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Metric density and approximate limits

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METRIC DENSITY AND APPROXIMATE LIMITS

by

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A. D. L.

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INTRODUCTION

The results of this thesis center around the idea of metric density and its relation to approximate limits. The concept of metric density and approximate limits originated in approximately 1915, and is due largely to Denjoy. The ideas and theorems in this thesis have been developed for Euclidean n-space, for the most part, by Denjoy, de la Vallée Poussin, and Klintchine.

In chapter I, R will denote a metric space, with metric ρ, S will be a subset of R, x_o will be a limit point of S, and f will denote a real-valued function with domain S. In this chapter we define $\liminf_{x \to x_o} f(x)$ and $\limsup_{x \to x_o} f(x)$, the limit inferior and limit superior of f at x_o, and develop properties of these concepts.

In chapter II, it is further assumed that R is a measure space, with a measure m defined on a σ-algebra $\mathcal{L}$. It is further assumed that R, as a metric space, is separable and dense - in - itself. (Further assumptions are listed in chapter II.) If $x_o \varepsilon R$, and if $S \varepsilon \mathcal{L}$, we define $L_S(x_o)$ and $U_S(x_o)$, the lower and upper metric densities of S at x_o. If these are equal, we say that the metric density of S exists at x_o, and the common value is denoted by $D_S(x_o)$. If $D_S(x_o) = 1$, x_o is called a point of density of S, and if $D_S(x_o) = 0$, x_o is called a point of dispersion of S. Properties of upper and lower metric density and metric density are established.

If $S \varepsilon \mathcal{L}$, if f is a measurable real-valued function with domain S, and if x_o is not a point of dispersion of S, we define $\lim_{x \to x_o} \text{ap inf } f(x)$ and $\lim_{x \to x_o} \text{ap sup } f(x)$, the approximate limit inferior and limit superior of f at x_o. If these are equal, we say that the approximate limit of f exists at x_o, and we denote the common value by $\lim_{x \to x_o} \text{ap } f(x)$. If x_o is in S and is a point of dispersion of S, we say that f(x) is approximately continuous at x_o. If x_o is in S and is not a point of dispersion of S, we say f(x) is approximately continuous at x_o in case $\lim_{x \to x_o} \text{ap } f(x) = f(x_o)$.

In Chapter II, similarities between approximate limits and ordinary limits, approximate continuity and ordinary continuity, and approximate derivatives and ordinary derivatives are discussed. It will also be pointed out where these concepts differ. Interspersed in this chapter are many examples illustrating the afore mentioned concepts.

Chapter III will be devoted to proving analogues of the classical Vitali and Lebesgue density theorems. These theorems will be proved for the measure space R, and it will again be assumed that R, as a metric space, is separable and dense - in - itself. Also S will again denote a measurable subset of R, and f will denote a measurable real-valued function with domain S.

In the thesis we will often use braces, $\{\ \}$, to indicate sets. In most cases, lower case Latin letters are used to indicate points and upper case Latin letters are used to indicate sets. Script letters are often used to denote collections of sets. We will also frequently encounter the following sets:

$N^*(p,\delta) = \{x \mid x \varepsilon R,\ \rho(p,x) < \delta\} - \{p\}$;

$N(p,\delta) = \{x \mid x \varepsilon R,\ \rho(p,x) < \delta\}$; and

$Q(p,\delta) = \{x \mid x \varepsilon R,\ \rho(p,x) \leq \delta\}$.

Also we frequently use (a,b) to denote $\{x \mid a < x < b\}$ and $[a,b]$ to denote $\{x \mid a \leq x \leq b\}$; $[a,b)$ and $(a,b]$ are defined in the obvious manner.

CHAPTER I

Section 1

In this chapter it is recalled that R is a metric space, S is a subset of R, f is a real-valued function defined on S, and x_o is a limit point of S.

Definition 1-1. We define the limit superior of f at the point x_o, written $\limsup_{x \to x_o} f(x)$, to be

$$\limsup_{x \to x_o} f(x) = \underset{\varepsilon>0}{\text{g.l.b.}}\ \underset{x\varepsilon S \cup N^*(x_o,\varepsilon)}{\text{l.u.b.}} f(x)$$

where the g.l.b. is taken with respect to all $\varepsilon>0$ and $N^*(x_o,\varepsilon)$ is a deleted open spherical ε - neighborhood of x_o. Similarly we define limit inferior of f at x_o to be

$$\liminf_{x \to x_o} f(x) = \underset{\varepsilon>0}{\text{l.u.b.}}\ \underset{x\varepsilon S \cap N^*(x_o,\varepsilon)}{\text{g.l.b.}} f(x)$$

Let $$A(\varepsilon) = \underset{x\varepsilon S \cap N^*(x_o,\varepsilon)}{\text{l.u.b.}} f(x)$$

and

$$B(\varepsilon) = \underset{x\varepsilon S \cap N^*(x_o,\varepsilon)}{\text{g.l.b.}} f(x) .$$

An immediate consequence of definition 1-1 is

$$\liminf_{x \to x_o} f(x) \leq \limsup_{x \to x_o} f(x).$$

Suppose this were not the case. Suppose

$$C = \underset{\varepsilon>0}{\text{l.u.b.}}\ B(\varepsilon) > \underset{\varepsilon>0}{\text{g.l.b.}}\ A(\varepsilon) = D.$$

Since $D < \frac{C+D}{2} < C$, there exists $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, such that $B(\varepsilon_1) > \frac{C+D}{2}$ and $A(\varepsilon_2) < \frac{C+D}{2}$. Let $\varepsilon_3 = \min(\varepsilon_1,\varepsilon_2)$; $\varepsilon_3 \leq \varepsilon_1$ and $\varepsilon_3 \leq \varepsilon_2$. $B(\varepsilon)$ is a decreasing function of ε, so that $B(\varepsilon_1) \leq B(\varepsilon_3)$. On the other hand, $A(\ )$ is an increasing function of ε, so that $A(\varepsilon_2) \geq A(\varepsilon_3)$. Now since $A(\varepsilon_3) \geq B(\varepsilon_3)$, we have $A(\varepsilon_2) \geq B(\varepsilon_1)$, a contradiction. Therefore $\liminf_{x \to x_o} f(x) \leq \limsup_{x \to x_o} f(x)$.

Before going to the first theorem, let us remark upon the extended real number system. This system consists of the real numbers E^1 and the two entities $+\infty$ and $-\infty$ which have the following properties:

1) $-\infty < +\infty$;
2) if $x \varepsilon E^1$, then $-\infty < x < +\infty$; and
3) if $x \varepsilon E^1$, $x + (+\infty) = +\infty + x = +\infty$ and

$$x + (-\infty) = -\infty + x = -\infty .$$

Theorem 1-1. $\limsup_{x \to x_o} f(x) = +\infty$ if and only if for all $\varepsilon > 0$ and for all real M, there exists an x_1, such that $x_1 \varepsilon N^*(x_o, \varepsilon) \cap S$ and $f(x_1) > M$.

Proof: To prove the sufficiency, suppose M is given and for each $\varepsilon > 0$ there exists an $x_1 \varepsilon S \cap N^*(x_o, \varepsilon)$ such that $f(x_1) > M$. Suppose $\varepsilon_1 > 0$. $A(\varepsilon_1) > M$. But this is true for every M, hence $A(\varepsilon_1) = +\infty$. But ε_1 was arbitrary, therefore $\text{g.l.b.}_{\varepsilon > 0} A(\varepsilon) = \limsup_{x \to x_o} f(x) = +\infty$.

For the necessity, suppose M is real, $\varepsilon > 0$, and $\limsup_{x \to x_o} f(x) = +\infty$. Then $A(\varepsilon_1) = +\infty$ for every ε which implies for any $\varepsilon > 0$ and for any real M, there exists an $x_1 \varepsilon N^*(x_o, \varepsilon) \cap S$ such that $f(x_1) > M$.

Example 1-1. Let $R = S =$ reals. Define

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 . \end{cases}$$

Then $\lim\limits_{x \to 0} \sup f(x) = +\infty$. Suppose $\varepsilon > 0$ and M is real. We may suppose $M > 0$. Let $\delta = \min(\varepsilon, \frac{1}{M})$. Let $x_1 = \delta/2$. Then $x_1 \varepsilon N^*(0,\delta) \subset S \cap N^*(0,\varepsilon)$ and $f(x_1) = \frac{1}{\delta/2} = \frac{2}{\delta} \geq 2M > M$, so $\lim\limits_{x \to 0} \sup f(x) = +\infty$.

A similar result may be proven with regard to limit inferior, namely

<u>Theorem</u> 1-2. $\lim\limits_{x \to x_o} \inf f(x) = -\infty$ if and only if for all $\varepsilon > 0$ and for all real M, there exists an $x_1 \varepsilon\, N^*(x_o,\varepsilon) \cap S$ such that $f(x_1) < M$.

Proof: Suppose M is real and for every $\varepsilon > 0$ there exists an $x_1 \varepsilon N^*(x_o,\varepsilon) \cap S$ such that $f(x_1) < M$. Suppose $\varepsilon_1 > 0$. Then $B(\varepsilon_1) < M$. But this is true for every M, hence $B(\varepsilon_1) = -\infty$. Further for any $\varepsilon > 0$, $B(\varepsilon) = -\infty$. Therefore $\lim\limits_{x \to x_o} \inf f(x) = -\infty$.

Conversely, suppose $\lim\limits_{x \to x_o} \inf f(x) = -\infty$. Then $B(\varepsilon) = -\infty$ for every $\varepsilon > 0$, i.e., $\underset{x \varepsilon S \cap N^*(x_o,\varepsilon)}{\text{g.l.b.}} f(x) = -\infty$, hence for any $\varepsilon > 0$ there exists $x_1 \varepsilon N^*(x_o,\varepsilon) \cap S$ such that $f(x_1) < M$.

It is clear that the statement and proof of theorem 1-2 is analogous to that of theorem 1-1; henceforth theorems and proofs will be given for limit superior only.

<u>Theorem</u> 1-3. $\limsup_{x \to x_0} f(x) = -\infty$ if and only if for every real M there exists an $\varepsilon > 0$ such that if $x_1 \varepsilon N^*(x_0, \varepsilon) \cap S$, then $f(x_1) < M$.

Proof: To prove the sufficiency suppose M is real and there exists $\varepsilon > 0$ such that if $x_1 \varepsilon N^*(x_0, \varepsilon) \cap S$, then $f(x_1) < M$. x_1 is arbitrary, so $\text{g.l.b.}_{\varepsilon > 0} A(\varepsilon) \leq A(\varepsilon) \leq M$. However, M is arbitrary; thus $\limsup_{x \to x_0} f(x) = -\infty$.

On the other hand suppose $\limsup_{x \to x_0} f(x) = -\infty$. Suppose M is real. Then $\text{g.l.b.}_{\varepsilon > 0} A(\varepsilon) < M$. This implies there exists $\varepsilon_1 > 0$ such that $A(\varepsilon_1) < M$, i.e., if $x_1 \varepsilon N^*(x_0, \varepsilon) \cap S$, then $f(x_1) < M$.

<u>Theorem</u> 1-4. Suppose L is real. $\limsup_{x \to x_0} f(x) = L$ if and only if the following two properties are satisfied for all $\varepsilon > 0$:

(1) $f(x) < L + \varepsilon$ for all x in some neighborhood $N^*(x_0, \delta) \cap S$; and

(2) $f(x) > L - \varepsilon$ for some x in every neighborhood $N^*(x_0, \delta) \cap S$.

Proof: Suppose $\varepsilon > 0$ and properties (1) and (2) are satisfied. By (1), $f(x) < L + \varepsilon$ for all x in some $N^*(x_0, \delta) \cap S$, hence $A(\delta) \leq L + \varepsilon$. Thus $\text{g.l.b.}_{\delta > 0} A(\delta) \leq L + \varepsilon$ for all $\varepsilon > 0$ and hence $\limsup_{x \to x_0} f(x) \leq L$. Applying condition (2) we have $A(\delta) \geq L - \varepsilon$ for every $\delta > 0$, i.e., $L - \varepsilon$ is a lower bound for $A(\delta)$ for all δ. Therefore

$\text{g.l.b.}_{\delta>0} A(\delta) \geq L - \varepsilon$; however $\varepsilon > 0$ is arbitrary, so $\limsup_{x \to x_0} f(x) \geq L$. This last inequality combined with the above gives us a sufficient condition for $\limsup_{x \to x_0} f(x)$

To prove the necessity, rather than suppose $\limsup_{x \to x_0} f(x) \geq L$, consider $\limsup_{x \to x_0} f(x) \geq L$ and $\limsup_{x \to x_0} f(x) \leq L$ separately. If $\limsup_{x \to x_0} f(x) \geq L$, then for all $\delta > 0$ there exists an x in each $N^*(x_0,\delta) \cap S$ such that $f(x) > L - \varepsilon$ since $\limsup_{x \to x_0} f(x) = \text{g.l.b.}\, A(\delta)$. This is precisely condition (2). Now suppose $\limsup_{x \to x_0} f(x) \leq L$, i.e., $\text{g.l.b.}_{\delta>0} A(\delta) \leq L < L + \varepsilon$. But this means $L + \varepsilon$ is not a lower bound for all $A(\delta)$; thus there exists $\delta_1 > 0$ such that $A(\delta_1) < L + \varepsilon$. Hence for all x in $N^*(x_0,\delta_1) \cap S$, $f(x) < L + \varepsilon$, which is condition (1).

In proving theorem 1-4 we have actually established a somewhat stronger theorem, namely

Theorem 1-5. $\limsup_{x \to x_0} f(x) \leq L$ if and only if for all $\varepsilon > 0$ there exists a neighborhood $N^*(x_0,\delta)$ such that for all $x \varepsilon N^*(x_0,\delta) \cap S$, $f(x) < L + \varepsilon$; and, $\limsup_{x \to x_0} f(x) \geq L$ if and only if for all $\varepsilon > 0$, $\delta > 0$, there exists $x_1 \varepsilon N^*(x_0,\delta) \cap S$ such that $f(x_1) > L - \varepsilon$.

Example 1-2. Let $R = S =$ reals. Show

$\lim_{x \to 0} \sup \cos \frac{1}{x} = 1$. Suppose $\varepsilon > 0$. Clearly condition (1) of theorem 1-4 is satisfied, since $\cos \frac{1}{x} < 1+\varepsilon$ for all $x \varepsilon N^*(0,\delta)$, if $\delta > 0$. On the other hand if $\delta > 0$, then there exists a positive integer N such if $n > N$, then $\frac{1}{2n\pi} < \delta$, i.e., $\frac{1}{2n\pi} \varepsilon N^*(0,\delta)$, and $\cos \frac{1}{\frac{1}{2n\pi}} = \cos 2n\pi = 1 > 1-\varepsilon$ and condition (2) of theorem 1-4 is satisfied. $\therefore \lim_{x \to 0} \sup \cos \frac{1}{x} = 1$.

Section II

Before considering further results let us recall the following definition.

<u>Definition</u> 1-2. Suppose $\{x_n\}$ is a sequence of points in R. Then $\lim_{n \to \infty} x_n = x$ if and only if for all $\varepsilon > 0$ there exists N such that if $n > N$, then $\rho(x_n,x) < \varepsilon$.

If $\{x_n\}$ is a sequence of extended real numbers we define $\lim_{n \to \infty} x_n = +\infty\ (-\infty)$ if and only if for every real M there exists N such that if $n > N$, then $x_n > M$ ($x_n < M$).

In this section we will discuss some theorems on limit superior and limit inferior in relation to sequences.

<u>Theorem</u> 1-6. $\lim_{x \to x_o} \sup f(x) = +\infty$ if and only if there exists $\{x_n\}$ with $x_n \varepsilon S$, $x_n \neq x_o$, $\lim_{n \to \infty} x_n = x_o$ and $\lim_{n \to \infty} f(x_n) = +\infty$.

Proof: Suppose there exists $\{x_n\}$ with $x_n \varepsilon S$,

$x_n \neq x_o$, $\lim_{x \to \infty} x_n = x_o$ and $\lim_{n \to \infty} f(x_n) = +\infty$. Suppose M is real. Since $\lim_{n \to \infty} f(x_n) = +\infty$, there exists N such that if $n > N$, then $f(x_n) > M$. Further, for every $\delta > 0$ there exists P such that if $m > P$, then $x_m \varepsilon N^*(x_o, \delta) \cap S$. Let $m > \max(N,P)$. Then $f(x_m) > M$. Therefore by theorem 1-1, $\lim_{x \to x_o} \sup f(x) = +\infty$.

On the other hand suppose $\lim_{x \to x_o} \sup f(x) = +\infty$. Again using theorem 1-1, we know that for all $\delta > 0$, for all real M, there exists $x_1 \varepsilon N^*(x_o, \delta) \cap S$ such that $f(x_1) > M$. Let $\delta_1 = 1$, $M = 1$; there exists $x_1 \varepsilon N^*(x_o, 1) \cap S$ such that $f(x_1) > 1$. Let $\delta_2 = \frac{1}{2}$, $M = 2$; there exists $x_2 \varepsilon N^*(x_o, \frac{1}{2}) \cap S$ such that $f(x_2) > 2$. In general let $\delta_n = \frac{1}{n}$, $M = n$; there exists $x_n \varepsilon N^*(x_o, \frac{1}{n}) \cap S$ such that $f(x_n) > n$.

In this manner we obtain a sequence $\{x_n\}$ such that $x_n \varepsilon S$, $x_n \neq x_o$ and $\lim_{x \to \infty} x_n = x_o$. Also we see if $n > M$, then $f(x_n) > n > M$ so that $\lim_{n \to \infty} f(x_n) = +\infty$.

For an illustration of this theorem, refer to example 1-1 and let the sequence $\{x_n\}$ be $\{\frac{1}{n}\}$.

<u>Theorem</u> 1-7. $\lim \sup f(x) = -\infty$ if and only if for all $\{x_n\}$ such that $x_n \varepsilon S$, $x_n \neq x_o$ and $\lim_{n \to \infty} x_n = x_o$, $\lim_{n \to \infty} f(x_n) = -\infty$.

Proof: The proof of the sufficiency will be by contradiction. Suppose

$$\limsup_{x \to x_o} f(x) = L, \ L \neq -\infty.$$

We want to show that there exists $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$ and $\{f(x_n)\}$ does not converge to $-\infty$.

If $L = +\infty$, then by theorem 1-6 there exists $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$, and $\lim_{n \to \infty} f(x_n) = +\infty$. Thus suppose L is real. Suppose $\delta > 0$. By theorem 1-4, for every $\delta > 0$ there exists $x_1 \varepsilon N^*(x_o, \delta) \cap S$ such that $f(x_1) > L - 1$. Consider those δ's of the form $\frac{1}{n}$. For every n there exists $x_n \varepsilon N^*(x_o, \frac{1}{n}) \cap S$ such that $f(x_n) > L - 1$. Thus we have a sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$. However $f(x_n)$ does not converge to $-\infty$, a contradiction.

For the necessity suppose $\limsup_{x \to x_o} f(x) = -\infty$, $\{x_n\}$, $x_n \varepsilon S$, $x_n \neq x_o$ and $\lim_{n \to \infty} x_n = x_o$. Let M be real. M is not a lower bound for $A(\delta)$ for all δ, therefore there exists $\delta_1 > 0$ such that $A(\delta_1) < M$. If $x \varepsilon N^*(x_o, \delta_1) \cap S$, then $f(x) < M$. There exists N such that if $n > N$, then $\rho(x_n, x_o) < \delta_1$, $x_n \varepsilon S$. Therefore if $n > N$ it follows that $f(x_n) < M$ and the proof is complete.

<u>Theorem</u> 1-8. $\limsup_{x \to x_o} f(x) = L$ if and only if

(1) there exists $\{x_n\}$ such that $x_n \varepsilon S$, $x_n \neq x_o$, $\lim_{n \to \infty} x_n = x_o$ and $\lim_{n \to \infty} f(x_n) = L$; and

(2) for every sequence $\{x_n\}$ for which $x_n \varepsilon S$, $x_n \neq x_o$,

$\lim_{n\to\infty} x_n = x_o$ and for which $\{f(x_n)\}$ converges,

then $\lim_{n\to\infty} f(x_n) \leq L$.

