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Oriented flow of rank 3 matroids

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ORIENTED FLOW OF RANK 3 MATROIDS

by

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Matthew R. Edmonds Mathematical Sciences

Oriented Flow of Rank 3 Matroids

Advisor: Jennifer McNulty

A parameter of oriented matroids called the oriented flow number is defined and studied. It is an extension of the concept of the circular chromatic number of a graph to oriented matroids. Oriented matroids can be realized as signed pseudosphere arrangements. When the rank of the matroid is 3, the pseudosphere arrangements take the form of line arrangements in the plane, in which the lines are not necessarily straight, and each pair of lines intersects exactly once. Rank 3 oriented matroids are studied in this setting. It had been conjectured that the oriented flow number of all rank 3 matroids is at most 4. This is shown to in fact be the case. This is shown first for uniform rank 3 matroids, and then the proof is extended to all rank 3 orientable matroids. The proof relies upon simple geometric considerations of arrangements and orientations of small numbers of lines in the plane. Larger arrangements are then viewed as unions of these smaller arrangements. The bound on the oriented flow number is then found by orienting the smaller arrangements in an optimal way.

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1 Background

1.1 Matroids

We assume a basic familiarity with the concepts of matroid theory. For a detailed treatment of the subject, see Oxley's *Matroid Theory* [6].

A matroid is a pair $\underline{M} = (E, \mathcal{S})$, where E is a finite set, called the *ground set* of $\underline{M}$, and $\mathcal{S} \subseteq 2^{|E|}$, satisfying any of several equivalent axiom systems. It is the definition of a matroid in terms of its circuits that we will primarily be interested in. This is presented below.

Definition 1.1. *A* matroid $\underline{M}$ *is an ordered pair* $(E, \underline{\mathcal{C}})$ *consisting of a finite set* E *and a collection* $\underline{\mathcal{C}}$ *of subsets of* E*, called circuits, satisfying the following three conditions:*

(C1) $\emptyset \notin \underline{\mathcal{C}}$.

(C2) If $\underline{C_1}$ *and* $\underline{C_2}$ *are members of* $\underline{\mathcal{C}}$ *and* $\underline{C_1} \subseteq \underline{C_2}$*, then* $\underline{C_1} = \underline{C_2}$.

(C3) If $\underline{C_1}$ *and* $\underline{C_2}$ *are distinct members of* $\underline{\mathcal{C}}$ *and* $e \in \underline{C_1} \cap \underline{C_2}$*, then there is a member* $\underline{C_3}$ *of* $\underline{\mathcal{C}}$ *such that* $\underline{C_3} \subseteq (\underline{C_1} \cup \underline{C_2}) \backslash e$.

If a single element $e \in E$ forms a circuit in $\underline{M}$, then e is called a *loop*.

An *independent set of* $\underline{M}$ is a subset of E that does not contain a circuit. A *basis* is a maximal independent set. If $A \subseteq E$, then *the rank of* A is $\rho(A) = \{|B \cap A| :$

B is a basis of $\underline{M}\}$. A *hyperplane* is a subset of E of rank $\rho(\underline{M}) - 1$.

If $\underline{\mathcal{C}}$ is the set of circuits of a matroid $\underline{M}$ with ground set E, then $\underline{\mathcal{C}}^* = \{E \backslash C : C \in \underline{\mathcal{C}}\}$ is the set of bases of a matroid on E. We call this matroid the *dual matroid of* $\underline{M}$ and denote it by $\underline{M}^*$. The circuits, independent sets, bases, circuits, loops and hyperplanes of $\underline{M}^*$ are called the cocircuits, coindependent sets, cobases, coloops and cohyperplanes of $\underline{M}$.

1.2 Oriented Matroids

The notation and definitions adopted in Sections 1.2 and 1.3 are all taken from *Oriented Matroids* by Björner et. al.[1], except for the information about pseudoline arrangements, which is from the work of Grünbaum [4, 5].

A *signed set* X is a set $\underline{X}$ together with a partition (X^+, X^-) of $\underline{X}$ into two subsets X^+ and X^- called the positive and negative elements of X. The set $\underline{X} = X^+ \cup X^-$ is the *support of* X. Two signed sets X and Y are equal if $X^+ = Y^+$ and $X^- = Y^-$. The *opposite* of a signed set X, denoted by $-X$, is the signed set with $(-X)^+ = X^-$ and $(-X)^- = X^+$. Given a signed set X and a set A, denote by $_{-A}X$ the signed set with $(_{-A}X)^+ = (X^+ \backslash A) \cup (X^- \cap A)$ and $(_{-A}X)^- = (X^- \backslash A) \cup (X^+ \cap A)$. We say that the signed set $_{-A}X$ is obtained from X by a *reorientation* on A.

Definition 1.2. *An* oriented matroid M *is an ordered pair* $(E, \mathcal{C})$ *consisting of a finite set* E *and a collection* $\mathcal{C}$ *of signed subsets of* E*, called oriented circuits, satisfying the following three conditions:*

(C0) $\emptyset \notin \mathcal{C}$;

(C1) $X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$;

(C2) For all $X, Y \in \mathcal{C}$*, if* $\underline{X} \subseteq \underline{Y}$*, then* $X = Y$ *or* $X = -Y$;

(C3) For all $X, Y \in \mathcal{C}$ *such that* $X \neq -Y$ *and* $e \in X^+ \cup Y^-$*, there is a* $Z \in \mathcal{C}$ *such that* $Z^+ \subseteq (X^+ \cup Y^+) \backslash \{e\}$ *and* $Z^- \subseteq (X^- \cup Y^-) \backslash \{e\}$.

An *orientation* of an (unoriented) matroid $\underline{M}$ is a signing of the ground set that satisfies the conditions (C1)-(C3). A matroid is *orientable* if it has an orientation. A matroid is *regular* if it is representable over all fields. Not all matroids are orientable; however, all regular matroids are.

Let M be an oriented matroid on a finite set E. Let $\mathcal{C}$ be the set of signed circuits of M. Let A be a subset of E, and let ${}_{-A}\mathcal{C} = \{{}_{-A}X : X \in \mathcal{C}\}$. Then it follows from the axioms that ${}_{-A}\mathcal{C}$ is also the set of circuits of an oriented matroid ${}_{-A}M$. We say that ${}_{-A}M$ is obtained from M by a *reorientation on* A. Two oriented matroids M and M' are *isomorphic up to reorientation* if their underlying matroids $\underline{M}$ and $\underline{M'}$ are isomorphic. Sets of all matroids that are isomorphic up to reorientation are called *reorientation classes* of oriented matroids. An orientable matroid may have several reorientation classes.

The collection of signed cocircuits of M also satisfies the axioms (C0)-(C3) above and forms the set of circuits of the dual oriented matriod $M^* = (E, \mathcal{C}^*)$.

1.3 Topological Representation Theorem

For $d \in \mathbb{Z}, d \geqslant -1$ let $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ denote the d-dimensional standard sphere. A subset S of S^d is called a *pseudosphere* if $S = h(S^{d-1})$ for some homeomorphism $h : S^d \to S^d$. If S_e is a pseudosphere of S^d, choosing one of the two components of $S^d \backslash S_e$ to be the *positive side* S_e^+ yields a *signed pseusosphere* $\vec{S}_e$. The *negative side* S_e^- equals $S^d \backslash (S_e \cup S_e^+)$. We define a *pseudosphere arrangement* $\mathcal{H} = (S_e)_{e \in E}$ to be a finite set of pseudospheres S_e in S^d such that

(A1) Every non-empty intersection $S_A = \bigcap_{e \in A} S_e$ is homeomorphic to a sphere of some dimension, for $A \subseteq E$; and

(A2) For every non-empty intersection S_A and every $e \in E$ such that $S_A \not\subseteq S_e$, the intersection $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

Every point $x \in S^d$ has an associated *sign vector* $X \in \{+, -, 0\}^E$, where X_e indicates whether x is on the positive side of S_e, the negative side of S_e, or lies on S_e. A *signed* *arrangement of pseudospheres* is a pseudosphere arrangement of signed pseudospheres. If $\mathcal{H}$ is a signed arrangement of pseudospheres, then let $\mathcal{C}(\mathcal{H})$ be the family of all sign vectors $X \in \{+,-,0\}^E$ that satisfy the following conditions:

(C1) $\bigcup_{e \in \underline{X}} S_e^{X_e} = S^d$, where $S_e^{X_e}$ denotes S_e^+ or S_e^-, and

(C2) the support $\underline{X} = \{e \in E : X_e \neq 0\}$ is minimal with property (C1).

The Topological Representation Theorem, proven by Folkman and Lawrence in 1978, describes the correspondence between oriented matroids and pseudosphere arrangements.

Theorem 1.1 (Topological Representation Theorem). [1]

(1) If $\mathcal{H} = (S_e)_{e \in E}$ is a signed arrangement of pseudospheres in S^d, then $\mathcal{C}(\mathcal{H})$ is the family of circuits of a rank $d+1$ simple oriented matroid on E.

(2) If $(E, \mathcal{C})$ is a rank $d+1$ simple oriented matroid, then there exists a signed arrangement of pseudospheres $\mathcal{H}$ in S^d such that $\mathcal{C} = \mathcal{C}(\mathcal{H})$.

(3) $\mathcal{C}(\mathcal{H}) = \mathcal{C}(\mathcal{H}')$ for two signed arrangements $\mathcal{H}$ and $\mathcal{H}'$ in S^d if and only if $\mathcal{H}' = h(\mathcal{H})$ for some self-homeomorphism h of S^d.

One powerful consequence of this theorem is that there is a one-to-one correspondence between equivalence classes of arrangements of pseudospheres in S^d and reorientation classes of simple rank $d+1$ oriented matroids.

When $d = 2$, the pseudosphere arrangements are pseudoline arrangements. A *pseudoline arrangement* in the real projective plane $\mathbb{P}^2$ is any family of simple closed curves in $\mathbb{P}^2$ such that every two curves have exactly one point in common, at which they cross each other. The arrangement determines a decomposition of $\mathbb{P}^2$ into open topological cells of dimensions 0, 1, and 2, respectively called vertices, line segments and faces of the arrangement. Two arrangements of pseudolines are *isomorphic* if and only if there exists an incidence-preserving one-to-one correspondence between the vertices, line segments and faces of one arrangement and those of the other. An arrangement of pseudolines is *stretchable* if it is isomorphic to an arrangement of straight lines. It is known that every pseudoline arrangement of at most 7 lines is stretchable [4]. An arrangement of straight lines in which no point belongs to more than two lines is called a *simple arrangement*. The different isomorphism types of simple arrangements of 7 lines or less are all known [4, 5], and shown in Figures 16 and 17 in the Appendix. In particular, up to isomorphism there is only one simple arrangement each of 1, 2, 3, 4 and 5 lines; 4 simple arrangements of 6 lines; and 11 simple arrangements of 7 lines. The numbers of isomorphism types for larger numbers of lines are not known. The uniform matroid $U_{3,n}$ is represented by pseudoline arrangements of n lines in which no point belongs to more than two lines. When $n \leqslant 7$, these arrangements are all isomorphic to the simple arrangements shown in Figures 16 and 17. We will make use of this fact later on.

1.4 Circular Chromatic Number in Graphs

The circular chromatic number of a graph G is a generalization of the chromatic number. Introduced by Vince in 1988 [7], it is denoted by $\chi^*(G)$, and defined as follows.

Definition 1.3. *For two integers $1 \leqslant d \leqslant k$, a (k,d)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0,1,2,\ldots,k-1\}$ such that $(x,y) \in E(G) \Rightarrow d \leqslant |c(x)-c(y)| \leqslant k-d$. The* circular chromatic number *of G, $\chi^*(G)$, is the infimum of those k/d for which there exists a (k,d)-coloring of G.*

Note that a $(k,1)$-coloring of G is just an ordinary k-coloring. Regarding the relation of $\chi^*(G)$ to the chromatic number $\chi(G)$, it can be shown that $\chi(G)-1 < \chi^*(G) \leqslant \chi(G)$.

Let k be a positive integer. A *k-flow* in a graph G is an orientation $\omega(G)$ together with a function $f : E(G) \to \{0,\pm1,\pm2,\ldots,\pm(k-1)\}$ such that the net flow $\sum_{vu\in E(G)} f(vu) - \sum_{uv\in E(G)} f(uv) = 0$ for each $v \in V(G)$. The *flow index* $\xi(G)$ is the smallest k for which G has a *nowhere-zero k-flow*, that is, a k-flow with $f(e) \neq 0$ for all $e \in E(G)$.

Goddyn et. al. [3] give an equivalent definition of the circular chromatic number by relating it to nowhere-zero flows in graphs. They define a (k,d)-flow and the star flow index of a graph G as follows.

Definition 1.4. *A* (k,d)-flow *in a graph* G *is a* k-*flow* $(\omega(G), f)$ *such that* $d \leqslant |f(e)| \leqslant (k-d)$ *for all* $e \in E(G)$. *The* star flow index $\xi^*(G)$ *is the infimum of those* k/d *for which there exists a* (k,d)-*flow in* G.

Vertex colorings and nowhere-zero flows of graphs are dual concepts. If G is a plane graph and G^d is its planar dual, then $\chi(G) = \xi(G^d)$.

1.5 Oriented Flow in Matroids

Let $\underline{M} = (E, C^*)$ be a regular matroid. An *integer flow* in $\underline{M}$ is an orientation of $\underline{M}$ together with a function $f : E \to \mathbb{Z}$ such that, for every cocircuit $B \in C^*$, $\sum_{e\in B+} f(e) = \sum_{e\in B-} f(e)$. A flow f is *nowhere-zero* if $f(e) \neq 0$ for all $e \in E$. For integers $0 < d < k$, a *(k,d)-flow* is an integer flow with values in the set $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$, and a *nowhere-zero k-flow* is a $(k,1)$-flow. The *star flow index* $\xi^*(\underline{M})$ is the infimum of k/d over all (k,d)-flows in $\underline{M}$, and the *flow index* $\xi(\underline{M})$ is the minimum k for which $\underline{M}$ has a nowhere-zero k-flow. If $\underline{M}$ has no coloops, then it is known that $\underline{M}$ has a nowhere-zero k-flow for some k.

Goddyn et. al. proved the following result [3].

Theorem 1.2. *A regular matroid* $\underline{M}$ *has a (k,d)-flow if and only if there exists an orientation of* $\underline{M}$ *such that, for any cocircuit* B, $\frac{d}{k-d} \leqslant \frac{|B^+|}{|B^-|} \leqslant \frac{k-d}{d}$.

This implies the following, which gives an equivalent definition of the star flow index of a regular matroid.

