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Axioms for geometry and analysis

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AXIOMS FOR GEOMETRY AND ANALYSIS

by

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IN APPRECIATION

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INTRODUCTION

"Every demonstrative science," says Aristotle, "must start from indemonstrable principles."[1] In mathematics these "indemonstrables" are called axioms, postulates, or assumptions. Aristotle adds, "otherwise, the steps of demonstration would be endless." The body of propositions representing a science constitutes a closed unit, and any effort to prove every proposition would result in a "vicious circle". Any of the propositions in a mathematical science can serve as the foundation so long as the rest of the propositions can be deduced from them. For the beginner in any logical science it is necessary to start with notions which he already understands or can easily acquire. This was Euclid's policy in his "Elements." This was also Hilbert's aim in his "Grundlagen der Geometrie." On the other hand Veblen's "Axioms for Geometry" assumes a tutored student with a developed skill in logical deduction.

Many sets of axioms have been worked out for geometry and analysis. Only a few are listed here, and those with the primary purpose of establishing the foundations of the mathematical sciences and the secondary purpose of displaying the variability of choice of foundations.

1. T. L. Heath, The Thirteen Books of Euclid (3 vols., London, 1908), I, p. 145.

Pure mathematics is sometimes classified into three branches; algebra, geometry, and analysis. For the purpose of this paper algebra and analysis are synonymous. There is no change in notation involved in passing from algebra to analysis, the introduction of the theory of limits being the chief distinction. We will therefore treat pure mathematics as only two sciences, geometry and analysis. To the student of analysis even this distinction fades.

GEOMETRY

History[1]

Like our number system, geometry had its origin before the dawn of recorded history. The Rhind Papyrus of the sixteenth century B. C. contains formulas for the areas of the rectangle, triangle, trapezoid, and circle. Egyptian progress in geometry was due to a need for it in surveying and architecture. Thales, a Greek, is reported to have learned Egyptian geometry and taken it to Greece. To Thales (about 600 B.C.), likewise, geometry was a practical science. It enabled him to measure the distance of a ship from shore. Pythagoras (about 540 B. C.) and his followers added much to the known science of geometry. They stated and proved many theorems, the most famous of which was the Pythagorean theorem. Hippocrates in his efforts to "square the circle" stated and proved many theorems pertaining to the circle. Plato about 400 B.C.), is credited with putting geometry on a sound logical basis. Archytas (about 350 B.C.), in his efforts to duplicate the cube, developed and proved several theorems pertaining mostly to mean proportionals.

Euclid (about 300 B.C.) was the master mind who assembled all the known theorems of geometry, added some, and using the logic of Plato, constructed the science of geometry. That his work was good is evidenced

1. This historical sketch follows in a general way D. E. Smith, History of Mathematics (2 vols., Boston, 1925), II, Chap. V.

by the fact that his book has been in use with very little change for 2200 years. He selected a few of the propositions to be used as fundamental statements without proof , and upon these built the whole science of geometry. Controversy centered on his fifth axiom[1] from the time of Euclid until almost the present. Critics were unanimously of the opinion that the fifth axiom could be proved a consequence of the other axioms. Modern mathematicians have further established the excellence of Euclid's work by showing that complete and consistent sciences of geometry can be constructed assuming a different fifth axioms.

No important additions were made to Euclid's geometry until in the seventeenth century Fermat and Descartes invented the analytic geometry. Analytic geometry, and later the application of the calculus to geometry, opened up large fields and added much to the science of geometry. Finally, in the nineteenth and twentieth centuries, mathematicians turned again to Euclid's method and established various logical foundations for the science of geometry.

Few substantial improvements were made in Euclid's axioms. The essential difference being that modern geometers chose to show that there is no one foundation for geometry.

1. See page 6.

EUCLID'S AXIOMS (1)

Euclid assumes the existence of various geometrical figures. He starts by defining them, probably intending to show with what his geometry shall deal. He has twenty-three such definitions. He then lists five postulates. These are his starting hypotheses for geometry. They are followed by five axioms which he considered obvious truths, true in any science. Modern philosophers prefer to consider nothing obviously ~~true~~ in any science. Axioms, like postulates, now serve only as starting hypotheses for a science. Euclid may then be said to have ten axioms as a foundation for his "Elements" and his fifth postulate is customarily called his fifth axiom.

Definitions

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilineal.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.

1. T. L. Heath, The Thirteen Books of Euclid, (3 vols., London, 1908), I, p. 153.

14. A figure is that which is contained by any boundary or boundaries.

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

16. And the point is called the centre of the circle.

17. A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semi-circle is the same as that of the circle.

19. Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, and isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same sides less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Axioms

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

HILBERT'S AXIOMS [1]

Hilbert leaves point, straight line, plane, between and congruent undefined. He relates them into a geometry by means of axioms. The axioms he lists in five groups which he proves to be mutually independent. Group One he calls the <u>axioms of combination</u>. Here he asserts the existence of points, lines, planes, and solids. Group Two he calls the <u>axioms of order</u>. Here he implies that the points of a straight line form a linearly ordered dense set. Group Three is <u>Euclid's parallel axiom</u> stated somewhat differently. Group Four he calls the <u>axioms of</u> congruence. They serve to establish the congruence of linear segments. Group Five he calls his <u>axiom of continuity</u>. Here he lists the Archimedian axiom and the axiom of completeness. Euclid stated the Archimedian axiom: "Two magnitudes are said to have a ratio, if they are such that a multiple of either may exceed the other." The axiom of completeness restricts the validity of the other axioms to a system made up only of points, straight lines and planes.

Hilbert has twenty-one axioms in all. He proves that they do not contain contradictions by submitting a geometry, known to be valid, that satisfies all of them. He proves that each group of axioms is independent of the others by submitting a geometry that fails to satisfy only that group.

