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There are many operators associated with a domain $\Omega \subset \mathbb{C}^n$ with smooth boundary $\partial\Omega$. There are two closely related projections that are of particular interest. The *Bergman projection* $\mathcal{B}$ is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace $L^2(\Omega) \cap \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is the space of all holomorphic functions on Ω. The *Szegö projection* $\mathcal{S}$ is the orthogonal projection of $L^2(\partial\Omega)$ onto the space $H^2(\Omega)$ of boundary values of elements of $\mathcal{O}(\Omega)$. On Ω, these projection operators have integral representations

$$\mathcal{B}[f](z) = \int_{\Omega} f(w)\, B(z,w)\, dw, \qquad \mathcal{S}[f](z) = \int_{\partial\Omega} f(w)\, S(z,w)\, d\sigma(w).$$

The distributions B and S are known respectively as the Bergman and Szegö kernels. In an attempt to prove that $\mathcal{B}$ and $\mathcal{S}$ are bounded operators on L^p, $1 < p < \infty$, many authors have obtained size estimates for the kernels B and S for *pseudoconvex* domains in $\mathbb{C}^n$.

In this thesis, we restrict our attention to the Szegö kernel for a large class of domains of the form $\Omega = \left\{(z,w) \in \mathbb{C}^2 : Im[w] > b\,(Re[z])\right\}$. Such a domain fails to be pseudoconvex precisely when b is not convex on all of $\mathbb{R}$. In an influential paper, Nagel, Rosay, Stein, and Wainger obtain size estimates for both kernels and sharp mapping properties for their respective operators in the convex setting. Consequently, if b is a convex polynomial, the Szegö kernel S is absolutely convergent off the diagonal only. Carracino proves that the Szegö kernel has singularities on *and off* the diagonal for a specific non-smooth, *non-convex* piecewise defined quadratic b. Her results are novel since very little is known for the Szegö kernel for non-pseudoconvex domains Ω. I take b to be an arbitrary even-degree polynomial with positive leading coefficient and identify the set in $\mathbb{C}^2 \times \mathbb{C}^2$ on which the Szegö kernel is absolutely convergent. For a polynomial b, we will see that the Szegö kernel is smooth off the diagonal if and only if b is convex. These results provide an incremental step toward proving the projection S is bounded on $L^p(\partial\Omega)$, $1 < p < \infty$, for a large class of non-pseudoconvex domains Ω.

Contents

Chapter 1

Introduction

1.1 Background

For a domain $\Omega \subset \mathbb{C}^n$, let $L^2(\Omega)$ denote the space of all square-integrable functions on Ω and $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω. We define the *Bergman space* to be $B^2(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$, which is the closed subspace of square-integrable holomorphic functions on Ω. Although there are several operators naturally associated with each domain in $\mathbb{C}^n$, two are of particular interest. The *Bergman projection* $\mathcal{B} = \mathcal{B}_\Omega$ is the orthogonal projection of $L^2(\Omega)$ onto the subspace $B^2(\Omega)$. If Ω has a smooth boundary $\partial\Omega$, the *Szegö projection* $\mathcal{S} = \mathcal{S}_\Omega$ is the orthogonal projection of $L^2(\partial\Omega)$ onto the space $H^2(\Omega)$ of boundary values of elements of $\mathcal{O}(\Omega)$. These projections have are defined on Ω and have integral representations

$$\begin{aligned}
\mathcal{B}[f](z) &= \int_\Omega f(w)\, B(z,w)\, dw & \qquad f \in L^2(\Omega) \\
\mathcal{S}[f](z) &= \int_{\partial\Omega} f(w)\, S(z,w)\, d\sigma(w) & \qquad f \in L^2(\partial\Omega),
\end{aligned}$$

where $z \in \Omega$, dw denotes the Lebesgue measure on Ω, and $d\sigma$ the Lebesgue measure on $\partial\Omega$. The distributions B and S are known respectively as the Bergman and Szegö kernels.

In order to prove the existence of the Bergman kernel, we start by fixing an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ for the space $B^2(\Omega)$. Such a basis exists since $B^2(\Omega)$ is a closed subspace of the separable Hilbert space $L^2(\Omega)$. Define the formal sum

$$B(z,w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}.$$

One shows that this converges uniformly on compact subsets of $\Omega \times \Omega$. Then by the Riesz-Fischer theorem, we have that $\overline{B(z,\cdot)} \in B^2(\Omega)$ for each $z \in \Omega$. Moreover, for each $g \in B^2(\Omega)$,

$$g(z) = \int_{\Omega} g(w)\, B(z,w)\, dw.$$

The function B is the *Bergman kernel*. The existence of the *Szegö kernel* follows from a similar argument, replacing $B^2(\Omega)$ with $H^2(\Omega)$.

Many authors have analyzed the Bergman and Szegö projections on specific domains with the hope of acquiring sharp size estimates for the kernels and showing that the projections are bounded on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. Korányi and Vági [14] showed that on generalized half-spaces and the ball in $\mathbb{C}^n$, the Szegö projection is bounded on $L^p(\partial\Omega)$, for $1 < p < \infty$. Phong and Stein [10] showed L^p-boundedness of the Szegö projection on all bounded, strictly-pseudoconvex domains. McNeal and Stein acquired regularity theorems for both projections on convex domains, see [11] and [12].

The Bergman and Szegö kernels themselves have also been studied extensively for a variety of domains. The most notable of such work is for the unit ball in $\mathbb{C}^n$. In this setting, the Bergman and Szegö kernels have the explicit form

$$B(z,w) = \frac{n!}{\pi^n}\frac{1}{(1 - z\cdot\overline{w})^{n+1}} \quad \text{and} \quad S(z,w) = \frac{(n-1)!}{2\pi^n}\frac{1}{(1 - z\cdot\overline{w})^{n}}$$

(see [5] for a full derivation). To see a kernel associated with a domain which lacks complete symmetry, we turn to Greiner and Stein [13]. They computed the Szegö kernel for the domain $H_k := \{(z, z_1) \in \mathbb{C}^2 : Im(z_1) > |z|^{2k}\}$, for any positive integer k. More specifically, if $\zeta := \left(z, t + i\left(|z|^{2k} + \mu\right)\right.$ and $\omega := \left(w, s + i\left(|w|^{2k} + \eta\right)\right.$, with $\mu, \eta > 0$, then the Szegö kernel is given by

$$\begin{aligned} S(\zeta,\omega) &= \frac{1}{4\pi^2}\left[\left(\left(\frac{i}{2}[s-t] + \frac{|z|^{2k}+|w|^{2k}}{2} + \frac{\mu+\eta}{2}\right) - z\overline{w}\right)^2\right. \\ &\qquad\qquad \left.\times\left(\frac{i}{2}[s-t] + \frac{|z|^{2k}+|w|^{2k}}{2} + \frac{\mu+\eta}{2}\right)^{(k-1)/k}\right]^{-1}. \end{aligned}$$

For general domains, finding an explicit formula can be a difficult, if not impossible, task.

Pseudoconvex domains with smooth boundary are an important, large class of domains containing the above. Such domains have the form $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, where the defining function $\rho \in C^\infty(\mathbb{C}^n)$ satisfies $\nabla\rho(z) \neq 0$ when $\rho(z) = 0$. Conceptually, one should think of pseudoconvexity as a generalization of geometric convexity that is invariant under biholomorphic mappings. Even though every pseudoconvex domain is geometrically convex, the converse is not always true without added boundary conditions. A geometrically convex domain is (weakly) pseudoconvex if it has a C^2-boundary [5]. An analytic definition of pseudoconvexity is given in Section (2.3).

The research done on the boundary behavior of the Bergman and Szegö kernels associated with pseudoconvex domains is also quite extensive. For the Bergman kernel, J.J. Kohn obtained the formula $B = I - \bar{\partial}^* N \bar{\partial}$, where N is the $\bar{\partial}$-Neumann operator. With this formula, connections between the $\bar{\partial}$-Neumann problem and the Bergman kernel were established. For more details, see [4] and [5]. By exploiting classical solutions to the $\bar{\partial}$-Neumann problem, N. Kerzmann showed in [3] that the Bergman kernel can be smoothly extended to $\left(\overline{\Omega} \times \overline{\Omega}\right) \setminus \triangle$ for strictly-pseudoconvex domains with smooth boundary, where $\triangle := \{(z, w) \in \partial\Omega \times \partial\Omega : \ z = w\}$ is the diagonal of the boundary. For pseudoconvex domains of finite type in $\mathbb{C}^2$, Nagel, Rosay, Stein, and Wainger obtain size estimates for the Bergman and Szegö kernels in terms of a "natural" non-isotropic metric described in [[16], NRSW; clarify]. Using the same non-isotropic metric, they define a large class of integral operators called non-isotropic smoothing (NIS) operators and show that the Szegö projection is an NIS operator of order 0. As a consequence, they show that the Szegö projection possesses certain mapping properties and is bounded on $L^p(\partial\Omega)$ for $1 < p < \infty$. They establish a relationship between the Bergman and Szegö kernels and show that results for the Szegö kernel can be obtained from those for the Bergman kernel. For pseudoconvex (model) domains with defining function ρ which is a subharmonic, non-harmonic polynomial on $\mathbb{C}$, [16] also comment that the kernels have a much simpler relationship,

$$B(z,w) = 2i\frac{\partial S}{\partial \overline{w}_2}(z,w).$$

More generally, on pseudoconvex domains, it has been shown that both kernels are smooth on $\Omega \times \Omega$, but not necessarily on $\overline{\Omega} \times \overline{\Omega}$, [[16]; clarify].

In contrast with the amount of work done on pseudoconvex domains, very little has been done concerning the Szegö kernel on non-pseudoconvex domains. An important exception is the recent work of Carracino [1]. She obtained detailed estimates for the Szegö kernel on the boundary of the non-smooth, non-pseudoconvex domain $\Omega = \left\{(x+iy, t+i\xi) \in \mathbb{C}^2 : \xi > b(x)\right\}$, where

$$b(x) = \begin{cases} (x+1)^2 & , x < -\frac{1}{2} \\ -x^2 + \frac{1}{2} & , -\frac{1}{2} \leq x \leq \frac{1}{2} \\ (x-1)^2 & , \frac{1}{2} < x. \end{cases} \tag{1.1.1}$$

Our goal is to extend the results from this model to the large class of non-pseudoconvex domains of the form

$$\Omega = \left\{(z_1 := x+iy, z_2 := t+i\xi) \in \mathbb{C}^2 \,:\, \xi > b(x)\right\}, \tag{1.1.2}$$

for which b is a real-valued, non-convex polynomial with positive leading coefficient. (In Chapter 2 we verify that these are the correct conditions on b to make Ω non-pseudoconvex.)

