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(I) INTRODUCTION

In analyzing the results of an experiment, the experimenter may wish to use least squares techniques to determine a functional relationship between values of some observed variable and values of an independent variable. To proceed, he sets up an experimental model involving one or more unknown parameters, takes a number of observations at certain values of the independent variable, and uses these observations to estimate the parameters by minimizing the squares of the difference between the observed values of the dependent variable and the estimated functional values. To use a simple example, he may consider the relationship to be linear and hence his model would be: $E(y) = ax + b$ where E denotes expected value and a and b are the unknown parameters. To estimate these parameters, he takes a sample of n observations and solves for a and b by minimizing $\sum_1^n (Y_i - ax_i - b)^2$. The reader is expected to have a general knowledge of these standard techniques.

If the results of an experiment follow the same algebraic relationship throughout the range of the independent variable, then the above method provides satisfactory estimates for the unknown parameters. However, many experiments are known to yield results which require the use of several submodels to describe the expected outcome in terms of a functional value of the independent variable, each submodel being appropriate throughout some specific range

of values of the independent variable. If so, it may be necessary to estimate the values of the independent variable (referred to as switching points) at which a change takes place from one phase to another and the proper parameters for the submodel describing each phase. For example, an experiment may yield results which indicate a linear relationship throughout lower values of the independent variable, but which also indicate that an exponential relationship is appropriate for values above some "switching point". In this case, instead of attempting to design a single phase model which will indicate the general regression (through some presumably happy medium such as a quadratic or a cubic polynomial), the techniques of multiple phase regression allow the experimenter to use appropriate submodels for the linear phase and for the exponential phase and to estimate the point at which the switch is made from one phase to the next. Thus the total model for the above experiment is:

$$E(y) = ax + b \quad x \leq S$$

$$= A \exp(-Bx) \quad x > S$$

where S denotes the unknown switching point which must be estimated from the data in addition to the parameters a , b , A , and B .

It is the aim of this paper to consider multiple phase regression techniques which will enable us to estimate the switching points and the appropriate parameters for each phase in a multiple phase problem. The method of least squares estimation subject to side conditions will be the

chief analytic basis for the results obtained. Several theorems are proved yielding a workable plan for solving this sort of problem and, since the primary purpose of this paper is to describe a technique for estimating the parameters in a multiple phase regression problem, only those theorems related directly to this end are proved. For reasons of simplicity of definition, most of the analysis is done for the case where there are presumed to be only two distinct phases, but these results can be generalized at the cost of extensive algebraic computation to a system having an arbitrary number of phases.

A two phase problem is solved as an illustration and a computer program to perform the computations is described in Section IV.

The submodels are linear and the program is listed in the appendix.

(II) STATEMENT OF THE PROBLEM

Let y_1, y_2, \dots, y_n denote a sample of independent observations taken at n ordered values of some variable x . The general multiple phase regression problem is to estimate a specified number k of switching points and the appropriate regression parameters for each of the $k + 1$ phases determined by these switching points. Approaching this in a manner similar to that used for regular least squares estimation, we have the system:

$$\begin{aligned} f(x) = E[y(x)] &= f_1(x, B_1) && x_1 \leq x \leq S_1 \\ &= f_2(x, B_2) && S_1 \leq x \leq S_2 \\ &= \cdot \\ &= \cdot \\ &= f_{k+1}(x, B_{k+1}) && S_k \leq x \leq x_n \end{aligned}$$

where B_i indicates the vector of parameters to be estimated for the functional relationship f_i in phase i , and S_i indicates the switching point from phase i to phase $i + 1$. These estimations are to be made in such a way that the overall function f is continuous and such that the switching point from phase i to phase $i + 1$ falls in the interval determined by the last point used in the estimation of B_i and the first point used in the estimation of B_{i+1} .

Each of the f_i 's is assumed to be expressible in the form: $f_i(x, B_i) = b_{i1}g_{i1}(x) + b_{i2}g_{i2}(x) + \dots + b_{iq_i}g_{iq_i}(x)$ where the unknown vector $B_i = (b_{i1}, b_{i2}, \dots, b_{iq_i})$ is to be

estimated and the g_{ij} 's are all known and differentiable functions of x for $i = 1, 2, \dots, q_i$. (these functions can be assumed to be linearly independent without loss of generality). Note that the requirement that the total function f be continuous throughout the range of the variable x is equivalent to requiring that it be continuous at the points where a switch occurs.

We are searching for a set of estimates $\{\bar{S}_1, \dots, \bar{S}_k; \bar{B}_1, \dots, \bar{B}_{k+1}\}$ which satisfy:

$$\text{CONSTRAINT (I): } f_j(\bar{S}_j, \bar{B}_j) = f_{j+1}(\bar{S}_j, \bar{B}_{j+1}) \text{ and}$$

$$\text{CONSTRAINT (II): } x_j(2) \leq \bar{S}_j \leq x_{j+1}(1)$$

where $x_j(2)$ and $x_{j+1}(1)$ denote the last sample point in phase j and the first sample point in phase $j + 1$ respectively.

As in all problems involving least squares techniques, we will need at least q observations in phase i if the dimension of B_i is q . If there are less than this number of observations, the least squares solution will not be unique and some adjustment to the model must be made such as decreasing the dimension of B_i or perhaps choosing that solution out of the many possible which best suits the individual experimenter's ideas as to what the results should be. The undesirability of having to do either is the same in multiple phase problems as it is in regular regression problems. However, for either case, the occurrence of this situation is rare enough that the acknowledgement of its possibility is the only reference which

will be made to it. Throughout this paper, it will be presumed that the dimension of B_1 is less than or equal to the number of observations in phase 1.

A least squares estimate made in phase 1 without regard to either constraint (I) or (II) will be referred to as a "local" least squares estimate and will be denoted by B_1^* . This local estimate will depend on what interval $[\bar{S}_1, \bar{S}_{1+1}]$ is considered pertinent to phase 1 since the addition or deletion of one or more sample points from the pertinent range will, in general, change the result of the regression. This dependence will not be indicated unless ambiguity exists since bulky notation results.

In view of the constraint (I), the switching point from phase j to phase $j + 1$ is also a "join point" for the functions f_j and f_{j+1} and these two terms will be used interchangeably throughout the paper. A "local" join point will merely be the true intersection of the local regression lines f_j^* and f_{j+1}^* and is denoted by S_j^* . It should be noted that this intersection may fail to occur or there may be several intersections if the regression is nonlinear.

The estimates considered to be the actual solution to the multiple phase problem will be denoted by \bar{B}_j and \bar{S}_j . It is apparent that the estimate of the function \bar{f}_j is completely determined by \bar{S}_j , \bar{B}_j , and \bar{S}_{j+1} .

In single phase regression the parameters are estimated by minimizing $\sum_{i=1}^n [y_i - \bar{f}(x_i, \bar{B})]^2$ over all possible values of the parameter vector B and a slight modification of this

technique will assist in determining the solutions in a multiple phase problem. In order to use an analysis of sums of squares in the multiple phase problem, we define the multiple phase residual to be $R(S_1, S_2, \dots, S_k; B_1, B_2, \dots, B_{k+1}) =$

$$R = \sum_{j=1}^{k+1} \sum_{i \in j(1)}^{j(2)} [y_i - f_j(x_i, B_j)]^2$$

Each term in the $\sum_{j=1}^{k+1}$ is simply the regular residual sum of squares in the j th phase if that phase is considered as an individual regression. In other words, it is the sum of squares over all i such that x_i is between the switching points S_j and S_{j+1} . Although the S_j 's do not appear explicitly in the equation for R , they are implicitly determined by the constraints (I) and (II). With respect to the index of summation for i in the double sum defining R , it should be noted that the first sample point in phase one is $x_{1(1)} = x_1$ and the last sample point in the $(k+1)$ st phase is $x_{k+1(2)} = x_n$. A solution to this problem is any set of estimates $\{\bar{S}_1, \bar{S}_2, \dots, \bar{S}_k; \bar{B}_1, \bar{B}_2, \dots, \bar{B}_{k+1}\}$ which yields a minimum value for R out of all sets of estimates which satisfy the constraints (I) and (II). Least squares estimates applied separately to each regime yield these estimates. Note, that we need only examine values of R for those sets of possible solutions which satisfy the constraints (I) and (II). If we denote this total collection of "admissible" estimate sets by H then we need only find the $\text{Min}_H (R)$.

The preceding discussion completely specifies the multiple phase problem with $k + 1$ phases. To proceed, now,

in the development of an analytic framework for the actual solutions to this problem, we will restrict ourselves to the case where the number of phases is presumed to be two. (The technique is adapted from a paper by D. Hudson[6]). Phase one in the two phase problem will be appropriate on the interval $[x_1, S]$ and phase two on the interval $[S, x_n]$ where S denotes the single switching point which we wish to estimate. The set H mentioned above is now the collection of all sets $\{B_1, B_2, S\}$ which satisfies the constraints (I) and (II). That is, we must have $f_1(\bar{S}, \bar{B}_1) = f_2(\bar{S}, \bar{B}_2)$ and $x_I \leq S \leq x_{I+1}$. (For simplicity we denote the last sample point in phase one by x_I and the first sample point in phase two by x_{I+1} ; the local least squares solutions will be denoted by $B_1^*(I)$, and $B_2^*(I)$; and the local join point will be denoted by $S^*(I)$.)