Corollary 1.1. *Let $\underline{M}$ be a regular matroid, with $\underline{C}^*$ the set of cocircuits. The star flow index $\xi^*(\underline{M})$ is the minimum over all orientations of $\underline{M}$ of*

$$\max_{B \in C^*} \operatorname{imbal}(B),$$

where the imbalance of B *is defined as*

$$\operatorname{imbal}(B) = \frac{|B|}{\min\{|B^+|, |B^-|\}}.$$

Noting that a (k,d)-coloring of a graph G induces a nowhere-zero k-flow in its cographic matroid $M^*(G)$, Goddyn et. al. [3] showed that $\chi^*(G) = \xi^*(M^*(G))$, so that

$$\chi^*(G) = \min_{\omega(G)} \max_{C \in C} \left\{ \frac{|C|}{|C^+|}, \frac{|C|}{|C^-|} \right\},$$

where the minimum is over all orientations $\omega(G)$ of $M(G)$.

Goddyn et. al. [2] generalized this definition of χ^* to oriented matroids, defining the *oriented flow number* of an oriented matroid $M = (E, C^*)$ to be

$$\phi_o(M) = \min_{\mathcal{O}} \max_{B \in C^*} \operatorname{imbal}(B),$$

where the minimum ranges over the set of reorientations $\mathcal{O}$ of M. Since reorientation classes of M correspond to equivalence classes of pseudosphere arrangements of M, we have an equivalent definition which will be more suited to our purposes.

Definition 1.5. *Let $\mathcal{H} = (H_e)_{e \in E}$ be an (unsigned) arrangement of pseudospheres with underlying matroid $\underline{M}$ such that $\underline{M}$ does not have a coloop. Let $\underline{\mathcal{C}}^*$ be the set of unsigned cocircuits. Then we define the* oriented flow number ϕ_o *of $\mathcal{H}$ to be*

$$\phi_o(\mathcal{H}) = \min_{\vec{\mathcal{H}}} \max_{B \in \mathcal{C}^*} \operatorname{imbal}(B), \tag{1}$$

where the minimum is taken over all signings of $\mathcal{H}$.

We also define the oriented flow number of an orientable matroid $\underline{M}$ as the minimum of $\phi_o(\mathcal{H})$ over all pseudosphere arrangements $\mathcal{H}$ of $\underline{M}$:

$$\phi_o(\underline{M}) = \min_{\mathcal{H}} \phi_o(\mathcal{H}).$$

Using probabilistic methods, Goddyn et. al. [2] proved the following bounds on $\phi_o(\mathcal{H})$.

Theorem 1.3. *If $\mathcal{H}$ is a pseudoline arrangement whose underlying matroid $\underline{M} = (E, \underline{\mathcal{C}}^*)$ is coloop-free and of rank 3, then $\phi_o(\mathcal{H}) \leqslant 17$, and furthermore, $|E| \geqslant 159 \Rightarrow \phi_o(\mathcal{H}) \leqslant 4$, and $|E| \geqslant 427 \Rightarrow \phi_o(\mathcal{H}) \leqslant 3$.*

Theorem 1.4. *If $\mathcal{H}$ is a pseudoline arrangement whose underlying matroid $\underline{M} = (E, \underline{\mathcal{C}}^*)$ is coloop-free and of rank $r \geqslant 4$, then $\phi_o(\mathcal{H}) \leqslant 14r^2 \ln r$.*

We explore the rank 3 case in depth, and show that $\phi_o(\mathcal{H}) \leqslant 4$ no matter the size of E.

2 Bounding the Oriented Flow Number of $U_{3,n}$

Let M be an oriented matroid on a finite set E, and let $\mathcal{H}$ be a projective pseudoline arrangement of M. Let C^* be the set of signed cocircuits of M. Let $\mathcal{F}$ be the set of faces of $\mathcal{H}$, and let $\mathcal{V}$ be the set of vertices of $\mathcal{H}$. If M has rank 3, each element of M is represented by a pseudoline in $\mathcal{H}$. We will refer to pseudolines simply as lines, with the understanding that the lines may be curved. Each signed cocircuit (or bond) B of M is represented by a vertex v_B in $\mathcal{H}$ in the following way: the elements of B are precisely those represented by lines not passing through v_B. We define the *degree of* v_B, denoted $\deg(v_B)$, as the number of lines passing through v_B. Then $|B| = |E| - \deg(v_B)$. Two vertices are *adjacent* if they lie on the same line l, and no other vertex on l lies between them. An *(open) line segment* s of l is the portion of l lying between two adjacent vertices. Let $\mathcal{S}$ be the set of line segments of $\mathcal{H}$. Two line segments are *adjacent* if their closures intersect at a vertex. Every line l is made up of the union of line segments and the vertices that separate them. Let p be any point in the plane. Define *the discrepancy of* p *in* $\mathcal{H}$, denoted by $\delta_{\mathcal{H}}(p)$, to be the sum of the orientations of the elements of M with respect to p in $\mathcal{H}$. Each line l contributes either 0, 1 or -1 to this sum, contributing 0 if and only if p lies on the line represented by l. If so, we say that l *contains* p. We call $|\delta_{\mathcal{H}}(p)|$ the *absolute discrepancy of* p *in* $\mathcal{H}$. Note that $|\delta_{\mathcal{H}}(p)| \leqslant |E|$. If F is any open face in $\mathcal{H}$, then define *the discrepancy of F in $\mathcal{H}$*, denoted by $\delta_{\mathcal{H}}(F)$, to be equal to $\delta_{\mathcal{H}}(p)$ for any $p \in F$. Two faces are *adjacent* if their closures intersect at a line segment. We call a face F an *n-face* if F is bounded by n line segments. For any two adjacent faces F_1 and F_2, $\delta_{\mathcal{H}}(F_2) = \delta_{\mathcal{H}}(F_1) \pm 2$ since F_1 and F_2 lie on the same side of all lines except the one that divides them. If s is a line segment separating adjacent faces F_1 and F_2, then define *the discrepancy of s in $\mathcal{H}$*, denoted by $\delta_{\mathcal{H}}(s)$, to be equal to $\delta_{\mathcal{H}}(p)$ for any p lying on s, and note that

$$\delta_{\mathcal{H}}(s) = \frac{\delta_{\mathcal{H}}(F_1) + \delta_{\mathcal{H}}(F_2)}{2}.$$

Let v_B be a vertex in $\mathcal{H}$ corresponding to a signed cocircuit B of M. Define *the discrepancy of B in $\mathcal{H}$*, denoted by $\delta_{\mathcal{H}}(B)$, to be equal to $\delta_{\mathcal{H}}(v_B)$. It follows from the definitions that $\delta_{\mathcal{H}}(B) = |B^+| - |B^-|$. When $\mathcal{H}$ is evident from the context it will be dropped from the notation and we will use the notation $\delta(p)$, $\delta(F)$, $\delta(s)$ and $\delta(B)$.

Since $|B| = |B^+| + |B^-|$,

$$|\delta(B)| = \big||B^+| - |B^-|\big| = |B| - 2\min\{|B^+|, |B^-|\}, \tag{2}$$

so that (1) becomes

$$\phi_o(\mathcal{H}) = \min_{\vec{\mathcal{H}}} \max_{B \in \mathcal{C}^*} \frac{2\,|B|}{|B| - |\delta(B)|}. \tag{3}$$

We will show that $\phi_o(\mathcal{H}) \leqslant 4$ for all pseudoline arrangements $\mathcal{H}$ whose underlying matroid M is coloop-free and has rank 3. The requirement that $\phi_o(\mathcal{H}) \leqslant 4$ is equiva-

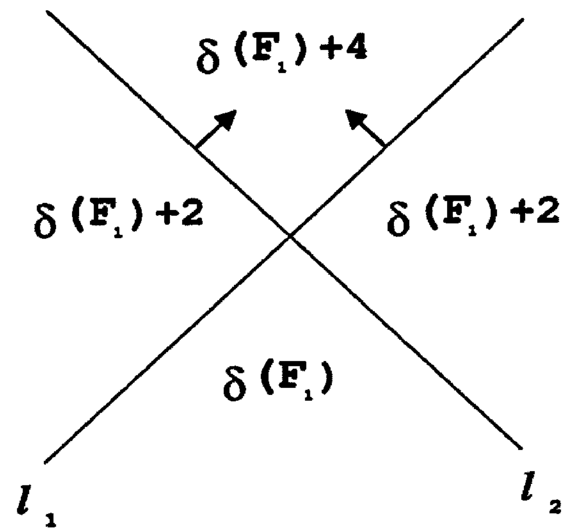

Figure 1: The relative discrepancy of adjacent faces in $U_{3,n}$.

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lent to the requirement that $|\delta(B)| \leqslant \frac{|B|}{2}$ for all $B \in \mathcal{C}^*$, since, given some signing of $\mathcal{H}$ that minimizes (3), we have:

$$
\begin{aligned}
& \phi_o(\mathcal{H}) \leqslant 4 \\
\iff & \frac{2\,|B|}{|B| - |\delta(B)|} \leqslant 4 \text{ for all } B \in \mathcal{C}^* \\
\iff & 2\,|B| \leqslant 4\,|B| - 4\,|\delta(B)| \text{ for all } B \in \mathcal{C}^* \\
\iff & |\delta(B)| \leqslant \frac{|B|}{2} \text{ for all } B \in \mathcal{C}^*. \qquad (4)
\end{aligned}
$$

We first examine the case where M is the uniform rank 3 matroid $U_{3,n}$, with $n \geqslant 4$ (when $n \leqslant 3$, every element in $U_{3,n}$ is a coloop and the ratio is undefined). Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$. Let $\mathcal{C}^*$ be the set of cocircuits and $\mathcal{F}$ the set of faces of $\mathcal{H}$. Let $B \in \mathcal{C}^*$ and let v be the vertex in $\mathcal{H}$ corresponding to B. Then v is the intersection of exactly 2 lines l_1 and l_2, and v is incident with four faces $F_1, \ldots, F_4$. The faces $F_1, \ldots, F_4$ and the vertex v lie on the same side of all lines except l_1 and l_2, so $\delta(F_1), \ldots, \delta(F_4)$ and $\delta(B)$ differ from each other only on account of the orientations of l_1 and l_2. Ordering $F_1, \ldots, F_4$ so that $\delta(F_1) \leqslant \delta(F_2) \leqslant \delta(F_3) \leqslant \delta(F_4)$, we must have $\delta(F_2) = \delta(F_3) = \delta(F_1) + 2$ and $\delta(F_4) = \delta(F_1) + 4$ since, as noted above, $\delta(F)$ differs by 2 for adjacent faces (see Figure 1). Now, l_1 and l_2 contribute 0 to $\delta(B)$, and they contribute 0 to $\delta(F_2)$ and $\delta(F_3)$ since F_2 and F_3 both lie on the positive side of one of l_1, l_2, and the negative side of the other. So $\delta(B) = \delta(F_2) = \delta(F_3)$, and furthermore, $\delta(B)$ is the average of $\delta(F_1), \ldots, \delta(F_4)$, since

$$\begin{aligned} \sum_{i=1}^{4} \frac{\delta(F_i)}{4} &= \frac{\delta(F_1) + 2(\delta(F_1) + 2) + (\delta(F_1) + 4)}{4} \\ &= \frac{4\delta(F_1) + 8}{4} \\ &= \delta(F_1) + 2 \\ &= \delta(B). \end{aligned} \tag{5}$$

Noting also that $|\delta(B)| = \max\{|\delta(F_1)|, |\delta(F_4)|\} - 2$, we have

$$\max_{B \in \mathcal{C}^*} |\delta(B)| = \max_{F \in \mathcal{F}} |\delta(F)| - 2. \tag{6}$$

Since each cocircuit B corresponds to a vertex v_B in $\mathcal{H}$, and $\delta(B) = \delta(v_B)$, we also have

$$\max_{v \in \mathcal{V}} |\delta(v)| = \max_{F \in \mathcal{F}} |\delta(F)| - 2. \tag{7}$$

Let F_1 be a face with maximum absolute discrepancy and F_2 be any face adjacent to F_1. Let s be the line segment separating F_1 and F_2. Then, since $|\delta(F_2)| = |\delta(F_1)| - 2$,

$$\begin{aligned} |\delta(s)| = \left|\frac{\delta(F_1) + \delta(F_2)}{2}\right| &\leqslant \left|\frac{\delta(F_1)}{2}\right| + \left|\frac{\delta(F_2)}{2}\right| \\ &= \left|\frac{\delta(F_1)}{2}\right| + \left(\left|\frac{\delta(F_1)}{2}\right| - 1\right) \\ &= |\delta(F_1)| - 1. \end{aligned}$$

Since $|\delta(s)|$ depends directly on $|\delta(F_1)|$, and F_1 is a face with maximum absolute discrepancy, we have

$$\max_{s \in \mathcal{S}} |\delta(s)| \leqslant \max_{F \in \mathcal{F}} |\delta(F)| - 1. \tag{8}$$

Note that (6) holds only when $\mathcal{H}$ is a pseudoline arrangement of $U_{3,n}$, while (8) holds for all pseudoline arrangements of rank 3 matroids.

Before proceeding further, we will need a few lemmas.

Lemma 2.1. *Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$, with $n \geqslant 4$. Let $\mathcal{F}$ be the collection of faces and C^* the set of cocircuits of $\mathcal{H}$. If $\mathcal{F}$ contains an n-face, then*

$U_{3,n}$ may be oriented so that

$$|\delta(F)| \leqslant \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd} \end{cases}$$

for all faces $F \in \mathcal{F}$; furthermore,

$$|\delta(B)| = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ 1 & \textit{if } n \textit{ is odd} \end{cases}$$

for all cocircuits $B \in C^*$.

