1957

A new approach to the teaching of arithmetic in the sixth grade

Maryanne McBride
The University of Montana

A NEW APPROACH

TO THE TEACHING OF ARITHMETIC

IN THE SIXTH GRADE

by

MARYANNE McBRIDE

B. A. Arizona State College, 1935

Presented in partial fulfillment

of the requirements for the degree

of

Master of Arts in Education

MONTANA STATE UNIVERSITY

1957

Approved by:

George H. Millis
Chairman, Board of Examiners

Ellis Waldron
Dean, Graduate School

AUG 1 9 1957
Date

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

THE PURPOSE OF THIS STUDY

The purpose of this study is to present an approach to the study of arithmetic which differs in many ways from the usual presentation in most textbooks.

The contention of this paper is that the intrinsic interest in arithmetic is to be found in the mathematical logic of the subject, rather than in the importance of its use in applications. In any discussion of the teaching of arithmetic in which the topic of abstract logic and precise definition arises, there are at least two questions that are asked. The first concerns the ability of a grade school pupil to grasp the ideas in abstract form and the second is the ability of the teacher to understand and present these ideas.

In this paper it is the aim of the writer to recount a method of teaching sixth grade pupils arithmetic from a mathematical point of view. The purpose is to show first what mathematics ought to be understood by the teacher and how the teacher might present this mathematics to the class in a meaningful way. This method has been carried out by the writer of this paper for the past three years. The results obtained at least in this one case were markedly better than those obtained by the usual methods presented in the average textbooks. It is the opinion of the writer that it might be highly profitable to investigate the entire area much further to find out what ideas should be presented in the seventh and eighth grades as well as in the intermediate grades to discover what might be done between arithmetic and algebra. The fundamental processes of algebra are, for the most part, to be found in arithmetic, and if the fundamental ideas are well understood by the student before he graduates from the eighth grade, the transition to handling unknowns might be made an easy one.

As to the preparation of the teacher, one course in the theory of arithmetic is very brief. If such a course could be coordinated with a subsequent course in methods using the ideas presented in the theory of arithmetic, it is felt that the average teacher would be in a position to present the methods outlined below in an adequate fashion.

In Chapter III there is a brief outline of the history of numbers which would prove to be of interest to both teacher and pupil and which sets mathematics in perspective with history, art, and sociology.

CHAPTER II

MATHEMATICAL DEFINITIONS AND THEIR EXPLANATION

Certainly there is no point in either student or teacher memorizing definitions without accompanying understanding. One must, at every stage, guard against formalized work associated with lack of comprehension. There is here provided a set of definitions written in a sophisticated way. The teacher, however, should be able to state them in simple terms so that the children can understand them.

I. MATHEMATICAL DEFINITIONS

Since the study of whole numbers and fractions will comprise a large part of the work of the sixth grade arithmetic course and since sixth grade pupils work with whole numbers as well as with fractions, the teacher should know the mathematical definitions of whole numbers and fractions. The wording of these mathematical definitions varies from book to book, but probably one of the most meaningful is that of L. M. Graves, "The Theory of Functions of Real Variables":[1]

---

[1]Lawrence M. Graves, "The Theory of Functions of Real Variables" (course offered in the University of Chicago, Summer Quarter, 1931), pp. 3-6. (Mimeographed.)

The Real Number System

A. The system of positive integers. We assume a set M of objects m and a relation $*$ with the properties:

1. To every m in M there corresponds an element m^* in M (called the successor of m).

2. There is an element m_o in M which is not the successor of an element of M.

3. There is at most one element of M having a given element of M as its successor.

4. If M_o is a subclass of M, if M_o contains the element m_o, and if furthermore M_o contains the successor of each of its elements, then M_o is identical with M.

Definition 1. Addition. The definition is inductive. For each m, $m + m_o = m^*$. For each m_1 and m_2, $m_1 + m_2^* = (m_1 + m_2)^*$.

Definition 2. The order relation. We say $m_1 > m_2$ in case there is an element m in M such that $m_1 = m_2 + m$.

Definition 3. Multiplication. The product $m_1 \times m_2$ or $m_1 m_2$ is defined as the sum of m_2 terms each equal to m_1.

Properties of the system $(M, *, +, \times, >)$.

1. The associative, commutative, and distributive laws hold for addition and multiplication.

2. If $m_1 > m_2$, then $m_1 + m > m_2 + m$ for every m.

3. Uniqueness of subtraction. For every m_1 and m_2 there is at most one element m in M such that $m_1 = m_2 + m$. When this element exists, it is denoted by $m_1 - m_2$.

4. If $m_1 > m_2$, then $m_1 m > m_2 m$ for every m.

5. If $m_1 > m_2$, then $m(m_1 - m_2) = mm_1 - mm_2$ for every m.

6. If $m_1 > m_2$ and $m_3 > m_4$, then $(m_1 - m_2)(m_3 - m_4) = m_1 m_3 + m_2 m_4 - m_1 m_4 - m_2 m_3$.

7. Uniqueness of division. If $m_1 > m_2$, then there is one and only one element m in M such that $m_2 m \leq m_1 < m_2(m^*)$.

8. If $m_1 = m_2 m$ and $m_1' = m_2' m'$, then m is greater than, equal to, or less than m', according as $m_1 m_2'$ is greater than, equal to, or less than $m_1' m_2$.

. . . .

B. <u>The system of positive fractions</u>. Consider the class of all pairs of positive integers (m, m').

Definition 4. <u>Equivalence of pairs</u>. $(m_1, m_1') \doteq (m_2, m_2')$ in case $m_1 m_2' = m_2 m_1'$.

Theorem 1. The relation of equivalence is reflexive, symmetric, and transitive, and hence divides the class of all pairs of positive integers into mutually exclusive and exhaustive sub-classes.

Definition 5. These sub-classes, denoted by f, are called <u>positive fractions</u>. That is, a positive fraction f is a class of equivalent pairs of positive integers, which includes <u>all</u> pairs equivalent to any one of its pairs. We say that f is a <u>maximal</u> class of equivalent pairs.

The class of all positive fractions is denoted by F.

Theorem 2. If $(m_1m_1' \overset{''}{=} (m_3,m_3')$ and $(m_2,m_2') \overset{''}{=} (m_4,m_4')$, then $(m_1m_2' + m_1'm_2, m_1'm_2') \overset{''}{=} (m_3m_4' + m_3'm_4, m_3'm_4')$ and $(m_1m_2, m_1'm_2') \overset{''}{=} (m_3m_4, m_3'm_4')$.

Definition 6. <u>Addition and multiplication</u>. If the fraction f_1 contains the pair (m_1,m_1') and f_2 contains (m_2,m_2'), then the sum $f_1 + f_2$ is defined to be the class containing the pair $(m_1m_2' + m_1'm_2, m_1'm_2')$, and the product f_1f_2 is the class containing $(m_1m_2, m_1'm_2')$. The sum and product are well-determined fractions by virtue of the last theorem.

Definition 7. <u>The order relation</u>. A fraction $f_1 > f_2$ in case f_1 contains a pair (m_1,m_1') and f_2 contains a pair (m_2,m_2') such that $m_1m_2' > m_2m_1'$.

Definition 8. A <u>group</u> is a class G of objects g with an operation $\#$, having the properties:

1. To each pair of elements g_1 and g_2 there corresponds a unique element of G denoted by $g_1 \# g_2$.

2. # is associative.

3. For every pair of elements g_1 and g_2 there exist elements g_3 and g_4 of G such that $g_1 \# g_3 = g_2$, $g_4 \# g_1 = g_2$.

Theorem 3. Every group (G, #) has also the properties:

4. G contains a unique unit element g_u such that $g \# g_u = g_u \# g = g$ for every g.

5. To every g there corresponds a unique element g^{-1} such that $g \# g^{-1} = g^{-1} \# g = g_u$.

6. The elements g_3 and g_4 of property 3 are necessarily unique.

Definition 9. A group (G,#) is <u>commutative</u> in case $g_1 \# g_2 = g_2 \# g_1$ for every g_1 and g_2.

Theorem 4. The class F of positive fractions with the operation of multiplication constitutes a commutative group. The class F with addition does not constitute a group.[2]

II. EXPLANATIONS OF MATHEMATICAL DEFINITIONS

The Real Number System

A. <u>The system of positive integers</u>. We assume a set of objects such as the positive whole numbers, 1, 2, 3, 4, 5, 6, 7 ------------ and an operation such as + .

1. For every object in the set there is another

---

[2]Lawrence M. Graves, "The Theory of Functions of Real Variables" (course offered in the University of Chicago, Summer Quarter, 1931), pp. 3-6. (Mimeographed.)

object which is called the successor of the first object. Example: $4 + 1 = 5$, $5 + 1 = 6$, etc.

2. There is an object in the set which is not the successor of any object of the set. Thus: 1 has no predecessor among the positive integers.

3. There is at most one object in the set having a given object of the set as its predecessor, thus: 7 has one and only one predecessor which is 6.

4. If there is a subclass of the set M, and if the subclass of the set M contains the first element and the successor of each of its elements, then the subclass is identical with the set M. Example: If the subclass contains 1 and the successor of each element such as 1, 2, 3, 4, 5, - - - - - - - then the subclass is identical with the set M.

Definition 1. The definition of addition is inductive. (This means reasoning from a study of particulars to general principles). For each object in the set, the operation may be performed upon it and the first object of the set, thus obtaining another object of the set. Example: $3 + 1 = 4$. Also for each object in the set and its successor, upon which the operation has been performed is equal to both elements upon which the operation has been performed. Example: $1 + 2 = 1 + 1^* = (1 + (1 + 1))$. Also, $5 + 5^* = (5 + 5)^* = 11$.

Definition 2. <u>The order relation</u>. We say one object of the set is greater than its successor, in case there is an object in the set such that the first object is equal to the second object of the set plus some other object in the set. Thus: $9 > 6$ since $9 = 6 + 3$ and 3 is an integer. Note that only one of the following conditions can hold at a time. Either $m < n$, or $m > n$ or $m = n$. The concept of a first element or last element can be defined in terms of this order. A first element m in a set of these objects is one which stands in the relation $m < n$ for all the objects in the set and there are no objects p in the set that stand in the relation $p < m$. A last element can be similarly defined with the order relation reversed. Well ordered means there is a first object in the set. Fractions are ordered but not well ordered.

Well ordered means there must be a least element. With finite integers there must be a last element and a first element.

Example of well ordered objects:

a. ____________________ .c

b. ____________________ .d

a + b. ____________________ .c + d

Definition 3. Multiplication. The product of 3×6 or $(3 \cdot 6)$ is defined as the sum of 6 terms each equal to 3. Example: $3 + 3 + 3 + 3 + 3 + 3 = 18$. Properties of the system are: the set, an operation, plus, times, and is greater than.

1. Associative means that no matter how elements are grouped for + and for x the results are the same. Thus: $(a + b) + c = a + (b + c)$ and $a \times b \times c = a \times (b \times c)$.

Commutative means that it makes no difference in what order elements are added or multiplied; the results are the same. Thus: $a + b = b + a$ and $a \times b = b \times a$.

The distributive law means that no matter which operation is performed first, the results will be the same. Thus: $2\ (5 + 4) = (2 \cdot 5 + 2 \cdot 4) = 2\ (9) = 18$.

2. If $6 > 4$, then $6 + 2 > 4 + 2$ and this holds for every element in the set.

3. Uniqueness of subtraction. For every (9) and (6) in the set there is at most one element (3) in the set such that $9 = 6 + 3$. When this element (3) exists it is denoted by $3 = 9 - 6$.

4. If $9 > 6$ then $(9 \cdot 3) > (6 \cdot 3)$ and this holds for every element such as (3).

5. If $8 > 6$, then $4\ (8-6) = (4 \cdot 8) - (4 \cdot 6) = 32 - 24$ for any element such as 4.

6. If $8 > 6$ and $7 > 3$, then $(8 - 6)\ (7 - 3) = (8 \cdot 7) + (6 \cdot 3) - (8 \cdot 3) - (6 \cdot 7)$ or $56 + 18 - 24 - 42 = 74 - 66 = 8$.