Proof: Condition (1) implies $\limsup_{x\to x_o} f(x) \geq L$. Since $\lim_{n\to\infty} x_n = x_o$, all but a finite number of terms of $\{x_n\}$ are in $N^*(x_o,\delta)$ for every $\delta > 0$. Suppose $\varepsilon > 0$. Then there exists n such that $x_n \varepsilon N^*(x_o,\delta) \cap S$ and $f(x_n) > L - \varepsilon$, since $\lim_{n\to\infty} f(x_n) = L$. Therefore $A(\delta) \geq L$ and $\limsup_{x\to x_o} f(x) = \text{g.l.b.}_{\delta>0}\, A(\delta) \geq L$.

On the other hand condition (2) implies $\limsup_{x\to x_o} f(x) \leq L$. Suppose this were not the case, i.e., $\limsup_{x\to x_o} f(x) = Q > L$. $A(\delta) \geq Q$ for all δ. $Q + \frac{1}{n}$ is not a lower bound for all the $A(\delta)$'s, so there exists δ_n such that $Q \leq A(\delta_n) < Q + \frac{1}{n}$. We may suppose $\delta_n < \frac{1}{n}$, for if not, then there exists a δ' such that $\delta' < \frac{1}{n}$ and $A(\delta') \leq A(\delta_n) < Q + \frac{1}{n}$. (Note that $A(\delta)$ is an increasing function). Thus $f(x) < Q + \frac{1}{n}$ for all $x \varepsilon N^*(x_o,\delta_n) \cap S$. $A(\delta_n) > Q - \frac{1}{n}$, therefore there exists $x_n \varepsilon N^*(x_o,\delta_n) \cap S$ such that $f(x_n) > Q - \frac{1}{n}$. Now

$$|f(x_n) - Q| < \frac{1}{n}$$

and further $\rho(x_o,x_n) < \frac{1}{n}$. We obtain a sequence $\{x_n\}$ with $x_n \neq x_o$, $\lim_{n\to\infty} x_n = x_o$ and $\lim_{n\to\infty} f(x_n) = Q > L$, a contradiction to condition (2). $\therefore$ $\limsup_{x\to x_o} f(x) \leq L$ as claimed and hence $\limsup_{x\to x_o} f(x) = L$.

To prove the necessity, let us suppose $\lim_{x \to x_o} \sup f(x) = L$. By theorem 1-4 there exists δ'_n, $0 < \delta'_n \leq \frac{1}{n}$ such that if $x \varepsilon N^*(x_o, \delta'_n) \cap S$, then $f(x) < L + \frac{1}{n}$. Also by theorem 1-4, for each $\delta > 0$, $\varepsilon > 0$, there exists $x \varepsilon N^*(x_o, \delta)$ such that $f(x) > L - \varepsilon$. In particular, for $\delta_1 \leq 1$, $\varepsilon = 1$, there exists $x_1 \varepsilon N^*(x_o, \delta_1) \cap S$ such that $f(x_1) > L - 1$; for $\delta_2 \leq \frac{1}{2}$, $\varepsilon = \frac{1}{2}$, there exists $x_2 \varepsilon N^*(x_o, \delta_2) \cap S$ such that $f(x_2) > L - \frac{1}{2}$; and in general for $\delta_n \leq \frac{1}{n}$, $\varepsilon = \frac{1}{n}$, there exists $x_n \varepsilon N^*(x_o, \delta_n) \cap S$ such that $f(x_n) > L - \frac{1}{n}$. Thus there exists a sequence $\{x_n\}$, $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$, $x_n \varepsilon S$ and $\lim_{n \to \infty} f(x_n) = L$, so condition (1) holds.

To show condition (2) is implied, suppose $\{x_n\}$ is such that $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$, $x_n \varepsilon S$ and $\{f(x_n)\}$ converges. Suppose $\lim_{x \to x_o} \sup f(x) \leq L$, i.e., $\text{g.l.b.}_{\delta > 0}\, A(\delta) \leq L < L + \varepsilon$. Then there exists a $\delta' > 0$ such that $A(\delta') < L + \varepsilon$ and $f(x) < L + \varepsilon$ for all $x \varepsilon N^*(x_o, \delta') \cap S$. There exists N such that if $n > N$, $x_n \varepsilon N^*(x_o, \delta') \cap S$. But then $f(x_n) < L + \varepsilon$. Therefore if $\{f(x_n)\}$ converges, $\lim_{n \to \infty} f(x_n) \leq L$ as we wanted to show.

As in the case of theorem 1-4, we have proved a stronger theorem in the proof of theorem 1-8, namely

<u>Theorem</u> 1-9. $\lim_{x \to x_o} \sup f(x) \leq L$ if and only if condition (2) in theorem 1-8 holds; and $\lim_{x \to x_o} \sup f(x) \geq L$

if and only if there exists $\{x_n\}$ with $x_n \varepsilon S$ such that $\lim_{n \to \infty} x_n = x_o$, $x_n \neq x_o$, and $\lim_{n \to \infty} f(x_n) = Q \geq L$.

Section III

<u>Definition</u> 1-3. Suppose y is real. Define

$C(y) = \{x | x \varepsilon S,\ f(x) > y\}$ and

$D(y) = \{x | x \varepsilon S,\ f(x) < y\}$.

<u>Definition</u> 1-4. Define

$G(x_o) = \{y | y$ is real, $x_o \varepsilon R$ and x_o is a limit point of $C(y)\}$.

Similarly define

$P(x_o) = \{y | y$ is real, $x_o \varepsilon R$ and x_o is a limit point of $D(y)\}$.

<u>Theorem</u> 1-10. If $y_1 < y_2$, then $C(y_2) \subset C(y_1)$ and $D(y_1) \subset D(y_2)$.

Proof: If $x \varepsilon C(y_2)$, then $f(x) > y_2 > y_1$, therefore $x \varepsilon C(y_1)$. If $x \varepsilon D(y_1)$, then $f(x) < y_1 < y_2$, so $x \varepsilon D(y_2)$.

<u>Theorem</u> 1-11. If $y_1 < y_2$ and $y_2 \varepsilon G(x_o)$, then $y_1 \varepsilon G(x_o)$

Proof: If $y_2 \varepsilon G(x_o)$, then x_o is a limit point of $C(y_2)$. But by theorem 1-10, $C(y_2) \subset C(y_1)$. Therefore x_o is a limit point of $C(y_1)$, i.e., $y_1 \varepsilon G(x_o)$.

A similar result can be established for $P(x_o)$.

It follows readily from theorem 1-11 that $G(x_o)$ is characterized as exactly as one of the following:

(1) the set of all reals;

(2) Φ ;

(3) $(-\infty, r)$, r real;

(4) $(-\infty, r]$, r real.

<u>Theorem</u> 1-12. $\limsup_{x \to x_o} f(x) = \text{l.u.b. } G(x_o)$.

Proof: Let H = l.u.b. $G(x_o)$. Suppose $\varepsilon > 0$. H is the least upper bound of $G(x_o)$, so $H-\varepsilon$ is not an upper bound. There exists k such that $H - \varepsilon < k$ and $k \varepsilon G(x_o)$. By theorem 1-11, $(H-\varepsilon)\varepsilon G(x_o)$ and thus x_o is a limit point of $C(H-\varepsilon)$. $C(H-\varepsilon) \cap N^*(x_o,\delta) \cap$ is not empty for all $\delta > 0$, so there exists $x_1 \varepsilon C(H-\varepsilon) \cap N^*(x_o,\delta)$. Now $x_1 \varepsilon N^*(x_o,\delta) \cap S$, and $f(x_1) > H-\varepsilon$, therefore, by theorem 1-5, $\limsup_{x \to x_o} f(x) \geq$

On the other hand, since H is an upper bound for $G(x_o)$, $(H + \frac{\varepsilon}{2}) \not\varepsilon G(x_o)$ and x_o is not a limit point of $C(H + \frac{\varepsilon}{2})$. Thus there exists a neighborhood $N^*(x_o,\delta)$ such that $C(H + \frac{\varepsilon}{2}) \cap N^*(x_o,\delta) = \Phi$. Thus if $x \varepsilon N^*(x_o,\delta) \cap S$, then $f(x) \leq H + \frac{\varepsilon}{2} < H + \varepsilon$. Appealing to theorem 1-5 again we see that $\limsup_{x \to x_o} f(x) \leq H$. Combining this result with the above, we get $\limsup_{x \to x_o} f(x) = H$.

CHAPTER II

Section I

<u>Definition</u> 2-1. Suppose R is any set and $\mathcal{L}$ is any σ-algebra of subsets of R. Let m be a countably additive, non-negative extended real-valued function defined on the sets of $\mathcal{L}$, i.e., let m be a measure on $\mathcal{L}$. Then we say that R, $\mathcal{L}$, and m form a <u>measure</u> <u>space</u>.

In this chapter and the following chapter we will be working with a particular kind of measure space; R will be a separable, dense-in-itself metric space and $\mathcal{L}$ will be a σ-algebra of subsets of R such that if $x_o \varepsilon R$, $\varepsilon > 0$, then $N(x_o, \varepsilon)$, an open spherical neighborhood of x_o, is a set of $\mathcal{L}$. S will denote a fixed measurable subset of R. Two additional assumptions which are made is first that if $\varepsilon > 0$, $x_o \varepsilon R$, then $0 < m(N(x_o, \varepsilon)) < +\infty$, and second, $m(\{x_o\}) = 0$. It is immediately seen that $m(\Phi) = 0$, since $m(N(x_o, \varepsilon)) = m(N(x_o, \varepsilon) \cup \Phi) = m(N(x_o, \varepsilon)) + m(\Phi)$. It is also seen that $\lim\limits_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} m(N(x_o, \varepsilon)) = 0$, which follows from the well known result in measure theory, that if $\{A_n\}$ is a decreasing sequence of sets and $m(A_k) < +\infty$ for some k, then $\lim\limits_{n \to \infty} m(A_n) = m(\bigcap\limits_{n=1}^{\infty} A_n)$.

When assumptions additional to those above are required, they will be explicitly stated.

We remark at this point on the requirement that R be separable and dense-in-itself. We assume this to insure that spherical neighborhoods do not reduce to points, in keeping with our requirement that $m(N(x_o,\varepsilon)) > 0$, $m(\{x_o\}) = 0$. R separable implies that if $G \subset R$, G open, then $G \varepsilon \mathcal{L}$.

Definition 2-2. Suppose R is a separable, dense-in-itself metric space and S is a measurable subset of R, i.e., $S \varepsilon \mathcal{L}$. Then we define

$$u_s(x_o, \lambda) = \underset{\substack{x_o \varepsilon N(p,\delta) \\ 0<\delta<\lambda \\ p \varepsilon R}}{\text{l.u.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))},$$

where the least upper bound is taken with respect to all $\delta > 0$ and $p \varepsilon R$ for which $\delta < \lambda$ and $x_o \varepsilon N(p,\delta)$. Similarly we define

$$ls(x_o, \lambda) = \underset{\substack{x_o \varepsilon N(p,\delta) \\ 0<\delta<\lambda \\ p \varepsilon R}}{\text{g.l.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))}.$$

Theorem 2-1. $l_s(x_o, \lambda) \leq u_s(x_o, \lambda)$ for $\lambda > 0$.

Definition 2-3. We define

$$U_s(x_o) = \underset{\lambda > 0}{\text{g.l.b.}}\ u_s(x_o, \lambda),$$

where the greatest lower bound is taken with respect to all $\lambda > 0$, to be the upper metric density of S at the point x_o. Analogously we define

$$L_s(x_o) = \underset{\lambda > 0}{\text{l.u.b.}}\ l_s(x_o, \lambda)$$

to be the lower metric density of S at the point x_o.

Since $S \cap N(p,\delta) \subset N(p,\delta)$,

$m(S \cap N(p,\delta)) \leq m(N(p,\delta)) \nleq 0$, and

$$0 \leq \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} \leq 1,$$

from which it follows that $0 \leq l_s(x_o,\lambda) \leq 1$ and $0 \leq u_s(x_o,\lambda) \leq 1$. Also as a direct consequence we get $0 \leq L_s(x_o) \leq 1$ and $0 \leq U_s(x_o) \leq 1$.

Theorem 2-2. If $\lambda_1 < \lambda_2$, then $u_s(x_o, \lambda_1) \leq u_s(x_o, \lambda_2)$ and $l_s(x_o,\lambda_2) \leq l_s(x_o, \lambda_1)$, i.e., u_s is an increasing function of λ and l_s is a decreasing function of λ.

Theorem 2-3. If $\delta > 0$, $\gamma > 0$, then

$$l_s(x_o,\delta) \leq u_s(x_o,\gamma).$$

Proof: Suppose $\alpha > 0$ such that $\delta > \alpha$, $\gamma > \alpha$. Then by theorem 2-2,

$$l_s(x_o,\delta) \leq l_s(x_o,\alpha) \leq U_s(x_o,\alpha) \leq U_s(x_o,\gamma).$$

Corollary. $0 \leq L_s(x_o) \leq U_s(x_o) \leq 1$.

Definition 2-4. The metric density of S is said to exist at a point x_o provided $L_s(x_o) = U_s(x_o)$, and we use $D_s(x_o)$ to denote the common value in case $L_s(x_o) = U_s(x_o)$. $D_s(x_o)$ is called the metric density of S at x_o.

Theorem 2-4. Suppose k is a real number. Then $U_s(x_o) \geq k$ if and only if for all $\varepsilon > 0$, and all $\lambda > 0$, there exists $\delta > 0$ and $p \varepsilon R$ such that $\delta < \lambda$, $x_o \varepsilon N(p,\delta)$, and $\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} > k-\varepsilon$.

Proof: Suppose $\varepsilon > 0$. $U_s(x_o) \geq k$ implies k is a lower bound for $u_s(x_o,\lambda)$ for all $\lambda > 0$. But

$$u_s(x_o,\lambda) = \underset{\substack{p \varepsilon R \\ x_o \varepsilon N(p,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} \geq k$$

implies there exists $\delta' > 0$ and $p\varepsilon R$ such that $\delta' < \lambda$, $x_o \varepsilon N(p,\delta')$, and $\frac{m(S \cap N(p,\delta'))}{m(N(p,\delta'))} > k-\varepsilon$.

On the other hand, suppose for all $\varepsilon > 0$ and for all $\lambda > 0$, there exists $\delta > 0$ and $p\varepsilon R$ such that $\delta < \lambda$, $x_o \varepsilon N(p,\delta)$ and $\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} > k-\varepsilon$. Then

$$u_s(x_o, \lambda) = \underset{\substack{p\varepsilon R \\ x_o \varepsilon N(p,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} > k-\varepsilon,$$

but $\varepsilon > 0$ is arbitrary, therefore $u_s(x_o, \lambda) \geq k$. But, further, $\lambda > 0$ is arbitrary, so $\underset{\lambda>0}{\text{g.l.b.}} u_s(x_o, \lambda) = U_s(x_o) \geq k$

<u>Theorem</u> 2-5. Suppose k is real. $U_s(x_o) \leq k$ if and only if for all $\varepsilon > o$, there exists $\lambda' > 0$ such that if $0 < \delta < \lambda'$, $p\varepsilon R$, $x_o \varepsilon N(p,\delta)$, then $\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$.

Proof: Suppose $\varepsilon > 0$ and $U_s(x_o) \leq k$. Then $\underset{\lambda>0}{\text{g.l.b.}} u_s(x_o, \lambda) \leq k < k + \varepsilon$, so $k+\varepsilon$ is not a lower bound for all $u_s(x_o, \lambda)$. Hence there exists $\lambda' > 0$ such that $u_s(x_o, \lambda') < k+\varepsilon$ which implies $\underset{\substack{p\varepsilon R \\ x_o \varepsilon N(p,\delta) \\ 0<\delta<\lambda'}}{\text{l.u.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$.

But this implies $\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$ for all δ and all $p\varepsilon R$ such that $0 < \delta < \lambda'$ and $x_o \varepsilon N(p,\delta)$.

Conversely, suppose for all $\varepsilon>0$ there exists $\lambda' > 0$ such that if $0 < \delta < \lambda'$, $p\varepsilon R$, $x_o \varepsilon N(p,\delta)$, then $\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$. Then

$$u_s(x_o\ \lambda') = \underset{\substack{p\varepsilon R \\ x_o\varepsilon N(p,\delta) \\ 0<\delta<\lambda'}}{\text{l.u.b.}} \frac{m(S\cap N(p,\delta))}{m(N(p,\delta))} \leq k+\varepsilon .$$

Therefore $U_s(x_o) = \underset{\lambda>0}{\text{g.l.b.}}\ u_s(x_o, \lambda) \leq k+\varepsilon$. But $\varepsilon>0$ is arbitrary, therefore $U_s(x_o) \leq k$.

<u>Theorem</u> 2-6. Suppose k is real. Then $L_s(x_o) \geq k$ if and only if for all $\varepsilon > 0$ there exists $\lambda' > 0$ such that if $0 < \delta < \lambda'$, $p\varepsilon R$, $x_o\varepsilon N(p,\delta)$, then $\frac{m(S\cap N(p,\delta))}{m(N(p,\delta))} > k-\varepsilon$.

The proof is similar to that of theorem 2-5 and will be omitted.

<u>Theorem</u> 2-7. Suppose k is real. Then $L_s(x_o) \leq k$ if and only if for all $\varepsilon > 0$ and for all $\lambda > 0$, there exists $\delta > 0$ and $p\varepsilon R$ such that $\delta < \lambda$, $x_o\varepsilon N(p,\delta)$ and $\frac{m(S\cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$.

Again the proof will be omitted since it is similar to that of theorem 2-4. With the aid of theorems 2-5 and 2-6 we get the following theorem.

<u>Theorem</u> 2-8. The metric density of S exists at x_o and is equal to k if and only if for all $\varepsilon > 0$ there exists $\lambda' > 0$ such that if $0 < \delta < \lambda'$, $p\varepsilon R$, $x_o\varepsilon N(p,\delta)$, then $k-\varepsilon < \frac{m(S\cap N(p,\delta))}{m(N(p,\delta))} < k+\varepsilon$.

Before going on, consider the following three examples.

<u>Example</u> 2-1. Consider the metric space R to be Euclidean 1- space and let $S = (0, +\infty)$. Let m be the usual Lebesgue measure. Let $x_o = 0$. Then as a consequence

of theorem 2-4, $u_s(0,\lambda) = 1$ for all $\lambda > 0$, and $U_s(0) = 1$. But on the other hand, by theorem 2-7, $l_s(0,\lambda) = 0$ for all $\lambda > 0$, and $L_s(0) = 0$ and hence the metric density of $(0, +\infty)$ does not exist at 0.

Example 2-2. Let R be Euclidean 1-space, let $S = (-1,1)$, and let $x_o = 0$. Let m be Lebesgue measure. As in the preceding example $U_s(0) = 1$, but also we find, $L_s(0) = 1$. Therefore $D_s(0)$ exists and is equal to one.

The next example is one in which the metric density exists, but is not equal to 1 or 0. We will be more able to appreciate the significance of this example after we have had the Lebesgue density theorem.

Before discussing this next example however, let us first establish the following

Lemma. Let $p, r \geq 0$, $q, s > 0$, where $\frac{r}{s} \leqq \frac{p}{q}$. Then $\frac{r}{s} \leqq \frac{p+r}{q+s} \leqq \frac{p}{q}$.

Proof: Suppose $\frac{r}{s} \leqq \frac{p}{q}$. Then $rq \leqq ps$ implies $rq + rs \leqq ps + rs$ and $\frac{r}{s} \leqq \frac{p+r}{q+s}$. Similarly $rq + pq \leqq ps + pq$ and $\frac{p+r}{q+s} \leqq \frac{p}{q}$.

Example 2-3. Let R be Euclidean 1- space and m ordinary Lebesgue measure. Let $x_o = 0$. Form $S \subset R$ in the following way. Define $S_1 = (-\frac{\pi^2}{6}, -\frac{\pi^2}{6} + \frac{1}{2}) \cup (\frac{\pi^2}{6} - \frac{1}{2}, \frac{\pi^2}{6})$ and in general

$$S_n = \left(-\frac{\pi^2}{6} + \sum_{k=1}^{n-1} \frac{1}{k^2}, -\frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{2n^2}\right) \cup \left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{2n^2}, \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}\right), \ n=2,3,\cdots$$

Let $S = \bigcup_{n=1}^{\infty} S_n$. Define intervals $I_1 = (0, \frac{\pi^2}{6})$,

$I_n = (0, \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2})$, $n = 2,3,$----.

