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Study of the Stieltjes Integral

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A STUDY OF THE STIELTJES INTEGRAL

by

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CHAPTER I

INTRODUCTION

The purpose of this paper is to make a study of the Stieltjes integral, more particularly the Lebesgue-Stieltjes integral, and to make a comparison of the Lebesgue integral and the Lebesgue-Stieltjes integral.

The Riemann integral receives some attention due to its relationship to the Lebesgue integral. The fundamental definitions and properties of the Riemann integral are stated.

The definitions and theorems of Variations of Functions, which are essential to the Lebesgue integral, are stated. Since the Lebesgue integral is the basis from which the Lebesgue-Stieltjes integral is derived it is given in more detail. Proof is given of such important theorems as those of Arzelà-Young, Egoroff as well as the Lebesgue Convergence theorem. A comparison is made of the Riemann integral and the Lebesgue integral.

The definitions and fundamental properties of the Riemann-Stieltjes integral are given. The Lebesgue-Stieltjes integral is developed in detail.

Some consideration is given the Perron integral, the Perron-Stieltjes integral, and the Denjoy integrals.

CHAPTER II

THE RIEMANN INTEGRAL

In considering the Riemann integral let us restrict attention to single real-valued bounded functions $f(x)$ defined on a bounded closed interval $a \leq x \leq b$. Consider a partition D of (a,b) into sub-intervals I_j. Let h_j be the length of I_j, let x_j be a point of the interval I_j, and $N(D)$ = greatest h_j. Let $S(D) = \sum_D f(x_j)\, h_j$.

Definition 1: A bounded function $f(x)$ is integrable on (a,b) in case the $\lim_{N(D)=0} S(D)$ exists. When it exists, this limit is denoted by $\int_a^b f(x)\, dx$.

Definition 2: Upper and lower integral. Let

$$M_j = \overline{\text{B}}_{\substack{X \\ I_j}} f(x), \qquad m_j = \underline{\text{B}}_{\substack{X \\ I_j}} f(x),$$

$$\overline{S}(D) = \sum_D M_j h_j, \quad \underline{S}(D) = \sum_D m_j h_j.$$

If $\lim_{N(D)=0} \overline{S}(D)$ exists, it is called the upper integral of $f(x)$, and is denoted by $\overline{\int_a^b} f(x)\, dx$. A similar definition and notation are used for the lower integral.

<u>Definition 3</u>: <u>Oscillation of a function</u>. For an arbitrary closed sub-interval (c,d) of (a,b), let M(c,d) = $\overline{B}$ f(x) for x in (c,d), m(c,d) = $\underline{B}$ f(x) for x in (c,d), O(c,d) = M(c,d) - m(c,d), O(x) = $\overline{\lim}_{x_1=x} f(x_1) - \underline{\lim}_{x_1=x} f(x_1) = \lim_{d=0} O(x-d, x+d)$, where O(c,d) is the oscillation of f(x) over (c,d).

<u>Definition 4</u>: A point set E has <u>Jordan Content</u> zero in case, for every number $\eta>0$ there is a finite set of intervals covering E, the sum of whose lengths is less than η. For every $\epsilon>0$, we shall denote the set of points in (a,b) at which $O(x) \geq \epsilon$ by E_ϵ.

<u>Theorem 1</u>: A bounded function f(x) is integrable on (a,b) if and only if, for every $\epsilon>0$ the set E_ϵ has Jordan Content zero.

<u>Definition 5</u>: A point set E has <u>Lebesgue measure</u> zero in case, for every $\eta>0$ there is a finite (or denumerably infinite) set of intervals covering E, the sum of whose lengths is less than η.

<u>Theorem 2</u>: <u>A bounded function</u> f(x) is integrable on (a,b) <u>if and only if the set D of points where</u> f(x) <u>is discontinuous has Lebesgue measure zero</u>.

Only those definitions and theorems have been given here which are fundamental to the further development of the ideas of integration. Any further references will be made by name to those definitions and theorems which can be found in any standard work on the topic.

CHAPTER III

VARIATIONS OF A FUNCTION

Suppose we have a real single-valued finite function $f(x)$ defined on a bounded interval $a \leq x \leq b$. Consider a partition D. Let I_j denote the segment $a_{j-1} \leq x \leq a_j$. $(j=1..n)$ Consider $S(D) = \sum_{j=1}^{n} [f(a_j) - f(a_{j-1})] = f(b) - f(a)$. Let $P(D)$ sum of the positive terms of $S(D) =$

$$\frac{1}{2} \sum_{j=1}^{n} \left\{ [f(a_j) - f(a_{j-1})] + \left| [f(a_j) - f(a_{j-1})] \right| \right\}.$$

Let $-N(D)$ be the sum of the negative terms of $S(D) =$

$$\frac{1}{2} \sum_{j=1}^{n} \left\{ [f(a_j) - f(a_{j-1})] - \left| [f(a_j) - f(a_{j-1})] \right| \right\}.$$

Then $P(D) + N(D) = \sum_{j=1}^{n} \left| f(a_j) - f(a_{j-1}) \right|$.

Let $p(a,b,f) = \overline{B}_D\, P(D)$. This is called the positive variation of f on (a,b).

Let $n(a,b,f) = \overline{B}_D\, N(D)$. This is the negative variation of f on (a,b).

Let $t(a,b,f) = \overline{B}_D \left\{ P(D) + N(D) \right\}$, which is the total variation, of f on (a,b).

<u>Definition 1</u>: If the total variation is finite, then f is said to be of bounded variation on (a,b). Such a function is not necessarily continuous.

<u>Theorem 1</u>: If $f(x)$ is of bounded variation on (a,b), then $f(x)$ may be written as the difference of two monotone increasing functions.

Theorem 2: The discontinuities of a function of bounded variation on a finite interval (a,b) are at most denumerably infinite.

Theorem 3: If $f(x)$ is of bounded variation on (a,b) so also is $|f(x)|$.

Theorem 4: If f_1, f_2 are of bounded variation, then $f_1 \pm f_2$, $f_1 f_2$ are also of bounded variation both on (a,b).

Theorem 5: Suppose f_1 and f_2 are of bounded variation and $G(x)$ is the greatest of and $g(x)$ is the smallest of $f_1(x)$, $f_2(x)$. Then $G(x)$ and $g(x)$ are also of bounded variation, where $G(x) = \frac{f_1(x) + f_2(x) + |f_1(x) - f_2(x)|}{2}$ and

$$g(x) = \frac{f_1(x) + f_2(x) - |f_1(x) - f_2(x)|}{2}.$$

CHAPTER IV

LEBESGUE INTEGRATION

In the present century Riemann's Integral has, for the purposes of theoretical investigations, been largely superseded by the more general formulation of Lebesgue. The theory of Lebesgue integration has as its foundation the conception of the measure of a set of points.

Consider the interval $\underline{I}$, $a < x < b$; or for $\underline{n}$ dimensions, $a_i < x_i < b_i$, $i = 1, 2, 3, \ldots, n$. The length of the interval, $L(I)$, is defined as $L(I) = b-a$. Let M denote a point set and I a sequence of intervals $\{I_n\}$, $n = 1,2,3,\ldots$. Let $\underline{I_n}$ be an open interval. The collection I covers M if each point of M belongs to at least one interval of the set. Then

$$L(I) = L(I_1) + L(I_2) + \text{-----} \quad \text{and } 0 < L(I) \leq +\infty$$

<u>Definition 1</u>: The exterior measure of M is $m_e(M) = \underline{B}\ L(I)$ for the coverings I of M.