1. David Hilbert, "<u>Grundlagen der Geometrie</u>," third edition, 1909. Quoted from the English translation by E. J. Townsend, <u>The Foundation of Geometry</u>, Chicago, 1910.

The Axioms

Group I

I, 1. Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.

Instead of "determine", we may also employ other forms of expression; for example, we may say A "lies upon" a, A, "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b have the point A in common," etc.

I, 2. Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.

I, 3. Three points A, B, C not situated in the same straight line always completely determine a plane, α. We write $ABC = \alpha$.

We employ also the expressions: A, B, C, "lie in" α; A, B, C, "are points of" α, etc.

I, 4. Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.

I, 5. If two points A, B of a straight line a lie in a plane α, then every point of a lies in α.

In this case we say: "The straight line a lies in the plane α," etc.

I, 6. If two planes α, β have a point A in common, then they have at least a second point B in common.

I, 7. Upon every straight line there exists at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.

Group II

The axioms of this group define the idea expressed by the word "between," and make possible, upon the basis of this idea, an order of sequence of the points upon a straight line, in a plane, and in space. The points of a straight line have a certain relation to one another which the word "between" serves to describe. The axioms of this group are as follows:

II, 1. If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.

II, 2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.

II, 3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

II, 4.(1) Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.

1. This axiom was proved by E. H. Moore to be a consequence of previously stated axioms. (Transactions of the American Mathematical Society, vol. III, 1902).

DEFINITION: We will call the system of two points A and B, lying upon a straight line, a segment and denote it by AB or BA. The points lying between A and B are called the points of the segment AB or the points lying within the segment AB. All other points of the straight line are referred to as the points lying outside the segment AB. The points A and B are called the extremities of the segment AB.

II, 5. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

Group III

The introduction of this axiom simplifies greatly the fundamental principles of geometry and facilitates in no small degree its development. This axiom may be expressed as follows:

III. In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a.

This straight line is called the parallel to a through the given point A.

Group IV

The axioms of this group define the idea of congruence or displacement.

Segments stand in a certain relation to one another which is described by the word "congruent."

IV, 1. If A, B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment A'B'. We indicate this relation by writing

$$AB \equiv A'B'.$$

Every segment is congruent to itself; that is, we always have

$$AB \equiv AB$$

We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in one and only one way.

IV, 2. If a segment AB is congruent to the segment A'B' and also to the segment A"B", then the segment A'B' is congruent to the segment A"B"; that is, if $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

IV, 3. Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and furthermore, let A'B' and B'C' be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$, we have $AC \equiv A'C'$.

DEFINITIONS: Let α be any arbitrary plane and h, k any two distinct half-rays lying in α and emanating from the point O so as to form a part of two different straight lines. We call the system formed by these two half-rays in h, k an angle and represent it by the symbol $\angle(h, k)$ or $\angle(k, h)$. From axioms II, 1-5, it follows readily that the half-rays h and k, taken together with the point O, divide the remaining points of the plane α into two regions having the following property: If A is a point of one region and B a point of the other, then every broken line joining A and B either passes through O or has a point in common with one of the half-rays h, k. If, however, A, A' both lie within the same region, then it is always possible to join these two points by a broken line which neither passes through O nor has a point in common with either of the half-rays h, k. One of these two regions is distinguished from the other in that the segment joining any two points of this region lies entirely within the region. The region so characterized is called the interior of the angle (h, k). To distinguish the other region from this, we call it the exterior of the angle (h, k). The half rays h and k are called the sides of the angle, and the point O is called the vertex of the angle.

IV, 4. Let an angle (h, k) be given in the plane α and let a straight line a' be given in a plane α'. Suppose also that, in the plane α', a definite side of the straight line a' be assigned. Denote by h' a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a'. We express this relation by means of the notation

$$\angle(h, k) \equiv \angle(h', k').$$

Every angle is congruent to itself; that is,

$$\angle(h, k) \equiv \angle(h, k).$$

or

$$\angle(h, k) \equiv \angle(k, h).$$

We say, briefly, that every angle in a given plane can be laid off upon a given side of a given half-ray in one and only one way.

IV, 5. If the angle (h, k) is congruent to the angle (h', k') and to the angle (h'', k''); that is to say, if $\angle(h, k) \equiv \angle(h', k')$ and $\angle(h, k) \equiv \angle(h'', k'')$, then $\angle(h', k') \equiv \angle(h'', k'')$.

Suppose we have given a triangle ABC. Denote by h, k the two half-rays emanating from A and passing respectively through B and C. The angle (h, k) is then said to be the angle included by the sides AB and AC, or the one opposite to the side BC in the triangle ABC. It contains all of the interior points of the triangle ABC and is represented by the symbol $\angle BAC$, or by $\angle A$.

IV, 6. If, in the two triangles ABC and $A'B'C'$, the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$ hold, then the congruences $\angle ABC \equiv \angle A'B'C'$ and $\angle ACB \equiv \angle A'C'B'$ also hold.

Group V

(1)This axiom makes possible the introduction into geometry of the idea of continuity. In order to state this axiom, we must first establish a convention concerning the equality of two segments. For this purpose, we can either base our idea of equality upon the axioms relating to the congruence of segments and define as "equal" the correspondingly congruent segments, or upon the basis of groups I and II, we may determine how, by suitable constructions, (a segment is to be laid off from a point of a given straight line so that a new, definite segment is obtained "equal" to it. In conformity with such a convention, the axiom of Archimedes may be stated as follows:

V, 1. Let A_1 be any point upon a straight line between the arbitrarily chosen points $\underline{A}$ and $\underline{B}$. Take the points A_2, A_3, A_4, ... so that $\underline{A_1}$ lies between A and $\underline{A_2}$, $\underline{A_2}$ between $\underline{A_1}$ and $\overline{A_3}$, $\underline{A_3}$ between $\underline{A_2}$ and $\underline{A_4}$, etc. Moreover, let the segments $\overline{AA_1}$, $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, ... be equal to one another. Then among this series of points, there always exists a certain point $\underline{A_n}$ such that $\underline{B}$ lies between $\underline{A}$ and $\underline{A_n}$.

