acyclic homomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An immediate consequence of Lemma 4.1 follows from Lemma 2.1 which states that $D \to H$ implies that $\chi(D) \le \chi(H)$.

Corollary 4.2. $\chi(D \times H) \le min\{\chi(D), \chi(H)\}$.

It is quite interesting that this corollary is so easy to prove for both digraphs and graphs, while in graph theory it is a much studied conjecture, formulated by Stephen T. Hedetniemi in 1966, that equality holds (see, e.g., [27]).

Hedetniemi's Conjecture 4.3. For any graphs G and H, the chromatic number χ satisfies $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

It is also an open question as to whether equality holds in the case of digraphs.

4.2 The cartesian product

The *cartesian product* $D \square H$ of two digraphs D, H is the digraph with $V(D) \times V(H)$ as its vertex set and an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2$ is an arc of D and $h_1 = h_2$, or $d_1 = d_2$ and $h_1 h_2$ is an arc of H. As with the direct product it is easy to see that the cartesian product is commutative.

Theorem 4.4. *Any digraphs D and H satisfy $\chi(D \square H) = max\{\chi(D), \chi(H)\}$.*

Proof. It is clear from the definition of the cartesian product that both D and H are isomorphic to a subdigraph of $D \square H$. Hence $\chi(D \square H) \geq \max\{\chi(D), \chi(H)\}$. For the reverse inequality we may assume without loss of generality that $\chi(D) \geq \chi(H)$. Let $f : V(D) \to \{0, 1, ..., \chi(D) - 1\}$ and $g : V(H) \to \{0, 1, ..., \chi(H) - 1\}$ be colorings of D and H respectively. Define $\varphi : V(D \square H) \to \{0, 1, ..., \chi(D) - 1\}$ by $\varphi(a, b) = f(a) + g(b) \bmod \chi(D)$. Suppose by way of contradiction that φ is not a proper coloring of $D \square H$. This implies that there exists a directed cycle $\overrightarrow{C_n}$ that is a subdigraph of $D \square H$ and $\varphi(a, b) = i$ for all $(a, b) \in V(\overrightarrow{C_n})$ and some $i \in \{0, 1, ..., \chi(D) - 1\}$. Let us write the vertices of $\overrightarrow{C_n}$ sequentially as

$$\Big((a_{1_0}, b_{1_0}), (a_{1_1}, b_{1_0}), (a_{1_2}, b_{1_0}), .., (a_{1_{k_1}}, b_{1_0}), (a_{1_{k_1}}, b_{1_1}), (a_{1_{k_1}}, b_{1_2}), (a_{1_{k_1}}, b_{1_3}), ...,$$

$$(a_{1_{k_1}}, b_{1_{j_1}}), (a_{2_0}, b_{1_{j_1}})(a_{2_1}, b_{1_{j_1}}), (a_{2_2}, b_{1_{j_1}}), ..., (a_{2_{k_2}}, b_{1_{j_1}}), (a_{2_{k_2}}, b_{2_0}), (a_{2_{k_2}}, b_{2_1}), ...\Big).$$

If a_{1_0} is the only D-coordinate in $\overrightarrow{C_n}$ then the H-coordinates of $\overrightarrow{C_n}$ reveal a directed cycle of length n in H. Thus for at least two vertices b_{1_s}, b_{1_t} of H we have $g(b_{1_s}) \neq g(b_{1_t})$ since g is a proper coloring of H. This leads to a contradiction as it implies that $\varphi(a_{1_0}, b_{1_s}) \neq \varphi(a_{1_0}, b_{1_t})$ since $g(b) \leq \chi(D) - 1$ for all $b \in V(H)$. If a_{1_0} is not the only D-coordinate then the definition of cartesian product forces the D-coordinates of $\overrightarrow{C_n}$ to form a closed directed walk in D. Thus for at least two of the vertices of this directed walk, say a_{ℓ_m} and a_{q_r}, we have $f(a_{\ell_m}) \neq f(a_{q_r})$ because f is a proper coloring of D. We may assume without loss of generality that $\ell \leq q$ and that q is minimal. (What we mean here is that q is the least integer such that there exists an r with $f(a_{\ell_m}) \neq f(a_{q_r})$.) If $\ell = q$ then for some $b \in V(H)$ the vertices (a_{ℓ_m}, b) and (a_{q_r}, b) belong to $\overrightarrow{C_n}$ and $\varphi((a_{\ell_m}, b)) \neq \varphi((a_{q_r}, b))$, which is a contradiction. Otherwise there exist two vertices b, b' of H such that

$$(a_{q-1_{k_{q-1}}}, b), (a_{q_1}, b), (a_{q_2}, b), ..., (a_{q_r}, b), ..., (a_{q_{k_q}}, b), (a_{q_{k_q}}, b')$$

is a contiguous subsequence of the vertices of $\overrightarrow{C_n}$. Since q is minimal we have $f(a_{q-1_{k_{q-1}}}) = f(a_{\ell_m}) \neq f(a_{q_r})$. This is a contradiction to $\overrightarrow{C_n}$ being monochromatically colored as it implies

that $\varphi((a_{q-1_{k_{q-1}}}, b)) \neq \varphi((a_{q_r}, b))$. Therefore φ is a proper coloring of $D \,\square\, H$ which finally implies that $\chi(D \,\square\, H) = \max\{\chi(D), \chi(H)\}$. $\square$

4.3 The lexicographic product

The *lexicographic product* $D \circ H$ of two digraphs D, H is defined to be the digraph with $V(D) \times V(H)$ as its vertex set and with an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2$ is an arc in D, or $d_1 = d_2$ and $h_1 h_2$ is an arc in H.

Lemma 4.5. *If D_2 is an acyclic digraph then the projection $\pi_{D_1} : V(H) \to V(D_1)$ is a homomorphism for every subdigraph H of $D_1 \circ D_2$.*

Proof. Let $(u, v)(x, y)$ be an arc of H. If $u \neq x$ then $ux = \pi_{D_1}((u, v))\pi_{D_1}((x, y))$ is an arc of D_1. Thus we assume that $u = x$ which implies that vy is an arc of D_2 and $\pi_{D_1}((u, v)) = \pi_{D_1}((x, y)) = u$. Since the subdigraph of $D_1 \circ D_2$ induced by the vertex set $F_u = \{(u, w) | w \in V(D_2)\}$ is isomorphic to D_2, the set $\pi_{D_1}^{-1}(u)$ is acyclic in H for all $u \in V(D_1)$. Therefore the projection π_{D_1} is a homomorphism. $\square$

Lemma 4.6. *If D, F, and H are digraphs with H homomorphic to F, then $D \circ H$ is homomorphic to $D \circ F$.*

Proof. Let $\varphi : V(H) \to V(F)$ be a homomorphism and define $\rho : V(D \circ H) \to V(D \circ F)$ by $\rho((d, h)) = (d, \varphi(h))$. For an arc $(d_1, h_1)(d_2, h_2)$ of $D \circ H$, either $d_1 d_2$ is an arc of D, in which case $\rho((d_1, h_1))\rho((d_2, h_2))$ is an arc in $D \circ F$, or $d_1 = d_2$ and $h_1 h_2$ is an arc in H. Assuming $h_1 h_2 \in A(H)$, either $\varphi(h_1)\varphi(h_2)$ is an arc of F or $\varphi(h_1) = \varphi(h_2)$. Hence, as $d_1 = d_2$, either $\rho((d_1, h_1))\rho((d_2, h_2))$ is an arc in $D \circ F$ or $\rho((d_1, h_1)) = \rho((d_2, h_2))$. Suppose, for some vertex (d, f) of $D \circ F$, that $\rho^{-1}((d, f))$ induces some n-cycle $\vec{C}_n$ in $D \circ H$. This implies from the definition of ρ that $\vec{C}_n = ((d, h_1), (d, h_2), ..., (d, h_n))$ where $\varphi(h_i) = f$ for $i = 1, 2, ..., n$.

Hence $(h_1, h_2, ..., h_n)$ is a cycle in H, completely mapped to f, which contradicts φ being a homomorphism. Therefore $\rho : V(D \circ H) \to V(D \circ F)$ is indeed a homomorphism. $\square$

Theorem 4.7. *If $\chi(H) = n$ then $\chi(D \circ H) = \chi(D \circ \overleftrightarrow{K}_n)$ for every digraph D.*

Proof. Since $\chi(H) = n$, the digraph H is homomorphic to $\overleftrightarrow{K}_n$ which implies that $D \circ H$ is homomorphic to $D \circ \overleftrightarrow{K}_n$ using Lemma 4.6. Thus $\chi(D \circ H) \leq \chi(D \circ \overleftrightarrow{K}_n)$. Let $m = \chi(D \circ H)$ and suppose that g is an m-coloring of $D \circ H$. Since $\chi(H) = n$ and the subdigraph of $D \circ H$ induced by $H_d := \{(d, h) | h \in V(H)\}$ is isomorphic to H, for all vertices d of D, there exist subsets A_d of $V(H)$ such that each $|A_d| = n$ and $g(d, h_1) \neq g(d, h_2)$ for all pairs of distinct $h_1, h_2 \in A_d$. Label the elements of A_d so that $A_d = \{h_1^d, h_2^d, ..., h_n^d\}$. Now define a mapping f from $V(D \circ \overleftrightarrow{K}_n)$ to $V(\overleftrightarrow{K}_m)$ by $f((d, i)) := g(d, h_i^d)$. By way of contradiction assume that there exists a color k such that a cycle $\vec{C}_\ell = ((d_1, \alpha_1), (d_2, \alpha_2), ..., (d_\ell, \alpha_\ell))$ is induced within the set $f^{-1}(k)$. We first note that by the definition of f, if $d_i = d_j$ then $f((d_i, \alpha_i)) \neq f((d_j, \alpha_j))$ for $\alpha_i \neq \alpha_j$. But f is monochromatic on $\vec{C}_\ell$, so this shows that the entries d_i within $\vec{C}_\ell$ must be distinct. Since $\vec{C}_\ell$ is a cycle in $D \circ \overleftrightarrow{K}_n$, it follows that $(d_1, d_2, ..., d_\ell)$ is a cycle in D. However this implies that $((d_1, h_{\alpha_1}^{d_1}), (d_2, h_{\alpha_2}^{d_2}), ..., (d_\ell, h_{\alpha_\ell}^{d_\ell}))$ is a cycle in $D \circ H$ which is a contradiction (of g being a coloring of $D \circ H$) since $g(d_i, h_{\alpha_i}^{d_i}) = f(d_i, \alpha_i)$ for all $i \in [\ell]$. Therefore $\chi(D \circ \overleftrightarrow{K}_n) \leq m = \chi(D \circ H)$ and hence $\chi(D \circ H) = \chi(D \circ \overleftrightarrow{K}_n)$. $\square$

Lemma 4.8. *If D and H are digraphs, then $D \circ H$ is homomorphically equivalent to $D \circ H^\bullet$.*

Proof. We first notice that $D \circ H^\bullet$ is homomorphic to $D \circ H$ because $D \circ H^\bullet$ is a subdigraph of $D \circ H$. In order to show that $D \circ H$ is homomorphic to $D \circ H^\bullet$, let ρ be a retraction from H to $H^\bullet$ and define $\phi : V(D \circ H) \to V(D \circ H^\bullet)$ by $\phi((d, h)) := (d, \rho(h))$. Suppose that $(d_1, h_1)(d_2, h_2) \in A(D \circ H)$. This implies that either $d_1 d_2 \in A(D)$ or $d_1 = d_2$ and $h_1 h_2 \in A(H)$. If $d_1 d_2 \in A(D)$ then $(d_1, \rho(h_1))(d_2, \rho(h_2)) \in A(D \circ H^\bullet)$ making ϕ a valid homomorphism. Hence assume that $d_1 = d_2$ and $h_1 h_2 \in A(H)$. Since ρ is a retraction, the inequality $\rho(h_1) \neq \rho(h_2)$ implies that $(d_1, \rho(h_1))(d_2, \rho(h_2)) \in A(D \circ H^\bullet)$, making ϕ a

valid homomorphism. If $\rho(h_1) = \rho(h_2) =: w$ then $\phi(d_1, h_1) = \phi(d_2, h_2) = (d_1, w)$. Since $\phi^{-1}((d_1, w)) = \{(d_1, x) \mid x \in \rho^{-1}(w)\}$ and $\rho^{-1}(w)$ is acyclic (in H), we see that $\phi^{-1}((d_1, w))$ is acyclic (in $D \circ H$). Hence ϕ is a homomorphism and $D \circ H$ is homomorphically equivalent to $D \circ H^\bullet$. □

It was proven in [13] that for two connected graphs G and H, the core of $G \circ H$ can be represented as the lexicographic product $G' \circ H^\bullet$, where G' is a subgraph of G which itself is a core. To close this chapter we present a directed analogue.