1.2 The Szegö kernel as an integral

On the boundary of Ω, where $\xi = b(x)$, we make the identification

$$\partial\Omega \ni (x+iy, t+ib(x)) \longleftrightarrow (x,y,t) \in \mathbb{R}^3. \tag{1.2.1}$$

Then a global generator for the set of tangential antiholomorphic vector fields, found in (2.3.1), can be identified with the smooth vector field on $\mathbb{R}^3$ given by $\overline{L} = \dfrac{\partial}{\partial x} + i\left[\dfrac{\partial}{\partial y} - b'(x)\dfrac{\partial}{\partial t}\right]$. Define

$$\mathcal{H}^2(\Omega) := \left\{ F \in \mathcal{O}(\Omega) \,:\, \sup_{\epsilon>0} \int_{\partial\Omega} |F(x+iy, t+ib(x)+i\epsilon)|^2 \;dx\,dy\,dt < \infty \right\}.$$

Then $\mathcal{H}^2(\Omega)$ can be identified with the space of all functions $f \in L^2(\mathbb{R}^3)$ which satisfy the differential equation $\overline{L}[f] = 0$ in the distributional sense. (See [18] for detailed discussions on $\mathcal{H}^p$ spaces.) Under this identification, the *Szegö projection operator* $\mathcal{S}$ is defined to be the orthogonal projection of $L^2(\mathbb{R}^3)$ onto the null space of $\overline{L}$.

It turns out that the projection

$$\mathcal{S}[f](x,y,t) = \iiint f(r,s,u)\, S[(x,y,t),(r,s,u)]\, dr\, ds\, du \tag{1.2.2}$$

associated with this Szegö kernel S is a singular integral operator. To motivate the delicate approach that one must take when dealing with such operators, we recall the Hilbert transform. This transform is given by the convolution of an L^p-function with the kernel $K(x) = \dfrac{1}{\pi x}$ for

$x \in \mathbb{R}$, i.e.,

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\ dy = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y)\ K(x-y)\ dy.$$

The transform is at best conditionally convergent on $L^p(\mathbb{R})$, for $1 < p < \infty$, since the kernel $K(x-y) = \dfrac{1}{x-y}$ has a singularity on the diagonal at $x = y$. Thus we should imagine the integral as a principal-value integral.

In [7], an explicit integral formula for the Szegö kernel has been obtained for domains of the form (1.1.2). If $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are elements of $\mathbb{C}^2$,

$$S(z,w) = c \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\tau e^{\eta\tau[z_1+\bar{w}_1]+i\tau[z_2-\bar{w}_2]}}{\int_{-\infty}^{\infty} e^{2\tau[\eta\lambda - b(\lambda)]}\, d\lambda}\, d\eta\, d\tau, \tag{1.2.3}$$

where c is an absolute constant. This thesis will follow in the footsteps of Haslinger [`?Haslinger:95`], Carracino [1], Halfpap, Nagel, and Wainger [18], all of whom take (1.2.3) as their starting point. After identifying the boundary with $\mathbb{R}^3$, the Szegö kernel becomes

$$S[(x,y,t),(r,s,u)] = c \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{\tau e^{\tau[i(t-u)+i\eta(y-s)-[b(x)+b(r)-\eta(x+r)]]}}{\int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda]}\, d\lambda}\, d\eta\, d\tau. \tag{1.2.4}$$

Unlike the Hilbert transform, it is not immediately clear where the Szegö kernel is smooth. It was shown in [[16]; clarify] that for any convex polynomial b, the Szegö kernel is only smooth off the diagonal $\Delta = \{[(x,y,t),(r,s,u)] : (x,y,t) = (r,s,u)\}$. As mentioned above, [16] obtained size estimates for the Szegö kernel of the form

$$|S[(x,y,t),(r,s,u)]| \leq C|B((x,y,t),\delta)|^{-1},$$

with corresponding estimates for its derivatives. In the above, δ is the non-isotropic distance between (x,y,t) and (r,s,u) induced by the domain Ω. For the Szegö kernel, the required non-isotropic singular-integral theory is obtained by substituting the Euclidean metric with a pseudometric which is non-isotropic. For a full description of this process, see [15] Balls and

Metrics for details.

For the non-pseudoconvex domain considered by Carracino, singularities exist on and off the diagonal. These singularities correspond to the non-unique location of the global minimum of the function $B_\eta(\lambda) = b(\lambda) - \eta\lambda$ as a function of λ, which appears in the denominator integral. She obtains precise estimates of the Szegö kernel near two critical locations off the diagonal, namely at points corresponding to $(x, r) = (\pm 1, \mp 1)$. At these critical points, Carracino shows that the lack of convexity of b causes the Szegö kernel to diverge. She then shows that away from these two points, the convexity of b yields results which parallel those found in the convex setting, i.e., singularities only when $x = r$.

The goal of this thesis is to show that for any even-degree polynomial with positive leading coefficient, singularities occur off the diagonal if and only if b is non-convex. As in Carracino's work, we should expect that the singularities will correspond to non-unique locations of the global minimum of B_η. We state these results more precisely in the next section.

1.3 Statement of the main theorems

We consider the integral defining S given in (1.2.4) for domains of the form (1.1.2) with b a non-convex, even-degree polynomial. We may assume without loss of generality that b' is monic. For each $\eta \in \mathbb{R}$, define $B_\eta(\lambda) := b(\lambda) - \eta\lambda$. Notice that by our choice of b, $\inf_{\lambda\in\mathbb{R}} B_\eta(\lambda)$ is finite, and it is attained at some $\lambda \in \mathbb{R}$. The collection of all such minimizing values is denoted Λ_η. Also, define the *Legendre transform* of b by

$$b^*(\eta) := \sup_{\lambda\in\mathbb{R}}(-B_\eta(\lambda)). \tag{1.3.1}$$

Let z and w be elements of $\mathbb{C}^2$ defined by

$$z = (z_1, z_2) = (x + iy, t + ib(x) + ih)$$
$$w = (w_1, w_2) = (r + is, u + ib(r) + ik).$$

Set

$$\Sigma := \{\, (z, w) : \ x, r \in \Lambda_\eta \text{ for some } \eta \in \mathbb{R} \,\} .$$

We have the following theorems:

Theorem 1.3.1. *If b is an even-degree polynomial with monic derivative, then the integral defining the Szegö kernel is absolutely convergent in the region in which*

$$h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

This is an open neighborhood of $(\overline{\Omega} \times \overline{\Omega}) \setminus \Sigma$. More generally, if i_1, j_1, i_2, and j_2 are non-negative integers, then

$$\partial_{z_1}^{i_1} \partial_{\bar{w}_1}^{j_1} \partial_{z_2}^{i_2} \partial_{\bar{w}_2}^{j_2} S(z, w) = c \int_{-\infty}^{\infty} \int_0^{\infty} e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} \frac{\eta^{i_1+j_1} \tau^{i_1+j_1+i_2+j_2+1}}{N(\eta, \tau)} \, d\eta \, d\tau$$

is absolutely convergent in the same region. (By Lemma 3.3.1, b^* is convex with super-linear growth; hence b^{**} is finite.)

By the definition of our domain, the point (z, w) is in $\left(\overline{\Omega} \times \Omega\right) \cup \left(\Omega \times \overline{\Omega}\right)$ if and only if $h + k > 0$. The following theorems are meant to address the case in which $h = k = 0$. They describe the points on the diagonal at which the Szegö kernel is divergent in terms of the local behavior of B_η.

Theorem 1.3.2. *If x and r are not in the same Λ_η for each $\eta \in \mathbb{R}$, the Szegö kernel and all of its partial derivatives converge absolutely.*

Theorem 1.3.3. *If x and r are both in Λ_{η_0} for some $\eta_0 \in \mathbb{R}$, then the Szegö kernel is not* *absolutely convergent.*

These two lemmas and a result given in Chapter 4 allow us to prove a biconditional statement relating the smoothness of the integral kernel off the diagonal to the convexity of b.

Corollary 1.3.4. *The Szegö kernel is smooth off of the diagonal if and only if b is a convex polynomial.*

Chapter 2

Domains Under Consideration

We begin the chapter by motivating our choice of the function b given in (2.0.1). Then we describe the manifold structure on $\partial\Omega$ and use it to prove the necessary conditions which b must satisfy to guarantee that Ω is non-pseudoconvex. Finally, the chapter closes with a section that explores how we may further restrict the form of b.

We study in domains of the form

$$\Omega = \left\{(z_1 := x + iy, z_2 := t + i\xi) \in \mathbb{C}^2 : \ \xi > b(x)\right\}, \tag{2.0.1}$$

where b is an even-degree, non-convex polynomial with monic derivative.

2.1 Our assumptions on b

To motivate this particular class of domains, we turn our attention to more recent endeavors. Nagel, Rosay, Stein, and Wainger studied unbounded model domains of the form (2.0.1), where b is a sub-harmonic, non-harmonic polynomial of degree m. In [7], a report of their joint work, Nagel reviews *spaces of homogenous type*, where he obtains the explicit integral formula for the Szegö kernel (1.2.4) and obtains sharp size estimates off the diagonal. Halfpap, Nagel, and Wainger showed that when b is a smooth, convex function with a point of infinite type satisfying certain growth conditions, the Bergman and Szegö kernels have singularities on $\Omega \times \Omega$ that are away from the diagonal. As mentioned above, Carracino investigated a non-pseudoconvex model domain Ω whose boundary was defined by a piecewise, non-smooth, non-convex quadratic. To build on Carracino's results, we have studied model domains of the from (2.0.1), where b is a non-convex quartic polynomial. We used the explicit formula for the Szegö kernel given in (1.2.4) and completely described the set on which the Szegö kernel is absolutely convergent. It was shown that singularities off the diagonal correspond to the existence of an inflection point of the quartic polynomial that defines the boundary.

This thesis is meant to compare and extend the results of Gilliam and Halfpap [2] to all even-degree, non-convex polynomials with monic derivative. We take our polynomial b to have the form

$$b(\lambda) = \frac{1}{2n}\lambda^{2n} + a_1\lambda^{2n-1} + \cdots + a_{2n-2}\lambda^2 + a_{2n-1}\lambda + a_{2n}, \tag{2.1.1}$$

where $n \geq 2$, and $a_i \in \mathbb{R}$, for $i = 1, \ldots, 2n$. In turn, obtaining results on domains for which b has this general form will provide a lens through which we can see the connection with the convex results.

2.2 Manifold structure

In order to understand the manifold structure, we view the boundary of Ω as a subset of $\mathbb{R}^4$, $\partial\Omega$ can be written as

$$\partial\Omega = \{(x, y, t, \xi) \in \mathbb{R}^4 : \rho(x, y, t, \xi) := b(x) - \xi = 0\}.$$

If b is an even-degree polynomial with monic derivative, then observe how the boundary is bounded in the negative-ξ direction since $\xi = b(x)$. Also for each cross section perpendicular to the y,t-axis, there are exactly two solutions whenever $\xi > M$, for $M := \max\{|b(\lambda)| : \lambda \in \mathbb{R} \text{ and } b'(\lambda) = 0\}$. This domain is a tube-like domain.

The tangent space of $\partial\Omega$ at $p \in \partial\Omega$ is given by

$$T_p(\partial\Omega) = \left\{L_p \in T_p(\mathbb{R}^4) : \; L_p[\rho] = 0\right\}, \tag{2.2.1}$$

where L_p is the smooth vector field $L = \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial t} + \alpha_4 \frac{\partial}{\partial \xi}\right)$ with $\alpha_j \in C^\infty(\mathbb{R}^4)$ evaluated at p. Forcing $\rho(x,y,t,\xi) = b(x) - \xi$ to be a solution to the equation $L_p[\rho] = 0$ yields a linear relationship on the coefficients,

$$L_p[\rho] = \alpha_1 b'(x) - \alpha_4 = 0 \qquad \Longleftrightarrow \qquad \alpha_4 = \alpha_1 b'(x),$$

which gives us a basis for the tangent space at p. The real tangent space at $p \in \partial\Omega$ written in terms of its basis is

$$T_p(\partial\Omega) = \left\langle \left(\frac{\partial}{\partial x} + b'(x)\frac{\partial}{\partial \xi}\right)_p, \; \frac{\partial}{\partial y}_p, \; \frac{\partial}{\partial t}_p \right\rangle,$$

which is a real hyperplane.