7. Uniqueness of division: If $10 > 2$, then there

is one and only one element (5) in the set such that $(2\cdot 5) \leqq 10 < 2(5^*)$.

8. If $10 = (5\cdot 2)$ and $18 = (6\cdot 3)$ then $2 >$, equal to or less than 3 according as $10\cdot 6$ is greater than, equal to or less than $(18\cdot 5)$.

B. The system of positive fractions. Consider the class of all pairs of positive integers. Thus: 3/4.

Definition 4. Equivalence relationship. Consider a set of elements a_i, a_j, a_k - - - - - - - - and a relation $\doteqdot$. The relation $\doteqdot$ is an equivalence relation provided:

(1) $a_i \doteqdot a_i$ for all i.

(2) $a_i \doteqdot a_j$ implies $a_j \doteqdot a_i$ for all i, j.

(3) $a_i \doteqdot a_j$ and $a_j \doteqdot a_k$ imply $a_i \doteqdot a_k$ for all i, j, k.

(1) is the reflexive property

(2) is the symmetric property

(3) is the transitive property

Definition 5. These sub-classes, denoted by f, are called positive fractions. Example of a fraction:

[1/2, 2/4, 3/6, 4/8, 5/10, 6/12 -------n/2n-------]

That is, a positive fraction is a class of equivalent ordered pairs of positive integers, which includes all pairs equivalent to any one of its pairs. We say, then, that a fraction is a maximal class of equivalent pairs.

Maximal means all.

The equivalence relation means $a \doteqdot a$ or $2/4 \doteqdot 2/4$.

If $a \doteqdot b$, then $b \doteqdot a$, or if $2/4 \doteqdot 4/8$, then $4/8 \doteqdot 2/4$. And if $a \doteqdot b$ and $b \doteqdot c$, then $a \doteqdot c$. Equivalence here means if $2/4 \doteqdot 4/8$ and if $4/8 \doteqdot 8/16$, then $2/4 \doteqdot 8/16$.

The class of all positive fractions is denoted by F.

Theorem 2. The theorem says: If $3/4 = 6/8$ and $2/7 = 6/21$, then $\frac{(3 \times 7) + (4 \times 2)}{4 \times 7} = \frac{(6 \times 21) + (8 \times 6)}{8 \times 21}$ or $\frac{21 + 8}{28} = \frac{126 + 48}{168}$ or $\frac{29}{28} = \frac{174}{168}$ and $\frac{3 \times 2}{4 \times 7} = \frac{6 \times 6}{8 \times 21}$ or $\frac{6}{28} = \frac{36}{168}$.

Definition 6. Addition and multiplication. This means that if fraction f_1 contains the pair 2/3 and if f_2 contains the pair 1/7, then $2/3 + 1/7 = \frac{(2 \cdot 7) + (3 \cdot 1)}{3 \cdot 7} = \frac{14 + 3}{21} = \frac{17}{21}$. This means that the fraction containing 2/3

[2/3, 4/6, 6/9, 8/12, 10/15, 12/18, 14/21, 16/24, 18/27, 20/30 --------2n/3n--------------------]

and the fraction containing 1/7

[1/7, 2/14, 3/21, 4/28, 5/35 --------n/7n------]

when added result in the fraction containing the sum of 14/21 and 3/21 which is 17/21.

[17/21, 34/42, 51/63, 68/84 ---------17n/21n---]

This result is unique, hence the use of common denominators. Also the product of the fraction 6/9

[6/9, 12/18, 18/27, 36/54 -------210/315-------- 6n/9n--------------------------------------]

and the fraction 5/35

[5/35, 10/70, 15/105, 20/140, 25/175, 30/210, 35/245, 40/280, 45/315 --------5n/35n -------]

result in the fraction containing the fraction.

[2/21, 4/42, 6/63, 8/84, 10/105 -------------- 30/315 -----------2n/21n--------------------]

Definition 7. <u>The order relation</u>. A fraction $f_1 > f_2$ in case f_1 contains a pair (m_1, m_1') and f_2 contains a pair (m_2, m_2') such that $m_1 m_2' > m_2 m_1'$. This means the pair m_1/m_1' (7/8) contained in f_1 is greater than the pair m_2/m_2' (1/3) contained in f_2 if $m_1 m_2'$ $(7 \cdot 3)$ is greater than $m_2 m_1'$ $(8 \cdot 1)$.

Definition 8. A group is a class G of objects g with an operation #, having the properties:

1. To each pair of elements g_1 and g_2 there corresponds a unique element of G denoted by $g_1 \# g_2$. This means that if the operation is performed upon any two of the elements of the group, the result will give an element of the group.

2. # is associative (The operation is associative) $(a + b) + c = a + (b + c)$.

Theorem 3. Every group (G, #) has also the properties:

4. G contains a unique unit element g_u such that $g \# g_u = g_u \# g = g$ for every g. This means the group must have an identity.

1 is identity for multiplication.

a/b x 1 = 1 x a/b = a/b.

o is identity for addition.

a/b + o = o + a/b = a/b.

5. To every g there corresponds a unique element g^{-1} such that $g \# g^{-1} = g^{-1} \# g = g_u$. This means the group must have an inverse. The inverse with respect to addition for a positive fraction is its counterpart in a negative fraction. Thus:

2/3 + (-2/3) = o.

The inverse for fractions for multiplication is the reciprocal (i.e.) 2/3 x 3/2 = 1.

6. The elements g_3 and g_4 of property 3 are necessarily unique.

Definition 9. A group (G, #) is <u>commutative</u> in case $g_1 \# g_2 = g_2 \# g_1$ for every g_1 and g_2. Thus: a/b + c/d = c/d + a/b; also, a/b x c/d = c/d x a/b.

Theorem 4. The class F of positive fractions with the operation multiplication constitutes a commutative group.

The class F with addition does not constitute a group.

CHAPTER III

PROCEDURES

Reading, writing, and arithmetic as they are taught in the first eight grades are, at the present time, under sharp criticism. The controversy consists of a long series of violent attacks on Schools of Education and Colleges of Education by people who may or may not be in educational work. These attacks become almost personal. Throughout the controversy, seldom is there an article that doesn't include the Teaching of Arithmetic. None propose a solution to the problem or even remedies or partial remedies. There may be no ideal solution, but there may be things we can do to improve the teaching of arithmetic in whatever ways we can.

In this connection, John W. Gardner reports:

> Concern over the mathematical incompetence of the average---and even above-average---American has become almost a national preoccupation. Science and industry cry in vain for more and better mathematicians. Ordinary businesses ask only that their employees be able to do simple arithmetic. Neither the extravagant nor the modest demands of society for mathematicians---or arithmeticians even---are being met. And public concern grows.[1]

Mr. Gardner continues by saying that the Educational Testing Service recently conducted a survey to determine

[1] John W. Gardner, "A National Weakness," American Mathematical Monthly, 63:396-399, June-July, 1956.

reasons why so many elementary and secondary school children fail in mathematics or drop out before their course is finished, and why others barely manage to struggle through. The following are some of the fundamental facts which motivated the Educational Testing Service to undertake its study.

Since 1910, according to a recent national survey,[2] there has been a consistent fall off in the proportion of high school students who take college preparatory mathematics. While it is true with the vast increase in the high school population that the character of the student population has also changed, this fails to explain why so many students begin mathematics courses in high school and do not complete them. A number of students drop elementary algebra without completing the first year; more drop out in the middle of their geometry course, and still more fail to complete intermediate algebra. A certain indifference toward the mathematics courses on the part of many young people might be suggested by these facts. However, it is, in many cases, a positive loathing for the subject rather than mild indifference that is to blame. Students have a poorer attitude toward mathematics than toward any other school subject according to nearly all of the available data. One survey showed that forty per cent of the students chose mathematics

---

[2]Gardner, loc. cit.

as their most disliked subject.

It was found that the incompetence in mathematics is prevalent and widespread even among those students who possess superior intelligence. A group of 526 students from one high school, all of whom had intelligence quotients of more than 114, were studied. Of this group, 135 either were retarded one or more years in mathematics or obtained grades of "C" or lower or dropped mathematics altogether. The mean intelligence quotient of this group was 123.[3]

Concerned over these findings, the Educational Testing Service committee acquired expert opinions and assembled data on two factors, teaching and curriculum, which many people believe lie at the heart of the problem. The committee sent out questionnaires to experienced persons in many fields, reviewed pertinent literature and interviewed students, and observed classroom procedures. Some astonishing facts concerning mathematics teaching in the United States were revealed.

It was found that one third of the states require no mathematics for certification of teachers who are preparing to teach mathematics, although all states required education courses for secondary mathematics teachers. In most cases, a student who anticipates entering elementary teaching can enter a teachers' college without any credits in secondary mathematics. Most states permit certification of teachers

---

[3]Gardner, loc. cit.

for teaching mathematics at the elementary level without their having had any mathematics courses at the college level.

> Under such circumstances, it is no surprise that one professor states: "Elementary teachers, for the most part, are ignorant of the mathematical basis of arithmetic. . . ." As for the secondary school teachers, one math professor interested in teacher education says, "They are not as good as our run-of-mine juniors."[4]

Mr. Gardner continues by saying that one observer while visiting sixty representative mathematics classes in various parts of the country, arrived at the conclusion that genuine and efficient learning was taking place in only eight of them. The same observer visited thirty-six elementary classrooms and got the inescapable impression that teaching was done in such a haphazard, routine way that any learning which took place was mostly accidental. He also decided during his observation of twenty-four high schools that the instruction there was no better and possibly even worse than on the elementary level.

The curriculum has undergone little change in spite of broad changes in the nature and knowledge of mathematics and in spite of the fact that many authorities have been suggesting changes since 1894. The result is that the curriculum is today alien in character to the real requirements for mathematics in today's world and to the interests of the students.

---

[4]Gardner, <u>loc</u>. <u>cit</u>.

The total picture given by the Educational Testing Service survey is not encouraging, but there are some indications of reform, especially in the curriculum. Educators and the public are the ones who must solve the problems. As the educators and public are given information such as that contained in the Educational Testing Service survey about the actual situation, more reforms may be made. The Educational Testing Service will soon publish a pamphlet summarizing the findings of the survey in the hope that such distribution of the facts will stimulate schools to take a good look at their mathematical programs to see what improvement can be made in them.

In his commentary regarding the crisis in mathematics, Mr. Gardner goes on to state that the problem facing our schools is a grave one and one that gives cause for alarm. It deserves the attention of all people who are interested in the future of America. This problem consists of the failure of our schools to supply more youngsters with a reasonable command of mathematics.[5]

The unprecedented advances in science and technology in recent years and the outstanding applications of these advances to the needs of industry and of national defense have caused an increased demand for intelligent and well-trained young men and women. This demand simply cannot be

---

[5]Gardner, loc. cit.

met unless extreme measures are instituted.

We do not face a shortage of talent but we do face a deficit of trained talent. We are not lacking in intelligent youth and, if good use were made of the available resources, we would not be in the present difficulty. Since the human race has, throughout history, been extremely wasteful of talent, it is hard for us to grasp the fact that we are placed in this dilemma. In previous generations the discovery and nourishment of talent has been largely accidental. Never before have the needs of society put such a premium upon gifted men and women or upon their training. Highly intelligent, highly trained men were never before considered one of the most marketable commodities. Even a generation ago this was true. Now the world needs more highly trained and highly intelligent men and women and it needs them very badly.

The nation needs men with mathematical and scientific proficiency. This need exists at all levels. Our need is not just for creative scientists at the top level.[6] Behind those scientists at the Nobel Prize level there exists a great army of capable and expertly trained men who do the testing and the acceptance or rejection of the new discoveries; men who do much of the routine testing and experimenting and who make many other contributions to the world

---

[6]Gardner, loc. cit.

of science. With these also work a very large group of well-trained and expert laboratory technicians and assistants.