Now we observe $\frac{m(S\cap I_n)}{m(I_n)} = \frac{\frac{1}{2}\sum_{r=n}^{\infty} \frac{1}{r^2}}{\sum_{r=n}^{\infty} \frac{1}{r^2}} = \frac{1}{2}$

for every n. We will use the intervals I_n together with theorem 2-8 to show $D_s(0) = \frac{1}{2}$.

Denote the right hand endpoint of I_n by y_n; $y_n = m(I_n) = \frac{\pi^2}{6} - \sum_{r=1}^{n-1} \frac{1}{r^2} = \sum_{r=n}^{\infty} \frac{1}{r^2}$. Note that $0 < y_{n+1} < y_n$, and $\lim_{n\to\infty} y_n = 0$. Suppose $\varepsilon > 0$. Choose N such that $N > \frac{1}{\varepsilon}$. Let $\lambda' = \frac{1}{2}y_N = \frac{1}{2}\cdot\sum_{r=N}^{\infty} \frac{1}{r^2}$. Consider any $\delta > 0$ such that $\delta < \lambda'$. Let $I = (p-\delta, p+\delta)$ where $-\delta < p < \delta$ so that $0\varepsilon I$. Define $I' = (p-\delta,0)$ and $I'' = (0,p+\delta)$. $m(I) = m(I') + m(I'')$. There exists $n \geq N$ such that $y_{n+1} \leq p+\delta < y_n$. Now

$$\frac{m(S\cap I'')}{m(I'')} \leq \frac{m(S\cap I_n)}{m(I_{n+1})} = \frac{\frac{1}{2}\sum_{r=n}^{\infty} \frac{1}{r^2}}{\sum_{r=n+1}^{\infty} \frac{1}{r^2}} = \frac{\frac{1}{2}(\frac{1}{n^2} + \sum_{r=n+1}^{\infty} \frac{1}{r^2})}{\sum_{r=n+1}^{\infty} \frac{1}{r^2}}$$

$$= \frac{1}{2}(1 + \frac{\frac{1}{n^2}}{\sum_{r=n+1}^{\infty} \frac{1}{r^2}}$$

$$< \frac{1}{2}(1 + \frac{2}{n})$$

$$\leq \frac{1}{2}(1 + \frac{2}{N}) = \frac{1}{2} + \frac{1}{N} < \frac{1}{2} + \varepsilon.$$

Note that $\frac{\frac{1}{n^2}}{\sum_{r=n+1}^{\infty} \frac{1}{r^2}} < \frac{2}{n}$ since

$$\sum_{r=n+1}^{\infty} \frac{1}{r^2} > \int_{n+1}^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_{n+1}^{b} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \left(-\frac{1}{x}\right)\Big|_{n+1}^{b}$$

$= \frac{1}{n+1}$. Thus $\dfrac{\frac{1}{n^2}}{\sum\limits_{r=n+1}^{\infty} \frac{1}{r^2}} < \frac{n+1}{n^2} \leq \frac{2}{n}$.

At the same time, however, $\frac{m(S \cap I'')}{m(I'')} > \frac{1}{2} - \varepsilon$, since $\frac{m(S \cap I'')}{m(I'')} \geq \frac{m(S \cap I_{n+1})}{m(I_n)} = \dfrac{\frac{1}{2} \sum\limits_{r=n+1}^{\infty} \frac{1}{r^2}}{\sum\limits_{r=n}^{\infty} \frac{1}{r^2}}$

$$= \frac{\frac{1}{2}}{1 + \dfrac{\frac{1}{n^2}}{\sum\limits_{r=n+1}^{\infty} \frac{1}{r^2}}} > \frac{\frac{1}{2}}{1 + \frac{2}{n}} \geq \frac{\frac{1}{2}}{1 + \frac{2}{n}} > \frac{1}{2}\left(1 - \frac{2}{N}\right) > \frac{1}{2} - \varepsilon.$$

Thus we have $\frac{1}{2} - \varepsilon < \frac{m(S \cap I'')}{m(I'')} < \frac{1}{2} + \varepsilon$. A similar argument shows $\frac{1}{2} - \varepsilon < \frac{m(S \cap I')}{m(I')} < \frac{1}{2} + \varepsilon$. But these last two inequalities, together with the lemma, give us

$$\frac{1}{2} - \varepsilon < \frac{m(S \cap I)}{m(I)} < \frac{1}{2} + \varepsilon$$

whenever I is an open interval with radius less than λ' and $0 \varepsilon I$. Thus $D_s(0) = \frac{1}{2}$.

<u>Theorem</u> 2-9. If R is a metric space and $x_o \varepsilon R$, then $D_R(x_o) = 1$.

Proof: Suppose $\delta > 0$. Suppose $x_o \varepsilon N(p,\delta)$. $R \cap N(p,\delta) = N(p,\delta)$, therefore

$$\frac{m(R \cap N(p,\delta))}{m(N(p,\delta))} = \frac{m(N(p,\delta))}{m(N(p,\delta))} = 1$$

and it follows that $D_R(x_o) = 1$.

Theorem 2-10. If S is a measurable subset of R, $x \varepsilon R$, then $U_S(x) + L_{\mathcal{C}S}(x) = 1$ and $L_S(x) + U_{\mathcal{C}S}(x) = 1$. Further, if $D_S(x)$ exists, then $D_{\mathcal{C}S}(x)$ exists, and $D_S(x) + D_{\mathcal{C}S}(x) = 1$.

Proof: Suppose $p \varepsilon R$. Then $N(p,\delta) = (S \cap N(p,\delta)) \cup (\mathcal{C}S \cap N(p,\delta))$. Since $(S \cap N(p,\delta)) \cap (\mathcal{C}S \cap N(p,\delta)) = \Phi$, we have

$$\frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} + \frac{m(\mathcal{C}S \cap N(p,\delta))}{m(N(p,\delta))} = 1 ;$$

therefore

$$\underset{\substack{p \varepsilon R \\ x_0 \varepsilon N(p,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(S \cap N(p,\delta))}{m(N(p,\delta))} = \underset{\substack{p \varepsilon R \\ x_0 \varepsilon N(p,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \left[1 - \frac{m(\mathcal{C}S \cap N(p,\delta))}{m(N(p,\delta))} \right]$$

$$= 1 + \underset{\substack{p \varepsilon R \\ x_0 \varepsilon N(p,\delta)) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \left[- \frac{m(\mathcal{C}S \cap N(p,\delta))}{m(N(p,\delta))} \right]$$

$$= 1 - \underset{\substack{p \varepsilon R \\ x_0 \varepsilon N(p,\delta) \\ 0<\delta<\lambda}}{\text{g.l.b.}} \frac{m(\mathcal{C}S \cap N(p,\delta))}{m(N(p,\delta))} .$$

Thus we have $u_S(x_0, \lambda) = 1 - l_{\mathcal{C}S}(x_0, \lambda)$.

But $U_S(x_0) = \underset{\lambda>0}{\text{g.l.b.}}\, u_S(x_0, \lambda) = \underset{\lambda>0}{\text{g.l.b.}}\, (1 - l_{\mathcal{C}S}(x_0, \lambda))$

$= 1 - \underset{\lambda>0}{\text{l.u.b.}}\, l_{\mathcal{C}S}(x_0, \quad) = 1 - L_{\mathcal{C}S}(x_0)$.

Similarly $L_S(x_0) + U_{\mathcal{C}S}(x_0) = 1$, since $U_{\mathcal{C}S}(x_0) + L_{\mathcal{C}(\mathcal{C}S)}(x_0) = 1$, from the preceding result.

Theorem 2-11. For any $p \varepsilon R$, $\lambda > 0$,

$u_{A \cup B}(p, \lambda) \leq u_A(p, \lambda) + u_B(p, \lambda)$.

Proof: Suppose $p \varepsilon N(a,\delta)$, $0 < \delta < \lambda$. Then

$$\frac{m((A \cup B) \cap N(q,\delta))}{m(N(q,\delta))} = \frac{m(A \cap N(a,\delta) \cup (B \cap N(q,\delta))}{m(N(q,\delta))}$$

$$\leq \frac{m(A \cap N(q,\delta))}{m(N(q,\delta))} + \frac{m(B \cap N(q,\delta))}{m(N(q,\delta))} .$$

As a further consequence we have $U_{A \cup B}(p) \leq U_A(p) + U_B(p)$. For suppose this were not the case, i.e., $U_{A \cup B}(p) > U_A(p) + U_B(p)$. Then there exists $\alpha > 0$ such that $\underset{\lambda > 0}{g.l.b.} u_{A \cup B}(p, \lambda) - \alpha > \underset{\lambda > 0}{g.l.b.} u_A(p, \lambda) + \underset{\lambda > 0}{g.l.b.} u_B(p, \lambda)$,

and hence there exist λ_1, λ_2 such that $\underset{\lambda > 0}{g.l.b.}\ u_{A \cup B}(p, \lambda) > u_A(p, \lambda_1) + u_B(p_1, \lambda_2)$. Let $\lambda * = \min(\lambda_1, \lambda_2)$. Then

$$u_{A \cup B}(p, \lambda *) \geq \underset{\lambda > 0}{g.l.b.} u_{A \cup B}(p, \lambda) > u_A(p, \lambda *) + u_B(p, \lambda *),$$

a contradiction.

Some further observations we are able to make are that if $D_A(x_o)$, $D_B(x_o)$, and $D_{A \cup B}(x_o)$ exist, then $D_{A \cup B}(x_o) \leq D_A(x_o) + D_B(x_o)$. To see this we need only observe that

$$D_{A \cup B}(x_o) = U_{A \cup B}(x_o) \leq U_A(x_o) + U_B(x_o) = D_A(x_o) + D_B(x_o).$$

Further, under the same conditions,

$$L_{A \cup B}(x_o) \leq L_A(x_o) + L_B(x_o).$$

<u>Theorem</u> 2-12. Suppose $A, B \varepsilon \mathcal{L}$ and $A \cap B = \phi$. Suppose $D_A(x_o)$ and $D_B(x_o)$ exist and are equal to a and b respectively. Then $D_{A \cup B}(x_o)$ exists and is equal to a + b.

Proof: Suppose $\varepsilon > 0$. Since $D_A(x_o) = a$, by theorem 2-8 there exists $\lambda_1 > 0$ such that if $p \varepsilon R$,

$x_o \varepsilon N(p,\delta)$, and

$0 < \delta < \lambda_1$, then

$$a - \frac{\varepsilon}{2} < \frac{m(A \cap N(p,\delta))}{m(N(p,\delta))} < a + \frac{\varepsilon}{2}.$$

Similarly, since $D_B(x_o) = b$, there exists $\lambda_2 > 0$ such that if $p \varepsilon R$, $x_o \varepsilon N(p,\delta)$, and $0 < \delta < \lambda_2$, then

$$b - \frac{\varepsilon}{2} < \frac{m(B \cap N(p,\delta))}{m(N(p,\delta))} < b + \frac{\varepsilon}{2} .$$

Let $\lambda = \min(\lambda_1, \lambda_2)$. Then if $\delta < \lambda$, we have

$$(a+b) - \varepsilon < \frac{m(A \cap N(p,\delta))}{m(N(p,\delta))} + \frac{m(B \cap N(p,\delta))}{m(N(p,\delta))}$$

$$= \frac{m((A \cup B) \cap N(p,\delta))}{m(N(p,\delta))} < (a+b) + \varepsilon .$$

Therefore by theorem 2-8, $D_{A \cup B}(x_o) = a+b$.

Another useful result is the following

<u>Theorem</u> 2-13. Suppose $A,B \varepsilon \mathcal{L}$, $A \subset B$, and $D_A(x_o) = a$ and $D_B(x_o) = b$. Then $D_{B-A}(x_o) = b-a$.

Proof: Since $B = A \cup (B-A)$, we have

$B \cap N(p,\delta) = (A \cap N(p,\delta)) \cup ((B-A) \cap N(p,\delta))$, and hence

$$\frac{m(B \cap N(p,\delta))}{m(N(p,\delta))} - \frac{m(A \cap N(p,\delta))}{m(N(p,\delta))} = \frac{m((B-A) \cap N(p,\delta))}{m(N(p,\delta))}$$

Using theorem 2-8, we can find a $\lambda > 0$ such that if

$0 < \delta < \lambda$, $p \varepsilon R$, $x_o \varepsilon N(p,\delta)$, then

$(b-a) - \varepsilon < \dfrac{m((B-A) \cap N(p,\delta))}{m(N(p,\delta))} < (b-a) + \varepsilon$, and thus

$D_{B-A}(x_o) = b-a$.

<u>Theorem</u> 2-14. Suppose $F,B \varepsilon \mathcal{L}$, $S = F \cup B$, and $F \cap B = \phi$. Suppose $D_S(x_o) = k$. Then $D_F(x_o)$ exists and $D_F(x_o) = k$ if and only if $D_B(x_o)$ exists and $D_B(x_o) = 0$.

Proof: Suppose $D_F(x_o)$ exists and $D_F(x_o) = k$.

Since $B = S-F$ and $F \subset S$, $D_B(x_o) = D_{S-F}(x_o) = D_S(x_o) - D_F(x_o) = k-k = 0$. Conversely, suppose $D_B(x_o)$ exists and $D_B(x_o) = 0$. Since $F = S-B$ and $B \subset S$, we have $D_F(x_o) = D_{S-B}(x_o) = D_S(x_o) - D_B(x_o) = k - 0 = k$.

At some point in the preceding discussion, one might ask the following question: If $D_A(x_o)$ exists and $D_B(x_o)$ exists, does it necessarily follow that $D_{A \cup B}(x_o)$ exists? The answer, suprisingly enough, is no. To see that this is the case, let us appeal again to example 2-3.

Let $A = S$ and define $B = \{x | x \varepsilon A, x > 0\} \cup \{x | x \varepsilon A, x < 0\}$. Then we have $D_A(0) = \frac{1}{2}$, $D_B(0) = \frac{1}{2}$. However, $U_{A \cup B}(0) = 1$ and $L_{A \cup B}(0) = \frac{1}{2}$, so that the metric density of $A \cup B$ does not exist at 0.

<u>Theorem</u> 2-15. Every point is a point of dispersion of ϕ.

Proof: If $x \varepsilon R$, then $D_R(x) = 1$, and $D_R(x) + D_\phi(x) = 1$, so $D_\phi(x) = 0$.

<u>Theorem</u> 2-16. If p is a point of dispersion of S, then p is a point of density of $\complement S$ and conversely.

Proof: This is a direct consequence of definition 2-5 and theorem 2-10.

In the remaining theorems of this section some properties of points of density and dispersion will be given in relation to unions, intersections, and inclusions of sets of $\mathcal{L}$.

<u>Theorem</u> 2-17. Suppose $p \varepsilon R$ and A and B are sets of $\mathcal{L}$. If $D_A(p) = 0$ and $D_B(p) = 0$, then $D_{A \cup B}(p) = 0$.

Proof: Suppose $p \varepsilon R$, $\lambda > 0$. Then

$u_{A \cup B}(p, \lambda) \leq u_A(p, \lambda) + u_B(p, \lambda)$.

Suppose $\varepsilon > 0$. There exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $u_A(p, \lambda_1) < \frac{\varepsilon}{2}$ and $u_B(p, \lambda_2) < \frac{\varepsilon}{2}$. Let $\lambda_3 = \min \lambda_1, \lambda_2$. Recalling that u_S is an increasing function of λ we have $u_{A \cup B}(p, \lambda_3) \leq u_A(p, \lambda_3) + u_B(p, \lambda_3) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore $U_{A \cup B}(p) = \underset{\lambda > 0}{\text{g.l.b.}}\ u_{A \cup B}(p, \lambda) < \varepsilon$. But ε is arbitrary, hence $U_{A \cup B}(p) = 0$ and the conclusion is $D_{A \cup B}(p) = 0$ as was desired.

<u>Corollary</u>. Suppose $p \varepsilon R$. If $A_i \varepsilon \mathcal{L}$ and $D_{Ai}(p) = 0$, $i = 1,2,---,n$, then $D_{\bigcup_{i=1}^{n} A_i}(p) = 0$.

Proof: The proof may be accomplished by induction.

<u>Theorem</u> 2-18. Suppose $p \varepsilon R$ and A, $B \varepsilon \mathcal{L}$. If $D_A(p) = 1$ and $D_B(p) = 1$, then $D_{A \cap B}(p) = 1$.

Proof: If p is a point of density of both A and B, then by theorem 2-16 p is a point of dispersion of both $\mathcal{C}A$ and $\mathcal{C}B$. By theorem 2-17, p is a point of dispersion of $\mathcal{C}A \cup \mathcal{C}B = \mathcal{C}(A \cap B)$. Applying theorem 2-16 again, we see that p is a point of density of $A \cap B$, i.e., $D_{A \cap B}(p) = 1$.

Corollary. Suppose $p \varepsilon R$. If $A_i \varepsilon \mathcal{L}$, and $D_{Ai}(p) = 1$, $i = 1,2,---,n$, then $D_{\bigcap_{i=1}^{n} Ai}(p) = 1$.

Proof: The proof is by induction and is analogous to the proof of the corollary of theorem 2-17.

Consideration of the following example will show that theorem 2-17 does not hold for countably infinite unions. Suppose R is the set of all reals and suppose m is Lebesgue measure and $p\varepsilon R$. Define

$$A_n = [p - \frac{1}{2^{n-1}}, p - \frac{1}{2^n}] \cup [p + \frac{1}{2^n}, p + \frac{1}{2^{n-1}}]$$

For each n, $D_{A_n}(p) = 0$; however $\bigcup_{n=1}^{\infty} A_n = [p-1, p+1] - \{p\}$ and p is a point of density of $\bigcup_{n=1}^{\infty} A_n$.

Similarly if we define $A_n = [p - \frac{1}{n}, p + \frac{1}{n}]$, we see that $D_{A_n}(p) = 1$ for each n but that $D_{\bigcap_{n=1}^{\infty} A_n}(p) = 0$, since $\bigcap_{n=1}^{\infty} A_n = p$. Thus we see that theorem 2-18 does not hold for countable intersections.

Theorem 2-19. Suppose $p\varepsilon R$ and $A, B\varepsilon \mathcal{L}$. If $A \supset B$, then $U_B(p) \leq U_A(p)$ and $L_B(p) \leq L_A(p)$. Hence if $D_A(p) = 0$, then $D_B(p) = 0$, while if $D_B(p) = 1$, then $D_A(p) = 1$.

Proof: Suppose $p\varepsilon N(q,\delta)$, $q\varepsilon R$. Since $A \cap N(q,\delta) \supset B \cap N(q,\delta)$ and

$$\frac{m(A \cap N(q,\delta))}{m(N(q,\delta))} \geq \frac{m(B \cap N(q,\delta))}{m(N(q,\delta))}$$, it follows that

$$u_A(p, \lambda) = \underset{\substack{p\varepsilon N(q,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(A \cap N(q,\delta))}{m(N(q,\delta))} \geq \underset{\substack{p\varepsilon N(q,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(B \cap N(q,\delta))}{m(N(q,\delta))}$$

$$= u_B(p, \lambda).$$

Hence $U_A(p) \leq U_B(p)$. Similarly $L_A(p) \geq L_B(p)$. Therefore if $D_A(p) = 0$, it follows that $D_B(p) = 0$.

On the other hand, it is also easily shown that if $D_B(p) = 1$, and if $B \subset A$, then $D_A(p) = 1$.

<u>Theorem</u> 2-20. Suppose $x_o \varepsilon R$ and S and A are measurable subsets of R. If $D_S(x_o) = 1$ and $D_{\complement S \cup A}(x_o) = 1$, then $D_A(x_o) = 1$, i.e., $D_{\complement S \cup A}(x_o) = D_S(x_o) + D_A(x_o)$ provided $D_{\complement S \cup A}(x_o) = D_S(x_o) = 1$.

Proof: Suppose $D_{\complement S \cup A}(x_o) = 1$. $D_S(x_o) = 1$ implies $D_{\complement S}(x_o) = 0$. Since $\complement S \cup (A \cap S) = \complement S \cup A$ and $\complement S \cap (A \cap S) = \Phi$, by theorem 2-12 we have

$$D_{\complement S \cup A}(x_o) = D_{\complement S \cup (A \cap S)}(x_o) = D_{\complement S}(x_o) + D_{A \cap S}(x_o)$$
$$= 0 + D_{A \cap S}(x_o) = 1.$$

But $A \cap S \subset A$. Therefore $D_A(x_o) = 1$.