Theorem 4.9. *If D is a connected symmetric digraph without loops and H is a connected digraph then the core of $D \circ H$ is $D' \circ H^\bullet$ where D' is a subdigraph of D which itself is a core.*

Proof. From Lemmas 3.7 and 4.8, it is enough to show that the core of $D \circ H^\bullet$ is $D' \circ H^\bullet$ with D' as described in the statement. Let $K = D \circ H^\bullet$ and ρ be a retraction from K to $K^\bullet$. For $d \in V(D)$ let F_d be the set $\{(d, h) \mid h \in V(H^\bullet)\}$. Let d be a vertex of D such that there exists a vertex h_1 in $H^\bullet$ with $(d, h_1) \in K^\bullet$ which implies that $\rho((d, h_1)) = (d, h_1)$. We first show that for any such d the image of F_d under ρ is contained in F_d. Assume there exists $h_2 \in V(H^\bullet)$ such that $\rho((d, h_2)) \neq (d, h_2)$ since if no such vertex exists we are done. Since $H^\bullet$ is connected (by Lemma 3.6), we may assume that either $h_1 h_2 \in A(H^\bullet)$ or $h_2 h_1 \in A(H^\bullet)$. Let $\rho((d, h_2)) = (x, y)$. If $h_1 h_2 \in A(H^\bullet)$ then $(d, h_1)(d, h_2) \in A(K)$ which implies that either $(d, h_1)(x, y) \in A(K^\bullet)$ or $(x, y) = (d, h_1)$. If $(x, y) = (d, h_1)$, then $\rho((d, h_2)) \in F_d$ and we are done. Hence assume that $(d, h_1)(x, y) \in A(K^\bullet)$ which implies that either $d = x$ (and we are done) or $dx \in A(D)$. Since D is symmetric, $xd \in A(D)$ which is a contradiction because this would imply that $(d, h_2)(x, y)$ and $(x, y)(d, h_2)$ are arcs in K making $\rho^{-1}((x, y))$ cyclic. The same argument holds when $h_2 h_1$ is an arc in $H^\bullet$. Thus the image of F_d under ρ is contained in F_d.

Since $K[F_d] \cong H^\bullet$ the restriction of ρ to F_d is a bijection which implies that $\rho(F_d) = F_d$. Hence $K^\bullet = D' \circ H^\bullet$ where D' is a subdigraph of D and D' is symmetric since $K^\bullet$ is an induced

subdigraph of K (by Lemma 3.3). By way of contradiction assume now that D' is not a core and let φ be a retraction from D' to a proper subdigraph J of D'. Define $\psi : V(D' \circ H^\bullet) \to V(J \circ H^\bullet)$ by $\psi((d,h)) = (\varphi(d), h)$. Suppose that $(d,h)(u,v) \in A(D' \circ H^\bullet)$. This implies that either $du \in A(D')$, or $d = u$ and $hv \in A(H^\bullet)$. If $du \in A(D')$ then $\varphi(d)\varphi(u) \in A(J)$ which implies that $\psi((d,h))\psi((u,v)) \in A(J \circ H^\bullet)$ or $\varphi(d) = \varphi(u)$, the latter being impossible since D' is symmetric. If $d = u$ and $hv \in A(H^\bullet)$ then $\psi((d,h))\psi((u,v)) \in A(J \circ H^\bullet)$. Hence ψ is a retraction which contradicts $K^\bullet$ being a core. Thus D' is a core. $\qquad\square$

We are now done introducing digraph products. In the next chapter we will see that the lexicographic product plays an integral role in being able to define the 'fractional chromatic number' in terms of homomorphisms.

Chapter 5

Fractional colorings of digraphs

In this chapter we look at the chromatic number of a digraph as the solution to an integer program in order to define the 'fractional chromatic number' for digraphs. We then find an alternate but equivalent definition using homomorphisms and the lexicographic product. We will also prove how this fractional chromatic number relates to other digraph invariants, namely the chromatic number and the circular chromatic number. Finally we show that the fractional version of Hedetniemi's Conjecture 4.3 is not true for digraphs.

5.1 Definition

Recall from the Chapter 2 that the directed analogue of the chromatic number of graphs that we are interested in is the '(acyclic) chromatic number' χ, namely the minimum number of acyclic sets needed to cover the vertex set of a given digraph. If we let Φ denote the set of acyclic subsets of the vertex set of a digraph D, it is easy to see that $\chi(D)$ is the optimum

value of the following integer linear program:

$$\min \sum_{A \in \Phi} x_A$$

$$\text{subject to} \quad \sum_{A \ni v} x_A = 1, \quad \text{for each } v \in V(D) \tag{5.1}$$

$$\text{and} \quad x_A \in \{0, 1\} \quad \text{for each } A \in \Phi.$$

This is similar to the integer linear program for the undirected case. Following the lead of the well-known undirected case, the *fractional (acyclic) chromatic number* $\chi_f(D)$ is the optimum value of the continuous relaxation of (5.1) above:

$$\min \sum_{A \in \Phi} x_A$$

$$\text{subject to} \quad \sum_{A \ni v} x_A = 1, \quad \text{for each } v \in V(D) \tag{5.2}$$

$$\text{and} \quad x_A \geq 0 \quad \text{for each } A \in \Phi.$$

Let D be a digraph and n a positive integer. A function f from $V(D)$ to the set $\binom{[n]}{k}$ of k-subsets from $[n]$ is a *k-tuple n-coloring* of D provided that for all $\beta \subseteq \binom{[n]}{k}$ with $\bigcap_{B \in \beta} B \neq \varnothing$, the union $\bigcup_{B \in \beta} f^{-1}(\{B\})$ is an acyclic subset of $V(D)$. For a moment let

$$\chi^*(D) = \inf\{n/k \mid D \text{ admits a } k\text{-tuple } n\text{-coloring}\}. \tag{5.3}$$

Notice that a trivial feasible solution to (5.2) is obtained by setting $x_{\{v\}} = 1$ for all $v \in V(D)$ and all other acyclic subsets are given weight zero. Also the objective function, $\sum_{A \in \Phi} x_A$, is bounded below by 1 and all the coefficients in (5.2) are rational. Thus many introductory linear programming books, see e.g. [8,31], verify that nothing is lost if we consider only rational feasible and optimal solutions to the linear program (5.2). We may assume that all the x_A have a common denominator k and here we will take n to be $n := \sum_{A \in \Phi} k x_A$. Hence for all $v \in V(D)$ we have $\sum_{A \ni v} k x_A = k$. This implies that there are n acyclic subsets of $V(D)$, with

repetition allowed, such that each vertex is in exactly k of these acyclic subsets. Label these n acyclic sets $A_1, A_2, ..., A_n$ and define $f : V(D) \to \binom{[n]}{k}$ by $f(v) = \{i \mid v \in A_i\}$. Suppose that $\beta \subseteq \binom{[n]}{k}$ with $\bigcap_{B \in \beta} B \neq \varnothing$. This implies that there exists $j \in [n]$ such that $j \in B$ for all $B \in \beta$. Hence $\bigcup_{B \in \beta} f^{-1}(\{B\}) \subseteq A_j$ and thus this union is acyclic. Therefore f is a k-tuple n-coloring of D. On the other hand assume that f is a k-tuple n-coloring of D, and, for $i \in [n]$, define $A_i = \{v \mid i \in f(v)\}$. Since f is a k-tuple n-coloring of D and $\bigcap_{v \in A_i} f(v) \supseteq \{i\}$, the union $\bigcup_{v \in A_i} f^{-1}(\{f(v)\}) \supseteq A_i$ is an acyclic subset of $V(D)$ (and thus so is A_i). Hence there are n such A_i's and each vertex of D is in exactly k A_i's. Thus we may set each $x_{A_i} = 1/k$ and all other acyclic subsets are given weight zero to give a feasible solution to the linear program (5.2) with value n/k.

Hence we first took an optimum solution to (5.2) with value n/k and built a k-tuple n-coloring from that solution. This implies that $\chi^* \leq \chi_f$. Then we took an n-tuple k-coloring and built a solution to (5.2) with value n/k giving the inequality in the other direction. Thus $\chi^* = \chi_f$ and since $\chi_f = n'/k'$ is rational implies that there is a k'-tuple n'-coloring of D, we need only the minimum rather than the infimum in (5.3). Therefore we have the following conclusion.

Theorem 5.1. *The fractional (acyclic) chromatic number $\chi_f(D)$ of a digraph D is the minimum rational number n/k such that D admits a k-tuple n-coloring.*

5.2 The lexicographic product and fractional colorings

For a given integer $k \geq 1$, we now define the *k-chromatic number* $\chi_k(D)$ of a digraph D to be the least integer n such that D admits a k-tuple n-coloring.

Theorem 5.2. *If D is a digraph and $k \geq 1$ an integer, then $\chi_k(D) = \chi(D \circ \overleftrightarrow{K}_k)$.*

Proof. First suppose that f is an n-coloring of $D \circ \overleftrightarrow{K}_k$. Define $g : V(D) \to \binom{[n]}{k}$ by $g(v) =$

$\{i \mid f(v,h) = i \text{ for some } h \in \overleftrightarrow{K}_k\}$. As f is a proper coloring we have $f(v,h_1) \neq f(v,h_2)$ when

$h_1 \neq h_2$. Thus $|g(v)| = k$ for all $v \in V(D)$. Assume that g is not a k-tuple n-coloring of

D. This implies that there exists a subset β of $\binom{[n]}{k}$ such that $\bigcap_{B \in \beta} B \neq \varnothing$ and $\bigcup_{B \in \beta} g^{-1}(B)$

contains a cycle, say $(v_1, v_2, ..., v_m)$, in D. Now $\bigcap_{B \in \beta} B \neq \varnothing$ implies that there exists $i \in [n]$

such that for all $v_j \in \bigcup_{B \in \beta} g^{-1}(B)$ there exists $h_j \in V(\overleftrightarrow{K}_k)$ such that $f(v_j, h_j) = i$. This

contradicts f being a proper coloring because $((v_1, h_1), (v_2, h_2), ..., (v_m, h_m))$ is a cycle in

$D \circ \overleftrightarrow{K}_k$ and a subset of $f^{-1}(i)$. Thus g is a k-tuple n-coloring of D. On the other hand,

assume that g is a k-tuple n-coloring of D. For each $d \in V(D)$, order the set $g(d)$ from least

to greatest and define $f : V(D \circ \overleftrightarrow{K}_k) \to [n]$ by $f(d, i) = i^{th}$ entry of $g(d)$. Suppose that

$((d_1, h_1), (d_2, h_2), ..., (d_m, h_m))$ is a cycle in $D \circ \overleftrightarrow{K}_k$ and a subset of $f^{-1}(j)$. This implies that

for all $\ell, q \in [m]$ we have $f(d_\ell, h_\ell) = f(d_q, h_q) = j$. Hence $d_\ell = d_q$ implies that $h_\ell = h_q$. Thus

using the definition of the lexicographic product we may conclude that $(d_1, d_2, ..., d_m)$ is a

cycle in D and a subset of $\bigcup_{\substack{B \in \binom{[n]}{k} \\ B \ni j}} g^{-1}(B)$, contradicting the fact that g is a k-tuple n-coloring.

Therefore f is an n-coloring of $D \circ \overleftrightarrow{K}_k$ and $\chi_k(D) = \chi(D \circ \overleftrightarrow{K}_k)$. $\qquad \square$

Corollary 5.3. *Any digraph D satisfies $\chi_f(D) = \min\{\chi(D \circ \overleftrightarrow{K}_k)/k \mid k = 1, 2, 3...\}$.*

Proof. It is clear from the preceding theorem that $\chi_f(D) = \inf\{\chi(D \circ \overleftrightarrow{K}_k)/k \mid k = 1, 2, 3...\}$

and since χ_f is the solution to a linear program with integer coefficients, χ_f is rational and

thus the infimum is attained. $\qquad \square$

Lemma 5.4. *If D and H are digraphs with $D \to H$, then $D \circ \overleftrightarrow{K}_n \to H \circ \overleftrightarrow{K}_n$ for every integer*

$n \geq 1$.

Proof. Suppose that $f : V(D) \to V(H)$ is a homomorphism. Define φ from $V(D \circ \overleftrightarrow{K}_n)$ to

$V(H \circ \overleftrightarrow{K}_n)$ by $\varphi((d, k)) := (f(d), k)$ and suppose that $(d_1, k_1)(d_2, k_2)$ is an arc of $D \circ \overleftrightarrow{K}_n$. We

have two cases to consider, depending on how this arc arose within $D \circ \overleftrightarrow{K}_n$.

Case 1: $d_1 = d_2$ and $k_1 k_2 \in A(\overleftrightarrow{K}_n)$. Here $(f(d_1), k_1)(f(d_2), k_2)$ is an arc in $H \circ \overleftrightarrow{K}_n$ by definition.

Case 2: $d_1 d_2 \in A(D)$. Here, either $f(d_1)f(d_2) \in A(H)$, which implies that $(f(d_1), k_1)(f(d_2), k_2) \in A(H \circ \overleftrightarrow{K}_n)$, or $f(d_1) = f(d_2)$. Thus either $(f(d_1), k_1) = (f(d_2), k_2)$ or $(f(d_1), k_1)(f(d_2), k_2) \in A(H \circ \overleftrightarrow{K}_n)$.

To confirm that φ is a homomorphism, it remains only to check that its fibers are acyclic. So suppose, by way of contradiction, that $\varphi^{-1}((h, i))$ contains a cycle $\vec{C}_m = ((d_1, i), (d_2, i), ..., (d_m, i))$. This implies that $(d_1, d_2, ..., d_m)$ is a cycle in D contained in $f^{-1}(h)$ which contradicts f being a homomorphism. $\qquad\square$

Using Corollary 5.3 and Lemma 5.4 we obtain the following result which gives us an important relationship between homomorphic digraphs and the fractional chromatic number.