Proof: Let $\mathcal{H}$ be a pseudoline arrangement of $\underline{U}_{3,n}$ that contains an n-face F_n. Let $\underline{C}^*$ be the set of unsigned cocircuits of $\underline{U}_{3,n}$. Without loss of generality, pick a line bounding F_n and orient it outwards. Proceed clockwise around the boundary of F_n, alternately orienting lines inwards and outwards with respect to F_n. Since F_n is an n-face, this completes the orientation of $\mathcal{H}$. Now, if n is even, $|\delta(F_n)| = 0$, and if n is odd, $|\delta(F_n)| = 1$. Let B be a cocircuit in $\mathcal{H}$, and let v_B be the vertex corresponding to B. Suppose v_B lies on the boundary of F_n, and n is even. Then v_B is the intersection of two lines, one oriented inwards with respect to F_n and one oriented outwards. So the four faces incident with B have δ-values -2, 0, 0 and 2, and $|\delta(B)| = 0$. Now suppose v_B lies on the boundary of F_n, and n is odd. Then $\mathcal{H}$ has one more line oriented outwards with respect to F_n than inwards, and $\delta(F_n) = -1$. If v_B is the

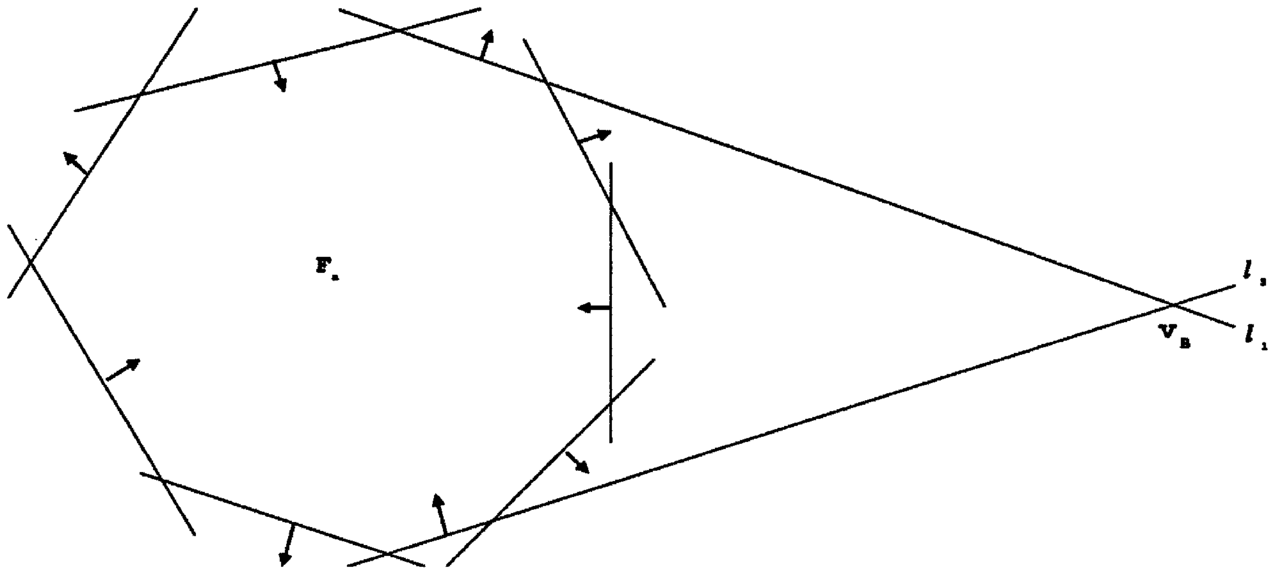

Figure 2: Here $n = 9$ and $k = 5$. The sign vector of v_B is given by $\bar{v}_B = (0, +, -, +, 0, -, +, -, +)$, and $\delta(B) = 1$.

---

intersection of an outward-oriented line and an inward-oriented line, then the four faces incident with v_B have δ-values $-3, -1, -1$ and 1, and $|\delta(B)| = 1$. If v_B is the intersection of two outward-oriented lines, then the four faces incident with v_B have δ-values -1, 1, 1 and 3, and $|\delta(B)| = 1$.

Suppose v_B does not lie on the boundary of F_n. Then v_B is still the intersection of two lines that lie on the boundary of F_n, since the boundary of F_n includes all lines in $\mathcal{H}$ (see Figure 2). Label the lines $l_1, \ldots, l_n$ of $\mathcal{H}$ so that v_B is the intersection of l_1 and l_k (here $k \neq 2$ and $k \neq n$). Without loss of generality, assume l_1 is oriented outwards with respect to F_n. The lines l_1 and l_k divide $\mathbb{P}^2$ into two projective half-planes. F_n lies entirely within one of these half-planes. The remaining lines of $\mathcal{H}$ are partitioned into $\{l_2, \ldots, l_{k-1}\}$ and $\{l_{k+1}, \ldots, l_n\}$. One of the sign vectors of $\{l_2, \ldots, l_{k-1}\}$ or $\{l_{k+1}, \ldots, l_n\}$ agrees with its sign vector relative to F_n, and the other is its negative. Without loss of generality, assume $\{l_{k+1}, \ldots, l_n\}$ is oriented the same direction with respect to v_B as it is with respect to F_n. Let $\bar{v}_B$ be the sign vector of B. If n is even, then $\bar{v}_B = (0, -, +, \ldots, \pm, 0, \mp, \ldots, -, +)$. Now l_k may be oriented outwards or inwards with respect to F_n, but in either case, $|\delta(B)| = 0$. If n is odd, assume that both l_1 and l_2 are oriented outwards with respect to F_n. Then $\bar{v}_B = (0, -, +, \ldots, \pm, 0, \mp, -, \ldots, +, -)$. Again, l_k may be oriented outwards or inwards with respect to F_n, but in either case, $|\delta(B)| = 1$. So, considering all possible cases, we have

$$|\delta(B)| = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{9}$$

for all $B \in \mathcal{C}^*$, implying by (6) that

$$|\delta(F)| \leqslant \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

for all $F \in \mathcal{F}$.

■

The next corollary follows easily.

Corollary 2.1. *If $n \geqslant 4$, then*

$$\phi_o(U_{3,n}) = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2n-4}{n-3} & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$. Let $\underline{C}^*$ be the set of unsigned cocircuits of $U_{3,n}$, and let $B \in \underline{C}^*$. First note that if n is even, the smallest $|\delta(B)|$ can be is 0, and if n is odd, the smallest $|\delta(B)|$ can be is 1. So,

$$\phi_o(\mathcal{H}) = \min_{\vec{\mathcal{H}}} \max_{B \in C^*} \frac{2\,|B|}{|B| - |\delta(B)|} \geqslant \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2|B|}{|B|-1} & \text{if } n \text{ is odd.} \end{cases}$$

Since $|B| = n - 2$ for all $B \in C^*$,

$$\phi_o(\mathcal{H}) \geqslant \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2n-4}{n-3} & \text{if } n \text{ is odd.} \end{cases}$$

For each $n \geqslant 4$, there is a pseudoline arrangement $\mathcal{H}'$ of $U_{3,n}$ with an n-face. This may be constructed by arranging the n lines of $U_{3,n}$ so they all border a single face. By Lemma 2.1, $\mathcal{H}'$ may be oriented so that the lower bounds of 0 and 1 for $|\delta(B)|$ are attained for all $B \in C^*$. So

$$\phi_o(\mathcal{H}') \leqslant \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2n-4}{n-3} & \text{if } n \text{ is odd} \end{cases}$$

and it follows that

$$\phi_o(U_{3,n}) = \min_{\mathcal{H}} \phi_o(\mathcal{H}) = \phi_o(\mathcal{H}').$$

■

We now develop notation for a partition of pseudolines. Let M be a coloop-free orientable matroid of rank 3. Let $\mathcal{H}$ be a pseudoline arrangement of M with orientation $\mathcal{O}$. Let $\mathcal{C}^*$ be the set of signed cocircuits of $\mathcal{H}$. Let $\Pi = (\Pi_1, \ldots, \Pi_k)$ be a partition of the lines $l_1, \ldots, l_n$ of $\mathcal{H}$. Each Π_i gives rise to a subarrangement $\mathcal{H}_i$ of $\mathcal{H}$. For each i, let $\mathcal{F}_i$ be the collection of faces of $\mathcal{H}_i$, let $\mathcal{S}_i$ be the collection of line segments of $\mathcal{H}_i$, let $\mathcal{V}_i$ be the collection of vertices of $\mathcal{H}_i$, and let $\mathcal{O}_i$ be the orientation of $\mathcal{H}$ restricted to Π_i. Now, each face $F \in \mathcal{F}$ is contained in some face $F_i \in \mathcal{F}_i$ for each i. Define $\delta_{\mathcal{H}_i}(F)$ to be equal to $\delta_{\mathcal{H}_i}(F_i)$. Let $(F_1, \ldots, F_k)$ be the faces of the subarrangements $\mathcal{H}_i$ that contain F, with each $F_i \in \mathcal{F}_i$. Then $F = F_1 \cap \cdots \cap F_k$ and $\delta_{\mathcal{H}}(F) = \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(F)$, implying

$$|\delta_{\mathcal{H}}(F)| = \left| \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(F) \right| \leqslant \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(F)| \, . \tag{10}$$

If s is a line segment in $\mathcal{H}$, then for each $\mathcal{H}_i$, s either lies in a face $F_i \in \mathcal{F}_i$, or lies on a line segment $s_i \in \mathcal{S}_i$. We define

$$\delta_{\mathcal{H}_i}(s) = \begin{cases} \delta_{\mathcal{H}_i}(F_i) & \text{if } v_B \text{ lies in a face } F_i \in \mathcal{F}_i \\ \delta_{\mathcal{H}_i}(s_i) & \text{if } v_B \text{ lies on a line segment } s_i \in \mathcal{S}_i. \end{cases}$$

Then $\delta_{\mathcal{H}}(s) = \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(s)$, and

$$|\delta_{\mathcal{H}}(s)| = \left|\sum_{i=1}^{k} \delta_{\mathcal{H}_i}(s)\right| \leqslant \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(s)|. \tag{11}$$

If $B \in \mathcal{C}^*$, then for each $\mathcal{H}_i$, the vertex v_B corresponding to B either lies in a face $F_i \in \mathcal{F}_i$, is a vertex $v_i \in \mathcal{V}_i$, or lies on a line segment $s_i \in \mathcal{S}_i$. So, we define

$$\delta_{\mathcal{H}_i}(B) = \delta_{\mathcal{H}_i}(v_B) = \begin{cases} \delta_{\mathcal{H}_i}(F_i) & \text{if } v_B \text{ lies in a face } F_i \in \mathcal{F}_i \\ \delta_{\mathcal{H}_i}(s_i) & \text{if } v_B \text{ lies on a line segment } s_i \in \mathcal{H}_i \\ \delta_{\mathcal{H}_i}(v_i) & \text{if } v_B \text{ is a vertex } v_i \in \mathcal{H}_i. \end{cases}$$

Then $\delta_{\mathcal{H}}(B) = \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(B)$. So, we also have

$$|\delta_{\mathcal{H}}(B)| = \left|\sum_{i=1}^{k} \delta_{\mathcal{H}_i}(B)\right| \leqslant \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(B)|. \tag{12}$$

We now use Lemma 2.1 to prove a bound on $|\delta(F)|$ in the particular case when $\mathcal{H}$ is a pseudoline arrangement of $U_{3,7}$. This will be used in the proof of Theorem 2.1.

Lemma 2.2. *Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,7}$, and let $\mathcal{F}$ be the collection of faces of $\mathcal{H}$. Then $\mathcal{H}$ may be oriented so that $|\delta(F)| \leqslant 3$ for all faces $F \in \mathcal{F}$.*

Proof: There are only 11 nonisomorphic pseudoline arrangements of $U_{3,7}$ [4]. These are shown in Figure 17. Figure 17a contains a 7-face, Figures 17b-e contain a 6-face, and Figures 17f-k contain neither a 6-face nor a 7-face. By Lemma 2.1, the arrangement containing a 7-face may be oriented so that it contributes at most 3 to $|\delta(F)|$ for all $F \in \mathcal{F}$. The arrangements that contain a 6-face can be partitioned into two pieces: the 6 lines bounding the 6-face and the one leftover line. By Lemma 2.1, the 6-line arrangement can be oriented so that it contributes at most 2 to $|\delta(F)|$ for all $F \in \mathcal{F}$, and the remaining line can be oriented arbitrarily, contributing 1 to $|\delta(F)|$ for all $F \in \mathcal{F}$. So, these arrangements can be oriented so that they contribute at most 3 to $|\delta(F)|$ for all $F \in \mathcal{F}$. The other 6 arrangements of $U_{3,7}$ are shown in Figures 3, 4 and 5. They have all been oriented so that $|\delta(F)| \leqslant 3$ for all $F \in \mathcal{F}$.

■

Consider $U_{3,4}$. Let $\mathcal{H}$ be the unique pseudoline arrangement of $U_{3,4}$ (up to isomorphism). It contains three 4-faces and four 3-faces and is shown in Figure 6. It is symmetric in the sense that each of the 4-faces is adjacent to the four 3-faces. Choose one of the 4-faces and orient the 4 lines bordering the face alternately with respect to this face. By Lemma 2.1, if the lines are oriented in this way, $|\delta(F)| \leqslant 2$ for all faces F and $|\delta(B)| = 0$ for all cocircuits B. By Corollary 2.1, this orientation minimizes $U_{3,4}$.

Our strategy for finding an upper bound for $\phi_o(U_{3,n})$, with $n > 4$, relies on the fact that there is only one pseudoline arrangement for $U_{3,4}$, and we know what the optimal way to orient it is. Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $U_{3,n}$.

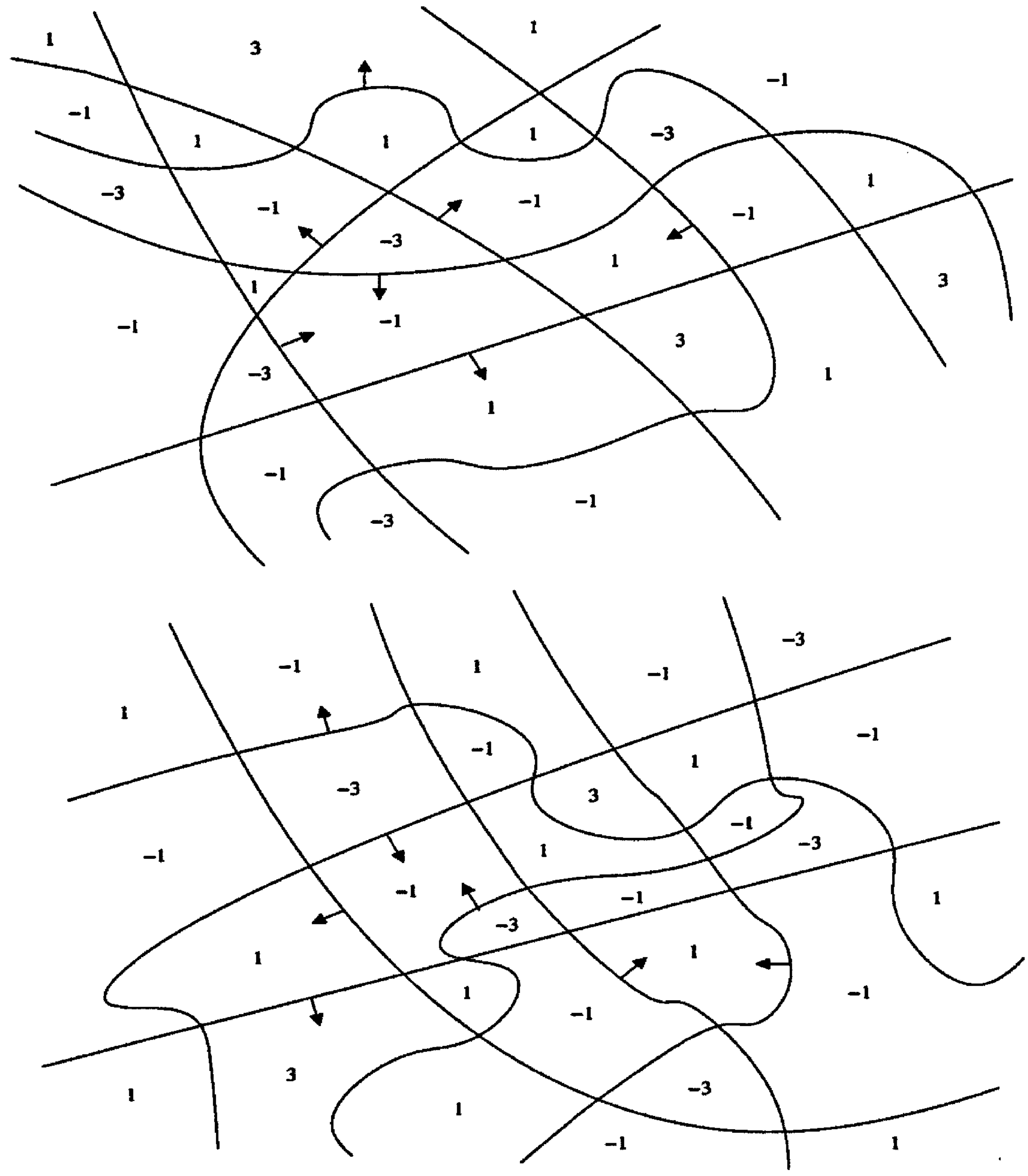

Figure 3: An orientation of two of the pseudoline arrangements of $U_{3,7}$ satisfying $|\delta(F)| \leqslant 3$ for all $F \in \mathcal{F}$. These correspond to Figures 17f and 17g. The values of $\delta(F)$ for each face are shown.