In industry the same situation exists. The inventors and others of their calibre stand in the spotlight but behind them is to be found an immense army of brilliant technical men who put their discoveries to work. Others, standing in the rear, are technicians and mechanics, rows and rows of them, who are also well-trained. Only recently have Americans learned that there must be such depth in the training if our technology has the vitality and sturdiness we expect of it. Several of the countries of the world can match us for competent men at the highest levels, but we outstrip most of them in intelligent, excellently trained men working at levels reaching back to the second, third, fourth, and fifth ranks.

Several years ago when our severe shortage of scientists became apparent, the situation caused serious national concern. Some of the best minds of the country were asked to find the reasons for the shortage and to find the remedy for it.

The reasons for the deficit are many and need not concern us here. Nevertheless, those studying the problem attempted to discover the remedy; they found mathematics to be the great stumbling block. Young people who have not had sufficient training in the early years in mathematics cannot be made into scientists and it was learned that those in possession of the suitable training were far too few.[7]

That situation was the motive for the study made by the Educational Testing Service. No attempt will here be made to place blame for the sad state of affairs. Such action would be foolish and wasteful. Our schools as a whole are doing a very good job under exceedingly difficult circumstances.

However, action is essential. Our energies must be used to attract a much greater number of competent young men and women into the teaching of mathematics and to strengthen and upgrade those who are now in this field. We must improve the preparation of those who teach.

The Carnegie Corporation granted $300,000 to the American Association for the Advancement of Science in the support of a program for the improvement of teaching science and mathematics. Another effort to draw national attention to the very critical problem is the Educational Testing Service study reported here.[8]

A similar viewpoint is expressed by Jacob S. Orleans and Edwin Wandt in their article giving the results of tests given to 1000 teachers and prospective teachers in widely separated geographical situations. While arithmetic may be viewed as a series of short-cuts which have been developed

---

[7]Gardner, loc. cit.

[8]Ibid.

to facilitate computation, it is unfortunate that there are many persons who know arithmetic only as a series of short-cuts without knowing the basic concepts and the processes for which they are the short-cuts.[9]

Summarizing the results of these tests, it was found that in one example at least twenty-five per cent of the teachers tested furnished answers which showed a lack of understanding of why the second partial product is moved over one place and another forty-nine per cent either clearly did understand the procedure or furnished answers that might be acceptable depending on further explanation. The remaining twenty-six per cent of the answers could hardly be acceptable explanations of the multiplication procedure in question. The results also showed that in general there are few processes, concepts, or relationships in arithmetic which are understood by a large per cent of teachers.

One explanation is that this lack of understanding has come about from having been taught a short-cut to a process as if it were the process itself. When a person learns only the short-cut, he learns it without understanding and learns it by rote. Later when he teaches it, he still generally knows it as rote.[10]

---

[9]Jacob S. Orleans and Edwin Wandt, "Do Teachers Understand Arithmetic? The 'Why' Is as Important as the 'How'," Elementary School Journal, 53: 501-507, May, 1953.

[10]Ibid.

Universal education and compulsory school attendance until age sixteen present a challenge to teachers who must teach large heterogeneous groups. These groups often contain children with extremely high intelligence and great curiosity and interest in learning as well as those with average intelligence and many who are extremely retarded both in intelligence and interest. Such a situation often appears to be almost an insurmountable challenge to the teacher. If the solution cannot be found, we can at least alleviate the situation by encouraging each student to achieve to the limit of his ability and maturity. In present-day school situations, there are many things which stand in the way of mathematical learning. One does not learn mathematics unless one has some understanding and appreciation of the ideas underlying the manipulation of figures and symbols. Also there must be a certain facility in manipulation which for some students is not easily acquired. Nevertheless, the pleasure and satisfaction in doing mathematics is lost unless the pupil is intrigued by the teacher into experiencing pleasurable and satisfying results and pleasant feelings from the results of working with numbers.

Although this study is restricted to the sixth grade, it is apparent here that many pupils have not mastered the fundamental skills and have troubles with multiplication tables and other skills. However, it is still possible to get them to take an interest in the ideas of mathematics which require the manipulative skill. It is believed that in most instances, that whenever the pupil has gained an interest in the ideas of mathematics, he has improved markedly in his manipulative skill without being conscious of his efforts in gaining that skill. Since this is an isolated problem, the possibilities are limited. This discussion cannot go back to the first five grades nor forward to the seventh and eighth grades. Thus it is necessarily isolated and restricted to what can be done or attempted at the sixth grade level. It is highly probable that in any grade there will be many levels of ability. Some pupils may advance at ten times the speed of others. Some may not attain much speed but nevertheless it is the belief of the writer that there is a possibility of making some headway with all.

Environment, social background, and attitutdes of the home may be important factors in detracting from or complementing the success of this or any other method that might be used.

Social attitudes which may exist in the community toward the purposes of the school may give rise to a very complex situation. There are some in every community who feel that the school is no more than a grandiose baby-sitting institution and who can hardly wait to get their children off to school so they will be free to do things they have planned. Then there are those whose interest in the school is coincidental with the success of the basket-ball team. There are others who are far-sighted and who are deeply concerned with the education of the child and his general intellectual development. Finally, there are those who are interested in their youngsters' learning only the things which they, the parents, deem useful or practical to the exclusion of everything else which is designated as frills and impractical rot. These are just as bad as those who don't want their children to learn anything.

Many others come to mind whose ideas influence the attitudes of the pupils, but the above will serve as examples to point out the many forces at work in any community which form certain attitudes on the part of the students and detract from or enhance the success of any teaching method.

In the above discussion we concerned ourselves with the mathematical background essential for the teacher of arithmetic. An attempt will here be made to show the part this information is to play in the instruction of the pupil. The object is to create in the student an interest in and an understanding of mathematics. To do so it is necessary to lead the student from applications of arithmetic to an interest in arithmetic, or as it has been popularly expressed, to get "from fruit to fractions." At no time should the working with numbers be divorced from interesting and practical examples, for if this should occur the meaning of arithmetic may well be lost and the entire study may become merely a matter of rote--which often happens.

On the other hand, if examples are to be the sole instrument of instruction, the pupil will never be brought to realize the importance of careful definitions and analytical reasoning. The teacher should strive to maintain a delicate balance between the two. Whatever textbook is used by the teacher, she will find it necessary on the one hand to seek further examples to better illustrate the ideas under consideration and on the other hand to restate definitions in more exact form, and to fill in gaps in the presentation. The teacher will find numerous occasions when she will be obliged to do both of these to give an adequate answer to pupils' questions, which at times are surprisingly penetrating.

As an example, let us consider the situation in which a pupil after pondering the definition of a fraction as outlined below asks for a definition of a whole number. Upon examining the formal definition given above, it is clear to the teacher that this definition is far beyond the comprehension of the sixth grade pupil. The question itself, however, is very reasonable. With the knowledge and understanding of the formal definition, the teacher would realize the difficulties involved and be prepared to set up an example which would accurately exhibit the ideas of the whole number system, and at the same time avoid the formidable terminology. Such an example might involve the drawing of five circles, five squares, and five triangles and then pointing out that the number of numbers in each collection is the same, and that this number is conveniently expressed by the number 5. Regardless of the nature of the element in the set, the number of elements might be so represented. The first positive whole number represents any set of one element, and finally whole numbers have the property that if any set is given, the next larger set can be obtained by adding one element to the given set. The teacher might then go on to point out that since it is always possible to add an element to any set, then there is no largest positive integer.

In order to sustain interest in this very important concept, the teacher might approach the whole idea by asking her pupils to write down the smallest positive number. Then ask them to try to write the largest positive number. With this start, the teacher can then proceed with an explanation of the whole number system. However, if the question does not arise, the teacher will know enough not to raise it himself.

The central idea to be comprehended by the pupils in the sixth grade is the understanding of fractions and their applications.

In his teaching the instructor must be able to give and to demand from the pupils accurate definitions and statements. The precision of definition carries over into algebra and particularly into geometry. This practice also makes pupils critical in their reading and thinking in other subjects.

The teacher should realize that basic concepts for seventh and eighth grade and high school mathematics are being laid down. Whatever is done here will be repeated unchanged in more complex situations, and if carefully done at this level confusion in later years may be avoided.

The major part of arithmetic expected of a sixth grade course by educational institutions concerns the comprehension of fractions and their applications. The study should be taught by a teacher who has an interest in the subject and who has more than a passing regard for the future as well as the present meaning of the subject. Explanations should be given with precision and exactness. Definitions should be accurate in a strict mathematical sense, yet they should be put into words which the children can understand.

The pupil must be led to an understanding of what a fraction is in such a fashion that he can fully comprehend the meaning of such a definition. Words used should always be those he can understand.

The writer attempted to do this in the following way. A search for a definition of a fraction was instituted. It was interesting to note that many textbooks avoid giving a definition of a fraction. The authors of such texts define a numerator and a denominator of a fraction by explaining that the denominator tells how many equal parts there are under consideration and the numerator tells how many of the equal parts are being used. The authors also discuss proper fractions, improper fractions and mixed numbers. After searching through a number of textbooks and finding no definition given, the pupils then turned to the dictionary. There they found no additional help. It was not an adequate definition from a mathematical point of view, and the pupils recognized the fact. The following is an example from the dictionary:

> frac'tion (frăk' shŭn) n. 1. A fragment; a scrap. 2. One or more parts of a unit. In simple fractions, the figure below the line (denominator) shows the total number of parts into which a unit is divided, the figure above the line (numerator) shows how many of these equal parts are taken. Thus the fraction 3/4 (read "three-fourths," or "3 divided by 4") indicates 3 of the 4 equal parts of the unit. See NUMBER, Table.[11]

This searching, with accompanying discussions of the relative merits of the definitions found, aroused intense interest and considerable thought.

At the outset it was made clear that there is a difference between a fraction and a fraction of something. The pupils had some background and previous experience in dividing objects such as apples, dollars, and rectangles into equal parts. Great effort was made to make certain that the pupils were helped to realize that 1/2 a cow, for instance, is not a fraction, but that the 1/2 may be a

---

[11] Webster's Elementary Dictionary, A Dictionary for Boys and Girls. New York: American Book Company, 1941. p. 256

symbol used to indicate one-half of a cow. Many other similar examples were given. This was followed by the ways that parts of things may be represented using the notations familiar to the students. One-half dollar may be represented by 1/2, by 2/4, and by 50/100. Fifty may be represented by five dimes, or 5/10 may be represented by two quarters or 2/4. If necessary, drawings of rectangles may be made to show equal parts of a whole, but the central idea is to gradually and as quickly as is feasible make the transition from concrete objects to abstract ideas. To the writer, this seemed important and logical, since the adult must constantly deal with abstract ideas and practice in this area seemed important even with young people. The class worked together with guidance from the teacher to formulate gradually such initial ideas as: a fraction of an apple is a part of an apple; a fraction of a yard is a part of a yard; a fraction of a foot may be represented by these symbols--2/3 foot, 1/2 foot, 3/4 foot, and others; a fraction of something is a part of something; a fraction of something may be represented by two numbers written in a certain order--one, the numerator, written above a horizontal line, and the other, the denominator, written below the horizontal line, as in 2/3 foot. Gradually, such statements were written, discussed, and used for the purpose of learning applications and leading up to abstract ideas until the idea was developed that since 1/2 may be represented as 2/4, 3/6, 4/8, etc., then these are all representations of the same fraction. Finally the concepts were made clearer and more understandable by more examples and by practice in writing sets using simple beginning representations at first, such as

[1/2, 2/4, 3/6, 4/8, 6/12 - - - - - - - - -]

until the following definition could be developed: a fraction is the set of all equal pairs of whole numbers written in a certain manner. Thus

[3/4, 6/8, 9/12 - - - - - - - - - - - - - -]

The next step was an extremely important one in many respects. By introducing the appropriate concepts and interpretations of this step to the student, it is possible for that student to lay the foundations for the operations of addition, subtraction, multiplication, and division of fractions; for work with ratio and percentage, as well as many of the operations in algebra and more advanced mathematics. The step itself was simply to get the pupils to think of a fraction as having an endless number of representatives. That is, to get them to think of the class

[1/3, 2/6, 3/9, 4/12, 5/15 - - - - - - - - -]

as a fraction, rather than to think of its first representative, 1/3 only, as a fraction. By having the pupils actually write sets, the relationship of each representative to its successor becomes well understood. Simple fractions should be used at first, such as

[1/2, 2/4, 3/6, 4/8 - - - - - - 20/40 - - -]

[1/3, 2/6, 3/9, 4/12- - - - - 20/60 - - -]

[1/4, 2/8, 3/12, 4/16 - - - - 20/80 - - -]

with which the pupils have become fairly familiar, but such a fraction as

[2/7, 4/14, 6/21, 8/28- - - -42/147 - - -]

should be introduced as soon as possible because of its value in equivalence understanding. Extreme caution was used to be sure, in the beginning, that the immediate successor of each first representative and each succeeding one was obtained by multiplying both the numerator and denominator of the first representative successively by 2, 3, 4, 5, and so on. For each set written during class, discussions were held to point out, and to be sure that the pupils understood, the relationships between the various representatives within a set. Thus by writing many sets, the students became familiar not only with the idea of equivalence but also with the process and the reasons for the process of changing representatives to higher terms, a process which some at least had learned by rote previously, but without understanding.