<u>Theorem 1</u>: For an arbitrary set M, $m_e(M)$ exists, and $0 \leq m_e(M) \leq +\infty$.

<u>Theorem 2</u>: If M is contained in M_2, then $m_e(M_1) \leq m_e(M_2)$.

<u>Theorem 3</u>: If M, $a \leq x \leq b$, is a bounded interval, then $m_e(M)$ is equal to the length of this interval.

<u>Theorem 4</u>: If M is bounded, then $m_e(M) < +\infty$; if furthermore M contains an interval, then $m_e(M) > 0$.

<u>Theorem 5</u>: The interval in a covering I of M may be restricted, without loss of generality, to be arbitrarily small intervals with rational endpoints.

Consider two point sets M and N. M + N denotes the set of all points belonging to M or N; MN the set of all points belonging to both M and N. Two point sets are equal, M = N, if the totality of points in one is the same as the totality of points in the other. −M denotes the set of all points x not belonging to M; CM the complement of M. The symbol $\subset$ denotes "is contained in" and the symbol $\supset$ denotes "contains." If $N \subset M$, then M−N is the set of points belonging to M and not to N. Let S be the set of all points. Then −M = S−M. (−M)(−N) = −(M+N). N −MN = (−M)N.

<u>Theorem 6</u>: If M_1, M_2, is a finite or infinite sequence of point sets and $M = M_1 + M_2 + \ldots\ldots$, then $m_e(M) \leq m_e(M_1) + m_e(M_2) + \ldots\ldots.$

<u>Definition 2</u>: <u>Distance between Two Point Sets</u>. Suppose M and N are non-vacuous point sets. Then d(M,N) = d(N,M) = <u>B</u> d(p,q), for p a point in M and q a point in N.

<u>Theorem 7</u>: If M and N are non-vacuous and d(M,N) = $\delta > 0$ then $m_e(M+N) = m_e(M) + m_e(N)$.

<u>Definition 3</u>: If M is a given point set and $m_e(N) = m_e(MN) + m_e(N-MN)$ for every point set N, then M is said to be <u>measurable</u>, and its measure is $m(M) = m_e(M)$.

Theorem 8: If M_1 and M_2 are such that $M_1 \cdot M_2 = 0$ and if one set is measurable, $m_e(M_1+M_2) = m_e(M_1) + m_e(M_2)$.

Theorem 9: If M is measurable, $-M$ is measurable also.

Theorem 10: If M_1 and M_2 are measurable, then $M_1 + M_2$ is also.

Theorem 11: If M_1 and M_2 are measurable, then $M_1 M_2$ is also.

Theorem 12: If M_1 and M_2 are measurable, then $M_1 - M_1 \cdot M_2$ is also.

Theorem 13: If M is an interval, it is measurable.

Lemma 1: If $M_1 \subset M_2 \subset M_3 \subset$ ---- is an increasing sequence of measurable point sets and $M = M_1 + M_2 +$ -----, then $\lim_{n=\infty} m_e(M_n N) = m_e(MN)$, for N an arbitrary point set.

Lemma 2: If $M_1 \supset M_2 \supset M_3 \supset$ ---- is a decreasing sequence of measurable point sets with product M, then $\lim_{n=\infty} m_e(M_n N) = m_e(MN)$, where N is an arbitrary point set with finite external measure.

Theorem 14: If $M_1 \supset M_2 \supset M_3 \supset$ --- is a decreasing sequence of measurable point sets with product M, then M is measurable.

Theorem 15: If $M_1 \subset M_2 \subset M_3 \subset$ --- is an increasing sequence of measurable point sets with the sum M, then M is measurable.

Theorem 16: If M_1, M_2, M_3, --- is any finite or infinite sequence of measurable point sets with the sum M, then M is measurable.

Theorem 17: If M is an open point set, M is measurable.

Theorem 18: If M is a closed set, M is measurable.

Theorem 19: If M_1, M_2, M_3, --- is a finite or infinite sequence of measurable point sets, then their product is measurable.

Theorem 20: If $m_e(M) = 0$, then M is measurable and $m(M) = 0$.

Theorem 21: If either M or N is measurable and $m_e(MN)$ is finite, then $m_e(M+N) = m_e(M) + m_e(N) - m_e(MN)$.

Theorem 22: If M_1, M_2, M_3, --- is a finite or infinite sequence of measurable point sets and $M_n \cdot M_m = 0$, $m \neq n$, then $m_e \left[\sum_1^\infty M_i\right] = \sum_1^\infty m_e(M_i)$.

Theorem 23: If M is a bounded measurable point set, there exists a closed set N contained in M such that $m(N) > m(M) - \epsilon$ for a preassigned positive ϵ.

Definition 4: Interior Measure. Suppose M is a given point set and N a measurable set contained in M. The interior measure of M is $m_i(M) = \overline{B}\ m(N)$ for measurable $N \subset M$.

Theorem 24: For any M, $m_i(M) \leq m_e(M)$; if M is measurable, $m_i(M) = m_e(M)$.

Theorem 25: $m_i(M)$ is the least upper bound of the measure of all closed point sets contained in M.

Theorem 26: If M is a point set with finite exterior measure and $m_i(M) = m_e(M)$, then M is measurable.

<u>Theorem</u> <u>27</u>: If $MN = 0$, $m_i(M) + m_i(N) \leq m_i(M+N)$.

Theorem 28: If M_1, M_2, M_3, --- is any finite or infinite sequence of sets such that $M_{n_1}M_{n_2} = 0$ if $n_1 \neq n_2$, then $m_i(\sum_1^\infty M_k) \geq \sum_1^\infty m_i(M_k)$.

<u>Theorem</u> <u>29</u>: If $MN = 0$, then $m_i(M+N) \leq m_i(M) + m_e(N) < m_e(M+N)$ and $m_i(M+N) \leq m_e(M) + m_i(N) < m_e(M+N)$.

<u>Theorem</u> <u>30</u>: If <u>M</u> is a point set with finite exterior measure and <u>N</u> is any measurable set with finite measure which contains <u>M</u>, then $m_i(M) = m(N) - m_e(N-MN) = m(N) - m_e(N-M)$.

<u>Theorem</u> <u>31</u>: If M is a bounded point set contained in a finite interval <u>I</u>, $\alpha < x < \beta$ then <u>M</u> is measurable if and only if $m_e(M) + m_e(I-M) = m_e(I) = m(I) = \beta - \alpha$.

<u>Theorem</u> <u>32</u>: (Arzelà-Young) If M_1, M_2, M_3, --- is an infinite sequence of measurable point sets such that for every integer <u>n</u> $m(M_n) \geq \epsilon > 0$ and $m_e(\sum_1^\infty M_n) < +\infty$, and if <u>N</u> is a set of points contained in infinitely many of the sets M_n, then <u>N</u> is measurable and $m(N) \geq \epsilon$.

<u>Proof</u>: Let $\overline{M}_n = \sum_{k=n}^{\infty} M_k$; $N = \prod_1^\infty \overline{M}_n$. Then $\overline{M}_1 \supset \overline{M}_2 \supset \overline{M}_3 \supset$ ---

By theorems 16 and 19 each $\overline{M}_n$ and <u>N</u> are measurable. Since $\overline{M}_n \supset M_n$, by theorem 2 $m(\overline{M}_n) \geq m(M_n) \geq \epsilon$. By lemma 2
$m(N) = \lim_{n=\infty} m(\overline{M}_n) \geq \epsilon$.