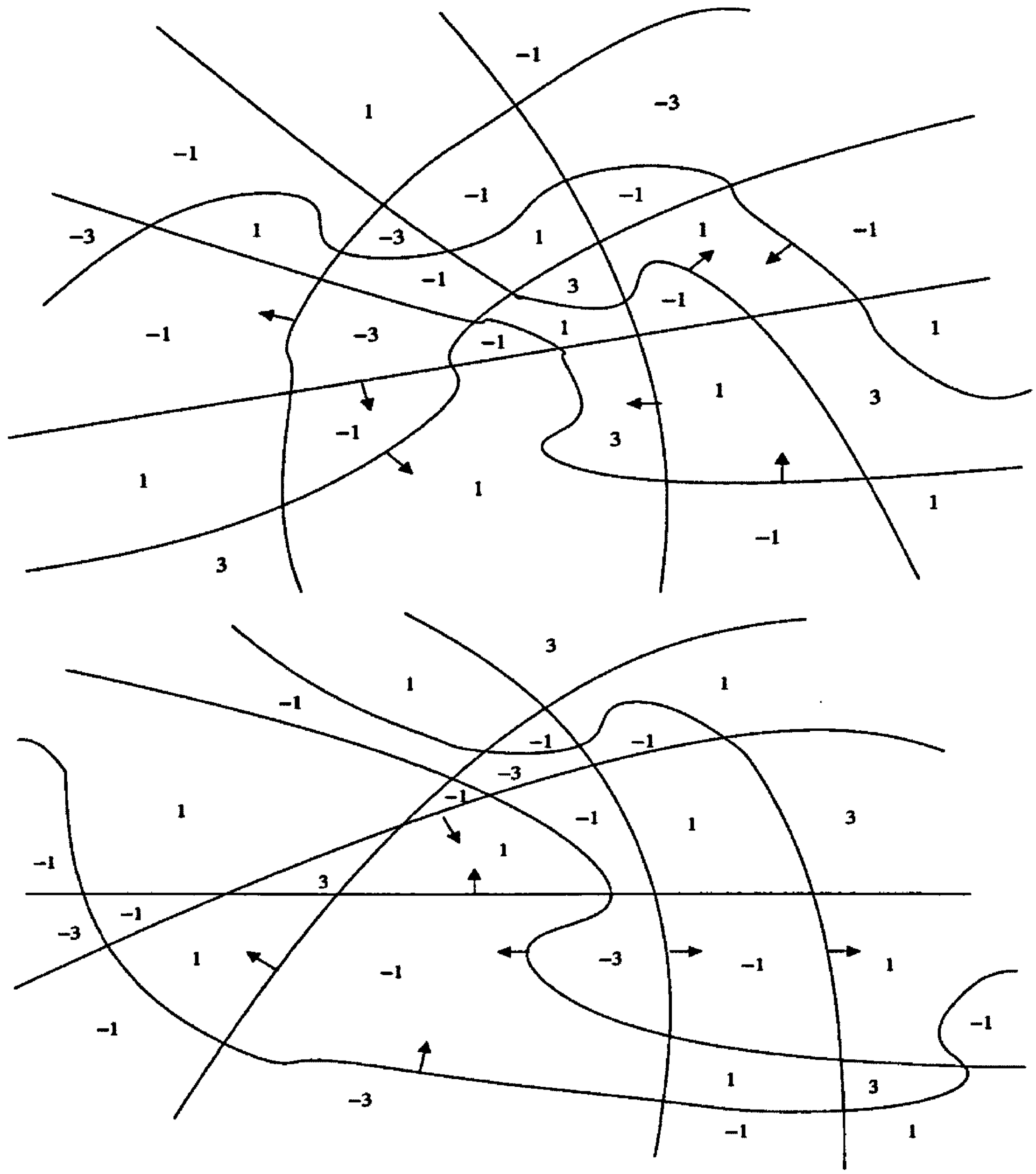

Figure 4: An orientation of two of the pseudoline arrangements of $U_{3,7}$, corresponding to Figures 17h and 17i, satisfying $|\delta(F)| \leqslant 3$ for all $F \in \mathcal{F}$.

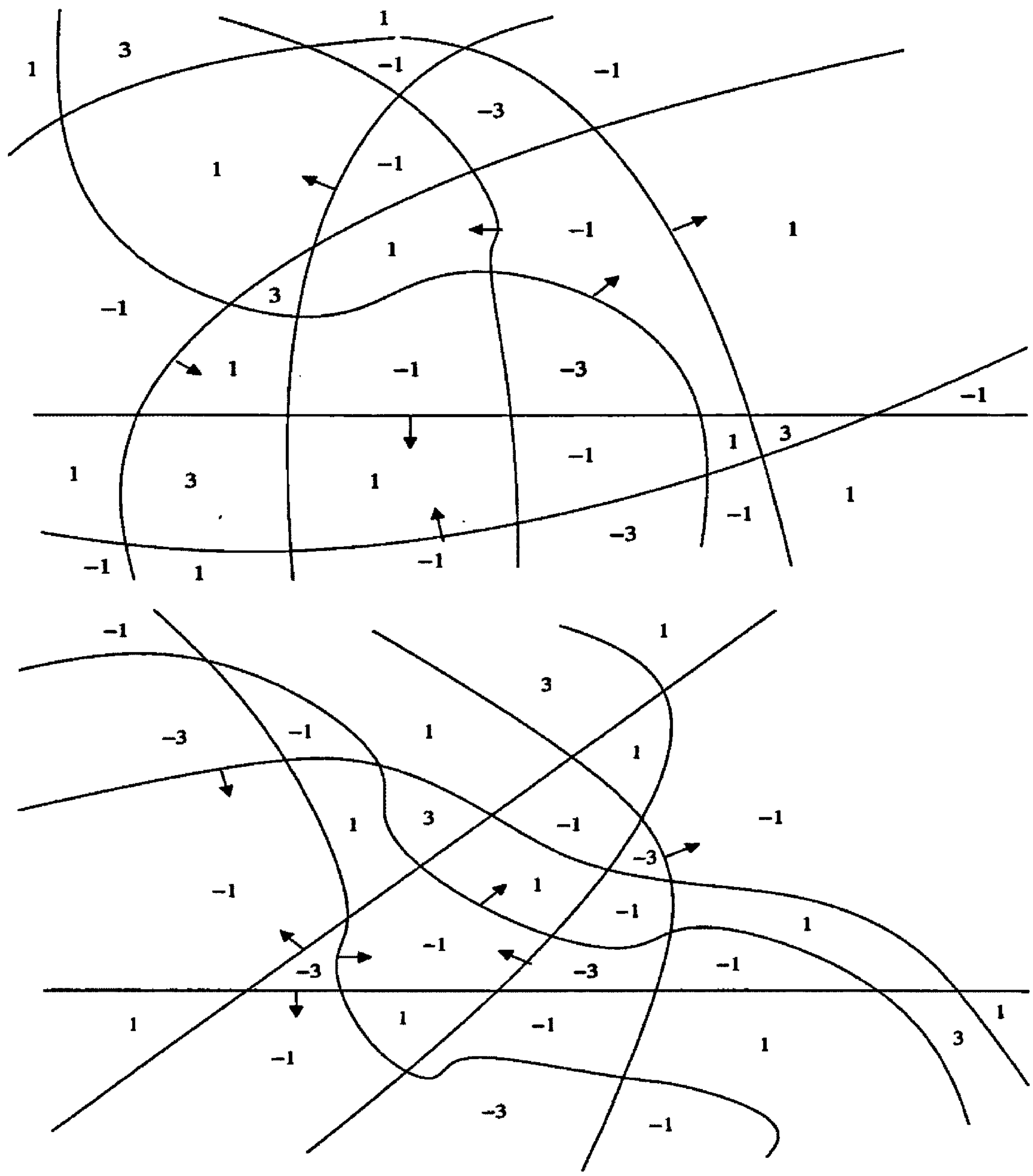

Figure 5: An orientation of two of the pseudoline arrangements of $U_{3,7}$, corresponding to Figures 17j and 17k, satisfying $|\delta(F)| \leqslant 3$ for all $F \in \mathcal{F}$.

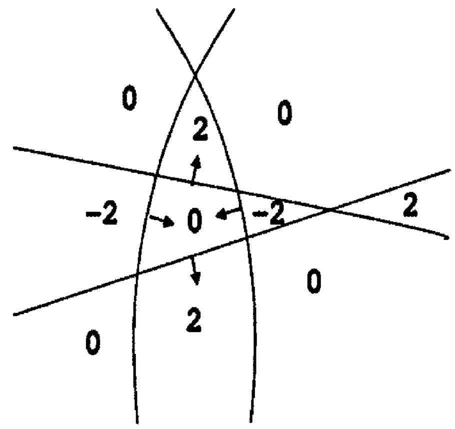

Figure 6: An orientation of the pseudoline arrangement of $U_{3,4}$ with minimal discrepancy.

Partition the lines of $\mathcal{H}$ into sets of four lines and orient them as described above, until there are either 0, 1, 2 or 3 lines remaining. Now, every face F in $\mathcal{H}$ lies in a face of each arrangement of the 4-line sets we have chosen. We can think of $\mathcal{H}$ as some number of 4-line arrangements laid on top of each other, plus up to 3 leftover lines. So $\delta(F)$ is the sum of $\delta(F_i)$ for each face F_i that contains F and lies in our choices of 4-line arrangements (ignoring, for the moment, the leftover lines). The manner in which we partition the lines is irrelevant since every subset of 4 lines in $\mathcal{H}$ is isomorphic to the pseudoline arrangement of $U_{3,4}$ shown in Figure 6. From above, $|\delta(F_i)| \leqslant 2$ for all $F_i \in \mathcal{F}(U_{3,4})$. Using (10), this choice of orientation will be sufficient to show $\phi_o(\mathcal{H}) \leqslant 4$ for all pseudoline arrangements with underlying matroid $U_{3,n}$.

We are now ready to prove the main result of this section.

Theorem 2.1. *Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $U_{3,n}$. Then $\phi_o(\mathcal{H}) \leqslant 4$ for all $n \geqslant 4$, with equality possible only if $n \equiv 2 \pmod 4$.*

Proof: Let $\mathcal{F}$ be the collection of faces of $\mathcal{H}$ and $\underline{\mathcal{C}}^*$ be the set of unsigned cocircuits of $U_{3,n}$. Note, if we can produce an orientation of $\mathcal{H}$ such that

$$\max_{F \in \mathcal{F}} |\delta(F)| \leqslant \frac{|B|+4}{2} \text{ for all } B \in \mathcal{C}^*, \tag{13}$$

then (6) implies

$$|\delta(B)| \leqslant \frac{|B|+4}{2} - 2 = \frac{|B|}{2} \text{ for all } B \in \mathcal{C}^*.$$

Thus, we have by (4),

$$\max_{F \in \mathcal{F}} |\delta(F)| \leqslant \frac{|B|+4}{2} \Rightarrow \phi_o(\mathcal{H}) \leqslant 4, \tag{14}$$

with $\phi_o(\mathcal{H}) = 4$ only if

$$\max_{F \in \mathcal{F}} |\delta(F)| = \frac{|B|+4}{2} \text{ for some } B \in \mathcal{C}^*.$$

To produce an orientation of $\mathcal{H}$ satisfying (13), we will partition and orient the lines of $\mathcal{H}$ as described in our discussion of strategy above. Four-line sets will be oriented as shown in Figure 6 until we are left with 0, 1, 2 and 3 leftover lines. This gives us four cases to consider: $n \equiv k \pmod 4$, with $k = 0, 1, 2$ or 3. We show that in each case, the resulting orientation satisfies (13). Note that $n = |B| + 2$.

Case 1: $n \equiv i \pmod 4$, with $i \in \{0, 1, 2\}$.

Partition the n lines of $\mathcal{H}$ into sets $\Pi_1, \ldots, \Pi_k, \Pi_{k+1}$, with $n = 4k + i$ and $|\Pi_j| = 4$ for $j \in \{1, \ldots, k\}$; and $|\Pi_{k+1}| = i$. Each Π_i (except Π_{k+1}) gives rise to a subarrangement $\mathcal{H}_i$ whose matroid is isomorphic to $U_{3,4}$. These subarrangements may all be oriented so that $|\delta_{\mathcal{H}_i}(F_i)| \leqslant 2$ for all $F_i \in \mathcal{F}_i$. The i leftover lines in Π_{k+1} may be oriented arbitrarily, contributing at most i to $|\delta(F)|$ for all $F \in \mathcal{F}$. So

$$\max_{F \in \mathcal{F}} |\delta(F)| \leqslant \sum_{i=1}^{k+1} |\delta_{\mathcal{H}_i}(F_i)| \leqslant 2\left(\frac{n-i}{4}\right) + i = \frac{n+i}{2} = \frac{|B| + 2 + i}{2}$$

$$\Rightarrow \phi_o(\mathcal{H}) \leqslant 4 \text{ by (14)},$$

with equality only if $i = 2$.