To say more, we introduce a map on $\mathbb{R}^4$ $(= T_p(\mathbb{R}^4))$, denoted by J. This map is called the *complex structure map*, which is a generalization of multiplication by i. It acts on the basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right\}$ in the following way:

$$J\left[\frac{\partial}{\partial x_j}\right] = \frac{\partial}{\partial y_j} \quad \text{and} \quad J\left[\frac{\partial}{\partial y_j}\right] = -\frac{\partial}{\partial x_j}.$$

The *holomorphic tangent space* at $p \in \partial\Omega$ is the largest J-invariant subspace of $T_p(\partial\Omega)$ and is given by

$$H_p(\partial\Omega) := T_p(\partial\Omega) \cap J\left[T_p(\partial\Omega)\right] = \left\langle \left(\frac{\partial}{\partial x} + b'(x)\frac{\partial}{\partial \xi}\right)_p, \ \left(\frac{\partial}{\partial y} - b'(x)\frac{\partial}{\partial t}\right)_p \right\rangle.$$

Notice that the real dimension of $H_p(\partial\Omega)$ is constant and independent of $p \in \partial\Omega$. A real manifold with this property is called a *CR manifold.* Hence $\partial\Omega$ is a CR manifold with real dimension 2.

For any vector space V, the *complexified vector space* $V^{\mathbb{C}}$ is defined by the sum $V^{\mathbb{C}} := V + iV$. Therefore the complexified tangent space at $p \in \partial\Omega$ is given by $T_p^{\mathbb{C}}(\partial\Omega) = T_p(\partial\Omega) + iT_p(\partial\Omega)$. The $-i$ eigenspace of $T_p^{\mathbb{C}}(\partial\Omega)$ is denoted $H_p^{(0,1)}(\partial\Omega)$ and is generated by the single vector $\overline{L}_p = \left(\frac{\partial}{\partial \bar{z}_1} + ib'\left(\frac{p+\bar{p}}{2}\right)\frac{\partial}{\partial \bar{z}_2}\right)_p$. We can use the Lie bracket, defined by $[\overline{L}^1, \overline{L}^2]_p := \overline{L}_p^1 L_p^2 - \overline{L}_p^2 L_p^1$, to construct additional generators for the complexified tangent space at p. Using $\overline{L}_p = \left(\frac{\partial}{\partial \bar{z}_1} + ib'\left(\frac{p+\bar{p}}{2}\right)\frac{\partial}{\partial \bar{z}_2}\right)_p$,

$$L_p = \left(\frac{\partial}{\partial z_1} - ib'\left(\frac{p+\bar{p}}{2}\right)\frac{\partial}{\partial z_2}\right)_p \quad \text{and} \quad \left[\,\overline{L}, L\,\right]_p = -\frac{1}{2}b''\left(\frac{p+\bar{p}}{2}\right)\left(\frac{\partial}{\partial \bar{z}_2} + \frac{\partial}{\partial z_2}\right)_p.$$

Under the operation of Lie brackets, one may verify that these are all the generators of the complexified tangent space at p, unless b'' vanishes at p (in which case $\left[\,L, \overline{L}\,\right]_p = 0$). Somewhat surprisingly, this allows us to recover the complexified tangent space at p,

$$T_p^{\mathbb{C}}(\partial\Omega) = \left\langle L_p, \overline{L}_p, \left[L_p, \overline{L}_p\,\right]\right\rangle.$$

2.3 Pseudoconvexity

We are in a position to state the analytic definition of pseudoconvexity. The Levi form, at a point $p \in \partial\Omega$, applied to a smooth antiholomorphic vector $\overline{L}_p = \left(\sum_{j=1}^n \xi_j \frac{\partial}{\partial \bar{z}_j}\right)_p$ is defined as $\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\xi_j\bar{\xi}_k$, where $\xi_j := \xi_j(p)$. We say that Ω is *strictly-pseudoconvex* if the Levi form at point p is strictly positive or negative definite, i.e., if $\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\xi_j\bar{\xi}_k > 0$ for all smooth, non-zero vectors satisfying the differential equation $L_p[\rho] = 0$. We say that Ω is *weakly-pseudoconvex* if the Levi form is semi-definite.

Lets determine which conditions on b are necessary to guarantee that our model domain (1.1.2) fails to be pseudoconvex. On this domain, the set of all antiholomorphic smooth vector fields that annihilate our defining function $\rho(z_1, z_2) := b\left(\frac{z_1+\overline{z}_1}{2}\right) + \frac{\bar{z}_2 - z_2}{2}$ is generated by one element, the tangential Cauchy-Riemann operator applied to $\partial\Omega$

$$\overline{L} = \frac{\partial}{\partial \bar{z}_1} - ib'\left(\frac{z_1 + \overline{z}_1}{2}\right)\frac{\partial}{\partial \bar{z}_2} \tag{2.3.1}$$

(see appendix for details). Accordingly, $\xi_1 = 1$ and $\xi_2 = -ib'$. Therefore the pseudoconvexity of Ω is equivalent to the condition

$$0 \leq \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\xi_j\bar{\xi}_k \;=\; \frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_1}(p)|\xi_1|^2 \;=\; \frac{1}{4}b''\left(\frac{z_1 + \bar{z}_1}{2}\right) = \frac{1}{4}b''(x)\,.$$

Thus Ω is pseudoconvex if and only if b is convex. Hence, for novel results, we take b to be non-convex.

We now further restrict the class of polynomials we consider. This process is used in the quartic setting, which is covered in Chapter 7. Also, this process highlights a slight "symmetry" that is occurring in the main integrand.

2.4 Additional restrictions on b

To start the section, recall that a simplification of the integral in question is

$$S[(x,0,0),(r,0,0)] = c\int_0^\infty \int_{-\infty}^\infty \frac{\tau e^{-\tau[b(x)+b(r)-\eta(x+r)]}}{\int_{-\infty}^\infty e^{-2\tau[b(\lambda)-\eta\lambda]}\,d\lambda}\,d\eta\,d\tau. \tag{2.4.1}$$

This integral is fairly complicated, so it might be to our advantage to apply linear transformations to b to obtain a new polynomial $\tilde{b}$, where $\tilde{b}$ vanishes to at least second order at the origin and the coefficient of the term of second-largest degree is zero. We say that $\tilde{b}$ is the *reduced* form of b.

To achieve this, we start by fixing a polynomial b of the form (2.1.1). In order to find the appropriate linear transformation needed to eliminate the degree $2n-1$ term, set $\lambda = x + \beta$. Then

$$\begin{aligned} b(x+\beta) &= \frac{1}{2n}(x+\beta)^{2n} + a_1(x+\beta)^{2n-1} + \cdots \\ &= \frac{1}{2n}\left(x^{2n} + 2n\beta x^{2n-1} + \cdots\right) + a_1\left(x^{2n-1} + (2n-1)\beta x^{2n-2} + \cdots\right) + \cdots \\ &= \frac{1}{2n}x^{2n} + (\beta + a_1)x^{2n-1} + \cdots . \end{aligned}$$

By taking $\beta = -a_1$, we eliminate the degree $2n-1$ term. Let us relabel the coefficients with b_i's so that

$$b(x-a_1) = \frac{1}{2n}x^{2n} + b_2 x^{2n-2} \cdots + b_{2n-2}x^2 + b_{2n-1}x + b_{2n},$$

where $b_1 = 0$. We claim that $\tilde{b}(x) = b(x-a_1) - b_{2n-1}x - b_{2n}$.

Consider the integrand of $S[(x-a_1,0,0),(r-a_1,0,0)]$. The exponent of the numerator is

$$-\tau\Big[b(x-a_1) + b(r-a_1) - \eta(x+r-2a_1)\Big].$$

Under the mapping $\eta \mapsto \eta + b_{2n-1}$, the numerator beomes

$$
\begin{aligned}
-\tau&\Big[b(x-a_1)+b(r-a_1)-(\eta+b_{2n-1})(x+r-2a_1)-2b_{2n}+2b_{2n}\Big]\\
&=-\tau\Big[b(x-a_1)+b(r-a_1)-\eta(x+r)+2\eta a_1-b_{2n-1}(x+r)+2a_1b_{2n-1}-2b_{2n}+2b_{2n}\Big]\\
&=-\tau\Bigg[\Big[b(x-a_1)-b_{2n-1}x-b_{2n}\Big]+\Big[b(r-a_1)-b_{2n-1}r-b_{2n}\Big]-\eta(x+r)\Bigg]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2\tau\Big[\eta a_1+a_1b_{2n-1}+b_{2n}\Big]\\
&=-\tau\Big[\tilde{b}(x)+\tilde{b}(r)-\eta(x+r)\Big]-2\tau\Big[\eta a_1-a_1b_{2n-1}+b_{2n}\Big]. \qquad (2.4.2)
\end{aligned}
$$

In order to compare, let us examine the exponent of the integrand of the λ-integral after the change of variable $\lambda \mapsto \lambda - a_1$,

$$
\begin{aligned}
-2\tau&\Big[b(\lambda-a_1)-(\eta+b_{2n-1})(\lambda-a_1)-b_{2n}+b_{2n}\Big]\\
&=-2\tau\Big[b(\lambda-a_1)-\eta\lambda+\eta a_1-b_{2n-1}\lambda+b_{2n-1}a_1-b_{2n}+b_{2n}\Big]\\
&=-2\tau\Bigg[\Big[b(\lambda-a_1)-b_{2n-1}\lambda-b_{2n}\Big]-\eta\lambda\Bigg]-2\tau\Big[\eta a_1+a_1b_{2n-1}+b_{2n}\Big]\\
&=-2\tau\Big[\tilde{b}(\lambda)-\eta\lambda\Big]-2\tau\Big[\eta a_1+a_1b_{2n-1}+b_{2n}\Big]. \qquad (2.4.3)
\end{aligned}
$$

If we set $C := -2\tau\Big[\eta a_1 + a_1 b_{2n-1} + b_{2n}\Big]$ and substitute (2.4.2) and (2.4.3), the integrand of $S[(x-a_1,0,0),(r-a_1,0,0)]$ reduces to

$$
\frac{\tau e^{-\tau\left[\tilde{b}(x)+\tilde{b}(r)-\eta(x+r)\right]+C}}{\int_{-\infty}^{\infty} e^{-2\tau\left[\tilde{b}(\lambda)-\eta\lambda\right]+C}\,d\lambda}=\frac{\tau e^{-\tau\left[\tilde{b}(x)+\tilde{b}(r)-\eta(x+r)\right]}}{\int_{-\infty}^{\infty} e^{-2\tau\left[\tilde{b}(\lambda)-\eta\lambda\right]}\,d\lambda},
$$

where $\tilde{b}$ vanishes to at least second order at the origin and the coefficient of the degree $2n-1$ term is zero.