We will suppose that $f(x)$, $g(x)$, $f_n(x)$ are single real valued functions for <u>x</u> on a measurable set <u>E</u> with finite measure. When several functions are used it will be assumed that they are defined on the same set <u>E</u>.

Let $E(f>c)$ be the set of points $\underline{x}$ of $\underline{E}$ for which $f(x) > c$. Similar definitions apply to $E(f \geqq c)$, $E(f < c)$, $E(f \leqq c)$, and $E(f=c)$.

Definition 1: Measurable Functions $f(x)$ is measurable if for an arbitrary constant $\underline{c}$ the point sets $E(f > c)$, $E(f \geqq c)$, $E(f < c)$, and $E(f \leqq c)$, are measurable. Any one of the four conditions for measurability implies the other three.

PROPERTIES OF MEASURABLE FUNCTIONS

Property 1: If $f(x)$ is measurable and $\underline{K}$ is a constant, $K \cdot f(x)$ and $K+f(x)$ are measurable.

Property 2: If $f(x)$ and $g(x)$ are measurable, then $E(f > g)$ is measurable for a particular $\underline{x}$ such that $f(x) > g(x)$, there exists a rational number r_x such that $f(x) > r_x > g(x)$.

Property 3: If $f(x)$ and $g(x)$ are measurable, $f(x) \pm g(x)$ is measurable $E(f \pm g > c) = E(f > c \mp g)$.

Property 4: If $f(x)$ and $g(x)$ are measurable, then $f(x) \cdot g(x)$ is measurable.

Property 5: If $\{ f_n(x) \}$ is a sequence of measurable functions on E, then the functions $\underline{F}(x) = \underline{\lim}_{n=\infty} f_n(x)$ and $\overline{F}(x) = \overline{\lim}_{n=\infty} f_n(x)$ are measurable.

Property 6: If $f(x)$ is continuous on an interval $\alpha \leqq x \leqq B$, then $f(x)$ is measurable.

Suppose $\underline{E}$ is a measurable point set of finite measure and f(x) is a single real-valued bounded function on $\underline{E}$. Then there exists an $\underline{M}$ such that $|f(x)| \leq M$, $x \subset E$. Let $\beta = \overline{B}_{x \subset E} f(x)$ and $\alpha = \underline{B}_{x \subset E} f(x)$. $\beta \leq M$, $\alpha \geq -M$. Let $D(y_i)$ be a partition of $\alpha\beta$, where $\alpha = y_0 \leq y_1 \leq \ldots\ldots \leq y_n = \beta$. Let ξ_i be an arbitrary value such that $y_{i-1} \leq \xi_i \leq y_i$. Form $\sum_1^n \xi_i\, m[E(y_{i-1} < f < y_i] = S(D)$.

Definition 2: f(x) is integrable (summable) on the point set E if $\lim_{N(D)=0} S(D)$ exists, and this limit is denoted $\int_E f(x)\, dx$. This is known as the Lebesgue Integral.

Let $\overline{S}(D) = \sum_1^n y_i m[E(y_{i-1} < f \leq y_i)]$ and $\underline{S}(D) = \sum_1^n y_{i-1}$ $m[E(y_{i-1} < f \leq y_i)]$. Then $\underline{S}(D) \leq S(D) \leq \overline{S}(D)$. Let $\overline{\int_E} f(x) = \lim_{N(D)=0} \overline{S}(D)$, if this limit exists. Let $\underline{\int_E} f(x) = \lim_{N(D)=0} \underline{S}(D)$, if this limit exists.

Definition 3: If f(x) = K on E, then for an arbitrary partition $\underline{D}$ $S(D) = K\, m(E)$. Hence $\int_E f(x) = K\, m(E)$. Let $I_e(x) = 1$ for $x \subset E$, $= 0$ for $x \not\subset E$. $I_e(x)$ is the characteristic function of the set E. Then $\int_E I_e(x) = m(E)$.

Theorem 1: If f(x) is a bounded measurable function on a set $\underline{E}$ of finite measure, then $\overline{\int_E} f(x)$ and $\underline{\int_E} f(x)$ exist and are equal.

Theorem 2: If f(x) is a bounded measurable function on a set $\underline{E}$ with finite measure, then f(x) is Lebesgue

) on $\underline{E}$.

<u>Theorem 3</u>: If $f(x)$ is a bounded measurable function on a set of finite measure and $\alpha = \underset{x \subset E}{\underline{B}} f(x)$, $\beta = \underset{x \subset E}{\overline{B}} f(x)$, then $\alpha m(E) \leqq \int_E f(x) \leqq \beta m(E)$.

<u>Theorem 4</u>: Suppose E has finite measure and $\{E_k\}$ is a finite or infinite sequence of measurable sets such that $E_n E_m = 0$, $n \neq m$, and $E = \sum_1^\infty E_k$. If $f(x)$ is a bounded measurable function on E, then $f(x)$ is summable on E, and $\int_E f(x) = \sum_1^\infty \int_{E_k} f(x)$.

<u>Theorem 5</u>: If $f(x)$ and $g(x)$ are bounded and measurable on E, $f(x) \leqq g(x)$ for x in E, then $\int_E f(x) \leqq \int_E g(x)$.

<u>Theorem 6</u>: If $f(x)$ and $g(x)$ are bounded and measurable on a set E of finite measure, then $\int_E f(x) + g(x) = \int_E f(x) + \int_E g(x)$.

<u>Theorem 7</u>: If K is a constant, then $\int_E Kf(x) = K \int_E f(x)$.

<u>Theorem 8</u>: If $f(x)$ is bounded and measurable on a set E of finite measure, then $|f(x)|$ is measurable on E and $\left|\int_E f(x)\right| \leqq \int_E |f(x)|$.

<u>Definition</u>: A proposition is said to hold almost everywhere on a set if it holds at every point except for a subset of Lebesgue measure zero.

<u>Theorem 9</u>: If $f(x)$ and $g(x)$ are bounded measurable functions on E and $f(x) = g(x)$ almost everywhere on E, then $\int_E f(x) = \int_E g(x)$.

<u>Theorem</u> <u>10</u>: If $f(x)$ is a bounded measurable function on $\underline{E}$, $f(x) \geqq 0$, and $\int_E f(x) = 0$, then $f(x) = 0$ almost everywhere on $\underline{E}$.

<u>Theorem</u> <u>11</u>: (Egoroff). If $\{f_n(x)\}$ is a sequence of finite valued functions which converge almost everywhere on $\underline{E}$ to a finite limit function, $f(x)$, then for $\epsilon > 0$ there exists a set $\overline{E}$ contained in $\underline{E}$ such that $m(\overline{E}) > m(E) - \epsilon$ and $\lim_{n=\infty} f_n(x) = f(x)$ uniformly in E.