Case 2: $n \equiv 3 \pmod 4$.

This case is not so straightforward. If the strategy followed in the preceding cases is used and the three leftover lines are oriented arbitrarily, one obtains an upper bound of $\phi_o(U_{3,n}) \leqslant 5$. So, instead of orienting 4-line sets until only 3 lines remain, we orient 4-line sets until 7 lines remain, and then consider the various pseudoline arrangements of $U_{3,7}$. Formally, we partition the n lines of $\mathcal{H}$ into sets $\Pi_1, \ldots, \Pi_k$, with

$n = 4k+3$, $|\Pi_j| = 4$ for $j \in \{1, \ldots, k-1\}$ and $|\Pi_k| = 7$. Each Π_i (except Π_k) gives rise to a subarrangement $\mathcal{H}_i$ whose matroid is isomorphic to $U_{3,4}$. As in Case 1, these subarrangements may all be oriented so that $|\delta_{\mathcal{H}_i}(F_i)| \leqslant 2$ for all $F_i \in \mathcal{F}_i$. Π_k gives rise to a subarrangement $\mathcal{H}_k$ whose matroid is isomorphic to $U_{3,7}$. By Lemma 2.2, $\mathcal{H}_k$ may be oriented so that $|\delta_{\mathcal{H}_k}(F_k)| \leqslant 3$ for all faces $F_k \in \mathcal{F}_k$. So we have,

$$\max_{F \in \mathcal{F}} |\delta(F)| \leqslant \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(F_i)| \leqslant 2\left(\frac{n-7}{4}\right) + 3 = \frac{n-1}{2} = \frac{|B|+1}{2}$$

$$\Rightarrow \phi_o(\mathcal{H}) < 4 \text{ by } (14).$$

In all cases, $\phi_o(\mathcal{H}) \leqslant 4$. Note also that equality is possible only if $n \equiv 2 \pmod 4$. In Theorem 2.2 we will show that equality can be attained.

■

The next corollary actually follows from the *proof* of Theorem 2.1. We state it here because we will need it in the proof of Theorem 3.1.

Corollary 2.2. *Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $U_{3,n}$. Then $\mathcal{H}$ may be oriented so that*

$$|\delta(F)| \leqslant \frac{n+2}{2}$$

for all $F \in \mathcal{F}$.

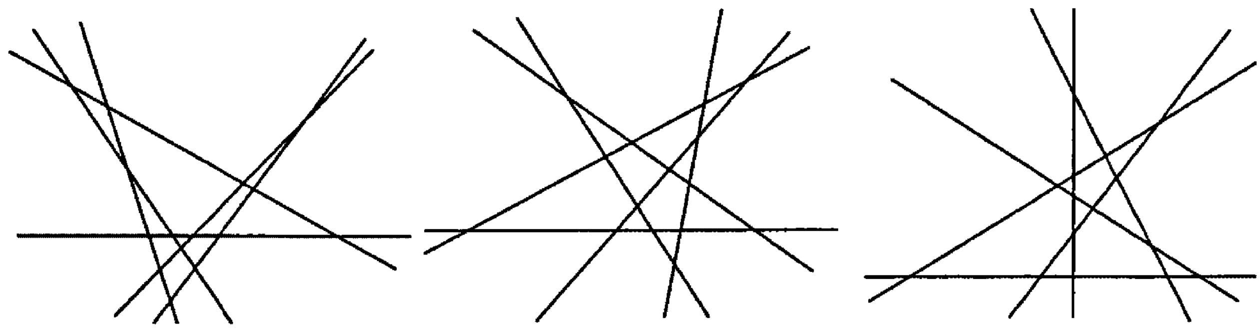

Figure 7: Three non-isomorphic pseudoline arrangements of $U_{3,6}$.

Proof: In Case 1 of the proof of Theorem 2.1, we showed $\mathcal{H}$ can be oriented so that $\max_{F\in\mathcal{F}} |\delta(F)| \leqslant \frac{n+i}{2}$, with $0 \leqslant i \leqslant 2$. In Case 2 we demonstrated an orientation with $\max_{F\in\mathcal{F}} |\delta(F)| \leqslant \frac{n-1}{2}$. Thus, $\mathcal{H}$ can be oriented so that $|\delta(F)| \leqslant \frac{n+2}{2}$ for all $F \in \mathcal{F}$.

■

Next we will show that the upper bound $\phi_o(\mathcal{H}) = 4$ can be attained for three of the four pseudoline arrangements of $U_{3,6}$. These are shown in Figure 7.

Theorem 2.2. *Let $\mathcal{H}$ be one of the pseudoline arrangements of $U_{3,6}$ shown in Figure 7. Then $\phi_o(\mathcal{H}) = 4$.*

Proof: Let $\mathcal{H}$ be one of the pseudoline arrangements of $U_{3,6}$ shown in Figure 7. We will show that no matter the orientation, $|\delta(F)| \geqslant 4$ for some $F \in \mathcal{F}$. Then by (6)

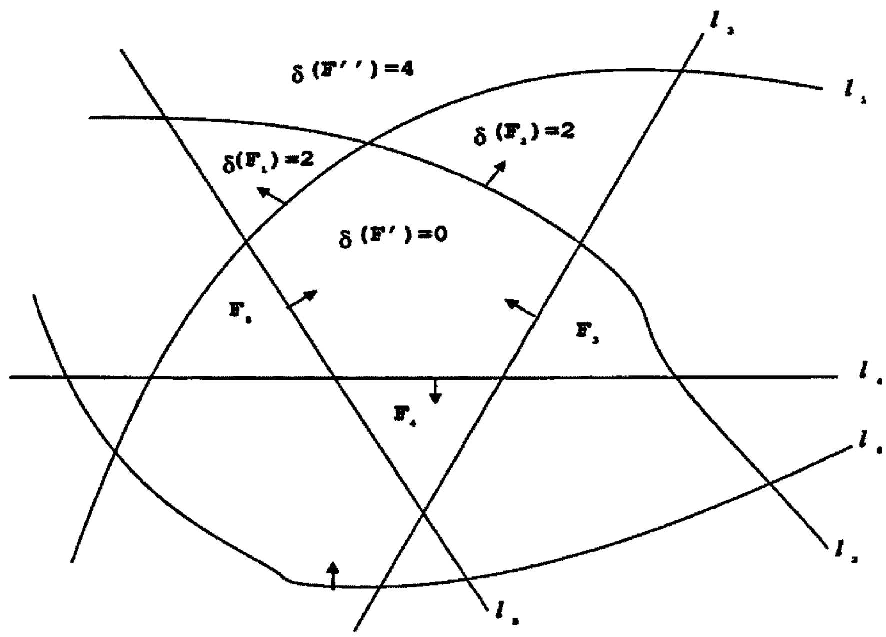

Figure 8: An orientation of $\mathcal{H}$ with $|\delta(F'')| = 4$.

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there exists $B \in \mathcal{C}^*$ such that $|\delta(B)| \geqslant 2$, so that

$$\phi_o(\mathcal{H}) = \min_{\vec{\mathcal{H}}} \max_{B \in \mathcal{C}^*} \frac{2\,|B|}{|B| - |\delta(B)|} \geqslant \frac{2 \cdot 4}{4 - 2} = 4. \tag{15}$$

Then by Theorem 2.1, $\phi_o(\mathcal{H}) = 4$.

In Figure 7 we see that each $\mathcal{H}$ contains at least one 5-face. Pick a 5-face $F' \in \mathcal{F}$. Pick a line bordering F' and label it l_1. Proceed cyclically around the boundary of F', labeling the four other lines bordering F' as l_2, l_3, l_4 and l_5 cyclically. Label the one remaining line as l_6. Let $F_1, \ldots, F_5$ be the faces of $\mathcal{H}$ adjacent to F' and incident respectively with $l_1, \ldots, l_5$.

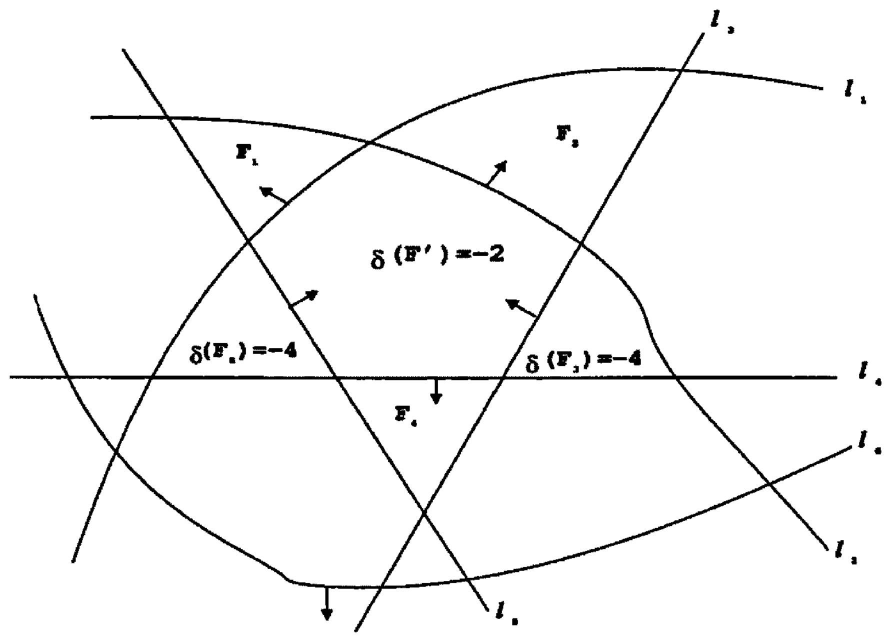

Figure 9: An orientation of $\mathcal{H}$ with $|\delta(F_3)| = |\delta(F_5)| = 4$.

We consider all possible orientations of the lines in $\mathcal{H}$ with respect to F'. Let k be the number of lines oriented in the same direction with respect to F'. Then $k = 3$, 4 or 5. Without loss of generality, say these k lines are oriented outwards. If $k = 5$, then $\delta(F') = -4$ or -6, depending on the orientation of l_6. Suppose $k = 4$. Then $\delta(F') = -2$ or -4, depending on the orientation of l_6. If $\delta(F') = -2$ and l_i is the inward-oriented line bordering F', then $\delta(F_i) = -4$. Now, suppose $k = 3$. Then at least two outward oriented lines, say l_1 and l_2, are adjacent on the border of F'. Let v be the vertex at the intersection of l_1 and l_2. Let F'' be the face adjacent to both F_1 and F_2, and incident with F' at v. Now, either l_6 is oriented inwards with respect to F', or it is oriented outwards. First suppose it is oriented inwards. Then $|\delta(F')| = 0$, but $|\delta(F_1)| = |\delta(F_2)| = 2$ and $|\delta(F'')| = 4$. (see Figure 8). Now suppose l_6 is oriented outwards with respect to F'. Then $\delta(F') = -2$. If l_i and l_j are the two inward-oriented lines bordering F', then $\delta(F_i) = \delta(F_j) = -4$ (see Figure 9).

In all cases, there exists a face $F \in \mathcal{F}$ such that $|\delta(F)| \geqslant 4$.

■

3 Bounding the Oriented Flow Number of All Rank 3 Orientable Matroids

We now consider the general case when M is a rank 3 matroid not equal to $U_{3,n}$. Then any pseudoline arrangement $\mathcal{H}$ of M contains vertices of degree greater than two. We want to find a way to partition and orient the lines in $\mathcal{H}$ so that $\delta(B) \leqslant \frac{|B|}{2}$ for each cocircuit B, to prove that $\phi_o(M) \leqslant 4$. The following lemma will be crucial for our proof.

Lemma 3.1. *Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$, with $n \geqslant 5$ and $n \neq 7$. Let F_1 be any face in $\mathcal{F}$. Then $\mathcal{H}$ may be oriented so that*

$$|\delta(F)| \leqslant \frac{n+2}{2} \text{ for all } F \in \mathcal{F} \text{ and } |\delta(F_1)| \leqslant \frac{n-2}{2}.$$

Proof: We will partition the lines of $\mathcal{H}$ into sets $\Pi_1, \ldots, \Pi_k$ in the manner described in the proof of Theorem 2.1. Since $n \geqslant 5$ and $n \neq 7$, $|\Pi_1| = 4$. In the proof of Theorem 2.1, we chose the set Π_1 arbitrarily. Here, we choose four lines that guarantee that F_1 lies in a 4-face of $\mathcal{H}_1$, and then orient them so that $|\delta_{\mathcal{H}_1}(F_1)| = 0$.

Let F_1 be an m-face in $\mathcal{H}$. If $m \geqslant 4$, choose Π_1 to be any four lines whose line segments bounding F_1 are consecutively adjacent. Then F_1 lies within a 4-face of $\mathcal{H}_1$ (see Figure 10). If $m = 3$, let l_1, l_2 and l_3 be the three lines bounding F_1. Consider

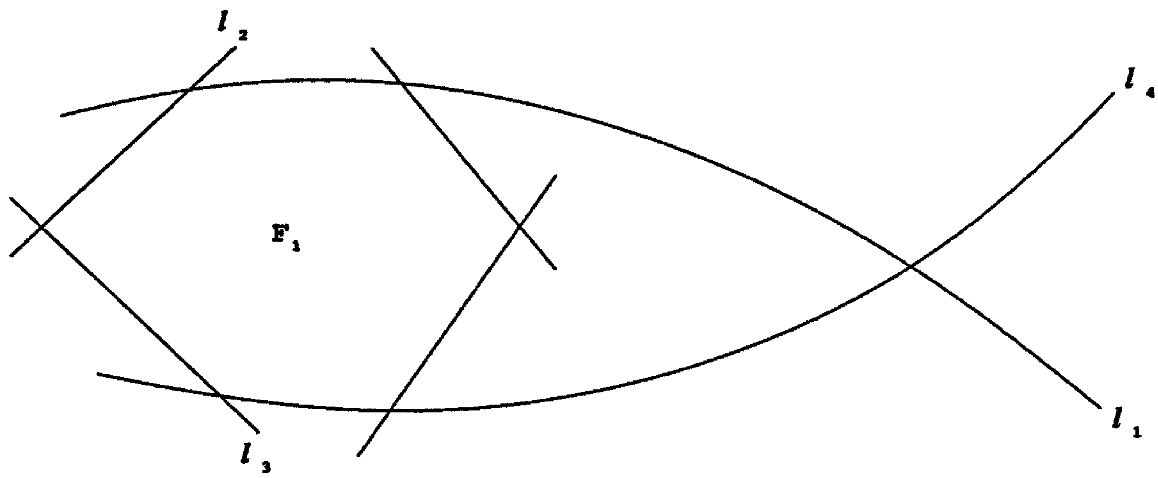

Figure 10: The case when $m \geqslant 4$.

the subarrangement $\mathcal{H}'$ generated by l_1, l_2, l_3 and any two other lines l_4 and l_5 in $\mathcal{H}$. Now, $\mathcal{H}'$ is isomorphic to the pseudoline arrangement of $U_{3,5}$ shown in Figure 11 [4]. Note that $\mathcal{H}'$ contains a single 5-face F_5. Since F_1 is a 3-face, F_1 must be one of the five faces adjacent to F_5. Suppose l_1 is the line segment separating F_1 and F_5. Let $\Pi_1 = \{l_2, l_3, l_4, l_5\}$. Then F_1 lies in a 4-face of $\mathcal{H}_1$. By the symmetry of $\mathcal{H}'$, the argument holds for all 3-faces in $\mathcal{H}'$.