It follows that $S[(x,0,0),(r,0,0)]$ is divergent on $\Big\{((x,0,0),(r,0,0)) : \ x,r\in\mathbb{R}\Big\}$ if and only if $S[(x-a_1,0,0),(r-a_1,0,0)]$ is divergent on the shifted set $\Big\{((x+a_1,0,0),(r+a_1,0,0)) :$

$x, r \in \mathbb{R}\big\}$. We may therefore assume W.L.O.G. our b has the form

$$b(\lambda) = \frac{1}{2n}\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n-2}\lambda^2, \tag{2.4.4}$$

where $n \geq 2$, and $a_i \in \mathbb{R}$, for $i = 2, \ldots, 2n-2$.

Chapter 3

Global Behavior of $\lambda(\eta)$ and $B_\eta(\lambda)$

In order to prove the main results of this thesis, it is essential to understand the behavior of each factor of the integrand of (1.2.4). Since these are exponential functions, it will be particularly useful to understand how the extreme values of the exponents and their location(s) vary with η. In this chapter, we study the asymptotic behavior for both in detail. These estimates will allow us to conclude that the η-integral in (1.2.4) is absolutely convergent at infinity.

3.1 Definitions and notation

We fix a polynomial b of the form (2.1.1). We may refer to the family $\mathcal{B} := \{B_\eta(\lambda) = b(\lambda) - \lambda\eta\}_{\eta\in\mathbb{R}}$ for brevity. Since the global minimum of each member of $\mathcal{B}$ is always attained at a finite number of points, we define the set containing these locations as well as its extreme elements.

Definition 3.1.1. *Let Λ_η be the set containing the location(s) of the global minimum of B_η as a function of λ. Moreover, let the largest and smallest elements of Λ_η be $\lambda(\eta)$ and $\sigma(\eta)$,* *respectively. Thus*

$$\Lambda_\eta := \{\sigma(\eta), \ldots, \lambda(\eta)\}$$

with $\sigma(\eta) \leq \cdots \leq \lambda(\eta)$. *Often we will think of* $\lambda(\eta)$ *and* $\sigma(\eta)$ *as fixed numbers instead of functions, so we set* $\lambda_\eta := \lambda(\eta)$ *and* $\sigma_\eta := \sigma(\eta)$ *for clarity. To distinguish between the function* λ *and the variable* λ, *we will often denote the function* $\lambda(\cdot)$. *Finally, the image of* $\lambda(\cdot)$ *is defined as* Λ.

Notice that $\sigma(\cdot)$ and $\lambda(\cdot)$ are well-defined functions on $\mathbb{R}$, and if the location of a global minimum is unique, $\sigma(\eta) = \lambda(\eta)$. In fact, $\sigma(\eta) = \lambda(\eta)$ if and only if $|\Lambda_\eta| = 1$, where $|\cdot|$ denotes the standard cardinality of a finite set.

3.2 Asymptotic behavior of $\lambda(\eta)$

We make some observations about $\lambda(\cdot)$, which follow from properties of b. The value $\lambda(\eta)$ is a critical point for B_η, i.e., it is a solution to the equation $B'_\eta(\lambda) = b'(\lambda) - \eta = 0$. Since b' is an odd-degree polynomial, there exists at least one solution to the equation $b'(\lambda) = \eta$ for each $\eta \in \mathbb{R}$. To find the values of η at which $\lambda(\eta)$ is the unique solution, recall the properties of b. The polynomial b is an even-degree, non-convex polynomial with positive leading coefficient, so it has at least one inflection point. Thus the set $\{|b'(\lambda)| : \lambda \in \mathbb{R} \text{ and } b''(\lambda) = 0\}$ is always non-empty and contains a finite number of elements; hence $M := \max\{|b'(\lambda)| : \lambda \in \mathbb{R} \text{ and } b''(\lambda) = 0\}$ is always finite. It follows that the equation $b'(\lambda) = \eta$ has a unique solution whenever $|\eta| > M$. By definition, $\lambda(\eta)$ is precisely the unique solution to the equation

$$\eta = \lambda^{2n-1} + \sum_{k=1}^{2n-1} b_k \lambda^{2n-1-k}, \tag{3.2.1}$$

where $b_k := a_k(2n-k)$. By applying the triangle inequality, the solution λ for each $|\eta| > M$ satisfies

$$\begin{aligned} |\eta| - |b_{2n-1}| &\leq |\eta - b_{2n-1}| \\ &= \left| \lambda^{2n-1} + \sum_{k=1}^{2n-2} b_k \lambda^{2n-1-k} \right| \\ &\leq |\lambda|^{2n-1} + \sum_{k=1}^{2n-2} |b_k| \, |\lambda|^{2n-1-k} \\ &\leq C \cdot \left(|\lambda|^{2n-1} + 1 \right) \end{aligned}$$

where $C := \left(1 + \sum_{k=1}^{2n-2} |b_k|\right)$. We conclude that as $|\eta| \to \infty$, the solution $|\lambda| \to \infty$. Also from (3.2.1),

$$\frac{\eta}{\lambda^{2n-1}} = 1 + \sum_{k=1}^{2n-1} b_k \lambda^{-k} \tag{3.2.2}$$

whenever a solution $\lambda \neq 0$. Thus if $|\lambda| \to \infty$, $|\eta| \to \infty$ and the sign of η and λ agree for all $|\lambda|$ sufficiently large. If we apply this and what we concluded just above to (3.2.2), then $\frac{\eta}{\lambda^{2n-1}} = 1 + o(1)$ as $\eta \to \pm\infty$ (or $\lambda \to \pm\infty$). By definition, we have just shown that $\eta \sim \lambda^{2n-1}$ as $\eta \to \pm\infty$ (or $\lambda \to \pm\infty$).

Indeed, we can say more. Notice that for any positive α and for x in, for example, $(-1/2, 1/2)$,

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + O(x^2) \\ &= 1 + x(\alpha + O(x)), \end{aligned} \tag{3.2.3}$$

so $(1+x)^\alpha = 1 + o(1)$ as $x \to 0$. For all $|\eta|$ (or $|\lambda|$) sufficiently large, $\sum_{k=1}^{2n-1} b_k \lambda^{-k} \in (-1/2, 1/2)$. By (3.2.3),

$$\frac{\eta^{\frac{1}{2n-1}}}{\lambda} = 1 + o(1)$$

as $\eta \to \pm\infty$ (or $\lambda \to \pm\infty$). By definition, we have just proven that

$$\eta^{\frac{1}{2n-1}} \sim \lambda \tag{3.2.4}$$

as $\eta \to \pm\infty$ (or $\lambda \to \pm\infty$). As a consequence, $\lim_{\eta\to\infty} \lambda(\eta) = \infty$ and $\lim_{\eta\to-\infty} \lambda(\eta) = -\infty$.

These estimates allow us to make an immediate observation.

Lemma 3.2.1. *For all $N \in \mathbb{N}$,*

$$\Lambda \cap (-\infty, -N) \neq \emptyset \quad \textit{and} \quad \Lambda \cap (N, \infty) \neq \emptyset.$$

3.3 Asymptotic behavior of $B_\eta(\lambda(\eta))$

As stated above, we want to derive asymptotic estimates that will allow us to establish convergence of the η-integral in equation (1.2.4) at infinity. Understanding how the global minimum of B_η varies for large $|\eta|$ will prove useful. Fortunately, the prior section will shed some light on these estimates; the asymptotic approximation of $B_\eta(\lambda_\eta)$ and the derivatives $B_\eta^{(j)}(\lambda_\eta)$ can be obtained from the asymptotic approximation of $\lambda(\cdot)$.

For each $\eta \in \mathbb{R}$, the value of B_η at the global minimum can be expressed as

$$\begin{aligned} B_\eta(\lambda_\eta) &= b(\lambda_\eta) - \eta\lambda_\eta \\ &= \frac{1}{2n}\lambda_\eta^{2n} + \sum_{k=1}^{2n-1} a_k \lambda_\eta^{2n-k} + a_{2n} - \eta\lambda_\eta. \end{aligned}$$

From estimate (3.2.4), $\lambda(\eta) = \eta^{\frac{1}{2n-1}}(1+o(1))$ as $|\eta| \to \infty$. By substitution and an application of the little-o property given in (3.2.3),

$$\begin{aligned}
B_\eta(\lambda(\eta)) &= \frac{1}{2n}\eta^{\frac{2n}{2n-1}}(1+o(1))^{2n} + \sum_{k=1}^{2n-1} a_k \left(\eta^{\frac{1}{2n-1}}(1+o(1))\right)^{2n-k} + a_{2n} - \eta^{\frac{2n}{2n-1}}(1+o(1)) \\
&= \frac{1}{2n}\eta^{\frac{2n}{2n-1}}(1+o(1)) + \sum_{k=1}^{2n-1} a_k \eta^{\frac{2n-k}{2n-1}}(1+o(1)) + a_{2n} - \eta^{\frac{2n}{2n-1}}(1+o(1)) \\
&= \left(\frac{1-2n}{2n}\right)\eta^{\frac{2n}{2n-1}}(1+o(1)) + \sum_{k=1}^{2n-1} a_k \eta^{\frac{2n-k}{2n-1}}(1+o(1)) + a_{2n} \\
&= \left(\frac{1-2n}{2n}\right)\eta^{\frac{2n}{2n-1}}\left[(1+o(1)) + \left(\frac{2n}{1-2n}\right)\left(\sum_{k=1}^{2n-1} a_k \eta^{\frac{-k}{2n-1}}(1+o(1)) + a_{2n}\eta^{\frac{-2n}{2n-1}}\right)\right] \\
&= \left(\frac{1-2n}{2n}\right)\eta^{\frac{2n}{2n-1}}(1+o(1))
\end{aligned}$$

as $|\eta| \to \infty$. By defintion,

$$B_\eta(\lambda(\eta)) \sim \left(\frac{1-2n}{2n}\right)\eta^{\frac{2n}{2n-1}} \tag{3.3.1}$$

as $|\eta| \to \infty$. Since $b^*(\eta) = -B_\eta(\lambda(\eta))$, we have proved

Lemma 3.3.1.

$$b^*(\eta) \sim \left(\frac{2n-1}{2n}\right)\eta^{\frac{2n}{2n-1}} \tag{3.3.2}$$

as $|\eta| \to \infty$.

From this, we arrive at another asymptotic approximation that will be useful in later chapters. For a fixed $\lambda_0 \in \mathbb{R}$,

$$b(\lambda_0) - \eta(\lambda_0) + b^*(\eta) \sim \left(\frac{2n-1}{2n}\right)\eta^{\frac{2n}{2n-1}} \tag{3.3.3}$$

as $|\eta| \to \infty$.

3.4 Asymptotic behavior of $b^{(j)}(\lambda_\eta)$

We now interpret $b^*(\eta)$ as the appropriate quantity to add to B_η to acquire a non-negative polynomial in λ, which vanishes to even order. In other words, for each fixed $\eta \in \mathbb{R}$, $b(\lambda) - \eta\lambda + b^*(\eta) \geq 0$, and each real zero has even multiplicity. With this observation, $B_\eta(\lambda) + b^*(\eta)$ has a Taylor series expansion about λ_η,

$$B_\eta(\lambda) + b^*(\eta) = \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda_\eta)}{j!} (\lambda - \lambda_\eta)^j. \tag{3.4.1}$$

It is now very clear why we must understand the asymptotic behavior of $b^{(j)}(\lambda_\eta)$.