<u>Proof</u>: Let E_o be a subset of $\underline{E}$ for which $\{f_n(x)\}$ tends to a finite limit function and $m(E_o) = m(E)$. For a given $\delta > 0$ let $E_{n\delta}$ be $E_o[\,|f_n(x) - f(x)| > \delta]$. For arbitrary $x \subset E_o$ and $\underline{n}$ sufficiently large $|f_n(x) - f(x)| < \delta$. Let $S_n = \sum_n^\infty E_{i\delta}$. Then $\prod_1^\infty S_n = 0$. But $S_n \supset S_{n+1} \supset \ldots\ldots$, and the S_i are measurable. Hence $\lim_{n=\infty} m(S_n) = m(\prod_1^\infty S_n)$ by lemma 2. For each $\delta > 0$ and $\eta > 0$ there is an integer $n(\delta, n)$ such that $m[S_n(\delta, n)] < \eta$. Let $\{\delta_k\} = \{\frac{1}{k}\}$ and $\{\eta_k\} = \{\frac{\epsilon}{2^k}\}$ for an arbitrary preassigned $\epsilon > 0$. Then $m[S_n(\frac{1}{k}, \frac{\epsilon}{2^k})] < \frac{\epsilon}{2^k}$. Let

$$S = \sum_1^\infty S_\eta \left(\frac{1}{k}, \frac{\epsilon}{2^k}\right).$$

Then $M(S) < \epsilon$ by theorem 16. Let $\overline{E} = E - S$. Then $m(\overline{E}) = m(E) - \epsilon$. For a given δ choose $\underline{k}$ so large that $\frac{\epsilon}{2^k} < \frac{\delta}{2}$. Then for $x \subset \overline{E} \subset E - \sum^k S_n$ there is an integer m_o such that $|f_n(x) - f(x)| < \frac{\delta}{2}$ and

$| f_{n+p}(x) - f(x) | < \frac{\delta}{2}$ for $n > m_o$. Hence $| f_{n+p}(x) - f_n(x) | < \delta$ for $x \subset \overline{E}$ and $n > m_o$.

<u>Theorem</u> <u>12</u>: (Lebesgue's Convergence Theorem). Suppose $\{ f_n(x) \}$ is a sequence of measurable functions on $\underline{E}$ and $| f_n(x) | \leq M$, and $\lim_{n=\infty} f_n(x) = f(x)$ almost everywhere on $\underline{E}$. Then $\lim_{n=\infty} \int_E f_n(x) = \int_E f(x)$.

<u>Proof</u>: Let E_o be $E[\lim_{n=\infty} f_n(x) = f(x)]$. Then $m(E-E_o) = 0$, or $m(E) = m(E_o)$. $f(x)$ is measurable on E_o by property 5, and hence measurable on $\underline{E}$. $| f(x) | \leq M$. Then $f(x)$ is summable on E_o and on E and $\int_{E_o} f(x) = \int_E f(x)$. By Egoroff's theorem, given $\epsilon > 0$ there is a set $\overline{E} \subset E_o$ such that $m(E_o) \geqq m(\overline{E}) > m(E_o) - \frac{\epsilon}{4M}$ and $\{ f_n(x) \}$ converges to $f(x)$ uniformly on $\overline{E}$. Choose n_o such that $| f_n(x) - f(x) | < \frac{\epsilon}{2m(E)}$ for $x \subset \overline{E}$ and $n \geqq n_o$. $\int_{E_o} | f(x) - f_n(x) | \leqq \int_{E_o} f_n(x) - f(x) = \int_{E_o - \overline{E}} | f_n(x) - f(x) | + \int_{\overline{E}} | f_n(x) - f(x) | \leqq 2M\frac{\epsilon}{4M} + m(\overline{E})\frac{\epsilon}{2m(E)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $\lim_{n=\infty} \int_{E_o} f_n(x) = \int_{E_o} f(x)$. Then $\lim_{n=\infty} \int_E f_n(x) = \lim_{n=\infty} \int_{E_o} f_n(x) + \int_{E-E_o} f_n(x) = \int_{E_o} f(x) = \int_E f(x)$.

CHAPTER V

A COMPARISON OF THE RIEMANN INTEGRAL AND THE LEBESGUE INTEGRAL

The definition of the definite integral as given by Riemann is very precise and leaves nothing to be desired in this respect. It is not only of interest from an historical point of view, but it still possesses great importance in Analysis. It will continue to be the basis upon which practical application of the Integral Calculus rests.

Several attempts were made to generalize the process of the Riemann integral, but Lebesgue first made progress in this matter. His theory of measure has in its turn, naturally led to further generalization.

The distinction between the Lebesgue integral and the Riemann integral rests essentially upon the difference between the two modes of dividing the domain of integration into sets of points.

The functions which are measurable in the sense of Lebesgue, and whose definition is closely related to that of measurable sets, form a very general class. This class includes, in particular, all the functions integrable in the Riemann sense.

The method of Lebesgue may be considered simpler than that of Riemann for it dispenses with the simultaneous introduction of the two extreme integrals, the lower and the upper.

Thus the Lebesgue method lends itself to an immediate extension to a certain class of unbounded functions, for instance, to all measurable functions of constant sign.

The Lebesgue integral renders it permissible to integrate term by term sequences and series of functions in certain general cases where passages to the limit under the integral sign were not allowed by the earlier methods of integration. The reason for this is found in the complete additivity of Lebesgue measure.

CHAPTER VI

THE RIEMANN-STIELTJES INTEGRAL

The notion of the integral of a bounded function f(x), defined in the linear interval (a,b), with respect to another function $\phi(x)$, defined in the same interval, is a generalization of the integral of a function f(x), with respect to the variable x. This notion was first introduced into Analysis by Stieltjes in connection with the theory of continued fractions.

Suppose f(x) and $\phi(x)$ are bounded functions, defined in the interval $(a \leq x \leq b)$. Consider a partition D of (a,b) into subintervals I_j. Let h_j be the length of I_j, let x_j be a point of the Interval I_j, and N(D) = greatest h_j. Let

$S(D) = \sum_D f(x_j)[\phi(x_j) - \phi(x_{j-1})]$.

If the functions f(x), $\phi(x)$ be such that the $\lim_{N(D)=0} S(D)$ exists, f(x) is said to have a <u>Stieltjes Integral</u> with respect to $\phi(x)$. When it exists, this limit is denoted by $\int_a^b f(x)d\phi(x)$. It will be spoken of as the Riemann-Stieltjes integral, or (RS) integral.

<u>Definition</u>: <u>Upper and lower integral</u>:

Let f(x) be any function bounded in $(a \leq x \leq b)$ and let $\phi(x)$ be a bounded monotone non-decreasing function, defined for the same interval. Consider a partition D of (a,b) as above. Let $M_j = \overline{B}_{x_{j-1}+0}^{x_j-0}(f(x))$, $m_j = \underline{B}_{x_{j-1}+0}^{x_j-0}(f(x))$ in the open interval (x_{j-1}, x_j).

$$\overline{S}(D) = \sum_D M_j\, [\phi(x_j-0) - \phi(x_{j-1}+0)] + \sum_D f(x_j)[\phi(x_j+0) - \phi(x_j-0)].$$

$$\underline{S}(D) = \sum_D m_j\, [\phi(x_j-0) - \phi(x_{j-1}+0)] + \sum_D f(x_j)[\phi(x_j+0) - \phi(x_j-0)].$$

If $\lim \overline{S}(D)$ exists, it is called the upper Riemann-Stieltjes Integral of f(x) with respect to $\phi(x)$, and is denoted by $\overline{\int_a^b} f(x)\, d\phi(x)$. A similar definition and notation are used for the lower Riemann-Stieltjes Integral.