Orient the lines of Π_1 alternately with respect to F_1, so that $\delta_{\mathcal{H}_1}(F_1) = 0$. This orientation is consistent with the method of orientation used in the proof of Theorem 2.1. By Corollary 2.2, $\mathcal{H}$ may now be oriented so that $|\delta(F)| \leqslant \frac{n+2}{2}$ for all $F \in \mathcal{F}$. For any given face F, the upper bound $\frac{n+2}{2}$ is attained precisely when $|\delta_{\mathcal{H}_i}(F)| = 2$

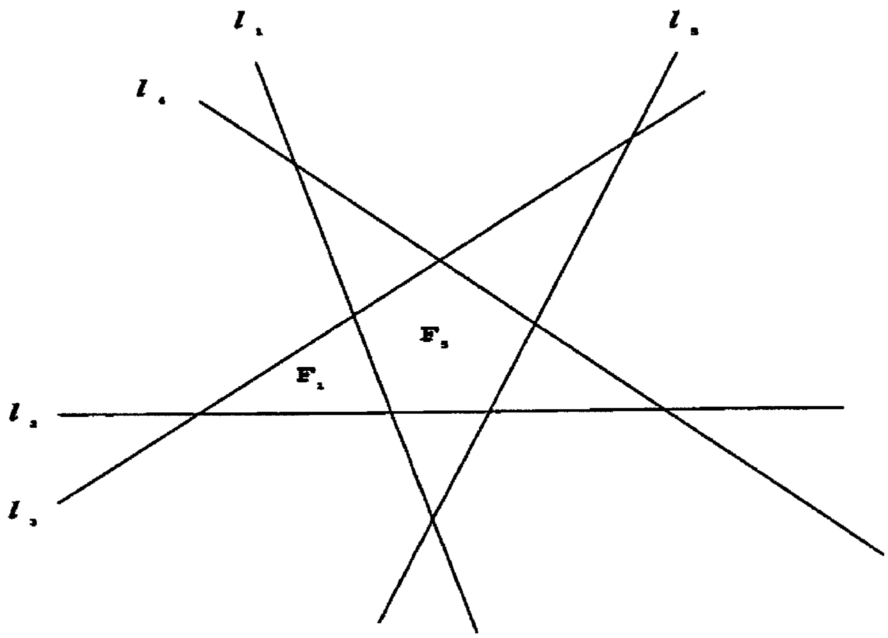

Figure 11: The case when $m = 3$.

for all i. Now, since $|\delta_{\mathcal{H}_1}(F_1)| = 0$, we have $|\delta(F_1)| \leqslant \frac{n+2}{2} - 2 = \frac{n-2}{2}$.

■

We now prove the main result of the thesis.

Theorem 3.1. *Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid M such that the rank of M is 3 and M does not have a coloop. Then $\phi_o(\mathcal{H}) \leqslant 4$.*

Proof: Let $l_1, \ldots, l_n$ be the lines of $\mathcal{H}$. We may assume that there exists a vertex in $\mathcal{H}$ with degree greater than 2; if not, then $\mathcal{H}$ is isomorphic to a pseudoline arrangement of $U_{3,n}$ for some $n \geqslant 4$, and $\phi_o(\mathcal{H}) \leqslant 4$ by Theorem 2.1.

We define a partition $\Pi = (\Pi_0, \Pi_1, \ldots, \Pi_k)$ of the lines of $\mathcal{H}$ as follows. Find a vertex v_1 in $\mathcal{H}$ of largest degree. If $\deg(v_1)$ is odd, let Π_1 be the set of lines intersecting at v_1. If $\deg(v_1)$ is even, let Π_1 be the set of lines intersecting at v_1 except one (that may be chosen arbitrarily). Let $\mathcal{H}_1$ be the subarrangement of $\mathcal{H}$ generated by Π_1. We will say that v_1 *gives rise* to $\mathcal{H}_1$. Now consider the arrangement $\mathcal{H} - \mathcal{H}_1$. Find a vertex v_2 of largest degree in $\mathcal{H} - \mathcal{H}_1$. If $\deg(v_2)$ is greater than 2, repeat the process; namely, if $\deg(v_2)$ is odd, let Π_2 be the set of lines intersecting at v_2, and if $\deg(v_2)$ is even, let Π_2 be the set of lines intersecting at v_2 except one. Let $\mathcal{H}_2$ be the subarrangement of $\mathcal{H}$ generated by Π_2.

Continue the process, defining the sets $\Pi_1, \ldots, \Pi_k$ until $\mathcal{H} \setminus \{\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k\}$ contains no vertices of degree greater than two. Let Π_0 be the set of remaining lines and $\mathcal{H}_0$ the corresponding subarrangement of $\mathcal{H}$. Let $m = |\Pi_0|$. Note that if $m \geqslant 3$, then $\mathcal{H}_0$ is isomorphic to a pseudoline arrangement of $U_{3,m}$. We will call the subarrangements $\mathcal{H}_1, \ldots, \mathcal{H}_k$ the *odd subarrangements of* $\mathcal{H}$ and $\mathcal{H}_0$ the *uniform subarrangement of* $\mathcal{H}$.

If $m \geqslant 4$, we orient the uniform subarrangement $\mathcal{H}_0$ as in the proof of Theorem 2.1. Let $F \in \mathcal{F}_0$. By Corollary 2.2,

$$|\delta_{\mathcal{H}_0}(F)| \leqslant \frac{m+2}{2}.$$

If $m \leqslant 3$, then no matter the orientation of $\mathcal{H}_0$, $|\delta_{\mathcal{H}_0}(F)| \leqslant m$ for all $F \in \mathcal{F}$. Since

$m \leqslant \frac{m+2}{2}$ if m equals 0, 1 or 2, we have

$$|\delta_{\mathcal{H}_0}(F)| \leqslant \begin{cases} \frac{m+2}{2} & \text{if } m \neq 3 \\ 3 & \text{if } m = 3. \end{cases} \tag{16}$$

Let s be a line segment in $\mathcal{H}_0$. By (8), $\max_{s \in \mathcal{S}_0} |\delta_{\mathcal{H}_0}(s)| \leqslant \max_{F \in \mathcal{F}_0} |\delta_{\mathcal{H}_0}(F)| - 1$, so that

$$|\delta_{\mathcal{H}_0}(s)| \leqslant \begin{cases} \frac{m}{2} & \text{if } m \neq 3 \\ 2 & \text{if } m = 3. \end{cases} \tag{17}$$

Let v be a vertex in $\mathcal{H}_0$. By (7), $\max_{v \in \mathcal{V}_0} |\delta_{\mathcal{H}_0}(v)| = \max_{F \in \mathcal{F}_0} |\delta_{\mathcal{H}_0}(F)| - 2$, so that

$$|\delta_{\mathcal{H}_0}(v)| \leqslant \begin{cases} \frac{m-2}{2} & \text{if } m \neq 3 \\ 1 & \text{if } m = 3. \end{cases} \tag{18}$$

The vertex v_B in $\mathcal{H}$ lies either in a face of $\mathcal{H}_0$ if it lies on zero lines of $\mathcal{H}_0$; a line segment of $\mathcal{H}_0$ if it lies on exactly one line of $\mathcal{H}_0$; or a vertex of $\mathcal{H}_0$ if it lies on two lines of $\mathcal{H}_0$. Let m_0 be the number of lines in $\mathcal{H}_0$ containing v_B. Note that $m_0 \in \{0, 1, 2\}$ since the degree of a vertex in $\mathcal{H}_0$ is at most 2. Combining (16), (17), and (18), we have

$$|\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(v_B)| \leqslant \begin{cases} \frac{m+2}{2} - m_0 & \text{if } m \neq 3; \\ 3 - m_0 & \text{if } m = 3. \end{cases} \tag{19}$$

This implies, furthermore, that

$$|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{m+3}{2} - m_0 \tag{20}$$

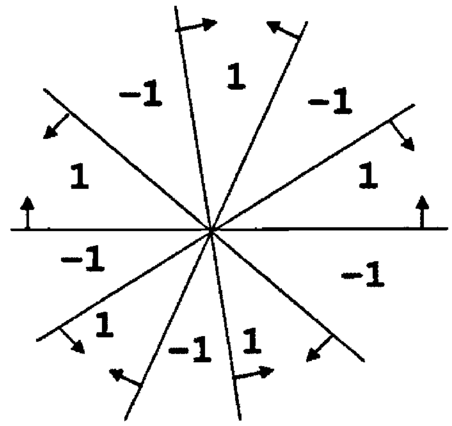

Figure 12: The method of orientation of odd subarrangements of $\mathcal{H}$.

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for all $m \geqslant 0$.

We orient the odd subarrangements $\mathcal{H}_i$, for $1 \leqslant i \leqslant k$, of $\mathcal{H}$ by alternating orientations as one rotates through the lines clockwise around v_i (see Figure 12). There are two possible ways to do this, yielding either $\delta_{\mathcal{H}_i}(F) = 1$ or $\delta_{\mathcal{H}_i}(F) = -1$ for any given face $F \in \mathcal{F}$. When we need to specify which orientation is to be used, we will do so. Suppose s is a line segment in $\mathcal{H}_i$ separating two faces $F_1, F_2 \in \mathcal{F}_i$. Then either $\delta_{\mathcal{H}_i}(F_1)$ or $\delta_{\mathcal{H}_i}(F_2)$ equals 1, and the other equals -1. Since $\delta_{\mathcal{H}_i}(s)$ is the average of $\delta_{\mathcal{H}_i}(F_1)$ and $\delta_{\mathcal{H}_i}(F_2)$, it follows that $\delta_{\mathcal{H}_i}(s) = 0$. Suppose v_B is a vertex in $\mathcal{H}$. For each i, v_B either equals v_i, or v_B lies in a face or a line segment of $\mathcal{H}_i$. For all i such that $v_B \neq v_i$, let k_i be the number of lines of $\mathcal{H}_i$ which contain v_B. So, if v_B lies in a face of $\mathcal{H}_i$, then $k_i = 0$, and if v_B lies in a line segment of $\mathcal{H}_i$, then $k_i = 1$. If

$v_B = v_j$ for some j, let $k_j = 0$. Then $k_i \in \{0,1\}$ for all i, and

$$|\delta_{\mathcal{H}_i}(B)| = |\delta_{\mathcal{H}_i}(v_B)| = \begin{cases} 1 & \text{if } k_i = 0 \text{ and } v_B \neq v_i \\ \\ 0 & \text{otherwise.} \end{cases}$$

Let $k_B = \sum_{i=1}^{k} k_i$. If $v_B \neq v_i$ for all i, then k_B is the number of distinct odd subarrangements of $\mathcal{H}$ containing v_B. If $v_B = v_j$ for some j, then k_B is the number of distinct odd subarrangements of $\mathcal{H}$, other than $\mathcal{H}_j$, which contain v_B.

In the proof of Theorem 2.1, we used the fact that since every vertex had degree two, $|B|$ was the same for all $B \in C^*$. In the general case we do not have that advantage. We will consider two cases: cocircuits whose corresponding vertices in $\mathcal{H}$ give rise to an odd subarrangement, and cocircuits that do not. Further subcases are needed depending on the values of k and m.

Case 1: $v_B = v_i$ for some $1 \leqslant i \leqslant k$, so v_B gives rise to an odd subarrangement. Then v_B is the intersection of all of the lines in some odd subarrangement, say Π_1; m_0 lines of the uniform subarrangement $\mathcal{H}_0$; and k_B lines of other distinct odd subarrangements (for example, see Figure 13). Order the odd subarrangements $\Pi_1, \ldots, \Pi_k$ so that $\Pi_2, \ldots, \Pi_{k_B+1}$ are the ones containing v_B. Note that $m_0 \in \{0,1\}$ since the partitioning of odd subarrangements leaves at most one line containing v_B in the uniform subarrangement. Furthermore, $m_0 = 0$ precisely when v_B lies in a face of $\mathcal{H}_0$, and

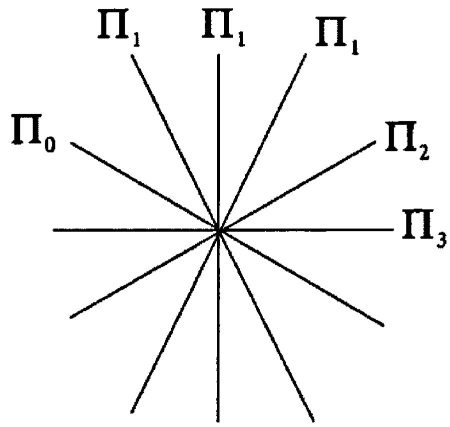

Figure 13: An example where $v_B = v_1$; here $m_0 = 1$ and $k_B = 2$. The labels indicate which subarrangement each line is contained in.

$m_0 = 1$ when v_B lies on a line segment of $\mathcal{H}_0$. Also, $0 \leqslant k_B \leqslant k-1$. Now, v_B misses $m - m_0$ lines of $\mathcal{H}_0$; all of the lines in $k - k_B - 1$ subarrangements; and all but one line in k_B subarrangements. So v_B misses at least $(m - m_0) + 2k_B + 3(k - k_B - 1)$ lines of $\mathcal{H}$, implying

$$|B| \geqslant (m - m_0) + 2k_B + 3(k - k_B - 1). \tag{21}$$

Since v_B lies on at least one line of $\mathcal{H}_i$ for $1 \leqslant i \leqslant k_B + 1$,

$$\sum_{i=1}^{k_B+1} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=1}^{k_B+1} 0 = 0.$$

Since v_B lies in a face of $\mathcal{H}_i$ for $k_B + 2 \leqslant i \leqslant k$,

$$\sum_{i=k_B+2}^{k} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=k_B+2}^{k} 1 = k - k_B - 1.$$

So,

$$|\delta(B)| \leqslant \sum_{i=0}^{k} |\delta_{\mathcal{H}_i}(B)| \leqslant |\delta_{\mathcal{H}_0}(B)| + k - k_B - 1. \tag{22}$$

We want to show that $|\delta(B)| \leqslant \frac{|B|}{2}$. Suppose

$$|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{1}{2}(m - m_0 + k + k_B - 1).$$

Then (22) implies that

$$\begin{aligned}|\delta(B)| &\leqslant \frac{1}{2}(m - m_0 + 3k - k_B - 3)\\ &= \frac{1}{2}((m - m_0) + 2k_B + 3(k - k_B - 1))\\ &\leqslant \frac{|B|}{2}.\end{aligned}$$

So,

$$|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{1}{2}(m - m_0 + k + k_B - 1) \Rightarrow |\delta(B)| \leqslant \frac{|B|}{2}. \tag{23}$$

We examine subcases for various values of m and k.