After much practice in the writing of sets, the pupils became more and more familiar with the idea of equivalence within a set and they also improved enormously in their knowledge of the multiplication tables. Some confessed they had never learned the major part of their multiplication tables until they became interested in writing sets. The interest ran high and many became interested in seeing how far they could continue writing a fraction. They found they often had to stop to check their multiplication because their products did not agree with those of a classmate. They also became curious about the largest and the smallest fractions. By trial and error and by discussion, it was decided that there is no largest or smallest fraction. The next step followed in logical order. The pupils were asked to decide, for instance, whether 4/10 belongs to the set whose first representative is 2/5; whether 8/32 belongs to the set whose first representative is 1/7. If not, they were to decide to what set it did belong. Such work gave practice, of course, in dividing both terms of the given representative by the same number, which consists of the process of reduction of representatives to their lowest terms. Continued practice increased understanding of equivalence relationships, increased facility in division of whole numbers, and produced the manner of finding the first or any desired representative of a class, when one representative was given. The pupils also found this very enjoyable. So enjoyable, indeed, that some carried their fractions out to several hundred representatives and compared their work.

Next, two fractions, such as the following, were written on the board:

[1/3, 2/6, 3/9, **4/12**, 5/15, 6/18 - - - - - - -
- - **20/60** - - - - - - - - - - - - - - - - - - n/3n]

[1/4, 2/8, **3/12**, 4/16, 5/20 - - - - - - - - -
- - **15/60** - - - - - - - - - - - - - - - - - - n/4n]

The pupils were asked to notice similarities and differences of parts of the two sets. It was noticed that no representatives within the same set had the same numerators or the same denominators, but that some representatives in one set had the same denominators as some representatives in the other set; for example: 20/60 and 15/60; also 4/12 and 3/12. To show how these facts can be used, the following examples were placed on the board:

2 chickens	4 hats
+ 4 chickens	+ 3 hats
6 chickens	7 hats

But what of

3 horses
+ 2 pigs
5 ?

By means of questions and answers and discussion, it was revealed that 3 horses and 2 pigs can only be added by making use of a common denominator, or common name, such as "animals." Thus the result of the addition is 5 animals. In the same manner, it was shown that adding the numerators together and adding the denominators together will not give a unique result.

Thus, even though 1/2 = 2/4

1/2 + 2/3 = 3/5 and 2/4 + 2/3 = 4/7

Addition by this method gives a reult that is not unique and hence the method is of no use. On the other hand, adding numerators only of representatives which have the same denominators yields a unique result. Thus

1/2 + 2/3 = 3/6 + 4/6 = 7/6, and
1/2 + 2/3 = 6/12 + 8/12 = 14/12, and
1/2 + 2/3 = 9/18 + 12/18 = 21/18, and
1/2 + 2/3 = 12/24 + 16/24 = 28/24.

[1/2, 2/4, 3/6, 4/8, 5/10, 6/12, 7/14, 8/16, 9/18, 10/20, 11/22, 12/24, - - - - -]
[2/3, 4/6, 6/9, 8/12, 10/15, 12/18, 14/21, 16/24, 18/27 - - - - - - - - - - - - - - - - - -]
[7/6, 14/12, 21/18, 28/24 - - - - - - - - - - -]

These examples also show that when we add fractions we add sets.

Thus the use of a common denominator appeared to be the only way that unique results could be obtained for addition of fractions. Many other examples were used to establish the fact that if addition means anything at all it means that we must always get the same result when we add the same two elements or equivalent elements, and that the reason for using a common denominator is that its use will give a unique result. Examples such as the two sets used on page 35 served to exemplify the fact that the least common denominator is desirable but not necessary, and also that addition of fractions means the addition of sets.

$$20/60 + 15/60 = 35/60 = 7/12$$

But $4/12 + 3/12 = 7/12$.

It was also shown by demonstration that considerable time and effort would be required to find even the least common denominator for some fractions such as $1/16 + 1/104$. This led logically into factoring of denominators, which is essentially the same as factoring of whole numbers. A good understanding of the divisibility of numbers is dependent on this process. It gives a better understanding of whole numbers and facilitates addition and subtraction of fractions. This factoring of whole numbers is as indispensable in algebra as multiplication and division are in arithmetic. It was felt, then, that elementary pupils should have a thorough foundation and practice in its use. If an elementary pupil knows that 21/6 may be factored thus:

$$\frac{21}{6} = \frac{\not{3}\cdot 7}{\not{3}\cdot 2} = \frac{7}{2}$$

then it is an easier step to take in algebraic manipulations such as:

$$\frac{21a}{6} = \frac{3\cdot 7a}{3\cdot 2} = \frac{7a}{2} \quad \text{and} \quad \frac{21}{6b} = \frac{3\cdot 7}{3\cdot 2\cdot b} = \frac{7}{2b}.$$

Further:

$$\frac{a^5}{4a^2} = \frac{a^3\cdot a^2}{4\cdot a^2} = \frac{a^3}{4}.$$

With such a foundation, freshmen in high school should be able to grasp the process for such examples as the following with very little trouble:

(a) $\frac{6m-13}{m^2-5m+6} - \frac{5}{m-3} = \frac{6m-13}{(m-3)(m-2)} - \frac{5}{m-3} =$

$\frac{6m-13-5(m-2)}{(m-3)(m-2)} = \frac{6m-13-5m+10}{(m-3)(m-2)} = \frac{m-3}{(m-3)(m-2)} = \frac{1}{m-2}.$

(b) $\frac{5}{b+4} + \frac{4}{b^2-16} = \frac{5}{b+4} + \frac{4}{(b-4)(b+4)} =$

$\frac{5(b-4)+4}{(b-4)(b+4)} = \frac{5b-20+4}{(b-4)(b+4)} = \frac{5b-16}{(b-4)(b+4)}.$

(c) $\frac{27}{x^2-81} + \frac{3}{2x+18} = \frac{27}{(x-9)(x+9)} + \frac{3}{2(x+9)} =$

$\frac{27\cdot 2+3(x-9)}{2(x-9)(x+9)} = \frac{54+3x-27}{2(x+9)(x-9)} = \frac{27+3x}{2(x+9)(x-9)} =$

$\frac{3\cancel{(x+9)}}{2\cancel{(x+9)}(x-9)} = \frac{3}{2(x-9)}$

Since Euclid's algorithm holds in the domain of whole numbers, factoring is unique. Factoring is absolutely essential in the study of algebra in high school. Factoring was taken up with the realization that considerable time might be needed to establish this important concept. First, a definition of factors was worked out following procedures similar to those shown above. A simple one such as the following was formulated. Factors of a whole number are those numbers which can be multiplied together to give that number. Numerous examples were given, such as:

$$8 = 2 \times 2 \times 2$$

$$9 = 3 \times 3$$

$$7 = 7 \times 1$$

It was also pointed out that a number is not completely factored until it is factored into its prime factors. This led to the explanation of the meaning of prime factors, and the need for using prime factors. It was shown by examples such as the ones below that incomplete factoring could produce a common denominator of 1440, but that complete factoring of each denominator would produce a common denominator of 240, which is the least common denominator and thus greatly simplify the work of adding the two representatives 2/48 and 1/30.

Example I. 2/48 + 1/30

Incomplete factoring:

8 /48
6 /6
1

3 /30
10 /10
1

8 x 6 = 48 10 x 3 = 30

Common denominator: 8 x 6 x 10 x 3 = 1440.

Example II. 2/48 + 1/30

Complete factoring:

2 /48
2 /24
2 /12
2 /6
3 /3
1

3 /30
2 /10
5 /5
1

3 x 2 x 2 x 2 x 2 = 48 3 x 2 x 5 = 30

Least common denominator: 2 x 2 x 2 x 2 x 3 x 5 = 240.

In this way a process for finding the least common denominator of two fractions was obtained and indirectly a new definition of a least common denominator was given. The least common denominator turned out to be a product of factors which contained each factor of the two denominators the maximum number of times that factor occurred in either denominator. Thus, in the above example, four 2's, one 3, and one 5 would be required.

After pupils have learned to add fractions as discussed above so that they have very little trouble with them, they are able to understand that addition of fractions means this kind of relationship:

$$\frac{2}{5} + \frac{9}{3} = \frac{(2x3)+(5x9)}{3x5} = \frac{6+45}{15} = \frac{51}{15}$$

and that:

$$\frac{2}{5} + \frac{9}{3} + \frac{7}{6} = \frac{(3x6x2)+(5x6x9)+(5x3x7)}{5x6x3} =$$

$$\frac{36+270+105}{90} = \frac{411}{90} = \frac{137}{30}.$$

Much practice was given in factoring and in the finding of prime numbers. This was followed by practice in the addition of sets, keeping before the pupils at all times the concept that they were adding sets rather than adding isolated numbers. The concept of the meaning of prime factors was also stressed and discussed throughout this part of the work.

Subtraction of sets was accomplished in the same manner. The only learning required other than that used in addition was that of borrowing. This again was given enough time to be accomplished well and with good understanding.

These operations on fractions provided many splendid opportunities to improve skills in the four fundamental operations in such painless fashion that the pupils were not aware of their own progress.

Multiplication of sets of fractions was much simpler. In this process a unique result could be obtained by multiplying any pair of numerators for a new numerator and multiplying the corresponding denominators for a new denominator. Here again, however, factoring expedited the work by the use of what is commonly called cancellation. The following examples are included to show how various forms of multiplication were explained.

First: Multiplication is an extended form of addition.

2 x 3 means 2 threes or 3 + 3 = 6

6 x 4 means (2 x 3) + (2 x 3) + (2 x 3) + (2 x 3) = 24.

Also:

6/8 x 1/2 means 1/2 as much as 6/8 x 1.

Thus:

6/8 x 1 = 6/8.   1/2 as much as 6/8 = 3/8.

However, the simplest way to obtain such a result is simply to factor the numerators and the denominators and multiply. Thus:

$$\frac{6}{8} \times \frac{1}{2} = \frac{3 \times \not{2} \times 1}{2 \times 2 \times 2 \times \not{2}} = \frac{3}{8}.$$

Subtraction and division of fractions are inverse operations to addition and multiplication and should be handled similarly.

Division, we know from experience with whole numbers, is the inverse of multiplication. So the use of a reciprocal of the divisor followed by multiplication will give a unique result for division of sets. However, if some prefer further explanation, the following illustrations with oral explanations should suffice:

$4 \div 2 = 2$. This means we are finding how many 2's there are in 4.

$4 \div 1 = 4$. This means we are finding how many 1's there are in 4.

$2/3 \div 1 = 2/3$. This means we are attempting to find how many 1's there are in 2/3. There are no 1's in 2/3; there is only 2/3 of 1 in 2/3.

$2/3 \div 1/3 = 2$. This means we are finding how many 1/3's there are in 2/3.

It will be noted that there are twice as many 1's in 4 as there are 2's in 4 because 2 is twice as big as 1. Therefore, provided the dividend remains the same, the larger the divisor, the smaller the quotient. Yet the same results may be obtained by simply inverting and multiplying.