Corollary 5.5. *If D and H are digraphs with $D \to H$, then $\chi_f(D) \leq \chi_f(H)$.*

5.3 Relationship with other digraph invariants

For graphs, the relationship between the fractional, circular, and regular chromatic numbers is well-known to be

$$\chi_f \leq \chi_c \leq \chi; \tag{5.4}$$

see e.g. [19, 28]. Hence one would hope that the analogous relationship holds for digraphs, and indeed it does. Recall from Section 2.2 that the directed graph $\vec{K}_{p/q}$, with $q < p$ (which we will assume for the remainder of this chapter), has vertex set $V(\vec{K}_{p/q}) = \{0, 1, 2.., p-1\}$ and arc set $A(\vec{K}_{p/q}) = \{ij \mid j - i \pmod{p} \in \{q, q+1, q+2, ..., p-1\}\}$; see also [3]. The circular chromatic number of a digraph D is $\chi_c(D) = \min\{p/q | D \to \vec{K}_{p/q}\}$. For digraphs, the second inequality in (5.4) was proved in [3]. Thus to complete the chain, it will be enough to show

that

$$\vec{K}_{p/q} \circ \overleftrightarrow{K}_q \to \overleftrightarrow{K}_p \tag{5.5}$$

and then apply Lemma 5.4. Indeed if $\chi_c(D) = p/q$ then D is homomorphic to $\vec{K}_{p/q}$ by the definition of χ_c and with (5.5), we obtain $D \circ \overleftrightarrow{K}_q \to \overleftrightarrow{K}_p$. Thus we'll have $\chi(D \circ \overleftrightarrow{K}_q) \le \chi(\overleftrightarrow{K}_p) = p$; this and Corollary 5.3 will establish that $\chi_f(D) \le p/q = \chi_c(D)$. Our next result fills in this missing piece (5.5).

Lemma 5.6. *For $q < p$, the product $\vec{K}_{p/q} \circ \overleftrightarrow{K}_q$ is homomorphic to $\overleftrightarrow{K}_p$.*

Proof. Define $f : V(\vec{K}_{p/q} \circ \overleftrightarrow{K}_q) \to V(\overleftrightarrow{K}_p)$ by $f(i,j) := i + j \pmod{p}$. Since the target digraph is complete, to show that f is a homomorphism it suffices to show that every fiber of f is acyclic. By way of contradiction suppose that there exists $k \in \{0, 1, ..., p-1\}$ such that $f^{-1}(k)$ contains (for some integer $n \ge 2$) an n-cycle $\vec{C}_n = ((i_0, j_0), (i_1, j_1), ..., (i_{n-1}, j_{n-1}))$. The definition of f then shows that

$$i_\alpha + j_\alpha \equiv i_\beta + j_\beta \equiv k \pmod{p} \text{ for all } \alpha, \beta \in \{0, 1, ..., n-1\}. \tag{5.6}$$

Notice that if $i_\alpha = i_\beta$ then $j_\alpha \equiv j_\beta \pmod{p}$ which implies that $j_\alpha = j_\beta$ because $0 \le j_\alpha, j_\beta \le q - 1 < p$. Similarly if $j_\alpha = j_\beta$ then $i_\alpha \equiv i_\beta \pmod{p}$ which implies that $i_\alpha = i_\beta$ since $0 \le i_\alpha, i_\beta \le p - 1$. Hence $i_\gamma \ne i_{\gamma+1}$ and $j_\gamma \ne j_{\gamma+1}$ for all $\gamma \in \{0, 1, 2, ..., n-1\}$ (where the subscripts are taken modulo n). Thus by the definition of the lexicographic product, $(i_0, i_1, i_2, ..., i_{n-1})$ is a cycle in $\vec{K}_{p/q}$. This implies, using the definition of $\vec{K}_{p/q}$, that $i_{\gamma+1} - i_\gamma \equiv \varphi_\gamma \pmod{p}$ where each $\varphi_\gamma \in \{q, q+1, ..., p-1\}$. Now (5.6), with $\alpha = \gamma$ and $\beta = \gamma + 1$, gives, again for each $\gamma \in \{0, 1, ..., n-1\}$, $i_\gamma + j_\gamma \equiv i_{\gamma+1} + j_{\gamma+1} \pmod{p}$. Hence $j_\gamma - j_{\gamma+1} \equiv \varphi_\gamma \pmod{p}$. Since $0 \le j_\gamma, j_{\gamma+1} \le q - 1$ and $\varphi_\gamma \in \{q, q+1, ..., p-1\}$ we may conclude that $j_{\gamma+1} > j_\gamma$ always holds. Hence we reach a contradiction because now $j_0 < j_1 < j_2 < \cdots < j_{n-1} < j_0$. Therefore f is a homomorphism as desired. $\qquad\square$

The conclusion of Lemma 5.6 together with our remarks preceding the statement of Lemma 5.6

now gives us the digraph analogue of (5.4).

Corollary 5.7. *Every digraph D satisfies* $\chi_f(D) \le \chi_c(D) \le \chi(D)$.

It is worth noting that unlike the circular chromatic number and the regular chromatic number, which satisfy the inequalities $\chi(D) - 1 < \chi_c(D) \le \chi(D)$ (see [3]), the fractional chromatic number can be arbitrarily less than the circular chromatic number. This is shown to be true in [7, 14] for graphs. Since χ_f, χ_c, and χ for digraphs are generalizations of the same parameters for graphs, the complete biorentations of the graphs in [7, 14] serve as examples in which the fractional chromatic number of a digraph is arbitrarily less than the chromatic number. We also note that Corollary 5.7 is sharp since $\chi_f(\overleftrightarrow{K}_n) = \chi_c(\overleftrightarrow{K}_n) = \chi(\overleftrightarrow{K}_n) = n$. It is natural at this point to be curious about which digraphs have equality amongst all three invariants and which do not. In order to begin answering these questions we need a few more lemmas.

Recall that for a digraph D, the values $\chi(D)$ and $\chi_f(D)$ may be cast as optimum values of linear programming problems; see (5.1) and (5.2). The linear program (5.2) however is not in standard form so consider the following LP:

$$
\begin{aligned}
&\min \sum_{A \in \Phi} x_A \\
\text{subject to} \quad &\sum_{A \ni v} x_A \ge 1, \quad \text{for each } v \in V(D) \quad\quad\quad (5.7)\\
\text{and} \quad &x_A \ge 0 \quad \text{for each } A \in \Phi.
\end{aligned}
$$

We claim that (5.2) and (5.7) are equivalent linear programs. Indeed any feasible solution of (5.2) is a feasible solution of (5.7). We now show that any feasible solution $\mathbf{x}$ to (5.7) can be perturbed into a feasible solution $\mathbf{x}''$ of (5.2) with the same objective function value. Consider a vertex $v \in V(D)$ such that $\sum_{v \ni A} x_A = b > 1$. Let $A_1, A_2, ..., A_t$ be a list of every acyclic subset of $V(D)$ containing v such that $x_{A_i} > 0$. Choose values $a_1, a_2, ..., a_t$ such that $a_i \le x_{A_i}$ and

$\sum_{i=1}^{t} a_i = b - 1$. Now define

$$
x'_A := \begin{cases}
x_A - a_i & \text{if } A = A_i, \\
x_A + a_i & \text{if } A = A_i - v \text{ and } A \neq \varnothing, \\
x_A & \text{otherwise.}
\end{cases}
$$

We now see that $\sum_{v \ni A} x'_A = \sum_{v \ni A} x_A - \sum_{i=1}^{t} a_i = 1$ and $\sum_{A \in \Phi} x_A = \sum_{A \in \Phi} x'_A$. Since D is finite we can inductively define $\mathbf{x}''$ such that it is a feasible solution to (5.2) and $\sum_{A \in \Phi} x_A = \sum_{A \in \Phi} x''_A$. Therefore (5.2) and (5.7) are equivalent.

Mimicking the well-known graph analogue (see e.g. [11, 28]) we now define the *fractional diclique number* ω_f of D to be the value of an optimum solution to the LP dual of (5.7) which is:

$$
\begin{aligned}
\max \quad & \sum_{v \in V(D)} y_v \\
\text{subject to} \quad & \sum_{v \in A} y_v \leq 1, \quad \text{for each } A \in \Phi \\
\text{and} \quad & y_v \geq 0 \quad \text{for each } v \in V(D).
\end{aligned}
\tag{5.8}
$$

This linear program is useful for obtaining lower bounds on $\chi_f(D)$ since LP theory (see e.g. [8, 31]) tells us that any feasible solution $\mathbf{y}$ to (5.8) satisfies $\sum_{v \in V(D)} y_v \leq \chi_f(D)$.

Lemma 5.8. *If A_D and A_H are acyclic subsets of $V(D)$ and $V(H)$ respectively, then $A_D \times A_H$ is an acyclic subset of $V(D \circ H)$.*

Proof. We first note that $D \circ H[A_D \times A_H] \cong D[A_D] \circ H[A_H]$ (This isomorphism is well-known and easy to check, in both the graph and digraph settings.). Thus, using Lemma 4.5 (which shows that the projection $D[A_D] \circ H[A_H] \to D[A_D]$ is a homomorphism) we see that $D \circ H[A_D \times A_H]$ is homomorphic to $D[A_D]$, which implies that $\chi(D \circ H[A_D \times A_H]) \leq \chi(D[A_D]) = 1$. Therefore $A_D \times A_H$ is acyclic. $\qquad\square$

Lemma 5.9. *If D and H are digraphs, then $\chi(D \circ H) \leq \chi(D) \cdot \chi(H)$.*

Proof. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are optimum solutions to the primal integer program (5.1) for D and H respectively. Define $\mathbf{x}$ by $x_A \coloneqq u_{A_D} \cdot v_{A_H}$ when $A = A_D \times A_H$, where A_D and A_H are acyclic subsets of $V(D)$ and $V(H)$ respectively, and $x_A \coloneqq 0$ otherwise. The preceding lemma ensures that $A_D \times A_H$ is an acyclic subset of $V(D \circ H)$. Hence we have for all $(d, h) \in V(D \circ H)$ that

$$\sum_{A \ni (d,h)} x_A = \sum_{A_D \ni d, A_H \ni h} x_{A_D \times A_H} = \sum_{A_D \ni d, A_H \ni h} u_{A_D} \cdot v_{A_H} = \sum_{A_D \ni d} u_{A_D} \cdot \sum_{A_H \ni h} v_{A_H} = 1,$$

since $\mathbf{u}$ and $\mathbf{v}$ are feasible solutions for their respective instances of (5.1). Thus $\mathbf{x}$ is feasible for LP (5.1) on $D \circ H$ and the value at $\mathbf{x}$ is $\chi(D) \cdot \chi(H)$. Therefore $\chi(D \circ H) \leq \chi(D) \cdot \chi(H)$. $\square$

Lemma 5.10. *If D and H are digraphs, then $\chi_f(D \circ H) = \chi_f(D) \cdot \chi_f(H)$.*

Proof. The proof that $\chi_f(D \circ H) \leq \chi_f(D) \cdot \chi_f(H)$ follows essentially verbatim the proof of Lemma 5.9, with the LP (5.2) in place of (5.1).

On the other hand assume that $\chi_f(D \circ H) = n/k$, which implies that

$$n \geq \chi_k(D \circ H) = \chi((D \circ H) \circ \overleftrightarrow{K}_k) = \chi(D \circ (H \circ \overleftrightarrow{K}_k))$$

using Theorem 5.1, Theorem 5.2, and the fact that the lexicographic product is associative (see, e.g., [14]). If we let $t = \chi(H \circ \overleftrightarrow{K}_k)$ then Corollary 5.3 implies that $\chi_f(H) \leq t/k$ and Theorem 4.7 implies that $n \geq \chi(D \circ (H \circ \overleftrightarrow{K}_k)) = \chi(D \circ \overleftrightarrow{K}_t)$. Hence Corollary 5.3 yields $\chi_f(D) \leq n/t$ which gives us $\chi_f(D \circ H) = n/k = n/t \cdot t/k \geq \chi_f(D) \cdot \chi_f(H)$. Putting the two bounds together yields the desired result. $\square$

It is interesting to note that even though the lexicographic product is not commutative, Lemma 5.10 implies that $\chi_f(D \circ H) = \chi_f(H \circ D)$ for any given digraphs D, H. When the same relation holds for χ in place of χ_f, we have our answer to the question of when (5.4) is sharp.

Theorem 5.11. $\chi_f(D) = \chi_c(D) = \chi(D)$ *if and only if* $\chi(D \circ H) = \chi(D) \cdot \chi(H)$ *for all digraphs* H.

Proof. In this proof, it will help to keep in mind the basic inequality $\chi_f \leq \chi_c \leq \chi$ from Corollary 5.7. Let H be a given digraph with $n \coloneqq \chi(H)$. If $\chi(D) = \chi_f(D)$ then Theorem 4.7 and Lemma 5.10 imply that $\chi(D \circ H) = \chi(D \circ \overset{\leftrightarrow}{K}_n) \geq \chi_f(D \circ \overset{\leftrightarrow}{K}_n) = \chi_f(D) \cdot \chi_f(\overset{\leftrightarrow}{K}_n) = \chi(D)n = \chi(D) \cdot \chi(H)$. Thus Lemma 5.9 yields $\chi(D \circ H) = \chi(D) \cdot \chi(H)$.