V, 2. To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

1. Axiom V, 1 introduces a weak type of continuity. A line must also satisfy the Dedikind Cut to be continuous. See page 26.

PIERI'S AXIOMS (1)

Pieri leaves only point and motion undefined. He groups points into a set S and postulates the results of various motions. Having defined the straight line in terms of motion and points he defines the plane as the set of all lines joining sides and points of the triangle formed by three non-collinear lines. He defines the sphere as the class of all points P which can be transformed into a point B by all the motions which leave A fixed. A is defined as the center of the sphere. From this definition the circle, midpoint of a straight line and distance can be defined. Axiom eleven defines perpendicular and asserts the uniqueness of a perpendicular from a point to a line. The first thirteen axioms define betweenness and line segments. Axiom sixteen is the triangle-transversal axiom. Axiom seventeen is a restatement of the Archimedean axiom.

The Axioms

1. The class S contains at least two distinct points.
2. Given any motion μ which establishes a correspondence between every point P and a point P', there exists another motion μ^{-1}, which makes every point P' correspond to P. The motion μ^{-1} is called the inverse of μ.
3. The resultant of two motions μ and ν performed successively is equivalent to a single motion.
4. Given any two distinct points A and B, there exists an effective motion which leaves A and B fixed.
5. If there exists an effective motion which leaves fixed three points A, B, C, then every motion which leaves A and B fixed leaves C fixed.

1. Mario Pieri, Della Geometria elementare come sistema ipotetico-deduttivo; monografia del punto e del moto, Memorie della R. Academia delle Scienze di Torino, (1899). The axioms are from J. W. Young, Fundamental Concepts of Algebra and Geometry, (New York, 1934), p. 155-163.

6. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are three non-collinear points and $\underline{D}$ is a point of the line $\underline{BC}$ distinct from $\underline{B}$, the plane $\underline{ABD}$ is contained in the plane $\underline{ABC}$.

7. If $\underline{A}$ and $\underline{B}$ are two distinct points, there exists a motion which leaves $\underline{A}$ fixed and transforms $\underline{B}$ into another point of the straight line $\underline{AB}$.

8. If $\underline{A}$ and $\underline{B}$ are distinct points, and if two motions exist which leaves $\underline{A}$ fixed and transform $\underline{B}$ into another point of the line $\underline{AB}$, the latter point is the same for both motions.

9. If $\underline{A}$, and $\underline{B}$ are two distinct points, there exists a motion which transforms $\underline{A}$ into $\underline{B}$ and which leaves fixed a point of the line $\underline{AB}$.

10. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are three non-collinear points, there exists a motion which leaves $\underline{A}$ and $\underline{B}$ fixed and which transforms $\underline{C}$ into another point of the plane $\underline{ABC}$.

11. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are three non-collinear points and $\underline{D}$ and $\underline{E}$ are points of the plane $\underline{ABC}$ common to the sphere $\underline{C_A}$ and $\underline{C_B}$, and distinct from $\underline{C}$, the two points $\underline{D}$ and $\underline{E}$ coincide.

12. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are non-collinear points, there exists at least one point not in the plane $\underline{ABC}$.

13. If $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are four points not in the same plane, there exists a motion which leaves $\underline{A}$ and $\underline{B}$ fixed and which transforms $\underline{D}$ into a point of the plane $\underline{ABC}$.

14. If $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are four distinct collinear points, the point $\underline{D}$ isnot a point of one and only one of the intervals $\underline{AB}$, $\underline{AC}$, $\underline{BC}$.

15. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are three collinear points, and $\underline{C}$ is between $\underline{A}$ and $\underline{B}$, no point can be between $\underline{A}$ and $\underline{C}$ and between $\underline{B}$ and $\underline{C}$ at the same time.

16. If $\underline{A}$, $\underline{B}$, $\underline{C}$ are three non-collinear points, every straight line of the plane $\underline{ABC}$ which has a point in common with the interval $\underline{AB}$ must also have a point in common with the interval $\underline{AC}$ or the interval $\underline{BC}$, provided the straight line does not pass through any of the points $\underline{A}$, $\underline{B}$, $\underline{C}$.

17. If $\underline{C}$ is any class of points contained in the interval $\underline{AB}$, there exists in this interval a point $\underline{X}$ such that no point of $\underline{C}$ is between $\underline{X}$ and $\underline{B}$, and such that for every point $\underline{Y}$ between $\underline{A}$ and $\underline{X}$, there is a point of $\underline{C}$ between $\underline{Y}$ and $\underline{X}$ or coincident with $\underline{X}$.

VEBLEN'S AXIOMS (1)

Veblen has twelve axioms based on the undefined terms point and order. The definitions given along with the axioms serve to show the manner in which the various concepts are introduced. Veblen has fewer axioms than Hilbert but the geometry is correspondingly more difficult to derive.

The Axioms

(1). There exist at least two distinct points.

(2). If points A, B, C are in the order A, B, C, they are in the order C, B, A.

(3). If points A, B, C, are in the order ABC, they are not in the order BCA.

(4). If points A, B, C are in the order ABC, then A is distinct from C.

(5). If A and B are any two distinct points, there exists a point C, such that A, B, C, are in the order ABC.

Def. 1. The line AB consists of A and B and all points X in one of the possible orders ABX, AXB, XAB. The points X in the order AXB constitute the segment AB. A and B are the end points of the segment.

(6). If C and D $(C \neq D)$ lie on the line AB, then A lies on the line CD.

(7) If there exist three distinct points, there exist three points A, B, C, not in the orders ABC, BCA, or CAB.

Def. 2. Three distinct points not lying on the same line are the vertices of a triangle ABC, whose sides are the segments AB, BC, CA, and whose boundery consists of its vertices and the points of its sides.