The role played by I in the evaluation of R for any admissible set is explicit, so we define the two phase residual to be

$$R(B_1, B_2, S, I) = R = \sum_{i=1}^I [y_i - f_1(x_i, B_1)]^2 + \sum_{i=I+1}^n [y_i - f_2(x_i, B_2)]^2.$$

The task of finding a set in H which yields a $\min_H (R)$ can be simplified somewhat by first examining subsets of H and then pooling the results. With this end in mind we define the following three subsets of H :

$$H_1 = \{ \{B_1, B_2, S\} \in H \mid x_I < S < x_{I+1} \text{ for some } I \text{ and } f_1^*(S, B_1) \neq f_2^*(S, B_2) \}$$

$$H_2 = \{ \{B_1, B_2, S\} \in H \mid S = x_I \text{ for some } I \}$$

$$H_3 = \{ \{B_1, B_2, S\} \in H \mid x_I < S < x_{I+1} \text{ for some } I \text{ and } f_1^*(S, B_1) = f_2^*(S, B_2) \}.$$

The set H_2 differentiates those solutions having a join point precisely at a sample point from the others while H_1 and H_3 breaks those solution sets yielding a join strictly between two sample points into those having equal values of the derivatives at the join point and those not having equal derivatives there. It is obvious that the sets H_1 are mutually exclusive and, furthermore, that $H = H_1 \cup H_2 \cup H_3$. (A fourth set H_4 is sometimes considered in which the solution set yields a join which is strictly greater than x_n or strictly less than x_1 but since either one of these cases is simply a degenerate one phase regression this set will not be considered to be contained in H). Thus, $\text{Min}_H (R) = \text{Min}[\text{min}_{H_1} (R), \text{min}_{H_2} (R), \text{min}_{H_3} (R)]$.

(Note that this minimum may not be unique and if so, all solution sets yielding the minimum must be considered as solutions). Following the notation of Hudson [6], solution sets in H_1 are said to yield a join of type 1; those in H_2 , a join of type 2, and those in H_3 , a join of type 3. It should be noted that if the submodels are both linear then the set H_3 will be empty since equality of the derivatives would imply a single phase regression. Before proving some theorems related to the various types of joins, we will now describe the problem in precise form. The problem yields the following system:

$$\begin{aligned} f(x) = E[y(x)] &= f_1(x, B_1) & x \leq S, \\ &= f_2(x, B_2) & S \leq x. \end{aligned}$$

Considering the case where x_1 is the last point in

phase one, let

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_I \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} y_{I+1} \\ y_{I+2} \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \quad \text{be the vectors of observations}$$

pertinent to phase one and phase two respectively, and let

$$B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ \cdot \\ \cdot \\ b_{1q} \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ \cdot \\ \cdot \\ b_{2r} \end{bmatrix} \quad \text{be the vectors of parameters}$$

to be estimated in phase one and phase two respectively.

(Note: we presume that $q \leq I$ and $r \leq n - I$ to allow for a unique least squares estimation in each phase). We can now write the system as:

$$(1) \quad E[y(x)] = f_1(x, B_1) = b_{11}g_1(x) + \dots + b_{1q}g_q(x) \quad x \leq S$$

$$(2) \quad \quad \quad = f_2(x, B_2) = b_{21}h_1(x) + \dots + b_{2r}h_r(x) \quad S \leq x$$

where the g_i , $i = 1, \dots, q$, and the h_i , $i = 1, \dots, r$, are all known differentiable and linearly independent functions of x . If we apply equations (1) and (2) specifically to the sample points used in the experiment, we can describe the system at these sample points in the following matrix notation:

$$(3) \quad E(Y_1) = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_q(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_q(x_2) \\ \vdots & \vdots & & \vdots \\ g_1(x_I) & g_2(x_I) & \dots & g_q(x_I) \end{bmatrix} B_1$$

and

$$(4) \quad E(Y_2) = \begin{bmatrix} h_1(x_{I+1}) & h_2(x_{I+1}) & \dots & h_r(x_{I+1}) \\ h_1(x_{I+2}) & h_2(x_{I+2}) & \dots & h_r(x_{I+2}) \\ \vdots & \vdots & & \vdots \\ h_1(x_n) & h_2(x_n) & \dots & h_r(x_n) \end{bmatrix} B_2$$

If we now denote the design matrices in equations (3) and (4) by M_1 and M_2 , we can write the system simply as

$$(5) \quad E(Y_1) = M_1 B_1$$

$$(6) \quad E(Y_2) = M_2 B_2.$$

It is presumed throughout this paper that M_1 and M_2 are of full rank.

To allow for more concise notation, if we let

$$Z = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

we can describe the system completely by

$$(7) \quad E(Y) = ZB.$$

With this notation, we get the following equation:

$$(8) \quad R(B_1, B_2, S, I) = R = (Y - ZB)' (Y - ZB).$$

The overall solution to our problem will thus be given by that solution set which minimizes this quantity and satisfies the constraints (I) and (II).

In order to satisfy constraint (I), we need $f_1(S, B_1) = f_2(S, B_2)$; this implies that $b_{11}g_1(S) + \dots + b_{1q}g_q(S) = b_{21}h_1(S) + \dots + b_{2r}h_r(S)$ or in vector notation, $[g_1(S), \dots, g_q(S)] \cdot B_1 = [h_1(S), \dots, h_r(S)] \cdot B_2$. However, the left hand vector on each side of this equation is simply a vector of real numbers for each S (the known functions evaluated at S). Hence, if we denote these vectors by Q_1 and Q_2 , we have simply

$$(9) \quad Q_1 B_1 = Q_2 B_2.$$

This reduces to $Q_1 B_1 - Q_2 B_2 = 0$ and upon combining,

$$(10) \quad [Q_1, -Q_2] \cdot \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0.$$

This is just a linear constraint on B for each S so we arrive finally at the equation $QB = 0$ for constraint (I). Thus, we wish to find $\text{Min}_B (Y - ZB)' (Y - ZB)$ subject to:

$$(11) \quad QB = 0 \text{ and}$$

$$\text{CONSTRAINT (II): } x_I \leq S \leq x_{I+1}.$$

Using this framework, we can proceed to prove the theorems necessary to justify our technique of solution.

III. THEOREMS

Solution sets which yield a join of type one have a simplifying quality which allows the experimenter to determine them simply by examining the local least squares estimates without regard to either constraint. To show this, we prove the following theorem:

THEOREM 1: If the solution to the two phase regression problem yields a join of type one between x_I and x_{I+1} , then $\bar{B}_1 = B_1^*(I)$, $\bar{B}_2 = B_2^*(I)$, and $\bar{S} = S^*(I)$. In other words, the estimates in the solution set are the local least squares estimates made with x_I assigned to the first phase.

Proof: It will suffice to show that the solution set obtained by minimizing R with respect to the constraint $f_1(\bar{S}, \bar{B}_1) = f_2(\bar{S}, \bar{B}_2)$ is the same as that set obtained without the constraint. Consider that we wish to minimize $R(B_1, B_2, S)$ subject to the constraint that $f_1(\bar{S}, B_1) - f_2(\bar{S}, B_2) = 0$. Since the join is of type one, we know that $f_1'(\bar{S}, B_1) - f_2'(\bar{S}, B_2) \neq 0$ so that this minimization problem lends itself to a Lagrangian multiplier attack. (Courant, pp. 192-199, [1]). Thus we are led to the equation $T(B_1, B_2, S, \mathcal{L}) = R(B_1, B_2, S) + 2\mathcal{L}[f_1(S, B_1) - f_2(S, B_2)]$ and the solution set which we seek will be contained in that set $(B_1, B_2, S, \mathcal{L})$ which yields the value zero simultaneously in each of the four equations obtained by differentiating T with respect to each of its arguments. We wish to solve:

$$(1) \quad \frac{\delta T}{\delta B_1} = 2 \sum_{i=1}^I [(f_1(x_i, \bar{B}_1) - y_i) \frac{\delta}{\delta B_1} f_1(x_i, \bar{B}_1)] + 2\mathcal{L} \frac{\delta}{\delta B_1} f_1(\bar{S}, \bar{B}_1) = 0$$

$$(2) \quad \frac{\delta T}{\delta B_2} = 2 \sum_{i=I+1}^n [(f_2(x_i, \bar{B}_2) - y_i) \frac{\delta}{\delta B_2} f_2(x_i, \bar{B}_2)] \\ + 2\mathcal{L} \frac{\delta}{\delta B_2} f_2(\bar{S}, \bar{B}_2) = 0$$

$$(3) \quad \frac{\delta T}{\delta \mathcal{L}} = 2[f_1(\bar{S}, \bar{B}_1) - f_2(\bar{S}, \bar{B}_2)] = 0$$

$$(4) \quad \frac{\delta T}{\delta S} = 2\mathcal{L}[f_1'(\bar{S}, \bar{B}_1) - f_2'(\bar{S}, \bar{B}_2)] = 0.$$

Examination of equation (4) and the fact that a join of type one implies that the derivatives of the f_i are not equal at \bar{S} yields that $\mathcal{L} = 0$. Therefore, equations one and two are reduced to the equations used in finding the local least squares solutions. In other words, $\bar{B}_1 = B_1^*(I)$ and $\bar{B}_2 = B_2^*(I)$. It follows that $S^*(I) = \bar{S}$ since if not, the regression parameters must be constrained to cause the f_i 's to join at a point other than their natural intersection and this would cause the residual $R(\bar{B}_1, \bar{B}_2, \bar{S})$ to increase and, in fact, to be larger than $R(\bar{B}_1^*, \bar{B}_2^*, S^*(I))$ which would contradict the fact that the solution set minimizes R out of all such sets.