<u>Theorem</u> 2-21. Suppose $A, B \varepsilon \mathcal{L}$. Suppose x_o is not a point of dispersion of B. Suppose $D_A(x_o) = 1$. Then $U_{A \cap B}(x_o) > 0$, i.e., x_o is not a point of dispersion of $A \cap B$.

Proof: Since $D_A(x_o) = 1$, $L_A(x_o) = 1$ and $U_{\complement A}(x_o) = 0$. Also since $\complement A \cap B \subset \complement A$, $U_{\complement A \cap B}(x_o) = 0$. By assumption, x_o is not a point of dispersion of B, so $U_B(x_o) > 0$. But now since $B = (A \cap B) \cup (\complement A \cap B)$, $0 < U_B(x_o) \leq U_{A \cap B}(x_o) + U_{\complement A \cap B}(x_o) = U_{A \cap B}(x_o)$. Therefore $U_{A \cap B}(x_o) > 0$ as was desired.

Section II

<u>Definition</u> 2-5. Suppose $S \varepsilon \mathcal{L}$, $p \varepsilon R$. Then p is said to be a <u>point of density</u> of S if the metric density of S exists at p and is equal to one. If the metric

density of S exists at p and is equal to zero, p is said to be a <u>point of dispersion</u> of S.

In the remainder of this chapter and in chapter III, it is again recalled that S is a measurable subset of R, and that f is a real-valued measurable function defined on S, i.e., $\{x|x\varepsilon S,\ f(x) > a\}\ \varepsilon\ \mathcal{L}$ for all real a. One can show that the following conditions on f are equivalent to measurability:

(1) $\{x|x\varepsilon S,\ f(x) < a\}\ \varepsilon\ \mathcal{L}$ for all real a;

(2) $\{x|x\varepsilon S,\ f(x) \geq a\}\ \varepsilon\ \mathcal{L}$ for all real a;

(3) $\{x|x\varepsilon S,\ f(x) \leq a\}\ \varepsilon\ \mathcal{L}$ for all real a.

<u>Definition</u> 2-6. Suppose y is real. Define $J(x_o) = \{y,|y$ real, x_o is not a point of dispersion of $C(y)\}$ (recall, for y real, $C(y) = \{s|s\varepsilon S,\ f(x) > Y\}$); and $I(x_o) = \{y,|y$ real, x_o is not a point of dispersion of $D(y)\}$ (recall, for y real, $D(y) = \{x|x\varepsilon S,\ f(x) < y\}$).

<u>Theorem</u> 2-22. Suppose a and b are real. (1) If $b < a$ and $a\varepsilon J(x_o)$, then $b\varepsilon J(x_o)$ and (2), if $a < b$ and $a\varepsilon I(x_o)$, then $b\varepsilon I(x_o)$.

Proof: (1) Suppose $b < a$ and $a\varepsilon J(x_o)$. Then $C(b)\supset C(a)$. If x_o is not a point of dispersion of C(a), then x_o is not a point of dispersion of C(b), i.e., if $a\varepsilon J(x_o)$, then $b\varepsilon J(x_o)$.

The proof of (2) is similar to the proof of (1).

It is seen by the above result that $J(x_o)$ and $I(x_o)$ are characterized the same as $G(x_o)$ and $P(x_o)$ respectively, where we recall that for $x_o \varepsilon R$, $G(x_o) = \{y | y \text{ real}, x_o \text{ is a limit point of } C(y)\}$, and $P(x_o) = \{y | y \text{ real}, x_o \text{ is a limit point of } D(y)\}$. In particular $J(x_o)$ assumes exactly one of the following forms:

(1) $(-\infty, \infty)$;

(2) ϕ ;

(3) $(-\infty, r)$, r real; or

(4) $(-\infty, r]$, r real.

An analogous characterization holds for $I(x_o)$.

<u>Theorem</u> 2-23. Suppose $p \varepsilon R$ and p is not a limit point of S. Then p is a point of dispersion of S, and hence $J(x_o) \subset G(x_o)$.

Proof: Suppose $\varepsilon > 0$. If p is not a limit point of S, there exists an open spherical neighborhood $N(p, \lambda)$ such that $N(p, \lambda) \cap S$ contains at most one point. Suppose $\delta < \frac{\lambda}{2}$, $q \varepsilon R$, and $p \varepsilon N(q,\delta)$. Then $N(q,\delta) \subset N(p, \lambda)$. We also have $m(N(q,\delta) \cap S) = 0$ and hence $\frac{m(N(q,\delta) \cap S)}{m(N(q,\delta))} = 0$.

$$\text{Thus } u_S(p,\tfrac{\lambda}{2}) = \underset{\substack{p \varepsilon N(q,\delta) \\ 0<\delta<\frac{\lambda}{2}}}{\text{l.u.b.}} \frac{m(S \cap N(q,\delta))}{m(N(q,\delta))} = 0 \text{ and } u_S(p) = 0.$$

This implies that $D_S(p)$ exists and is equal to zero, since $0 \leq L_S(p) \leq U_S(p) = 0$. Hence p is a point of dispersion of S. Equivalently, if p is not a point of dispersion of S, then p is a limit point of S. In particular, if $S = C(y)$, then $J(x_o) \subset G(x_o)$.

Section III

In this section we will again suppose R is a separable, dense-in-itself metric space, S is a measurable subset of R, and f is a real-valued measurable function defined on S.

Definition 2-7. Suppose $x_0 \varepsilon R$ and x_0 is not a point of dispersion of S. We define l.u.b. $\{y \mid y \text{ real}, x_0$ is not a point of dispersion of $C(y)\}$, i.e., l.u.b. $J(x_0)$, to be the approximate limit superior of f(x) at x_0 and write $\lim_{x \to x_0} \text{ap sup } f(x)$. Similarly we define g.l.b. $\{y \mid y \text{ real}, x_0$ is not a point of dispersion of $D(y)\}$, i.e., g.l.b. $I(x_0)$, to be the approximate limit inferior of f(x) at x_0 and write $\lim_{x \to x_0} \text{ap inf } f(x)$.

Definition 2-8. Suppose x_0 is not a point of dispersion of S. If $\lim_{x \to x_0} \text{ap inf } f(x)$ and $\lim_{x \to x_0} \text{ap sup } f(x)$ are equal, then the approximate limit of f(x) is said to exist at x_0 and is defined to be the common value. We write $\lim_{x \to x_0} \text{ap } f(x)$ for the approximate limit of f at x_0, if it exists.

By considering the following function, we will find that the approximate limit may exist, while the ordinary limit does not exist. Let R = S = [0,1]. Let

$x_o = \frac{1}{2}$ and let m be Lebesgue measure. Define

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational, } x\varepsilon[0,1], \\ 0 \text{ if } x \text{ is irrational, } x\varepsilon[0,1] . \end{cases}$$

The ordinary limit of f(x) as $x \to \frac{1}{2}$ does not exist, since

$$\limsup_{x \to \frac{1}{2}} f(x) = \underset{\delta>0}{\text{g.l.b.}} \ \underset{\substack{x\varepsilon[0,1] \\ x\varepsilon N(\frac{1}{2},\delta)}}{\text{l.u.b.}} f(x) = 1, \text{ and}$$

$$\liminf_{x \to \frac{1}{2}} f(x) = \underset{\delta>0}{\text{l.u.b.}} \ \underset{\substack{x\varepsilon[0,1] \\ x\varepsilon N(\frac{1}{2},\delta)}}{\text{g.l.b.}} = 0.$$

However $\underset{x \to \frac{1}{2}}{\lim \text{ ap sup}} f(x) = \text{l.u.b. } J(x_o = \text{l.u.b.}(-\infty,0) = 0,$

and $\underset{x \to \frac{1}{2}}{\lim \text{ ap inf}} f(x) = \text{g.l.b. } I(x_o) = \text{g.l.b.}(0,\infty) = 0.$

Therefore $\underset{x \to \frac{1}{2}}{\lim \text{ ap}} f(x) = 0.$

It is an immediate consequence of theorem 2-16 that $\underset{x \to x_o}{\lim \text{ ap sup}} f(x) \leq \underset{x \to x_o}{\lim \text{ sup}} f(x)$. Analogously it may be shown that $\underset{x \to x_o}{\lim \text{ ap inf}} f(x) \geq \underset{x \to x_o}{\lim \text{ inf}} f(x)$.

<u>Theorem</u> 2-23. Suppose f is defined on S and x_o is not a point of dispersion of S. Then $\underset{x \to x_o}{\lim \text{ ap inf}} f(x)$ $\leq \underset{x \to x_o}{\lim \text{ ap sup}} f(x)$, i.e., g.l.b. $I(x_o) \leq$ l.u.b. $J(x_o)$.

Proof: Deny the theorem, i.e., suppose g.l.b. $I(x_o) >$ l.u.b.$J(x_o)$. Let g.l.b.$I(x_o) = X$ and l.u.b.$J(x_o) = Y$. Let a and b be real numbers such that $Y < a < b < X$.

Since $a \notin J(x_o)$, x_o is a point of dispersion of $C(a)$. Also $b \notin I(x_o)$, so x_o is a point of dispersion of $D(b)$. Now $C(a) \cup D(b) = S$, and since x_o is a point of dispersion of both $C(a)$ and $D(b)$, it is also a point of dispersion of the union, and hence of S. This is a contradiction and thus the theorem is verified.

Theorem 2-24. Suppose x_o is not a point of dispersion of S. If $\lim_{x \to x_o} f(x)$ exists, then $\lim_{x \to x_o} \text{ap } f(x)$ exists and is equal to $\lim_{x \to x_o} f(x)$.

Proof: Since $\lim_{x \to x_o} f(x)$ exists we have

$$\lim_{x \to x_o} \text{ap sup } f(x) \leq \lim_{x \to x_o} \sup f(x) = \lim_{x \to x_o} \inf f(x)$$

$$\leq \lim_{x \to x_o} \text{ap inf } f(x).$$

In the next few theorems $\lim_{x \to x_o} \text{ap sup } f(x)$, $\lim_{x \to x_o} \text{ap inf } f(x)$, and $\lim_{x \to x_o} \text{ap } f(x)$ will be characterized in terms of the sets $C(\alpha)$ and $D(\beta)$.

Definition 2-9. Define $E(y,\varepsilon) = \{x \mid x \varepsilon S, |f(x)-y| < \varepsilon\}$, where y is real

From this definition it is seen that $E(y,\varepsilon) = D(y+\varepsilon) \cap C(y-\varepsilon)$.

Theorem 2-25. Suppose A is real and $D_S(x_o) = k > 0$. Then $\lim_{x \to x_o} \text{ap } f(x) = A$ if and only if $D_{E(A,\varepsilon)}(x_o) = k$ for all $\varepsilon > 0$.

Proof: Suppose $\varepsilon>0$. Suppose $D_S(x_o)=k$ and $\lim_{x \to x_o} \text{ap} f(x)=A$.
Now $A + \frac{\varepsilon}{2} \notin J(x_o)$, so x_o is a point of dispersion of $C(A+\frac{\varepsilon}{2})$, i.e., $D_{C(A+\frac{\varepsilon}{2})}(x_o) = 0$. Similarly $D_{D(A-\frac{\varepsilon}{2})}(x_o) = 0$. But $S - E(A,\varepsilon) \subset D(A-\frac{\varepsilon}{2}) \cup C(A+\frac{\varepsilon}{2})$, therefore $D_{S-E(A,\varepsilon)}(x_o) = 0$. On the other hand $D_{S-E(A,\varepsilon)}(x_o) = D_S(x_o) - D_{E(A,\varepsilon)}(x_o)$, therefore $D_{E(A,\varepsilon)}(x_o) = k$.

Conversely, suppose A real, $D_S(x_o) = k$, and $D_{E(A,\varepsilon)}(x_o) = k$ for all $\varepsilon>0$. Then $D_{S-E(A,\varepsilon)}(x_o) = 0$. Now $(S - E(A,\varepsilon)) \supset D(A-\varepsilon) \cup C(A+\varepsilon)$ and hence $D_{D(A-\varepsilon)}(x_o)$ $= D_{C(A+\varepsilon)}(x_o) = 0$. Therefore $A + \varepsilon \notin J(x_o)$, $A - \varepsilon \notin I(x_o)$, and hence $\lim_{x \to x_o} \text{ap } f(x) = A$.

Corollary. Suppose A is real and x_o is a point of density of S. Then $\lim_{x \to x_o} \text{ap } f(x)$ exists and is equal to A if and only if x_o is a point of density of $E(A,\varepsilon)$ for every $\varepsilon>0$.

<u>Definition</u> 2-10. Define
$K(x_o) = \{B \mid B \subset R,\ B \varepsilon\ \ ,$ and x_o is a point of density of $B\}$.

<u>Theorem</u> 2-26. Suppose x_o is a point of density of S. Then $\lim_{x \to x_o} \text{ap sup } f(x) \leq A$ if and only if $D(A+\varepsilon) \varepsilon K(x_o)$ for all $\varepsilon>0$.

Proof: Suppose $\lim_{x \to x_o} \text{ap sup } f(x) \leq A$.
l.u.b. $J(x_o) \leq A$, so A is an upper bound for $J(x_o)$. Thus $A + \varepsilon/2\ \varepsilon\ J(x_o)$ for all $\varepsilon>0$ and x_o is a point of dispersion of $C(A+\varepsilon/2)$. $C(A+\varepsilon/2) \supset D(A+\varepsilon) \cap S$, therefore x_o is a point

of dispersion of $\mathcal{C}D(A+\varepsilon)\cap S$ and a point of density of $\mathcal{C}(S\cap \mathcal{C}D(A+\varepsilon))$, and hence, by theorem 2-20, a point of density of $D(A+\varepsilon)$, i.e., $D(A+\varepsilon)\varepsilon K(x_o)$ for all ε, and the necessity is proved.

Conversely, suppose $D(A+\varepsilon)K(x_o)$ for all $\varepsilon>0$. $D(A+\varepsilon)\subset \mathcal{C}C(A+\varepsilon)$ and we conclude x_o is a point of dispersion of $C(A+\varepsilon)$ for all $\varepsilon>0$. $A+\varepsilon \not\in J(x_o)$, therefore

$$\lim_{x\to x_o} \text{ap sup } f(x) = \text{l.u.b. } J(x_o) \leq A+\varepsilon,$$

and $\lim_{x\to x_o} \text{ap sup } f(x) \leq A$.

A result analogous to the preceding theorem is: $\lim_{x\to x_o} \text{ap inf } f(x) \leq A$ if and only if $C(A-\varepsilon)\varepsilon K(x_o)$ for all $\varepsilon>0$, assuming x_o is a point of density of S.

<u>Theorem</u> 2-27. Suppose x_o is a point of density of S. Then $\lim_{x\to x_o} \text{ap sup } f(x) \geq A$ if and only if whenever $G\varepsilon K(x_o)$, $\varepsilon>0$, it follows that $S\cap G\cap C(A-\varepsilon) \neq \Phi$.

Proof: Let us first deny the necessity, i.e., suppose there exists $G\varepsilon K(x_o)$, $\varepsilon>0$, and suppose $S\cap G\cap C(A-\varepsilon) = \Phi$. Suppose $x_1 \varepsilon S\cap G$. Then $x_1 \not\varepsilon C(A-\varepsilon)$, so $x_1 \varepsilon\ \ C(A-\varepsilon)$; hence $G\cap S\subset \mathcal{C}C(A-\varepsilon)\cap S$. Now x_o is a point of density of each set on the left, hence it is a point of density of $\mathcal{C}C(A-\varepsilon)\cap S$ and a point of dispersion of $C(A-\varepsilon)$. Therefore $A-\varepsilon \not\in J(x_o)$ and $A-\varepsilon$ is an upper bound for $J(x_o)$ which implies

$$\lim_{x\to x_o} \text{ap sup } f(x) \leq A-\varepsilon < A,$$

a contradiction. This proves the necessity.

For the sufficiency, let us again deny the result. Suppose for all $G \varepsilon K(x_o)$ and all $\varepsilon > 0$, $G \cap S \cap C(A-\varepsilon) \neq \Phi$ and suppose $\lim_{x \to x_o} \text{ap sup } f(x) < A$. Choose $\varepsilon > 0$ such that $\lim_{x \to x_o} \text{ap sup } f(x) < A - 2\varepsilon$. Then $A-\varepsilon \notin J(x_o)$. Therefore x_o is a point of dispersion of $C(A-\varepsilon)$ and a point of density of $\mathcal{C}C(A-\varepsilon)$. Thus $\mathcal{C}C(A-\varepsilon) \varepsilon K(x_o)$. Therefore $S \cap \mathcal{C}C(A-\varepsilon) \cap C(A-\varepsilon) \neq \Phi$, a contradiction, and $\lim_{x \to x_o} \text{ap sup } f(x) \geq A$ as asserted.

In a similar manner we can prove the following two theorems.

<u>Theorem</u> 2-28. Suppose x_o is a point of density of S. Then $\lim_{x \to x_o} \text{ap sup } f(x) \geq A$ if and only if whenever $G \varepsilon K(x_o)$, $\varepsilon > 0$, $\delta > 0$, it follows that $S \cap N(x_o, \delta) \cap G \cap C(A-\varepsilon) \neq \Phi$.

<u>Theorem</u> 2-29. Suppose x_o is a point of density of S. Then $\lim_{x \to x_o} \text{ap sup } f(x) \geq A$ if and only if whenever $G \varepsilon K(x_o)$, $\varepsilon > 0$, $\delta > 0$, it follows that $S \cap N^*(x_o, \delta) \cap G \cap C(A-\varepsilon) \neq \Phi$ (recall $N^*(x_o, \delta)$ is a deleted spherical δ-neighborhood of x_o).

We see that in theorem 2-29 we have a result analogous to the result found in chapter I for $\lim_{x \to x_o} \sup f(x) \geq A$. Namely, $\lim_{x \to x_o} \sup f(x) \geq A$ if and only if for every $\varepsilon > 0$, $\delta > 0$, it follows that $S \cap N^*(x_o, \delta) \cap C(A-\varepsilon) \neq \Phi$.

<u>Theorem</u> 2-30. Suppose A is real and f is a

measurable function. Suppose there exists k, $1 \leq k < +\infty$, such that if $q \varepsilon R$, then $m(N(q,5r)) \leq km(N(q,r))$. Suppose $D_S(x_o) = \ell > 0$. Then $\lim\limits_{x \to x_o} ap\, f(x) = A$ if and only if there exists F such that $F \varepsilon \mathcal{L}, F \subset S$, and $D_F(x_o) = \ell$ and $\lim\limits_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = A$.

Proof: Let us first prove the sufficiency. Suppose there exists F such that $D_F(x_o) = \ell$ and $\lim\limits_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = A$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \varepsilon N(x_o, \delta) \cap F$, then $|f(x) - A| < \varepsilon$. Thus $N(x_o \delta) \cap F \subset E(A, \varepsilon) \subset S$. We first observe that $D_{F \cap N(x_o, \delta)}(x_o) = \ell$. For, $D_{N(x_o, \delta)}(x_o) = 1$, so $D_{\complement N(x_o, \delta)}(x_o) = 0$ and hence $D_{F \cap \complement N(x_o, \delta)}(x_o) = 0$. But now $\ell = D_F(x_o) = D_{F \cap N(x_o, \delta)}(x_o) + D_{F \cap\ N(x_o, \delta)}(x_o)$

$$= D_{F \cap N(x_o, \delta)}(x_o).$$

Next we observe that $D_{E(A, \varepsilon)}(x_o) = \ell$, since $U_{E(A, \varepsilon)}(x_o) \leq U_S(x_o) = \ell$ and $L_{E(A, \varepsilon)}(x_o) \geq L_{F \cap N(x_o, \delta)}(x_o) = \ell$. Therefore by theorem 2-25, $\lim\limits_{x \to x_o} ap\, f(x) = A$.