(8). If three distinct points A, B, C, do not lie on the same line, and D and E are two points in the orders BCD and CEA, then a point F exists in the order AFB and such that D, E, F, lie on the same line.

Def. 5. A point O is in the interior of a triangle if it lies on a segment, the end points of which are points of different sides of the triangle. The set of such points O is the interior of the triangle.

Def. 6. If A, B, C form a triangle, the plane ABC consists of all points collinear with any two points of the sides of the triangle.

1. Oswold Veblen, Axioms for Geometry, in Transactions of the American Mathematical Society, Vol. 5 (1904) p. 346.

(9). If there exist three points not lying in the same line, there exists a plane ABC such that there is a point D not lying in the plane. ABC.

Def. 7. If A, B, C, and D are four points not lying in the same plane, they form a tetrahedron ABCD whose faces are the interiors of the triangles ABC, BCD, CDA, DAB (if the triangle exist) whose vertices are the four points A, B, C, and D and whose edges are the segments AB, BC, CD, DA, AC, BD. The points of faces, edges, and vertices constitute the surface of the tetrahedron.

Def. 8. If A, B, C, and D are the vertices of a tetrahedron, the space ABCD consists of all points collinear with any two points of the faces of the tetrahedron.

(10). If there exist four points neither lying in the same line nor lying in the same plane, there exists a space ABCD such that there is no point E not collinear with two points of the space, ABCD.

(11). If there exists an infinitude of points, there exists a certain pair of points A, C such that if $[\sigma]$ is any infinite set of segments of the line AC, having the property that each point which is A, C or a point of the segment AC is a point of a segment σ, then there is a finite subset $\sigma_1, \sigma_2, \ldots\ldots\ldots\ldots\ldots \sigma_n$ with the same property.

(12). If a is any line of any plane α there is some point C of α through which there is not more than one line of the plane α which does not intersect a.

Comparison

Euclid defines a point (Def. 1) but builds his geometry without postulating its existence or any quality it might possess. Modern geometers prefer to leave the notion of point undefined but postulate its existence.

Euclid defines a straight line (Def. 4) and then postulates its existence. (Ax. 1). Hilbert considers it a fundamental notion, but postulates its uniqueness. Pieri postulates the existence of a planary motion which leaves two points fixed (Ax. 4) and defines a line as the set of all points that remain fixed in such a motion. Veblen postulates the existence of an ordered set of points (Ax. 5) and defines a line as such a set.

A plane surface is defined by Euclid but considered fundamental enough to not warrant postulating. Hilbert, assuming the existence of points, postulates the uniqueness of the plane determined by three non-collinear points (Ax. I, 3). Pieri and Veblen establish the existence of non-collinear points by axioms and define the plane in terms of three such points (After Ax. 6 and Ax. 8 respectively).

Angles are defined by Euclid (Def. 8) and the equality of all right angles is postulated (Ax. 4). He further establishes the congruence of identical figures in axiom 9. Hilbert defines angle as the system of two half-lines emanating from one point and postulates their congruence (Ax. IV, 4). Pieri and Veblen define angles and their congruence in terms of point relations that have been established by axioms.

Euclid in his postulate 3 established the continuity of space by saying "any center and distance". This is his nearest approach to the continuity of a line. Hilbert in V, 1 and Pieri in 17 present a type of continuity. Veblen in 11 states his axiom of continuity in the form usually known as the Heine-Borel proposition.

Euclid's postulate 5, the parallel axiom, has had an interesting history.(1) Euclid was apparently apprehensive of it for he avoided using it until the 29th theorem when it could no longer be avoided. Several Greek commentators attacked the propriety of using it as an axiom and tried to deduce it from the other postulates and axioms. In the eighteenth century an Italian, Saccheri, attacked its independence by assuming the axiom false and developing a geometry that would contradict itself somewhere. He never succeeded in showing a contradiction but thought he did. In the nineteenth century Bolyai and Lobatchevsky working independently made the assumption that there is an infinite number of lines through a point parallel to a line. On this hypothesis they built a complete logical science of geometry, of which Euclidean geometry was a limiting case. This established the independence of Euclid's 5th axiom. Later Reimann built a geometry on the assumption that there exists no line through a point not on a given line parallel to the given line. All these geometries satisfy our perception of space as nearly as we are able to observe. Therefore, there is no question as to which is true. But Euclidean geometry admits of easier development

1. This paragraph follows in a general way J. W. Young, Fundamental Concepts of Algebra and Geometry, (New York, 1934), chap III.

so all modern geometers have included Euclid's 5th postulate in some form in their axiomsfor geometry.

Euclid and Hilbert postulate the congruence of figures regardless of their continuity. Pieri and Veblen postulate the existence of coincident points and define congruence in terms of order among these points.

Euclid proves in his geometry that the diagonals of a parallelogram bisect each other, tacidly assuming that they meet. Hilbert in II, 5, Pieri in 16, and Veblen in 8 present the so-called triangle-tranversal axiom from which Euclids assumption can be proved.

Euclid has only ten axioms, but he assumes some things, sub rosa, which are now preferably stated explicitly. Hilbert has twenty-one axioms not all of which are entirely independent. Pieri has seventeen axioms, probably independent of each other. Veblen has only twelve axioms, and they are mutually independent. His approach seems the most logical from the standpoint of primitiveness of concept. Add a few axioms to simplify some of the proofs, and Veblen's set would afford the best method for building upon known concepts. Compare for example Hilbert's undefined terms point, line, plane, between, and congruent; Pieri's undefined terms point and motion and Veblen's undefined terms of point and order. Obviously Veblen has selected the simplest fundamental notions upon which to base his axioms.