Since $b^{(j)}$ is polynomial in λ, for each $j = 2, \ldots, 2n$, the technique used in the prior section will give us the desired asymptotic estimates. $\lambda_\eta \sim \eta^{\frac{1}{2n-1}}$ as $|\eta| \to \infty$, so

$$b^{(j)}(\lambda_\eta) \ \sim \ \frac{(2n-1)!}{(2n-j)!}\eta^{\frac{2n-j}{2n-1}} \tag{3.4.2}$$

as $|\eta| \to \infty$.

With these results, we turn our attention to the local behavior of the location(s) of the global minimum of B_η. This is a bit more delicate and will be invaluable in describing the singularities of the Szegö kernel, on and off the diagonal.

Chapter 4

Properties of $\lambda(\eta)$

In this chapter, we build on the prior by analyzing the local behavior of $\lambda(\cdot)$. This analysis will establish the set on which the Szegö kernel fails to be absolutely convergent. We take a heuristic approach to understanding the function $\lambda(\cdot)$ and start by posing some basic questions that should be resolved: Given that $\lambda(\eta)$ is a location of the global minimum of B_η, what is the relationship between $\lambda(\cdot)$ and b'? If so, does $\lambda(\cdot)$ inherit any smoothness from b'? What is the structure of Λ?

4.1 Properties I

Recall that for each $\eta \in \mathbb{R}$, $\lambda(\eta)$ is a critical point of $B_\eta(\lambda) = b(\lambda) - \eta\lambda$. Because of this, we first explore the relationship between the functions λ and b'_Λ.

Lemma 4.1.1. *The map $\eta \mapsto \lambda(\eta)$ is injective.*

Proof. For each $\eta \in \mathbb{R}$, $\lambda(\eta)$ satisfies

$$\sup_{\lambda \in \mathbb{R}} \{\eta\lambda - b(\lambda)\} = \eta\lambda(\eta) - b(\lambda(\eta)).$$

As mentioned, $\lambda(\eta)$ is a critical point of B_η, so $\lambda(\eta)$ satisfies $b'(\lambda(\eta)) = \eta$. Thus b' is a left inverse for the function $\lambda(\cdot)$, and it follows that $\lambda(\cdot)$ is injective. $\square$

For concision, define b'_Λ to be the restriction of b' to Λ, The prior proof foreshadowed the following lemma.

Lemma 4.1.2. *$b'_\Lambda : \Lambda \longrightarrow \mathbb{R}$ and $\lambda : \mathbb{R} \longrightarrow \Lambda$ are inverses.*

Proof. We showed in the proof of Lemma 4.1.1 that b'_Λ is a left inverse for $\lambda(\cdot)$. Thus it remains to prove $\lambda(b'_\Lambda(\omega)) = \omega$ for all $\omega \in \Lambda$.

Fix $\omega \in \Lambda$. There exists a unique $\eta_\omega \in \mathbb{R}$ such that $\omega = \lambda(\eta_\omega)$. Since this is a critical point of $\lambda \mapsto b(\lambda) - \eta_\omega\lambda$,

$$b'_\Lambda(\omega) = b'_\Lambda(\lambda(\eta_\omega)) = \eta_\omega.$$

Hence

$$\lambda(b'_\Lambda(\omega)) = \lambda(\eta_\omega) = \omega.$$

$\square$

With a relationship between the two functions established, we can say more about $\lambda(\cdot)$ by considering b'_Λ.

Lemma 4.1.3. *b'_Λ is increasing, i.e., if $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 < \lambda_2$, then $b'_\Lambda(\lambda_1) < b'_\Lambda(\lambda_2)$.*

Proof. Fix $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 < \lambda_2$. Since the map $\eta \mapsto \lambda(\eta)$ is a function, this guarantees the existence of $\eta_1, \eta_2 \in \mathbb{R}$ satisfying $\eta_1 \neq \eta_2$ and $\lambda_1 = \lambda(\eta_1)$, $\lambda_2 = \lambda(\eta_2)$. Also, since b'_Λ is the inverse of $\lambda(\cdot)$,

$$\eta_i = b'_\Lambda(\lambda(\eta_i)) = b'(\lambda_i), \quad i = 1, 2. \tag{4.1.1}$$

Thus to prove the lemma we must show that $\eta_1 < \eta_2$.

Suppose, on the contrary, $\eta_2 < \eta_1$. By the definition of λ_i, $B_{\eta_2}(\lambda_2) < B_{\eta_2}(\lambda_1)$. Since $\lambda_1 < \lambda_2$, $\lambda_1 - \lambda_2 < 0$. Thus

$$\begin{aligned} B_{\eta_1}(\lambda_2) - B_{\eta_1}(\lambda_1) &= b(\lambda_2) - \lambda_2\eta_1 - b(\lambda_1) + \lambda_1\eta_1 \\ &= (-\lambda_2 + \lambda_1)[\eta_2 + (\eta_1 - \eta_2)] + b(\lambda_2) - b(\lambda_1) \\ &= (-\lambda_2 + \lambda_1)(\eta_1 - \eta_2) + B_{\eta_2}(\lambda_2) - B_{\eta_2}(\lambda_1) \\ &< 0, \end{aligned}$$

so $B_{\eta_1}(\lambda_2) < B_{\eta_1}(\lambda_1)$. This contradicts the fact that $B_{\eta_1}(\lambda) = b(\lambda) - \lambda\eta_1$ attains its global minimum at $\lambda = \lambda_1$. As desired, we conclude $\eta_2 > \eta_1$. □

By inverse properties, we get the following.

Corollary 4.1.4. *The function $\lambda : \mathbb{R} \longrightarrow \Lambda$ is increasing.*

As one might suspect, $\lambda(\cdot)$ inherits some "smoothness" from b'_Λ, as will be discussed in the following sections. In the meantime, we establish more local properties of .

4.2 Properties II

In order to understand the structure of Λ, we consider the set of all $\eta \in \mathbb{R}$ on which the location of the global minimum of B_η is not unique.

Definition 4.2.1. *Let $\mathcal{C}$ be the set on which $\sigma(\eta) < \lambda(\eta)$. In terms of Λ_η,*

$$\mathcal{C} := \{\eta \in \mathbb{R} : \ |\Lambda_\eta| > 1\}.$$

From the statement at (3.2.1), $\mathcal{C}$ is a bounded subset of $\mathbb{R}$. Below, we will prove $\mathcal{C}$ is empty if and only if b is convex. Since most of the statements below become trivial without the assumption that $\mathcal{C} \neq \emptyset$, we will often assume this without commment. Also, each element $c \in \mathcal{C}$ has a corresponding interval $[\sigma_c, \lambda_c) \subset \mathbb{R}$ that is closely related to a subset of the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$ on which the Szegö kernel is absolutely convergent. Because of this, we will call each non-empty interval $[\sigma_\eta, \lambda_\eta)$ a *safe zone.*

This section will lay the ground work for a full description of Λ, which can be concisely written in terms of the safe zones. Recall that, for each $\eta \in \mathbb{R}$, Λ_η is the set containing the location(s) of the global minimum of B_η.

Lemma 4.2.2. *Take $c \in \mathcal{C}$. Then if $\omega \in (\sigma_c, \lambda_c) \setminus \Lambda_c$, there exists no $\eta \in \mathbb{R}$ for which $\omega \in \Lambda_\eta$.*

Proof. Since ω is not a location of the global minimum of B_c,

$$b(\omega) - \omega c > b(\sigma_c) - \sigma_c c \quad \text{and} \quad b(\omega) - \omega c > b(\lambda_c) - \lambda_c c.$$

Clearly, $\lambda_c - \omega > 0$. Thus if $\eta > c$,

$$\begin{aligned} B_\eta(\omega) - B_\eta(\lambda_c) &= b(\omega) - \eta\omega - b(\lambda_c) + \eta\lambda_c \\ &= (-\omega + \lambda_c)[c + (\eta - c)] + b(\omega) - b(\lambda_c) \\ &= [\lambda_c - \omega][\eta - c] + B_c(\omega) - B_c(\lambda_c) \\ &> 0. \end{aligned}$$

Similarly, for $\eta < c$, $\sigma_c - \omega < 0$ and $B_\eta(\omega) - B_\eta(\sigma_c) > 0$. We conclude that there is no $\eta \in \mathbb{R}$

for which ω is the location of the global minimum of B_η. $\square$

We have just provided the requisite information to show that the safe zones are mutually disjoint. A concise representation of Λ is predicated on the following.

Lemma 4.2.3. *For $\eta_1, \eta_2 \in \mathbb{R}$ with $\eta_1 \neq \eta_2$, the intervals $[\sigma_{\eta_1}, \lambda_{\eta_1}]$ and $[\sigma_{\eta_2}, \lambda_{\eta_2}]$ are disjoint.*

Proof. Since $\lambda(\cdot)$ is increasing, we may assume $\eta_1, \eta_2 \in \mathcal{C}$ with $\eta_1 < \eta_2$; hence $\lambda_{\eta_1} < \lambda_{\eta_2}$. It will suffice to show $\lambda_{\eta_1} < \sigma_{\eta_2}$.

Suppose that $\lambda_{\eta_1} \in [\sigma_{\eta_2}, \lambda_{\eta_2}]$. By Lemma 4.2.2, $\lambda_{\eta_1} \in \Lambda_{\eta_2}$. Therefore $b'_\Lambda(\lambda_{\eta_1})$ satisfies

$$\eta_1 = b'_\Lambda(\lambda_{\eta_1}) = \eta_2,$$

which contradicts $\eta_1 \neq \eta_2$. Thus $\lambda_{\eta_1} < \sigma_{\eta_2}$, and the given sets are disjoint. $\square$

By the lemma, the image of $\lambda(\cdot)$ takes a slightly more transparent form,

$$\Lambda = \Big(\bigcup_{\eta \in \mathbb{R}} \Lambda_\eta \Big) \setminus \bigcup_{c \in \mathcal{C}} [\sigma_c, \lambda_c).$$

In order to clarify this description even further, it would be helpful if we could say something about the cardinality of $\mathcal{C}$.

Lemma 4.2.4. *Let deg$(b) = 2n \geq 4$. For the set $\mathcal{C} = \{\eta \in \mathbb{R} : \ |\Lambda_\eta| > 1\}$, $|\mathcal{C}| \leq n - 1$.*

Proof. If $c \in \mathcal{C}$, then there are at least two distinct points, σ_c and λ_c, at which the polynomial B_c attains a global minimum. I claim there are at least two inflection points contained in the interval (σ_c, λ_c).

For completeness, we prove the claim. By choice of b, B''_c is a non-constant polynomial.

Since σ_c and λ_c are two distinct locations of the global minimum, there exists an $\epsilon > 0$ such that the following properties hold:

1. $B_c''(\lambda) > 0$ whenever $0 < |\sigma_c - \lambda| < \epsilon$ or $0 < |\lambda_c - \lambda| < \epsilon$,
2. $B_c'(\lambda) > 0$ for all $\lambda \in (\sigma_c, \sigma_c + \epsilon)$,
3. $B_c'(\lambda) < 0$ for all $\lambda \in (\lambda_c - \epsilon, \lambda_c)$.