In introducing decimals, pupils had already become familiar with the fact that our whole numbers are based on 10's. From experiences with money, they also knew that $.01 means one cent, or 1/100 of a dollar and that $.10 means ten cents or 10/100 of a dollar. They soon were able to understand that decimals are representatives of fractions written in a new way: 1/10 = .1; 10/100 = .10; 1/100 = .01.

It was shown that any representative of a fraction may be written as a decimal. The pupils were used to thinking of 1/2 dollar as \$.50. It can be shown how 1/2 is changed to .50 by dividing the numerator by the denominator. Then 3/7 may be experimented with:

```
       .4285714
  7/ 3.0000000
     2 8
       20
       14
        60
        56
         40
         35
          50
          49
           10
            7
            30
            28
             20
```

With a little help, the pupils were able to discover that the remainders in the subtractions are 2, 6, 4, 5, 1 and 3; since a remainder must be smaller than the divisor 7, these are the only digits that can appear as remainders. When these have all appeared, the remainders start repeating in the same sequence. Every representative of a fraction will either terminate or repeat when changed to decimal form.

Foundations of percentage can be made understandable to sixth grade students. It was explained that <u>percent</u> means <u>by</u> <u>the</u> <u>hundred</u>. Therefore, in expressing percent, we choose a representative from a fraction which has 100 in the denominator. Thus 5% means 5/100, $37\frac{1}{2}\%$ means $37\frac{1}{2}/100$, and so on.

Since much is done with comparison of numbers and fractional parts in the sixth grade curriculum, the concept of ratio was introduced. Ratio of two numbers is merely an unspecified representative of a fraction; when this idea was once understood, number relationships and simple ratios gave little trouble.

Percent is a word coming from the Latin word *per centum* meaning *by the hundred*. Therefore, in expressing per cent, we chose a representative from a fraction which has 100 in the denominator. Thus writing the fractions [1/2 - - -], [1/3 - - -], and [1/4 - - -]:

[1/2, 2/4, 3/6, 4/8 - - - - - - 50/100 -]

[1/3, 2/6, 3/9 - - - 33 1/3/100 -] (100/300 = $\frac{100/3}{100}$

[1/4, 2/8, 3/12 - - - 25/100 - - - -]

Thus 1/2 may also mean 50/100 and .50 and 50%. 1/4 may also mean 25/100 and .25 and 25%. 5% means 5/100 and 37-1/2% means $37\frac{1}{2}$/100.

By this time the pupils had become accustomed to the fact that "of" in arithmetic means "x". So they readily understood that 5% of 40 meant 5/100 x 40, or .05 x 40, and that 33-1/3% of 60 meant 33 1/3/100 x 60, or 100/300 x 60.

The class used in this study had during the year been presented many simple statements to solve of the following type:

Tell what number N stands for:

N x 17 = 612.	N - 313 = 644
336 ÷ N = 4.	835 + N = 900

After a number of such examples, the pupils became accustomed to thinking of N as an unknown number. Such work was of definite assistance in introducing percentage. There are essentially only three types of percentage problems. An example of the first type is: What is 25/100 of 40? Step by step, this was taught, by means of previous understandings, in the following manner:

(a) What is 25% of 40?

(b) What is 25/100 of 40?

(c) In arithmetic, "of" means "x".

(d) So what is 25/100 x 40?

(e) $\frac{25}{100} \times 40 = \frac{\overset{1}{\cancel{25}}}{\underset{4}{\cancel{100}}} \times \frac{\overset{10}{\cancel{40}}}{1} = 10$

(f) Now let's assume that "what" means an unknown something.

(g) Replace "what" with N. N is 25/100 of 40.

(h) N is 10.

However, it is felt that in the future follow-up of this experiment linear equations will be introduced and simple linear equations will be taught. Then all three types of percentage problems can be introduced.

The three types are shown here:

(a) N is 25/100 of 40. Find N.

(b) 10 is N% of 40. Find N.

(c) 25% of N is 10. Find N.

Reading of floor plans, making scale drawings and graphs, finding perimeters and areas of rectangles and squares and finding volumes of rectangular solids are all practical applications of the use of whole numbers and of fractions. These applications were all used in connection with this study as sufficient understanding developed.

Casting out nines was taught not only because it is an interesting and enjoyable concept, but because it gives the speediest and easiest methods of checking the four fundamental processes. This is particularly true for checking multiplication and division.

The practice of casting out nines is based on the concept of congruence classes. Two integers A and B are said to be congruent modulo, the integer M, if their difference is a multiple of M. That is, $A-B = KM$ where K is an integer.

If $A \equiv B \pmod{M}$ and $C \equiv D \pmod{M}$, then $A + C \equiv B + D \pmod{M}$ and $AC \equiv BD \pmod{M}$. This says that if A and C are divided by the modulus M, the sum of the remainders is congruent to the remainder of the sum.

Thus		$29 \equiv 2 \pmod{9}$
and		$32 \equiv 5 \pmod{9}$
then	$29 + 32 = 61$	$\equiv 7 \pmod{9}$.
Also	29×32	$\equiv 10 \equiv 1 \bmod 9$.

The process of casting out nines is based on the property:

$$10 \equiv 1 \pmod 9$$

$$10^2 \equiv 1 \pmod 9$$

$10^n \equiv 1 \pmod 9$ from the above property.

Hence $x10^n \equiv x \pmod 9$.

Thus $3877 = 3\cdot10^3 + 8\cdot10^2 + 7\cdot10^1 + 7\cdot10^0$,

and so

$$3\cdot10^3 \equiv 3 \pmod 9$$

$$8\cdot10^2 \equiv 8 \pmod 9$$

$$7\cdot10^1 \equiv 7 \pmod 9$$

$$7\cdot10^0 \equiv 7 \pmod 9$$

Hence $3877 \equiv 3 + 8 + 7 + 7 \pmod 9$

This may be used to check calculation as shown in the following example:

$$94 \times 62 = 5828.$$

$$4 \times 8 = 32 \equiv 5 \text{ mod } 9.$$

$$5828 = 23 \equiv 5 \text{ mod } 9.$$

Therefore the multiplication is probably correctly performed.

In connection with reading and writing numbers, pupils need to know how to write numbers to the base 10. This is an aid to understanding place value. Thus in base 10, 372 is

$$10^2\cdot3 + 10^1\cdot7 + 10^0\cdot2 = 372.$$

Writing numbers to base 2 may be shown to create interest and to arouse curiosity. Thus 372 to the base 2 would be

		Remainders
2	/372	
2	/186	0
2	/93	0
2	/46	1
2	/23	0
2	/11	1
2	/5	1
2	/2	1
2	/1	0
	0	1

$1\cdot2^8 + 0\cdot2^7 + 1\cdot2^6 + 1\cdot25 + 1\cdot2^4 + 0\cdot2^3 + 1\cdot2^2 + 0\cdot2^1 + 0\cdot2^0 = 256 + 0 + 64 + 32 + 16 + 0 + 4 + 0 + 0 = 372.$

372 to base 2 is 101, 110,100.

A simpler example is 8 to base 2:

2	/8	
2	/4	0
2	/2	0
2	/1	0
	0	1

$1\cdot2^3 + 0\cdot2^2 + 0\cdot2^1 + 0\cdot2^0 = 8.$

8 to base 2 is 1000.

CHAPTER IV

EVALUATION OF PROCEDURES

I. TESTING MATERIALS AND RESULTS

The data in Table I show the increments of gain in grade placement in arithmetic reasoning, in arithmetic fundamentals and in total arithmetic, as measured by the California Achievement Test, Intermediate Battery.[1] They also show each pupil's rank in achievement based on total arithmetic achievement. The median achievement for each category is given. Normal increment gain is one school year or an increment of 1.0.

[1]Earnest W. Tiegs and Willis W. Clark, "California Achievement Tests Elementary Grades 4-5-6," (Los Angeles: California Test Bureau, 5916 Hollywood Boulevard, 1951).

TABLE I.

INCREMENTS OF GAIN IN GRADE PLACEMENT IN ARITHMETIC BY SIXTH GRADE STUDENTS, 1956-1957

Student	Rank Based On Total Increment	Arithmetic Reasoning: Original Grade Placement	Arithmetic Reasoning: Increment Of Gain	Arithmetic Fundamentals: Original Grade Placement	Arithmetic Fundamentals: Increment Of Gain	Total Arithmetic: Original Grade Placement	Total Arithmetic: Increment Of Gain
22	1	6.9	2.1	6.4	3.6	6.5	3.0
8	2	6.0	2.3	6.1	2.7	6.1	2.6
21	3	6.9	2.6	6.6	2.0	6.7	2.2
4	4	5.6	2.7	6.0	1.9	6.1	2.0
16	5	5.0	1.9	4.9	2.1	5.0	2.0
7	6	7.6	.5	6.5	2.1	6.5	1.9
10	7	8.3	-.2	5.5	2.1	6.1	1.7
13	8	7.2	1.5	6.5	1.5	6.6	1.6
15	9	7.6	1.4	8.2	1.8	8.1	1.4
11	10	7.4	.5	6.6	1.5	6.8	1.3
6	11	10.0	0.0	7.7	1.8	8.2	1.3
5	12	7.9	.2	6.8	1.5	7.1	1.3
3	13	7.2	1.5	5.7	1.4	6.1	1.3
1	14	5.2	1.7	6.7	.3	5.6	1.3
9	15	7.4	1.6	6.9	1.1	7.2	1.1
12	16	5.0	.4	5.2	1.5	5.2	1.1
14	17	6.2	1.7	6.6	.9	6.5	1.1
18	18	5.7	.5	5.4	1.1	5.5	1.0
19	19	7.2	.7	5.8	1.2	6.2	1.0
17	20	7.4	0.0	7.0	.8	7.1	.6
20	21	7.4	-.2	6.3	.9	6.6	.6
2	22	7.6	-.7	6.7	0.0	6.9	-.1
Median		7.2	1.05	6.5	1.5	6.5	1.3

Table does not include pupil #23 who dropped out and pupil #24 who entered last two weeks of third quarter. Increment 1.8 means equivalent to 1.8 school years. 2.1 is equivalent to 21 school months. Rank is based on increments of gain in Total Arithmetic.

The data in Table II show the increments of gain in grade placement in reading vocabulary, in reading comprehension and in total reading, as measured by the California Achievement Test, Intermediate Battery.[2] They also show each pupil's rank in achievement based on total reading. The median achievement in each category is given.

---

[2]Tiegs and Clark, loc. cit.

TABLE II.

INCREMENTS OF GAIN IN GRADE PLACEMENT IN READING BY SIXTH GRADE STUDENTS, 1956-1957

		Vocabulary		Comprehension		Total Reading	
Student	Rank Based On Total Increment	Original Grade Placement	Increment Of Gain	Original Grade Placement	Increment Of Gain	Original Grade Placement	Increment Of Gain
11	1	6.2	2.1	6.2	2.6	6.2	2.3
10	2	6.7	1.2	5.3	2.2	6.0	1.8
2	3	3.1	1.8	4.4	1.8	3.7	1.7
6	4	6.7	1.2	7.9	2.6	7.2	1.6
8	5	4.1	1.5	5.3	1.8	4.6	1.6
13	6	7.4	.9	7.5	2.0	8.8	1.3
7	7	5.3	1.4	4.9	1.0	5.1	1.2
4	8	6.7	1.6	6.9	1.0	6.8	1.2
14	9	7.6	1.4	8.8	0.0	8.0	1.0
16	10	5.7	1.5	6.2	.7	6.0	1.0
1	11	5.8	1.8	6.0	.2	6.0	.9
15	12	7.6	0.0	7.1	2.4	7.5	.8
21	13	6.2	.5	7.5	1.3	6.7	.8
9	14	7.4	.5	6.9	1.0	7.2	.7
12	15	4.3	.9	5.1	.6	4.7	.7
18	16	5.5	0.0	4.7	1.3	5.2	.5
20	17	4.7	.4	5.3	.6	5.0	.4
3	18	5.0	-.3	5.4	1.2	5.2	.2
17	19	4.5	.1	5.7	0.0	5.0	.1
22	20	7.9	-.7	7.5	1.3	7.8	0.0
5	21	7.4	.2	8.8	-1.3	7.9	-.3
19	22	6.5	-.3	7.5	-.4	6.9	-.3
Median		6.2	.9	6.2	1.1	6.1	.85

Table does not include pupil #23 who dropped out and pupil #24 who entered last two weeks of third quarter. Increment 1.8 means equivalent to 1.8 school years. 2.1 is equivalent to 21 school months. Rank is based on increments of gain in Total Reading.