ANALYSIS

Historical

Analysis is purely an arithmetic method or operation, finding its justification in the science of numbers. However, the first approach to the method of analysis was made in the field of geometry long before numbers, as then understood, could handle the method. The Greeks invented the method of exhaustion in the fifth century B. C. As an example, they found the area of a circle by inscribing a polygon then enlarging the inscribed figure by successive doubling of the number of sides until a limit had been sufficiently approached. In principle they set up an infinite converging bounded sequence and assumed its sum had a definite limit. It was consistent in the Greeks to assume that such a series had a limit because they naturally believed that space was continuous. They never adequately explained how one was to completely exhaust an area with a variable sum that approached that area as a limit but never quite reached that limit. Archimedes in 225 B. C. proved rigorously by the method of exhaustion, that the area of a parabolic segment is four thirds of the triangle with the same base and vertex or two-thirds of the circumscribed parallelogram. In each case he proved that the area could be neither more nor less than the area which that formula gives. Therefore, the area given by that formula is the true area.

In the seventeenth century Kepler and Cavalieri made the next approach toward the method of analysis. Their theory was that space and lines were made up of "indivisibles". A surface, for instance, is made up of lines (the indivisibles). An infinite number of these lines are summed to obtain the area of the surface. Cavalieri showed that the area of a triangle was one-half the area of a parallelogram with the same base and altitude as follows:[1] Calling the smallest indivisible element of the triangle 1, the next larger 2, the next 3, and so on to n the base. The area of the triangle is therefore $1+2+3 \ldots\ldots\ldots + n$, or $\frac{1}{2}n(n+1)$. But each element of the parallelogram is n, and there are n of them as in the triangle, and so the area is n^2. Then the ratio of the area of the triangle to the area of the parallelogram is $\frac{1}{2}n(n+1) : n^2 = \frac{1}{2}(1+1/n)$. But $\frac{1}{2}(1+1/n) = \frac{1}{2}$ as $n \rightarrow \infty$. The method of indivisibles provided a shorthand treatment for the method of exhaustion but still lacked definite proof that the limit sought existed. Neither was it shown that the indivisibles existed. There were also certain other naive assumptions that we need not describe here.

Leibniz invented the notation that is used today. He indicated the sum of Cavalieri's indivisibles by the integral sign, $\int$, and the inverse operation by d. In 1676 he published a manuscript containing such statements as $\overline{dx^3} = 3x^2$, $\overline{d\sqrt{x}} = \frac{1}{\sqrt{x}}$.

Newtons works, published in 1687 and 1704, show two methods used for analysis. He first used the method of indivisibles. In order to show that his infinitesimals existed he changed from the method of indivisibles to that of fluxions. This method can be pictured geometrically as a point flowing along a curve. He finds the ratio of its y velocity to its x velocity at any point on the curve, assuming

1. Smith, History of Mathematics, Ginn and Co., Vol. II, p. 677

that a moving body has a definite velocity at every instant of time. He thus avoided an existence proof for his two infinitesimals. He interpreted the ratio geometrically like modern mathematicians do as the limiting slope of a secant through two points on a curve as the distance between the points becomes small. Newton called integration the method of quadrature, and the solution of differential equations he called the inverse method of tangents.

Newton and Leibniz devised an analysis that worked in most cases. Their method was weak in that no one had shown that the number system was continuous, a necessary property of the domain of the variable.

Since the time of Newton and Leibniz the number system has been enlarged to include all its possible limits. The very small constant, as Leibniz conceived the infinitesimal, has gone into disrepute to be replaced with Newton's theory of limits. Newton's theory is still held that as two variables approach limiting values, if the ratio of their rates of change approaches a limit, this limit has a definite value.

Genetic Development

Following is a list of definitions wherein the fundamental notions of analysis are developed. The system of real numbers so far as needed is built up by the "genetic" method.

A set (class, assemblage, body) we will leave undefined. It represents a fundamental idea. All things possessing a common characteristic are said to constitute a set.

One-to-one correspondence also represents a fundamental notion. Counting objects is the process of establishing a one-to-one correspondence between the objects and the system of positive integers.

Counting can not be logically defined in more fundamental terms: Its validity must be granted to afford a starting point in mathematics.

When an element belongs to a set it possesses the characteristic necessary to define it as a member of that set.

A subset, $[a_1]$,[1] of a set $[a]$ is a set such that every element, a_1, of $[a_1]$ belongs to the set $[a]$.

If set $[a]$ can be put into one-to-one correspondence with set $[b]$ then the sets $[a]$ and $[b]$ are said to be equivalent.

The set $[n]$ of all equivalent sets is symbolized by n which is called the cardinal number of every one of the equivalent sets.

Of two sets $[a]$ and $[b]$, if every element of $[a]$ can be put into one-to-one correspondence with elements of $[b]$ but every element of $[b]$ can not be put into one-to-one correspondence with elements of $[a]$ then the

1. The symbol $[a]$ to represent the set $a_1, a_2, a_3, \ldots\ldots a_n$ is due to Veblen and Lennes, Infinitesimal Analysis, (New York, 1907).

cardinal number of [a] is said to be less than that of [b]. Set [a] is said to be equivalent to a part of [b].

Designating the cardinal numbers of sets [a] and [b] by a and b, the relation a less than b is indicated $a < b$.

If sets [a] and [b] are equivalent then $a = b$; otherwise $a \neq b$.

The definition given here for "less than" precludes more than one of the relations, $a = b$, $b < a$, and $a < b$ being true.

The set of cardinal numbers [n] can now be put into one-to-one correspondence with the positive integers [n'] in such a way that of any two elements of [n], c, d in the relation $c < d$ the corresponding elements of numbers c', d' of [n'] are in the relation $c' < d'$.

A set such that of any two of its elements a and b, $a = b$, $a < b$, or $b < a$ is said to be an ordered set.

Given two sets [a] and [b], form a set [c] such that every element of [a] and [b] is an element of [c] and every element of [c] is an element of [a] or [b]. Then of their cardinal numbers, $a + b = c$.

The set [c] is obviously unique regardless of order of elements. Then $a + b = b + a$.