Now suppose that $\chi(D \circ H) = \chi(D) \cdot \chi(H)$ for all digraphs H. Corollary 5.3 implies that there exists a positive integer m such that $\chi_f(D) = \chi(D \circ \overset{\leftrightarrow}{K}_m)/m$. Hence $\chi_f(D) = \chi(D) \cdot \chi(\overset{\leftrightarrow}{K}_m)/m = \chi(D)$. $\qquad\square$

5.4 Fractional version of Hedetniemi's Conjecture

We saw in Section 4.3 that the lexicographic product behaves similarly for graphs and digraphs. However this is not always the case for the direct product, defined on p. 23. Consider the following result, a fractional version of Hedetniemi's Conjecture (4.3) for graphs.

Theorem 5.12. *For any graphs G and H, the fractional chromatic number χ_f satisfies $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.*

This theorem was recently proved by Zhu [40].

We now prove two propositions which together show that the fractional version of Hedetniemi's Conjecture (4.3) for digraphs is not true.

Proposition 5.13. $\chi_f(\vec{C}_n) = 1 + 1/(n-1)$.

Proof. Corollary 5.7 and [3] imply that $\chi_f(\vec{C}_n) \leq \chi_c(\vec{C}_n) = 1 + 1/(n-1)$. For the other direction, we define the canonical fractional clique **y** (i.e. feasible solution to LP (5.8)) by

$\mathbf{y}(i) = 1/(n-1)$ for all $i \in V(\vec{C}_n)$. LP duality gives $\chi_f(\vec{C}_n) \geq \omega_f(\vec{C}_n)$, but the right side here is at least the objective function of (5.8) evaluated at $\mathbf{y}$, namely $1 + 1/(n-1)$. Thus indeed $\chi_f(\vec{C}_n) \geq 1 + 1/(n+1)$, and the result is proved. $\qquad\square$

Proposition 5.14. *For any two coprime integers m and n, $\vec{C}_m \times \vec{C}_n \cong \vec{C}_{mn}$.*

Proof. Let $V(\vec{C}_m) = \{0, 1, \ldots, m-1\}$ and $V(\vec{C}_n) = \{0, 1, \ldots, n-1\}$, both with the canonical arc set. The definition of the direct product implies that $(i, j)(s, t) \in A(\vec{C}_m \times \vec{C}_n)$ if and only if $s - i \equiv 1 \pmod{m}$ and $t - j \equiv 1 \pmod{n}$. Consider an algorithm starting with the vertex $(0, 0)$ and adding 1 to each coordinate at each step (with the addition taken modulo m and n respectively). Notice that each step in the algorithm represents an edge in $\vec{C}_m \times \vec{C}_n$. Hence if the algorithm reaches every vertex before returning back to $(0, 0)$, we'll have shown that $\vec{C}_m \times \vec{C}_n$ contains a spanning subdigraph isomorphic to $\vec{C}_{mn}$. Suppose that the algorithm reaches $(0, 0)$ on the kth step. This implies that both m and n divide k, and since they are coprime, $mn | k$, so that $k \geq mn.$. Since there are mn ordered pairs, k is at most mn. Thus $k = mn$, and the algorithm returns to $(0, 0)$ on the mn^{th} step. As noted above, we now see that $\vec{C}_m \times \vec{C}_n$ contains a (spanning) copy of $\vec{C}_{mn}$. To see that in fact $\vec{C}_m \times \vec{C}_n \cong \vec{C}_{mn}$, notice that each vertex of $\vec{C}_m \times \vec{C}_n$ has outdegree 1, implying that $|A(\vec{C}_m \times \vec{C}_n)| = mn = |\vec{C}_{mn}|$, so that the containment $\vec{C}_m \times \vec{C}_n \supseteq \vec{C}_{mn}$ is not proper. $\qquad\square$

Using Proposition 5.13 we see that $\chi_f(\vec{C}_{mn}) = 1 + 1/(mn - 1)$, $\chi_f(\vec{C}_m) = 1 + 1/(m-1)$, and $\chi_f(\vec{C}_n) = 1 + 1/(n-1)$. Putting these facts together with Proposition 5.14, we see that $\chi_f(\vec{C}_m \times \vec{C}_n)$ is strictly less than both $\chi_f(\vec{C}_m)$ and $\chi_f(\vec{C}_n)$. Therefore the generalization to digraphs of the fractional version (Theorem 5.12) of Hedetniemi's Conjecture is not true.

We now have a working definition of the fractional chromatic number χ_f of a digraph. We have seen that χ_f is intimately tied to the lexicographic product and homomorphisms. Also we now understand how it relates to other digraph parameters, namely the chromatic number and the circular chromatic number. Though there is more to explore about χ_f, we choose to move on to Chapter 6 and the main results of this dissertation.

Chapter 6

New constructions and cores

This chapter finally brings us to our main results. First we construct highly chromatic digraphs without short cycles in (the proof of) Theorem 6.1. Furthermore we will see that, in fact, the digraphs constructed are cores. Then in Section 6.2 we prove the deepest result (Theorem 6.9) of this dissertation: for any pair n, k of integers both exceeding one, we construct uniquely n-colorable digraphs with digirth equal to k.

6.1 Highly chromatic digraphs without short cycles

In this section, we *construct* digraphs with arbitrarily large digirth and chromatic number. In fact, the construction strengthens the probabilistic result in [3] because it produces a digraph with digirth k and chromatic number n for each pair k, n of integers exceeding one. It is also of interest that unlike the analogous graph constructions in [20], [21], and [24], our construction is primitively recursive in n. Additionally Theorem 6.3 establishes that these digraphs belong to the special class of cores.

Theorem 6.1. *For any given integers k and n exceeding one, there exists an n-chromatic digraph D with $\vec{g}(D) = k$.*

Proof. For $n = 2$, the directed k-cycle will suffice. For $n \geq 2$, we proceed by induction on n and suppose that we have already constructed a digraph D_n with chromatic number n, digirth k, and $V(D_n) = \{d_1, d_2, ..., d_m\}$. We now define D_{n+1}.

For each $i \in [m]$ let D_n^i be a digraph with vertex set $V(D_n^i) = \{(d_1, i), (d_2, i), ..., (d_m, i)\}$ which is isomorphic to D_n in the natural way. Next construct m directed paths P_{d_i}, for $1 \leq i \leq m$, each of length $k - 2$, with vertex sets $\{(d_i, p_1), (d_i, p_2), ...(d_i, p_{k-1})\}$ and arc sets $A(P_{d_i}) := \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \mid j \in [k - 2]\}$. Now define m digraphs $H(n, i)$, for $1 \leq i \leq m$, in the following manner. The vertex sets are $V(H(n, i)) := V(D_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(H(n, i)) := A(D_n^i) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d, i)(d_i, p_1)} \mid d \in V(D_n) \right\} \cup \left\{ \overrightarrow{(d_i, p_{k-1})(d, i)} \mid d \in V(D_n) \right\}.$$

Finally, we define D_{n+1} to be the digraph with

$$V(D_{n+1}) := \bigcup_{i=1}^{m} V(H(n, i))$$

and

$$A(D_{n+1}) := \bigcup_{i=1}^{m} A(H(n, i)) \cup \left\{ \overrightarrow{(d_i, p_\ell)(d_j, p_h)} \mid d_i d_j \in A(D_n) \text{ and } \ell, h \in [k - 1] \right\}.$$

In order to illustrate this construction, we include in Figure 6.1 a diagram of D_3 with $k = 3$. All double-tailed arrows represent numerous arcs in the diagram. The double-tailed arrows running horizontally indicate an arc from every vertex at the tail to every vertex at the head. The double-tailed arrows running up and down rindicate an arc from every vertex at the tail to one vertex at the head and from one vertex at the tail to every vertex at the head respectively. The three remaining diagrams in this section follow similar schematics.
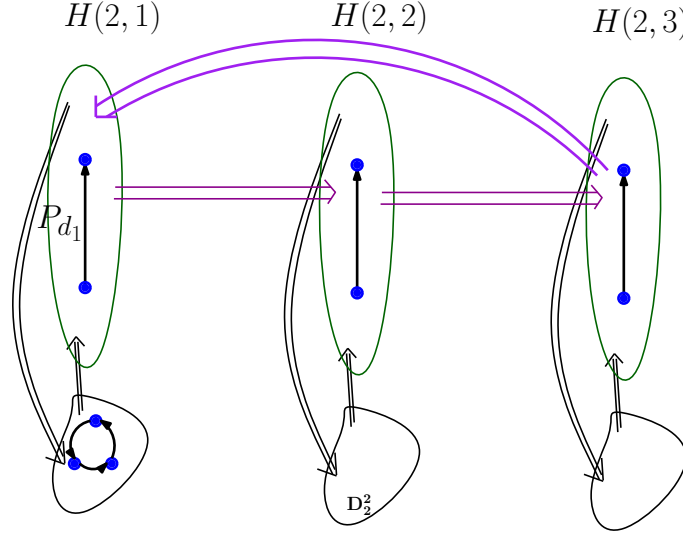
Figure 6.1: D_3 with $k = 3$

We first show that the digirth of D_{n+1} is k. Since the digirth of each $H(n,i)$ is k and there are no arcs from D_n^i to D_n^j for $j \neq i$, any cycle containing a vertex from some D_n^i has length exceeding $k-1$. Hence it suffices to show that the subdigraph Σ of D_{n+1} induced by the vertices of the P_{d_i}'s has digirth exceeding $k-1$. Because there exists an acyclic homomorphism $\psi : V(\Sigma) \to V(D_n)$ (sending every vertex in P_{d_i} to d_i), we have $\vec{g}(\Sigma) \geq \vec{g}(D_n) = k$. Therefore $\vec{g}(D_{n+1}) = k$.

It is clear that $\chi(D_{n+1}) \geq \chi(D_n) = n$ since D_n is isomorphic to a subdigraph of D_{n+1}. If D_{n+1} is n-chromatic, then there exists an acyclic homomorphism $\sigma : V(D_{n+1}) \to V(\overleftrightarrow{K}_n)$. To set up the contradiction we are about to derive, fix a σ 'color' $\alpha \in V(\overleftrightarrow{K}_n)$. Since D_n^i is isomorphic to D_n, the function σ maps $V(D_n^i)$ onto $V(\overleftrightarrow{K}_n)$ for all $i \in [m]$. Every vertex in D_n^i is in a cycle with the vertices of P_{d_i}, which implies that there exists a vertex $v_i \in P_{d_i}$ such that $\sigma(v_i) \neq \alpha$. The subdigraph Λ of D_{n+1} induced by $\{v_1, v_2, .., v_m\}$ is isomorphic to D_n. This contradicts the fact that D_n has chromatic number n since σ, now seen to avoid α on $V(\Lambda)$, effectively maps $V(\Lambda)$ to $V(\overleftrightarrow{K}_{n-1})$ acyclically. Thus $\chi(D_{n+1}) \geq n + 1$. We now show that $\chi(D_{n+1}) = n + 1$ by giving an acyclic homomorphism from D_{n+1} to $\overleftrightarrow{K}_{n+1}$. Let ζ be an acyclic homomorphism from D_n to $\overleftrightarrow{K}_n$. Define a mapping $\phi : V(D_{n+1}) \to V(\overleftrightarrow{K}_{n+1})$ as follows. For

vertices $(d_j, i) \in V(D_n^i)$, let

$$\phi((d_j, i)) = \begin{cases} \zeta(d_j) & \text{if } \zeta(d_j) \neq \zeta(d_i), \\ n+1 & \text{if } \zeta(d_j) = \zeta(d_i). \end{cases}$$

For vertices $(d_i, p) \in V(P_{d_i})$, define $\phi((d_i, p)) = \zeta(d_i)$.

As the target digraph of ϕ is complete, to show that ϕ is an acyclic homomorphism, it will suffice to show that each fiber of ϕ is acyclic. The fibers of $\phi|_{V(D_n^i)}$ are acyclic for all $i \in [m]$ because they are identical, up to relabeling, to the fibers of ζ. This implies that the fibers of $\phi|_{V(H(n,i))}$ are acyclic since $\phi(V(P_{d_i})) \cap \phi(V(D_n^i)) = \varnothing$. Hence it suffices to show that the restriction of ϕ to $\Sigma \left(= D_{n+1} \left[\bigcup_{i=1}^{m} V(P_{d_i}) \right] \right)$ is an acyclic homomorphism. Let $\psi : V(\Sigma) \to V(D_n)$ be defined as above and notice that $\phi|_{V(\Sigma)} = \zeta \circ \psi$ since $\phi((d_i, p)) = \zeta(d_i)$ for all vertices $(d_i, p) \in V(P_{d_i})$. As ζ and ψ are acyclic homomorphisms, so too is their composition $\phi|_{V(\Sigma)}$. Therefore ϕ is an acyclic homomorphism which finally implies that $\chi(D_{n+1}) = n+1$. $\square$

The preceding theorem (and proof) will also appear in [29]. We remark here on the number of vertices in the D_n's. We see from the construction that we have the recurrence relation:

$$|V(D_n)| = |V(D_{n-1})|^2 + (k-1) \cdot |V(D_{n-1})| \text{ with } |V(D_2)| = k.$$

It is easy to confirm that $|V(D_n)| = 2^{2^{n-1}} - 1$ for $k = 3$. For a general k we do not have a closed form. However we may inductively argue that $|V(D_n)|$ is $O\left(2^{2^{n-3}} \cdot k^{2^{n-2}} \right)$ after observing that $|V(D_2)| = k$ and $|V(D_3)| = 2k^2 - k$.