Although the results from a reading test may not have any bearing on arithmetic learning, it was felt that reading vocabulary and reading comprehension are necessary to good performance in interpreting and comprehending written work in arithmetic. Some implications of correlation between the tests result. Therefore, the results of the reading test are included.

II. EVALUATION

By examination of Table I, it may be noted that although average achievement may have existed in some instances for previous years the median increments in most cases for the current year are more than average. The inference can't be strong and if results were treated statistically they would probably show no significant differences between the increments of the year under consideration and previous years. However, increments for total arithmetic in all but three cases are average or better than average. Of the three that failed to make average gain, two made fair gains. Of these two, one was still above the normal grade level in total arithmetic and made a good gain in arithmetic fundamentals. The other also had a good gain in arithmetic fundamentals.

The characteristics of this test may be considered. The intermediate battery includes one test for fourth, fifth and sixth grades. The sixth grade is at the top of this battery. Although the scoring takes this fact into consideration, the type of examples given in the test seem to have been largely those which fit the traditional methods of teaching. In general the type of examples used in this test would not be a challenge to the reasoning ability of a child taught by the methods described in this paper. This may imply that if tests better fitted to the type of work taught during the year were given the results may have been different. Also, if intelligence quotients had been included in the data other inferences would have been possible. It may be worthy of note that in all but three cases the increments of gain were large and above average either in both arithmetic reasoning and in arithmetic fundamentals or the increment of gain was very large either in arithmetic fundamentals or in arithmetic reasoning. Of the three who did not make large gains in either category two of them made gains that were nearly average in arithmetic fundamentals.

However, if we look at the increments of gain in arithmetic fundamentals it will be noticed that for nearly all cases which showed small gains in comprehension, large gains were made in arithmetic fundamentals. This could imply that those who gained little in comprehension were those who needed improvement in vocabulary and in arithmetic fundamentals so desperately that little comprehension was possible until gains were accomplished in those areas.

Comparable comparisons can be noticed by comparing the increments of gain in reading vocabulary and reading comprehension in Table I. In general this table also shows that although many students made average or better gains in both areas, many made larger gains in one area than in the other. This also may imply that many of those who made small gains in comprehension were handicapped because of vocabulary. Attention may be directed to particular students and their accomplishments on the two tests. Student number 2 is a case in point. This student failed to make desirable gains in arithmetic but his original grade placement in reading vocabulary was 3.1. His increment gain was 1.8, almost two full years of gain. His original grade placement in reading comprehension was 4.4; again his increment of gain was 1.8. His total reading original grade placement was 3.7 and total reading increment of gain was 1.7. He ranked third in the class for total gains in reading but his present total reading grade placement is still only 5.4 when it should be at least 6.8. This same student ranked last for total gains in arithmetic. Such comparisons may help to explain low accomplishment in arithmetic. This student struggled constantly and conscientiously with vocabulary and reading comprehension and had little time or energy left for work in arithmetic. Other similar comparisons could be made.

Some implications might be drawn that would indicate that some students who were under average in ability were capable of improvement in vocabulary and in arithmetic fundamentals, since much can be accomplished in those skills by rote learning and by imitation and memory while comprehension requires more reasoning ability.

The results as a whole can mean little statistically because the number tested was too small and there was no control group and because the experiment could not be continued for an extended period of time.

In the total arithmetic column, only three failed to make normal gains and seventeen made gains ranging from thirteen months up to three school years. Five of the seventeen made gains ranging from two to three school years. Further study of results brings out the facts that on the basis of total scores in all three areas there had been fifteen instances of scores below grade level at the end of the fifth grade and only six such scores at the end of the sixth grade. In four of those six cases, the deficits were lessened.

Throughout the experiment, comprehension was stressed. Prime consideration was given to what seemed to be best for the student.

CHAPTER V

CONCLUSIONS

The conclusions to be drawn from this experiment must be considered only tentative for three reasons. First, the experiment has not been thoroughly tested under various conditions. The study was not conducted over a long period of time. Second, this experience has been an isolated one. Classes preceding and following this one have had no associations with or practice in these procedures while in the other grades. The teachers of those classes have known little of these procedures. Therefore, little is known of the carry-over from one class to another. This is only a very critical part of what should be a larger study. Third, the enthusiasm of the teacher, alone, may have been enough to arouse more than usual effort and accomplishment on the part of some students. At the same time, the same enthusiasm and the departures from traditional methods may have had the opposite effect on some others.

However, the interest of the class in general, their attitude of questioning and inquiry, their recognition of underlying principles as they recur from time to time as work progressed, and their enjoyment of the fundamental operations as skills were acquired, seemed to show some gains in basic understanding and much gain in manipulation.

The average sixth grader seems to have no difficulty in making the transition between abstract ideas and the application of these ideas. Ground-work could be laid for other grades if more teachers cared to take advantage of what the sixth graders had learned and if the program were continued and expanded.

A side effect, and an amusing one, arose when the pupils began writing sets. Those pupils who had previously depended almost wholly upon their parents or older brothers and sisters to take most of the responsibility for their work were suddenly thrown on their own.

It might be profitable to investigate the other seven grades, to find the facility of various age groups to grasp and enjoy abstract ideas. The attention of mathematicians has just recently been focused on this problem. Similar investigations have been going on at the University of Illinois. Professional mathematicians are only now becoming conscious of the need for such work in the grade school.

Since teachers may be inclined to follow the book through thick or thin, perhaps a new series of textbooks is in order.

BIBLIOGRAPHY

BIBLIOGRAPHY

Almack, John C. Research and Thesis Writing. Chicago: Houghton Mifflin Company, 1930.

Ball, W. W. R. Mathematical Recreations. New York: Macmillan and Company, 1896.

Banks, J. Houston. Elements of Mathematics. New York: Allyn and Bacon Inc., 1956.

Birkhoff, Garrett, and Saunders MacLane. A Survey of Modern Algebra. New York: Macmillan Company, 1941.

Courant, Richard and Herbert Robbins. What is Mathematics? New York: Oxford University Press, 1941.

Fessenden, Seth A., Roy Ivan Johnson and P. Merville Larson. The Teacher Speaks. New York: Prentice-Hall Inc., 1954.

Gardner, John W. "A National Weakness," American Mathematical Monthly, 63:396-399, June-July, 1956.

Graves, Lawrence M. "The Theory of Functions of Real Variables." Course offered in the University of Chicago, Summer Quarter, 1931. (Mimeographed.)

Hogben, Lancelot. Mathematics for the Million. New York: W. W. Norton and Company, Inc., 1937.

Hogben, Lancelot. The Wonderful World of Mathematics. Garden City, New York: Garden City Books, 1955.

Karpinski, Louis C. Numbers Through the Ages and Around the World. A Century of Progress Wonder Library Monograph. Chicago: Colortext Publications, n.d.

Kershner, R. B. and L. R. Wilcox. The Anatomy of Mathematics. New York: The Ronald Press Company, 1950.

Kraitchik, Maurice. Mathematical Recreations. New York: W. W. Norton and Company, Inc., 1942.

Lieber, Lillian R. Galois and the Theory of Groups: A Bright Star in Mathesis. Lancaster, Pennsylvania: The Science Press Printing Company, 1932.

Lieber, Lillian R. The Education of T. C. Mitts. New York: W. W. Norton and Company, Inc., 1944.

Orleans, Jacob S. and Edwin Wandt. "Do Teachers Understand Arithmetic? The 'Why' is as Important as the 'How'," Elementary School Journal, 53:501-507, May, 1953.

Peany, Katherine B. "Protractor - Ruler - Compass. Eighth Grade Pupils Integrate Art and Arithmetic," School Arts, 52:211, February, 1953.

Smith, David Eugene. The Wonderful Wonders of One - Two - Three. New York: McFarlane, Warde, McFarlane, 1937.

Tiegs, Earnest W. and Willis W. Clark. "California Achievement Tests Elementary Grades 4-5-6," Los Angeles: California Test Bureau, 1951.

Turner, J. Hilton. "The Partial Answer to an Inevitable Question; Roman Elementary Mathematics. The Operations," The Classical Journal, 47: no. 2, November, 1951.

Upensky, J. V. and M. A. Heaslet. Elementary Number Theory. New York: McGraw-Hill Book Company, Inc., 1939.

Webster's Elementary Dictionary, A Dictionary for Boys and Girls. New York: American Book Company, 1941.

Weiss, Marie J. Higher Algebra. New York: John Wiley and Sons, Inc., 1949.

APPENDIXES

APPENDIX A. HISTORY OF MATHEMATICS

I. NUMBERS THROUGH THE AGES AND AROUND THE WORLD[1]

The use of numbers has indeed played an important part in the advance of knowledge and civilization. In the gradual development of the sciences of mathematics and arithmetic, which are based upon numbers, many different peoples have shared. Famous European and Asiatic nations, of the past and of the present, as well as the country of Egypt in Africa, have all made contributions. Our own continent, also, has done much to aid in the development of the science of numbers. Today the entire world with its science, industry and commerce, is benefiting from the progress which has been made in the use of numbers during the last twenty-five hundred years.

Aztecs and Mayas

Numbers on the American Continent

The simplest way to represent any small number is to use a corresponding number of dots or lines. This was the method employed by the Aztecs of old Mexico.

. . . .

To show large numbers some simpler system is necessary. The natives of Mexico, who went barefoot, counted on their toes as well as their fingers, and thus used twenty as the basis of their number system, instead of ten. This was also the system common with the Indians of North America.

The natives of Peru used quipus, or knots on strings to show different figures. A small knot stood for one, a large knot ten.

The Mayas of Central America kept an extensive record of days, using a modified 20 system; that is, progressing from 20 to 360, to 20x360, to 20x20x360, and so on with higher multiples. This was the most highly developed arithmetic in America before the coming of Columbus.

---

[1]Louis C. Karpinski, Numbers Through the Ages and Around the World, a Century of Progress Wonder Library Monograph, Chicago: Colortext Publications, n.d.

Egyptian Numbers

Centuries before the birth of Christ, the Egyptians developed symbols for language and numbers. The land of the pyramids was probably the first place this was done.

To represent the numbers from one to ten the Egyptians first used picture symbols, which are known as hieroglyphics. Since it proved difficult to use these simple pictures in representing large numbers, there were developed special symbols for 10, 100, 1,000, 10,000, and 100,000.

The priests made such progress in arithmetic, geometry, and numbers that is was necessary to introduce more compact numbers called hieratic numerals. Without arithmetic and geometry, the Egyptians could not have built the pyramids.

Babylonians

Whenever you tell the time of day, you really pay a tribute to the ancient Babylonians, for they worked out the method.

Since the skies over Babylon were very clear, the people studied the stars. In the study of astronomy the use of angles is necessary, so the Babylonians divided the circle into 360 degrees and studied the sizes of the various chords in a circle. For some reason, they showed a great fondness for 60, having special symbols for 60; 3600, or 60x60; 216,000, or 60x60x60.

The knowledge acquired in the study of numbers by both the Egyptians and the Babylonians played an important part in the development of mathematics in Greece.

Greeks

At first the Greeks used picture symbols for numbers, just as did the Egyptians. But later on, they worked out the idea of using the initials of the Greek words for 5, 10, 100, 1,000, to represent the corresponding numbers.

It was the Greeks who early developed the theory of numbers, especially in the relationship of squares and cubes and other numerical facts suggested by geometry. Because of the attention which the Greeks gave to the subjects of arithmetic and algebra and geometry, progress in the study of physics and astronomy was made possible.