On the other hand suppose $\lim\limits_{x \to x_o} ap\, f(x) = A$. By Theorem 2-25, $D_{E(A, \varepsilon)}(x_o) = \ell$ for all $\varepsilon > 0$. In particular, $D_{E(A, \frac{1}{n})}(x_o) = \ell$ for every positive integer n. For each n there exists $\alpha_n > 0$ such that if $p \varepsilon R$, $0 < \delta < \alpha_n$, $x_o \varepsilon N(p, \delta)$ then $\dfrac{m(E(A, \frac{1}{n}) \cap N(p, \delta))}{m(N(p, \delta))} > \ell - \dfrac{1}{n}$.

Define a decreasing sequence $\{\gamma_n\}$ by

$\gamma_n = \min(\alpha_1, ---, \alpha_n)$. Define the sequence $\{\beta_n\}$ by $\beta_n = \frac{1}{n}\cdot\gamma_n$. It is easily shown that $\{\beta_n\}$ has the following properties:

(1) $\{\beta_n\}$ is a strictly decreasing sequence, $\beta_n \leqq \gamma_n \leqq \alpha_n$.

(2) $\lim_{n \to \infty} \beta_n = 0$.

(3) If $0 < \lambda < \beta_n$, if $p \varepsilon R$, and if $x_o \varepsilon N(p, \lambda)$, then
$$\frac{m(E(A,\frac{1}{n}) \cap N(p, \lambda))}{m(N(p, \lambda))} > \ell - \frac{1}{n} .$$

Associated with the sequence $\{\beta_n\}$ is the sequence $\{\eta_n\}$ where $\eta_n = m(N(x_o, \beta_n))$. $\{\eta_n\}$ is a decreasing sequence with $\eta_n > 0$, and $\lim_{n \to \infty} \eta_n = 0$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < \mu < \delta$, then $m(N(x_o, \mu)) < \varepsilon$. For each n there exists μ_n, with $\mu_{n-1} > \mu_n$, such that $m(N(x_o, \mu_n)) < \frac{\eta_n}{kn}$, and hence
$$\frac{m(N(x_o, \mu_n))}{m(N(x_o, \beta_n))} < \frac{\eta_n}{kn}\Big/\eta_n = \frac{1}{kn} .$$

Define $F_n = (N(x_o, \beta_n) - N(x_o, \mu_{n+1})) \cap E(A, \frac{1}{n})$ and $F = \bigcup_{n=1}^{\infty} F_n$. We first show $\lim_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = A$. Suppose $\varepsilon > 0$.

Choose N so that $\frac{1}{N} < \varepsilon$. Let $\delta = \mu_{N+1}$. Suppose $x \varepsilon N(x_o, \delta)$ and $x \varepsilon F$. If $N \geq n$, then $0 < \mu_{N+1} \leqq \mu_{n+1}$ and $x \not\varepsilon F_n$, i.e., $\rho(x, x_o) < \mu_{n+1}$. Hence there exists an $n > N$ such that $x \varepsilon F_n$. Therefore $x \varepsilon E(A, \frac{1}{n})$. Therefore $|f(x) - A| < \frac{1}{n} < \frac{1}{N} < \varepsilon$, i.e., $\lim_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = A$.

It remains to show $D_F(x_o) = \ell$. Suppose $\varepsilon > 0$ is given. Choose N so that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $\delta = \frac{\beta_N}{2}$. Suppose $x_o \varepsilon N(p, \lambda)$, $0 < \lambda < \delta$. There exists an m such that $\frac{\beta_{m+1}}{2} \leq \lambda < \frac{\beta_m}{2}$ and $m \geq N$. By the triangle inequality, $N(p, \lambda) \subset N(x_o, \beta_m)$. Hence,

$$F \cap N(p, \lambda) \supset F_m \cap N(p, \lambda)$$
$$= (N(x_o, \beta_m) - N(x_o, \mu_{m+1})) \cap E(A, \tfrac{1}{m}) \cap N(p, \lambda)$$
$$= (N(p, \lambda) - N(x_o, \mu_{m+1})) \cap E(A, \tfrac{1}{m}).$$

Also we have

$$N(p, \lambda) \cap E(A, \tfrac{1}{m})$$
$$\subset \left\{[N(p, \lambda) - N(x_o, \mu_{m+1})] \cap E(A, \tfrac{1}{m})\right\} \cup N(x_o, \mu_{m+1})$$
$$= (F_m \cap N(p, \lambda)) \cup N(x_o, \mu_{m+1}).$$

We now observe that $N(x_o, \beta_{m+1}) \subset N(p, 3\lambda)$, so $m(N(x_o, \beta_{m+1})) \leq m(N(p, 3\lambda)) \leq k m(N(p, \lambda))$.

Now

$$\ell - \frac{1}{m} < \frac{m(N(p, \lambda) \cap E(A, \frac{1}{m}))}{m(N(p, \lambda))}$$
$$\leq \frac{m(F_m \cap N(p, \lambda))}{m(N(p, \lambda))} + \frac{m(N(x_o, \mu_{m+1}))}{m(N(p, \lambda))} .$$

Also

$$\frac{m(N(x_o, \mu_{m+1}))}{m(N(p, \lambda))} < \frac{m(N(x_o, \beta_{m+1}))}{k(m+1)} \Bigg/ \frac{m(N(x_o, \beta_{m+1}))}{k}$$
$$= \frac{1}{m+1} < \frac{1}{N} .$$

From these last two inequalities we find

$$\frac{m(F_m \cap N(p, \lambda))}{m(N(p, \lambda))} + \frac{1}{N} > \ell - \frac{1}{N} \text{, and hence}$$

$$\frac{m(F \cap N(p, \lambda))}{m(N(p, \lambda))} \geq \frac{m(F_m \cap N(p, \lambda))}{m(N(p, \lambda))} > \ell - \frac{2}{N} > \ell - \varepsilon .$$

Finally $l_F(x_o,\delta) \geq \ell - \varepsilon$, and so $L_F(x_o) \geq \ell - \varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that $L_F(x_o) \geq \ell$. At the same time, since $F\subset S$, $U_F(x_o) \leq U_S(x_o) = \ell$. Thus $D_F(x_o)$ exists, $D_F(x_o) = \ell$, and the necessity is established.

Corollary 1. Under the conditions of the theorem, if $D_S(x_o) = 1$, then $\lim_{x \to x_o} \text{ap}\, f(x) = A$ if and only if there exists F, such that $F\subset S$, $F\varepsilon K(x_o)$, $\lim_{\substack{x \to x_o \\ x\varepsilon F}} f(x) = A$.

Corollary 2. Under the conditions of the theorem, $\lim_{x \to x_o} \text{ap}\, f(x) = A$ if and only if there exists $B\subset S$ such that $D_B(x_o) = 0$ and $\lim_{\substack{x \to x_o \\ x\varepsilon S-B}} f(x) = A$.

Proof: The proof is immediate. For the sufficiency, since $B\subset S$ and $D_B(x_o) = 0$, we have $D_{S-B}(x_o) = \ell$. Let F of the theorem be S-B. Then $\lim_{\substack{x \to x_o \\ x\varepsilon F}} f(x) = A$ and hence $\lim_{x \to x_o} \text{ap}\, f(x) = A$.

Conversely, suppose $\lim_{x \to x_o} \text{ap}\, f(x) = A$. Then we know there exists F such that $F\subset S$, $D_F(x_o) = \ell$, and $\lim_{\substack{x \to x_o \\ x\varepsilon F}} f(x) = A$. Let $B = S-F$. Then $D_B(x_o) = 0$, $F = S-B$, and $\lim_{\substack{x \to x_o \\ x\varepsilon S-B}} f(x) = A$.

We remark on the assumption made in the above theorem concerning the existence of a k such that

$m(N(q,5r) \leq km(N(q,r))$. For the purpose of proving the theorem, it would have been sufficient to assume there exists k', $1 \leq k' < \infty$, such that if $q \varepsilon R$, and if $r>0$, then such that $m(N(q,3r)) \leq k'm(N(q,r))$. In chapter III, it will be necessary to assume the existence of k, such that if $q \varepsilon R$, and if $r>0$, then $m(Nq,5r)) \leq km(N(q,r))$, and for this reason, 5 was used rather than 3.

Concerning the above remark, it is further pointed out that either the existence of k' or k as described above would be sufficient to prove the theorem.

For since $\dfrac{m(N(q,9r))}{m(N(q,3r))} \leq k'$, and $\dfrac{m(N(q,3r))}{m(N(q,r))} \leq k'$,

it follows that $\dfrac{m(N(q,5r))}{m(N(q,r))} \leq \dfrac{m(N(q,9r))}{m(N(q,r))} \leq k'^2$.

It is possible to establish the following stronger

<u>Theorem</u> 2-31. Suppose A is real and f is a measurable function. Suppose there exists k, $1 \leq k < +\infty$, such that if $q \varepsilon R$, then $m(Nq,5r)) \leq km(N(q,r))$. If x_o is not a point of dispersion of S, then $\lim_{x \to x_o} \text{ap } f(x) = A$ if and only if there exists B such that $B \varepsilon \mathfrak{A}$, $B \subset S$, $D_B(x_o) = 0$, and

$$\lim_{\substack{x \to x_o \\ x \varepsilon S-B}} f(x) = A.$$

Section IV

This section will be devoted to theorems and examples on approximate limits and approximate continuity. There will in many cases be analogues with the theorems and proofs for ordinary limits and continuity. For this reason, many of the proofs of the theorems will not be given. However,as might be expected, we will be able to find many cases where no analogue exists. For example, continuity on a compact set implies boundedness, but approximate continuity on a compact set does not insure boundedness.

As before, let us suppose R is separable, dense-in-itself, $S \in \mathcal{L}$, f and g are measurable functions defined on S, and that x_0 is not a point of dispersion of S.

<u>Theorem</u> 2-32. Suppose A and B are real and f and g are defined on S. If x_0 is not a point of dispersion of S, and if $\lim_{x \to x_0} \text{ap } f(x) = A$ and $\lim_{x \to x_0} \text{ap } g(x) = B$, then

$\lim_{x \to x_0} \text{ap } (f(x) \pm g(x)) = A \pm B.$

Proof: Since the approximate limits of f(x) and g(x) exist at x_0, there exist sets G and H contained in S such that $D_G(x_0) = D_H(x_0) = 0$ and

$$\lim_{\substack{x \to x_0 \\ x \in S-G}} f(x) = A \quad , \quad \lim_{\substack{x \to x_0 \\ x \in S-H}} g(x) = B.$$

But $(S-G) \cup (S-H) = S - (G \cup H)$ and $\lim_{\substack{x \to x_0 \\ x \in S-(G \cup H)}} (f(x)+g(x)) = A + B.$

Further we note that $D_{G \cup H}(x_0) = 0$ and hence by theorem 2-31,

$\lim_{x \to x_0} \text{ap } (f(x)+g(x)) = A + B.$

A similar argument may be given for the case

$\lim_{x \to x_0} \text{ap } (f(x)-g(x)).$

<u>Theorem</u> 2-33. Under the conditions of theorem 2-32, if $\lim_{x \to x_0} \text{ap } f(x) = A$ and $g(x) = cf(x)$, then $\lim_{x \to x_0} \text{ap } g(x) = cA.$

Theorem 2-34. Under the conditions of theorem 2-32, if $\lim ap\, f(x) = A$ and $\lim_{x \to x_o} ap\, g(x) = B$, then $\lim_{x \to x_o} ap\, f(x)g(x) = AB$.

Theorem 2-35. Under the conditions of theorem 2-32 and the assumptions $g(x) \neq 0$, $B \neq 0$, $\lim_{x \to x_o} ap\, \frac{f(x)}{g(x)} = \frac{A}{B}$.

Definition 2-11. Suppose $x_o \varepsilon S$. If x_o is a point of dispersion of S, we define f to be approximately continuous at x_o. If x_o is not a point of dispersion of S we define f to be approximately continuuous at x_o in case $\lim_{x \to x_o} ap\, f(x)$ exists and $\lim_{x \to x_o} ap\, f(x) = f(x_o)$.

In case $D_S(x_o) = \ell > 0$, $x_o \varepsilon S$ and the conditions of theorem 2-30 are satisfied, it is an immediate consequence that $\lim_{x \to x_o} ap\, f(x) = f(x_o)$ if and only if there exists $F \subset S$ such that $\lim_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = f(x_o)$, and $D_F(x_o) = \ell$.

Also, clearly, if $D_S(x_o) = \ell > 0$, if $x_o \varepsilon S$, then f is approximately continuous at x_o if and only if there exts B such that $B \subset S$, $B \varepsilon \mathcal{L}$, $D_B(x_o) = 0$, and $\lim_{\substack{x \to x_o \\ x \varepsilon S-B}} f(x) = f(x_o)$.

By utilizing theorem 2-31, one can establish that the following theorem is true.

Theorem 2-36. If $x_o \varepsilon S$, and if there exists k, $1 \leq k < +\infty$, such that if $q \varepsilon R$, $m(N(q,5r)) \leq km(N(q,r))$, then f is approximately continuous at x_o if and only if there exists B such that $B \varepsilon \mathcal{L}$, $B \subset S$, $D_B(x_o) = 0$, and

$$\lim_{\substack{x \to x_o \\ x \varepsilon (S-B) \cup \{x_o\}}} f(x) = f(x_o).$$

Example. Let $R = S = \{x \mid x \text{ is real}\}$. Let m be ordinary Lebesgue measure. Define

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational}, \\ 1 \text{ if } x \text{ is irrational}. \end{cases}$$

Case 1. Suppose x_o is irrational. Let

$F = \{x \mid x \varepsilon R, \ x \text{ irrational}\}$; $F \varepsilon K(x_o)$.

$$\lim_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = 1 = f(x_o).$$

Hence f(x) is approximately continuous at all irrational points.

Case 2. Suppose x_o is rational. Then $\lim_{\substack{x \to x_o \\ x \varepsilon F}} f(x) = 1$.

Thus $\lim_{x \to x_o} \text{ap } f(x) = 1$, but $f(x_o) = 0 \neq 1$, and thus f(x) is not approximately continuous at rational points.

This example illustrates that a function f(x) may be approximately continuous at a point, although it is not continuous there.

Theorem 2-37. Suppose x_o is not a point of dispersion of S. If f(x) and g(x) are approximately continuous at x_o, then the following functions are approximately continuous at x_o:

(1) $f(x) \pm g(x)$, (2) $f(x)g(x)$, (3) $\frac{f(x)}{g(x)}$ provided

$g(x) \neq 0$, $g(x_o) \neq 0$.

Proof: (1) Since $f(x)$ and $g(x)$ are approximately continuous at x_o, there exist sets G and H such that $D_G(x_o) = D_H(x_o) = 0$ and $\lim_{\substack{x \to x_o \\ x \in S-G}} (f(x) = f(x_o)$, $\lim_{\substack{x \to x_o \\ x \in S-H}} g(x) = g(x_o)$.

Now $\lim_{\substack{x \to x_o \\ x \in S-(G\cup H)}} (f(x) \pm g(x)) = \lim_{\substack{x \to x_o \\ x \in S-(G\cup H)}} f(x) \pm \lim_{\substack{x \to x_o \\ x \in S-(G\cup H)}} g(x) = f(x_o) \pm g(x_o)$.

Therefore $\lim_{x \to x_o} \text{ap} (f(x) \pm g(x)) = f(x_o) \pm g(x_o)$.

The proofs of (2) and (3) can also easily be given.

It is not true under the conditions of the above theorem that composition of approximately continuous functions gives rise to an approximately continuous function, as the following counterexample will show.

Let R = S = reals. Let $x_o = 0$ and m be Lebesgue measure. Define

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}),\ n = 0,1.2.---; \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2^n},\ n = 0,1,2,---; \\ 0 & \text{otherwise.} \end{cases}$$

Now $f(x)$ is approximately continuous at $x = 0$, and in fact is continuous at $x = 0$, since $\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x)$, and hence $\lim_{x \to 0} f(x) = f(0) = 0$.

We next show $g(x)$ is approximately continuous at

$x = 0$. First we have $\lim_{x \to 0} \text{ap} \sup g(x) = \text{l.u.b. } J(0)$ $= \text{l.u.b.} \{y | 0 \text{ is not a point of dispersion of } C(y)\}$ $= \text{l.u.b.}(-\infty, 0) = 0$. Similarly, $\lim_{x \to 0} \text{ap} \inf g(x) = \text{g.l.b.} I(0)$ $= \text{g.l.b.} \{y | 0 \text{ is not a point of dispersion of } D(y)\}$ $= \text{g.l.b.}[0, \infty) = 0$. Therefore $\lim_{x \to 0} \text{ap } g(x) = 0 = g(0)$, so $g(x)$ is approximately continuous at $x = 0$. But now we find

$$g(f(x)) = \begin{cases} 1 \text{ if } x \varepsilon (0,1); \text{ and} \\ 0 \text{ otherwise} \end{cases}$$

is not approximately continuous at $x = 0$. To see this we need only note that $\lim_{x \to 0} \text{ap} \sup f(x) = \text{l.u.b.} \{y | 0 \text{ is not a point of dispersion of } C(y)\} = \text{l.u.b.}(-\infty, 1) = 1$, while $g(f(0)) = g(0) = 0$ and we cannot possibly have $\lim_{x \to 0} \text{ap } g(f(x)) = 0$. Thus we conclude that $g(f(x))$ is not approximately continuous at $x = 0$, even though g is approximately continuous at $f(0)$ and f is continuous at 0.

It is pointed out here that the above example can be modified so that f is continuous everywhere, and so that g is approximately continuous at 0 and continuous everywhere else, but yet $g(f(x))$ is still not approximately continuous at 0.

It is natural to ask the following question: If $f(x)$ is approximately continuous at every point of a compact set $S \subset R$, is $f(x)$ bounded there? We know the answer to this question is yes if f(x) is continuous; however, the following counterexample will show that approximate comtinuity is not a sufficient condition for f(x) to be bounded on S, even if S is compact.

Example. Let R be the reals and S = [0,1]. Let m be Lebesgue measure. Define f(x) (see figure 1.) on [0,1] in the following way:

$$f(x) = \begin{cases} 0 \text{ if } x = 0; \\ 0 \text{ if } x\varepsilon(\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{2n}}); \\ 2^{3n+1} x + 2^{(n+1)}(1-2^{n+1}) \text{ for } x\varepsilon[\frac{1}{2^{n-1}} - \frac{1}{2^{2n}}, \frac{1}{2^{n-1}} - \frac{1}{2^{2n+1}}); \\ -2^{3n+1} x + 2^{2(n+1)} \text{ for } x\varepsilon[\frac{1}{2^{n-1}} - \frac{1}{2^{2n+1}}, \frac{1}{2^{n-1}}] . \end{cases}$$

It is easily seen that f(x) is continuous on (0,1], and hence approximately continuous on (0,1]. Further f(x) is approximately continuous at 0, as the following argument will establish.

