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Tensor algebras and harmonic analysis

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TENSOR ALGEBRAS AND HARMONIC ANALYSIS

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CONTENTS

CHAPTER I

INTRODUCTION

Harmonic analysis is about linear spaces and how a group acts upon them. More particularly, harmonic analysis is about how the linear space "breaks up" under action by the group, and about the subspaces that are invariant under action by the group. In this paper, the linear spaces we consider are Banach algebras. For these algebras the closed invariant subspaces are the closed ideals.

An important part of harmonic analysis is spectral synthesis, and it is to this topic that we address ourselves. A Banach algebra will be said to be of spectral synthesis in case its closed ideals, i.e., its closed invariant subspaces, can be determined in a very predictable manner. Some very important algebras that are of spectral synthesis will be examined. However, the principal theorem proved in this paper, Malliavin's Theorem, is negative in the sense that it proves the existence of a large and important class of algebras that are not of spectral synthesis.

In the course of our study of spectral synthesis we will study the rudiments of the theory of commutative Banach algebras which allow us to present portions of classical Fourier analysis in a very natural way. For example, Weiner's Theorem is an easy corollary to results on Banach algebras applied to the Banach algebra of functions with absolutely convergent Fourier series.

The use of tensor algebra techniques in the study of harmonic analysis was introduced by N. Th. Varapoulos. In this paper, tensor algebras are introduced without recourse to the general theory of tensor algebras of topological spaces. The definition of tensor algebra is restricted to two factors, although the results are readily generalized.

The elements of Fourier analysis on locally compact abelian groups are presented, and Fourier analysis on the classical groups, the circle group, the integers, and the real numbers, is treated. The Lebesgue spaces are discussed and L_1 is given special emphasis.

In the last chapter, a complete proof of the counterexample of Schwartz is given. Finally, the tools developed earlier, the results of Banach algebras, Fourier analysis on locally compact abelian groups, and the tensor algebra techniques of N. Th. Varapoulos are all brought together in the proof of Malliavin's Theorem.

CHAPTER II

ELEMENTARY THEORY OF BANACH ALGEBRAS

1. Definitions and Examples.

Definition: A is a normed algebra in case A is a normed linear space with norm $\|\cdot\|$ and an associative algebra over the complex numbers, C, such that $\|xy\| \leq \|x\| \|y\|$ for all x, $y \in A$. A is a Banach algebra in case A is a complete normed algebra. A is a commutative Banach algebra in case the multiplication in A is commutative.

If A has a multiplicative identity, we denote it by 1 and we denote scalar multiples of 1 by the corresponding complex number. Without loss of generality [5, p.197] we require that whenever A has 1, $\|1\| = 1$.

Examples: The field of complex numbers, C, with $\|x\| = |x|$ is a commutative Banach algebra with identity.

Let X be a compact Hausdorff space. Let C(X) be the algebra of all continuous complex-valued functions on X, with pointwise multiplication and addition. Let C(X) be normed by the sup-norm: $\|f\|_{C(X)} = \sup_{x \in X} f(x)$. C(X) is a commutative Banach algebra with identity.

Let A be a Banach algebra and let L(A) be the algebra of all linear operators on A with pointwise addition and standard multiplication, endowed with the operator norm: $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$. L(A) is a (non-commutative) Banach algebra with identity.

This paper is principally concerned with commutative Banach algebras with identity.

2. Resolvent and Spectrum.

<u>Definition</u>: Let A be a Banach algebra with identity. An element x A is <u>invertible</u> if there exists an element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = 1$. The set of all invertible elements is denoted by <u>A^{-1}</u>. The <u>resolvent of x</u>, R(x), is the set $\{\lambda \in C: (\lambda - x) \in A^{-1}\}$. The <u>spectrum of x</u>, sp(x), is the set $C \setminus R(x)$. Let U be an open subset of C. A function g mapping U into A is <u>analytic</u> in case g is locally a convergent power series, i.e., for all $\lambda \in U$ there exists a neighborhood $N(\lambda)$ of λ such that $\mu \in$

$N(\lambda)$ implies $g(\mu) = \sum_{n=1}^{\infty} a_n x^n$ for $a_n \in C$ and $x \in A$.

The aim of these definitions is to present an important result, the Gelfand- Mazur Theorem. To this end we examine the spectrum of x and the resolvent of x in greater detail.

Lemma 1: Let A be a commutative Banach algebra with identity, and let $x \in A$ and $\lambda \in C$. If $||x|| < |\lambda|$ then $\lambda \in R(x)$ and $(\lambda - x)^{-1} = \sum_{k=0}^{\infty} x^k/\lambda^{k+1}$.

Proof: Set $y = x/\lambda$ so that $||y|| < 1$. Then all we must show is that $1 \in R(y)$ and $(1-y)^{-1} = \sum_{k=0}^{\infty} y^k$. Set $Y_n = \sum_{k=0}^{k=n} y^k$. Then for $m < n$, we have $||Y_n - Y_m|| = ||\sum_{k=m+1}^{k=n} y^k|| \leq \sum_{k=m+1}^{k=n} ||y||^k$, using the additive and multiplicative properties of the norm. Since $||y|| < 1$, Y_n is a Cauchy sequence and has a limit in A, say $Y = \sum_{k=0}^{\infty} y^k$. Now $(1-y)Y_n = Y_n(1-y) = 1-y^{n+1}$. But $y^{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$ so that $(1-y)Y = Y(1-y) = 1$.

Note that this lemma also says that for $\lambda \in C$, $\lambda \in sp(x)$ implies $||x|| \geq |\lambda|$, i.e., sp(x) is contained in a disc of radius $||x||$. This lemma further asserts that if $||x-1|| < 1$ then $x \in A^{-1}$ and $x^{-1} = \sum_{k=0}^{\infty} (1-x)^k$. For if $||x-1|| < 1$, $[1-(1-x)]^{-1} = x^{-1} = \sum_{k=0}^{\infty} (1-x)^k$ by direct application of the lemma.

Lemma 2: Let $x \in A$ be invertible and let $y \in A$ satisfy $\|y-x\| < \|x^{-1}\|^{-1}$. Then y is invertible and $y^{-1} = x^{-1}\sum_{k=0}^{\infty} (1-x^{-1}y)^k$.

Proof: $\|1-x^{-1}y\| \leq \|x^{-1}\| \, \|x-y\| < 1$. If we apply the above observation to $x^{-1}y$ the proof is complete.

Lemma 3: The spectrum of x, sp(x), is non-empty and is compact in the usual topology of the complex numbers. The function $\lambda \longrightarrow (\lambda - x)^{-1}$ is analytic on R(x).

Proof: We have already noted that sp(x) is contained in a disc of radius $\|x\|$. Suppose then that $\lambda_0 \in R(x)$. The series $\sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k (\lambda_0 - x)^{-(k+1)} = h(\lambda)$ is analytic in the disc $\{\lambda \in C: |\lambda - \lambda_0| < \|(\lambda_0 - x)^{-1}\|^{-1}$. From Lemma 2 we have

$$\begin{aligned}(\lambda - x)^{-1} &= (\lambda_0 - x)^{-1}\sum_{k=0}^{\infty} (1 - \lambda_0 - x)^{-1}(\lambda - \lambda_0 + \lambda_0 - x)^k \\ &= \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k(\lambda_0 - x) \\ &= h(\lambda).\end{aligned}$$

Consequently R(x) is open and $\lambda \longrightarrow (\lambda - x)^{-1}$ is analytic on R(x). We must show that R(x) is not the entire complex plane. Assume that R(x) = C. Then $\lambda \longrightarrow (\lambda - x)^{-1}$ is an entire function. As $|\lambda| \longrightarrow \infty$ $(x - \lambda)^{-1} = |\lambda|^{-1}\|(x/ \; -1)^{-1}\| \longrightarrow |\lambda|^{-1} \longrightarrow 0$. We can argue from Liouville's Theorem, which says that every bounded entire function is a constant, that $(x-\lambda)^{-1} = 0$. This is clearly impossible, so that $R(x)$ is not the entire complex plane, and $sp(x)$ is non-empty.

Theorem 4: (Gelfand-Mazur) If A is a commutative Banach algebra with identity and A is also a field, then A is isometrically isomorphic to the field of complex numbers.

Proof: Every commutative Banach algebra with identity has a subalgebra which is isometrically isomorphic to the field of complex numbers. All that must be shown is that A consists only of scalar multiples of 1. If $x \in A$ there exists a $\lambda \in sp(x)$ by Lemma 3. Now $\lambda - x \notin A^{-1}$ so that $\lambda - x = 0$ since the only non-invertible element of a field is 0. Consequently, $\lambda = x$ and the theorem is proved.