Stieltjes established the existence of the integral for the case in which f(x) is continuous in (a,b) and $\phi(x)$ is a function of bounded variation in the same interval.

<u>Theorem 1</u>: If f(x), $\phi(x)$ be any two functions for which $\int_a^b f(x)\, d\phi(x)$ exists, then $\int_a^b \phi(x)\, df(x)$ exists and the two integrals satisfy the relation

$$\int_a^b f(x)\, d\phi(x) + \int_a^b \phi(x)\, df(x) = f(b)\,\phi(b) - f(a)\,\phi(a).$$

<u>Theorem 2</u>: If f(x) be bounded in (a,b), and $\phi(x)$ be of bounded variation in the same interval, the necessary and sufficient condition that f(x) should have a Riemann-Stieltjes integral with respect to $\phi(x)$ is that the variation of $\phi(x)$ over the set of points of discontinuity of f(x) should be zero.

<u>Theorem 3</u>: If $f_1(x)$, $f_2(x)$ both have Riemann-Stieltjes integrals with respect to the monotone function $\phi(x)$, then $f_1(x) + f_2(x)$ is integrable (RS), with respect to $\phi(x)$ and $\int_a^b \{f_1(x) + f_2(x)\}\, d\phi(x) = \int_a^b f_1(x)\, d\phi(x) + \int_a^b f_2(x)\, d\phi(x)$.

Another property of the Riemann-Stieltjes integral is that

$$\int_a^b f(x)\, d\phi(x) = \int_a^c f(x)\, d\phi(x) + \int_c^b f(x)\, d\phi(x).$$

CHAPTER VII

THE LEBESGUE-STIELTJES INTEGRAL

There is an analogy between additive functions of bounded variation of an interval and additive functions of sets. This analogy will be presented in this paper, by associating a function U^* of a set with each additive function U of bounded variation of an interval. We shall suppose that the functions of an interval are defined in the whole space of N dimensions.

A figure is defined as a set expressed as the sum of a finite number of intervals. U(I) is then defined as a function of an interval on a figure R [or, in an open set G] if U(I) is a finite real number uniquely defined for each interval I contained in R [or in G].

A function of an interval U(I) is said to be additive on a figure R_0 or in G, if $U(I_1 + I_2) = U(I_1) + U(I_2)$ whenever I_1, I_2 and $I_1 + I_2$ are intervals contained in R_0 [or in G] and I_1 and I_2 are not overlapping.

The upper variation of U on R_0 is the upper bound of U(R) for figures $R \subset R_0$. We shall denote it by $\overline{W}(U;R_0)$. The definition for the lower variation is similar. Lower variation is denoted by $\underline{W}(U;R_0)$. The number $\overline{W}(U;R_0) + |\underline{W}(U;R_0)|$, which is non-negative, will be called absolute variation of U on R_0 and we denote it by $W(U;R_o)$. If $W(U;R_o) < +\infty$ the function U is said to be of bounded variation on R_o.

Suppose given in the first place, a non-negative additive function U of an interval. For any set E we denote by $U^*(E)$ the lower bound of the sums $\sum_k U(I_k)$, where $\{I_k\}$ is an arbitrary sequence of intervals such that $E \subset \sum I_k^o$. For an arbitrary additive function U of bounded variation, with the upper and lower variations W_1 and W_2, we denote by W_1^* and $(-W_2)^*$ the functions of a set that correspond to the non-negative functions W_1 and $-W_2$. Then we write by definition, $U^* = W_1^* - (-W_2)^*$. The function U^* is thus defined for all sets and is finite for bounded sets.

When U is non-negative, U^* is an outer measure in the sense of Carathéodory. That is, it fulfills the three conditions for Carathéodory measure. The first two conditions are obvious but the third condition requires proof.

Let A and B be any two sets whose distance does not vanish, and let ϵ be a positive number. There is than a sequence $\{I_n\}$ of intervals such that $[A+B] \subset \sum_n I_n^o$ and $\sum_n U(I_n) \leq U^*(A+B) + \epsilon$. We may clearly suppose that the intervals of the sequences have diameters less than $\rho(A,B)$. We then have $U^*(A) + U^*(B) \leq U^*(A+B)$. Then $U^*(A) + U^*(B) = U^*(A+B)$ which establishes the third condition.

The function U^*, determined by a non-negative function of an interval, itself determines, since it is an outer Carathéodory measure, the class $\mathcal{L}_U$ of the sets measurable with respect to U^* and the process of integration (U^*). To simplify the notation, we shall omit the asterisk and write simply $\mathcal{L}_U$ for $\mathcal{L}_{U^*}$, integral (U) for integral (U^*), measure U of a set instead of measure (U^*), $\int_E fdU$ instead of $\int_E fdU^*$, and so on. This slight change of notation will not cause any confusion, since the measure U^* is determined uniquely by the function of the interval U.

When U is a general additive function of an interval, of bounded variation, we shall understand by $\mathcal{L}_U$ the common part of the classes $\mathcal{L}_{W_1}$ and $\mathcal{L}_{-W_2}$, where W_1 and W_2 denote respectively the upper and lower variations of U. A function f(x) of a point will be termed integrable (U) on a set E, if f(x) is integrable (W_1) and $(-W_2)$ simultaneously. By the integral (U) of f(x) we shall mean the number $\int_E fdW_1 - \int_E fd(-W_2)$. We shall write it $\int_E fdU$ as in the case of a non-negative function U. <u>This integration with respect to an additive function of bounded variation of an interval is called the Lebesgue-Stieltjes integration or simply LS-integration</u>. In the case of the integration over an interval $I = [a,b]$ in R_1 [straight line], we frequently write $\int_a^b fdU$ for $\int_I fdU$.

When the function U is continuous, every indefinite integral $\phi(U)$ vanishes, together with the function U^*, on the boundary of any figure. Consequently an indefinite integral with respect to a continuous function U of bounded variation of an interval is additive not only as function of a set ($\mathcal{L}_U$) but also as function of an interval.

Originally, these notions and the theorems that follow from them referred, not to additive functions of an interval, but to functions of a real variable. We can establish a correspondence between functions of a real variable and additive functions of a linear interval. This correspondence will render it immaterial which of the two kinds of functions is considered.

To do this, let f(x) be an arbitrary finite function of a real variable on the interval I_0. Let us term increment of f(x) over any interval I = (a,b) contained in I_0, the difference f(b) - f(a). Thus defined the increment is an additive function of a linear interval $I \subset I_0$, and corresponds in a unique manner to the function f(x). Conversely, if we are given any additive function F(I) of a linear interval I, this in itself defines, except for an additive constant, a finite function of a real variable f(x) whose increments on the interval I coincides with the corresponding values of the function F(I).

We shall understand by upper, lower, and absolute variations of a function of a real variable f(x) on an interval I, the upper, lower, and absolute variations of the increment of f(x) over I. We shall denote these numbers by the symbols $\overline{W}(f;I)$, $\underline{W}(f;I)$ and $W(f;I)$ respectively.

A finite function will be termed of bounded variation on an interval I_0 if its increment is a function of an interval of bounded variation on I_0. Similarly, the function is absolutely continuous or singular if its increment is absolutely continuous or singular.

Thus we can see that the difference between the definitions adapted for functions of an interval and for functions of real variables is only formal. Other definitions for the functions of a real variable can be set up by a proper modification of those for functions of an interval. In various cases it is more convenient to operate on functions of a real variable than on additive functions of an interval.