We need the next result as a tool for strengthening Theorem 6.1.

Lemma 6.2. *For $k = \vec{g}(D_n)$, every arc of D_n is in a directed k-cycle.*

Proof. We notice that the assertion is true for $n = 2$ because $D_2 \cong \vec{C}_k$ and proceed by induction. Next assume its truth for D_n and let uv be an arc in D_{n+1}. If, for an $i \in [m]$, uv is an arc in

D_n^i, which is isomorphic to D_n, then we may use the inductive hypothesis to see that uv is in a k-cycle. Another easy case is when uv is an arc of some P_{d_i} since uv is in a k-cycle for all $i \in [m]$ by our construction. Similarly, for all $i \in [m]$, our construction implies that uv is in a k-cycle when either $u \in V(D_n^i)$ and $v = (d_i, p_1)$ or $v \in V(D_n^i)$ and $u = (d_i, p_{k-1})$. The last case to inspect is when $u \in V(P_{d_i})$ and $v \in V(P_{d_j})$ for $i, j \in [m]$ with $i \neq j$. In this case, by our construction, uv is an arc in D_{n+1} if and only if $d_i d_j$ is an arc in D_n. Thus $d_i d_j$ is in a k-cycle of D_n by the induction hypothesis. Finally this in turn implies that uv is in a k-cycle of D_{n+1} and the proof is complete. $\qquad\square$

We now are prepared to say something substantially stronger about the digraphs constructed in the proof of Theorem 6.1: they are cores. This suggests that there is a sort of minimality to this construction. For the proof, we need to recall a definition. A digraph D is *strongly connected* if for every pair u, v of distinct vertices in D, there exists a directed walk from u to v and from v to u, i.e., if every vertex of D is reachable from every other vertex in D. It is straightforward to see that the digraphs constructed for proving Theorem 6.1 are strongly connected.

Theorem 6.3. *For any given integers $n > 1$ and $k > 2$ the digraph D_n constructed in Theorem 6.1 is a core with $\vec{g}(D_n) = k$.*

Proof. It is clear that D_2 is a core since $D_2 \cong \vec{C}_k$ so that we may proceed by induction. Assume that D_n is a core with $n \geq 2$ and let $\varphi : V(D_{n+1}) \to V(D_{n+1})$ be a retraction (defined on p. 16). Define Γ to be the image of φ and thus our goal is to show that $\Gamma = V(D_{n+1})$. Since φ is a retraction and Γ induces a subdigraph of D_{n+1}, the function φ must map k-cycles (i.e. shortest cycles) to k-cycles. Thus

$$\text{if two vertices } u, v \text{ are in the same } k\text{-cycle then } \varphi(u) \neq \varphi(v). \tag{6.1}$$

Thus Lemma 6.2 implies that all arcs uv of D_{n+1} satisfy $\varphi(u) \neq \varphi(v)$. In other words φ is a

non-contracting homomorphism (defined on p. 7).

We first proceed to show that for the subdigraph $\Sigma \left(= D_{n+1} \left[\bigcup_{i=1}^{m} V(P_{d_i}) \right] \right)$ of D_{n+1}, as defined in the proof of Theorem 6.1, the image of φ restricted to $V(\Sigma)$ is contained in $V(\Sigma)$. Let $u \in V(P_{d_i})$ and $v \in V(D_n^j)$, for some i, j, and we will show that $\varphi(u) \neq v$. This and the fact that D_n is strongly connected will suffice to show that $\varphi(V(\Sigma)) \subseteq V(\Sigma)$, for we can repeat our argument below as necessary to force every such $\varphi(u)$ into $V(\Sigma)$. If $i = j$, then the construction of D_{n+1} puts u and v together in a k-cycle and hence they cannot be in the same fiber of φ. This proves that $\varphi(u) \neq v$ for otherwise, with φ being the identity on Γ, we'd have $\varphi(u) = \varphi \circ \varphi(u) = \varphi(v)$, contradicting (6.1). Notice that there exists a directed path from d_i to d_j for all $i, j \in [m]$ because D_n is strongly connected. For the case $i \neq j$, we proceed by induction on the distance, s, from d_i to d_j in D_n. Assume that this distance is $s+1$ and that for every $r \in [m]$ with $0 \leq \overrightarrow{\text{dist}(d_i, d_r)} \leq s$ we have $\varphi(u) \neq z$ for all $z \in V(D_n^r)$. By assumption there is a path $P = (d_i, d_{i+1}, ..., d_{i+s}, d_j)$ in D_n which by our construction implies that $u(d_{i+1}, p_\ell)$ is an arc in D_{n+1} for all $\ell \in [k-1]$. Hence the induction hypothesis implies that $\varphi(u) \neq \varphi((d_{i+1}, p_\ell))$ for all $\ell \in [k-1]$. Thus if $\varphi(u) = v$ then $v\varphi(d_{i+1}, p_\ell) \in A(D_{n+1})$ for all $\ell \in [k-1]$ because φ is a non-contracting homomorphism. However the induction hypothesis again implies that $\varphi(d_{i+1}, p_\ell) \notin V(D_n^j)$ which forces $\varphi(d_{i+1}, p_\ell)$ to be (d_j, p_1) for every $\ell \in [k-1]$. This cannot happen because $k > 2$ and (d_{i+1}, p_ℓ) and (d_{i+1}, p_t) are in a k-cycle together for all $\ell, t \in [k-1]$ with $\ell \neq t$. Therefore the restriction of φ to $V(\Sigma)$ is indeed contained in $V(\Sigma)$.

The next step is to show that $D_{n+1}[\Gamma]$ cannot be a subdigraph of $D_{n+1} - H(n, i)$ for any $i \in [m]$. By way of contradiction we assume that there is an i such that $D_{n+1}[\Gamma]$ is a subdigraph of $D_{n+1} - H(n, i)$. Choose exactly one vertex v_j from each P_{d_j} and define Λ to be the subdigraph of D_{n+1} induced by the v_j's which by our construction is isomorphic to D_n. Define $\psi : V(\Sigma) \rightarrow V(\Lambda)$ by $\psi(d_\ell, p_s) = v_\ell$ for all $\ell \in [m]$ and $s \in [k-1]$. Consider the homomorphism $\psi|_\zeta \circ \varphi|_{V(\Lambda)} : V(\Lambda) \rightarrow V(\Lambda)$ where $\zeta := \text{Im}(\varphi|_{V(\Lambda)})$. (Note that these restrictions compose because ψ is defined on $\zeta \subset \text{Im}(\varphi|_{V(\Sigma)}) \subseteq V(\Sigma)$.) Since we're under the assumption that the

image of φ is contained in $V(D_{n+1} - H(n,i))$, there exists an $\alpha \in [m]$ such that $\varphi(v_\alpha) \neq v_\alpha$. Thus v_α is not in the image of $\psi|_\zeta \circ \varphi|_{V(\Lambda)}$ which implies that $\psi|_\zeta \circ \varphi|_{V(\Lambda)}$ is not a bijection. Thus by Lemma 3.1 the digraph Λ is not a core. This contradicts the induction hypothesis that D_n is a core because Λ is isomorphic to D_n. Therefore $D_{n+1}[\Gamma]$ cannot be a subdigraph of $D_{n+1} - H(n,i)$ for any $i \in [m]$.

We now show that

$$\text{for all } i \in [m] \text{ there exists a } j \in [k-1] \text{ such that } \varphi((d_i, p_j)) = (d_i, p_j) \in \Gamma. \qquad (6.2)$$

By way of contradiction assume that Γ is contained in $V(D_{n+1} - P_{d_i})$. This follows from the negation of (6.2) because the definition of retraction implies that

$$\text{for any vertices } u, v \in V(D_{n+1}), \text{ if } \varphi(u) = v \text{ then } \varphi(v) = v.$$

Notice that for all $\alpha \in V(D_n^i)$ the arc $\varphi(\alpha)\varphi(d_i, p_1)$ is in $D_{n+1}[\Gamma]$ because $\alpha(d_i, p_1)$ is an arc in D_{n+1}. However, in the construction of D_{n+1}, the only arcs from D_n^i to Σ are those from D_n^i to (d_i, p_1). Thus $\varphi(\alpha) \in V(D_{n+1} - H(n,i))$ for every $\alpha \in V(D_n^i)$. However this and the fact that φ is a retraction imply that Γ is contained in $V(D_{n+1} - H(n,i))$ which contradicts the preceding paragraph. Thus we've established (6.2). This fact will now allow us to show that for every $i \in [m]$ the vertices (d_i, p_1) and (d_i, p_{k-1}) are sent to the same P_d by φ. We again proceed by way of contradiction and assume that for two distinct vertices u_1, u_{k-1} of D_n, we have $\varphi(d_i, p_1) \in V(P_{u_1})$ and $\varphi(d_i, p_{k-1}) \in V(P_{u_{k-1}})$. Considering an $\alpha \in V(D_n^i)$ we see that both $\varphi(d_i, p_{k-1})\varphi(\alpha)$ and $\varphi(\alpha)\varphi(d_i, p_1)$ must be in the arc set of $D_{n+1}[\Gamma]$ since φ is a non-contracting homomorphism. The preceding sentence and our construction of D_{n+1} thus imply that $\varphi(\alpha) \in V(\Sigma)$, say $\varphi(\alpha) \in V(P_{u_0})$, because $u_1 \neq u_{k-1}$. Similarly $\big(\varphi(\alpha), \varphi(d_i, p_1), \varphi(d_i, p_2), ..., \varphi(d_i, p_{k-1})\big)$ is a k–cycle in $D_{n+1}[\Gamma]$. This implies that $\big(u_0, u_1, ..., u_{k-1}\big)$ is a cycle in D_n. We may thus assume that the u_j's are distinct for otherwise $\big(u_0, u_1, ..., u_{k-1}\big)$ would contain a cycle of length less than k. Now (6.2) shows that there

exists $j \in [k-1]$ such that $u_j = d_i$. If we let $\ell = j - 1 \pmod{k}$, then the preceding sentence

implies that $u_\ell d_i \in A(D_n)$. Thus for all $r, s \in [k-1]$ the arc $(u_\ell, p_r)(d_i, p_s)$ is in D_{n+1} and

(6.2) implies that $\varphi(d_i, p_\ell) = (u_\ell, p_t)$ for some $t \in [k-1]$. Hence (u_ℓ, p_t) is in the image of φ

implying that $\varphi((u_\ell, p_t)) = (u_\ell, p_t)$ because φ is a retraction (φ is the identity on its image).

This contradicts the fact that φ is a non-contracting homomorphism. Thus for every $i \in [m]$

the vertices (d_i, p_1) and (d_i, p_{k-1}) are indeed sent to the same P_d by φ.

We next show that in fact for every $i \in [m]$ all the vertices of P_{d_i} are sent to the same

P_d by φ for some $d \in V(D_n)$. By way of contradiction assume that some $s \in [k-1]$ and

$\ell \in \{2, 3, ..., k-2\}$ satisfy $\varphi(d_i, p_\ell) = (v, p_s)$ while $\varphi(d_i, p_1) \in V(P_u)$, where u and v are distinct

vertices of D_n. The preceding paragraph implies that (d_i, p_{k-1}) is sent to P_u as well. As P_{d_i} is

a directed path of length $k - 2$, the image under φ of $V(P_{d_i})$ induces a directed path of length

not exceeding $k - 2$. However since $\varphi(d_i, p_1), \varphi(d_i, p_{k-1}) \in V(P_u)$ and $u \neq v$, the preceding

sentence implies that there is a cycle in D_n containing u and v which has length less than k.

This contradiction lets us deduce that for every $i \in [m]$ all the vertices of P_{d_i} are sent to P_d by

φ for some $d \in V(D_n)$. This implies that the restriction of φ to $V(\Sigma)$ is the identity because of

(6.2). For $u \in V(D_n^i)$, the image $\varphi(u)$ lies in neither D_n^j, for $j \neq i$, nor P_{d_i} because we already

know that φ fixes P_{d_i}. Also, $\varphi(u) \in P_{d_j}$ with $d_j \neq d_i$ implies that $d_j d_i$ and $d_i d_j$ are arcs in D_n

because $u(d_i, p_1)$ and $(d_i, p_{k-1})u$ are arcs in D_{n+1}. Hence we see that for all $u \in V(D_n^i)$ the

image $\varphi(u)$ lies in D_n^i. This implies that the restriction of φ to each $V(D_n^i)$ is a retraction of

D_n^i to itself. Since each D_n^i is isomorphic to D_n, a core by induction, each D_n^i is also a core.