Q.E.D.

The ease with which the above theorem can be proved for joins of type one verifies the assertion that our analysis can be simplified a great deal through considering the joins separately. Perhaps the implications of the above theorem can be made more explicit through the statement of the following corollary:

If there is no $S^*(I)$ such that $x_I < S^*(I) < x_{I+1}$, then it is not possible that there is a join of type one between x_I and x_{I+1} . The proof of this corollary is by contradiction. Assuming that there is a join of type one between x_I and x_{I+1} would imply that \bar{S} was between these two points. However, by the theorem we know that $\bar{S} = S^*(I)$ which contradicts the fact that there is no $S^*(I)$ lying strictly between x_I and x_{I+1} .

Q.E.D.

As a result of the theorem and corollary we may now state that if there is no "natural" join of type one in the interval (x_I, x_{I+1}) then there can be no type one join after applying the constraints, and thus it is not necessary to use iterative numerical techniques to test for joins of this kind at all points in the interval.

This result is all that is necessary to fully examine the system for a join of type one. The experimenter need only examine the local regressions for the cases where a switching point is possible and evaluate R for each of those cases yielding a natural join in the proper interval. The solution set which yields a minimum value from these several computations will be that set which yields a $\text{Min}(R)$.

H_1

It may happen, that there is no natural join in the correct interval and even if so, there may be a solution set in H_2 or H_3 which will yield a smaller value of R so we must now consider some techniques for treating these cases.

If the solution set is in H_2 , we have $\bar{S} = x_I$ for some particular I and the question arises as to how x_I should be treated with respect to the local regressions. At first glance, it seems that we should assign the point x_I to the right hand regression, compute R with the estimates constrained to cause a join at x_I , assign x_I to the left hand regression, compute new estimates constrained to cause a join at x_I , and then choose that set of estimates which yields the smallest value of R as being the "best" of the two. The situation is surprisingly simpler than this, however, and the following theorem establishes that we may assign x_I to either the right or left hand regression and still get the same results.

THEOREM 2: If the join is of type two then the estimates of B_1 and B_2 are the same whether the join point is assigned to the left hand regression or to the right hand regression.

Proof: We have $\bar{S} = x_I$ for some particular I . Developing the problem along lines similar to those used in section II,

we have:

$$\text{II (1)} \quad E(y) = f_1(x, B_1) = \sum_{i=1}^{q_1} b_{1i} g_i(x) \quad x \leq x_I$$

$$\text{II (2)} \quad = f_2(x, B_2) = \sum_{i=1}^{q_2} b_{2i} h_i(x) \quad x \geq x_I$$

and, letting M_1 and M_2 denote the respective design matrices, we wish to compare the results obtained by using

$$M_1 = \begin{bmatrix} g_1(x_1) & g_2(x_2) & \dots & g_{q_1}(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_{q_1}(x_2) \\ \vdots & \vdots & & \vdots \\ g_1(x_I) & g_2(x_I) & \dots & g_{q_1}(x_I) \end{bmatrix} \quad \text{and}$$

$$M_2 = \begin{bmatrix} h_1(x_{I+1}) & h_2(x_{I+1}) & \dots & h_{q_2}(x_{I+1}) \\ h_1(x_{I+2}) & h_2(x_{I+2}) & \dots & h_{q_2}(x_{I+2}) \\ \vdots & \vdots & & \vdots \\ h_1(x_n) & h_2(x_n) & \dots & h_{q_2}(x_n) \end{bmatrix}$$

with the results obtained by using

$$M_1' = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_{q_1}(x_1) \\ \vdots & \vdots & & \vdots \\ g_1(x_{I-1}) & \dots & \dots & g_{q_1}(x_{I-1}) \end{bmatrix} \quad \text{and}$$

$$M_2' = \begin{bmatrix} h_1(x_I) & h_2(x_I) & \dots & h_{q_2}(x_I) \\ \vdots & \vdots & & \vdots \\ h_1(x_n) & h_2(x_n) & \dots & h_{q_2}(x_n) \end{bmatrix}$$

The constraint that $f_1(x_I, B_1) = f_2(x_I, B_2)$ can be written as

$$\begin{aligned} & [(g_1(x_I), g_2(x_I), \dots, g_{q_1}(x_I)) - (h_1(x_I), -h_2(x_I), \dots, \\ & \quad - h_{q_2}(x_I))] \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0. \end{aligned}$$

Following the notation of II(10), this is simply

$$(Q_1, -Q_2) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0.$$

Letting

$$Z_1 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} M_1^* & 0 \\ 0 & M_2^* \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \text{and } Q = (Q_1, -Q_2),$$

if we can show that $\text{Min}_B [(Y - Z_1 B)'(Y - Z_1 B)]$ subject to $QB = 0$ is the same as $\text{Min}_B [(Y - Z_2 B)'(Y - Z_2 B)]$ subject to $QB = 0$, we are done.

$$\text{Let } T = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \overrightarrow{Q_1} & & & \overleftarrow{Q_2} & & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} = (Z_1 - Z_2) \text{ and proceed.}$$

$$\begin{aligned} \text{Min}_B [(Y - Z_1 B)'(Y - Z_1 B)] &= \text{Min}_B [(Y - (Z_2 + T)B)'(Y - (Z_2 + T)B)] \\ &= \text{Min}_B [(Y - Z_2 B - TB)'(Y - Z_2 B - TB)]. \end{aligned}$$

In view of the constraint $QB = 0$, we have that $TB = (0, 0, 0, \dots, 0)'$ so that the last Min is equivalent to $\text{Min}_B [(Y - Z_2 B)'(Y - Z_2 B)]$ subject to $QB = 0$.

Q.E.D.

This theorem allows us to assign the join point x_I to whichever phase is convenient and, in view of the techniques which we will now develop for determining the actual estimates yielding a join of type two, this is a considerable advantage. Before proving the theorem which will allow us to compute these estimates, we will prove a necessary identity.

Recall, that we have II(8): $R = (y - ZB)'(Y - ZB)$ which is the residual sum of squares for any B . Using the techniques of single phase analysis, the local solutions are given by $B^* = (Z'Z)^{-1}Z'Y$ and the local residual sums of squares are given by $R^* = (Y - ZB^*)'(Y - ZB^*)$. We wish to establish a relationship between the residual for the local least squares estimates and the residual for an arbitrary vector B . Namely, $(Y - ZB)'(Y - ZB) = (Y - ZB^*)'(Y - ZB^*) + (B^* - B)'Z'Z(B^* - B)$.

Proof: $(Y - ZB) = Y - ZB^* + ZB^* - ZB = (Y - ZB^*) + Z(B^* - B)$.

This is a vector equality, so we may take the transpose of each side times the original vector and still retain equality.

$$\begin{aligned} \text{Thus, } (Y - ZB)'(Y - ZB) &= [(Y - ZB^*)' \\ &\quad + (B^* - B)'Z'][(Y - ZB^*) + Z(B^* - B)] = \\ (1) \quad (Y - ZB^*)'(Y - ZB^*) &+ (B^* - B)'Z'Z(B^* - B) \\ &\quad + (B^* - B)'Z'(Y - ZB^*) \\ &\quad + (Y - ZB^*)'Z(B^* - B). \end{aligned}$$

Examining the third term in this sum, we get $B^*'Z'Y - B^*'Z'ZB^* - B'Z'Y + B'Z'ZB^*$ and substituting $B^* = (Z'Z)^{-1}Z'Y$ into this we have $[(Z'Z)^{-1}Z'Y]'Z'Y - [(Z'Z)^{-1}Z'Y]'(Z'Z)(Z'Z)^{-1}Z'Y - B'Z'Y + B'(Z'Z)(Z'Z)^{-1}Z'Y$. All of these terms cancel, yielding zero. Further, the fourth term in equation (1) is simply the transpose of the third so it is also zero, and we have:

$$(Y - ZB)'(Y - ZB) = (Y - ZB^*)'(Y - ZB^*) + (B^* - B)'Z'Z(B^* - B).$$

Q.E.D.

Using this identity we can now prove the following which will yield a technique for computing the constrained estimates

from the local least squares estimates. With \bar{B} , B^* , Z , and Q defined as in section II and letting $C = Z'Z$ we have:

THEOREM 3: If the local least squares estimates are given by B^* then the least squares estimates \bar{B} subject to the constraint that $f_1(x', B_1) = f_2(x', B_2)$ [note that this is $QB = 0$] for any x' are given by:

$$\bar{B} = B^* - C^{-1}Q' (QC^{-1}Q')^{-1}QB^*.$$

Proof: We wish to find $\text{Min}_B (Y - ZB)'(Y - ZB)$ subject to $QB = 0$.