3. The Maximal Ideal Space.

Let A be a commutative Banach algebra with identity. Denote by M_A the set of all maximal ideals of A. M_A is called the maximal ideal space of A. We start the discussion of the maximal ideal space with a lemma about commutative rings.

Lemma 5: Let A be a commutative ring with identity. Every proper ideal of A is contained in a maximal ideal.

Every non-invertible element of A is contained in a maximal ideal. An ideal I of A is maximal if and only if A/I is a field.

Proof: Let I be a proper ideal of A. Let S be the set of all proper ideals of A that contain I, and let S be inductively ordered by inclusion. Then the hypotheses of Zorn's Lemma are satisfied for S and Zorn's Lemma gives us a maximal ideal that contains I.

If $x \in A$ is not invertible, then $xA = \{xa: a \in A\}$ is a proper ideal of A. Since $x \in xA$ and xA is contained in a maximal ideal of A, x is contained in a maximal ideal of A.

Suppose that A/I is a field. Then A/I has only two ideals, $\{0\}$ and A/I. Now there is a one-to-one correspondence between the ideals of A/I and the ideals of A that contain I. Thus I is maximal. On the other hand, suppose I is maximal. Then A/I can have no proper ideals. Hence A/I is a field.

In case A is a commutative Banach algebra with identity and I is a maximal ideal of A, we wish to show that A/I is isometrically isomorphic to C. If A is an algebra and I an ideal of A, then A/I is also an algebra. If A is a normed algebra and I is a closed ideal of A then A/I is a normed algebra with quotient norm: $\|x+I\| = \inf\{\|y\| : x+I = y+I, \text{i.e., } x-y \in I\}$. If A is also complete, then so is A/I. Thus all that remains to be shown is that if I is a maximal ideal of A then I is closed.

Theorem 6: Let A be a commutative Banach algebra with identity and let I be a proper ideal of A. Then the closure of I, $\bar{I}$, is a proper ideal of A. Consequently all maximal ideals are closed.

Proof: Let I be a proper ideal of A. Clearly, $\bar{I}$ is an ideal of A. Since I is proper, $x \in I$ implies x is not invertible. Thus as was observed after Lemma 1, $\|x-1\| \geq 1$ for all $x \in I$ and hence for all $x \in \bar{I}$. Thus $1 \notin \bar{I}$ so that $\bar{I}$ is proper. In case I is maximal, $\bar{I}$ is proper implies $I = \bar{I}$, that is, all maximal ideals are closed.

Corollary 7: If A is a commutative Banach algebra with identity and I is a maximal ideal of A, A/I is isometrically isomorphic to the field of complex numbers.

Proof: Lemma 5 and Theorem 6 combined with the Gelfand-Mazur Theorem give the desired result.

<u>Definition</u>: Let A be a commutative Banach algebra with identity and let ϕ be a non-zero linear functional on A. ϕ is a <u>multiplicative linear functional</u> in case $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$, that is, ϕ is an algebra homomorphism of A into C.

<u>Lemma 8</u>: If ϕ is a multiplicative linear functional then ϕ is continuous and $\|\phi\| = 1$.

<u>Proof</u>: If $x \in A$ and $|\lambda| > \|x\|$ then $\lambda - x$ is invertible by Lemma 1. Now $\phi(\lambda - x) \neq 0$. Hence $|\phi(x)| \leq \|x\|$ so that ϕ is continuous. Since $\|\phi\| = \sup_{\|x\| \leq 1} |\phi(x)|$ and $\phi(1) = 1$, we have $\|\phi\| = 1$.

<u>Theorem 9</u>: Let A be a commutative Banach algebra with identity. There is a one-to-one correspondence between the multiplicative linear functionals on A and the maximal ideals of A.

<u>Proof</u>: If ϕ is a multiplicative linear functional, denote the kernel of ϕ by $\ker\phi$. If $x \in \ker\phi$ $\phi(x) = 0$. If $x \in \ker\phi$ and $y \in A$, $\phi(xy) = \phi(x)\phi(y) = 0$ so that $xy \in \ker\phi$, that is, $\ker\phi$ is an ideal. Since $\phi(1) = 1$ $\ker\phi$ is proper and since $A/\ker\phi \cong C$ $\ker\phi$ is maximal. On the other hand, suppose that I is a maximal ideal of A. Then since $A/I = C$, the projection

$\phi: A \rightarrow A/I$ is a complex-valued homomorphism of A with kernel I.

Following Gamelin, we hereafter identify each maximal ideal with the multiplicative linear functional that it determines.

Definition: Let A be a commutative Banach algebra with identity. The conjugate (or dual) space of A is the set of all continuous linear functionals on A and is denoted A*.

We are seeking to topologize M_A. Obviously $M_A \subseteq A^*$, so that M_A can inherit topologies from A*. A* has two important topologies, the norm topology and the weak-star topology. The weak-star topology is the weakest topology, in the sense that the collection of open sets is minimal, such that the mapping $L \rightarrow L(x)$ is continuous on A* for each $x \in A$, $L \in A^*$. Choose $L_0 \in A^*$ and $x_1, x_2, \ldots, x_n \in A$ and $\epsilon > 0$. A basic weak-star open neighborhood, U, of L_0 is given by the formula
$U = \{L \in A^*: |L(x_i) - L_0(x_i)| < \epsilon \quad, i = 1, 2, \ldots, n\}$.
A weak-star open set is any union of such basic neighborhoods.

Theorem 10: M_A is a compact Hausdorff space with the weak-star topology.

Proof: M_A is a subset of the unit ball of A^* which is a compact set by Alaoglu's Theorem. If $\{\phi_\lambda\} \subseteq M_A$ and $\phi_\lambda(x) \longrightarrow \phi(x)$ for all $x \in A$ then $\phi_\lambda(xy) = \phi_\lambda(x)\,\phi_\lambda(y) \longrightarrow \phi(x)\,\phi(y) = \phi(xy)$. Hence M_A is closed and consequently compact. If $\phi_1 \neq \phi_2$ there exists $x \in A$ such that $\phi_1(x) \neq \phi_2(x)$. Thus $|\phi_1(x) - \phi_2(x)| > \epsilon$ for some real number $\epsilon > 0$. Set $U_1 = \{\psi \in M_A : |\phi_1(x) - \psi(x)| < \epsilon/2\}$ and $U_2 = \{\psi \in M_A : |\phi_2(x) - \psi(x)| < \epsilon/2\}$. Then $U_1 \cap U_2 = \emptyset$ and $\phi_1 \in U_1$, $\phi_2 \in U_2$. Hence M_A is Hausdorff.

4. The Gelfand Transform.

Definition: Let A be a commutative Banach algebra with identity and let $x \in A$. The Gelfand transform $\hat{x}$ of x is a complex-valued function on M_A given by the formula: $\hat{x}(\phi) = \phi(x)$, for all $\phi \in M_A$. The algebra of all Gelfand transforms is denoted $\hat{A}$ and is called the Gelfand space of A.

Lemma 11: The Gelfand transform $x \longrightarrow \hat{x}$ is an algebra homomorphism of A onto $\hat{A}$. The functions in $\hat{A}$ are continuous and $\hat{A}$ separates the points of M_A and contains the constants. The Gelfand transform is norm-decreasing, that is, $\|\hat{x}\|_{M_A} \leq \|x\|$.

Proof: Since $\widehat{x+y}(\phi) = \phi(x+y) = \hat{x}(\phi) + \hat{y}(\phi)$, and $\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}(\phi)\hat{y}(\phi)$, for all $x, y \in A$, $\phi \in M_A$ it is easy to see that the Gelfand transform is an algebra homomorphism. Also $\hat{A}$ contains the constants since $\hat{1}(\phi) = 1$, for all $\phi \in M_A$. The weak-star topology is defined to be the weakest topology such that the Gelfand transform is continuous. From the proof of Lemma 8 we have that $|\phi(x)| \leq \|x\|$ for all $\phi \in M_A$, $x \in A$, i.e., $|\hat{x}(\phi)| = |\phi(x)| \leq \|x\|$. Thus $\|\hat{x}\|_{M_A} = \sup_{\phi \in M_A} |\hat{x}(\phi)| \leq \|x\|$. Finally if $x(\phi_1) = x(\phi_2)$ for all $x \in A$, then $\phi_1(x) = \phi_2(x)$ for all $x \in A$, i.e., $\phi_1 = \phi_2$. Thus A separates the points of M_A.

Lemma 12: If $x \in A$ then the spectrum of x, sp(x), coincides with the range of $\hat{x}$, $\hat{x}(M_A)$.