We want to show $\lim_{x \to 0} \text{ap } f(x) = 0$. Consider $J(0) = \{y | y \text{ real, } 0 \text{ is not a point of dispersion of } C(y)\}$. If y = 0, $C(y) = [0,1] - f^{-1}(\{0\})$. We will show 0 is a point of dispersion of $[0,1] - f^{-1}(\{0\})$, so that $0 \notin J(0)$. We first note that $D_{[0,1]-f^{-1}(\{0\})}(0) = 0$, since if n is an arbitrary positive integer, then for any open interval I, with $0\varepsilon I$ and $m(I) < \frac{1}{2^n}$, it follows that

$$\frac{m(([0,1]-f^{-1}(\{0\}))\cap I)}{m(I)} < \frac{\sum_{k=n+1}^{\infty} \frac{1}{2^{2k}}}{\frac{1}{2^{n+1}}} = \frac{1}{3} \cdot \frac{1}{2^{n-1}} .$$

Figure 1

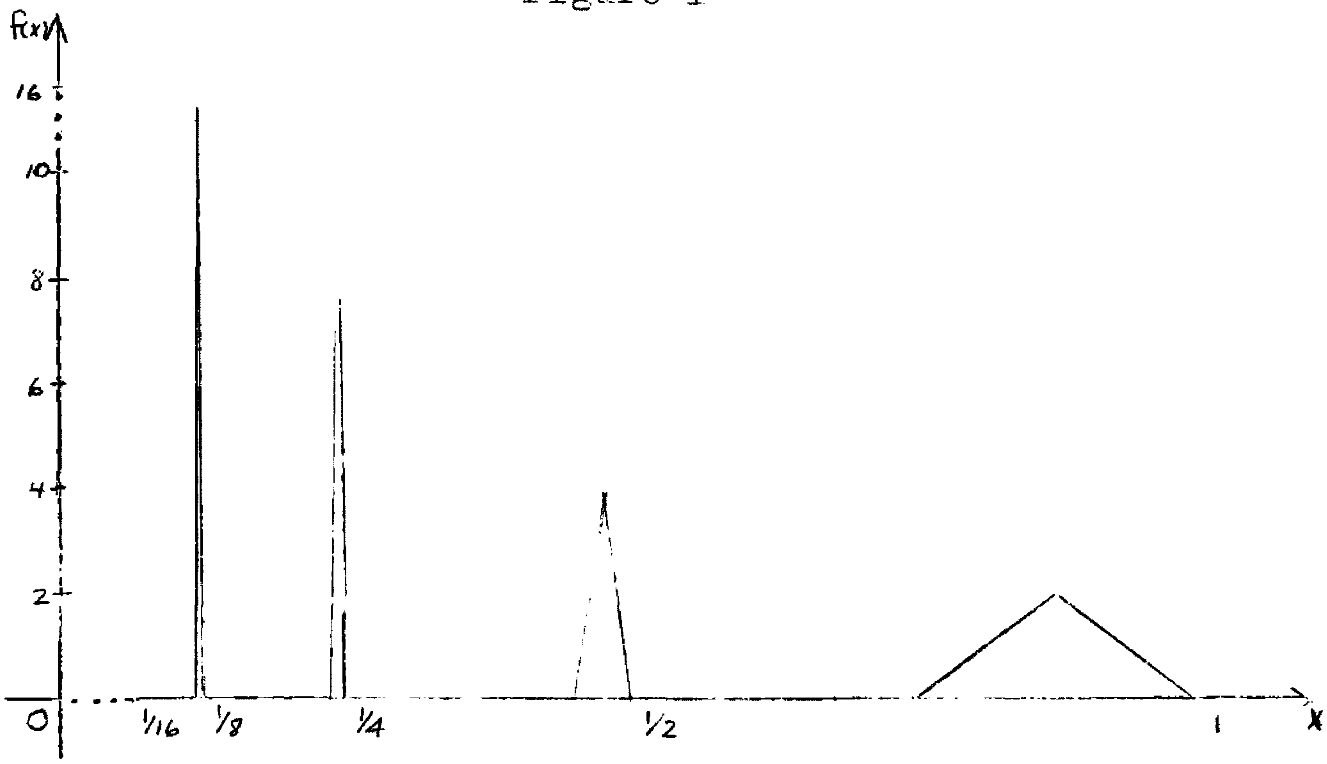
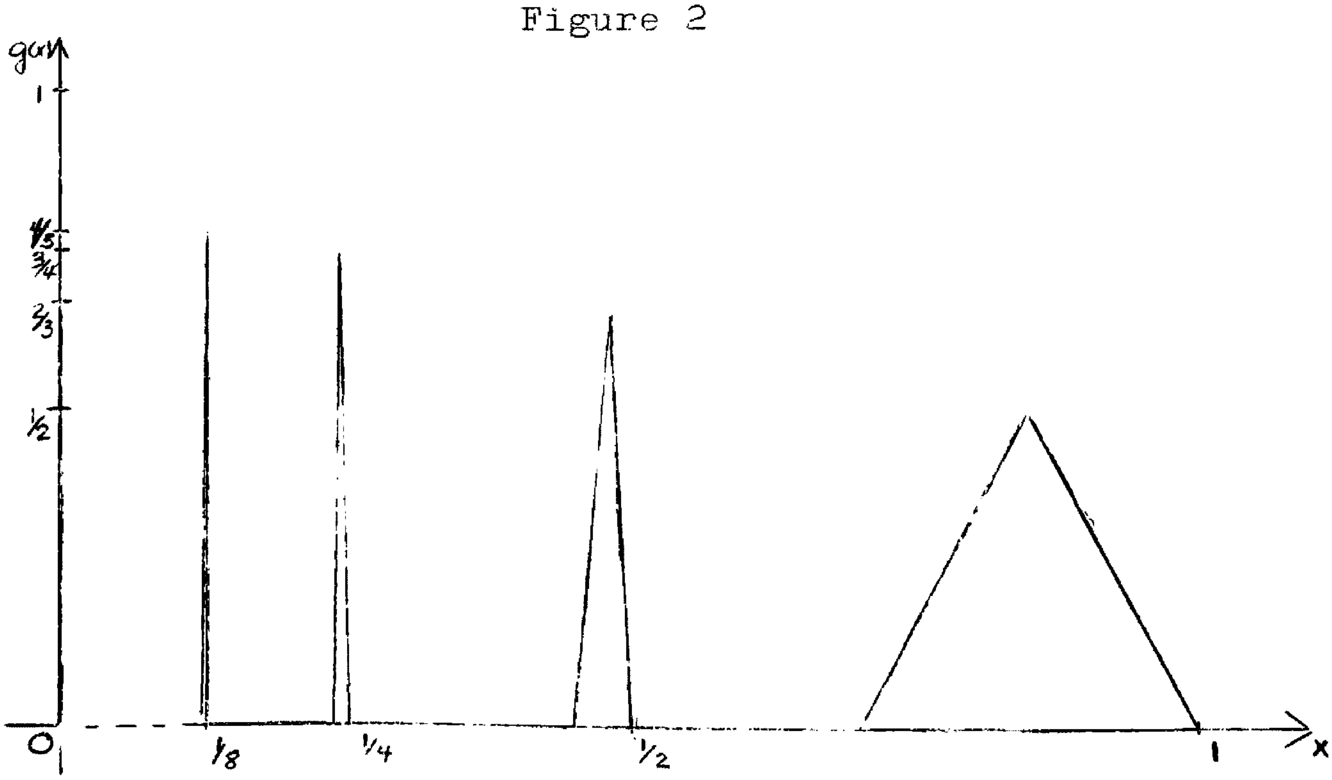


Figure 2



Thus $0 \notin J(0)$.

If $y < 0$, then $C(y) = [0,1]$ and 0 is not a point of dispersion of $[0,1]$, thus $y \in J(0)$ if $y < 0$. Therefore $\lim\limits_{x \to 0} \text{ap} \sup f(x) = \text{l.u.b. } J(0) = \text{l.u.b.}(-\infty, 0) = 0$.

Let us now consider

$I(0) = \{y \mid 0 \text{ is not a point of dispersion of } D(y)\}$.

If $y \leq 0$, then $D(y) = \Phi$ so that $y \notin I(0)$ if $y \leq 0$. Therefore $\lim\limits_{x \to x_0} \text{ap} \inf f(x) \geq 0$. Thus we conclude the approximate limit exists at 0 and $\lim\limits_{x \to 0} \text{ap} f(x) = 0 = f(0)$, i.e., $f(x)$ is approximately continuous at 0 and hence on $[0,1]$. However it is clear $f(x)$ is not bounded on $[0,1]$, since at the midpoint of any interval of the form $[\frac{1}{2^{n-1}} - \frac{1}{2^{2n}}, \frac{1}{2^{n-1}}]$, i.e., at $\frac{1}{2^{n-1}} - \frac{1}{2^{2n+1}}$, we have $f(x) = 2^n$, i.e., $f(\frac{1}{2^{n-1}} - \frac{1}{2^{2n+1}}) = 2^n$.

In a similar manner, one may construct an approximately continuous function f which is bounded on a compact subset $S \subset R$, but for which $\text{g.l.b.}_{t \in S} f(t) < f(x) < \text{l.u.b.}_{t \in S} f(t)$ for each $x \in S$, in other words, f does not assume a maximum or minimum on S. In fact, by modifying the above example, we can show that a function can be approximately continuous and bounded on $[0,1]$, but not have a maximum there.

Define (see figure 2.)

$$g(x) = \begin{cases} 0 \text{ if } x = 0; \\ 0 \text{ if } x \varepsilon [\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{2n}}]; \\ (\frac{n}{n+1})(\frac{1}{2^n} f(x) \text{ for } x \varepsilon [\frac{1}{2^{n-1}} - \frac{1}{2^{2n}}, \frac{1}{2^{n-1}}], \\ \text{where } f(x) \text{ is as in the preceding example.} \end{cases}$$

We see that the maximum obtained by $g(x)$ on the interval $[\frac{1}{2^{n-1}} - \frac{1}{2^{2n}}, \frac{1}{2^{n-1}}]$ is $\frac{n}{n+1}$. Thus $g(x)$ is bounded above by 1, and in fact, $\underset{t\varepsilon[0,1]}{\text{l.u.b.}}\, g(t) = 1$. Also $g(x)$ is approximately continuous on $[0,1]$. But, $g(x)$ does not have a maximum on $[0,1]$.

Remark. In chapter III, it will be shown that if f is a measurable function on a measurable subset S of R satisfying suitably restricted conditions, i.e., the conditions of the Lebesgue density theorem, then f is approximately continuous almost everywhere on S, i.e., approximately continuous everywhere on S except for a set of measure 0.

It will be convenient to have a theorem which gives a necessary and sufficient condition for a measurable function to be approximately continuous at a point x_o when examining the theorem of the above remark. This is the content of the following

Theorem 2-38. Suppose $x_o \varepsilon S$, $D_S(x_o) = \ell > 0$. Then a measurable function $f(x)$ is approximately continuous at x_o if and only if for every pair of real numbers k_1 and k_2 such that $k_1 < f(x_o) < k_2$, the set $\{x | x\varepsilon S, k_1 < f(x) < k_2\}$ has metric density ℓ at x_o.

Proof: Suppose $D_{\{x|x\varepsilon S, k_1<f(x)<k_2\}}(x_o) = \ell$ for all k_1, k_2 for which $k_1 < f(x_o) < k_2$. By theorem 2-25, it will suffice to show $D_{E(f(x_o),\varepsilon)}(x_o) = \ell$ for all $\varepsilon>0$. Suppose $\varepsilon>0$. Let $k_1 = f(x_o) - \varepsilon$, $k_2 = f(x_o) + \varepsilon$. Then $D_{\{x|x\varepsilon S, |f(x)-f(x_o)|<\varepsilon\}}(x_o) = \ell$. But $\{x|x\varepsilon S, |f(x)-f(x_o)|<\varepsilon\} = E(f(x_o),\varepsilon)$, i.e., $D_{E(f(x_o),\varepsilon)}(x_o) = \ell$ and hence $\lim_{x \to x_o} \text{ap } f(x) = f(x_o)$.

Conversely suppose $\lim_{x \to x_o} \text{ap } f(x) = f(x_o)$ and suppose $k_1<f(x_o)<k_2$. Then $D_{\{x|x\varepsilon S, |f(x)-f(x_o)|<\varepsilon\}}(x_o) = \ell$ for all $\varepsilon>0$, since $D_{E(f(x_o),\varepsilon)}(x_o) = \ell$. Let $\varepsilon = \min(k_2 - f(x_o), f(x_o) - k_1)$. Then

$\{x|x\varepsilon S, f(x_o) - \varepsilon<f(x)<f(x_o) + \varepsilon\}$

$\subset \{x|x\varepsilon S, k_1<f(x)<k_2\}$, so $L_{\{x|x\varepsilon S, k_1<f(x)<k_2\}}(x_o) \geqq \ell$.

However at the same time, $\{x|x\varepsilon S, k_1<f(x)<k_2\} \subset S$, so $U_{\{x|x\varepsilon S, k_1<f(x)<k_2\}}(x_o) \leqq \ell$, so $D_{\{x|x\varepsilon S, k_1<f(x)<k_2\}}(x_o) = \ell$. Thus this theorem yields a criterion for approximate continuity.

Another question which arises is: If f is approximately continuous on a connected set, does f have the intermediate value property? This question can be answered affirmatively if our metric space R is the reals and m is Lebesgue measure. This somewhat surprising result and its elegant proof were first given by de la Valleé Poussin.

We remark that it can easily be shown for Euclidean 1-space and Lebesgue measure, that metric density is the same if closed spherical neighborhoods are used in the definition, rather than open spherical neighborhoods. In the remainder of this chapter, and in chapter III, we will find it convenient to use this remark. In chapter III, we prove this remark for our measure space R.

In proving the above mentioned result we will use the following

<u>Lemma</u>. Suppose E is a measurable set contained in $[a,b]$ and m is Lebesgue measure. Define $g(x) = m(E\cap[a,x])$ for $x\varepsilon[a,b]$. Suppose $x_o\varepsilon(a,b)$. If $D_E(x_o) = k$, then $g'(x_o) = k$ and conversely.

Proof: Suppose $\varepsilon>0$ and $D_E(x_o) = k$. There exists $\delta>0$ such that if I is a closed interval, $x_o\varepsilon I$, and $m(I)<\delta$, then

$$k - \varepsilon < \frac{m(E\cap I)}{m(I)} < k + \varepsilon.$$

Consider the difference quotient

$$\frac{g(x_o) - g(x)}{x_o - x} = \frac{m(E\cap[a,x_o]) - m(E\cap[a,x])}{x_o - x}.$$

For convenience suppose $x_o > x$. Let $I = [x,x_o]$. Since $[a,x_o] = [a,x]\cup[x,x_o]$ and

$E\cap[a,x_o] = (E\cap[a,x])\cup(E\cap[x,x_o])$, we have

$m(E\cap[a,x_o]) - m(E\cap[a,x]) = m(E\cap[x,x_o]) = m(E\cap I)$.

Therefore $$\frac{m(E\cap[a,x_o]) - m(E\cap[a,x])}{x_o - x} = \frac{m(E\cap I)}{m(I)} ,$$

and we conclude $g'(x_o) = k$. It is clear that if $x_o < x$, the result is the same. Thus if $D_E(x_o) = k$, then $g'(x_o) = k$. Conversely it is seen that $g'(x_o) = k$ implies $D_E(x_o) = k$.

Theorem 2-39. If $f(x)$ is approximately continuous on $[a,b]$ and if $f(a) < 0$, $f(b) > 0$, then there exists $c \varepsilon (a,b)$ such that $f(c) = 0$.

Proof: The proof is by contradiction. Let f, measurable, be an approximately continuous function on $[a,b]$ such that $f(a) < 0$, $f(b) > 0$, and $f(x) \neq 0$ for any $x \varepsilon (a,b)$. Define

$$A = \{x | x \varepsilon [a,b],\ f(x) < 0\},$$
$$B = \{x | x \varepsilon (a,b],\ f(x) > 0\};$$

$A, B \varepsilon \mathcal{L}$, $A \cup B = [a,b]$, and $A \cap B = \Phi$.

The following claim is made: if $x_o \varepsilon A \cap (a,b)$, then $D_A(x_o) = 1$, and hence $D_B(x_o) = 0$. Since f is approximately continuous at x_o,

$E(f(x_o),\varepsilon) = \{x | x \varepsilon [a,b],\ |f(x) - f(x_o)| < \varepsilon\} \varepsilon K(x_o)$ for all $\varepsilon > 0$. Let $\varepsilon = -f(x_o) > 0$. Then

$E(f(x_o), -f(x_o)) = \{x | x \varepsilon [a,b], |f(x) - f(x_o)| < -f(x_o)\}$
$= \{x | x \varepsilon [a,b],\ 2f(x_o) < f(x) < 0\} \subset A$. Therefore $D_A(x_o) = 1$ and $D_B(x_o) = 0$ for $x_o \varepsilon A$ as claimed. A similar claim holds for $x_o \varepsilon B \cap (a,b)$, that is, $x_o \varepsilon B \cap (a,b)$ implies $D_B(x_o) = 1$ and $D_A(x_o) = 0$.

Let $g(x) = m(A \cap [a,x])$ for $x \varepsilon [a,b]$. If $x \varepsilon A \cap (a,b)$, then $g'(x)$ exists and $g'(x) = 1$ by the lemma, while if

$x \varepsilon B \cap (a,b)$, then $g'(x)$ exists and $g'(x) = 0$. We will show this leads to a contradiction of the mean value theorem for derivatives, namely, that if $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \varepsilon (a,b)$ such that $g'(c) = \frac{g(b) - g(a)}{b - a}$. We will show in particular that $0 < \frac{g(b) - g(a)}{b - a} < 1$, contradicting $g'(c) = 0$ or $g'(c) = 1$ for all points c in (a,b).

It is observed that $A = D(0)$ and $B = C(0)$. Since $g(a) = 0$ and $g(b) = m(A)$, we have $\frac{g(b) - g(a)}{b - a} = \frac{m(A)}{b-a}$. Suppose $m(A) = 0$. Then $m(B) = m(C(0)) = m([a,b]) = b-a$. The claim is made that a is not a point of dispersion of $C(0)$, and hence $0 \varepsilon J(a)$. Consider an open interval I with a as its midpoint. Then $\frac{m(C(0) \cap I)}{m(I)} = \frac{1}{2}$ and $U_{C(0)}(a) \geq \frac{1}{2}$. Thus a is not a point of dispersion of $C(0)$, so $0 \varepsilon J(a)$. But then $\lim_{x \to a} \text{ap sup } f(x) \geq 0$ in contradiction to the facts that f is approximately continuous at a and $f(a) < 0$.

In a similar fashion, if we suppose $m(B) = 0$, we are led to a contradiction, and hence $m(B) > 0$. Therefore it follows that

$$0 < \frac{m(A)}{b-a} = \frac{g(b) - g(a)}{b - a} < 1$$

and the proof is complete.

<u>Theorem</u> 2-40. Suppose $f(x)$ is bounded and Lebesgue measurable on $[a,b]$. Let $F(x) = \int_{[a,x]} f(t)dt$, where $\int_{[a,x]} f(t)dt$ is the Lebesgue integral. If $x_0 \varepsilon (a,b)$

and f is approximately continuous at x_o, then F(x) has a derivative at x_o and $F'(x_o) = f(x_o)$.

Proof: Since f is bounded on [a,b], there exists M such that $|f(x)| \leq M$ on [a,b]. Suppose $x_o \varepsilon (a,b)$ and suppose f is approximately continuous at x_o. If $\varepsilon > 0$, $E(f(x_o),\varepsilon) \varepsilon K(x_o)$. Let

$J = E(f(x_o),\varepsilon/2) = \{x | x \varepsilon [a,b], |f(x)-f(x_o)| < \varepsilon/2\}$.

$J \varepsilon K(x_o)$, so $L_J(x_o) = 1 > 1 - \frac{\varepsilon}{4M}$. Let $Q(p,\delta) = \{x | x \varepsilon R, \rho(p,x) \leq \delta\}$. There exists N, such that

$$l_J\left(x_o, \tfrac{1}{2N}\right) = \underset{\substack{x_o \varepsilon Q(p,\delta) \\ 0<\delta<\frac{1}{2N}}}{\text{g.l.b.}} \frac{m(J \cap Q(p,\delta))}{m(Q(p,\delta))} > 1 - \frac{\varepsilon}{4M},$$ i.e., if

$x_o \varepsilon Q(p,\delta)$ and $m(Q(p,\delta)) = 2\delta < \frac{1}{N}$, then

$m(J \cap Q(p,\delta)) > (1 - \frac{\varepsilon}{4M}) M(Q(p,\delta))$ and

$m(\complement J \cap Q(p,\delta)) < \frac{\varepsilon}{4M} m(Q(p,\delta))$.

Let $\lambda = \frac{1}{N} > 0$. Suppose $x_o < x < x_o + \lambda$, $x \varepsilon [a,b]$. Then

$$\left|\frac{F(x) - F(x_o)}{x - x_o} - f(x_o)\right| = \left|\frac{\int_{[a,x]} f(t)dt - \int_{[a,x_o]} f(t)dt}{x - x_o} - f(x_o)\right|$$

$$= \left|\frac{\int_{[x_o,x]} f(t)dt}{x - x_o} - f(x_o)\right| .$$

Let $p = \frac{x+x_o}{2}$ and let $\delta = \frac{x-x_o}{2}$. Then $Q(p,\delta) = [x_o,x]$ and $x_o \varepsilon Q(p,\delta)$. Further, since $x < x_o + \lambda$, $x - x_o < \lambda$, and $\delta = \frac{x-x_o}{2} < \frac{\lambda}{2} = \frac{1}{2N}$.