As is true in the case of other integrals, we can state the necessary and sufficient conditions for a function to be LS-integrable. Those conditions are (1) The function must be Borel measurable and bounded or (2) The function must be measurable and equally absolutely continuous.

It is necessary to consider step functions and jump functions in connection with the Lebesgue-Stieltjes integral.

We shall consider the integral $\int f d\alpha$ to be a LS-integral for α non-decreasing and continuous. Then we consider the integrals $\int d\sigma$ and $\int |\sigma_m - \sigma_n| \, d\alpha \to 0$. We shall suppose these functions are defined for all points of a measurable set E of Δ. Let the characteristic function of the set E be φ_E. Then $f(E) = \int \varphi_E \, d\alpha$. The theory can be built up in the Lebesgue sense by a systematic treatment analagous to that for the Lebesgue integral.

Now let α be a non-decreasing function. Then we may write $\alpha = \Gamma + \Theta$, where Γ is a continuous non-decreasing (step) function and Θ is a non-decreasing (jump) function.

<u>Definition 1</u>: The exterior measure $g_e^{\Gamma}(E) = \underset{E<\Sigma\Delta Y}{\underline{B}} \sum \Delta\Gamma$.

Then we can define convergence almost everywhere, convergence approximately, and convergence almost uniformly, each with respect to Γ.

For the stepfunctions we have:

$$\int \sigma \, d\Gamma = \sum_{1}^{1} \sigma_1 \Delta_1 \Gamma \text{ and } \sigma(\sigma_1, \sigma_2) = \int |\sigma_1 - \sigma_2| \, d\Gamma.$$

Hence we get completeness.

We shall next consider the non-decreasing jump function which satisfies the definition of a jump function. That is

$$\Theta(x) = \sum_{a<c\leq x} [\Theta(c) - \Theta(c-0)] + \sum_{a\leq c<x} [\Theta(c+0) - \Theta(c)].$$

Now for the function f to be Lebesgue integrable with respect to Θ the $\sum f(c)[\Theta(c+0) - \Theta(c-0)]$ must converge absolutely. Then by definition the integral $\int f \, d\Theta =$

$\sum f(c)[\Theta(c+0) - \Theta(c-0)]$.

Definition 2: A function f that is Lebesgue integrable with respect to α is Lebesgue integrable with respect to Γ and θ.

Then since the function f is Lebesgue integrable with respect to α the LS $\int f d\alpha = \int f d\Gamma + \int f d\theta$.

Definition 3: We shall define measure here to be: $g^{\theta}(E) = \int_E \phi_E \, d\alpha$ where ϕ is a Lebesgue integrable function with respect to α.

Definition 4: The exterior measure of θ is

$$g^{\theta}(E) = \sum_{c^E} [\theta(c+0) - \theta(c-0)].$$

Then $g_e^{\alpha}(E) = g_e^{\Gamma}(E) + g_e^{\theta}(E)$; also $g_e^{\alpha}(\sum E) \leq \sum g_e^{\alpha}(E)$.

We may now treat in a similar manner many of the theorems for the Lebesgue integral for the Lebesgue-Stieltjes integral. We have a theorem for the Lebesgue integral:- If $f^L \leq g^L$ then $\int f dx \leq \int g dx$. Written for the Lebesgue-Stieltjes integral it reads: Suppose α non-decreasing and f_1 and f_2 are Lebesgue integrable functions with respect to α, then $\int f_1 dx \leq \int f_2 dx$.

We shall consider $g^{\theta}(E) = g_e^{\theta}(E)$. Since $\sum [\theta(c+0) - \theta(c-0)]$ is absolutely convergent, additivity follows.

Definition 5: h(x) is absolutely continuous with respect to α if for every $\epsilon > 0$ there is a $\delta > 0$ such that $g^{\alpha}(E) < \delta$ it is true that $|h(E)| < \epsilon$.

Now we have the theorem for the Lebesgue integral: If f is a Lebesgue integrable function then the integral $\int_E f dx$ is absolutely continuous. Treated for the LS-integral it reads: If f is a Lebesgue integrable function with respect to α then the integral $\int_E f dx$ is α-absolutely continuous. We will include the proof. We may take $f \geqq 0$ and $f_n = f \wedge n$. Then there is an n such that $\int_{\ell} f d\alpha - \int f_n d\alpha < \frac{\epsilon}{2}$. Hence, by Levi's Theorem $\int_E (f-f_n) d\alpha < \frac{\epsilon}{2}$. Then for $g^{\alpha}(E)$ small enough $\int_E f_n d\alpha \leqq n$ if $g^{\alpha}(E) \leqq \frac{\epsilon}{2}$. This is true for special points.

<u>Theorem 1</u>: If α is non-decreasing, and the distance function is $\int |f_1 - f_2| d\alpha$, the Lebesgue space is complete with respect to α.

<u>Proof</u>: If α is continuous this is true at once. Suppose $\alpha = \Gamma + \theta$. Then we shall consider that if $\int |f_n - f_m| d\alpha \rightarrow 0$, then $\int |f_n - f_m| d\Gamma \rightarrow 0$ and $\int |f_n - f_m| d\theta \rightarrow 0$. Then there is a function f that is a Lebesgue integrable function with respect to Γ such that $\int |f_n - f| d\Gamma \rightarrow 0$. Also $\int |f_n - f_m| d\theta = \sum_c |f_n(c) - f_m(c)| \, [\theta(c+0) - \theta(c-0)]$. Hence $|f_n(c) - f_m(c)| \rightarrow 0$ for every jump. There is an $\overline{f}$ defined at the jumps c such that $f_n(c) \rightarrow \overline{f}(c)$. Now take

$$\sum_{c=c_1}^{c_p} |f(c) - f_n(c)| \, [\theta(c+0) - \theta(c-0)] \leqq \sum_{c_1}^{c_p} |f(c) - f_m(c)| \, [\theta(c+0) - \theta(c-0)] + \sum_c |f_m(c) - f_n(c)| \, [\theta(c+0) - \theta(c-0)] \leqq$$

$$\sum_{c_1}^{c_p} |f(c) - f_m(c)| \, [\theta(c+0) - \theta(c-0)] + \epsilon \text{ for } {m \atop n} > N_{\epsilon}.$$ Therefore

$\sum_{c_1}^{c\ p} |f(c)-f_n(c)| [\theta(c+0)-\theta(c-0)]$ for $n > M_\epsilon$ and p an arbitrary value. Therefore

$\int |f - f_n| \, d\theta \to 0$ and $\int |f - f_n| \, d\Gamma \to 0$. Then $\int |f - f_m| d\alpha \to 0$ and we get completeness.

<u>Theorem 2</u>: The set of all step functions is dense on L_α-space with distance $= \int |f_1 - f_2| \, d\alpha$.

<u>Proof</u>: Point steps are allowed. Then for every $\epsilon > 0$ there is a σ^{S_0} such that $\int |\sigma - f| \, d\Gamma < \frac{\epsilon}{2}$ and an M such that $|\sigma| \leq M$. Let $\sigma_1 = f$ at a sufficiently large number of jumps of θ so that for the remaining jumps

$\sum^* [\theta(c+0) - \theta(c-0)] < \frac{\epsilon}{4M}$ and $\sum^* |f(c)| [\theta(c+0) - \theta(c-0)] < \frac{\epsilon}{4}$.