Given two sets [a] and [b], form a set [c] by associating each element of [a] with every element of [b]. Then of their cardinal numbers, $ab = c$.

Associating each element of [b] with every element of [a] obviously brings the elements together in the same pairs as associating each element of [a] with every element of [b]. Therefore, $ab = ba$.

Given three sets [a], [b], [c], form a fourth set [d] such that every element of [a], [b], or [c] is an element of [d] and every element of [d] is an element of [a], [b], or [c]. Then the cardinal number of set [d] is unique regardless of the order in which they were combined. Therefore, $\underline{(a + b) + c = a + (b + c)} = \underline{a + b + c}$.

Given three sets [a], [b], [c], form the set [ab] then associate each element of [ab] with every element of [c]. A brief inspection will show that the same triples will appear had the set [bc] been formed and each element of [a] associated with every element of [bc]. Therefore $\underline{(ab)c = a(bc)}$.

Given three sets [a], [b], [c], form a new set [d] such that each element of [a] is associated with every element of [b] and [c]. The set [d] is evidently unique. Stated in cardinal numbers $\underline{ab + ac = d = a(b + c)}$.

These definitions have established a system of positive integers and the primary rules of operation. Any set which can be put into one-to-one correspondence with the set of positive integers is said to be <u>denumerable</u> or simply <u>numerable</u>.

Two integers $\underline{a}$ and $\underline{b}$ may be said to constitute the <u>rational fraction</u> $\underline{a/b}$ when properly associated.

Of two rational fractions $\underline{a/b}$ and $\underline{a'/b'}$ if $\underline{ab' = a'b}$ then $\underline{a/b = a'/b'}$, if $\underline{ab' < a'b}$ then $\underline{a/b < a'/b'}$, if $\underline{a'b < ab'}$ then $\underline{a'/b' < a/b}$. The set of rational fractions is thus by definition an ordered set. It can be easily shown that the set of rational fractions is numerable.

From our rules of operation if $a/b < a'/b'$, then $a/b < \frac{a+a'}{b+b'} < a'/b'$. Therefore, between every pair of rational fractions there is another number. An ordered set possessing this property is said to be dense.

The rule of operation $a + b = c$ defines negative numbers and zero. b is negative when $c < a$, and b is zero when $a = c$.

We have here built up roughly the system of rational numbers as it is known and used. When the Greeks finallly admitted fractions they had the positive part of this system as their arithmetic. Because a number existed between every pair of numbers the system would admit of very small numbers and thus appeared to correspond with the properties of space. Unlike space, however, this system of numbers is not continuous.

Given a set $[a]$ divided into two (non-empty) subsets ($[a_1]$ and $[a_2]$ such that for every element a_1 of $[a_1]$ and a_2 of $[a_2]$, $a_1 < a_2$ and such that every element of $[a]$ is an element of $[a_1]$ or $[a_2]$. Then there is an element X which divides the two subsets. This statement is called the Dedekind Cut. A dense set which satisfies the Dedekind Cut is continuous. The element X is called the upper limit of subset $[a_1]$. Applied to the set of rational numbers the Dedekind Cut addd an indefinitely large set of numbers to the system, for the number X need not be a rational number. This continuous set is called the real number system.

(1) A *variable* is a symbol that represents any one of a set of elements. A variable in analysis is a symbol representing any one of a set of numbers.

A *constant* is a symbol that represents a set of only one element. A constant in analysis is a number.

Any one element of a set represented by a variable is called a *value* of the variable.

Given two variables x and y. If to each value of x there corresponds one and only one value of y then y is said to be a one-*valued function* of x.

Given two variables x and y. If to each value of x there corresponds a set of values of y then y is said to be a *many-valued* function of x.

In analysis, if to every number represented by the variable x there corresponds one or more numbers represented by the variable y, then y is said to be a single-or many-valued function respectively of x.

If to any value of x and any value of y there correspond one and only one value of z then z is said to be a single-valued function of x and y.

A *segment* $\overline{ab}$ is the set $[x]$ of all elements such that $a < x < b$.

A *neighborhood* of an element a is the segment $\overline{cd}$ such that $c < a < d$.

The element a is said to be a *limit* of the set $[e]$ if there are elements of $[e]$ other than a in every neighborhood of a.

1. First stated in this form by Veblen and Lennes, in *Infinitesimal Analysis*, (New York, 1907) p. 44.

The set of rational numbers together with the irrationals defined by the Dedekind Cut constitutes a set which contains all its limit points.

If a is a limit of the set represented by the variable x, then x is said to approach a upon the set.

If the number 0 is the limit of a function of x as x approaches a, that function is an infinitesimal.

The fundamental notions of analysis have been developed historically and intuitionally thus far. Now after a brief historical sketch we shall give a rigorous development of the number system by David Hilbert and E. V. Huntington.

ARITHMETIC

Historical

Ethnology of primitive cultures indicates that the positive integers were the first numbers to appear in civilization. The more backward races show more or less ability to count though that may be the extent of their arithmetic ability. At the dawn of written history, there were several systems of symbols for the positive integers in existence. Our system originated with the Hindus, all ten digits appearing for the first time in the year 876.

Fractions, also, had appeared by the dawn of recorded history, doubtless growing out of a need for them in commerce. Early writings of the Babylonians, Egyptians, Chinese, and Mayans show frequent use of the fraction. In the third century B. C. they first came to be regarded as true numbers. Our own method of writing fractions, excepting the bar between numerator and denominator, probably originated with the Hindus, about the fourth century A. D.

Incommensurable ratios were noticed by the Greeks in their studies of geometry. The Pythagoreans supposedly proved the incommensurability of $\sqrt{2}$. Our present notation, the radical sign appeared first in France in 1494. The trancendental number π was met in efforts to express the length of the circumference of a circle. Approximations for π are given in the early writings of the four civilizations mentioned above: Babylonian, Egyptian, Chinese and Mayan. The trancendence of π was first proved in 1882 by F. Lindemann of Germany.