Now $\left| \dfrac{F(x) - F(x_o)}{x - x_o} - f(x_o) \right| = \left| \dfrac{\int_{Q(p,\delta)} f(t)dt}{m(Q(p,\delta))} - f(x_o) \right|$

$$= \left| \frac{\int_{Q(p,\delta)} f(t)dt - \int_{Q(p,\delta)} f(x_o)dt}{m(Q(p,\delta))} \right|$$

$$= \left| \frac{\int_{Q(p,\delta)} (f(t) - f(x_o))dt}{m(Q(p,\delta))} \right| \leq \frac{\int_{Q(p,\delta)} |f(t) - f(x_o)|dt}{m(Q(p,\delta))}$$

$$= \frac{\int_{Q(p,\delta)\cap J} |f(t) - f(x_o)|dt}{m(Q(p,\delta))} + \frac{\int_{Q(p,\delta)\cap \complement J} |f(t)-f(x_o)|dt}{m(Q(p,\delta))}$$

$$\leq \frac{\frac{\varepsilon}{2} m(Q(p,\delta))}{m(Q(p,\delta))} + \frac{2Mm(\ J\cap Q(p,\delta))}{m(Q(p,\delta))}$$

$< \frac{\varepsilon}{2} + 2M\cdot\frac{\varepsilon}{4M} = \varepsilon$. Hence $F(x)$ has a right hand derivative at x_o equal to $f(x_o)$. Thus $F'(x_o)$ exists and $F'(x_o) = f(x_o)$.

In light of the preceding theorem, one might wonder whether, if boundedness and measurability are replaced by summability, the theorem still holds true? Unfortunately the answer is no, as the following counterexample illustrates.

<u>Example</u>. Let R be Euclidean 1-space and let m be Lebesgue measure. Let f be defined on $[-1,1]$ as follows:

$$f(x) = \begin{cases} 2^n \text{ for } (\frac{1}{2^n}) - (\frac{1}{2^{2n+1}}) \leq x \leq \frac{1}{2^n}, \ n=0,1,2---, \\ 0 \text{ otherwise.} \end{cases}$$

We note first that f is summable on $[-1,1]$, since

$$\int_{[-1,1]} f(x)dx = \lim_{n\to\infty} \int_{[-1,1]} (f(x))_o^n \, dx + \lim_{n\to\infty} \int_{[-1,1]} (f(x))_{-n}^o dx$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{2^k}{2^{2k+1}} + 0 = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{2^{k+1}} = 1,$$

where $(f(x))_o^n = \begin{cases} n \text{ if } f(x) \geq n, \\ f(x) \text{ if } 0 \leq f(x) \leq n, \text{ and} \\ 0 \text{ if } f(x) \leq 0, \end{cases}$

and $(f(x))_n^o = \begin{cases} - n \text{ if } f(x) \leq - n, \\ f(x) \text{ if } - n \leq f(x) \leq 0, \text{ and} \\ 0 \text{ if } f(x) \geq 0. \end{cases}$

We further note that $f(x)$ is approximately continuous at $x = 0$. For, if J is a closed interval such that $\frac{1}{2^{n+1}} < M(J) \leq \frac{1}{2^n}$ and $0 \varepsilon J$, then

$$\frac{m(S \cap J)}{m(J)} \leq \frac{\sum_{k=n}^{\infty} \frac{1}{2^{2k+1}}}{\frac{1}{2^{n+1}}} = \frac{(\frac{1}{3})(\frac{1}{2^{2n-1}})}{\frac{1}{2^{n+1}}} = \frac{1}{3} (\frac{1}{2^{n-2}})$$

$\to 0$ as $n \to \infty$.

Define $F(x)$ as before, i.e., $F(x) = \int_{[-1,x]} f(t)dt$ for $x \varepsilon [-1,1]$. Suppose first that $0 < x \leq 1$. Then

$$\left|\frac{F(x) - F(x_o)}{x - 0}\right| = \left|\frac{\int_{[-1,x]} f(t)dt - \int_{[-1,0]} f(t)dt}{x}\right|$$

$= \left|\frac{\int_{[0,x]} f(t)dt}{x}\right|$. There exists n such that $\frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}$. Then

$$\frac{\int_{[0,x]} f(t)dt}{x} \geq \frac{\sum_{k=n+1}^{\infty} \frac{2^k}{2^{2k+1}}}{\frac{1}{2^n}} = \frac{\sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}}}{\frac{1}{2^n}} = \frac{1}{2} .$$

Hence we see that if F has a right hand derivative, it is at least $\frac{1}{2}$, while $f(0) = 0$. Hence we see the theorem does not hold true if boundedness and measurability are replaced by summability.

CHAPTER III

The principal results in this chapter are the Vitali covering theorem and Lebesgue density theorem. As before, R will be a separable, dense-in-itself metric space, S a measurable subset of R, $\mathcal{L}$ a σ-algebra of subsets of R, m a measure defined on $\mathcal{L}$, and f a measurable function on S. There are some inherent problems involved in proving these two theorems for a measure space which do not arise in a Euclidean space. As an example of such a problem, suppose x and x' are distinct points in R. Generally $m(N(x,\delta)) \neq m(N(x',\delta))$. This problem and others will be discussed in detail when they arise.

When working with metric density in a Euclidean space one often finds it convenient to know that if the metric density exists at a point, then it is the same if either open neighborhoods or closed neighborhoods are used in the definition. It is recalled, in fact, that near the end of chapter II we took the liberty of using this convenience. For Euclidean n-space it can readily be shown that this is the case. This is also a desired convenience in the case of a measure space.

Define $Q(x_o,\lambda) = \{x | x\varepsilon R, \rho(x,x_o) \leq \lambda\}$. We note that $Q(x_o,\lambda)\varepsilon\mathcal{L}$, since $Q(x_o,\lambda) = \bigcap_{n=1}^{\infty} N(x_o,\lambda+\frac{1}{n})$, and $m(Q(x_o,\lambda)) = \lim_{n\to\infty} m(N(x_o,\lambda+\frac{1}{n}))$. Also,

$N(x_o, \lambda) = \bigcup_{n=N}^{\infty} Q(x_o, \lambda - \frac{1}{n})$, where N is such that $\lambda > \frac{1}{N}$, and $m(N(x_o,\lambda)) = \lim_{n \to \infty} m(Q(x_o, \lambda - \frac{1}{n}))$.

Let $u_S(x_o, \lambda)$ and $U_S(x_o)$ be defined as before. Define $u'_S(x_o,\lambda) = \underset{\substack{p\varepsilon R \\ x_o\varepsilon Q(p,\delta) \\ 0<\delta<\lambda}}{\text{l.u.b.}} \frac{m(S\cap Q(p,\delta))}{m(Q(p,\delta))}$, and $U'_S(x_o) = \underset{\lambda>0}{\text{g.l.b.}}\ u'_S(x_o,\lambda)$. Define $l'_S(x_o,\ \ )$ and $L'_S(x_o)$ in an analogous fashion. If $L'_S(x_o) = U'_S(x_o)$, let $D'_S(x_o)$ stand for the common value, i.e., $D'_S(x_o)$ will stand for the metric density of S, with respect to the closed spherical neighborhoods $Q(p,\delta)$, at the point x_o.

<u>Theorem</u> 3-1. Let $x_o \varepsilon R$ and let S be a measurable subset of R. Then $D'_S(x_o)$ exists if and only if $D_S(x_o)$ exists and $D'_S(x_o) = D_S(x_o)$ if either exists.

Proof: Suppose $u_S(x_o,\lambda) = k$ and $\varepsilon>0$. There exists $\delta>0$ and $p\varepsilon R$ such that $x_o \varepsilon N(p,\delta)$, $0 < \delta < \lambda$, and $\frac{m(S\cap N(p,\delta))}{m(N(p,\delta))} > k - \varepsilon$. There exists $a > 0$ such that

$\frac{m(S\cap N(p,\delta))-a}{m(N(p,\delta))} > k - \varepsilon$. We know that

$\lim_{\substack{\alpha \to \delta \\ \alpha < \delta}} m(S\cap Q(p,\alpha)) = m(S\cap N(p,\delta))$, so there exists δ' such

that $0 < \delta' < \delta$ and $m(S\cap N(p,\delta)) - m(S\cap Q(p,\delta')) < a$, i.e., $m(S\cap N(p,\delta)) - a < m(S\cap Q(p,\delta'))$. But now we have

$$\frac{m(S\cap Q(p,\delta'))}{m(Q(p,\delta'))} \geq \frac{m(S\cap Q(p,\delta'))}{m(N(p,\delta))} > \frac{m(S\cap N(p,\delta))-a}{m(N(p,\delta))} > k - \varepsilon.$$

Therefore $u'_S(x_o,\lambda) > k - \varepsilon$ for every $\varepsilon>0$, so $u'_S(x_o,\lambda) \geq k = u_S(x_o,\lambda)$.

On the other hand, suppose $u_S'(x_o, \lambda) = k$. Suppose $\varepsilon>0$. There exists $\delta>0$ and $p\varepsilon R$ such that $x_o \varepsilon Q(p,\delta)$, $o<\delta<\lambda$, and $\frac{m(S\cap Q(p,\delta))}{m(Q(p,\delta))} > k - \varepsilon$. There exists $b > 0$ such that $\frac{m(S\cap Q(p,\delta))}{m(Q(p,\delta))+b} > k - \varepsilon$. Since $\lim_{\substack{\alpha \to \delta \\ \alpha > \delta}} m(N(p,\alpha)) = m(Q(p,\delta))$, there exists δ' such that $0 < \delta < \delta' < \lambda$ and $m(N(p,\delta')) - m(Q(p,\delta)) < b$, i.e., $m(N(p,\delta')) < m(Q(p,\delta)) + b$. But now we have

$$\frac{m(S\cap N(p,\delta'))}{m(N(p,\delta'))} \geq \frac{m(S\cap Q(p,\delta))}{m(N(p,\delta'))} \geq \frac{m(S\cap Q(p,\delta))}{m(Q(p,\delta))+b} > k - \varepsilon.$$

Therefore $u_S(x_o,\lambda) > k - \varepsilon$ for every $\varepsilon > 0$, and hence $u_S(x_o,\lambda) \geq k = u_S'(x_o,\lambda)$. Hence $u_S(x_o,\lambda) = u_S'(x_o,\lambda)$. Since $\lambda > 0$ is arbitrary, we conclude that $U_S(x_o) = U_S'(x_o)$.

In a similar fashion it can be shown that $L_S(x_o) = L_S'(x_o)$, or alternatively it can be shown that $U_{\mathcal{C}S}'(x_o) + L_S'(x_o) = 1$, and since $U_{\mathcal{C}S}'(x_o) = U_{\mathcal{C}S}(x_o)$ it follows that $L_S'(x_o) = L_S(x_o)$. Thus we see that $D_S'(x_o)$ exists if and only if $D_S(x_o)$ exists, and $D_S'(x_o) = D_S(x_o)$ if either exists. Henceforth we will use $D_S(x_o)$ for the metric density of S at x_o, whether open neighborhoods, or closed neighborhoods are used.

We will find the following lemma a useful aid in proving the Vitali covering theorem.

Lemma. Suppose G is a bounded subset of R. Suppose there is a positive real number t such that if $x_1 \varepsilon G$, $\lambda > 0$, then $m(Q(x_1,\lambda)) \leq tm(Q(x_1,\frac{\lambda}{5}))$. Then, if $\varepsilon>0$, there exists $\delta>0$ such that if $x_1 \varepsilon G$, $\gamma>0$, and

$m(Q(x_1, \gamma)) < \delta$, then $\gamma < \varepsilon$.

Proof: Since G is bounded, there exists $x_o \varepsilon G$ and $\alpha > 0$ such that $G \subset Q(x_o, \alpha)$. If $x_1 \varepsilon G$, then $Q(x_o, \alpha) \subset Q(x_1, 2\alpha)$. Let $\beta = 2\alpha$. Then $m(Q(x_1, \beta)) = m(Q(x_1, 2\alpha)) \geq m(Q(x_o, \alpha)) = a > 0$. Now $m(Q(x_1, \frac{\beta}{5})) \geq \frac{1}{t} m(Q(x_1, \beta)) \geq \frac{a}{t}$. Also $m(Q(x_1, \frac{\beta}{25})) \geq \frac{1}{t} m(Q(x_1, \frac{\beta}{5})) \geq \frac{a}{t^2}$, and in general, $m(Q(x_1, \frac{\beta}{5^n})) \geq \frac{a}{t^n}$.

Suppose $\varepsilon > 0$. There exists n such that $\frac{\beta}{5^n} < \varepsilon$. Let $\delta = \frac{a}{t^n}$. Then if $m(N(x_1, \gamma)) < \delta = \frac{a}{t^n}$, it follows that $\gamma < \frac{\beta}{5^n} < \varepsilon$, for, if $\gamma \geq \frac{\beta}{5^n}$, then $m(N(x_1, \gamma)) \geq \frac{a}{t^n}$ in contradiction to our choice of δ.

Definition 3-1. Suppose $S \subset R$. A collection $\mathcal{A}$ of closed spherical neighborhoods forms a Vitali Covering of S if whenever $p \varepsilon S$, $\varepsilon > 0$, there exists a $Q \varepsilon \mathcal{A}$ such that $p \varepsilon Q$ and the radius of Q is less than ε. Denote the radius of Q by rad(Q).

Theorem 3-2. (Vitali). Suppose $S \varepsilon \mathcal{L}$ and $S \subset G$, G open and bounded. Suppose that there is a positive real number t such that if $x_1 \varepsilon G$, $\lambda > 0$, then $m(Q(x_1, \lambda)) \leq t m(Q(x_1, \frac{\lambda}{5}))$. Suppose that $\mathcal{A}$ is a Vitali cover of S. Then either there exists $I_1, I_2, ---, I_k$ such that $I_i \cap I_j = \emptyset$ if $i \neq j$, $I_i \varepsilon \mathcal{A}$ and $I_i \subset G$, and $S \subset \bigcup_{i=1}^{k} I_i$, or else there exists $I_1, ---, I_k, ---$, with $I_i \cap I_j = \emptyset$ if $i \neq j$, $I_i \varepsilon \mathcal{A}$, $I_i \subset G$, and such that, if $\varepsilon > 0$, there exists k such that $m(S-\bigcup_{j=1}^{k} I_j) < \varepsilon$. Further, $m(S-\bigcup_{j=1}^{\infty} I_j) = 0$.

Proof: Let $\mathcal{H} = \{A \mid A\varepsilon\mathcal{A},\ A\cap S \neq \phi,\ \text{rad}(I) < 1,\ A\subset G\}$. $\mathcal{H}$ forms a Vitali covering of S. For, suppose $p\varepsilon S$, $\varepsilon>0$. We may suppose $\varepsilon<1$. Since $\mathcal{A}$ is a Vitali covering of S, there exists $B'\varepsilon\mathcal{A}$ such that $p\varepsilon B'$ and $\text{rad}(B') < \varepsilon$. Further, we can choose this B' so that $B'\subset G$. We are able to do this, since G is an open set containing S, and hence there exists $\delta>0$, $0<\delta<\varepsilon$, such that $N(p,\delta)\subset G$. Since $\mathcal{A}$ is a Vitali cover of S, there exists $A\varepsilon\mathcal{A}$, such that $\text{rad}(A)<\delta/2$, and $p\varepsilon A$. Also, $A\subset N(p,\delta)\subset G$. Let this A be B'. Now we observe that $B'\varepsilon\mathcal{H}$, since $B'\varepsilon\mathcal{A}$, $B'\cap S\neq\phi$, $\text{rad}(B')<\varepsilon$, and $B'\subset G$, thus $\mathcal{H}$ forms a Vitali covering of S.

Choose $I_1\varepsilon\mathcal{H}$ so that for every $A\varepsilon\mathcal{H}$, $2\ \text{rad}(I_1)>\text{rad}(A)$. We are able to do this because the radii of the sets in $\mathcal{H}$ have a least upper bound. If $S\subset I_1$, we are done.

If there exists $x\varepsilon S-I_1$, then $x\varepsilon\mathcal{C}I_1$, which is open, so there exists $I\varepsilon\mathcal{H}$ such that $I_1\cap I = \phi$. Choose $I_2\varepsilon\mathcal{H}$ such that $I_1\cap A = \phi$, $2\ \text{rad}(I_2) > \text{rad}(A)$. Suppose $I_1, I_2, ---, I_{k-1}$ have already been chosen, mutually disjoint, $I_j\varepsilon\mathcal{H}$, $j = 1,2,---,k-1$, and such that $2\ \text{rad}(I_j) > \text{rad}(A)$ for every $A\varepsilon\mathcal{H}$ with $A\cap(\bigcup_{n=1}^{j-1} I_n) = \phi$, $j = 2,3,---,k-1$. Also, if $A\varepsilon\mathcal{H}$, then $2\ \text{rad}(I_1) > \text{rad}(A)$. If $S\subset\bigcup_{j=1}^{k-1} I_j$, we are done. If there exists $x\varepsilon S - \bigcup_{n=1}^{k-1} I_n$, then x is in

$\mathcal{C}\bigcup_{n=1}^{k-1} I_n$, which is open. Therefore there exists an open spherical neighborhood $N(x,\delta) \subset \mathcal{C}\bigcup_{n=1}^{k-1} I_n$ such that $N(x,\delta) \cap \bigcup_{n=1}^{k-1} I_n = \Phi$. There exists $A \varepsilon \mathcal{H}$ such that $x \varepsilon A$ and $\mathrm{rad}(A) < \delta/2$. Then $A \subset N(x,\delta)$ and $A \cap (\bigcup_{n=1}^{k-1} I_n) = \Phi$. Among these closed spherical neighborhoods A, there exists one, call it I_k, such that $2\mathrm{rad}(I_k) > \mathrm{rad}(A)$ for all $A \varepsilon \mathcal{H}$ with $A \cap (\bigcup_{n=1}^{k-1} I_n) = \Phi$. By finite induction, either there exists k such that $S - \bigcup_{n=1}^{k} I_n = \Phi$, in which case the theorem is established, or else there exists a sequence $\{I_n\}$, $n = 1,2,\text{---},k,\text{---}$, of disjoint neighborhoods in $\mathcal{H}$ such that if k is an integer, $k \geq 2$, and if $A \varepsilon \mathcal{H}$ and if $A \cap (\bigcup_{n=1}^{k-1} I_n) = \Phi$, then $2\ \mathrm{rad}(I_k) > \mathrm{rad}(A)$. Also, if $A \varepsilon \mathcal{H}$, then $2\ \mathrm{rad}(I_1) > \mathrm{rad}(A)$.

Since G is bounded, S is bounded. Thus there exists $x_o \varepsilon S$ and $\alpha > 0$ such that $S \subset N(x_o,\alpha)$. Let $M = N(x_o,\alpha+2)$. Suppose $x \varepsilon I_k$. Since $I_k \cap S \neq \Phi$, there exists y such that $y \varepsilon I_k \cap S$. Then $\rho(x_o,y) < \alpha$. Also $\rho(x,y) \leq 2$, since $\mathrm{rad}(I_k) < 1$, and hence $\rho(x_o,x) < \alpha + 2$, so $x \varepsilon M$. Hence $I_k \subset M$ and $\bigcup_{k=1}^{\infty} I_k \subset M$. Now we have $\sum_{k=1}^{\infty} m(I_k) = m(\bigcup_{k=1}^{\infty} I_k) \leq m(M) < +\infty$. (Note here, that M, as an open sphere, has positive finite measure by assumption.) Thus $m(I_k) \to 0$ as $k \to \infty$.