Let $\sigma_1 = \sigma$ elsewhere. We still have $\int |\sigma_1 - f| \, d\Gamma < \frac{\epsilon}{2}$.

Then $\int |\sigma_1 - f| \, d\theta = \sum^* |\sigma_1(c)| [\theta(c+0) - \theta(c-0)] +$
$\sum^* |f(c)| [\theta(c+0) - \theta(c-0)] < \frac{\epsilon}{4} + \frac{\epsilon}{4}$.

We will treat one more theorem of Lebesgue for Lebesgue-Stieltjes. The theorem for Lebesgue is: Suppose that the Lebesgue integrable functions $f_n \to f$ approximately and $\int_E f_n dx$ are equally absolutely continuous, then the function f is a Lebesgue integrable function and $\int_E f_n dx \to \int_E f dx$ uniformly for E measurable. Treated for Lebesgue-Stieltjes it reads: Suppose the Lebesgue integrable functions with respect to α $f_n \to f$ approximately with respect to α and that $\int_E f_n dx$ is equally absolutely continuous with respect to α then the function f is a Lebesgue integrable function with respect to α and $\int_E f_n d\alpha \to \int_E f d\alpha$ uniformly for E α-measurable.

<u>Proof</u>: From equal absolutely continuity we have for every $\epsilon > 0$ there is a $\delta > 0$ such that the measure $g^\alpha(E) < \delta$

then $\int_E |f_n|\, d\alpha < \epsilon$. By approximate convergence we have an N such that $\frac{m}{n} > N$ it is true that $g^\alpha(E_{mn\epsilon})$, where $E_{mn\epsilon}$ is the set $E[|f_m - f_n| > \epsilon]$. At the points where α has jumps $f_n \to f$. Therefore for $\frac{m}{n} > N$ it is true that $\int_\Delta |f_m - f_n|\, d\alpha \leq$

$$\int_{E_{mn\epsilon}} \{|f_m| + |f_n|\} d\alpha + \int_{\Delta - E_{mn\epsilon}} |f_m - f_n|\, d\alpha \leq 2\epsilon + 2\epsilon[\alpha(b) - \alpha(a)].$$

Now there is a LS-integrable function $\bar{f}$ such that $\lim \int_\Delta |f_n - f|\, d\alpha = 0$. Hence $f_n \to \bar{f}$ approximately with respect to α. Therefore the f is a Lebesgue integrable function with respect to α.

As has already been pointed out there is a marked analogy between additive functions of bounded variation on an interval and additive functions of sets. All of the essential properties of the ordinary Lebesgue integral, except at most those implying the process of derivation hold for the Lebesgue-Stieltjes integral.

However, the Lebesgue-Stieltjes integral is not, in general, an additive function of an interval.

Since there is such a close analogy between the two integrals we can treat many of the theorems for Lebesgue integration for the Lebesgue-Stieltjes integral. We have treated some in this manner.

CHAPTER VIII

THE PERRON INTEGRAL and THE PERRON-STIELTJES INTEGRAL

Perron introduced a new definition of an integral based on major and minor functions. It does not require the theory of measure. In its original form this definition concerned only integration of bounded functions, but it has now been extended to unbounded functions.

Moreover, the Perron integral may be regarded as a synthesis of the two fundamental conceptions of integration. One of these corresponds to the idea of the definite integral as limit of certain approximating sums. The other one corresponds to the idea of the indefinite integral as a primitive function.

The notions of major and minor functions, and their applications to Lebesgue integration are discussed for arbitrary spaces. In defining the Perron integral, and the Perron-Stieltjes integral, we shall limit ourselves to functions of one real variable.

We shall suppose we are given a regular sequence $\mathfrak{N} = \{\mathfrak{N}_k\}$ of nets of intervals in a space R_m and a function of an interval F in R_m. $\mathcal{Q}$ denotes any interval containing x and belonging to one of the nets $\mathfrak{N}$.

<u>Definition 1</u>: We shall call <u>upper derivate</u> of F at a point x with respect to the sequence of nets $\mathfrak{N}$ the upper limit of the ratio $F(Q)/|Q|$ as $\delta(Q)\to 0$. By symmetry we define similarly the <u>lower derivate</u> of F at x with respect to the sequence of nets $\mathfrak{N}$. We shall denote these two derivates by $(\mathfrak{N})\overline{F}(x)$ and $(\mathfrak{N})\underline{F}(x)$. When they are equal at a point x, their common value will be denoted by $(\mathfrak{N})F'(x)$ and called the derivate of F at x with respect to the sequence of nets $\mathfrak{N}$.

<u>Definition 2</u>: A system of intervals will be called a <u>normal set</u> in the space R_m, when it consists of the closed intervals $[a_k^{(1)}, a_{k+1}^{(1)}; a_k^{(2)}, a_{k+1}^{(2)}; \text{---}; a_k^{(m)}, a_{k+1}^{(m)}]$ for $k = 0, \pm 1, \pm 2, \text{----}$, which are determined by systems of numbers $a_k^{(i)}$ subject to the condition $a_k^{(i)} < a_{k+1}^{(i)}$, for $i = 1, 2, \text{----}, m$ and $k = \text{----}, -1, 0, +1, \text{----}$, and $\lim_{k\to\pm\infty} a_k^{(i)} = \pm\infty$. A regular sequence of normal nets will be termed normal sequence.

<u>Definition 3</u>: An additive function of an interval F is termed <u>Major</u> [<u>Minor</u>] <u>function</u> of a function of a point f on a figure R_0 if, at every point x of this figure, $-\infty \neq \underline{F}_s(x) \geq f(x)[+\infty \neq \overline{F}_s(x) \leq f(x)]$. Then it follows that if the functions of an interval U and V are respectively a major and a minor function of a function on a figure R_0, their difference U-V, is monotone non-negative on R_0.

Theorem 1: If f is a summable function, then, for each $\epsilon > 0$, the function f has an absolutely continuous major function U, and an absolutely continuous minor function V such that, for each interval I, $0 \leq U(I) - \int_I f(x)\,dx \leq \epsilon$ and $0 \leq \int_I f(x)\,dx - V(I) \leq \epsilon$.

Let U be any major function of f on R_0 and let V be any minor function of f on R_0. A function of a real variable, f, is termed integrable in the sense of Perron on a figure R_0 in R_1, if f has both major and minor functions on R_0, and if the lower bound of the numbers $U(R_0)$, and the upper bound of the numbers $V(R_0)$ are equal. The common value of the two bounds is then called the definite Perron integral, of f on R_0, and denoted by $\mathcal{T}\int_{R_0} f(x)\,dx$. For a function f on a figure R_0 to be integrable it is necessary and sufficient that for each $\epsilon > 0$ there should exist a major function U and a minor function V of f on R_0 such that $U(R_0) - V(R_0) < \epsilon$.

Since the function U-V is monotone non-decreasing for every major function U and every minor function V of f, then every function which is $\mathcal{T}$-integrable on a figure R_0, is also on every figure $R \subset R_0$. The function of an integral $F(I) = \mathcal{T}\int_I f(x)\,dx$, thus defined for every interval $I \subset R_0$, is called an indefinite Perron integral of f on R_0. Thus F(I) is an additive function of an interval. A function of a real variable is termed indefinite $\mathcal{T}$-integral [major, minor function] of a function f, if this is the case for the function of an interval determined by it.