Negative numbers presented themselves to the late Greeks in the solution of algebraic equations. The Hindus in the seventh century were first to recognize them as numbers. The minus sign originated with Tycho Brahe of Denmark in 1598.

The operations addition and subtraction are fundamental to the system of positive integers, synonymous with counting. Multiplication and division, though less fundamental, were recognized in the earliest writings of civilization. The extraction of roots appeared as stated before, among the Greek geometers. The relations "equals" and "less than" are fundamental notions necessary to the system of positive integers and granted, consciously or unconsciously, whenever numbers are used. As the number system expanded from positive integers to the system now in use, the various operations and relations were applied to each new branch. Their symbols, as used today, were invented in Europe in the fifteenth, sixteenth and seventeenth centuries.

HILBERT'S AXIOMS[1]

Hilberts axioms for the real number system contain the undefined terms number, $+$, $\cdot$, and $>$. There are seventeen in all. No effort was made to obtain a minimum number of them so they were not mutually independent.

THEOREMS OF CONNECTION (1-12).

1. From the number a and the number b, there is obtained by "addition" a definite number c, which we express by writing $a + b = c$ or $c = a + b$.
2. There exists a definite number, which we call 0, such that, for every number a, we have $a + 0 = a$ and $0 + a = a$.
3. If a and b are two given numbers, there exists one and only one number x, and also one and only one number y, such that we have respectively $a + x = b$, $y + a = b$.
4. From the number a and the number b, there may be obtained in another way, namely, by "multiplication", a definite number c, which we express by writing $ab = c$ or $c = ab$.
5. There exists a definite number, called 1, such that, for every number a, we have $a \cdot 1 = a$ and $1 \cdot a = a$.
6. If a and b are any arbitrarily given numbers, where a is different from 0, then there exists one and only one number x and also one and only one number y such that we have respectively $ax = b$, $ya = b$.

If a, b, c are arbitrary numbers, the following laws of operation always hold:

7. $a + (b + c) = (a + b) + c$.
8. $a + b = b + a$
9. $a(bc) = (ab)c$
10. $a(b + c) = ab + ac$
11. $(a + b)c = ac + bc$
12. $ab = ba$.

THEOREMS OF ORDER (13 - 16)

13. If a, b are any two distinct numbers, one of these, say a, is always greater ($>$) than the other. The other number is said to be the smaller of the two. We express this relation by writing $a > b$ and $b < a$.
14. If $a > b$ and $b > c$, then is also $a > c$.
15. If $a > b$, then is also $a + c > b + c$ and $c + a > c + b$.
16. If $a > b$ and $c > 0$, then is also $ac > bc$ and $ca > cb$.

1. David Hilbert, Grundlagen

THEOREMS OF ARCHIMEDES (17)

17. If a, b are any two arbitrary numbers, such that $a > 0$ and $b > 0$, it is always possible to add a to itself a sufficient number of times so that the resulting sum shall have the property that $a + a + a + \cdots\cdots + a > b$.

HUNTINGTON'S AXIOMS[1]

Huntington has prepared sets of axioms for various systems. His set for the real number system consists of only fourteen which he proves are mutually independent. The set given here is for the system of real and complex numbers. This set was chosen because it offers a complete foundation for the number system of which the science of analysis treats. Huntington calls the complete number system the set of complex numbers and classifies the real number system as a subset of the complex numbers. The set of complex numbers admits of the operations of addition and multiplication. The subset of real numbers admits of the relation of order. The undefined terms $\underline{K}$, $\underline{C}$, $\underline{+}$, $\underline{\cdot}$, $\underline{\leq}$ correspond respectively to the complex numbers, the real numbers and the relations $\underline{+}$, $\underline{X}$, $\underline{\leq}$ as ordinarily understood.

DEFINITIONS

Definition 1. If there is a uniquely determined element z such that $a + z = z$, the z is called the zero-element, or zero.

Definition 2. If there is a unique zero-element z (see definition 1), and if there is a uniquely determined element u, different from zero, and such that $u \cdot u = u$, the u is called the unit-element, or unity.

Definition 3. If there is a unique zero-element z (definition 1), and if a given element a determines uniquely an element a' such that $a + a' = z$, then a' is called the negative of a, and is denoted by -a.

1. E. V. Huntington, Set of Postulates for Ordinary Complex Algebra, in Transactions of the American Mathematical Society, 6 (1905) p. 222.

Definition 4. If there is a unique zero-element $\underline{z}$ and a unique unit-element $\underline{u}$- (see definition 1 and 2), and if a given element $\underline{a}$, different from $\underline{z}$, determines uniquely an element $\underline{a}''$ such that $a \cdot a'' = u$, then $\underline{a}''$ is called the reciprocal of $\underline{a}$, and is denoted by $\underline{1/a}$.

AXIOMS

The first seven postulates, giving the general laws of operation in the system, are to be understood to hold only in so far as the elements, sums and products involved are elements of $\underline{K}$.

POSTULATE I, 1. $a + b = b + a$.

POSTULATE I, 2. $(a + b) + c = a + (b + c)$.

POSTULATE I, 3. If $a + b = a + b'$, then $b = b'$.

POSTULATE I, 4. $a \cdot b = b \cdot a$.

POSTULATE I, 5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

POSTULATE I, 6. If $a \cdot b = a \cdot b'$, and $a + a \neq a$, then $b = b'$.

POSTULATE I, 7. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

POSTULATE I, 8. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{K}$, then $a + b$ is an element of $\underline{K}$.

POSTULATE I, 9. There is an element $\underline{x}$ in $\underline{K}$ such that $x + x = x$.

POSTULATE I,10. If there is a unique zero-element $\underline{z}$ in $\underline{K}$ (see definition 1), then for every element $\underline{a}$ in $\underline{K}$ there is an element $\underline{a}'$ in K, such that $a + a' = z$.