In view of the identity just proved, this is equivalent to finding $\text{Min}_B [(Y - ZB^*)'(Y - ZB^*) + (B^* - B)'C(B^* - B)]$ subject to $QB = 0$. The first term is constant for all B so our problem reduces to finding $\text{Min}_B (B^* - B)'C(B^* - B)$ subject to the

constraint. We now have a minimization problem subject to a constraint and since any extremum must be a minimum we shall proceed to solve it using a Lagrangian multiplier \mathcal{L} . Let
 (1) $S = (B^* - B)'C(B^* - B) + \mathcal{L}QB$. Differentiating with respect to each of the elements b_i which compose the vector B , we have the following system of simultaneous equations in matrix form:

$$(2) \quad \frac{\delta S}{\delta B} = -C(B^* - B) + \mathcal{L}Q' = 0, \text{ which implies}$$

(3) $C(B^* - B) = \mathcal{L}Q' = Q'\mathcal{L}$. Now, C is a positive definite matrix (since $C = Z'Z$ and Z is of full rank) and hence C^{-1} exists. Thus we have $C^{-1}C(B^* - B) = C^{-1}Q'\mathcal{L}$ which simplifies to

$$(4) \quad (B^* - B) = C^{-1}Q'\mathcal{L}.$$

Multiplying both sides of (4) by Q yields: $Q(B^* - B) = QC^{-1}Q'\mathcal{L}$

and since $QC^{-1}Q'$ is a real number, we get

$$(5) \quad (QC^{-1}Q')^{-1}Q(B^* - B) = \mathcal{L}.$$

Using this value in (3) yields

$$(6) \quad C(B^* - B) = Q'(QC^{-1}Q')^{-1}Q(B^* - B) \\ = Q'(QC^{-1}Q')^{-1}QB^* - Q'(QC^{-1}Q')^{-1}QB.$$

Now, the constraint is that $QB = 0$ so we have simply

$$(7) \quad C(B^* - B) = Q'(QC^{-1}Q')^{-1}QB^*. \text{ Multiplication by } C^{-1} \text{ yields}$$

$$(8) \quad B^* - B = C^{-1}Q'(QC^{-1}Q')^{-1}QB^* \text{ which yields the desired}$$

conclusion, $B = B^* - C^{-1}Q'(QC^{-1}Q')^{-1}QB^*$. Q.E.D.

For purposes of using the equation derived above, it should be noted that, for a particular x' , QB^* and $QC^{-1}Q'$ are just two real numbers say a and b . If we use this, the complicated formula derived can be simplified by evaluating each of the above numbers and setting $B = B^* - (a/b)C^{-1}Q'$.

We can now compute the constrained estimates, but since we are searching for a minimum value of R , we must evaluate this residual for each of the estimate sets thus obtained. This task can be accomplished through the use of the following:

Corollary: With a and b defined as above, the Residual Sum of Squares is equal to the Local Residual Sum of Squares plus a^2/b . That is, $R = R^* + a^2/b$.

Proof: $R = (Y - ZB)'(Y - ZB) = Y'Y - B'Z'Y - Y'ZB + B'Z'ZB = Y'Y - (B^{*'} - a/bQC^{-1})Z'Y - Y'Z(B^* - a/bC^{-1}Q') + (B^* - a/bQC^{-1})Z'Z(B^* - a/bC^{-1}Q') = Y'Y - B^{*'}Z'Y - Y'ZB^* + B^{*'}Z'ZB^* + a/b[QC^{-1}Z'Y + Y'ZC^{-1}Q' - B^{*'}Z'ZC^{-1}Q' - QC^{-1}Z'ZB^*] + a^2/b^2[QC^{-1}Z'ZC^{-1}Q']$. But, $B^* = C^{-1}Z'Y$ and $Z'Z = C$, so the

coefficient of a/b is zero. Also, the coefficient of a^2/b^2 reduces to $QC^{-1}Q'$ which is what we defined b to be so we have $(Y - ZB)'(Y - ZB) = (Y - ZB*)'(Y - ZB*) + (a^2/b^2) \cdot b$ or $R = R* + a^2/b$. Q.E.D.

Notice, that as a result of this corollary it is possible to evaluate the residual sum of squares which would result from constraining the estimates without ever computing the actual constrained estimates. For purposes of manual calculations this would be desirable but for use on a high speed computer it is not practical, and the above corollary is used simply as a formula to evaluate R after the estimates have been made.

The theorems which we have proved so far are sufficient to allow us to search completely for joins of type one or type two. However, no such nice results have been derived for joins of type three. Fortunately, it appears that this type of join occurs infrequently and, for that matter, need not be considered as a possibility if the submodels are linear. If it happens, however, that a join of this type must be searched for, then the experimenter is forced to use some technique of successive approximation throughout the interval under consideration. This method is somewhat tedious at best, but the following can simplify the job greatly:

THEOREM 4: If the residual sum of squares for the local least squares solutions with x_I assigned to the first phase is greater than the residual sum of squares computed for any

acceptable type one join in previous steps, then it is not possible that a join of type three will occur between x_I and x_{I+1} .

Proof: If the local join point $S^*(I)$ is not an element of $[x_I, x_{I+1}]$, then the residual obtained when the B_i are constrained to cause a join anywhere in that interval will be greater than the original and hence will be greater than at least one acceptable join of type one or type two. Hence, the solution could not lie in $[x_I, x_{I+1}]$. If it happens that $S^*(I)$ is in the interval $[x_I, x_{I+1}]$, then the added constraint that the derivatives must be equal certainly could not reduce the original Local Residual and hence, again, a join of type three in that interval would have to yield a larger value of R than some other acceptable solution set. Q.E.D.

As a result of this theorem, the number of intervals which will have to be examined for joins of type three can be reduced considerably since any interval which yields a local solution satisfying the above hypothesis can be omitted.

A brief discussion of a numerical attack to solve for joins of type three will be given in the next section.

Using the results from this section, we can now outline a specific approach to solving the two phase regression problem.

IV. PROCEDURE

Consider a sample of n observations y_1, \dots, y_n taken at ordered values of an independent variable x . We wish to search throughout the interval (x_1, x_n) for a switching point of one of the three types discussed and, theoretically, this requires the examination of each of the intervals $[x_1, x_2)$, $[x_2, x_3)$, \dots , $[x_{n-1}, x_n)$. In practice, however, this is generally not necessary since it may happen that the experimenter has prior knowledge concerning the general location of the switching point which allows him to restrict his attention to some subinterval of $[x_1, x_n]$. This prior knowledge may be the result of either a technical restriction on the switching point or observing obvious trend lines in a scatter diagram. Let us suppose that, in this discussion, the switching point is known to fall in the interval $[x_k, x_{k+j})$. The intervals $[x_k, x_{k+1})$, $[x_{k+1}, x_{k+2})$, \dots , $[x_{k+j-1}, x_{k+j})$ must be searched individually and we proceed with this task by examining each of them for joins of type one and type two simultaneously. (A brief discussion for joins of type three will follow). Let $B_1^*(I)$, $B_2^*(I)$, $S^*(I)$, and $R^*(I)$ be associated, as usual, with the local estimates and let these same symbols with the $*$ replaced by a $**$ denote the estimates and computations associated with estimates which are constrained to cause a join of type two at x_1 .

We wish to complete the following table:

x_I	$B_1^*(I)$	$B_2^*(I)$	$S^*(I)$	$R^*(I)$	$B_1^{**}(I)$	$B_2^{**}(I)$	$S^{**}(I)$	$R^{**}(I)$
-------	------------	------------	----------	----------	---------------	---------------	-------------	-------------

for all pertinent values of x_I , from which the minimum value of R over the sets H_1 and H_2 can be determined simply by examining the columns headed R^* and R^{**} .

Begin by estimating $B_1^*(k)$ and $B_2^*(k)$, calculating $S^*(k)$, and then determining if $S^*(k)$ is in the interval (x_k, x_{k+1}) . If $S^*(k)$ is in the correct interval, compute $R^*(k)$ in the usual fashion. If it is not, theorem one implies that there can be no join of type one in this interval and we will use the convention of setting $R^*(k)$ equal to infinity. We can now fill the first five columns in the above table for the case when $I = k$. To compute the necessary figures for the last four columns, consider, first, that if $S^*(k)$ is in the "right" place then constraining the estimates to cause a join at x_k must increase the residual sum of squares and hence $R^{**}(k)$ cannot yield a minimum and it is unnecessary to compute any of these estimates so we may simply set $R^{**}(k)$ equal to plus infinity. If $S^*(k)$ is not in the proper interval, we proceed as described in theorem 3 to calculate the constrained estimates from the local estimates and using the corollary to theorem three we can compute $R^{**}(k)$ directly from the original value of $R^*(k)$. Enter these figures in the last four columns of the table. We are now

finished with the interval $[x_k, x_{k+1})$ and can proceed to the interval $[x_{k+1}, x_{k+2})$ which is treated identically. Continuing in this manner until we have examined all the intervals, the table is completed.