Proof: Suppose $\lambda \in sp(x)$. Then $\lambda - x$ is not invertible and by Lemma 5 $\lambda - x$ belongs to some maximal ideal of A. By Theorem 9 this maximal ideal is the kernel of a multiplicative linear functional, say ϕ. Then $\phi(\lambda - x) = 0$ implies $\phi(x) = \hat{x}(\phi) = \lambda$. Hence, $\lambda \in \hat{x}(M_A)$. On the other hand, suppose $\lambda \in x(M_A)$. Choose $\phi \in M_A$ such that $\hat{x}(\phi) = \lambda$. Then $\phi(x - \lambda) = 0$ so that $x - \lambda$ is not invertible and $\lambda \in sp(x)$.

5. Two Examples and Weiner's Theorem.

Example: If X is a compact Hausdorff space and C(X) is the algebra of all continuous complex-valued functions on X then C(X), endowed with the sup-norm is a commutative Banach algebra with identity. For every $x \in X$, the evaluation homomorphism ϕ_x defined by $\phi_x(f) = f(x)$ for all $f \in C(X)$ belongs to $M_{C(X)}$. On the other hand, every $\phi \in M_{C(X)}$ is an evaluation homomorphism ϕ_x at some point $x \in X$. For suppose $\phi \in M_{C(X)}$ is distinct from each ϕ_x. Then for every $x \in X$ there is $f_x \in C(X)$ such that $f_x(x) \neq 0$ and $\phi(f_x) = 0$. Then $|f_x|^2 > 0$ in a neighborhood of x and $\phi(|f_x|^2) = \phi(f_x)\phi(\overline{f_x}) = 0$. Since X is compact, choose $x_1, x_2, \ldots, x_n$ X such that $|f_{x_1}|^2 + |f_{x_2}|^2 + \ldots + |f_{x_n}|^2 = g$ is positive on all of X. Then g is invertible so $\phi(g) \neq 0$. This contradiction proves that $\phi = \phi_x$ for some $x \in X$. Thus $x \longrightarrow \phi_x$ is a homeomorphism between X and $M_{C(X)}$.

Example: Let ℓ^1 be the set of sequences $a = \{a_n\}_{n=-\infty}^{\infty}$ with norm $\|a\| = \sum_{n=-\infty}^{\infty} |a_n| < \infty$. With pointwise addition and convolution ($c = a*b$, $c_m = \sum_{k=-\infty}^{\infty} a_{n-k} b_k$) as multiplication ℓ^1 is a commutative Banach algebra with identity. Let $e_n \in \ell^1$ be the sequence nth entry is 1 and all other entries are 0. e_0 is the identity for ℓ^1. Since $e_n * e_k = e_{n+k}$, $e_n^{-1} = e_{-n}$. Also $(e_1)^n = e_n$ so that ℓ^1 is generated by e_1 and e_{-1}. For $a \in \ell^1$ $a = \sum_{n=-\infty}^{\infty} a_n e_n$ where $a_n \in C$.

Choose $\lambda \in C$ such that $|\lambda| = 1$. Then ϕ_λ defined by $\phi_\lambda(a) = \sum_{n=-\infty}^{\infty} a_n \lambda^n$ is a multiplicative linear functional on ℓ^1. Choose $\phi \in M_{\ell^1}$. Then $|\phi(e_1)| \leq \|e_1\| = 1$ and $|\phi(e_{-1})| = |\phi(e_1)|^{-1} \leq \|e_{-1}\| = 1$. Hence $|\phi(e_1)| = 1$, and $\phi = \phi_{\phi(e_1)}$ on the generators of ℓ^1. Thus M_{ℓ^1} can be identified with the unit circle. Set $\lambda = e^{i\theta}$. For $a \in \ell^1$ $\hat{a}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{i\theta}$. Consequently, ℓ^1 is the family of continuous functions on the unit circle with absolutely convergent Fourier series.

<u>Lemma 13</u>: Let A be a commutative Banach algebra with identity and let $x_1, x_2, \ldots, x_n \in A$. Then either $\phi(x_i) = 0, 1 \leq i \leq n$, for some $\phi \in M_A$ or else there exist $y_1, y_2, \ldots, y_n \in A$ such that $\sum_{i=1}^{i=n} x_i y_i = 1$.

<u>Proof</u>: Let I be the ideal generated by $x_1, x_2, \ldots, x_n$ that is $I = \left\{ \sum_{i=1}^{i=n} x_i y_i : y_i \in A, 1 \leq i \leq n \right\}$. If I is proper, by Lemma 5 I is contained in a maximal ideal which is the kernel of a multiplicative linear functional, say ϕ. Then $\phi(x_i) = 0$ for $1 \leq i \leq n$. If I is not proper $1 \in I$, i.e., $\sum_{i=1}^{i=n} x_i y_i = 1$ for some $y_i \in A$, $1 \leq i \leq n$.

Theorem 14: (Weiner) If $f_1, f_2, \ldots, f_n$ are continuous functions on the unit circle with absolutely convergent Fourier series and no common zero, then there exist $g_1, g_2, \ldots, g_n$ continuous functions on the unit circle with absolutely convergent Fourier series such that $\sum_{i=1}^{i=n} f_i g_i = 1$.

Proof: In the example above we have identified $\widehat{\ell^1}$ with the family of continuous functions on the unit circle which have absolutely convergent Fouier series. Thus $f_i = \hat{a}_i$ for some $a_i \in \ell^1$, $1 \leq i \leq n$. The ideal generated by the a_i must be all of ℓ^1 or else by Lemma 13 $f_i(e^{i\theta}) = \hat{a}_i(e^{i\theta}) = 0$ for some θ, $0 \leq \theta \leq 2\pi$, i.e., the f_i have a common zero. Thus there exist $b_i \in \ell^1$, $1 \leq i \leq n$, such that $\sum_{i=1}^{i=n} a_i b_i = 1$. Set $g_i = b_i$. Then we have $\sum_{i=1}^{i=n} f_i g_i(e^{i\theta}) = \sum_{i=1}^{i=n} \hat{a}_i \hat{b}_i(e^{i\theta}) = \widehat{\sum_{i=1}^{i=n} a_i b_i}(e^{i\theta}) = \hat{1}(e^{i\theta}) = e^{i\theta}$, since the Gelfand transform is an algebra homomorphism.

Lemma 15: Let $x \in A$. Let h be a complex-valued function which is defined and analytic on $x(M_A) = sp(x)$. Then there exists $g \in A$ such that $\hat{g} = h \circ \hat{x}$.

Proof: The Cauchy integral formula gives us that $h(z_0) = 1/2\pi i \int_T h(z)/z-z_0 \, dz$, $z_0 \in sp(x)$, for an appropriate contour T around sp(x). We define $g = 1/2\pi i \int_T h(z)/z-x \, dz$. Then $\phi(g) = \hat{g}(\phi) = 1/2\pi i \int_T h(z)/z-\phi(x) \, dz = h(\phi(x))$. Thus $\hat{g} = h \circ \hat{x}$.

Theorem 16: (Weiner-Levy) If f is a continuous function on the unit circle with absolutely convergent Fourier series and if g is a function analytic in a neighborhood of the range of f, then $g \circ f$ has absolutely convergent Fourier series.

Proof: As in Weiner's Theorem above, $f = \hat{a}$ for some $a \in \ell^1$. Now the range of f is equal to $\hat{a}(M_{\ell^1}) = sp(a)$ and g is analytic in a neighborhood of sp(a). By Lemma 15 there exists $b \in \ell^1$ such that $\hat{b} = g \circ \hat{a} = g \circ f$. Thus $g \circ f$ is a continuous function on the unit circle with absolutely convergent Fourier series.

6. The Spectral Radius Formula.

Definition: Let A be a commutative Banach algebra with identity, and let $x \in A$. The spectral radius of x is $\sup \{|\lambda| : \lambda \in sp(x)\} = \|x\|_{M_A}$.

Theorem 17: The spectral radius of $x \in A$ is given by $\|x\|_{M_A} = \lim_n \|x^n\|^{1/n}$.