Arabic Arithmetic

800 A. D. to 1500 A. D.

The Arabs studied not only the number systems of the Greeks, the Babylonians, and the Hindus, but also all of the developments made earlier in mathematics. The material thus gathered, the Arabs combined in excellent textbooks on arithmetic, algebra, and trigonometry, which for several hundred years formed the basis for the study of mathematics in Europe.

The Arabic arithmetic, using Hindu numerals with the zero, was translated from Arabic into Latin and later into Hebrew, French, Italian, and German. The earliest Arabic arithmetic based upon Hindu sources was written about 825 A.D. by Al-Khowarizmi. The Latin translation of this arithmetic was called algorismus, the word being formed from the name Al-Khowarizmi.

Among Arabic mathematical terms which we still use today are the words "algorism," "algebra," "zero," and "cipher." The English word "fraction," which means "broken" number, really comes from the Arabic word which has a similar meaning. In Latin the word for "fraction" signifies "small."

European Arithmetic

From 1100 A. D. to 1500 A. D.

In the twelfth century the first translations of Arabic arithmetical works were made into the Latin. Before the invention of printing, which occurred about 1450, extensive treatises on the Hindu-Arabic arithmetic were available in manuscript form. These manuscripts were in Latin, Hebrew, Italian, German, French, and English. There were even treatises in Icelandic.

Tangible Arithmetic. About 1300 A.D. the abacus of the Romans was developed in Europe, and was used to about 1500 A.D. This was reckoning on lines with counters, called "reckoning pennies," or in Latin projectilia. The familiar expressions "cast an account" and "carry one," both trace back to the line reckoning, or abacus, where counters are actually carried.

On the lowest line of the abacus a single counter represents units, and no more than four coins are placed upon any one line. To represent five units a single counter is placed in the space above the lowest line. On the second line the counters represent tens, on the third hundreds, and on the fourth thousands. In the second space, a counter represents five tens, and so on up. . . .

Roman Arithmetic

Both the Greeks and the Romans used a board with hollow grooves in which they placed little stones to represent numbers. In an abacus, beads on a wire frame were used for the same purpose. This form of counting machine is still used today in China, in Japan, and in Russia.

Roman Symbols for Fractions

The Roman symbols for fractions are still used by druggists the world over. Physicians also employ these symbols, which are not understood by most people.

The earliest Roman symbols were not letters. Later, C, the initial letter of centum (100), and M, the initial letter of milia (1,000), were used to represent those figures.

Hindu Numerals

In India, the first progress made in the use of numbers and in arithmetic was much the same as in Egypt and Greece. Probably some sort of abacus was also used there.

It was some great Hindu genius who had the brilliant idea of using the zero symbol to represent nothing, and to combine with it the ordinary Hindu symbols for the numbers from one to nine. With the use of the zero symbol, tens, hundred, thousands, and other powers of ten are represented by position. Thus, for instance, the figure "2" represents two units, but when a zero is placed after, two tens are represented.

This use of the zero symbol was the most important advance ever made in the science of computation. The great development of arithmetic followed upon the invention of this symbolism.

The Hindus applied the principles of arithmetic in figuring interest and compound interest, and in calculating areas and the volume of geometric figures. Many other types of problems which still appear in American arithmetics were worked out by the Hindus. Especially did they make great advance in the treatment of common or vulgar fractions.

The First Arithmetic Printed in Europe

In 1478, in the little town of Treviso in Italy, there was published a commercial arithmetic. It was the first systematic treatise on the subject to appear in print. The author's name was not given. At that time the cities of Italy were great commercial centers, which explains the interest shown there in arithmetic.

The moneys used then--ducats, crowns, marks, francs, livres, florins--were as much of a problem as are our various currencies of today. There was perhaps even more confusion in Italy, since each one of the large cities there issued its own money, thus causing great confusion in exchange.

Many of the words found in our business arithmetics today have come down to us from old Italian terms which were used in the early arithmetics:

Per cent	pro cento
Debtor	debbitore
Numerator	numeratore
Denominator	denominatore
Interest	interesse
Discount	sconto

First Arithmetic Printed in English

In the year 1537 an unknown arithmetician published at St. Albans the first printed explanation in English of the Hindu-Arabic numerals. A similar work appearing in 1539 added "in hole numbers or in broken" to its title. The expression "broken" numbers, meaning fractions, appears in many an early English arithmetic text.

The first really popular English arithmetic was published in London in 1542 by Roberte Recorde, an English physician. It was entitled "The Grounde of Artes: Teaching the woorke and practise of Arithmetike, both in whole numbres and Fractions, after a more easyer and exacter sorte, than anye lyke hath hytherto beene set forth."

Recorde's arithmetic continued in use for nearly two hundred years. Proportion or the "golden rule of three," both "direct" and "reverse," and the "double rule of three," with variations, the rule of fellowship, alligation, and the "rule of falsehood" are the principal applications studied.

Besides the fundamental operations of addition, subtraction, multiplication, and division as applied to integers and fractions, Recorde teaches "accounting by Counters" on lines. This is partly devised "for them that cannot write and reade," but Recorde notes that it may be useful for any of "them that can do both but have not at some time their pen or tables ready with them."

The terms in Recorde's arithmetic became the accepted terminology of English arithmetic and were used in the American colonies. Many of the problems were also widely copied even in later arithmetics. All of his works were in the form of a dialogue carried on between the Master and the Pupil.

. . . .

Decimal Fractions

The greatest advance in arithmetic, after the invention of the place system with the zero of the Hindus, was the invention of decimal fractions. It was Simon Stevin, of Bruges, in Belgium, who wrote (1585) the first systematic treatise on decimal fractions.

In his book, called "Disme," (our "dime"), Stevin explained that if weights and measures were also changed to the ratio 10, as 10 inches in a foot and 10 ounces to the pound, then all the applications of arithmetic could be simplified by using decimals. By the simple use of the decimal point, our number system is extended to the right, giving tenths, hundredths, and thousandths. Multiplication and division are performed exactly as in whole numbers, save that the decimal point must be correctly placed.

French children now enjoy much simpler arithmetic than do American children, for the French metric system puts weights and measures on a decimal basis.

Progress of Arithmetic in America

In 1556, a young Spaniard Juan Diez Freyle published in the city of Mexico a work giving the value of silver and gold of different degrees of refinement. This work, dedicated to "Jesus . . . and to our Lady" was the first arithmetical treatise of the new world.

Almost exactly one hundred and fifty years later--in 1705--the first printer of New York, William Bradford, decided that an American arithmetic was necessary. With the generous help of two textbooks imported from England, Bradford constructed the arithmetical part of a "Secretary's Guide," called "The Young Man's Companion."

"The American Instructor: or, Young Man's Best Companion" was published in 1748 by Benjamin Franklin's firm. This English work (the word "American" in the title was added by our Benjamin), "containing Spelling, Reading, Writing and Arithmetick," . . . "Merchants Accompts," "Shop and Book-keeping" and even "how to Pickle and Perserve" was widely popular in America until the end of the century.

Another text, Dilworth's "The Schoolmaster's Assistant, being a Compendium of Arithmetic, both practical and theoretical" was used from 1773 to 1825.

After the American Revolution, Nicholas Pike of Newburyport in Massachusetts put into circulation the first really popular, "made in America" arithmetic.

Less than one hundred years ago Pike's "Complete Arithmetic" and his more popular "Abridgement" were still in use in New England and New Jersey.

Daboll's "Schoolmaster's Assistant" first appeared about 1800 but was still in use when Abraham Lincoln went to district school. During the Civil War period, Warren Colburn of New England, to supplement the ordinary instruction in numbers, designed a little book, "Intellectual Arithmetic upon the Inductive Method of Instruction," full of questions to be used for oral arithmetic. This was intended, and doubtless the intention was realized, to worry the young children just as the regular arithmetics troubled the older ones.

The boy and girl who look around them in the world today will see in every branch of science and industry and commerce developments which have been made possible only by arithmetic and the higher mathematics. The young American who wishes to contribute something to the development of new inventions must make up his mind to get the foundation of arithmetic without which no progress in scientific lines is possible.

II. ROMAN NUMERALS

This article is an attempt to answer this question: How did the Romans work problems with their numerals? This is not an easy question to answer, because the material available on the subject is very meager. However, the author apparently has done a great deal of research on the subject and has considerable authority for his beliefs in the matter. He says that practical calculation was a part of the Roman's life from his early youth.[2] He then proceeds to show how some of the operations were performed. Thus, to subtract 126 from 378 by grouping the numerals and by the elimination of the symbols occurring in both numerals, the result would be found thus:

CCC	LXX	VIII
C	XX	VI
CC	L	II

With Arabic numerals, on the other hand, a knowledge of subtraction tables would be necessary for performing the operation. For addition of the same numbers the above process would be reversed. Multiplication can be done in the same way, except that the person doing the multiplication must remember that it is not the numerals that we multiply but the numbers they represent. Thus at least the three operations -- addition, subtraction, and multiplication -- can be performed with the Roman numerals. They are, of course, inferior to Arabic numerals although they do have some advantages. The great superiority of the Arabic numerals is due to the fact that thanks to the use of zero to indicate a blank space, each numeral obtains a part of

[2] J. Hilton Turner, "The Partial Answer to an Inevitable Question; Roman Elementary Mathematics. The Operations," The Classical Journal, 47: no. 2, November, 1951.

its value from its position.

Another method of computation used by the Romans was finger counting. The author pictures many of the finger positions used to represent certain Roman numerals and explains to some extent how the various operations were performed. The diagrams on the hand and finger positions used by the deaf and dumb as a means of communication are given.

The other means of computation taken up by this author is the use of the abacus. The Romans did not invent the abacus but they did at first use pebbles placed on the sand in similar positions to the beads on an abacus to do calculation. Some of the oldest of these are two in Rome and Paris about 5 by $3\frac{1}{2}$ inches in size. The author shows by diagrams and by explanation exactly how addition, subtraction, multiplication and (in a limited way) division may be performed on an abacus.

III. THE WONDERFUL WORLD OF MATHEMATICS

It is believed that the earliest men and women like ourselves lived about twenty-five thousand years ago. They first did their counting by putting down one rock for one person or thing, two rocks or possibly indicating two fingers for two persons or things, and quantity beyond three was just "a heap." Early man's first written record was just a notch on a tree or a stroke on a rock to mark the passage of a day, or the killing of a deer.[3]

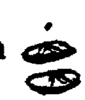
Gradually by watching the locations of the sun, moon and stars, men slowly and gradually learned to understand and record the passage of time and were able to divide it roughly into days, months, and years. Thus they learned the first elements of astronomy, and had greater need for more mathematical symbols for recording these and other events which they observed; as time went on, these ancient people depended upon a few specialists to record and to report the time of the rising and setting of the moon and later of the sun. This gradually led to the possibility of predicting the seasons and the length of the days and to some extent the weather. Gradually these wise men who were able to work out calendars and to predict seasons and weather became people of special importance. Farmers gladly provided them with a living, so that they could devote their entire time to foretelling the seasons. These men became a ruling class and often they were also priests who mixed magic with their calendar-making arts, but they did their job with considerable accuracy and with great skill.

The first written numbers were made by clipping notches in wood or stone to record the passing of events. Such writing was done about five thousand years ago in both Egypt and Mesopotamia. Although these two countries were

[3]Lancelot Hogben, The Wonderful World of Mathematics, Garden City, New York: Garden City Books, 1955.

miles apart, their number systems seemed to have started in the same way, but the mathematicians of Egypt wrote on papyrus, a paper made of reeds, while those of Mesopotamia wrote on soft clay tablets, which later were hardened in the sun. Thus the shapes of their numbers were different, but both used simple strokes for ones, and different marks for tens and higher numbers.

Three thousand years after this, the Romans still used strokes for the numbers from one to four. They used new symbols in the form of letters for fives, tens, fifties, and higher numbers. We know these as Roman numerals. About the same time, the Chinese used a different sign for ten, but continued to use strokes for the first three numbers.