If $B_c''(\lambda) \geq 0$ for all $\lambda \in (\sigma_c, \lambda_c)$, then B_c' is non-decreasing on (σ_c, λ_c). Thus $B_c'(\omega_1) \leq B_c'(\omega_2)$ whenever $\omega_1, \omega_2 \in (\sigma_c, \lambda_c)$ with $\omega_1 < \omega_2$. For $\omega_1 \in (\sigma_c, \sigma_c + \epsilon)$ and $\omega_2 \in (\lambda_c - \epsilon, \lambda_c)$, properties 2 and 3 yield

$$0 < B_c'(\omega_1) \leq B_c'(\omega_2) < 0,$$

which is a contradiction. Thus there exists some $\omega_3 \in (\sigma_c, \lambda_c)$ such that $B_c''(\omega_3) < 0$. By property 1 and the intermediate value property applied to B_c'', there exists at least two inflection points ω_1, ω_2 satisfying $\omega_1 \in (\sigma_c, \omega_3)$ and $\omega_2 \in (\omega_3, \lambda_c)$. Thus there are at least two inflection points contained in the interval (σ_c, λ_c).

Since the number of inflection points cannot exceed $2n - 2$ (the degree of b'') and the safe zones are disjoint, the cardinality of $\mathcal{C}$ cannot exceed $\frac{1}{2}(2n-2)$, i.e., $|\mathcal{C}| \leq n - 1$. □

Having shown that $\mathcal{C}$ is a finite set, we have the following obvious consequence which will be useful in the near future.

Corollary 4.2.5. *There exists $\epsilon > 0$ such that $|\Lambda_\eta| = 1$ for all $\eta \in \mathbb{R}$ and $c \in \mathcal{C}$ satisfying $0 < |\eta - c| < \epsilon$.*

Proof. This is simply a consequence of the fact that since $\mathcal{C}$ is a discrete subset of $\mathbb{R}$, $\mathcal{C}$ is closed; hence $\mathcal{C}^c$ is open. □

We close this chapter with a section that exploits the relationship between $\lambda(\cdot)$ and b'_Λ, and the disjoint safe zones.

4.3 Properties III

We start with a proposition that will assist us in showing that $\lambda(\cdot)$ is densely distributed between consecutive safe zones. Then a complete description of Λ will follow from the continuity of $\lambda(\cdot)$. From a full description of the image Λ, the non-convexity of b is shown to be biconditionally related to the existence of a safe zone. All of the facts above will be exploited, and we will end the section by pointing out additional features of $\lambda(\cdot)$.

The following proposition presents an ideal setting to invoke the tube lemma.

Lemma 4.3.1 (The tube lemma, [6])**.** *Consider the product space $X \times Y$, where Y is compact. If U is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then U contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.*

Proposition 4.3.2. *Let $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with*

$$F(x_0, y) > 0$$

for some $x_0 \in \mathbb{R}$ and all $y \in [a, b]$. Then there exists some $\epsilon > 0$ such that, for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ and all $y \in [a, b]$,

$$F(x, y) > \delta/2 > 0,$$

where $\delta := \min_{y \in [a,b]} F(x_0, y)$.

Proof. First note that for fixed $x_0 \in \mathbb{R}$, $F(x_0, \cdot)$ is a continuous function of one variable. Since $\{x_0\} \times [a, b]$ is compact in $\mathbb{R} \times \mathbb{R}$ and F is continuous, its image is compact. Hence the minimum δ defined above is attained and satisfies $\delta > 0$.

Define the open set $V := (\delta/2, \infty)$. Since F is continuous, $F^{-1}(V)$ is open in $\mathbb{R}\times\mathbb{R}$ and it contains $\{x_0\}\times[a,b]$ since $F(x_0,y)\geq\delta$ for all $y\in[a,b]$. Consider the subspace $\mathbb{R}\times[a,b]$. In the subspace topology, there is an open subset of $\mathbb{R}\times[a,b]$ containing the slice $\{x_0\}\times[a,b]$, i.e.,

$$\{x_0\}\times[a,b] \;\subseteq\; F^{-1}(V)\cap\Big(\mathbb{R}\times[a,b]\Big) \;\subseteq\; \mathbb{R}\times[a,b].$$

Since $[a,b]$ is compact, we can apply the tube lemma to get a neighborhood say B of x_0 in $\mathbb{R}$ so that

$$\{x_0\}\times[a,b] \;\subseteq\; B\times[a,b] \;\subseteq\; F^{-1}(V)\cap\Big(\mathbb{R}\times[a,b]\Big) \;\subseteq\; F^{-1}(V).$$

B is a neighborhood of x_0 in $\mathbb{R}$, so we may shrink B (if necessary) and assume $B = B(x_0,\epsilon)$ for some $\epsilon > 0$. Thus we have an $\epsilon > 0$ such that, for all $x\in B(x_0,\epsilon)$ and all $y\in[a,b]$,

$$F(x,y)\in V \iff 0<\delta/2<F(x,y).$$

$\square$

The following lemma illustrates the density of $\lambda(\mathbb{R})$ near the boundary of each safe zone. This fact is essential and provides one of the last steps in obtaining a concise representation of Λ.

Lemma 4.3.3. *Fix $\eta_0\in\mathbb{R}$ and any real number α. Then*

1. *if $\alpha<\sigma_{\eta_0}$, then $(\alpha,\sigma_{\eta_0})\cap\Lambda\neq\emptyset$.*
2. *if $\alpha>\lambda_{\eta_0}$, then $(\lambda_{\eta_0},\alpha)\cap\Lambda\neq\emptyset$.*

Proof. Consider the family $\mathcal{B} = \{B_\eta(\lambda) = b(\lambda)-\eta\lambda\}_{\eta\in\mathbb{R}}$, and recall that $|\Lambda_{\eta_0}| = 1 \iff \sigma(\eta_0) = \lambda(\eta_0)$.

We start by proving the first statement. Since σ_{η_0} and λ_{η_0} are points at which the function B_{η_0} attains its global minimum,

$$B_{\eta_0}(\omega) > B_{\eta_0}(\sigma_{\eta_0}) \tag{4.3.1}$$

for all $w < \sigma_{\eta_0}$. Consider the continuous function

$$F(\eta, \omega) := B_\eta(\omega) - B_\eta(\sigma_{\eta_0}).$$

For fixed $N \in \mathbb{N}$ (to be determined later) satisfying $-N < \alpha$, $F(\eta_0, \omega) > 0$ for all $\omega \in [-N, \alpha]$, by inequality (4.3.1). We now apply Proposition 4.3.2. For $\delta_N := \min\limits_{\omega \in [-N,a]} F(\eta_0, \omega)$, there exists an $\epsilon_N \in (0, 1)$ such that

$$F(\eta, \omega) > \delta_N/2 > 0 \quad \Longleftrightarrow \quad B_\eta(\omega) > \delta_N/2 + B_\eta(\sigma_{\eta_0}) > B_\eta(\sigma_{\eta_0}) \tag{4.3.2}$$

for all $\eta \in (\eta_0 - \epsilon_N, \eta_0 + \epsilon_N)$ and all $\omega \in [-N, \alpha]$.

In Section 3.2, we proved that $\eta \sim \lambda^{2n-1}$ as $\lambda \to -\infty$, with $\lambda \in \Lambda$. Thus if $\lambda(\eta) \to -\infty$, $\eta \to -\infty$. It follows that there exists some N_0 such that whenever $\lambda \in \Lambda \cap (-\infty, -N_0)$ $(\neq \emptyset$, by Lemma 3.2.1), η satisfies $\eta < \eta_0 - \epsilon_N$. The above N satisfying $-N < \alpha$ was arbitrary. By setting $N := \min\{N_0, \alpha - 1\}$, we get a $\delta_N > 0$ and $\epsilon_N \in (0, 1)$ satisfying inequality (4.3.2) and

$$\lambda(\eta) < -N \quad \Longrightarrow \quad \eta \notin (\eta_0 - \epsilon_N, \eta_0 + \epsilon_N). \tag{4.3.3}$$

To summarize, inequality (4.3.2) showed that whenever $\eta \in (\eta_0 - \epsilon_N, \eta_0)$, $\lambda(\eta) \notin [-N, \alpha]$. Also by implication (4.3.3), $\lambda(\eta) < -N$ implies $\eta \notin (\eta_0 - \epsilon_N, \eta_0)$. Therefore, it must be that $\lambda(\eta) > \alpha$ if $\eta \in (\eta_0 - \epsilon_N, \eta_0)$. Finally, by the monotonicity of $\lambda(\cdot)$ and the fact that safe zones are non-overlapping, $\lambda(\eta) < \sigma(\eta_0)$ for each $\eta \in (\eta_0 - \epsilon_N, \eta_0)$. Thus for each $\eta \in (\eta_0 - \epsilon_N, \eta_0)$, we have $\lambda(\eta) \in (\alpha, \sigma_{\eta_0})$, which yields the desired result.

The proof of the second statement follows by a similar argument. $\square$

All of the hard work is now complete, and the desired facts may be proved.

Lemma 4.3.4. *The map $\lambda : \mathbb{R} \setminus \mathcal{C} \longrightarrow \Lambda \setminus \lambda[\mathcal{C}]$ is a homeomorphism.*

Proof. By Lemma 4.1.2 and the continuity of b'_Λ, we just need to show $\lambda(\cdot)$ is continuous on the open subset $\mathbb{R} \setminus \mathcal{C}$. Let $\{\eta_k\}_k$ be a sequence converging to η_0 in $\mathbb{R} \setminus \mathcal{C}$. Since η_0 is not in the finite set $\mathcal{C}$, there exists an $\epsilon_1 > 0$ such that $|\Lambda_\eta| = 1$ for all $\eta \in B(\eta_0, \epsilon_1)$. Without loss of generality, we may assume $\{\eta_k\}_k \subset B(\eta_0, \epsilon_1)$. Suppose there exists a subsequence and an $\epsilon_2 > 0$ such that $\lambda(\eta_{k_j}) \notin B(\lambda(\eta_0), \epsilon_2)$ for all $j \in \mathbb{N}$. By Lemma 4.3.3, there exists η_1, η_2 such that

$$\lambda(\eta_0) - \epsilon_2 < \lambda(\eta_1) < \lambda(\eta_0) < \lambda(\eta_2) < \lambda(\eta_0) + \epsilon_2.$$

Thus for all $j \in \mathbb{N}$, $\lambda(\eta_{k_j}) \notin (\lambda(\eta_1), \lambda(\eta_2))$; hence $\eta_{k_j} \notin (\eta_1, \eta_2)$ by (4.1.1) and Lemma 4.1.3. This contradicts the fact that $\eta_{k_j} \to \eta_0$. We conclude that $\lambda(\eta_k) \to \lambda(\eta_0)$, so $\lambda(\cdot)$ is continuous. □

Notice that the prior proof can be used to show more. If $\{\eta_k\}_k \subset \mathbb{R} \setminus \mathcal{C}$ were a sequence converging from the right to some $c \in \mathcal{C}$, then the same argument gives $\lim_{k\to\infty} \lambda(\eta_k) = \lambda(c)$. In other words, we have the following:

Corollary 4.3.5. *The map $\lambda : \mathbb{R} \longrightarrow \Lambda$ is continuous from the right.*

We characterize the image $\Lambda := \{\lambda(\eta) : \ \eta \in \mathbb{R}\}$ in terms of the safe zones. In the fourth-degree setting, it will be shown that the image can be characterized in terms of the coefficients of b. Unfortunately, factoring an arbitrary even-degree polynomial is not an easy task, so we will use the tools we have. In particular, let us use what have acquired about the safe zones.

Theorem 4.3.6. $\Lambda = \mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma_c, \lambda_c)$, *where* $\mathcal{C} = \{\eta \in \mathbb{R} : |\Lambda_\eta| > 1\}$.