If the submodels are both straight lines or if the possibility of a join of type three can be legitimately ignored, we are done and the solution set $[\bar{B}_1, \bar{B}_2, \bar{S}]$ is simply that set associated with the smallest of all values listed for R^* and R^{**} . If, on the other hand, we must search for joins of type three, we first eliminate those intervals which could not contain a join of type three using theorem 4 and in the remaining intervals must apply some combination of numerical techniques and least squares estimation to determine the best location for a join of type three. The following discussion briefly describes how one might proceed with this problem. Let the interval (x_q, x_{q+1}) be a possible location for a join of type three and begin by dividing it into 100 segments (or 10 or 1000 segments depending upon the degree of accuracy desired). We must now systematically fit $B_1(q, k)$, $B_2(q, k)$ subject to $f_1(z_k, B_1) = f_2(z_k, B_2)$ and $f_1'(z_k, B_1) = f_2'(z_k, B_2)$ where z_k denotes the k th point in the subdivision and then compute $R(q, k)$ for each of these constrained estimates for all $k = 1, 2, \dots, 100$. Repeat this in all intervals where a join of type three is possible, choose the smallest value of $R(t, k)$, compare it to the minimum obtained in the table and if it is smaller, there is a join of type three. If it is

larger choose the appropriate solution set from the table and the problem is finished.

V. AN APPLICATION

James H. Veghte of the Aerospace Medical Research Laboratories has conducted an experiment in which he measured the amount of oxygen consumed by Gray Jays at various temperature levels. The results of this experiment indicate that a two phase regression would be appropriate and Mr. Veghte has generously supplied his data for this problem.

The nature of the experiment and the techniques which were used are best explained by the following quote from his article, "Thermal and Metabolic Responses of the Gray Jay to Cold Stress":

"Skin and Cloacal temperatures and oxygen consumption were measured in this 12-month study of gray jays to determine the thermoregulatory responses which enable a native Alaskan bird to withstand the extreme cold environment of subarctic regions.....

Adult birds were captured near Fairbanks, Alaska in a simple bait trap. Most birds were subjected to cold-stress experiments within 24 hours of capture to determine their natural response. Usually the experiments were conducted at the same time each day, but, due to difficulties of capture, this schedule could not always be followed.....The birds' oxygen consumption was determined with an open-circuit system. An airflow rate of 64 ml/min was maintained through a small plexiglass metabolic chamber by adjusting the air resistance of the system. A 200-300 cm. air sample was passed in through the chamber, a desiccant, a continuously recording F-3 Beckman oxygen analyzer, a wet flowmeter, and a variable-speed vacume pump.....

The metabolic chamber was placed in a thermostatically controlled cold box. Its air temperature was recorded by means of a thermocouple placed near the inlet. The sampling flow rate was sufficient to maintain uniform temperatures within the metabolic chamber. The temperature of the metabolic chamber was continuously decreased from 25° to -50° C. over a period of 4 hours...The monthly metabolic data were grouped according to seasons to discern the average seasonal oxygen-consumption response of the gray jay to the

environmental temperature profile. Data were calculated and plotted at 5-minute intervals.....The seasonal critical temperatures were determined to compare these data with Scholander's. The critical temperatures were determined by two techniques: (1) by employing Scholander's method in which the average cloacal temperature of the birds in the thermoneutral zone, $41.6^{\circ} \pm 0.3^{\circ}\text{C}$. S.D., was connected with the best fit of the consumption data, and (2) the least squares method in which the regression line equation was determined for all oxygen-consumption values between 6° and -50° C. or the values below the thermoneutral zone. The intercept of these two lines determine the critical temperature."

The critical temperature referred to by Mr. Veghte is that temperature at which the birds are presumed to have reached their basal metabolism level (i.e., that temperature at which they are no longer breathing to keep warm, but simply to maintain bodily functions). This point on the temperature scale is the "switching point" which we wish to determine using the methods outlined in this paper.

The rate of oxygen consumption once this basal metabolism level has been reached can be presumed (as Mr. Veghte did) to be relatively constant so that f_2 (our regression function in phase 2) will simply be an estimate of a constant. In phase one, a linear model was considered appropriate and the data (see page 35) were handled through a program like that listed in the appendix with the second phase constrained to be constant instead of arbitrarily linear. Note that we need only search for joins of type one or type two. The results for each season are listed, in table form, below.

SUMMER

x_I	b_{11}^*	b_{12}^*	b_{21}^*	s^*	R^*	b_{11}^{**}	b_{12}^{**}	b_{21}^{**}	s^{**}	R^{**}
-1.5	-.028	3.26	2.34	33.53	$+\infty$	-.043	2.62	2.69	-1.5	7.04
2.5	-.028	3.25	2.26	35.20	$+\infty$	-.041	2.71	2.61	2.5	5.88
5.1	-.029	3.21	2.19	35.20	$+\infty$	-.040	2.76	2.56	5.1	5.19
8.0	-.030	3.16	2.15	33.56	$+\infty$	-.039	2.81	2.50	8.0	4.50
10.8	-.031	3.12	2.10	32.37	$+\infty$	-.038	2.85	2.44	10.8	3.94
13.8	-.032	3.08	2.07	31.23	$+\infty$	-.037	2.89	2.38	13.8	3.46
16.8	-.033	3.05	2.04	30.29	$+\infty$	-.036	2.92	2.31	16.8	3.10
19.4	-.034	3.03	2.00	30.63	$+\infty$	-.036	2.95	2.25	19.4	2.87
22.3	-.034	3.02	1.95	31.81	$+\infty$	-.035	2.97	2.18	22.3	2.70
22.6	-.034	3.00	1.94	31.06	$+\infty$	-.035	2.97	2.17	22.6	2.68

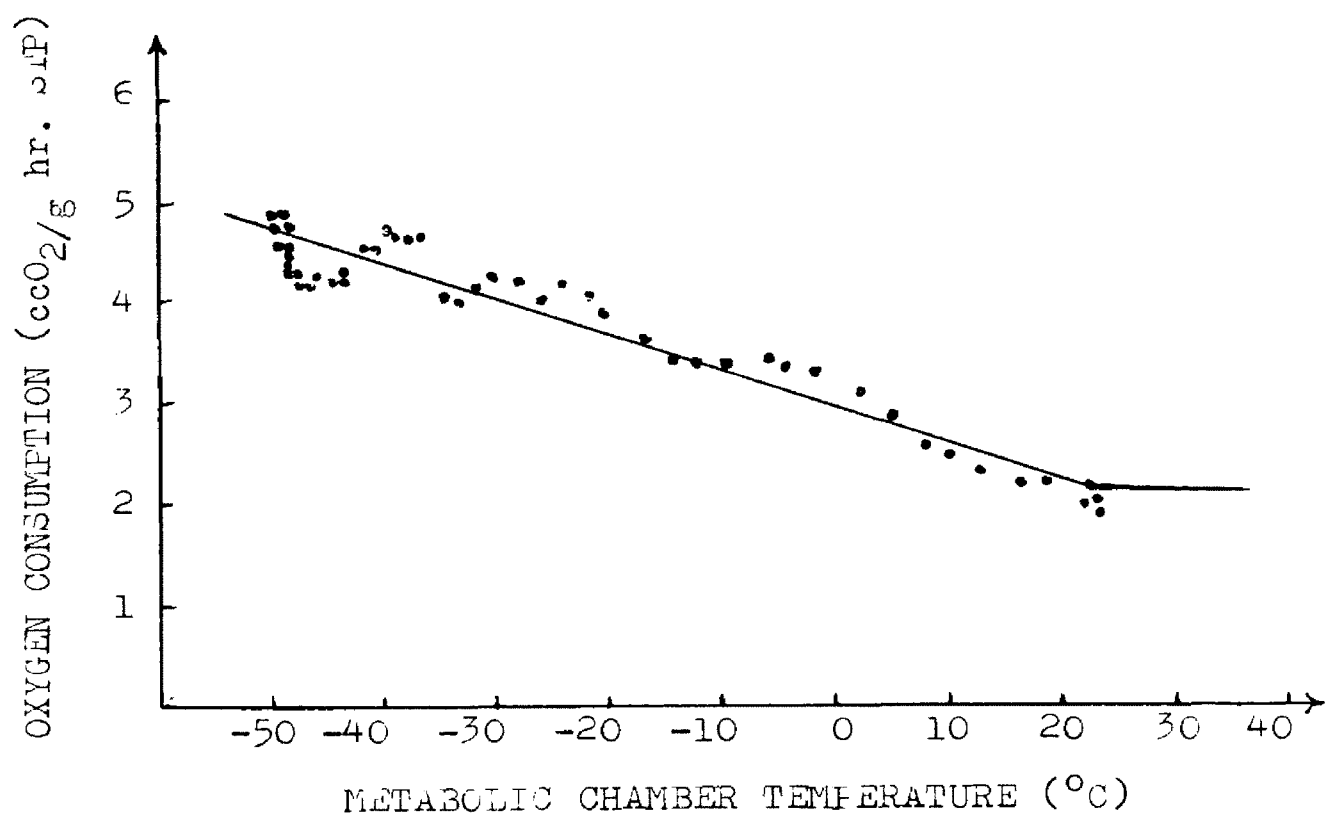
$\text{Min}(R) = 2.68$

The join is of type two.