Case 1.1: $m = 3$. Then $|\delta_{\mathcal{H}_0}(B)| \leqslant 3 - m_0$ by (19). If $k \geqslant 4$, then $3 - m_0 \leqslant \frac{1}{2}(3 - m_0 + k + k_B - 1)$, and $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23). Suppose $k < 4$. The unique uniform arrangement $\mathcal{H}_0$ with 3 lines has exactly four 3-faces. It may be oriented so that $|\delta(F^*)| = 3$ for any chosen face F^*, and $|\delta(F)| = 1$ for the other three faces. If $k < 4$, there must be at least one face F^* in $\mathcal{H}_0$ that does not contain v_i for all i. Reorient $\mathcal{H}_0$ so that $|\delta(F^*)| = 3$ and $|\delta(F)| = 1$ for the other three faces. If $m_0 = 0$, then v_B falls within a face F of $\mathcal{H}_0$, and $|\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(F)| = 1$. Then, since $k \geqslant 1$ and $m = 3$,

$$\frac{1}{2}(m - m_0 + k + k_B - 1) \geqslant \frac{3}{2} \geqslant 1 = |\delta_{\mathcal{H}_0}(B)|,$$

and $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23).

Suppose $m_0 = 1$. Then v_B lies on a line segment s of $\mathcal{H}_0$, and $|\delta_{\mathcal{H}_0}(B)| \leqslant 2$. If $k \geqslant 3$, then $2 \leqslant \frac{1}{2}(m - m_0 + k + k_B - 1)$, and $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23). Suppose $k = 1$. Let F_1 and F_2 be the two faces in $\mathcal{H}_0$ bordering s. Reorient $\mathcal{H}_0$ so that $\delta(F_1) = 1$ and $\delta(F_1) = -1$. Then $\delta_{\mathcal{H}_0}(B) = \delta_{\mathcal{H}_0}(s) = 0$, and $0 \leqslant \frac{1}{2}(m - m_0 + k + k_B - 1) = 1 + \frac{1}{2}k_B$, implying $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23). Suppose $k = 2$. Then $|B| \geqslant 5 - k_B \geqslant 4$ by (21). Now,

$$\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_1}(B) + \delta_{\mathcal{H}_2}(B).$$

Without loss of generality, suppose v_B gives rise to $\mathcal{H}_1$. Then $\delta_{\mathcal{H}_1}(B) = 0$. Now, $\delta_{\mathcal{H}_0}(B) = -2, 0$ or 2, depending on the orientation of $\mathcal{H}_0$, and

$$|\delta_{\mathcal{H}_2}(B)| = \begin{cases} 0 & \text{if } k_B = 1 \\ 1 & \text{if } k_B = 0. \end{cases}$$

If $\delta_{\mathcal{H}_0}(B) = 0$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_2}(B)| \leqslant 1 < \frac{|B|}{2}.$$

If $|\delta_{\mathcal{H}_0}(B)| = 2$ and $k_B = 1$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 2 \leqslant \frac{|B|}{2}.$$

Suppose $|\delta_{\mathcal{H}_0}(B)| = 2$ and $k_B = 0$. We reorient $\mathcal{H}_2$ to "balance" the discrepancy of v_B. That is, if $\delta_{\mathcal{H}_0}(B) = 2$, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$, and if $\delta_{\mathcal{H}_0}(B) = -2$, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$. Then

$$|\delta(B)| = |\delta_{\mathcal{H}_0} + \delta_{\mathcal{H}_2}| = 1 < \frac{|B|}{2}.$$

Case 1.2: $m \neq 3$. Then $|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{m+2}{2} - m_0$ by (19). If $k \geqslant 3$, then $\frac{m+2}{2} - m_0 \leqslant \frac{1}{2}(m - m_0 + k + k_B - 1)$, and $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23).

Suppose $k \in \{1, 2\}$. There are various subcases, depending on the value of m, and most fixing a special orientation.

Case 1.2a: $m = 0$. Then $\mathcal{H}$ is partitioned into odd subarrangements only. We may assume $k = 2$ since if $k = 1$, then M has rank 2. Now, $|B| \geqslant 3k - k_B - 3$ by (21), and $|\delta(B)| \leqslant k - k_B - 1$ by (22). Since $k = 2$,

$$\begin{aligned} 1 &\leqslant k + k_B \\ \Rightarrow \frac{1}{2} &\leqslant \frac{1}{2}k + \frac{1}{2}k_B \\ \Rightarrow k - k_B - 1 &\leqslant \frac{3}{2}k - \frac{1}{2}k_B - \frac{3}{2} \\ \Rightarrow |\delta(B)| &\leqslant \frac{|B|}{2}. \end{aligned}$$

Case 1.2b: $m = 1$. Then $\mathcal{H}_0$ consists of a single line. We may assume $k = 2$ since if $k = 1$, then M contains a coloop or has rank 2. Now, no matter the orientation of

$\mathcal{H}_0$, $|\delta_{\mathcal{H}_0}(B)|$ equals 1 if $m_0 = 0$ (so v_B lies in a face of $\mathcal{H}_0$), and equals 0 if $m_0 = 1$ (so v_B lies on the single line in $\mathcal{H}_0$). So $|\delta_{\mathcal{H}_0}(B)| = 1 - m_0$. Then

$$1 - m_0 \leqslant 1 - \frac{m_0}{2} + \frac{k_B}{2} = \frac{1}{2}(1 - m_0 + 2 + k_B - 1),$$

implying $|\delta(B)| \leqslant \frac{|B|}{2}$ by (23).

Case 1.2c: $m = 2$. Then $\mathcal{H}_0$ has two faces F_1, F_2 and two possible orientations. Under one orientation, $|\delta_{\mathcal{H}_0}(F_1)| = 0$ and $|\delta_{\mathcal{H}_0}(F_2)| = 2$. Under the other orientation, $|\delta_{\mathcal{H}_0}(F_1)| = 2$ and $|\delta_{\mathcal{H}_0}(F_2)| = 0$. If $k = 1$, then v_B gives rise to the only odd subarrangement. Now, v_B cannot lie on a line segment of $\mathcal{H}_0$ since, if it did, M would contain a coloop. So $\mathcal{H}_0$ can be reoriented so that $|\delta_{\mathcal{H}_0}(F)| = 0$ for whichever face $F \in \mathcal{F}_0$ contains v_B, so that $|\delta_{\mathcal{H}_0}(B)| = 0$, hence $|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 0$ and $|\delta(B)| \leqslant \frac{|B|}{2}$.

Suppose $k = 2$. Then $|B| \geqslant 5 - m_0 - k_B \geqslant 3$ by (21). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two odd subarrangements of $\mathcal{H}$. Without loss of generality, suppose v_B gives rise to $\mathcal{H}_1$. Now, $|\delta_{\mathcal{H}_0}(B)| \leqslant 2$, $|\delta_{\mathcal{H}_1}(B)| = 0$, and $|\delta_{\mathcal{H}_2}(B)| \leqslant 1$. Suppose $|\delta_{\mathcal{H}_2}(B)| = 0$. If $m_0 = 0$, then $|B| \geqslant 4$, and

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| \leqslant 2 \leqslant \frac{|B|}{2}.$$

If $m_0 = 1$, then v_B lies on a line segment of $\mathcal{H}_0$, and $|\delta_{\mathcal{H}_0}(B)| = 1$. Then

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 1 < \frac{3}{2} \leqslant \frac{|B|}{2}.$$

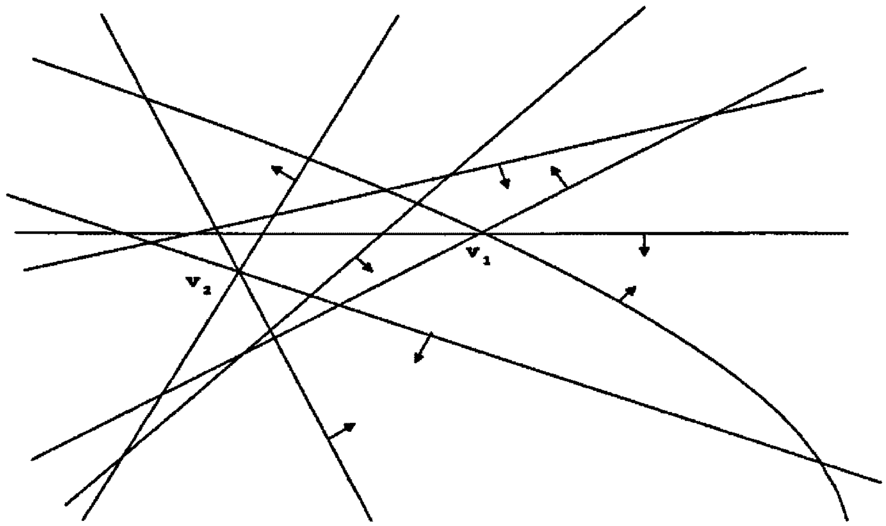

Figure 14: $\mathcal{H}_0$ is shown in bold. $\mathcal{H}_2$ is oriented to balance the discrepancy of v_1.

Now, suppose $|\delta_{\mathcal{H}_2}(B)| = 1$. We reorient $\mathcal{H}_2$ to "balance" the discrepancy of v_B. So, if $\delta_{\mathcal{H}_0}(B)$ is positive, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$, and if $\delta_{\mathcal{H}_0}(B)$ is negative, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$ (see Figure 14). If $\delta_{\mathcal{H}_0}(B) = 0$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_2}(B)| = 1 < \frac{3}{2} \leqslant \frac{|B|}{2}.$$

Suppose $\delta_{\mathcal{H}_0}(B)$ is positive. Then $\delta_{\mathcal{H}_0}(B) = 2$ if v_B lies in a face of $\mathcal{H}_0$ and 1 if v_B lies on a line segment of $\mathcal{H}_0$. Orient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$. Then

$$\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_1}(B) + \delta_{\mathcal{H}_2}(B) \leqslant 2 + 0 - 1 = 1,$$

and $|\delta(B)| < \frac{3}{2} \leqslant \frac{|B|}{2}$. If $\delta_{\mathcal{H}_0}(B)$ is negative, orient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$, and the result is the same.

Case 1.2d: $m = 4$. Then $\mathcal{H}_0$ is isomorphic to the unique pseudoline arrangement of $U_{3,4}$. Note that $\mathcal{H}_0$ contains a 4-face. Orient $\mathcal{H}_0$, following the procedure in Lemma 2.1, so that $|\delta_{\mathcal{H}_0}(F)| \leqslant 2$ for all faces $F \in \mathcal{F}_0$. Then $|\delta_{\mathcal{H}_0}(B)| \leqslant 2 - m_0$. Since $k \geqslant 1$,

$$\begin{aligned} k &\geqslant 1 - (m_0 + k_B) \\ \Rightarrow \frac{1}{2} - \frac{m_0}{2} - \frac{k_B}{2} &\leqslant \frac{k}{2} \\ \Rightarrow |\delta_{\mathcal{H}_0}(B)| \leqslant 2 - m_0 &\leqslant \frac{3}{2} - \frac{m_0}{2} + \frac{k_B}{2} + \frac{k}{2} \\ \Rightarrow |\delta_{\mathcal{H}_0}(B)| &\leqslant \frac{1}{2}(4 - m_0 + k_B + k - 1) \\ \Rightarrow |\delta(B)| &\leqslant \frac{|B|}{2} \text{ by (23).} \end{aligned}$$

Case 1.2e: $m = 7$. Then, by Lemma 2.2, $\mathcal{H}_0$ may be oriented so that $|\delta_{\mathcal{H}_0}(F)| \leqslant 3$ for all faces $F \in \mathcal{F}$. Then $|\delta_{\mathcal{H}_0}(B)| \leqslant 3 - m_0$. Since $k \geqslant 1$,

$$\begin{aligned} k &\geqslant -m_0 - k_B \\ \Rightarrow -\frac{m_0}{2} - \frac{k_B}{2} &\leqslant \frac{k}{2} \\ \Rightarrow |\delta_{\mathcal{H}_0}(B)| \leqslant 3 - m_0 &\leqslant 3 - \frac{m_0}{2} + \frac{k}{2} + \frac{k_B}{2} \\ \Rightarrow |\delta_{\mathcal{H}_0}(B)| &\leqslant \frac{1}{2}(7 - m_0 + k + k_B - 1) \end{aligned}$$

$$\Rightarrow |\delta(B)| \leqslant \frac{|B|}{2} \text{ by (23).}$$

Case 1.2f: $m \geqslant 5$ and $m \neq 7$. First suppose $k = 1$. Then v_B gives rise to the only odd subarrangement of $\mathcal{H}$. Necessarily, $k_B = 0$. Suppose that $m_0 = 0$, so that v_B lies in a face of $\mathcal{H}_0$. Then by Lemma 3.1, $\mathcal{H}_0$ may be reoriented so that $|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{m-2}{2}$. Since $|B| = m$ and $|\delta_{\mathcal{H}}(B)| = |\delta_{\mathcal{H}_0}(B)|$, we have

$$|\delta(B)| \leqslant \frac{m-2}{2} < \frac{m}{2} = \frac{|B|}{2}.$$

Now suppose $k = 1$ and $m_0 = 1$, so that v_B lies on a line segment s of $\mathcal{H}_0$. Then $|B| = m - 1$. Let l be the line in $\mathcal{H}$ containing s. Then v_B lies in a face of $\mathcal{H}_0 \backslash l$. If $m \geqslant 6$ then $|\mathcal{H}_0 \backslash l| \geqslant 5$, and by Lemma 3.1, $\mathcal{H}_0 \backslash l$ can be reoriented so that

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0 - l}(B)| \leqslant \frac{(m-1)-2}{2} = \frac{m-3}{2} < \frac{m-1}{2} = \frac{|B|}{2}.$$

Suppose $m = 5$. Then $\mathcal{H}_0$ is isomorphic to the unique pseudoline arrangement of $U_{3,5}$. Note that $\mathcal{H}_0$ contains a 5-face, so by Lemma 2.1, $\mathcal{H}_0$ can be reoriented so that $|\delta_{\mathcal{H}_0}(F)| \leqslant 3$ for all $F \in \mathcal{F}_0$, implying $|\delta_{\mathcal{H}_0}(s)| \leqslant 2$. So,

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(s)| \leqslant 2 = \frac{|B|}{2}.$$

Now suppose $k = 2$. Then $|B| \geqslant m + 3 - k_B - m_0$ by (21). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two odd subarrangements of $\mathcal{H}$. Without loss of generality, suppose v_B gives rise to

$\mathcal{H}_1$. Then $\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_2}(B)$. By (19), we have

$$|\delta_{\mathcal{H}_0}(B)| \leqslant \frac{m+2}{2} - m_0.$$

Note also that $|\delta_{\mathcal{H}_2}(B)| = 1 - k_B$. So, if $k_B = 1$ then

$$|\delta(B)| \leqslant \frac{m+2}{2} - m_0 \leqslant \frac{m}{2} + 1 - \frac{m_0}{2} \leqslant \frac{|B|}{2}.$$

If $k_B = 0$ then, as in Case 1.2c, reorient each odd subarrangement to "balance" the discrepancy of the vertex corresponding to the other. If $\delta_{\mathcal{H}_0}(B) = 0$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_2}(B)| = 1 < \frac{m}{2} + \frac{3}{2} - \frac{m_0}{2} \leqslant \frac{|B|}{2}$$

since $m \geqslant 5$. If $\delta_{\mathcal{H}_0}(B)$ is positive, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$, and if $\delta_{\mathcal{H}_0}(B)$ is negative, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$. Then by (22),

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| - 1,$$

so that

$$|\delta(B)| \leqslant \left(\frac{m+2}{2} - m_0\right) - 1 = \frac{m}{2} - m_0 < \frac{m}{2} + \frac{3}{2} - \frac{m_0}{2} \leqslant \frac{|B|}{2}.$$

This completes the proof of Case 1. The cases in which $\mathcal{H}$ was reoriented are all disjoint: these are cases 1.1, 1.2c, and 1.2f. In all cases, the reorientation was done in a manner consistent with the method of proof of Theorem 2.1, so the bounds on

$\delta_{\mathcal{H}_0}(B)$ which we used are all valid. $\mathcal{H}$ is not reoriented in Case 2, so there will be no ambiguity regarding the orientation of $\mathcal{H}$.