Proof: For any positive integer n and for $\phi \in M_A$ $|\hat{x}(\phi)| = |\hat{x}^n(\phi)|^{1/n} \leq \|x^n\|^{1/n}$. Hence $\|x\|_{M_A} \leq \liminf_n \|x^n\|^{1/n}$. Let L be any continuous linear functional on A. Define $h(\lambda) = L((\lambda - x)^{-1})$ for $\lambda \notin sp(x)$. Now L is analytic on R(x) and $h(\lambda) = \sum_{n=0}^{\infty} L(x^n)/|\lambda|^{n+1}$ is convergent for $|\lambda| > \|\hat{x}\|_{M_A}$. By the uniform boundedness principle $\sup_n \|x^n\|/|\lambda|^{n+1} = M$ for some real number M. Consequently, $\limsup_n \|x^n\|^{1/n} \leq \limsup_n M^{1/n}|\lambda|^{1+1/n} = |\lambda|$. Since this inequality holds whenever $|\lambda| \geq \|x\|_{M_A}$ we have $\limsup_n \|x^n\|^{1/n} \leq \|\hat{x}\|_{M_A}$. Combining with the above inequality we have $\|\hat{x}\|_{M_A} \leq \liminf_n \|x^n\|^{1/n} \leq \limsup_n \|x^n\|^{1/n} \leq \|\hat{x}\|_{M_A}$. Hence $\lim_n \|x^n\|^{1/n}$ exists and equals $\|\hat{x}\|_{M_A}$.

Corollary 18: The Gelfand transform $x \to \hat{x}$ is an isometry if and only if $\|x\|^2 = \|x^2\|$ for all $x \in A$.

Proof: If $\|x^2\| = \|x\|^2$ for all $x \in A$ then $\|x^{2n}\| = \|x\|^{2n}$ for all $n \geq 1$. Hence $\|x\| = \|x^{2n}\|^{1/2n} \to \|x\|_{M_A}$. On the other hand if $x \to \hat{x}$ is an isometry $\|x^2\| = \|\hat{x}^2\|_{M_A} = \|\hat{x}\|^2_{M_A} = \|x\|^2$.

The spectral radius formula gives a description of the kernel of the Gelfand transform: $\hat{x} = 0$ if and only if $\lim_n \|x^n\|^{1/n} = 0$. If $\lim_n \|x^n\|^{1/n} = 0$ then x is called a generalized nilpotent. On the other hand $\hat{x} = 0$ if and only if $\phi(x) = 0$ for all $\phi \in M_A$, i.e., x belongs to the intersection of all maximal ideals.

Definition: Let A be a commutative Banach algebra with identity. The intersection of all maximal ideals of A is called the (Jacobson) radical of A. A is called semi-simple in case the radical of $A = \{0\}$, that is, A is semi-simple in case the Gelfand transform is an isomorphism.

Let I be an ideal of A. The set of all maximal ideals that contain I is called the hull of I. Alternatively, the hull of I is the set of all $\phi \in M_A$ such that $\phi(x) = 0$ for all $x \in I$. The hull of I is a closed subset of M_A.

Let E be a subset of M_A. The kernel of E is the intersection of all ideals that belong to E. Alternatively, the kernel of E is the set of all $x \in A$ such that $\hat{x}(\phi) = 0$ for all $\phi \in E$.

Definition: A is regular on M_A in case for every closed subset E of M_A and for every $\phi \in M_A \setminus E$ there is an $x \in A$ such that $\hat{x}(\phi) = \phi(x) = 0$ while $\hat{x}(\phi) = 0$ for all $\phi \in E$. If A is regular on M_A we say that A is a regular Banach algebra.

Finite unions of hulls are hulls and arbitrary intersections of hulls are hulls so that the hulls of the ideals of A can be taken as the closed sets for a topology on M_A. For any subset E of M_A the closure of E, $\overline{E}$, in this topology is the hull of the kernel of E. Hence this topology is called the hull-kernel topology.

Theorem 19: The weak-star topology on M_A and the hull-kernel topology on M_A coincide if and only if A is a regular Banach algebra.

Proof: We have already noted that every hull is a weak-star closed subset of M_A. What we must show then is that if A is a regular Banach algebra every weak-star closed subset of M_A is a hull. Let E be a closed subset of M_A. Since A is a regular Banach algebra for every $\phi \in M_A \setminus E$ there exists $x \in A$ such that $\hat{x}(\phi) \neq 0$ but $\hat{x} = 0$ on E. Then $\{x \in A: \hat{x} = 0 \text{ on } E\}$ is an ideal of A and E is its hull. On the other hand, if the topologies coincide and E is a closed subset of M_A, E is the hull of an ideal I. Now if $\psi \in M_A \setminus E$ there exists $x \in I$ such that $\hat{x}(\psi) \neq 0$ while $\hat{x}$ vanishes on E. Thus A is regular on M_A.

7. Spectral Synthesis.

<u>Definition</u>: Let A be a regular, semi-simple commutative Banach algebra with identity. Let E be a closed subset of M_A. Define the following sets:

$I(E) = \{x \in A: \hat{x}(\phi)=0 \text{ for all } \Phi \in E\}$

$I_0(E) = \{x \in A: \hat{x} = 0 \text{ on a neighborhood of } E\}$

$J(E) = \overline{I_0(E)}$, the closure of $I_0(E)$.

E is a <u>set of spectral synthesis</u> in case $J(E) = I(E)$.

Note that I(E) is the kernel of E and that I(E) is the largest ideal that has E for its hull and $I_0(E)$ is the smallest ideal that has E for its hull so that $I_0(E) \subseteq J(E) \subseteq I(E)$. To illustrate this definition we return to an earlier example. Let X be a compact Hausdorff space, and let C(X) be the set of all continuous complex-valued functions on X. We have already identified $M_{C(X)}$ and X. We will show that every closed subset of $M_{C(X)}$ is a set of spectral synthesis. Let E be a closed subset of $M_{C(X)}$. Since $J(E) \subseteq I(E)$ all we must show is that $I(E) \subseteq J(E)$. Choose $f \in I(E)$, so that $f(x) = 0$ for all $x \in E$. Let U be any open neighborhood of f, say $U = \{g \in C(X): \|f-g\| < \epsilon\}$, for some real number $\epsilon > 0$. Then $U \setminus \{f\} \cap I_0(E) \neq \emptyset$, i.e., $f \in \overline{I_0(E)} = J(E)$ and E is a set of spectral synthesis.

We can in fact show more about the ideals of $C(X)$. We will show that every ideal is dense in the kernel of its hull. Let H be an ideal of $C(X)$ and let $Z(H)$ be the hull of H. Then $I_0(Z(H)) \subseteq H \subseteq I(Z(H))$ since $I_0(E)$ is the smallest ideal that has $Z(H)$ for a hull and $I(Z(H))$ is the largest ideal that has $Z(H)$ for a hull. Since $Z(H)$ is a closed subset of $M_{C(X)}$ $Z(H)$ is a set of spectral synthesis. Thus $\overline{I_0(Z(H))} = I(Z(H))$ so that $\overline{H} = I(Z(H))$. In words, H is dense in the kernel of its hull. In case H is a closed ideal of $C(X)$, H is the kernel of its hull, that is, H is the intersection of the maximal ideals of $C(X)$ that contain H. In this example we see that if E is a closed subset of $M_{C(X)}$ the kernel of E, $I(E)$, is the only closed ideal of $C(X)$ that has E for its hull. Thus the closed subsets of $M_{C(X)}$ completely determine the closed ideals of $C(X)$.

Spectral synthesis is frequently defined in another non-equivalent fashion. Let A be a regular semi-simple commutative Banach algebra with identity. A closed subset E of M_A is said to be a <u>set of spectral synthesis in the second sense</u> in case the kernel of E is the only closed ideal H of A such that the hull of H is E.

If E is a set of spectral synthesis in the second sense then E is a set of spectral synthesis. What we must show is that the hull of J(E) is E. Now M_A is a compact Hausdorff space so that we can find a closed subset F of M_A such that E is a subset of the interior of F. Since A is regular we can choose $x \in A$ such that $\hat{x}(\phi) \neq 0$ and $x(F) = 0$ for every $\phi \in M_A \setminus E$. Since E is a subset of the interior of F, $x \in I_0(E)$. Since this is true for all $\phi \in M_A \setminus E$ the hull of J(E) must be E. Thus $J(E) = I(E)$ so E is a set of spectral synthesis.

In general, it is not true that if E is a set of spectral synthesis then E is a set of spectral synthesis in the second sense. In the next section we will examine a case in which the two definitions are equivalent.

Definition: A regular semi-simple commutative Banach algebra is said to be of spectral synthesis in case every closed subset of M_A is a set of spectral synthesis. Spectral synthesis is said to fail in A in case A is not of spectral synthesis.

Note in the above example C(X) is of spectral synthesis. One of the principal goals of this paper is to prove Malliavin's Theorem, which says that spectral synthesis fails for an important class of Banach algebras.

8. Some Basic Facts of Fourier Analysis.

Definition: A group G is an LCA group in case G is a locally compact abelian group. If G is an LCA group a character of G is a continuous homomorphism of G into the group of complex numbers of modulus 1 (the circle group). The set of all characters of G, denoted Γ, is an abelian group under pointwise multiplication, and is called the dual group of G or the character group of G.