Proof. Let $\mathcal{C} := \{c_i\}_{i=1}^k$ with $c_i < c_{i+1}$. To prove this result, we exploit the fact that the continuous image of a connected set is connected. By Lemmas 4.1.4, 4.2.3, and 4.3.4, we have

$$\lambda[(-\infty, c_1)] \subseteq (-\infty, \sigma_{c_1}), \quad \lambda[[c_k, \infty)] \subseteq [\lambda_{c_k}, \infty), \quad \lambda\,[[c_i, c_{i+1})] \subseteq [\lambda_{c_i}, \sigma_{c_{i+1}}),$$

for all $c_i \in \mathcal{C}$. Since the above sets are all connected intervals, it suffices to show that each set containment is actually equality. This can be done by invoking Lemmas 3.2.1 and 4.3.3. □

One might wonder why the Inverse Function Theorem was not invoked earlier, or at all. Recall the theorem:

Theorem 4.3.7 (Inverse Function Theorem)**.** *Let f be a real-valued, continuously differentiable function on $(\alpha, \beta) \subset \mathbb{R}$ with f' nonzero on (α, β). Then f is invertible, and the inverse is continuously differentiable on $f[(\alpha, \beta)]$.*

Two questions had not been addressed: Does the image Λ contain any open intervals? What happens at the points in Λ where b''_Λ vanishes? Here is an example where Λ contains every open interval and yet one still cannot use the Inverse Function Theorem on the entire domain Λ of b'_Λ.

Example 4.3.8. *Consider the convex polynomial $b(x) = \frac{1}{4}x^4$. By Lemma 4.3.4, $\Lambda = \mathbb{R}$. If $\eta = 0$, then $\lambda(0) = 0$. Also, $b'(x) = \eta \iff x^3 = \eta$. Using the notation above, $b'_\Lambda(x) = x^3$ which is invertible on $\mathbb{R}$, however the inverse is not differentiable at the origin. Notice that $\frac{d}{dx} b'_\Lambda(x) = 3x^2$. Since $0 \in \Lambda_0 = \{0\}$, we cannot apply the inverse function theorem on the whole domain.*

After removing any additional zeros of b''_Λ, we can apply Theorems 4.3.6 and the Inverse Function Theorem to get the anticipated diffeomorphism.

Lemma 4.3.9. *Define $\mathcal{N} := \{\eta \in \mathbb{R} : b''_\Lambda(\lambda(\eta)) = 0\}$. Then the function $\lambda : \mathbb{R} \setminus (\mathcal{C} \cup \mathcal{N}) \longrightarrow \Lambda$ is C^∞, with derivative*

$$\lambda'(\eta) = \frac{1}{b''(\lambda(\eta))}.$$

Theorem 4.3.6 also allows us to relate the non-convexity of b to the existence of a safe zone. Later, this connection will yield singularities of the Szegö kernel off the diagonal in $\mathbb{R}^3 \times \mathbb{R}^3$, but there is much more work to be done.

Corollary 4.3.10. *Let $b(x)$ be given by (2.1.1). Then $|\Lambda_\eta| = 1$ for all $\eta \in \mathbb{R}$ if and only if b is a convex polynomial.*

Proof. Suppose there exists an $\eta_0 \in \mathbb{R}$ such that $|\Lambda_{\eta_0}| > 1$. Then the polynomial B_{η_0} would have competing global minima; hence B_{η_0} would have inflection points. Since the inflection points of B_{η_0} and b coincide, b is non-convex.

To show the converse, suppose $b(\lambda)$ is non-convex. Then there exists some $\omega \in \mathbb{R}$ such that $b''(\omega) < 0$. Hence, there is some interval (α, β) containing ω such that $b''(\lambda) < 0$ whenever $\lambda \in (\alpha, \beta)$. This implies that b' is strictly decreasing on the interval (α, β). By Lemmas 4.1.3 and 4.3.6, $b'_\Lambda(\lambda)$ is increasing on the restricted domain $\Lambda = \mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma_c, \lambda_c)$. Since the safe zones are non-overlapping, the interval (α, β) must be contained in a safe zone $[\sigma_{\eta_0}, \lambda_{\eta_0})$. Thus for $\eta = \eta_0$, we have $|\Lambda_{\eta_0}| > 1$. $\square$

The prior proof sheds some light on the location of the inflection points of b. As mentioned, B_η and b have the same inflection points, which are independent of $\eta \in \mathbb{R}$. The location of the global minimum cannot be an inflection point, so all inflection points must be interior to a safe zone.

The image Λ also reveals the location of the critical points of b relative to the safe zones. To see this, start by fixing any $\eta_0 \in \mathbb{R}$. If there exists an $\omega \in \mathbb{R}$ such that $\eta_0 = b'(\omega)$ and $\omega \neq \lambda_{\eta_0}$, then $\omega \in [\sigma_c, \lambda_c)$ for some $c \in \mathcal{C}$. Suppose the contrary, $\omega \notin [\sigma_c, \lambda_c)$ for all $c \in \mathcal{C}$. Then $\omega = \lambda_{\eta_1}$ for some $\eta_1 \in \mathbb{R}$ because $\lambda(\cdot)$ maps $\mathbb{R}$ bijectively to $\mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma_c, \lambda_c)$. Since $\omega = \lambda_{\eta_1}$,

$$\eta_0 = b'(\omega) = b'(\lambda_{\eta_1}) = \eta_1.$$

Thus $\eta_0 = \eta_1$, which yields a contradiction since $\omega \neq \lambda_{\eta_0}$. If we set $\eta_0 = 0$, then $\omega = \lambda(0)$ or ω is in a safe zone.

By the last two paragraphs, we have proven the following.

Corollary 4.3.11. *Except for $\sigma(0)$ and $\lambda(0)$, all critical points and inflection points of b are contained in the interior of some safe zone.*

Corollary 4.3.12. *Every element of the family $\mathcal{B}$ is convex on $(-\infty, \sigma_{\delta_-}) \cup (\lambda_{\delta_+}, \infty)$, where $\delta_+ := \max\{\lambda : \lambda \in \mathcal{C}\}$ and $\delta_- := \min\{\lambda : \lambda \in \mathcal{C}\}$.*

Proof. Start by observing $B''_\eta = b''$, which is independent of η. By Corollary 4.3.11, all the inflection points of b, hence B_η, are in the interval $[\sigma_{\delta_-}, \lambda_{\delta_+})$. Since B''_η is an even-degree polynomial with positive leading coefficient, B''_η is non-negative on the desired set, for each in $\eta \in \mathbb{R}$. □

To close this section, let us take a slight detour. By the results of Chapter 2, we may assume b vanishes to even order at the origin. If we assume this, an interesting consequence arises. Admittedly, this result is not essential to the thesis, but its consequence will surface in the non-convex quartic setting.

Do something with the following!

If we define $\mathcal{D} := \mathcal{C} \cup \mathcal{N}$, then $(b^*)' = \lambda(\cdot)$ on $\mathcal{D}$ by Lemma 5.2.3. Since b^* is defined on $\mathbb{R}$, b^* has a critical point at $\eta_0 \in \mathcal{D}$ if and only if $(b^*)'(\eta_0) = \lambda(\eta_0) = 0$, which is unique since $\lambda(\cdot)$ is increasing. Thus the global minimum of b^* occurs at $b^*(\eta_0) = -B_{\eta_0}(\lambda(\eta_0)) = -B_{\eta_0}(0) = -b(0)$, which equals zero if we assume $b(0) = b'(0) = 0$.

Lemma 4.3.13. *Suppose b is an even-degree polynomial with monic derivative, vanishing to second order at the origin.*

1. *If $\lambda(0) \neq 0$ and $b(\lambda_0) = 0$, then $\lambda(\eta)$ is bounded away from zero $[0, \infty)$,*

2. If $\lambda(0) \neq 0$ and $b(\lambda_0) \neq 0$, then $\lambda(\eta)$ is bounded away from zero on $\mathbb{R}$.

Proof. In the first case, zero is a critical point and $\lambda(0) \neq 0$, so zero is in some safe zone according to Lemma 4.3.11. It follows that $0 \in [\sigma_0, \lambda_0)$; hence the desired result holds.

In the second case, $B_0(\lambda_0) = b(\lambda_0) < 0$ because $B_0(\lambda_0) \leq B_0(0) = b(0)$ and $b(\lambda_0) \neq 0$. Recall that $b(\lambda_0) = -b^*(0)$ and $b^*(0) < b^*(\eta)$ for all $\eta \neq 0$, from above. From the first case, we know that zero is in some safe zone, say $0 \in [\sigma_c, \lambda_c)$. Since the safe zones do not overlap, we just need to show $\sigma_c < 0$. Suppose $\sigma_c = 0$. Then

$$-b^*(c) = B_c(\lambda_c) = B_c(\sigma_c) = B_c(0) = 0.$$

On the other hand, we know $b^*(c) > b^*(0) > 0$, which yields a contradiction. It follows that $\sigma_c < 0$, and the desired result follows. $\square$

Chapter 5

Legendre Transform

Until this point, not much has been said about the Legendre transform, besides the definition given in Chapter 1. Through a brief exposition, this chapter is designed to cover elementary properties of this maximizing transform, whose geometric properties were used to find several results proved in Chapter 4. Here we look at the definition, basic properties, an example, and a factorization lemma. In relation to this thesis, the factorization lemma is the only necessary result of this chapter. Without interrupting the flow, a reader that is only interested in the new results of this thesis may skip to the last lemma.

5.1 Introduction

In order to motivate this chapter, we reflect on the integral in question with a slight rearrangement of an exponent,

$$S[(x,0,0),(r,0,0)] = c\int_0^{\infty}\int_{-\infty}^{\infty} \frac{\tau e^{-\tau[(b(x)-\eta x)+(b(r)-\eta r)]}}{\int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda]}\,d\lambda}\,d\eta\,d\tau.$$

As commented on several times before, estimating the size of this integral can become a tractable task if we know more about the integrand and the behavior of the exponents. Extensive theory has been developed in an attempt to estimate integrals with an exponential integrand, e.g., Laplace's Method and stationary phase approximation. It is well understood that the main contribution to an exponential integrand comes from the global maxima of the exponent as they vary over their real parameter space (See the introduction to Chapter 6).

In our setting, the Legendre transform is a tool to help identify these extreme values. From the integral above, this becomes apparent by noticing that both exponents are multiples of $B_\eta(\lambda) = b(\lambda) - \eta\lambda$. If b is an even-degree polynomial with positive leading coefficient, then, for each $\eta_0 \in \mathbb{R}$, the global minimum of B_{η_0} is finite. Thus the maximum of $(-B_\eta)$, over the real parameter space λ, is a function of η. In Chapter 3, we defined the function $\lambda(\cdot)$ to be the largest location in which the global minimum of B_η is achieved. So, at each $\eta \in \mathbb{R}$, the value $-B_\eta(\lambda(\eta))$ is precisely the maximum value that $(-B_\eta)$ can achieve with the real parameter λ. But what is this function $-B_\eta(\lambda(\eta))$? And is there a terse definition? To define this more generally, we turn to the Legendre transform.