$\bar{s} = 22.6$

$\bar{B}_1 = (-.035, 2.97); \bar{B}_2 = (2.17)$

$f_1(T) = -.035T + 2.97; f_2(T) = 2.17$



FALL

x_I	b_{11}^*	b_{12}^*	b_{21}^*	S^*	R^*	b_{11}^{**}	b_{12}^{**}	b_{21}^{**}	S^{**}	R^{**}
-1.5	-.059	2.90	2.40	8.44	∞	-.068	2.51	2.61	-1.5	4.20
2.5	-.058	2.93	2.35	10.02	∞	-.065	2.67	2.51	2.5	3.26
5.1	-.058	2.92	2.32	10.41	∞	-.062	2.76	2.45	5.1	2.82
8.0	-.058	2.94	2.29	11.16	∞	-.060	2.85	2.37	8.0	2.52
10.8	-.057	2.97	2.24	12.87	2.35	---	---	---	---	∞
13.8	-.056	3.01	2.19	14.52	2.36	---	---	---	---	∞
16.8	-.056	3.03	2.17	15.52	∞	-.055	3.05	2.13	16.8	2.41
19.4	-.054	3.07	2.11	17.57	∞	-.054	3.09	2.04	19.4	2.53
22.3	-.053	3.11	2.03	20.13	∞	-.053	3.13	1.95	22.3	2.67
22.6	-.052	3.14	1.84	24.80	∞	-.053	3.13	1.93	22.6	2.68

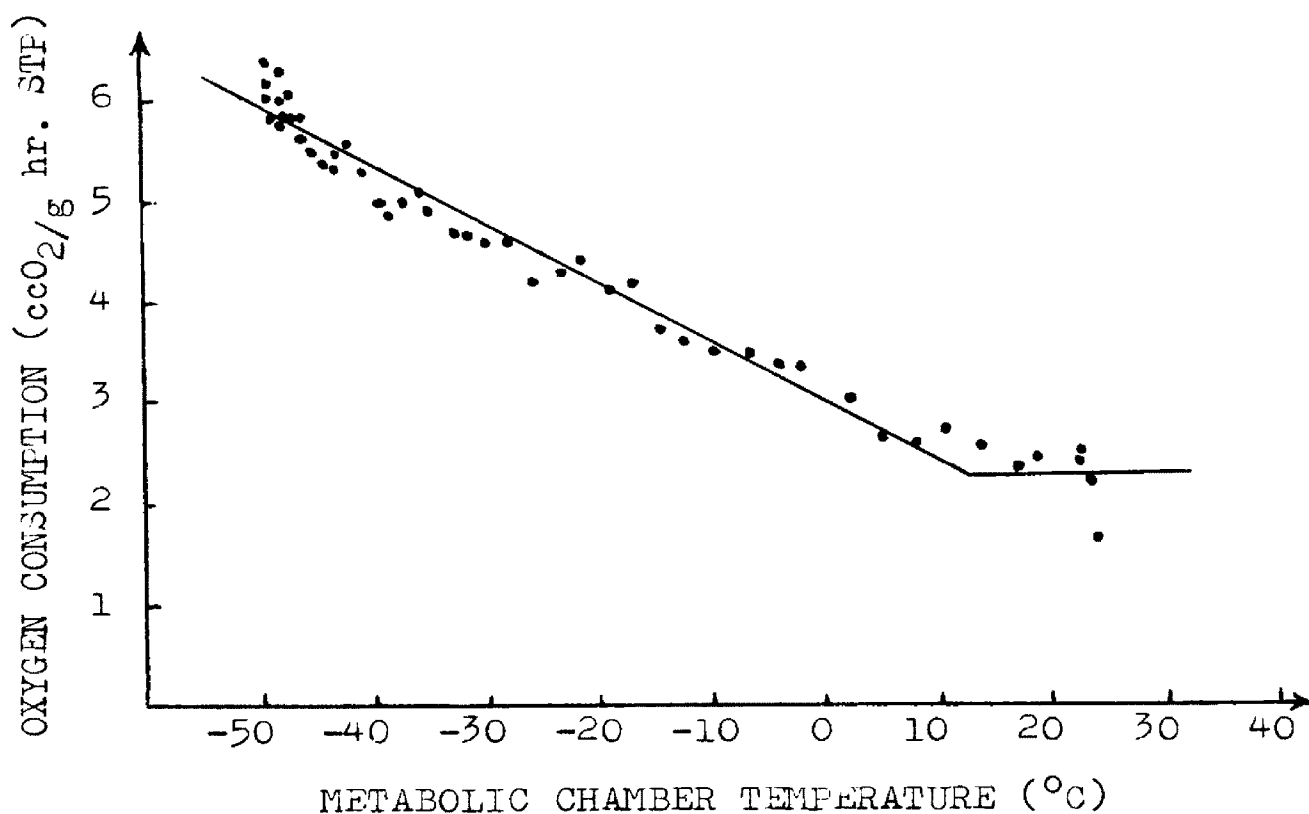
$$\text{Min } (R) = 2.35$$

The join is of type one.

$$\bar{s} = 12.87$$

$$\bar{B}_1 = (-.057, 2.97); \quad \bar{B}_2 = (2.24)$$

$$f_1(T) = -.057T + 2.97; \quad f_2(T) = 2.17$$



WINTER

x_I	b_{i1}^*	b_{i2}^*	b_{21}^*	S^*	R^*	b_{i1}^{**}	b_{i2}^{**}	b_{21}^{**}	S^{**}	R^{**}
-1.5	-.058	2.46	2.14	5.51	∞	-.064	2.19	2.29	-1.5	1.84
2.5	-.058	2.46	2.12	5.88	∞	-.060	2.35	2.20	2.5	1.29
5.1	-.057	2.49	2.09	6.97	1.11	---	---	---	---	∞
8.0	-.056	2.51	2.09	7.44	∞	-.056	2.52	2.07	8.0	1.12
10.8	-.056	2.52	2.09	7.76	∞	-.054	2.59	2.01	10.8	1.24
13.8	-.055	2.56	2.09	8.67	∞	-.053	2.65	1.93	13.8	1.52
16.8	-.054	2.61	2.07	9.95	∞	-.051	2.71	1.85	16.8	1.83
19.4	-.053	2.65	2.07	11.23	∞	-.050	2.75	1.78	19.4	2.21
22.3	-.051	2.71	2.01	13.88	∞	-.049	2.79	1.69	22.3	2.59
22.6	-.050	2.75	1.95	16.08	∞	-.049	2.79	1.68	22.6	2.62

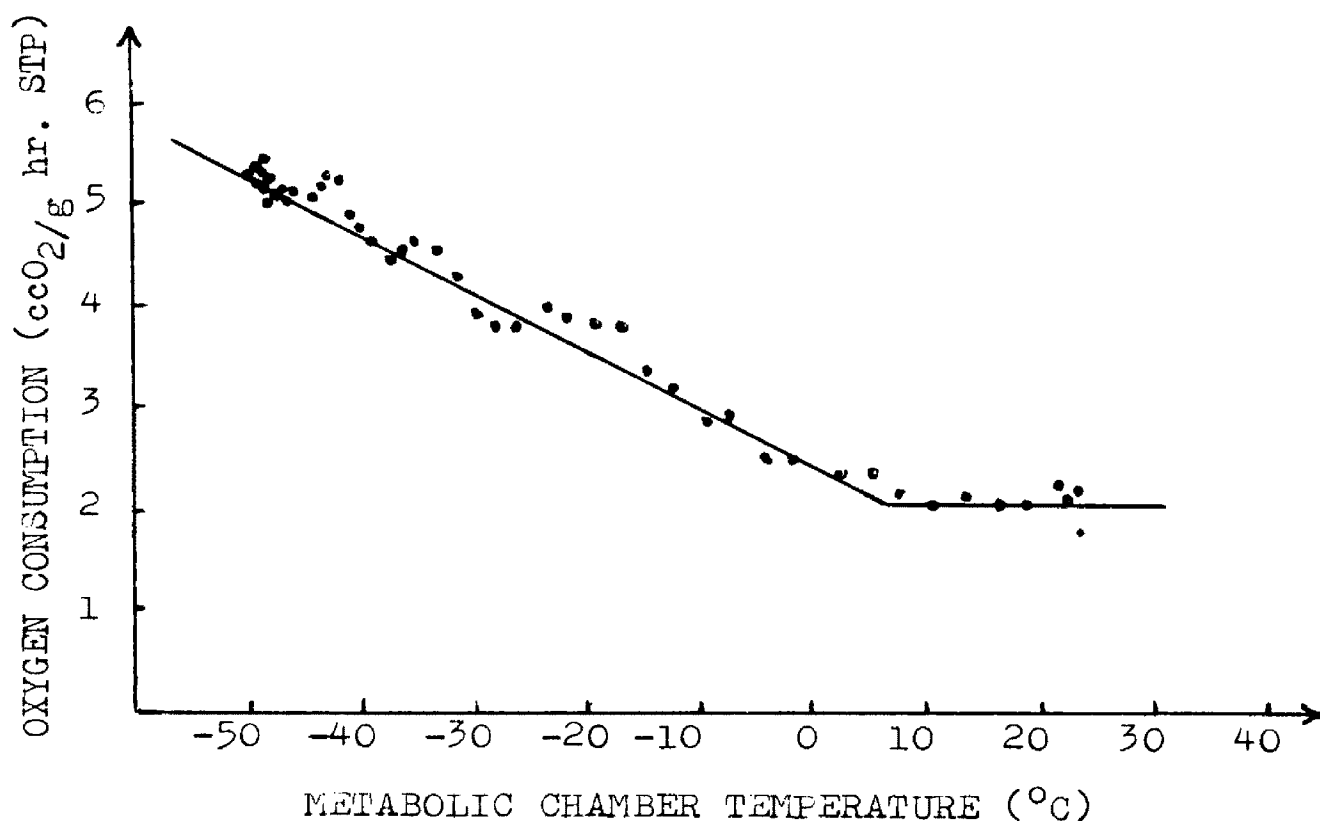
$$\text{Min}(R) = 1.11$$

The join is of type one.