If G is an LCA group with normalized Haar measure and $0 < p < \infty$, $L_p(G)$ is the space of all Borel measurable functions f on G such that $\int_G |f|^p dx < \infty$. A norm for $L_p(G)$ is given by $\|f\|_p = (\int_G |f|^p dx)^{1/p}$. If we identify the functions in $L_p(G)$ that differ only on a null set $L_p(G)$ is a Banach space.

For f, $g \in L_p(G)$, the convolution of f and g, $f*g$, is defined by

$$f*g(x) = \int_G f(xy)g(y^{-1})dy = \int_G f(y)g(y^{-1}x)dy \quad ;$$

with convolution as multiplication $L_p(G)$ is a Banach algebra.

$L_\infty(G)$ is the space of all bounded measurable functions on G.

A norm for $L_\infty(G)$ is given by $\|f\|_\infty = \operatorname{ess\,sup}_{x \in G} |f(x)| = \inf \{a: \{x: |f(x)| > a\} \text{ is a null set}\}$.

We are principally concerned with $L_1(G)$. In general, $L_1(G)$ does not have an identity. However we do have the following theorem.

<u>Theorem 20</u>: If G is a discrete LCA group then $L_1(G)$ has an identity.

<u>Proof</u>: In case G is discrete each point of G is an open set and has equal Haar measure. If e is the identity of G, define $u(x) = 0$ if $x \neq e$, $u(x) = 1$ if $x = e$. Then u is the identity of $L_1(G)$, for $f*u(x) = \int_G f(y)u(y^{-1}x)dy = \sum_{y \in G} f(y)u(y^{-1}x) = f(x)$.

The converse to this theorem is also true [7, p.30].

If G is an LCA group we can identify the maximal ideal space of $L_1(G)$ and Γ by $f \to \hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx$ for $\chi \in \Gamma$. With the topology of uniform convergence on compact sets Γ is an LCA group and is in fact homeomorphic to $M_{L_1(G)}$. Thus if G is discrete $L_1(G)$ is a commutative Banach algebra with identity so that Γ is compact by Theorem 10.

The following examples treat the LCA groups that are the objects of study of classical Fourier analysis.

Examples: Let G be the additive group of real numbers R. Every character of R is of the form $s \to e^{ist}$ for some real number t. Thus if G = R, then $G = \Gamma = R$, that is, R is its own dual group. For $t \in R$, $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}dx$, the usual Fourier transform of the line.

If G is the circle group T $\chi(x) = e^{ixt}$ and $\chi(x+2\pi) = e^{ixt}$ for all $\chi \in \Gamma$ so that t must be an integer. Thus Γ is the group of integers Z and $\hat{f}(t) = 1/2\pi \int_{-\pi}^{\pi} f(e^{i\theta})e^{-it\theta}d\theta$.

If G is the group of integers Z $\chi(n) = e^{int}$. Thus $\chi(1) = e^{it}$ so that $\chi \to e^{it}$ is an isomorphism between the dual group of Z and T. $\hat{f}(e^{it}) = \sum_{n=-\infty}^{\infty} f(n)e^{-int}$, the classical Fourier series.

The above examples, particularly the last two, illustrate the Pontryagin Duality Theorem which states that if G is an LCA group and Γ is its dual group then G is the dual group of Γ. Thus it makes sense to identify the maximal ideal space of $L_1(\Gamma)$ with G. If G is compact then Γ is discrete [7, p. 9].

Thus if G is compact $L_1(\Gamma)$ has an identity. Furthermore, $L_1(\Gamma)$ is regular and semi-simple [7, p.30] so that we can examine spectral synthesis in $L_1(\Gamma)$. In $L_1(\Gamma)$ the two definitions of spectral synthesis are equivalent [7, p.161].

In our attempt to prove Malliavin's Theorem, we will consider an algebra which is isometrically isomorphic to $L_1(\Gamma)$, namely $A(G) = \{f \in C(G): f = \hat{F}$ for some $F \in L_1(\Gamma)\}$. $A(G)$ will be given the norm defined by $\|f\|_{A(G)} = \|F\|_1 = \int_\Gamma F(u)du$, rather than the norm of $C(G)$. Using $A(G)$ and some tensor algebra techniques to be introduced in the next chapter we will be able to prove Malliavin's Theorem, which can now be stated: Spectral synthesis fails in $A(G)$ for every non-discrete compact abelian group G.

CHAPTER III

TENSOR ALGEBRAS

1. Definition and Elementary Properties of Tensor Algebras.

The definition we will give of a tensor algebra is a special case of the definition of tensor algebras for topological spaces in general.

Definition: Let X_1 and X_2 be two compact spaces, and set $X = X_1 \times X_2$, the product space, and let $C(X)$ be the space of all continuous complex-valued functions on X. The tensor algebra $V(X) = C(X_1) \hat{\otimes} C(X_2)$ is the set of all functions f in $C(X)$ of the form:

(3.1) $$f(x_1,x_2) = \sum_{j=1}^{\infty} a_j g_j(x_1) h_j(x_2), \quad (x_1,x_2) \in X,$$
where $g_j \in C(X_1)$ and $h_j \in C(X_2)$ have sup-norm at most one and
$$T = \inf \sum_{j=1}^{\infty} |a_j| < \infty,$$

where the infimum is taken over all possible decompostions of f such that $\sum_{j=1}^{\infty} |a_j| < \infty$.

We can norm V(X) by setting $\|f\|_{V(X)} = T$. With this norm it is easy to verify that V(X) is a Banach algebra. The subalgebra of V(X) whose elements admit a decomposition of the form (3.1) with only a finite number of terms is denoted $C(X_1) \otimes C(X_2)$ and $V(X) = C(X_1) \hat{\otimes} C(X_2)$ is the completion of $C(X_1) \otimes C(X_2)$.

We have already seen in Chapter II that $M_{C(X_1)} = X_1$ and $M_{C(X_2)} = X_2$. We now have the following lemma.

Lemma 1: $M_{V(X)} = X = X_1 \times X_2$.

Proof: A point in X, $x = (x_1, x_2)$, defines the homomorphism $f \longrightarrow f(x_1, x_2)$. On the other hand, every complex-valued homomorphism on $C(X_1) \hat{\otimes} 1 \cong C(X_1)$ which must be an evaluation homomorphism associated with a point of X_1 say x_1. Similarly, every complex-valued homomorphism on V(X) defines a homomorphism on $1 \hat{\otimes} C(X_2)$ which must be an evaluation homomorphism of some point in X_2 say x_2. Thus if w is a complex-valued homomorphism on V(X), then $w(f) = w(\sum_{j=1}^{\infty} a_j g_j h_j) = \sum_{j=1}^{\infty} a_j w(g_j) w(h_j) = \sum_{j=1}^{\infty} a_j g_j(x_1) h_j(x_2) = f(x_1, x_2)$. Hence $M_{V(X)} = X_1 \times X_2 = X$.

Lemma 2: $V(X)$ is a regular Banach algebra.

Proof: Let D be a closed subset of $X = X_1 \times X_2$ and let $x = (x_1, x_2)$ be a point of X such that $x \in X \setminus D$. Let $N(x)$ be a neighborhood of x such that $D \cap N(x) = \emptyset$. Set $N_1 = \{y_1 : (y_1, y_2) \in N(x)\}$ and $N_2 = \{y_2 : (y_1, y_2) \in N(x)\}$. There exists $f_1 \in C(X_1)$ such that $f_1(x_1) = 1$ and $f_1 = 0$ off N_1. Similarly, there exists $f_2 \in C(X_2)$ such that $f_2(x_2) = 1$ and $f_2 = 0$ off N_2. Note that f_1 and f_2 are such that for all $(t_1, t_2) \in X \setminus N(x)$ either $f_1(t_1) = 0$ or $f_2(t_2) = 0$. Define $f(t_1, t_2) = f_1(t_1) f_2(t_2)$. Then $f \in V(X)$ and $f(x_1, x_2) = 1$ while $f|_D = 0$. Thus the algebra $V(X)$ is regular.

2. The Linear Mappings P and M.

In case $X_1 = X_2 = G$ is a compact abelian group we simply write $V(G) = C(G) \boxtimes C(G)$.

We define two linear mappings $\overline{M}$ and $\overline{P}$, $\overline{M}: C(G) \longrightarrow C(G \times G)$ and $\overline{P}: C(G \times G) \longrightarrow C(G)$ as follows:

(3.2) $\overline{M}(f)(x,y) = f(x+y)$ for $f \in C(G)$, $x, y \in G$ and

$\overline{P}(g)(x) = \int_G g(x-y, y)\,dy$, for $g \in C(G \times G)$, $x \in G$

where dy indicates integration with respect to normalized Haar measure on G.