From theorem 1 we see that every function which is integrable in the sense of Lebesgue on a figure R_o, is so in the sense of Perron, and its definite Lebesgue and Perron integrals over R_o are equal.

Theorem 2: Every $\mathcal{T}$-integrable function is measurable, and is almost everywhere finite and equal and equal to the derivative of its indefinite integral.

Theorem 3: Every function f which is $\mathcal{T}$-integrable and almost everywhere non-negative on a figure R_o, is summable on this figure.

This theorem shows that although the Perron integration is more general than Lebesgue integration, the two processes are completely equivalent in the case of integration of functions of constant sign.

In considering the Perron-Stieltjes integral we shall restrict ourselves to finite functions. Suppose we are given two finite functions f and G. An additive function of an interval U will be termed major function of f with respect to G on an interval I_o, if to each point x these correspond to a number $\epsilon > 0$ such that $U(I) \geq f(x)G(I)$ for every interval I containing x and of length less than ϵ. The definition of minor function with respect to G is symmetrical. Now following the method for establishing the Perron integral with the help of the notions of major and minor functions with respect to G, we define Perron-Stieltjes integration, or $\mathcal{J}$S-integration with respect to any finite function G whatever. The $\mathcal{J}$S-integral of a function G on an interval $I_0=[a,b]$ will be denoted by $(\mathcal{J}S) \int_{I_0} f(x)\, dG(x)$, or by $(\mathcal{J}S) \int_a^b f(x)\, dG(x)$.

The criterion for $\mathcal{J}$S-integrability of a function is entirely similar to that $\mathcal{J}$-integrability.

If $G(x) = x$ for every point x $\mathcal{J}$S-integration with respect to G coincides with $\mathcal{J}$-integration. Thus the Perron-Stieltjes integral includes the ordinary Perron integral, at any rate as regards integration of finite functions. The Perron-Stieltjes integral includes also the Lebesgue-Stieltjes integral. But the definite Perron-Stieltjes and Lebesgue-Stieltjes integrals are not always equal, even for a function f integrable in both senses. This is due to the fact that the indefinite integral of Lebesgue-Stieltjes is not in general an additive function of an interval.

CHAPTER IX

THE DENJOY INTEGRALS

We may regard the Lebesgue integral as a special modification of the conception of the integral due to Newton. We define it as follows:

(L) A function of a real variable f is integrable if there exists a function F such that $F'(x) = f(x)$ at almost all points x, and F is absolutely continuous.

The function F (then uniquely determined apart from an additive constant) is the indefinite integral of the function f.

This is a descriptive definition of the Lebesgue integral; that is, it is based on differential properties of the indefinite integral and therefore, connected with the Newtonian notion of Primitive.

The definition (L) constitutes a modification of that of the integral of Newton, in two directions. We have a generalization which enables us to disregard sets of measure zero in the fundamental relation $F'(x) = f(x)$. There is an essential restriction which excludes all but the absolutely continuous functions from the domain of continuous primitive functions considered.

Although it is not possible to wholly remove the second modification from the definition (L) it is possible to replace it by much weaker conditions. The corresponding generalizations of the notion of absolute continuity give rise to extensions of the Lebesgue integral, known as the integrals $\mathcal{D}_*$ and $\mathcal{D}$ of Denjoy. We shall consider two generalizations of absolutely continuous functions: the functions which are generalized absolutely continuous in the restricted sense or ACG_*, and those which are generalized absolutely continuous in the wide sense or ACG. If, in the definition (L) we replace the condition of absolutely continuous functions by the conditions that the function F is ACG_*, or ACG respectively, we obtain the descriptive definitions of the integrals $\mathcal{D}_*$ and $\mathcal{D}$. The second definition requires a simultaneous generalization of the notion of derivative, to which is assigned the name of approximate derivative, which corresponds to approximate continuity.

Definition 1: A finite function F will be termed absolutely continuous in the wide sense on a set E, or absolutely continuous on E, or simply absolutely continuous (AC), if given any $\epsilon > 0$ there exists an $\eta > 0$ such that for every sequence of non-overlapping intervals $\{[a_k, b_k]\}$ whose end-points belong to E, the inequality $\sum_k (b_k - a_k) < \eta$ implies $\sum_k |F(b_k) - F(a_k)| < \epsilon$.

Definition 2: A function F will be termed generalized absolutely continuous function in the wide sense on E, or generalized absolutely continuous function on E, or finally ACG on E if F is continuous on E and if E is the sum of a finite or enumerable sequence of sets E_n on each of which F is AC.

<u>Definition 3</u>: A finite function F is said to be absolutely continuous in the restricted sense on a bounded set E, or to be AC_* on E, if F is bounded on an interval containing E and if to each $\epsilon>0$ there corresponds an $\eta>0$ such that, for every finite sequence of non-overlapping intervals $\{I_k\}$ whose end-points belong to E, the inequality $\sum_k |I_k|<\eta$ implies $\sum_k O(F;I_k)<\epsilon$.

<u>Definition 4</u>: A function will be termed generalized absolutely continuous on a set E, or ACG_* on E, if the function is continuous on E and if the set E is expressible as the sum of a sequence of bounded sets on each of which the function is AC_*.

The essential ideas for the Denjoy integrals have already been sketched. We will now complete them. A function of a real variable f will be termed $\mathcal{D}$-integrable on an interval I = [a,b] if there exists a function F which is ACG on I and which has f for its approximate derivative almost everywhere. The function F is then called indefinite $\mathcal{D}$-integral of f on I. Its increment F(I) = F(b)-F(a) over the interval I is termed definite $\mathcal{D}$-integral of f over I and is denoted by $\mathcal{D}\int_I f(x)\,dx$ or $\mathcal{D}\int_a^b f(x)\,dx$.

Similarly, a function f will be termed $\mathcal{D}$-integrable on an interval I = [a,b], if there exists a function F which is ACG_* on I and which has f for its ordinary derivative almost everywhere. The function F is then called indefinite $\mathcal{D}_*$-integral of f on I; the difference $F(I) = F(b)-F(a)$ is termed definite $\mathcal{D}_*$-integral of f over I and denoted by $(\mathcal{D}_*)\int_I f(x)\,dx$ or by $(\mathcal{D}_*)\int_b^a f(x)\,dx$.

The integrals $\mathcal{D}$ and $\mathcal{D}_*$, are often given the names of Denjoy integrals in the wide sense, and in the restricted sense respectively. The first of these is also termed Denjoy-Khintchine integral, and the second, Denjoy-Perron integral, for the latter is equivalent to the Perron integral as we have defined them.

The fundamental relations between the Denjoy and Lebesgue processes are given in the following:

<u>Theorem 1°</u>: A function f which is $\mathcal{D}$-integrable on an interval I is necessarily also $\mathcal{D}_*$-integrable on I and we have $(\mathcal{D})\int_I f\,dx = (\mathcal{D}_*)\int_I f\,dx$.

<u>2°</u>: A function f which is Lebesgue integrable on an interval I is necessarily $\mathcal{D}_*$-integrable on I and we have $(\mathcal{D}_*)\int_I f\,dx = \int_I f\,dx$.

<u>3°</u>: A function which is $\mathcal{D}$-integrable and almost everywhere non-negative on an interval I is necessarily Lebesgue integrable on I.

Thus we see for functions of constant sign the Denjoy processes are equivalent to those of Lebesgue.

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--------Notes taken from Lectures on Modern Theories of Integration, at the University of Chicago, 1938.