Thus, the restriction of φ to each $V(D_n^i)$, for $i \in [m]$, is the identity. Therefore, we've finally

established that $\Gamma = V(D_{n+1})$ and hence by induction reached the conclusion that the D_n's

are cores. $\qquad \square$

6.2 Uniquely *n*-colorable digraphs without short cycles

We now reach the section containing the main result of this dissertation. The proof of Theorem 6.9 constructs uniquely n-colorable digraphs with digirth k for any pair n, k of suitable integers. This result is a constructive version of the probabilistic proof appearing in [15] and is analogous to the undirected construction appearing in [38]. For the proof of Theorem 6.9 we first need to prove a few lemmas and to construct a few other digraphs related to the D_n of Section 6.1. The first of these digraphs, denoted B_n, is a spanning subdigraph of D_n. We will define B_n inductively and start by setting B_2 to be the path of length $k - 1$. We now define B_{n+1} from B_n. Suppose that $V(B_n) = \{d_1, d_2, ..., d_m\} = V(D_n)$ and set $V(B_{n+1}) = V(D_{n+1})$. For $i \in [m]$ let B_n^i be B_n tagged with an i. Now define m digraphs $F(n, i)$, for $1 \le i \le m$, in the following manner. The vertex sets are $V(F(n, i)) := V(B_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(F(n, i)) := A(B_n^i) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d, i)(d_i, p_1)} \mid d \in V(D_n) = V(B_n) \right\}.$$

Finally, we define B_{n+1} to be the digraph with

$$V(B_{n+1}) := \bigcup_{i=1}^{m} V(F(n, i))$$

and

$$A(B_{n+1}) := \bigcup_{i=1}^{m} A(F(n, i)) \cup \left\{ \overrightarrow{(d_i, p_\ell)(d_j, p_h)} \mid d_i d_j \in A(B_n) \text{ and } \ell, h \in [k-1] \right\}.$$

It may be helpful for the reader to view Figure 6.2 for an example of this construction.

Lemma 6.4. B_n *is acyclic for all* n.

Proof. We proceed by induction and notice first that B_2 is acyclic as it is just a directed path. Now, assuming that B_n is acyclic, we see that each subdigraph B_n^i of B_{n+1} is acyclic by our induction hypothesis. Thus there does not exist a cycle in B_{n+1} containing a vertex from any B_n^i since there are no arcs from any vertex of the P_{d_i}'s to any vertex of the B_n^i's. Since the

$F(2,1)$ $\qquad\qquad$ $F(2,2)$ $\qquad\qquad$ $F(2,3)$
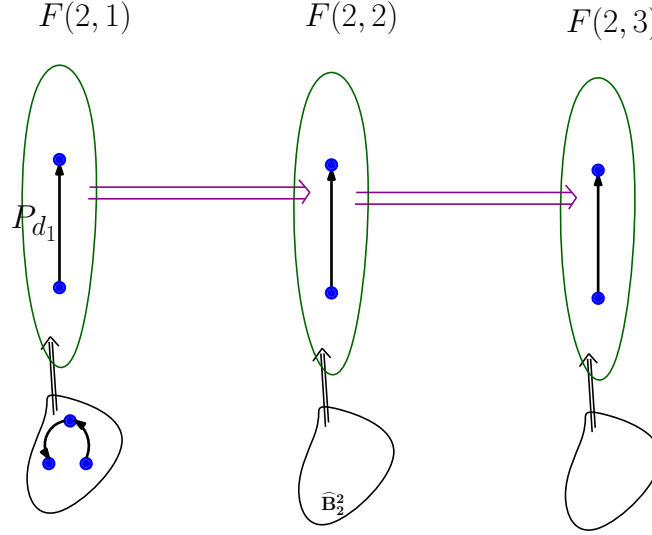
Figure 6.2: B_3 with $k = 3$

subdigraph $\widehat{\Sigma}$ of B_{n+1} induced by the vertices of the P_{d_i}'s is homomorphic to B_n via projection onto the first coordinate, we see that $\vec{g}(\widehat{\Sigma}) \geq \vec{g}(B_n) = \infty$. Therefore B_{n+1} is also acyclic. $\qquad\square$

Lemma 6.5. *If $\left(\alpha_0, \alpha_1, ..., \alpha_{k-1}\right)$ is a shortest cycle in D_n, then there exists a unique $\ell \in \{0, 1, \ldots, k-1\}$ such that the arc $\alpha_\ell \alpha_{\ell+1}$ is in $A(D_n) \smallsetminus A(B_n)$.*

Proof. Proceeding by induction, again we see that the statement is true for B_2. Thus by our induction hypotheses the statement is true for any shortest cycle contained in any D_n^i. Since all shortest cycles containing a vertex (d, i) from a D_n^i and a vertex from a P_{d_j} have the form $\left((d, i), (d_i, p_1), .., (d_i, p_{k-1})\right)$, the unique arc in $A(D_{n+1}) \smallsetminus A(B_{n+1})$ for such cycles is $(d_i, p_{k-1})(d, i)$. Thus it remains to show that the statement is true for shortest cycles contained in Σ (defined on p. 43). So let us now suppose that the cycle $\left(\alpha_0, \alpha_1, ..., \alpha_{k-1}\right)$ is contained in Σ; as we have seen in Theorem 6.1 we may assume that $\alpha_j = (d_j, p_{r_j})$, where $(d_0, d_1, ..., d_{k-1})$ is a shortest cycle in D_{n-1}. Thus the induction hypothesis yields that there exists a unique ℓ such that $d_\ell d_{\ell+1}$ is in $A(D_{n-1}) \smallsetminus A(B_{n-1})$. Therefore $\alpha_\ell \alpha_{\ell+1}$ is the unique arc of $\left(\alpha_0, \alpha_1, ..., \alpha_{k-1}\right)$ in $A(D_n) \smallsetminus A(B_n)$, and the proof is complete. $\qquad\square$

Next we construct digraphs D'_n with digirth k. For $n = 2$, the directed k-cycle will suffice. For $n \geq 2$, we proceed by induction on n and suppose that we have already constructed a digraph D'_n with digirth k and $V(D'_n) = \{d_1, d_2, ..., d_m\}$. We now define D'_{n+1}.

For each $i \in [m]$ let $\widehat{D}^i_n$ be a digraph with vertex set $V(\widehat{D}^i_n) = \{(d_1, i), (d_2, i), ..., (d_m, i)\}$ which is isomorphic to D'_n in the natural way. Next construct m directed paths P_{d_i}, for $1 \leq i \leq m$, each of length $k - 2$, with vertex sets $\{(d_i, p_1), (d_i, p_2), ...(d_i, p_{k-1})\}$ and arc sets $A(P_{d_i}) = \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \mid j \in [k-2]\}$. Now define m digraphs $H'(n, i)$, for $1 \leq i \leq m$, in the following manner. The vertex sets are $V(H'(n, i)) := V(\widehat{D}^i_n) \cup V(P_{d_i})$, and the arc sets are

$$A(H'(n,i)) := A(\widehat{D}^i_n) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d,i)(d_i, p_1)} \mid d \in V(D'_n) \right\} \cup \left\{ \overrightarrow{(d_i, p_{k-1})(d,i)} \mid d \in V(D'_n) \right\}.$$

Finally, we define D'_{n+1} to be the digraph with

$$V(D'_{n+1}) := \bigcup_{i=1}^{m} V(H'(n, i))$$

and

$$A(D'_{n+1}) := \bigcup_{i=1}^{m} A(H'(n, i)) \cup \left\{ \alpha\beta \mid \alpha \in V(H'(n,i)), \beta \in V(H'(n,j)) \text{ and } d_i d_j \in A(D'_n) \right\}.$$

Figure 6.3 is included on the following page in order to clarify the construction of D'_n.

It is clear from the construction of D'_n that D_n is a spanning subdigraph of D'_n for all n which implies that $\chi(D'_n) \geq \chi(D_n) = n$ and $\vec{g}(D'_n) \leq \vec{g}(D_n)$.

Lemma 6.6. *For each integer $n \geq 2$, the digraph D'_n has digirth $\vec{g}(D'_n) = \vec{g}(D_n) = k$.*

Proof. We observe that D'_2 has digirth k and proceed by induction. Using the induction hypothesis, we see that $\vec{g}(\widehat{D}^i_n) = k$ for all $i \in [m]$, which combined with the construction of D'_{n+1} implies that $\vec{g}(H'(n, i)) = k$ for all $i \in [m]$. Thus we need only consider cycles which contain vertices α and β where $\alpha \in V(H'(n, i))$, $\beta \in V(H'(n, j))$ and $i \neq j$. But this implies,
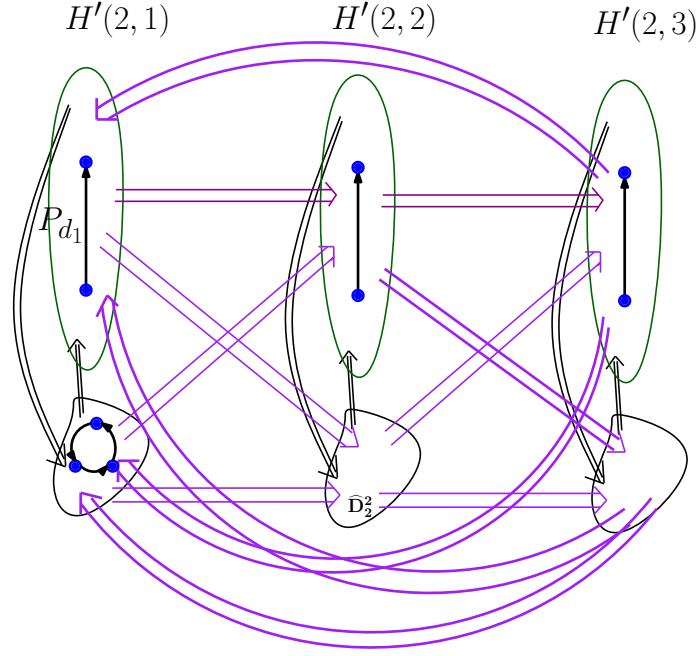
Figure 6.3: D_3' with $k = 3$

from the construction of D_{n+1}', that there exists a path in D_n' from d_i to d_j and from d_j to d_i. Thus the induction hypothesis also implies that the cycle containing α and β has length at least k. Combining these observations, we see that $\vec{g}(D_{n+1}') = k$, and induction gives the lemma. $\qquad\square$

Finally we define B_n' inductively and start by letting B_2' be the path of length $k-1$. We now define B_{n+1}' from B_n'. Suppose that $V(B_n') = \{d_1, d_2, ..., d_m\} = V(D_n')$ and set $V(B_{n+1}') = V(D_{n+1}')$. For $i \in [m]$ let $\widehat{B}_n^i$ be B_n' tagged with an i. Now define m digraphs $F'(n, i)$, for $1 \le i \le m$, in the following manner. The vertex sets are $V(F'(n, i)) \coloneqq V(\widehat{B}_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(F'(n, i)) \coloneqq A(\widehat{B}_n^i) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d, i)(d_i, p_1)} \,\middle|\, d \in V(D_n) = V(B_n') \right\}.$$

Finally, we define B'_{n+1} to be the digraph with

$$V(B'_{n+1}) := \bigcup_{i=1}^{m} V(F'(n,i))$$

and, for all $s, t \in [m]$,

$$A(B'_{n+1}) := \bigcup_{i=1}^{m} A(F'(n,i)) \qquad \cup \left\{ \overrightarrow{(d_i, p_\ell)(d_j, p_h)} \mid d_i d_j \in A(B'_n) \text{ and } \ell, h \in [k-1] \right\}$$
$$\cup \left\{ \overrightarrow{(d_s, i)(d_t, j)} \mid d_i d_j \in A(B'_n) \right\} \quad \cup \left\{ \overrightarrow{(d_s, i)(d_j, p_h)} \mid d_i d_j \in A(D'_n) \text{ and } h \in [k-1] \right\}.$$

It is clear from the construction that B_n is a spanning subdigraph of B'_n. Figure 6.4 may help bring to light some of the nuances of this construction.
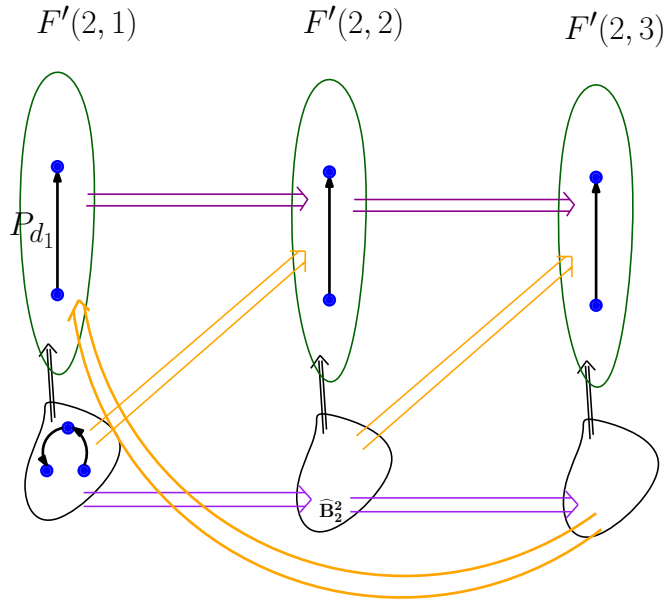

Figure 6.4: B'_3 with $k = 3$

Lemma 6.7. B'_n *is acyclic for all* n.