POSTULATE I,11. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{K}$, then $\underline{a \cdot b}$ is an element of $\underline{K}$.

POSTULATE I, 12. If there is a unique zero-element, $\underline{z}$, in $\underline{K}$ (see definition 1), then there is an element $\underline{y}$ in $\underline{K}$, different from $\underline{z}$, and such that $y \cdot y = y$.

POSTULATE I,13. If there is a unique zero-element, $\underline{z}$, and a unique unity-element, u, different from $\underline{z}$, in $\underline{K}$ (see definition 1 and 2), then for every element $\underline{a}$ in $\underline{K}$, provided $a \neq z$, there is an element $\underline{a}''$ in $\underline{K}$ such that $a \cdot a'' = u$.

The Postulates I: 1 - 13 make the class $\underline{K}$ a field with respect to $\cdot$ and $+$.

POSTULATE II, 1. If $\underline{a}$ is an element of $\underline{C}$, then $\underline{a}$ is an element of $\underline{K}$.

POSTULATE II, 2. The class $\underline{C}$ contains at least one element.

POSTULATE II, 3. If $\underline{a}$ is an element of $\underline{C}$, then there is an element $\underline{b}$ in $\underline{C}$, such that $a \neq b$.

POCSULATE II, 4. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{C}$, then $a + b$, if it exists in $\underline{K}$ at all, is an element of $\underline{C}$.

POSTULATE II, 5. If $\underline{a}$ is an element of $\underline{C}$, then its negative, $\underline{-a}$ (see definition 3), if it exists in $\underline{K}$ at all is an element of $\underline{C}$.

POSTULATE II, 6. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{C}$, then $a \quad b$, if it exists in $\underline{K}$ at all, is an element of $\underline{C}$.

POSTULATE II, 7. If $\underline{a}$ is an element of $\underline{C}$, then its reciprocal $\underline{1/a}$ (see definition 4), if it exists in $\underline{K}$ at all, is an element of $\underline{C}$.

The Postulates II: 1 - 7, taken with the postulates I: 1 - 13, make the sub-class $\underline{C}$, like the class $\underline{K}$, a field with respect to $+$ and $\cdot$.

POSTULATE III, 1. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{C}$, and $a \neq b$, then either $a < b$ or else $a > b$.

POSTULATE III, 2. If $a < b$, then $a \neq b$.

POSTULATE III, 3. If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are elements of $\underline{C}$, and if $a < b$ and $b < c$, then $a < c$.

POSTULATE III, 4. If Γ is a non-empty subclass in $\underline{C}$, and if there is an element $\underline{b}$ in $\underline{C}$ such that $\alpha < b$ for every element $\underline{\alpha}$ of Γ, then there is an element $\underline{X}$ in $\underline{C}$ having the following two properties with regard to the sub-class Γ:

1°) if $\underline{\alpha}$ is an element of Γ, then $\alpha < X$ or $\alpha = X$; while

2°) if $\underline{x'}$ is any element of $\underline{C}$ such that $x' < X$, there is an element ξ in Γ such that $\xi > x'$.

The Postulates III: 1 - 4 and II: 2 - 3, taken with the redundant postulate III, 5 (which is here omitted), make the sub-class $\underline{C}$ a one-dimensional continuum with respect to $\underline{<}$, in the sense defined by Dedekind.

POSTULATE IV, 1. If $\underline{a}$, $\underline{x}$, $\underline{y}$, $\underline{a + x}$, and $\underline{a + y}$ are elements of $\underline{C}$, and $x < y$, then $a + x < a + y$, whenever $a + x \neq a + y$.

POSTULATE IV, 2. If $\underline{a}$, $\underline{b}$, and $\underline{a \cdot b}$ are elements of $\underline{C}$, and $a > z$ and $b > z$, then $a \cdot b > z$ (where $\underline{z}$ is the zero-element of Definition 1).

The twenty-six postulates of groups I - IV make the sub-class $\underline{C}$ equivalent to the class of all real numbers with respect to $+$, $\cdot$, — and $<$.

POSTULATE V, 1. If $\underline{K}$ is a field with respect to $+$ and $\cdot$, then there is an element $\underline{i}$ in $\underline{K}$ such that $i \cdot i = -u$, where $-u$ is the negative of the unit-element of the field (see Definitions 2 and 3).

POSTULATE V, 2. If $\underline{K}$ and also $\underline{C}$ are fields with respect to $+$ and $\cdot$, and if there is an element $\underline{i}$ such that $i \cdot i = -u$ (see Postulate V, 1), then for every element $\underline{a}$ in $\underline{K}$ there are elements $\underline{x}$ and $\underline{y}$ in $\underline{C}$ such that $x + (i \cdot y) = a$.

These twenty-eight postulates make the class $\underline{K}$ equivalent to the class of all (ordinary) complex numbers with respect to $+$, $\cdot$, and $\underline{<}$.

GEOMETRY AND ANALYSIS

Assume that we have a complete system built up for the analysis of real numbers. Suppose we define any set of three such numbers as a point, and the set of all sets of three numbers that satisfy an equation of the form $ax + by + cz + d = o$ as a plane, and the set of all sets which satisfy two such equations as lines. Further, suppose we describe the relation that exists when no numbers satisfy the two equations as parallel planes, and define the set of all sets that satisfy an equation of the form $(x - a)^2 + (y - b)^2 + (z - c)^2 + d = o$ as a sphere. The set of sets of numbers that satisfy both the equation for a plane and the equation for a sphere we can define as a circle. It is obvious that we can obtain a complete geometry from our analysis structure merely by definition. Furtheremore, the operations in analysis remain valid in geometry. Linear order can be defined in the geometry in such a way that the relations of order of analysis hold without change in geometry.

To construct a geometry for restricted relativity it is only necessary to define a point as any set of four numbers. To extend the geometry to mechanics the point is defined as a set of n numbers.

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