The most amazing of all the early number systems was the one used by the Mayas, of Central America. These people were completely cut off from all civilizations of the Old World, yet these remarkable Mayas could write any number by using one of three signs, a dot, a stroke, and a kind of an oval. With the dots and strokes only, they could build up any number from one to nineteen. Thus ____ = 5, and the dot = one. Then  became 19. By adding one oval below any number, they multiplied it by twenty, thus: . = 1,  = 20. Adding a second oval would again multiply the number by twenty. When the Mayas reckoned time, however, they adjusted this system by adding a second oval, and they multiplied the number by 18 instead of 20. Then  meant not 400 (1 x 20 x 20) but 260 (1 x 20 x 18). We can easily understand that they used their signs this way to adjust to the moon calendar of 360 days. The Mayas eventually produced a sun calendar of 365 days. When they recorded days which were carved on stone columns, they used special numerals shaped like human faces.

The early builders of Egypt usually drew scale plans on clay tablets before building began. In building the pyramids, they used measures which were based on proportions of a man's body. Thus, 1 foot became the length of a man's foot. The width of four digits or four fingers became one palm, or the width of 1 palm. Seven palms, or the length from a man's elbow to the tops of his fingers became one cubit, and so on. They checked that their walls were built at right angles by means of a plumb line. They learned that right angles can be drawn by drawing equal arcs from any two spots on a straight line, and joining the points where the arcs cross. Before collecting taxes, the Egyptian official first measured the area of each field. Irregular fields were first marked off into triangles, then the area was found.[4]

The Egyptians found north and south by marking the sun's noon shadow, which points due north in locations north of the Tropic of Cancer. A line drawn at right angles to the noon shadow located east and west.

---

[4]Hogben, op. cit., pp. 16-17.

Most ancient peoples used a base of ten. The Egyptians and Mesopotamians used the abacus, using grooves in sand instead of wires as are used today. The value of each pebble in the first groove to the right was one. The second groove to the left was 10 x 1, and the third 10 x 10 x 1, and the fourth 10 x 10 x 10 x 1. It is a well known fact that Japanese businessmen still use the abacus with remarkable skill. The astronomers of Babylon were careful observers of eclipses and much interested in the square and the circle.

The Phoenicians are famous as one of the most successful of early peoples in the arts of sea-faring and trading. They early became acquainted with the sea, and gained knowledge from their voyages concerning the earth and the heavens, which would never occur to people living inland. For instance, these people of Tyre and Sidon soon learned that a part of a distant view is hidden from view by the curved surface of the earth, and therefore they concluded that the earth is spherical. They also studied the stars and night, and learned much about astronomy. Hoever, their greatest contributions to civilization were probably not in the field of map-making or astronomy, but in a new sort of writing. The Phoenicians developed an alphabet of about twenty-six symbols similar to the one used today.[5]

---

[5]Hogben, op. cit., p. 29.

This soon replaced the older systems of Greece and other countries, which had used a vast number of picture symbols for words and ideas. Because of this great achievement, it was easy thereafter to master the art of reading, and the written word was no longer a mystery.

When the Greeks had mastered the new alphabet writing and their trade had brought them knowledge of astronomy and navigation, and wealth in money as well as a wealth of knowledge from other lands, and when the wealth enabled the free citizens to use slaves for most of their day-to-day work, they had time for study and argument.

Pythagoras was a Greek mathematician who mixed both magic and religion with his instruction in mathematics. He tried to keep his mathematical knowledge as a secret among a small group of his students. However, he was a pioneer in the teaching of mathematics, and is, of course, remembered for proving that the square on the longest side of a right-angled triangle is equal to the sum of the squares drawn on the two shorter sides.

Another famous Greek was Euclid, who wrote a series of textbooks which have proved to be the best sellers of all times. The Greeks then made great strides in the use of the triangle, and of angles, and made great advances in geometry and surveying.

While the world was saddled with the number system of the Greeks, there could be no simple table for multiplication such as very young children learn to use today. The mathematicians of Egypt did have scrolls of some tables to avoid doing all their calculations on a counting frame, but a scroll would need to be almost endless to hold all the numbers used for our multiplications in modern astronomy.

We, today, use a kind of shorthand in mathematics. Thus: $A = \frac{bh}{2}$. But for the Greeks, such a shorthand was almost impossible because they used every letter of the alphabet to stand for a different number.[6] The Roman merchant might do some recording of numbers, but the laborious task of calculating was one for a slave working with an abacus. Before the Roman Empire and other parts of western Europe could make any real progress in the art of calculation and science, it had to have help. This help came from an Eastern civilization. It came from the invention and use of the zero. The civilization was one of the world's oldest; it grew up in the valley of the River Indus in India. Several hundred years before the Romans rose to power, the mathematicians of India had found a close value for π . The people learned their first lessons in mathematics through astronomy, which has always been a gateway to time-reckoning and to temple-building.

Some unknown person in India, perhaps a counting clerk, made the marvelous discovery which makes modern science possible. He used the right-hand column of the

---

[6]Hogben, op. cit., p. 41

abacus to stand for units only, the next column to the left to stand for tens only, and so on. To indicate an empty column he at first used a dot and later a zero was used to replace the dot. Thus, so simple a device as learning to use a symbol for an empty column enabled us to write 22, 202, 2002, 2020, or 222 so that there could be no doubt which was which. Whereas, before the abacus limited the size of numbers that could be written or used in calculation. The mathematicians of India began to think more about fractions and to write them as we do. In about 500 years after this revolutionary event, India had mathematicians that solved problems which had baffled the most famous scholars of other countries. Varahamihira could forecast the position of planets. Aryabhata stated a rule for finding square roots and gave the value of π as 3.1416 which is still considered accurate enough for most purposes even today.

APPENDIX B. ORIGINAL DATA USED FOR TABLES I AND II

Pupil	Arithmetic Reasoning 5th Grade 1956 Total	Arithmetic Reasoning 6th Grade 1957 Total	Increase Over 1956
1	5.2	6.9	1.7
2	7.6	6.9	
3	7.2	8.7	1.5
4	5.6	8.3	2.7
5	7.9	8.1	.2
6	10.0	10.0	
7	7.6	8.1	.5
8	6.0	8.3	2.3
9	7.4	9.0	1.6
10	8.3	8.1	
11	7.4	7.9	.5
12	5.0	5.4	.4
13	7.2	8.7	1.5
14	6.2	7.9	1.7
15	7.6	9.0	1.4
16	5.0	6.9	1.9
17	7.4	7.4	
18	5.7	6.2	.4
19	7.2	7.9	.7
20	7.4	7.2	
21	6.9	9.5	2.6
22	6.9	9.0	2.1

Pupil	Arithmetic Fundamentals 5th Grade 1956 Total	Arithmetic Fundamentals 6th Grade 1957 Total	Increase Over 1956
1	6.7	7.0	.3
2	6.7	6.7	
3	5.7	7.1	1.4
4	6.0	7.9	1.9
5	6.8	8.3	1.5
6	7.7	9.5	1.8
7	6.5	8.6	2.1
8	6.1	8.8	2.7
9	6.9	8.0	1.1
10	5.5	7.6	2.1
11	6.6	8.1	1.5
12	5.2	6.7	1.5
13	6.5	8.0	1.5
14	6.6	7.5	.9
15	8.2	10.0	1.8
16	4.9	7.0	2.1
17	7.0	7.8	.8
18	5.4	6.5	1.1
19	5.8	7.0	1.2
20	6.3	7.2	.9
21	6.6	8.6	2.0
22	6.4	10.0	3.6

Pupil	Total Arithmetic 5th Grade 1956 Total	Total Arithmetic 6th Grade 1957 Total	Increase Over 1956
1	5.6	7.0	1.3
2	6.9	6.8	
3	6.1	7.4	1.3
4	6.1	8.1	2.0
5	7.1	8.4	1.3
6	8.2	9.5	1.3
7	6.5	8.4	1.9
8	6.1	8.7	2.6
9	7.2	8.3	1.1
10	6.1	7.8	1.7
11	6.8	8.1	1.3
12	5.2	6.3	1.1
13	6.6	8.2	1.6
14	6.5	7.6	1.1
15	8.1	9.5	1.4
16	5.0	7.0	2.0
17	7.1	7.7	.6
18	5.5	6.5	1.0
19	6.2	7.2	1.0
20	6.6	7.2	.6
21	6.7	8.9	2.2
22	6.5	9.5	3.0

Pupil	Percentile Rank Arithmetic Reasoning	Percentile Rank Arithmetic Fundamentals	Percentile Rank Total Arithmetic
1	50	60	60
2	50	50	50
3	95	60	70
4	90	80	85
5	85	90	90
6	99	95	95
7	85	90	90
8	90	95	90
9	95	85	90
10	85	75	80
11	85	85	85
12	15	50	30
13	95	90	85
14	85	70	75
15	95	99	95
16	50	60	60
17	70	80	75
18	30	40	40
19	85	60	60
20	60	60	60
21	95	90	95
22	95	99	95

Pupil	Reading Vocabulary 5th Grade 1956 Total	Reading Vocabulary 6th Grade 1957 Total	Increase Over 1956
1	5.8	7.6	1.8
2	3.1	4.9	1.8
3	5.0	4.7	
4	6.7	8.3	1.6
5	7.4	7.6	.2
6	6.7	7.9	1.2
7	5.3	6.7	1.4
8	4.1	5.6	1.5
9	7.4	7.9	.5
10	6.7	7.9	1.2
11	6.2	8.3	2.1
12	4.3	5.2	.9
13	7.4	8.3	.9
14	7.6	9.0	1.4
15	7.6	7.6	
16	5.7	7.2	1.5
17	4.5	4.6	.1
18	5.5	5.5	
19	6.5	6.2	
20	4.7	5.1	.4
21	6.2	6.7	.5
22	7.9	7.2	

Pupil	Reading Comprehension 5th Grade 1956 Total	Reading Comprehension 6th Grade 1957 Total	Increase Over 1956
1	6.0	6.2	.2
2	4.4	6.2	1.8
3	5.4	6.6	1.2
4	6.9	7.9	1.0
5	8.8	7.5	
6	7.9	10.5	2.6
7	4.9	5.9	1.0
8	5.3	7.1	1.8
9	6.9	7.9	1.0
10	5.3	7.5	2.2
11	6.2	8.8	2.6
12	5.1	5.7	.6
13	7.5	9.5	2.0
14	8.8	8.8	
15	7.1	9.5	2.4
16	6.2	6.9	.7
17	5.7	5.7	
18	4.7	6.0	1.3
19	7.5	7.1	
20	5.3	5.9	.6
21	7.5	8.8	1.3
22	7.5	8.8	1.3

Pupil	Total Reading 5th Grade 1956 Total	Total Reading 6th Grade 1957 Total	Increase Over 1956
1	6.0	6.9	.9
2	3.7	5.4	1.7
3	5.2	5.4	.2
4	6.8	8.0	1.2
5	7.9	7.6	
6	7.2	8.8	1.6
7	5.1	6.3	1.2
8	4.6	6.2	1.6
9	7.2	7.9	.7
10	6.0	7.8	1.8
11	6.2	8.5	2.3
12	4.7	5.4	.7
13	7.5	8.8	1.3
14	8.0	9.0	1.0
15	7.5	8.3	.8
16	6.0	7.0	1.0
17	5.0	5.1	.1
18	5.2	5.7	.5
19	6.9	6.6	
20	5.0	5.4	.4
21	6.7	7.5	.8
22	7.8	7.8	

Pupil	Percentile Rank Reading Vocabulary	Percentile Rank Reading Comprehension	Percentile Rank Total Reading
1	70	30	50
2	10	30	15
3	5	50	15
4	85	80	80
5	70	70	70
6	80	99	90
7	50	25	40
8	20	60	30
9	80	80	80
10	80	70	75
11	85	90	90
12	15	20	15
13	85	95	90
14	95	90	95
15	70	95	85
16	60	50	60
17	5	20	10
18	20	30	20
19	30	60	50
20	10	25	15
21	50	90	70
22	60	90	75