Case 2: $v_B \neq v_i$ for $1 \leqslant i \leqslant k$. Then v_B does not give rise to an odd subarrangement, and v_B lies on m_0 lines contained in Π_0, and k_B lines contained in distinct odd subarrangements (for example, see Figure 15). Consider the orientation of $\mathcal{H}$ given in Case 1. Reorder the odd subarrangements $\mathcal{H}_1, \ldots, \mathcal{H}_k$ so that the first k_B of these are the ones containing v_B. We must have $0 \leqslant m_0 \leqslant 2$, $0 \leqslant k_B \leqslant k$, and $m_0 + k_B \geqslant 2$. Now, v_B misses $m - m_0$ lines of the uniform subarrangement $\mathcal{H}_0$. There are at least 3 lines in each odd subarrangement, and v_B misses all the lines in $k - k_B$ subarrangements, and all but one line in k_B subarrangements. So v_B misses at least $(m - m_0) + 2k_B + 3(k - k_B)$ lines of $\mathcal{H}$, implying

$$|B| \geqslant (m - m_0) + 2k_B + 3(k - k_B). \tag{24}$$

Since v_B lies on a line segment of $\mathcal{H}_i$ for $1 \leqslant i \leqslant k_B$,

$$\sum_{i=1}^{k_B} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=1}^{k_B} 0 = 0.$$

Since v_B lies on a face of $\mathcal{H}_i$ for $k_B + 1 \leqslant i \leqslant k$,

$$\sum_{i=k_B+1}^{k} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=k_B+1}^{k} 1 = k - k_B.$$

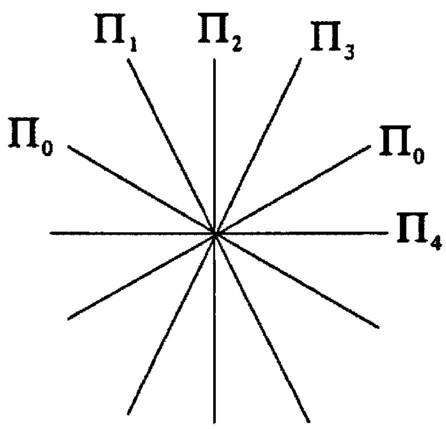

Figure 15: An example where $v_B \neq v_i$ for all i; here $m_0 = 2$ and $k_B = 4$. The labels indicate which subarrangement each line is contained in.

Using the last two relations, and (20) to bound $|\delta(\mathcal{H}_0)|$, we find that

$$|\delta(B)| \leqslant \sum_{i=0}^{k} |\delta_{\mathcal{H}_i}(B)| \leqslant \left(\frac{m+3}{2} - m_0\right) + (k - k_B). \tag{25}$$

Comparing (24) and (25), we see that $|\delta(B)| \leqslant \frac{|B|}{2}$ provided that

$$\left(\frac{m+3}{2} - m_0\right) + (k - k_B) \leqslant \frac{1}{2}(m - m_0) + k_B + \frac{3}{2}(k - k_B).$$

But the latter inequality is equivalent to $m_0 + k_B \geqslant 3 - k$, which follows from $m_0 + k_B \geqslant 2$ and $k \geqslant 1$.

This completes the proof of the theorem. In all cases, $|\delta(B)| \leqslant \frac{|B|}{2}$, so by (4), $\phi_o(\mathcal{H}) \leqslant 4$.

■

4 Potential Topics of Future Research

There are many outstanding questions about the oriented flow number. First, can we find a natural bound for $\phi_o(\mathcal{H})$ in higher ranks? It does not seem that the geometric methods used here could be applied to higher rank matroids, except possibly to rank 4 matroids. In rank 3, how does the upper bound on $\phi_o(\mathcal{H})$ decrease as the number of elements in the matroid increases? Goddyn et. al. [2] showed that $\phi_o(\mathcal{H}) \leqslant 3$ when $|E| \geqslant 427$. Perhaps this can be improved. Also, are there any other arrangements $\mathcal{H}$, other than the three shown here, for which $\phi_o(\mathcal{H}) = 4$?

The methods of proof used here suggest algorithms for generating relatively "balanced" orientations of line arrangements. There may be applications for this which are completely unrelated to matroid theory.

A Appendix: Simple Arrangements of Small Numbers of Lines

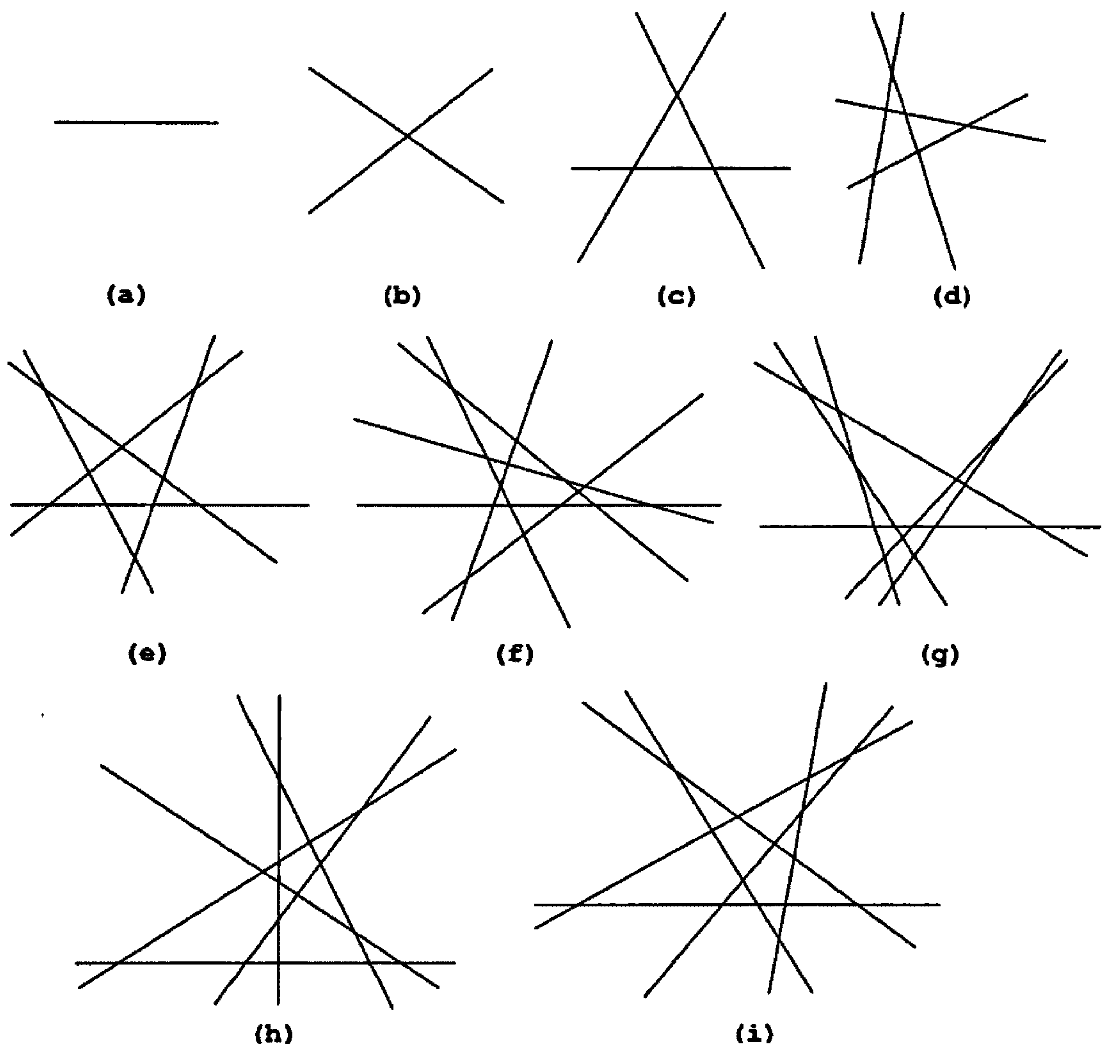

Figure 16: The nonisomorphic simple arrangements of at most 6 lines.

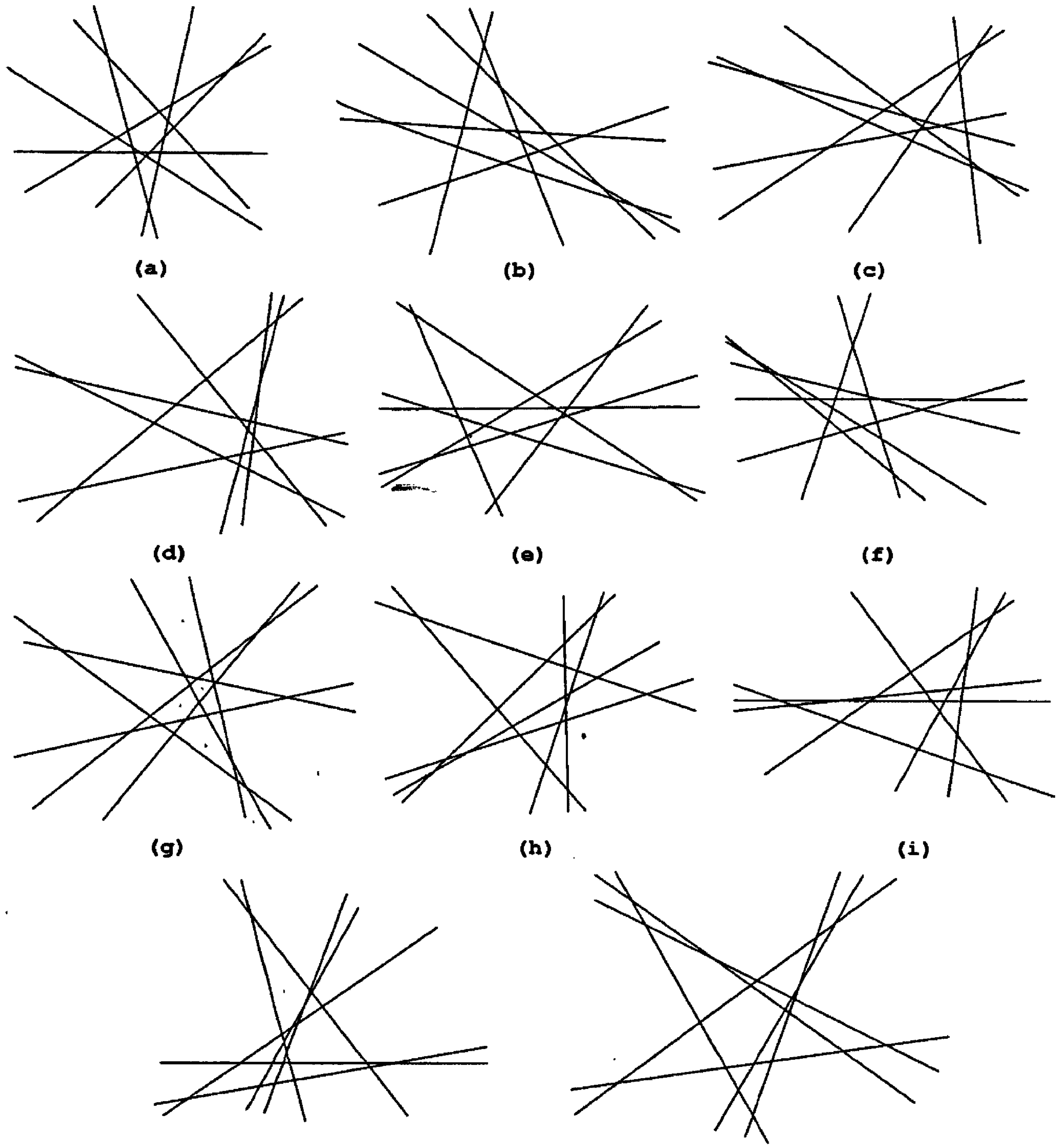

Figure 17: The nonisomorphic simple arrangements of 7 lines.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented Matroids*, Cambridge University Press, Cambridge, 1993.

[2] L. Goddyn, P. Hliněný, W. Hochstättler, *Balanced Signings of Oriented Matroids and Chromatic Number*, unpublished manuscript, 2003.

[3] L. Goddyn, M. Tarsi, C.Q. Zhang, *On (k,d)-colorings and fractional nowhere-zero flows*, J. Graph Theory **28** (1998), 155-161.

[4] B. Grünbaum, *Convex Polytopes*, J. Wiley & Sons Ltd., London, 1967.

[5] B. Grünbaum, *Arrangements and Spreads*, American Mathematical Society, Providence, Rhode Island, 1972.

[6] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[7] X. Zhu, *Circular chromatic number: a survey*, Discrete Mathematics **229** (2001), 371-410.