Suppose $\varepsilon > 0$. There exists n such that $\sum_{k=n+1}^{\infty} m(I_k) < \varepsilon/t$, where t is described in the statement of the theorem. If k is an integer, $k \geq n + 1$, let A_k be the closed spherical neighborhood with the same center as I_k, and such that $rad(A_k) = 5\, rad(I_k)$. Then $m(A_k) \leq t m(I_k)$. Also

$$m\left(\bigcup_{k=n+1}^{\infty} A_k\right) \leq \sum_{k=n+1}^{\infty} m(A_k) \leq t \sum_{k=n+1}^{\infty} m(I_k) < \varepsilon.$$

As before, if $x \varepsilon S - \bigcup_{k=1}^{n} I_k$, then $x \varepsilon \mathcal{C} \bigcup_{k=1}^{n} I_k$, which is open, so there exists $\gamma > 0$ such that $N(x, \gamma) \cap (\bigcup_{k=1}^{n} I_k) = \Phi$. Choose $A \varepsilon \mathcal{H}$ with $x \varepsilon A$, $rad(A) < \frac{\gamma}{2}$. Then $A \subset N(x, \gamma)$, $x \varepsilon A$, and $A \cap (\bigcup_{k=1}^{n} I_k) = \Phi$. But A is not disjoint from all the I_k, since $rad(A) > 2\, rad(I_k)$ for k sufficiently large. This follows from the fact that $m(I_k) \rightarrow 0$ as $k \rightarrow \infty$, and from the previous lemma, i.e., if $\xi > 0$, there exists $\delta > 0$ such that if $x \varepsilon G$, $\eta > 0$, and $m(Q(x, \eta)) < \delta$, then $\eta < \xi$. Let m be the first positive integer such that $A \cap I_m \neq \Phi$. Note that $m > N$. For this integer m, $rad(A) < 2\, rad(I_m)$. This follows since $A \varepsilon \mathcal{H}$, $A \cap (\bigcup_{k=1}^{m-1} I_k) = \Phi$, and if $rad(A) \geq 2\, rad(I_m)$, then I_m could not join the sequence $\{I_k\}$. Since $rad(A) < 2\, rad(I_m)$ and $A \cap I_m \neq \Phi$, it follows that $A \subset A_m$. To verify that $A \subset A_m$, suppose $p \varepsilon A$. Since $A \cap I_m \neq \Phi$, there exists $q \varepsilon A \cap I_m$, and $\rho(p,q) \leq 2\, rad(A) < 4\, rad(I_m)$. Let r be the center of I_m (r is also the center of A_m). Then $\rho(q,r) \leq rad(I_m)$. But now we have

$\rho(p,r) \leq \rho(p,q) + \rho(q,r) < 4\, rad(I_m) + rad(I_m)$
$= 5\, rad(I_m) = rad(A_m)$. Therefore $A \subset A_m$. Because $x \varepsilon A_m$

and $m > n$, it follows that $x \in \bigcup_{k=n+1}^{\infty} A_k$. Hence

$S - \bigcup_{k=1}^{n} I_k \subset \bigcup_{k=n+1} A_k$ and $m(S - \bigcup_{k=1}^{n} I_k) \leq m(\bigcup_{k=n+1}^{\infty} A_k) < \varepsilon$.

In addition we also have

$m(S - \bigcup_{k=1}^{\infty} I_k) = 0$, since $S - \bigcup_{k=1}^{\infty} I_k \subset S - \bigcup_{k=1}^{n} I_k$ for all n, so $m(S - \bigcup_{k=1}^{\infty} I_k) \leq m(S - \bigcup_{k=1}^{n} I_k)$ for all n. If n is sufficiently large, $m(S - \bigcup_{k=1}^{n} I_k) < \varepsilon$, so $m(S - \bigcup_{k=1}^{\infty} I_k) < \varepsilon$ for all $\varepsilon > 0$. Therefore $m(S - \bigcup_{k=1}^{\infty} I_k) = 0$.

The Vitali theorem just proved was not the most general; we made important use of the fact that S was contained in G, and that G was open and bounded. In the next theorem it will be shown the conclusion of theorem 3-1 holds true when S is not necessarily bounded.

<u>Theorem</u> 3-3. (Vitali). Suppose R is separable and dense-in-itself. Suppose $S \subset R$, $S \in \mathcal{L}$, and suppose there is a positive real number t such that if $x_1 \in R$, $\lambda > 0$, then $m(N(x_1, \lambda)) \leq t\, m(N(x_1, \frac{\lambda}{5}))$. Suppose $\mathcal{A}$ is a Vitali cover of S. Then there exists a sequence $\{I_n\}$ (possibly finite) of disjoint sets of $\mathcal{A}$ such that $m(S - \bigcup_{n=1}^{\infty} I_n) = 0$.

Proof: Suppose $x_o \in R$. Let $S_1 = N(x_o, 1) \cap S$. Since $S_1 \in \mathcal{L}$ and $S_1 \subset N(x_o, 1)$, which is open and bounded, by theorem 3-2, there exist disjoint sets $I_1^1, I_2^1, ---, I_{k_1}^1$ in $\mathcal{A}$, each contained in $N(x_o, 1)$, and such that $m(S_1 - \bigcup_{j=1}^{k_1} I_j^1) < \frac{1}{2}$.

Let $S_2 = N(x_o,2) \cap S - \bigcup_{j=1}^{k_1} I_j^1$. $S_2 \varepsilon \mathcal{L}$ and $S_2 \subset N(x_o,2) - \bigcup_{j=1}^{k_1} I_j^1$, which is open and bounded, so again by theorem 3-2, there exist disjoint sets $I_1^2, I_2^2, ---, I_{k_2}^2$ of $\mathcal{A}$, each contained in $N(x_o,2) - \bigcup_{j=1}^{k_1} I_j^1$, and such that $m(S_2 - \bigcup_{j=1}^{k_2} I_j^2) < \frac{1}{4}$. Let $S_3 = N(x_o,3) \cap S - (\bigcup_{m=1}^{2} \bigcup_{j=1}^{k_m} I_j^m)$. Applying the same reasoning to S_3 as was applied to S_1 and S_2, $S_3 \varepsilon \mathcal{L}$ and $S_3 \subset N(x_o,3) - (\bigcup_{m=1}^{2} \bigcup_{j=1}^{k_m} I_j^m)$, and we find $I_1^3, I_2^3, ---, I_{k_3}^3$ in $\mathcal{A}$, mutually disjoint and contained in $N(x_o,3) - (\bigcup_{n=1}^{2} \bigcup_{j=1}^{k_m} I_j^m)$, such that $m(S_3 - \bigcup_{j=1}^{k_3} I_j) < \frac{1}{8}$

Using induction and continuing in this manner, for each positive integer n we let $S_n = N(x_o,n) \cap S - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m)$. $S_n \varepsilon \mathcal{L}$ and $S_n \subset N(x_o,n) - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m)$, which is open and bounded, so by theorem 3-2, there exist disjoint sets $I_1^n, I_2^n, ---, I_{k_n}^n$ of $\mathcal{A}$, each contained in $N(x_o,n) - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m)$, and such that $m(S_n - \bigcup_{j=1}^{k_n} I_j^n) < \frac{1}{2^n}$. We emphasize that this is true for each positive integer n.

We next observe the following point set identities. For each positive integer n,

$(N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)) \subset (N(x_o,n)\cap S - (\bigcup_{m=1}^{n} \bigcup_{j=1}^{k_m} I_j^m))$,

and in particular, for t a positive integer, $t \geq n$,

$(N(x_o n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)) \subset (N(x_o,t)\cap S - (\bigcup_{m=1}^{t} \bigcup_{j=1}^{k_m} I_j^m))$

$= ([N(x_o,t)\cap S - (\bigcup_{m=1}^{t-1} \bigcup_{j=1}^{k_m} I_j^m)] - \bigcup_{j=1}^{k_t} I_j^t) = (S_t - \bigcup_{j=1}^{k_t} I_j^t)$.

Therefore,

$m((N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m))) \leq m(S_t - \bigcup_{j=1}^{k_t} I_j^t) < \frac{1}{2^t}$.

But since t is an arbitrary positive integer greater than or equal to n, $m((N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m))) = 0$. We note that this last relationship holds for each positive integer n.

Now, we observe $S = S\cap(\bigcup_{n=1}^{\infty} N(x_o,n))$

$= \bigcup_{n=1}^{\infty}(N(x_o,n)\cap S)$, and $S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m) = S\cap \mathcal{C}(\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)$

$= (\bigcup_{n=1}^{\infty} N(x_o,n)\cap S)\cap \mathcal{C}(\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)$

$= \bigcup_{n=1}^{\infty}(N(x_o,n)\cap S\cap \mathcal{C}(\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m))$

$= \bigcup_{n=1}^{\infty}(N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m))$

Therefore $m(S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m) \leq m(\bigcup_{n=1}^{\infty}[N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)])$

$\leq \sum_{n=1}^{\infty} m(N(x_o,n)\cap S - (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} I_j^m)) = 0$.

It remains to show that $I_j^m \cap I_i^n = \phi$ if $m \neq n$

and $j \neq i$. Suppose $m \neq n$ and $m < n$. Suppose $x \varepsilon I_i^n$. Then, from the way in which I_i^n was chosen,

$$x \varepsilon N(x_o, n) - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m), \text{ so } x \notin I_j^m \text{ for any } j = 1,2,---,k_m,$$

and for any $m < n$. Conversely, if $x \varepsilon I_j^m$, then $x \notin N(x_o,n) - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m)$, and since $I_i^n \subset N(x_o,n) - (\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{k_m} I_j^m)$, $i = 1,2,---, k_n$, $x \notin I_i^n$. Also if $m = n$ and $j \neq i$, then $I_j^m \cap I_i^m = \phi$. Thus we have a sequence of disjoint sets, call it $\{I_n\}$, such that $I_n \varepsilon \mathcal{H}$ and $m(S - \bigcup_{n=1}^{\infty} I_n) = 0$.

The remark is made at this time that if G is an open subset of R, then $G \varepsilon \mathcal{L}$. This is because R is separable, R has an enumerable dense subset A, and hence an enumerable open base formed by taking all open spherical neighborhoods of positive rational radii of all points $a \varepsilon A$. Thus G may be written as an enumerable union of open spherical neighborhoods, each of which is a measurable set. Therefore G is measurable, as an enumerable union of measurable sets.

<u>Theorem</u> 3-4. If R is a separable dense-in-itself metric space and S is a measurable subset of R, and k is any real number, then the set of points at which the lower metric density of R is less than k is measurable.

Proof: We may suppose $0 < k \leq 1$. Consider the set G_{nm} of points of R which are contained in open neighborhoods of radius less than $\frac{1}{n}$, in which the relative measure of S is less than $k - \frac{1}{m}$, where m and n are positive integers. G_{nm} is an open set for all m and n, and hence measurable.

We want to show the set of points at which the lower metric density of S is less than k is $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm}$.

Suppose $x \varepsilon \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm}$. Then there exists m such that $x \varepsilon G_{nm}$, $n = 1,2,---$, and there exists $\delta_n > 0$, $p_n \varepsilon R$, such that $x \varepsilon N(p_n, \delta_n)$, $0 < \delta_n < \frac{1}{n}$ and $\frac{m(S \cap N(p_n,\delta_n))}{m(N(p_n,\delta_n))} < k - \frac{1}{m}$.

For each n, $l_S(x, \frac{1}{n}) < k - \frac{1}{m}$.

Also $L_S(x) = \underset{\delta>0}{\text{l.u.b.}}\ l_S(x,\delta) = \lim_{\frac{1}{n} \to 0} l_S(x, \frac{1}{n}) \leq k - \frac{1}{m} < k$.

Therefore $L_S(x) < k$ for all $x \varepsilon \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm}$.

On the other hand, suppose $L_S(x) < k$. Then there exists m such that $L_S(x) < k - \frac{1}{m}$.

$L_S(x) = \underset{\delta>0}{\text{l.u.b.}}\ l_S(x,\delta)$, so $l_S(x,\frac{1}{n}) < k - \frac{1}{m}$ for all n.

But $l_S(x,\frac{1}{n}) = \underset{\substack{p\varepsilon R \\ x\varepsilon N(p,\delta) \\ 0<\delta<\frac{1}{n}}}{\text{g.l.b.}} \frac{m(X \cap N(p,\delta))}{m(N(p,\delta))} < k - \frac{1}{m}$.

Thus for each n there exists an open spherical neighborhood $N(p_n,\delta_n)$ such that $p_n \varepsilon R$, $x \varepsilon N(p_n,\delta_n)$, $0 < \delta_n < \frac{1}{n}$, and $\frac{m(S \cap N(p_n,\delta_n))}{m(N(p_n,\delta_n))} < k - \frac{1}{m}$. Therefore $x \varepsilon G_{nm}$ for each n, and hence $x \varepsilon \bigcap_{n=1}^{\infty} G_{nm}$ and $x \varepsilon \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm}$. Thus $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm} = \{x | x \varepsilon R, L_S(x) < k\}$. But $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} G_{nm}$ is measurable as an enumerable union of enumerable intersections of open sets, so $\{x \mid x \varepsilon R,\ L_S(x) < k\} \varepsilon \mathcal{L}$.

<u>Theorem</u> 3-5. (Lebesgue density theorem). Suppose R is separable and dense-in-itself and S is a measurable subset of R. Suppose that for every $\varepsilon > 0$ there exists an open set G, such that $S \subset G$ and $m(G-S) < \varepsilon$. Let $\mathcal{A}$ be a Vitali covering of S, and suppose there is a positive real number t such that if $x \varepsilon R$, $\lambda > 0$, then $m(N(x, \lambda)) \leq tm(N(x, \frac{\lambda}{5}))$. Then the metric density of S exists and is equal to one at every point of S except for a set of measure zero.

Proof: The proof is by contradiction. Suppose the theorem is not true, i.e., suppose there is a measurable set $S \subset R$, and suppose for every $\varepsilon > 0$ there exists an open set G such that $m(G-S) < \varepsilon$, and the set of points of S at which the lower metric density is less than one has positive measure.

Let $T = \{x \mid x \varepsilon S,\ L_S(x) < 1\}$. By theorem 3-4, $T \varepsilon \mathcal{L}$, since $T = S \cap \{x \mid x \varepsilon R, L_S(x) < 1\} \varepsilon \mathcal{L}$. By assumption, $m(T) > 0$. Define $T_n = \{x \mid x \varepsilon S,\ L_S(x) < 1 - \frac{1}{n}\}$. For some n, $m(T_n) = k > 0$, since $T_n \subset T$, $\bigcup_{n=1}^{\infty} T_n = T$, $m(\bigcup_{n=1}^{\infty} T_n) = m(T)$, and $0 < m(\bigcup_{n=1}^{\infty} T_n) \leq \sum_{n=1}^{\infty} m(T_n)$. We will show that $m(T_n) = k > 0$ for an integer n leads to a contradiction.

Let G be an open set containing S such that $m(G-S) < \frac{k}{n}$. Suppose $x \varepsilon T_n$. $L_S(x) < 1 - \frac{1}{n}$ for $x \varepsilon T_n$, so

$l_S(x,\delta) < 1 - \frac{1}{n}$ for every $\delta>0$ and in particular for those δ's of the form $\frac{1}{m}$, m a positive integer. For each m, there exists a closed spherical neighborhood Q_{mx} such that $Q_{mx} \subset G$, $x \varepsilon Q_{mx}$, $\mathrm{rad}(Q_{mx}) < \frac{1}{m}$, and $\frac{m(S \cap Q_{mx})}{m(Q_{mx})} < 1 - \frac{1}{n}$. For every $x \varepsilon T_n$ we get such a sequence $\{Q_{mx}\}$, $m = 1,2,---$, such that $x \varepsilon Q_{mx}$, $Q_{mx} \subset G$, and $\frac{m(S \cap Q_{mx})}{m(Q_{mx})} < 1 - \frac{1}{n}$, $m = 1,2,---$, and $\lim_{m \to \infty} \mathrm{rad}(Q_{mx}) = 0$. Let

$$\mathcal{Q} = \{Q_{mx} | x \varepsilon T_n,\ m = 1,2,---\}.$$

$\mathcal{Q}$ covers T_n in the Vitali sense, i.e., $\mathcal{Q}$ is a set of closed apherical neighborhoods, and for every $x \varepsilon T_n$ and every $\varepsilon > 0$, there exists $Q \varepsilon \mathcal{Q}$ such that $x \varepsilon Q$ and $\mathrm{rad}(Q) < \varepsilon$.

By the Vitali theorem, T_n is covered except for a set of measure 0 by a countable number of disjoint closed spherical neighborhoods P_1, P_2, --- in $\mathcal{Q}$. Let $H = \bigcup_{m=1}^{\infty} P_m$. Then $H \subset G$, since $P_m \subset G$, $m = 1,2,---$. Also, $m(H) \geq k$ since H covers all of T_n except for a set of measure 0. Now $\frac{m(S \cap P_m)}{m(P_m)} < 1 - \frac{1}{n}$, for $P_m \varepsilon \mathcal{Q}$, so $\frac{m(\mathcal{C}S \cap P_m)}{m(P_m)} > \frac{1}{n}$ and $m(\mathcal{C}S \cap P_m) > \frac{1}{n} m(P_m)$. Therefore $m(\mathcal{C}S \cap H) = m(\bigcup_{m=1}^{\infty} (\mathcal{C}S \cap P_m))$

$$= \sum_{m=1}^{\infty} m(\mathcal{C}S \cap P_m) > \sum_{m=1}^{\infty} \frac{1}{n} m(P_m) = \frac{1}{n} \sum_{m=1}^{\infty} m(P_m)$$

$$= \frac{1}{n} m(H) \geq \frac{k}{n}.$$ But, $G \supset H$, so

$$\mathcal{C}S \cap G \supset \mathcal{C}S \cap H \text{ and } m(\mathcal{C}S \cap G) \geq m(\mathcal{C}S \cap H) > \frac{k}{n},$$

i.e., $m(G-S) > \frac{k}{n}$, a contradiction. Therefore the theorem must be true.

In the statement of the theorem we assumed the existence of an open set $G\supset S$ such that $m(G-S) < \varepsilon$ for any $\varepsilon>0$. We note that if S is open, the theorem is trivially true, since in this case the metric desnity of S is 1 at every point of S. If S is closed and $x_o \varepsilon R$, define $S_n = S\cap Q(x_o,n)$ for each positive integer n. S_n is closed and bounded. Suppose $\varepsilon>0$. For each n there exists G_n, G_n open, such that $S_n \subset G_n$ and $m(G_n - S_n) < \frac{\varepsilon}{2^n}$. Let $G = \bigcup_{n=1}^{\infty} G_n$. G is open and $S\subset G$. Also

$m(G-S) = m(\bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{\infty} S_n) \leq m(\bigcup_{n=1}^{\infty} (G_n - S_n))$

$\leq \sum_{n=1}^{\infty} m(G_n - S_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$. Thus again we can find an open set G, such that $S\subset G$, and $m(G-S) < \varepsilon$.

With the aid of the Lebesgue density theorem we may establish the remark made in section IV of chapter II concerning approximate continuity.

<u>Theorem</u> 3-6. Suppose the conditions of the Lebesgue density theorem are satisfied. Let f(x) be a measurable function on S. Then f(x) is approximately continuous at almost all points of S.

Proof: Let P be the set of points of S at which f is not approximately continuous. For every $x_o \varepsilon P$, there are rationals r_1 and r_2 such that $r_1 < f(x_o) < r_2$ and $\{x | x\varepsilon S,\ r_1 < f(x) < r_2\} \notin K(x_o)$. Define $P_{r_1 r_2} = \{x | x\varepsilon S,\ r_1 < f(x) < r_2\}$. $x_o \varepsilon P_{r_1 r_2}$ and x_o is not a point of density of $P_{r_1 r_2}$.

Define $C_{r_1r_2} = \{x \mid x \varepsilon P_{r_1r_2},\ P_{r_1r_2} \notin K(x)\}$. $x_o \varepsilon C_{r_1r_2}$.

Now $P_{r_1r_2} \varepsilon \mathcal{L}$, and $C_{r_1r_2} \varepsilon \mathcal{L}$ by theorem 3-4. By the Lebesgue density theorem, $m(C_{r_1r_2}) = 0$. Consider

$$\bigcup_{\substack{r_1<r_2 \\ r_1,r_2 \text{ rational}}} C_{r_1r_2}$$

. As the countable union of sets of measure 0, $m\left(\bigcup_{\substack{r_1<r_2 \\ r_1,r_2 \text{ rational}}} C_{r_1r_2}\right) = 0$. If $x_o \varepsilon P$, then there exists r_1, r_2 rational, such that $r_1 < f(x_o) < r_2$ and $P_{r_1r_2} \notin K(x_o)$, so $x_o \varepsilon P_{r_1r_2}$ and $x_o \varepsilon C_{r_1r_2}$. Therefore

$$P \subset \bigcup_{\substack{r_1<r_2 \\ r_1,r_2 \text{ rational}}} C_{r_1r_2}$$

and $m(P) = 0$.

REFERENCES

Goffman, C., _Real Functions_, New York: Rinehart, 1953.

Munroe, M. E., _Introduction to Measure and Integration_, Cambridge, Mass., Addison-Wesley, 1953.

de la Valleé Poussin, C., _Intégrales de Lebesgue, Fonctions d' Ensemble, Classes de Baire_, Paris: 1934.