Lemma 3: $\overline{P} \circ \overline{M}$ is the identity on $C(G)$.

Proof: Choose $f \in C(G)$. $\overline{P} \circ \overline{M}(f)(x) = \overline{P}(\overline{M}(f))(x) = \int_G \overline{M}(f)(x-y,y)dy = \int_G f(x)dy = f(x)$.

Let Γ be the dual of the comact group G. Recall that $A(G) = \{f \in C(G): f = \hat{F}$ for some $F \in L_1(\Gamma)\}$. Now Γ is discrete since G is compact and so $f \in A(G)$ implies $f(x) = \hat{F}(x) = \sum_{\chi \in \Gamma} a_\chi \chi(x)$ where $\|f\|_{A(G)} = \sum_{\chi \in \Gamma} |a_\chi|$. We define M to be the restriction of $\overline{M}$ to A(G) and we define P to be the restriction of $\overline{P}$ to V(G).

Lemma 4: M is continuous and norm-decreasing.

Proof: Let $f \in A(G)$ and write $f = \sum_{\chi \in \Gamma} a_\chi \chi(x)$ where $\sum_{\chi \in \Gamma} |a_\chi| < \infty$. Now $M(f)(x,y) = \sum_{\chi \in \Gamma} a_\chi \chi(x+y) = \sum_{\chi \in \Gamma} a_\chi \chi(x)\chi(y)$ since M is linear and characters are homomorphisms. Thus $M(f) \in V(G)$ and $\|M(f)\|_{V(G)} \leq \sum_{\chi \in \Gamma} |a_\chi| = \|f\|_{A(G)}$, that is, M is norm-decreasing.

Lemma 5: P is continuous and norm-decreasing. Furthermore, P maps V(G) onto A(G).

Proof: Let $f \in V(G)$ and write $f(x,y) = \sum_{j=0}^{\infty} a_j g_j(x) h_j(y)$. Evaluate $P(f)(x) = \int_G \sum_{j=0}^{\infty} a_j g_j(x-y) h_j(y)dy = \sum_{j=0}^{\infty} a_j (g_j * h_j)(x)$. Now g_j and h_j belong to $L_2(G)$ and we can argue that $g_j * h_j \in A(G)$ with the aid of the Plancherel Theorem [7, p.26]. Also $\|g_j * h_j\|_{A(G)} \leq \|g_j\|_2 \|h_j\|_2 \leq \|g_j\|_\infty \|h_j\|_\infty \leq 1$, so that

$\|P(f)\|_{A(G)} \leq \|f\|_{V(G)}$. Thus P is continuous and is norm-decreasing. Since $P \circ M$ is the identity on A(G) P must map V(G) onto A(G).

Also since $P \circ M$ is the identity on A(G) and $\|M\| \leq 1$ and $\|P\| \leq 1$ we see that M is an isometry. Hence we can study V(G) to deduce properties of A(G) and vice versa. As an example of this we have the following theorem.

<u>Theorem 6</u>: If a closed subset E of G is not a set of spectral synthesis for A(G), then the closed set $E^* = \{(x,y): x+y \in E\}$ is not a set of spectrsl synthesis for V(G).

<u>Proof</u>: Since E is not a set of spectral synthesis for A(G) there exist a real number $a > 0$ and a function $f \in I(E) \subseteq A(G)$ such that $\|f-g\|_{A(G)} > a$ for all $g \in J(E)$. M(f) is then a function in $I(E^*)$. Now every neighborhood of E^* contains a neighborhood U of the form $\{(x,y): x+y \in W$, where W is a neighborhood of $E\}$. Let u be a function that vanishes on U. Then $P(u)(x) = \int_G u(x-y,y)dy =$ 0 whenever x is in W, i.e., $P(u) \in I_0(E)$ whenever $u \in I_0(E^*)$. Since $\|M(f)-u\|_{V(G)} \geq \|P \circ M(f)-P(u)\|_{A(G)} = \|f-P(u)\|_{A(G)} > a$, E^* is not a set of spectral synthesis for V(G).

3. Kronecker Sets.

Definition: Let G be a locally compact group. A subset E of G is said to be independent in case for every choice of k distinct points in E, $x_1, x_2, \ldots, x_k$, and k integers, $n_1, n_2, \ldots, n_k$, $\sum_{j=1}^{j=k} n_j x_j = 0$ implies $n_j x_j = 0$ for all $j \leq k$. A subset E of G is said to be a Kronecker set in case every continuous unitary function on E can be uniformly approximated by the characters of G, i.e., for every unitary function f on E and for every real number $\epsilon > 0$ there exists $\chi \in \Gamma$ such that $\sup_{x \in E} |f(x) - \chi(x)| < \epsilon$. In groups of finite order there are no non-empty Kronecker sets but the following definition provides analogous objects of interest. A subset E of G is said to be of type K_p where p is a natural prime if $\{\chi|_E : \chi \in \Gamma\} = \{f \in C(E) : f^p = 1\}$.

Lemma 7: Kronecker sets and sets of type K_p are independent.

Proof: Let E be a Kronecker set. Choose $x_1, x_2, \ldots x_k$ in E and any integers $n_1, n_2, \ldots, n_k$. Suppose $\sum_{j=1}^{j=k} n_j x_j = 0$. Then $\chi(\sum_{j=1}^{j=k} n_j x_j) = 1 = \sum_{j=1}^{j=k} \chi(x_j)^{n_j}$. Now every function f on E which can be uniformly approximated on E by the characters of G must satisfy $\sum_{j=1}^{j=k} f(x_j)^{n_j} = 1$. Since E is a Kronecker set, $\sum_{j=1}^{j=k} a_j^{n_j} = 1$ for arbitrary complex numbers a_j of modulus 1. Hence $n_j = 0$ for all $1 \leq j \leq k$.

If E is of type K_p then $\sum_{j=1}^{j=k} f(x_j)^{n_j} = 1$ must be satisfied by every continuous function f on E of modulus 1. Thus $n_j = 0 (\mathrm{mod}\ p)$ for $1 \leq j \leq k$.

Let E be a subset of an LCA group G and set $A(E) = \{f|_E : f \in A(G)\}$. Endow A(E) with the quotient norm, i.e., $\|f\|_{A(E)} = \inf\{\|F\|_{A(G)} : F = f \text{ on } E\}$. Then A(E) can be canonically identified with A(G)/ I(E) and $M_{A(E)} = E$. If E is a compact Kronecker set, it can be shown that A(E) and C(E) are isometric [7, p.113].

<u>Theorem 8</u>: Let G be a compact group. If K_1 and K_2 are closed subsets of G such that $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2$ is a Kronecker set (or type K_p set), then $A(K_1+K_2)$ is isometrically isomorphic to $C(K_1) \boxtimes C(K_2)$.

<u>Proof</u>: $K_1 \cup K_2$ is Kronecker and hence is independent. Thus we have the canonical map $K = K_1 \times K_2 \longrightarrow K_1+K_2 = E$ which permits us to identify A(E) with a subalgebra A'(K) of C(K) endowed with the norm of A(E). That is, $A'(K) = \{F \in C(K): F(k_1,k_2) = f(k_1+k_2) \text{ where } f \in A(E) \text{ and } (k_1,k_2) \in K\}$, and $\|F\|_{A'(K)} = \|f\|_{A(E)}$. The remainder of this proof is broken into two parts. First, $A'(K) \subseteq V(K) = C(K_1) \boxtimes C(K_2)$ and the injection map is norm-decreasing.

If $F \in A'(K)$, then for every real number $\epsilon > 0$ there exists a function $g = \sum_{\chi\in\Gamma} a_\chi \chi$ in $A(G)$ such that $F(k_1,k_2) = g(k_1+k_2)$ and $\|g\|_{A(G)} = \sum_{\chi\in\Gamma} |a_\chi| \leq \|F\|_{A(E)} + \epsilon$. Now $F(k_1,k_2) = \sum_{\chi\in\Gamma} a_\chi \chi(k_1+k_2) = \sum_{\chi\in\Gamma} a_\chi \chi(k_1)\chi(k_2)$ so that $F \in V(K)$ and $\|F\|_{V(K)} \leq \sum_{\chi\in\Gamma} |a_\chi| \leq \|F\|_{A'(K)} + \epsilon$. Hence $\|F\|_{V(K)} \leq \|F\|_{A'(K)}$.