$$\bar{s} = 6.97$$

$$\bar{B}_1 = (-.057, 2.49); \quad \bar{B}_2 = (2.09)$$

$$f_1(T) = -.057T + 2.49; \quad f_2(T) = 2.09$$



SPRING

x_I	b_{i1}^*	b_{i2}^*	b_{21}^*	S^*	R^*	b_{i1}^{**}	b_{i2}^{**}	b_{21}^{**}	S^{**}	R^{**}
-1.5	-.069	2.63	2.19	6.37	∞	-.078	2.27	2.38	-1.5	5.97
2.5	-.069	2.63	2.16	6.75	∞	-.073	2.46	2.27	2.5	5.04
5.1	-.069	2.64	2.14	7.35	4.67	---	---	---	---	∞
8.0	-.068	2.68	2.11	8.43	4.68	---	---	---	---	∞
10.8	-.066	2.73	2.06	10.21	∞	-.066	2.75	2.04	10.8	4.76
13.8	-.065	2.78	2.01	11.89	∞	-.064	2.83	1.94	13.8	4.94
16.8	-.064	2.82	2.00	12.90	∞	-.062	2.89	1.85	16.8	5.21
19.4	-.063	2.85	2.02	13.04	∞	-.061	2.94	1.76	19.4	5.52
22.3	-.062	2.89	2.02	14.01	∞	-.060	2.98	1.65	22.3	5.96
22.6	-.061	2.94	1.95	16.39	∞	-.060	2.98	1.64	22.6	6.01

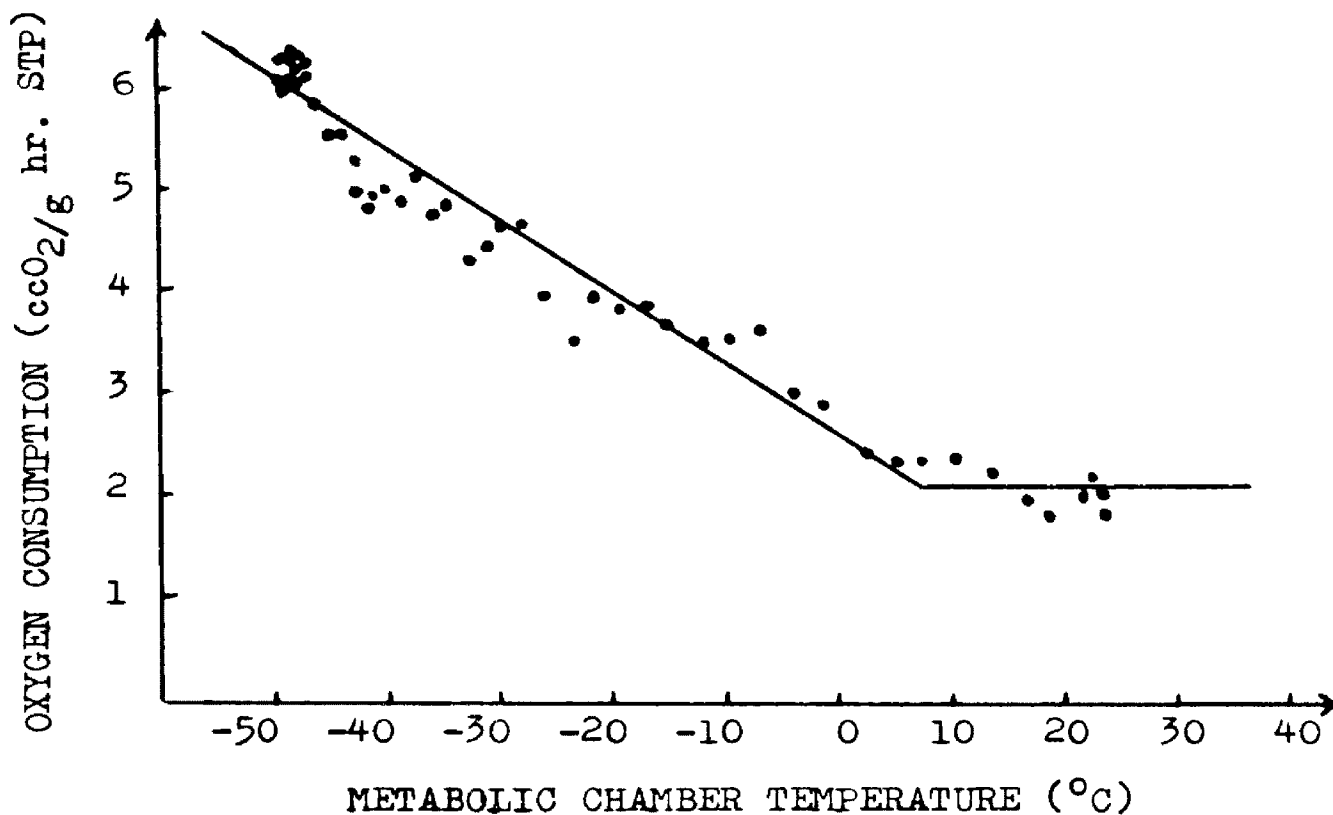
$$\text{Min}(R) = 4.67$$

The join is of type one.

$$\bar{S} = 7.35$$

$$\bar{B}_1 = (-.069, 2.64); \quad \bar{B}_2 = (2.14)$$

$$f_1(T) = -.069T + 2.64; \quad f_2(T) = 2.14$$



To compare the results obtained using the multiple phase regression technique with those obtained by Sholander and Veghte (see page 29), consider the following table:

ESTIMATES OF CRITICAL TEMPERATURES ($^{\circ}\text{C}$)

	Sholander's Method	Multiple Phase Regression Method	Modified Least Squares Method (Veghte)
Winter	5	6.97	7
Spring	9	7.35	7
Summer	5	22.6	36
Fall	6	12.87	9

DATA

Temp C.

-----OXYGEN CONSUMPTION-----

	SPRING	SUMMER	FALL	WINTER
-49.6	6.05	4.82	6.32	5.24
-49.5	6.19	4.91	6.11	5.30
-49.4	6.42	4.79	5.96	5.27
-49.1	6.40	4.57	5.77	5.19
-49.0	6.15	4.62	5.70	5.38
-48.6	6.15	4.78	5.77	5.27
-48.4	6.47	4.46	6.22	5.01
-47.9	6.28	4.34	5.98	5.10
-47.4	6.34	4.24	5.73	5.15
-46.9	6.18	4.19	5.73	5.07
-46.3	5.90	4.19	5.54	5.12
-45.3	5.63	4.26	5.43	5.06
-44.7	5.64	4.23	5.30	5.00
-43.4	5.38	4.23	5.28	5.14
-43.1	5.07	4.31	5.40	5.22
-41.9	4.87	4.52	5.47	5.25
-40.8	5.07	4.54	5.21	4.89
-39.7	5.12	4.75	4.90	4.76
-38.7	4.92	4.68	4.76	4.63
-37.5	5.18	4.68	4.92	4.44
-36.3	4.80	4.69	5.04	4.51
-34.8	4.88	4.04	4.81	4.69
-33.2	4.35	4.04	4.59	4.54
-31.5	4.50	4.13	4.57	4.28
-29.6	4.69	4.22	4.52	3.89
-27.8	4.69	4.22	4.50	3.77
-26.0	4.01	4.06	4.14	3.78
-23.6	3.56	4.19	4.24	3.96
-21.5	3.97	4.09	4.35	3.85
-19.3	3.86	3.87	4.07	3.80
-16.8	3.94	3.62	4.09	3.74
-14.6	3.72	3.43	3.64	3.33
-12.2	3.60	3.37	3.57	3.13
-9.6	3.60	3.36	3.47	2.87
-6.9	3.69	3.40	3.39	2.89
-4.0	3.10	3.35	3.29	2.51
-1.5	2.92	3.28	3.30	2.46
2.5	2.45	3.09	2.95	2.33
5.1	2.38	2.85	2.60	2.38
8.0	2.38	2.57	2.55	2.16
10.8	2.45	2.45	2.64	2.06
13.8	2.32	2.31	2.52	2.12
16.8	2.09	2.20	2.31	2.14
19.4	1.89	2.21	2.40	2.11
22.3	2.02	2.17	2.35	2.24
23.6	2.18	1.97	2.41	2.12
23.5	2.06	2.00	2.12	2.14
24.0	1.83	1.87	1.56	1.76