Proof. It is easy to see that B'_2 is acyclic and thus we continue by induction. Once again the induction hypothesis and the construction of B'_{n+1} imply that $F'(n,i)$ is acyclic for every $i \in [m]$. Since there is no arc from any vertex of a P_{d_i} to any vertex of a $\widehat{B}^j_n$, it suffices

to consider the subdigraph of B'_{n+1} induced by the vertices of the P_{d_i}'s and the subdigraph induced by the $\widehat{B}^i_n$'s. Both of these subdigraph s are homomorphic to B'_n and thus are acyclic by induction. Therefore B'_{n+1} is acyclic. $\qquad\square$

We may now define directed graphs $D'_m \ast \overleftrightarrow{K}_n$ which will be shown to be uniquely n-colorable with digirth equal to $\vec{g}(D'_m)$. The vertex set of $D'_m \ast \overleftrightarrow{K}_n$ is $V(D'_m \ast \overleftrightarrow{K}_n) = V(D'_m) \times V(\overleftrightarrow{K}_n)$ and there is an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2 \in A(D'_m)$ and $h_1 h_2 \in A(\overleftrightarrow{K}_n)$, or $d_1 d_2 \in A(B'_m)$ and $h_1 = h_2$. It is worth noting that the direct product $D'_m \times \vec{K}_n$ is a spanning subdigraph of $D'_m \ast \overleftrightarrow{K}_n$. The first two properties to notice about $D'_m \ast \overleftrightarrow{K}_n$ are that it has digirth at least $\vec{g}(D'_m)$ and is n-colorable because the projections are homomorphisms. (The projection onto $\overleftrightarrow{K}_n$ being a homomorphism relies on the fact that B'_m is acyclic.) We now introduce some notation for future use. Let the vertices of $\overleftrightarrow{K}_n$ be $0, 1, ..., n-1$ and, for $t \in V(\overleftrightarrow{K}_n)$, let $H^t(m-1, i)$ be the set of vertices $V(H'(m-1, i)) \times \{t\}$. Similarly define $P^t_{d_i}$ and $D^t(m-1, i)$ to be $V(P_{d_i}) \times \{t\}$ and $V(\widehat{D}^i_{m-1}) \times \{t\}$ respectively. Lastly define $\Omega^n(m-1, j)$ to be the subdigraph of $D'_m \ast \overleftrightarrow{K}_n$ induced by $\bigcup_{i=0}^{n-1} D^i(m-1, j)$. The next lemma provides the linchpin to proving that $D'_m \ast \overleftrightarrow{K}_n$ is uniquely n-colorable when $m > n$.

Lemma 6.8. *If $n \le m-1$ and $j \in \{1, 2, ..., |V(D'_{m-1})|\}$, then the chromatic number of $\Omega^n(m-1, j)$ is n.*

Proof. First we show that there exists an m-coloring ϕ_m of D_m such that $\alpha\beta \in A(D_m) \setminus A(B_m)$ implies that $\phi_m(\alpha) \ne \phi_m(\beta)$. It is easy to see that ϕ_2 exists and thus we may proceed by induction. Define a mapping $\phi_m : V(D_m) \to V(\overleftrightarrow{K}_m)$ as follows. For vertices $(d_j, i) \in V(D^i_{m-1})$, let

$$\phi_m((d_j, i)) = \begin{cases} \phi_{m-1}(d_j) & \text{if } \phi_{m-1}(d_j) \ne \phi_{m-1}(d_i), \\ m-1 & \text{if } \phi_{m-1}(d_j) = \phi_{m-1}(d_i). \end{cases}$$

For vertices $(d_i, p) \in V(P_{d_i})$, define $\phi_m((d_i, p)) = \phi_{m-1}(d_i)$.

Now suppose that $\alpha\beta \in A(D_m) \smallsetminus A(B_m)$ which implies that either α, β are vertices of some D^i_{m-1}, or α is a vertex of some P_{d_i} and β is a vertex of D^i_{m-1}, or α is a vertex of some P_{d_i} and β is a vertex of some P_{d_j}. In the case where α, β are vertices of some D^i_{m-1}, we may assume that $\alpha = (d_j, i)$ and $\beta = (d_h, i)$, where $d_j d_h$ is in $A(D_{m-1}) \smallsetminus A(B_{m-1})$. Thus using the induction hypothesis we see that $\phi_{m-1}(d_j) \neq \phi_{m-1}(d_h)$ which implies that $\phi_m(\alpha) \neq \phi_m(\beta)$. In the second case, where α is a vertex of some P_{d_i} and β is a vertex of D^i_{m-1}, it is clear from the definition of ϕ_m that $\phi_m(\alpha) \neq \phi_m(\beta)$. In the last case, in which α is a vertex of some P_{d_i} and β is a vertex of some P_{d_j}, we may assume that $d_i d_j$ is in $A(D_{m-1}) \smallsetminus A(B_{m-1})$. Thus the inductive hypotheses and the definition of ϕ_m imply that $\phi_m(\alpha) \neq \phi_m(\beta)$. Therefore ϕ_m is an m-coloring of D_m such that $\alpha\beta \in A(D_m) \smallsetminus A(B_m)$ implies that $\phi_m(\alpha) \neq \phi_m(\beta)$.

We now define Γ to be the subdigraph of $D'_{n+1} * \overleftrightarrow{K}_n$ induced by the set $\{((\alpha, 1), \phi_n(\alpha)) | (\alpha, 1) \in V(D^1_n)\}$ and notice that Γ is a subdigraph of $\Omega^n(n, 1)$ and the digraph induced by $D^i(m-1, 1)$ is isomorphic to the digraph induced by $D^i(m-1, j)$ for all $j \in \{1, 2, ..., |V(D'_{m-1})|\}$. Consider the mapping $\rho : V(D_n) \rightarrow V(\Gamma)$ defined by $\rho(\alpha) = ((\alpha, 1), \phi_n(\alpha))$. Since ϕ_n is an n-coloring of D_n (i.e. a homomorphism to $\overleftrightarrow{K}_n$) ρ is well-defined and bijective. We now suppose that $\alpha\beta \in A(D_n)$ in order to show that ρ is in fact a homomorphism. The first case is when $\alpha\beta$ is an arc in B_n which implies that $(\alpha, 1)(\beta, 1) \in A(B'_{n+1})$ and thus $((\alpha, 1), \phi_n(\alpha))((\beta, 1), \phi_n(\beta)) \in A(\Gamma)$ whether or not $\phi_n(\alpha) = \phi_n(\beta)$ (because of the definition of our $*$-product). The second case is when $\alpha\beta$ is an arc in D_n but not B_n. From the preceding paragraph we know that this implies that $\phi_n(\alpha) \neq \phi_n(\beta)$. Also the construction of D'_{n+1} implies that $(\alpha, 1)(\beta, 1)$ is an arc in D'_{n+1}. Hence $((\alpha, 1), \phi_n(\alpha))((\beta, 1), \phi_n(\beta))$ is an arc in Γ and ρ is a homomorphism. This now implies that $\chi(\Gamma) \geq n$ and in fact $\chi(\Gamma) = n$ since we saw above that $D'_{n+1} * \overleftrightarrow{K}_n$ is n-colorable. Recall that Γ is a subdigraph of $\Omega^n(n, 1)$. Since, for $m > n$, the digraph induced by $D^i(n, 1)$ is isomorphic to a subdigraph of the digraph induced by $D^i(m-1, 1)$ for all $i \in \{0, 1, \ldots, n-1\}$, the digraph Γ is isomorphic to a subdigraph of $\Omega^n(m-1, 1)$. Therefore the chromatic number of $\Omega^n(m-1, j)$ is n for all $j \in \{1, 2, ..., |V(D'_{m-1})|\}$. $\qquad\square$

Finally we have all the necessary tools to prove the deepest result of this dissertation.

Theorem 6.9. *For every integer $n \geq 2$, the digraph $D'_m * \overleftrightarrow{K}_n$ is uniquely n-colorable whenever $n \leq m - 1$.*

Proof. We have seen that the canonical projection $\pi : V(D'_m * \overleftrightarrow{K}_n) \to V(\overleftrightarrow{K}_n)$ is a surjective homomorphism and thus $D'_m * \overleftrightarrow{K}_n$ is n-colorable. Now suppose that there exists another surjective homomorphism $\psi : V(D'_m * \overleftrightarrow{K}_n) \to V(\overleftrightarrow{K}_n)$ and we will show that ψ is a composition of π with an automorphism of $\overleftrightarrow{K}_n$. Notice that since the target digraph is $\overleftrightarrow{K}_n$ this amounts to showing that $\psi((\alpha, i)) = \psi((\beta, i))$ for all vertices α, β of D'_m and $i \in V(\overleftrightarrow{K}_n)$. In other words we need only show that the fibers of ψ are a relabeling of the fibers of π. The preceding lemma has the direct consequence that for all $j \in \{1, 2, ..., |V(D'_{m-1})|\}$ and $\gamma_s \in V(\overleftrightarrow{K}_n)$ there exists an $\alpha \in V(\Omega^n(m-1, j))$ such that $\psi(\alpha) = \gamma_s$ because $\Omega^n(m-1, j)$ is n-chromatic. For each such j and γ_s, let α^s_j be such that $\alpha^s_j \in V(\Omega^n(m-1, j))$ and $\psi(\alpha^s_j) = \gamma_s$.

Consider two vertices d_0 and d_1 of D_{m-1} such that there is an arc from d_0 to d_1 in D_{m-1}. Lemmas 6.2 and 6.5 imply that there exists a cycle $(d_0, d_1, ..., d_{k-1})$ in D_{m-1} and there exists a unique $\ell \in [k]$ such that $d_\ell d_{\ell+1} \in A(D_{m-1}) \smallsetminus A(B_{m-1})$. Thus for all $\beta_h \in V(\widehat{D}^h_{m-1})$ the sequence $((\beta_0, i_0), (\beta_1, i_1), ..., (\beta_{k-1}, i_{k-1}))$ is a cycle in $D'_m * \overleftrightarrow{K}_n$ whenever $i_\ell \neq i_{\ell+1}$. Hence for all $s \in [n]$ and some $i \in [n]$ (which depends on s) the vertices α^s_ℓ and $\alpha^s_{\ell+1}$ lie in $D^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ respectively, for otherwise $(\alpha^s_0, \alpha^s_1, ..., \alpha^s_{k-1})$ would be a monochromatic cycle with respect to ψ. Similarly, supposing that $\alpha^s_\ell \in D^i(m-1, \ell)$,

$$\nexists \nu \in D^r(m-1, \ell+1) \text{ with } \psi(\nu) = \gamma_s \text{ nor a vertex } \mu \in H^r(m-1, \ell) \text{ with } \psi(\mu) = \gamma_s \text{ when } r \neq i,$$

$$(6.3)$$

for otherwise $(\alpha^s_0, \alpha^s_1, ..., \alpha^s_\ell, \nu, \alpha^s_{\ell+2}, ..., \alpha^s_{k-1})$ and $(\alpha^s_0, \alpha^s_1, ..., \alpha^s_{\ell-1}, \mu, \alpha^s_{\ell+1}, \alpha^s_{\ell+2}, ..., \alpha^s_{k-1})$ would be monochromatic cycles with respect to ψ. Now consider the set $\{i \in [n]|$ there exists an s such that $\alpha^s_\ell \in D^i(m-1, \ell)\}$ and suppose that the size of this set is less than n. This implies that for all $s \in [n]$ there does not exist an $\alpha^s_\ell \in D^1(m-1, \ell)$, where

the 1 is taken without loss of generality. This implies further that no $\alpha_{\ell+1}^s$, for $s \in [n]$, lies in $D^1(m-1, \ell+1)$ as we've established that for each fixed $s \in [n]$, the vertices α_ℓ^s and $\alpha_{\ell+1}^s$ lie in $D^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ respectively (i.e. share the same superscript i here). However there exists some $s_1 \in [n]$ and $\beta \in D^1(m-1, \ell)$ such that $\psi(\beta) = \gamma_{s_1}$ because every vertex is sent to some color, and we just concluded that $\alpha_{\ell+1}^{s_1}$ cannot be in $D^1(m-1, \ell+1)$. Hence we reach a contradiction because this leads to the cycle $(\alpha_0^{s_1}, \alpha_1^{s_1}, ..., \alpha_{\ell-1}^{s_1}, \beta, \alpha_{\ell+1}^{s_1}, ..., \alpha_{k-1}^{s_1})$ being monochromatic with respect to ψ. Thus we may conclude that for all $i \in [n]$ there exists an $\alpha_\ell^s \in D^i(m-1, \ell)$. Suppose that $\alpha_\ell^{s_i} \in D^i(m-1, \ell)$. Appealing to (6.3), we now see that for every $i \in [n]$, when $r \neq i$, there does not exist a vertex in $D^r(m-1, \ell+1)$ nor a vertex in $H^r(m-1, \ell)$ either of which is colored γ_{s_i}. Therefore $H^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ are both monochromatic of the same color with respect to ψ.