Second, $V(K) \subseteq A'(K)$ and the injection map is norm-decreasing. If $F \in V(K)$ for every real number $\epsilon > 0$, F can be written $F(k_1,k_2) = \sum_{j=1}^{\infty} a_j g_j(k_1) h_j(k_2)$ where $\|g_j\|_{C(K_1)} \leq 1$, $\|h_j\|_{C(K_2)} \leq 1$ and $\sum_{j=1}^{\infty} |a_j| \leq \|F\|_{V(K)} + \epsilon$. Since $K_1 \cup K_2$ is Kronecker (or of type K_p) so are K_1 and K_2. Consequently, $A(K_i)$ and $C(K_i)$ are isometric for $i = 1, 2$. Thus there exist for each positive integer j two functions in $A(G)$, g'_j and h'_j, equal respectively to g_j on K_1 and h_j on K_2 such that $g'_j(k_1) = \sum_{\chi\in\Gamma} b_\chi \chi(k_1)$ and $h'_j(k_2) = \sum_{\psi\in\Gamma} c_\psi \psi(k_2)$ and $\sum_{\chi\in\Gamma} |b_\chi| \leq 1+\epsilon$ and $\sum_{\psi\in\Gamma} |c_\psi| \leq 1+\epsilon$. $F(k_1,k_2)$ is thus written: $F(k_1,k_2) = \sum_{\chi,\psi\in\Gamma} a_{\chi,\psi} \chi(k_1)\psi(k_2)$ where $\sum_{\chi,\psi\in\Gamma} |a_{\chi,\psi}| \leq \|F\|_{V(K)}(1+2\epsilon+\epsilon^2)$. Set $\chi\cup\psi$ equal to the continuous function on $K = K_1 \times K_2$ that is equal to χ on K_1 and ψ on K_2. Now $\chi\cup\psi(k_1,k_2) = \chi(k_1)\psi(k_2)$ so that $\chi\cup\psi$ is unitary. Since $K_1 \cup K_2$ is Kronecker there exists a character θ of G such that $\|\chi\cup\psi - \theta\| < \eta$, for every real number $\eta > 0$. Also $\|\chi(k_1)\psi(k_2)-\theta(k_1)\theta(k_2)\|_{V(K)} \leq 2\eta$. Thus if we set $f_\eta(k_1,k_2) = \sum_{\chi,\psi\in\Gamma} a_{\chi,\psi}\,\theta(k_1)\theta(k_2)$, then $f_\eta \in A'(K)$, $\|f_\eta\|_{A'(K)} \leq \|F\|_{V(K)}(1+2\epsilon + \epsilon^2)$ and $\|F-f_\eta\|_{V(K)} \leq 2\eta\,\|F\|_{V(K)}(1+2\epsilon + \epsilon^2)$. If we choose a sequence $\{\eta\}_i$ of real numbers tending toward zero we see that F is the sum of a series of functions converging in $A'(K)$, i.e., $\|F-f_\eta\|_{V(K)} \longrightarrow 0$ as $\eta \longrightarrow 0$. Thus F belongs to $A'(K)$. Finally, $\|f_\eta\|_{A''(K)} \leq \|F\|_{V(K)}(1+2\epsilon + \epsilon^2)$ implies that $\|F\|_{A''(K)} \leq \|F\|_{V(K)}$.

4. Perfect Sets and Cantor Sets.

<u>Definition</u>: A set is said to be <u>perfect</u> in case it is non-empty, compact, and has no isolated points. A topological space is said to be <u>totally disconnected</u> in case every open set is also closed. A <u>Cantor set</u> is a perfect metric totally disconnected set.

All Cantor sets are homeomorphic to the Cantor set of the real line. The group $D_\omega = \prod_{i=0}^{\infty} (Z_2)_i$, the product of countably many times of the group of two elements is a Cantor set. Our interest in Cantor sets and in D is explained by the fact that every locally compact abelian group contains a Cantor set which is a Kronecker set or a type K_p set [7, p.100].

CHAPTER IV

MALLIAVIN'S THEOREM

1. Introduction.

We are now nearly ready to prove Malliavin's Theorem that if G is a non-discrete compact abelian group, then spectral synthesis fails in A(G). We have already seen that if G is an infinite compact group then G contains a compact Cantor set K which is also a Kronecker set or a type K_p set. K can be partitioned into K_1 and K_2 both Cantor sets such that $K_1 \cap K_2 = \emptyset$. If we set $E = K_1 + K_2$, then A(E) is isomorphic to $V(D_\infty) = C(D_\infty) \hat{\otimes} C(D_\infty)$ since K_1 and K_2 are Cantor sets. If a closed subset F of E is not a set of spectral synthesis for A(E), _a fortiori_ F is not a set of spectral synthesis for A(G). Hence to prove Malliavin's Theorem all we must show is that spectral synthesis fails in A(E). Since A(E) is isomorphic to $V(D_\infty)$ we can make one final reduction: Malliavin's Theorem is proved if spectral synthesis fails in $V(D_\infty)$. To prove this we need the counterexample of Schwartz.

2. The Counterexample of Schwartz.

Let R be the additive group of real numbers and let $R^3 = R \times R \times R$, three-dimensional Euclidean space with dual group R^3. Let $E = \{z \in R^3 : |z| = (z_1^2+z_2^2+z_3^2)^{1/2} = 1\}$. The counterexample of Schwartz [8] states that E is not a set of spectral synthesis for $A(R^3)$.

Theorem 1: Spectral synthesis fails in $A(R^3)$. In particular E is not a set of spectral synthesis for $A(R^3)$.

Proof: We have already mentioned in Chapter II that in $L_1(G)$ which is isometrically isomorphic to A(G) the two definitions of spectral synthesis are equivalent. In our case $G = R^3$ and to prove the theorem we will show that E is not a set of spectral synthesis in the second sense for $L_1(R^3)$.

Let W be the set of all complex-valued infinitely differentiable functions on R^3. Then W is a subset of

$A(R^3)$ [7, p.165] . If $g \in W$, $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3) \in R^3$ then $g(y) = \hat{f}(y) = \int_{R^3} f(x)e^{-ix\cdot y}dx$ where $f \in L_1(R^3)$ and $x\cdot y = x_1y_1 + x_2y_2 + x_3y_3$. By Leibnitz' Rule, $\partial g/\partial y_1\ (y) = \int_{R^3} -ix_1f(x)e^{-ix\cdot y}dx$.

Let J be the set of all $f \in L_1(R^3)$ such that $f \in W$ and $\hat{f}(y) = 0$ on E. Let I be the set of all $f \in J$ such that $\partial \hat{f}/\partial y_1 = 0$ on E. Let $\bar{I}$ and $\bar{J}$ be the closures of I and J respectively. Now $Z(\bar{I}) = Z(\bar{J}) = E$. We will show $\bar{I} \neq \bar{J}$ by constructing a bounded linear functional S on $L_1(R^3)$ such that $S|_{\bar{I}} = 0$ and $S|_{\bar{J}} \neq 0$, i.e., S annihilates I but not J. Let u be the unit mass, uniformly distributed over E and set $\hat{u}(x) = \int_E e^{ix\cdot y}du(y)$. Fix $x \in R^3$ and let r be the Euclidean distance from x to 0. Then $\hat{u}(x) = 1/4\int_0^{2\pi}d\phi \int_0^{\pi} e^{ir\cos\theta}d\theta = (\sin r)/r$, by changing to spherical coordinates. Thus $|x_1\hat{u}(x)| \leq 1$ for all $x = (x_1,x_2,x_3)$ in R^3. Let S be defined by $S(f) = \int_{R^3} f(x)x_1\hat{u}(-x)dx$ so that S is a bounded linear functional on $L_1(R^3)$. If $f \in W$ then $x_1f \in L_1(R^3)$. So

$$\begin{aligned} S(f) &= \int_{R^3} f(x)x_1\hat{u}(-x)dx \\ &= \int_{R^3} x_1f(x) \int_E e^{-ix\cdot y}du(y)\, dx \\ &= \int_E \int_{R^3} x_1f(x)e^{-ix\cdot y}\, dx\, du(y) \\ &= \int_E \partial \hat{f}/\partial y_1\, du. \end{aligned}$$

Thus $S(f) = 0$ for all $f \in I$. However it is obvious that there are functions in J on which S does not vanish. Hence $\bar{I} \neq \bar{J}$ and spectral synthesis fails in $A(R^3)$.

Actually to prove Malliavin's Theorem we need the fact that spectral synthesis fails in $A(T^3)$ where T^3 is the three-dimensional torus. Since T^3 is locally isomorphic to R^3, the above proof can be modified to show that spectral synthesis fails in $A(T^3)$ [10, pp.95ff] .

3. Malliavin's Theorem.

Malliavin's Theorem will be proved when we establish the following result.

<u>Theorem 2</u>: Spectral synthesis fails in $V(D_\infty)$.