REFERENCES

- [1] Courant, R., Differential and Integral Calculus, Vol. 2. New York: Interscience Publishers, Inc., 1950.
- [2] Cramer, H., Mathematical Methods of Statistics. Princeton: Princeton University Press, 1946.
- [3] Dixon, W. J., and Massey, F. J., Introduction to Statistical Analysis. New York: McGraw-Hill, 1957.
- [4] Feller, W., An Introduction to Probability Theory and its Applications, Vol. 2. New York: John Wiley and Sons, Inc., 1966.
- [5] Fraser, D. A. S., Statistics, an Introduction. New York: John Wiley and Sons, Inc., 1958.
- [6] Hudson, D. J., "Fitting Segmented Curves Whose Join Points Have to be Estimated," Journal of the American Statistical Association, (1966), pp. 1097-1129.
- [7] Dastebaum, M., "A Confidence Interval on the Abscissa of the Point of Intersection of Two Fitted Linear Regressions," Biometrics, 15 (1959), pp. 323-324.
- [8] Lasiewski, R. C., "Oxygen Consumption of Torpid, Resting, Active, and Flying Hummingbirds," Physiological Zoology, 36 (1963), pp. 122-139.
- [9] Lasiewski, R. C., and Dawson, W. R., "Physiological Responses to Temperature in the Common Nighthawk," Condor, 66 (1964), pp. 477-490.
- [10] Page, E. S., "On Problems in Which a Change in Parameter Occurs at an Unknown Point," Biometrika, 44 (1957), pp. 248-252.
- [11] Page, E. S., "A Test for a Change in Parameter Occuring at an Unknown Switching Point," Biometrika, 42 (1955), pp. 523-527.
- [12] Plackett, R. C., Principles of Regression Analysis. Oxford at the Clarendon Press, 1960.
- [13] Quandt, R. E., "The Estimation of the Parameters of a Linear Regression System Obeying Two Separate Regimes," Journal of the American Statistical Association, 53 (1958), pp. 873-880.

- [14] Quandt, R. E., "Tests of the Hypothesis that a Linear Regression System Obeys Two Separate Regimes," Journal of the American Statistical Association, 55 (1960), pp. 324-330.
- [15] Rao, C. R., Linear Statistical Inference and its Applications. New York: John Wiley and Sons, Inc., 1965.
- [16] Robison, D. E., "Estimates of the Points of Intersection of Two Polynomial Regressions," Journal of the American Statistical Association, 59 (1964), pp. 214-224.
- [17] Scheffe, H., The Analysis of Variance. New York: John Wiley and Sons, Inc., 1959.
- [18] Sprent, P., "Some Hypotheses Concerning Two Phase Regression Lines," Biometrics, 17 (1961), pp. 634-645.
- [19] Veghte, J. H., "Thermal and Metabolic Responses of the Gray Jay to Cold Stress," Physiological Zoology, 37 (1964), pp. 316-328.
- [20] Veghte, J. H., and Herneid, C. F., "Radiometric Determination of Feather Insulation and Metabolism of Arctic Birds," Physiological Zoology, 38 (1965)
- [21] Walker, H. M., and Lev, J., Elementary Statistical Methods. New York: Holt, 1958.

APPENDIX

```

C      A PROGRAM TO COMPUTE TWO PHASE REGRESSIONS

      DIMENSION A(8,8),B(8,1),IROW(8),JCOL(8),B1(10),B2(10),
1 B3(10),B4(10),BB1(10),BB2(10),BB3(10),BB4(10),R1(10),
1 R2(10),ALPHA(10),S(10),T(10),X(100),Y(100)

      COMMON A,B,IROW,JCOL

1001  READ 100, M
C      THIS IS THE TOTAL NUMBER OF OBSERVATIONS
100  FORMAT (12)
      READ 101, (X(I),Y(I), I = 1,M)
C      X(I) IS INDEPENDENT VAR., Y(I) IS DEPENDENT VAR.
101  FORMAT (5X,F18.8,10X,F18.8)
      READ 102, II,JJ
C      II and JJ REPRESENT THE SUBSCRIPTS OF THE INDEPENDENT
C      VARIABLE WHICH DETERMINE THE RANGE TO BE EXAMINED FOR
C      A SWITCHING POINT. (JJ - II)LESS THAN 10.
102  FORMAT (5X,I2,5X,I2)
      DO 1  I = II,JJ
      L = I + 1 - II
      DO 2  K = 1,8
      B(K,1) = 0.
      DO 3  J = 1,8
3     A(K,J) = 0.
2     CONTINUE
      DO 4  J = 1,I
      A(1,1) = A(1,1) + X(J)**2
      A(1,2) = A(1,2) + X(J)

```

```

      B(1,1) = B(1,1) + X(J)*Y(J)
4  B(2,1) = B(2,1) + Y(J)
      A(2,1) = A(1,2)
      A(2,2) = I
      K = I + 1
      DO 5 J = K,M
      A(3,3) = A(3,3) + X(J)**2
      A(3,4) = A(3,4) + X(J)
      B(3,1) = B(3,1) + X(J) * Y(J)
5  B(4,1) = B(4,1) + Y(J)
      A(4,3) = A(3,4)
      N = M - I
      A(4,4) = N
      CALL MAINV (A,4,B,1,DET,IFS)

```

C MAINV IS A CANNED SUBPROGRAM

```

      BB1(L) = B(1,1)
      BB2(L) = B(2,1)
      BB3(L) = B(3,1)
      BB4(L) = B(4,1)

```

C THIS REMEMBERS THE I'TH LOCAL LEAST SQUARES SOLUTIONS

```

      R1(L) = 0.
      R2(L) = 0.
      DO 6 J = 1,I
6  R1(L) = R1(L) + (Y(J) - B(1,1)*X(J) - B(2,1))**2
      DO 7 J = K,M
7  R2(L) = R2(L) + (Y(J) - B(3,1)*X(J) - B(4,1))**2

```

```

C      THIS HAS COMPUTED LOCAL RESIDUAL SUMS OF SQUARES
      ALPHA(L) = (BB4(L) - BB2(L))/(BB1(L) - BB3(L))
C      THIS HAS COMPUTED THE INTERSECTION OF THE TWO LINES
C      THE FOLLOWING TESTS TO SEE IF IT IS IN THE RIGHT PLACE
      IF(ALPHA(L) - X(I)) 17,17,18
18 IF (ALPHA(L) - X(I + 1)) 19,17,17
19 S(L) = R1(L) + R2(L)
      T(L) = 9999.
      B1(L) = 0.
      B2(L) = 0.
      B3(L) = 0.
      B3(L) = 0
      GO TO 1
17 SS = X(I)*(B(1,1) - B(3,1)) + (B(2,1) - B(4,1))
      TT = (A(1,1)+A(3,3))*X(I)**2+2.*(A(2,1)+A(4,3))*X(I) +
1 A(2,2) + A(4,4)
      S(L) = 999.
      T(L) = R1(L) + R2(L) + SS**2/TT
      B1(L) = BB1(L) -SS/TT*(A(1,1)*X(I) + A(1,2))
      B2(L) = BB2(L) -SS/TT*(A(1,2)*X(I) + A(2,2))
      B3(L) = BB3(L) +SS/TT*(A(3,3)*X(I) + A(3,4))
      B4(L) = BB4(L) +SS/TT*(A(4,3)*X(I) + A(4,4))
1 CONTINUE
      N = JJ - II + 1
C      THE FOLLOWING PUNCHES THE RESULTS FOR EACH STEP.
      PUNCH 109, (X(I), I = II, JJ)
      PUNCH 109, (BB1(I), I = 1, N)

```

C THIS PUNCHES THE COEFFICIENT FOR X IN PHASE ONE
PUNCH 109, (BB2(I), I = 1,N)

C THIS PUNCHES THE CONSTANT TERM FOR PHASE ONE
PUNCH 109, (BB3(I), I = 1,N)

C THIS PUNCHES THE COEFFICIENT FOR X IN PHASE TWO
PUNCH 109, (BB4(I), I = 1,N)

C THIS PUNCHES THE CONSTANT TERM IN PHASE TWO.
PUNCH 110,(R1(I), I = 1,N)

C THIS PUNCHES THE LOCAL RESIDUAL FROM PHASE ONE
PUNCH 110,(R2(I), I = 1,N)

C THIS PUNCHES THE LOCAL RESIDUAL FROM PHASE TWO
PUNCH 109, (X(I), I = II,JJ)
PUNCH 109, (ALPHA(I), I = 1,N)

C THIS PUNCHES THE LOCAL JOIN POINT FOR EACH STEP.
MM = II + 1
MMM = JJ + 1
PUNCH 109, (X(I), I = MM,MMM)
PUNCH 111, (S(I), I = 1,N)

C THIS PUNCHES THE TOTAL SUM OF THE SQUARED RESIDUALS
C IF A JOIN OF TYPE ONE IS NOT POSSIBLE, S(I) = 9999.
C THE FOLLOWING FOUR STATEMENTS PUNCH THE CONSTRAINED
C ESTIMATES FOR EACH STEP. THEY ARE 0 IF A JOIN OF TYPE
C ONE WAS POSSIBLE IN THE INTERVAL.
PUNCH 109, (B1(I), I = 1,N)
PUNCH 109, (B2(I), I = 1,N)
PUNCH 109, (B3(I), I = 1,N)
PUNCH 109, (B4(I), I = 1,N)

```
PUNCH 111, (T(I), I = 1,N)
C     THIS PUNCHES THE TOTAL RESIDUAL SUM OF SQUARES USING
C     THE CONSTRAINED ESTIMATES.    T(I) = 9999. IF A JOIN OF
C     TYPE ONE WAS POSSIBLE
      GO TO 1001
109  FORMAT (10F8.4/)
110  FORMAT (10F8.3/)
111  FORMAT (10F8.2/)
      END
```