Now suppose, in order to reach a contradiction, that for some color γ_{s_1} there exists a vertex $\mu \in D^{t_1}(m-1, \ell+2)$ with $\psi(\mu) = \gamma_{s_1}$, where $D^{t_2}(m-1, \ell+1)$ is colored γ_{s_1} and $t_1 \neq t_2$. This implies that there does not exist a vertex $\beta \in V(P_{\ell+1}^{t_2})$ such that $\psi(\beta) = \gamma_{s_1}$, for otherwise the cycle $(\alpha_0^{s_1}, \alpha_1^{s_1}, ..., \alpha_\ell^{s_1}, \beta, \mu, \alpha_{\ell+3}^{s_1}..., \alpha_{k-1}^{s_1})$ would be monochromatic with respect to ψ. Hence there exist vertices $\beta_2, \beta_3 \in V(P_{\ell+1}^{t_2})$ such that $\psi(\beta_2) = \gamma_{s_2}$, $\psi(\beta_3) = \gamma_{s_3}$, and s_1, s_2 and s_3 are distinct because every vertex in $V(\Omega^n(m-1, \ell+1)) \setminus D^{t_2}(m-1, \ell+1)$ is in a cycle with the vertices in $P_{\ell+1}^{t_2}$. This implies that there does not exist a vertex $\nu \in V(\Omega^n(m-1, \ell+2)) \setminus D^{t_2}(m-1, \ell+2)$ such that $\psi(\nu) = \gamma_{s_i}$, for $i = 2$ or $i = 3$, for otherwise the cycle $(\alpha_0^{s_i}, \alpha_1^{s_i}, ..., \alpha_\ell^{s_i}, \beta_i, \nu, \alpha_{\ell+3}^{s_i}, ..., \alpha_{k-1}^{s_i})$ would be monochromatic with respect to ψ. However this implies that $\Omega^n(m-1, \ell+2) \setminus D^{t_2}(m-1, \ell+2)$, which is isomorphic to $\Omega^{n-1}(m-1, \ell+2)$, is $(n-2)$-colored, contradicting Lemma 6.8. Thus for all $i \in [n]$, the sets $D^i(m-1, \ell+1)$ and $D^i(m-1, \ell+2)$ are monochromatic of the same color which in turn implies that $P_{\ell+1}^i$ is monochromatic of the same color. Therefore we may inductively argue that all $H^i(m-1, j)$, for $j \in [k]$, are monochromatically colored the same. As D_m is strongly connected, this implies that $\psi((\alpha, i)) = \psi((\beta, i))$ for all vertices α, β of D_m' and $i \in V(\overleftrightarrow{K}_n)$, and as we noted in the first paragraph of this proof, this is enough to show that $D_m' \star \overleftrightarrow{K}_n$ is uniquely n-colorable. $\square$

Recall from p. 54 and Lemma 6.6 that $\vec{g}(D'_m * \overleftrightarrow{K}_n) \geq \vec{g}(D'_m) = k$. In the preceding proof we encountered a number of directed k-cycles in $D'_m * \overleftrightarrow{K}_n$ implying that $\vec{g}(D'_m * \overleftrightarrow{K}_n) = \vec{g}(D'_m)$. We also note that we were able to construct D'_m with $\vec{g}(D'_m) = k$ for any pair m, k of integers exceeding one. Therefore our proof of Theorem 6.9 constructs a uniquely n-colorable digraph with digirth k for every pair n, k of integers both exceeding one. Thus we now have a constructive and more precise version of (an important case of) the main theorem appearing in [15].

Chapter 7

Future directions

Let us begin where we finished. In Chapter 6 we constructed uniquely n-colorable digraphs with large digirth. Another way to describe these digraphs is that they are uniquely $\overleftrightarrow{K}_n$-colorable (and have large digirth). Noting that $\overleftrightarrow{K}_n$ is a core, and with an eye to the nonconstructive results of [15], we would like to construct digraphs with arbitrarily large girth which are uniquely H-colorable for any core H. In fact I feel confident that the construction is done, but the proof still eludes me. Consider the following conjecture concerning the digraphs D'_m constructed for Theorem 6.9, and notice that $\vec{g}(D'_m \ast H) \geq \vec{g}(D'_m)$.

Conjecture 7.1. For all cores H and some constant c, the digraph $D'_m \ast H$ is uniquely H-colorable for $m > c \cdot \chi(H)$.

In Section 2.3 we saw that that the directed complete rational graphs $\vec{K}_{p/q}$ ($p \geq q$) are cores. Notice that if D is a digraph which is uniquely $\vec{K}_{p/q}$-colorable, then $\chi_c(D) = p/q$. Thus a direct corollary of Conjecture 7.1 would be a construction of digraphs with circular chromatic number p/q and arbitrarily large girth for any rational number $p/q \geq 1$.

Another related problem of interest would be to construct highly chromatic digraphs with arbitrarily large undirected girth. Also recall from Chapter 2 that we still do not have a prim-

itively recursive (in n) construction of n-chromatic graphs with large girth nor a construction of uniquely H-colorable graphs with arbitrarily large girth for general cores H. Note that Zhu's constructive theorem [38] concerned only odd girth.

The fractional chromatic number of a graph (see Chapter 5) gives rise to a famous graph: the *Kneser graph* $K(n,k)$ defined to have vertex set $\binom{[n]}{k}$, with two vertices adjacent exactly when the corresponding k-subsets are disjoint. An alternate but equivalent definition of the fractional chromatic number of a graph G is:

$$\chi_f(G) = \min\{n/k \mid G \to K(n,k)\};$$

cf. our original definition on p. 31. Thus another construction that would be of interest is a directed analogue of the Kneser graphs.

Let us return to Hedetniemi's Conjecture 4.3. In Chapter 5 we saw that though the fractional version of this conjecture holds for graphs (Theorem 5.12), it fails to hold for digraphs (p. 40). It would be of interest to find for any two digraphs D and H a lower bound on $\chi_f(D \times H)$ in terms of $\min\{\chi_f(D), \chi_f(H)\}$. Are there classes of digraphs such that $\chi_f(D \times H) = \min\{\chi_f(D), \chi_f(H)\}$? Are there classes of digraphs such that $\chi(D \times H) = \min\{\chi(D), \chi(H)\}$? Are there digraphs such that $\chi(D \times H) < min\{\chi(D), \chi(H)\}$? Each of these questions is worthy of pursuit.

The final problem I propose here involves the circular chromatic number of a digraph. Recall Theorem 5.11:

$$\chi_f(D) = \chi_c(D) = \chi(D) \text{ if and only if } \chi(D \circ H) = \chi(D) \cdot \chi(H) \text{ for all digraphs } H.$$

What are specific examples of digraphs D such that $\chi_f(D) = \chi_c(D) = \chi(D)$? Can we find examples or, better, classify digraphs such that $\chi_c(D) = \chi(D)$?

As often happens in mathematics, new results breed more questions than they answer. We choose here to end.

Bibliography

[1] H. L. Abbott and B. Zhou, *The star chromatic number of a graph*, J. Graph Theory **17** (1993), no. 3, 349–360. MR1220995 (94g:05032)

[2] Jørgen Bang-Jensen and Gregory Gutin, *Digraphs: Theory, algorithms and applications*, Springer Monographs in Mathematics, Springer-Verlag London Ltd. London, 2001. MR1798170 (2002e:05002)

[3] Drago Bokal, Gašper Fijavž, Martin Juvan, P. Mark Kayll, and Bojan Mohar, *The circular chromatic number of a digraph*, J. Graph Theory **46** (2004), no. 3, 227–240. MR2063373 (2005b:05085)

[4] Béla Bollobás and Norbert Sauer, *Uniquely colourable graphs with large girth*, Canad. J. Math. **28** (1976), no. 6, 1340–1344. MR0429621 (55 #2632)

[5] J. A. Bondy and Pavol Hell, *A note on the star chromatic number*, J. Graph Theory **14** (1990), no. 4, 479–482. MR1067243 (91i:05052)

[6] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. MR2368647 (2009c:05001)

[7] Peng-An Chen, *A new coloring theorem of Kneser graphs*, J. Combin. Theory Ser. A **118** (2011), no. 3, 1062–1071. MR2763055 (2012d:05141)

[8] Vašek Chvátal, *Linear programming*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Company, New York, 1983. MR717219 (86g:90062)

[9] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38. MR0102081 (21 #876)

[10] Tomás Feder, Pavol Hell, and Bojan Mohar, *Acyclic homomorphisms and circular colorings of digraphs*, SIAM J. Discrete Math. **17** (2003), no. 1, 161–169 (electronic). MR2033312 (2004m:05098)

[11] Chris Godsil and Gordon Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001. MR1829620 (2002f:05002)

[12] D. Greenwell and L. Lovász, *Applications of product colouring*, Acta Math. Acad. Sci. Hungar. **25** (1974), 335–340. MR0357199 (50 #9667)

[13] Geňa Hahn and Claude Tardif, *Graph homomorphisms: structure and symmetry*, Graph symmetry (Montreal, PQ, 1996), 1997, pp. 107–166. MR1468789 (99c:05091)

[14] Richard Hammack, Wilfried Imrich, and Sandi Klavžar, *Handbook of product graphs*, Second Edition, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2011. With a foreword by Peter Winkler. MR2817074 (2012i:05001)

[15] Ararat Harutyunyan, P. Mark Kayll, Bojan Mohar, and Liam Rafferty, *Uniquely D-colourable digraphs with large girth*, Canad. J. Math. **64** (2012), no. 6, 1310–1328. MR2994666

[16] Ararat Harutyunyan and Bojan Mohar, *Strengthened Brooks' theorem for digraphs of girth at least three*, Electron. J. Combin. **18** (2011), no. 1, Paper 195, 11pp. MR2853052

[17] Z. Hedrlín and A. Pultr, *Relations (graphs) with given finitely generated semigroups*, Monatsh. Math. **68** (1964), 213–217. MR0168684 (29 #5942)

[18] Pavol Hell and Jaroslav Nešetřil, *The core of a graph*, Discrete Math. **109** (1992), no. 1-3, 117–126. Algebraic graph theory (Leibnitz, 1989). MR1192374 (93m:05163)

[19] _____, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and its Applications, vol. 28, Oxford University Press, Oxford, 2004. MR2089014 (2005k:05002)

[20] Igor Kříž, *A hypergraph-free construction of highly chromatic graphs without short cycles*, Combinatorica **9** (1989), no. 2, 227–229. MR1030376 (91c:05083)

[21] L. Lovász, *On chromatic number of finite set-systems*, Acta Math. Acad. Sci. Hungar. **19** (1968), 59–67. MR0220621 (36 #3673)

[22] MathSciNet, *accessed 3/31/2014*, ©2014 American Mathematical Society.

[23] Jaroslav Nešetřil, *On uniquely colorable graphs without short cycles*, Časopis Pěst. Mat. **98** (1973), 122–125, 212. MR0325437 (48 #3784)

[24] Jaroslav Nešetřil and Vojtěch Rödl, *A short proof of the existence of highly chromatic hypergraphs without short cycles*, J. Combin. Theory Ser. B **27** (1979), no. 2, 225–227. MR546865 (81b:05048)

[25] V. Neumann Lara, *The dichromatic number of a digraph*, J. Combin. Theory Ser. B **33** (1982), no. 3, 265–270. MR693366 (84i:05057)

[26] Gert Sabidussi, *Graph derivatives*, Math. Z. **76** (1961), 385–401. MR0130186 (24 #A53)

[27] Norbert Sauer, *Hedetniemi's conjecture—a survey*, Discrete Math. **229** (2001), no. 1-3, 261–292. Combinatorics, graph theory, algorithms and applications. MR1815610 (2002b:05063)

[28] Edward R. Scheinerman and Daniel H. Ullman, *Fractional graph theory*, Dover Publications Inc. Mineola, NY, 2011. A rational approach to the theory of graphs, With a foreword by Claude Berge, Reprint of the 1997 original. MR2963519

[29] Michael Severino, *A short construction of highly chromatic digraphs without short cycles*, Contributions to Discrete Mathematics (to appear).

[30] Eckhard Steffen and Xuding Zhu, *Star chromatic numbers of graphs*, Combinatorica **16** (1996), no. 3, 439–448. MR1417353 (97g:05084)

[31] James K. Strayer, *Linear programming and its applications*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1989. MR1007600 (90h:90112)

[32] Leo Tolstoy, *Anna Karenina*, Doubleday and Company, Inc. 1946.

[33] Peter Ungar and Blanche Descartes, *Advanced Problems and Solutions: Solutions: 4526*, Amer. Math. Monthly **61** (1954), no. 5, 352–353. MR1528740

[34] A. Vince, *Star chromatic number*, J. Graph Theory **12** (1988), no. 4, 551–559. MR968751 (90c:05098)

[35] Guanghui Wang and Guizhen Liu, *Paths, cycles and circular colorings in digraphs*, Theoret. Comput. Sci. **410** (2009), no. 21-23, 1982–1985. MR2519286 (2010f:05090)

[36] Xuding Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory **16** (1992), no. 6, 557–569. MR1189048 (93i:05066)

[37] ———, *Uniquely H-colorable graphs with large girth*, J. Graph Theory **23** (1996), no. 1, 33–41. MR1402136 (97e:05089)

[38] ———, *Construction of uniquely H-colorable graphs*, J. Graph Theory **30** (1999), no. 1, 1–6. MR1658558 (99j:05078)

[39] ———, *Circular chromatic number: a survey*, Discrete Math. **229** (2001), no. 1-3, 371–410. Combinatorics, graph theory, algorithms and applications. MR1815614 (2002a:05116)

[40] ———, *The fractional version of Hedetniemi's conjecture is true*, European J. Combin. **32** (2011), no. 7, 1168–1175. MR2825542 (2012j:05331)

[41] A. A. Zykov, *On some properties of linear complexes*, Amer. Math. Soc. Translation **1952** (1952), no. 79, 33. MR0051516 (14,493a)