<u>Proof</u>: We have just seen in the preceeding section that spectral synthesis fails in $A(T^3)$. In view of Theorem 3.6 spectral synthesis must also fail in $V(T^3)$.

We now define a mapping d from D_∞ onto T^3 as the composite $d = d''\circ d'$ where $d': D_\infty \longrightarrow I^3 = [0,1]^3$ is defined by $d'((a_1,a_2,a_3,\ldots)) = (0.a_1a_4a_7\ldots,\ 0.a_2a_5a_8\ldots,\ 0.a_3a_6a_9\ldots)$ here the elements of I^3 are written in their binary expansions and $d'': I^3 \longrightarrow T^3$ is defined by $d''((x_1,x_2,x_3)) = (e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3})$. The mapping d is surjective since both d' and d'' are. Also d is continuous and it preserves normalized Haar measure. That is, if E is a subset of D_∞ and if m represents normalized Haar measure on D_∞ and if m' represents normalized Haar measure on T^3 then $m(E) = m'(d(E))$. The mapping d is also injective if we remove from D_∞ a set of null measure, i.e., a countable set. The mapping d thus permits us to identify canonically $L_\infty(D_\infty)$ and $L_\infty(T^3)$ and hence to define a canonical isometric mapping from $C(T^3)$ onto $C(D_\infty)$. That is, if $f \in C(D_\infty)$ then $f = g \circ d$ for some $g \in C(T^3)$. Thus we can write:

$$C(T^3) \hat{\otimes} C(T^3) \xrightarrow{s} C(D_\infty) \hat{\otimes} C(D_\infty) \xrightarrow{s'} L_\infty(D_\infty) \hat{\otimes} L_\infty(D_\infty) \cong L_\infty(T^3) \hat{\otimes} L_\infty(T^3) \text{ or}$$

$$V(T^3) \xrightarrow{s} V(D_\infty) \xrightarrow{s'} V' = L_\infty(D_\infty) \hat{\otimes} L_\infty(D_\infty) \cong L_\infty(T^3) \hat{\otimes} L_\infty(T^3),$$

where $s' \circ s$ is the canonical injection of $V(T^3)$ into V'. We will consider elements of V' as being in $L_\infty(D_\infty) \hat{\otimes} L_\infty(D_\infty)$ or as being in $L_\infty(T^3) \hat{\otimes} L_\infty(T^3)$ as is appropriate. Because of the definition of tensor products both s and s' have operator norm at most one.

We want to show now that s' is an isometry. $C(D_\infty)$ contains an approximating identity, i.e., a sequence of functions $\{p_j\}$ in $C(D_\infty)$ such that each p_j is positive, $\int_{D_\infty} p_j(y)dy = 1$ for each j and the support of p_j tends to $\{0\}$ as j tends to infinity. Thus there exists a sequence of mappings $\overline{p}_n$ defined by $\overline{p}_n(f) = p_n * f$, each mapping $L_\infty(D_\infty)$ into $C(D_\infty)$ such that $\overline{p}_n(f) \to f$ for all $f \in C(D_\infty)$. Similarly there exists a sequence of mappings $\bar{p}_n \otimes \bar{p}_n$ defined by $\bar{p}_n \otimes \bar{p}_n(f \otimes g) = \bar{p}_n(f) \otimes \bar{p}_n(g)$ each mapping $V' = L_\infty(D_\infty) \hat{\otimes} L_\infty(D_\infty)$ into $V(D_\infty)$. Now if s' were not an isometry there would exist a function F in $V(D_\infty)$ such that $\| \bar{p}_n \otimes \bar{p}_n \, s'(F) \|_{V(D_\infty)} \leq \| s'(F) \|_{V'} < \|F\|_{V(D_\infty)}$. When n goes to infinity $\bar{p}_n \otimes \bar{p}_n \circ s''(F) \longrightarrow F$ and we obtain $\|F\|_{V(D_\infty)} \leq \|s''(F)\|_{V'} < \|F\|_{V(D_\infty)}$, which is clearly impossible. Thus s' must be an isometry.

If we utilize an approximating identity P_n in $C(T^3)$ and the mappings

$\bar{P}_n : L_\infty(T^3) \longrightarrow C(T^3)$ defined by $\bar{P}_n(f) = P_n * f$, $f \in L_\infty(T^3)$

$\bar{P}_n \otimes \bar{P}_n : V'' \longrightarrow V(T^3)$ defined by $\bar{P}_n \otimes \bar{P}_n(f \otimes g) = \bar{P}_n(f) \otimes \bar{P}_n(g)$, $f \otimes g \in V'$

we can apply the same argument <u>mutatis mutandis</u> to show that $s' \circ s$ is an isometry. Hence so is s.

To help prevent confusion the following diagram is presented:

$$\begin{array}{ccccc} V' & \xrightarrow{\bar{P}_n \otimes \bar{P}_n} & V(T^3) & & \\ \downarrow & & \downarrow s & & \\ V'' & \xrightarrow{\bar{p}_n \otimes \bar{p}_n} & V(D_\infty) & \xrightarrow{s'} & V'. \end{array}$$

Both s and s' are isometries and both $\bar{P}_n \otimes \bar{P}_n$ and $\bar{p}_n \otimes \bar{p}_n$ have operator norm at most one.

Corresponding to s there is a continuous bijective mapping $\bar{s}$ from $M_{V(D_\infty)} = D_\infty \times D_\infty$ onto $M_{V(T^3)} = T^3 \times T^3$ defined by $\widehat{\bar{s}(x,y)}(f) = s(f)(x,y)$ for $(x,y) \in D_\infty \times D_\infty$ and $f \in V(T^3)$. Now spectral synthesis fails in $V(T^3)$ so there exist a closed subset E of $T^3 \times T^3 = M_{V(T^3)}$, a real number $a > 0$ and a function f in $I(E) \subseteq V(T^3)$ such that $\| f-h \|_{V(T^3)} > a$, for each $h \in J(E)$. Consider $E^* = \bar{s}^{-1}(E)$, a closed subset of $D_\infty \times D_\infty = M_{V(D_\infty)}$. We will show that E*is not a set of spectral synthesis for $V(D_\infty)$. If $f \in I(E)$ then $s(f) \in I(E^*)$. Let g be any function in $J(E^*)$. Since s' is an isometry $\|s(f)-g\|_{V(D_\infty)} = \| s' \circ s(f) - s'(g) \|_{V'}$. Thus

$\| (\bar{P}_n \otimes \bar{P}_n)(s' \circ s)(f) - (\bar{P}_n \otimes \bar{P}_n) \circ s'(g) \|_{V(T^3)} \leq \|s(f)-g\|_{V(D_\infty)}$

since $\bar{P}_n \otimes \bar{P}_n$ has operator norm at most one. Set

$$f_n = (\bar{P}_n \otimes \bar{P}_n)(s' \circ s)(f) \text{ and}$$
$$g_n = (\bar{P}_n \otimes \bar{P}_n) \circ s'(g).$$

Note that both f_n and g_n are in $V(T^3)$. Now $\|f_n - f\|_{V(T^3)}$ tends to 0 as n tends to infinity so that if we choose m sufficiently large $\|f_m - f\|_{V(T^3)} \leq a/2$. We can also require that m be sufficiently large so that g_m be close to 0 in a neighborhood of E, i.e., $\|f-g_m\|_{V(T^3)} \geq a$. Gathering our inequalities we have

$\|s(f)-g\|_{V(D_\infty)} \geq \| f_m - g_m \|_{V(T^3)} \geq \|f_m + f - f - g_m\|_{V(T^3)} \geq$

$\|f-g_m\|_{V(T^3)} - \|f_m-f\|_{V(T^3)} \geq a-a/2 = a/2 > 0.$
Hence E* is not a set of spectral synthesis for $V(D_\infty)$ and spectral synthesis fails in $V(D_\infty)$.

This concludes the proof of Malliavin's Theorem. For completeness we will restate Malliavin's Theorem and summarize the reductions made to accomplish the proof.

Theorem 3: (Malliavin) Let G be a non-discrete compact abelian group. Then spectral synthesis fails in A(G).

Proof: If G is a non-discrete compact abelian group, G contains a Cantor set K that is also a Kronecker set or a type K_p set. K can be decomposed into K_1 and K_2, both Cantor sets, such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 = \emptyset$. Then $A(K_1+K_2) = C(K_1) \hat{\otimes} C(K_2) = C(D_\infty) \hat{\otimes} C(D_\infty) = V(D_\infty)$. Now $V(D_\infty)$ is not of spectral synthesis, so that $A(K_1+K_2)$ is not of spectral synthesis and hence spectral synthesis fails in A(G).

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