Definition 5.1.1. *Let f be a real-valued function of a real variable. The* Legendre transform of f, *denoted f^*, is given by*

$$f^*(\eta) := \sup_{\lambda \in \mathbb{R}} (\eta\lambda - f(\lambda)).$$

Notice that even for well-behaved functions, the transform can take on infinite values. For example, the Legendre transform is infinite everywhere whenever f is a concave function with an upper bound. One can verify that the Legendre transform of a real-valued function f is finite everywhere if and only if f has super-linear growth.

Like the Fourier transform, the Legendre transform is an operator which produces a function of a different variable. Unlike many transforms that consist of integration with a kernel, the Legendre transform uses maximization as its transformation procedure. This supremum may be interpreted in several, equally useful ways. For a fixed $\eta_0 \in \mathbb{R}$, the obvious interpretation of f^* is the supremum of the difference between the line $y = \eta_0\lambda$ and $f(\lambda)$. If f is convex and differentiable with solution λ_0 to the equation $f'(\lambda) = \eta_0$, then we can interpret $f^*(\eta_0)$ as the negative of the y-intercept of the tangent line to the graph of f that has slope η_0. The convexity of f guarantees the uniqueness of the y-intercept, $-f^*$. If f is an even-degree polynomial with positive leading coefficient, then our desired interpretation is

$$f^*(\eta) = (-1)\min_{\lambda\in\mathbb{R}}(f(\lambda) - \eta\lambda).$$

5.2 Properties

Let us work through some elementary properties of the transform.

Lemma 5.2.1. *The function f^* is convex.*

Proof. Given $\eta_0, \eta_1 \in \mathbb{R}$ and $t \in [0,1]$,

$$\begin{aligned}(t\eta_0 + (1-t)\eta_1)\,\lambda - f(\lambda) &= t\eta_0\lambda + (1-t)\eta_1\lambda - tf(\lambda) - (1-t)f(\lambda) \\ &= t\,(\eta_0\lambda - f(\lambda)) + (1-t)\,(\eta_1\lambda - f(\lambda))\,.\end{aligned}$$

We now take the supremum over the real parameter λ. By the subadditivity,

$$f^*(t\eta_0 + (1-t)\eta_1) \leq tf^*(\eta_0) + (1-t)f^*(\eta_1)$$

for each $t \in [0,1]$. By definition, f^* is convex. $\square$

In order to see the process that one might take to compute f^*, we calculate the Legendre transform of Chistine's non-convex, piecewise quadratic b.

Example 5.2.2. *Consider*

$$b(\lambda) = \begin{cases} (\lambda+1)^2 & ,\lambda < -\frac{1}{2} \\ -\lambda^2 + \frac{1}{2} & ,-\frac{1}{2} \leq \lambda \leq \frac{1}{2} \\ (\lambda-1)^2 & , \frac{1}{2} < \lambda. \end{cases}$$

One can easily verify that b has a continuous derivative, so we can apply basic calculus techniques to find b^*. The derivative of the map $\lambda \mapsto F_\eta(\lambda) := \eta\lambda - b(\lambda)$ is given by

$$F'_\eta(\lambda) = \begin{cases} -2\lambda - 2 + \eta & ,\lambda < -\frac{1}{2} \\ 2\lambda + \eta & ,-\frac{1}{2} \leq \lambda \leq \frac{1}{2} \\ -2\lambda + 2 + \eta & ,\frac{1}{2} < \lambda. \end{cases}$$

Independently of η, the map F_η has the same inflection points as b, which is concave on the interval $(-\frac{1}{2}, \frac{1}{2})$. Hence the global maximum of F_η is only achieved by some value(s) in an extreme interval.

By examining the function F_η, one finds that for each $\eta \in (-1, 1)$, the function has two local maxima occurring at $\lambda = \frac{\eta-2}{2}$ and $\lambda = \frac{\eta+2}{2}$. By direct evaluation, the only time F_η has a non-unique location where the global maximum occurs is when $\eta = 0$. It follows that

1. *if $\eta \leq 0$, then the global maximum occurs when $\lambda = \frac{\eta-2}{2}$;*

2. *if $\eta \geq 0$, then the global maximum occurs when $\lambda = \frac{\eta+2}{2}$.*

By substitution, we get the Legendre transform

$$b^*(\eta) = \begin{cases} \frac{1}{4}\eta^2 - \eta & ,\eta \leq 0 \\ \frac{1}{4}\eta^2 + \eta & ,\eta \geq 0. \end{cases}$$

The region above the graph of b^* is the intersection of the convex region above the graph of $\frac{1}{4}\eta^2 - \eta$ with the convex region above the graph of $\frac{1}{4}\eta^2 + \eta$. Hence it is convex, and b^* is a convex funciton. □

For an addition example, see Example **??** in the appendix.

As a consequence of its convexity, f^* is Lipschitz continuous on each closed subset of $\mathbb{R}$; hence it is differentiable almost everywhere. For additional details and properties, see [9].

In order to say more, additional hypotheses on our function f are needed. Since this general theory is not essential to the thesis, we will prove the remaining results for the special case in which b is an even-degree polynomial with positive leading coefficient. Our choice of b has the consequence that $\lambda(\eta)$, the largest location of the global minimum of B_η, is smooth off of a discrete set $\mathcal{D}$, by Lemma 4.3.9. Because of this,

Lemma 5.2.3. $(b^*)' = \lambda(\cdot)$ *on* $\mathbb{R} \setminus \mathcal{D}$.

Proof. By Definition 3.1.1, $b^*(\eta) = \eta\lambda(\eta) - b(\lambda(\eta))$. Also, $\lambda(\eta)$ satisfies $b'(\lambda(\eta)) = \eta$. Since $\lambda(\cdot)$ is smooth on $\mathbb{R} \setminus \mathcal{D}$,

$$\begin{aligned} (b^*)'(\eta) &= \eta\lambda'(\lambda) + \lambda(\eta) - b'(\lambda(\eta)) \cdot \lambda'(\eta) \\ &= \eta\lambda'(\lambda) + \lambda(\eta) - \eta\lambda'(\eta) \\ &= \lambda(\eta). \end{aligned}$$

Thus $(b^*)' = \lambda(\cdot)$ on $\mathbb{R} \setminus \mathcal{D}$. □

An interesting result follows with the added condition that b is convex.

Lemma 5.2.4. *For any non-constant, convex function b, the Legendre transform is an invo-*

lution, i.e.,

$$(b^*)^*(\eta) = b(\eta) \qquad \textit{for all } \eta \in \mathbb{R}.$$

Proof. First, we show that $(b^*)^*(x_0) \leq b(x_0)$ for each $x_0 \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$. Since $(b^*)^*(x_0) = \sup_{\eta\in\mathbb{R}}(\eta x_0 - b^*(\eta))$, there exists an $\eta_0 \in \mathbb{R}$ for each $\epsilon > 0$ satisfying

$$\begin{aligned} (b^*)^*(x_0) &= \sup_{\eta\in\mathbb{R}}(\eta x_0 - b^*(\eta)) \\ &\leq \eta_0 x_0 - b^*(\eta_0) + \epsilon \\ &= \eta_0 x_0 - \sup_{x\in\mathbb{R}}(\eta_0 x - b(x)) + \epsilon \\ &\leq \eta_0 x_0 - \eta_0 x_0 + b(x_0) + \epsilon \\ &= b(x_0) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have $(b^*)^*(x_0) \leq b(x_0)$.

To obtain a reverse inequality, let us assume that $(b^*)^*(x_0) < b(x_0)$. Since b is convex, there exists a line with slope η_0 and $y-$intercept c that separates the point $(x_0, (b^*)^*(x_0))$ from the graph $y = b(x)$. This line satisfies

$$(b^*)^*(x_0) < \eta_0 x_0 + c < b(x_0) \ \text{ and } \ \eta_0 x + c < b(x), \quad \text{for all } x \in \mathbb{R}.$$

Therefore, $\eta_0 x - b(x) < -c$ for all $x \in \mathbb{R}$, which implies $b^*(\eta_0) < -c$. Also from above,

$$\begin{aligned} \eta_0 x_0 + c &> \sup_{\eta\in\mathbb{R}}(x_0\eta - b^*(\eta)) \\ &\geq \eta_0 x_0 - b^*(\eta_0), \end{aligned}$$

so $b^*(x_0) > -c$ This contradicts $b^*(\eta_0) < -c$. Thus we must of had equality so that such a separation could not exist. Since x_0 was arbitrary, $(b^*)^* = b$. $\square$

One should notice that the convexity of b was the only hypothesis required for the above result.

5.3 Factoring lemma

We conclude this chapter with a lemma, which is a key component of this thesis. The lemma is a generalization of work done by Halfpap, see [ ]. Using results from Chapter 4, we can extend her conclusion to the higher-degree setting.

For fixed $\omega \in \mathbb{R}$, we define $A_\omega(\eta) := b(\omega) - \eta\omega + b^*(\eta)$.

Lemma 5.3.1. *Suppose that b is an even-degree polynomial with monic derivative. Then for each $\eta_0 \in \mathbb{R}$ and $\omega \in \Lambda_{\eta_0}$, A_ω can be factored as*

$$A_\omega(\eta) = (\eta - \eta_0)F_\omega(\eta)$$

on the interval (η_0, ∞), where F_ω is continuous from the right and bounded on each finite, non-empty interval (η_0, N).

Proof. Let $\omega \in \Lambda_{\eta_0}$. Then $\lambda(\eta_0) := \lambda_{\eta_0} \geq \omega$. Since $\lambda(\cdot)$ monotone, $\lambda_\eta > \omega$ if and only if $\eta > \eta_0$. By definition, λ_η is a solution to the equation

$$b^*(\eta) = \eta\lambda_\eta - b(\lambda_\eta),$$

which shows that A_ω is non-negative. On the interval $(\eta_0, +\infty)$,

$$\begin{aligned}
A_\omega(\eta) &= b(\omega) - \eta\omega + b^*(\eta) \\
&= b(\omega) - \eta\omega + \eta\lambda_\eta - b(\lambda_\eta) \\
&= \eta(\lambda_\eta - \omega) + b(\omega) - b(\lambda_\eta) - \eta_0(\lambda_\eta - \omega) + \eta_0(\lambda_\eta - \omega) \\
&= (\eta - \eta_0)(\lambda_\eta - \omega) - (\lambda_\eta - \omega)^2 \left[\frac{b(\lambda_\eta) - b(\omega) - \eta_0(\lambda_\eta - \omega)}{(\lambda_\eta - \omega)^2}\right] \\
&=: (\eta - \eta_0)(\lambda_\eta - \omega) - (\lambda_\eta - \omega)^2 \phi_\omega(\eta).
\end{aligned}$$

Notice that

$$\phi_\omega(\eta) = \frac{b(\lambda_\eta) - \eta_0\lambda_\eta - [b(\omega) - \eta_0\omega]}{(\lambda_\eta - \omega)^2} = \frac{B_{\eta_0}(\lambda_\eta) - B_{\eta_0}(\omega)}{(\lambda_\eta - \omega)^2}.$$

Since $\omega \in \Lambda_{\eta_0}$, the minimality of $B_{\eta_0}(\omega)$ yields $B_{\eta_0}(\lambda_\eta) \geq B_{\eta_0}(\omega)$ for all $\eta \in \mathbb{R}$. Therefore ϕ_ω is non-negative on the interval $(\eta_0, +\infty)$. Since $\lambda(\cdot)$ is continuous from the right, so is ϕ_ω. It follows that for each $\eta \in (\eta_0, +\infty)$,

$$A_\omega(\eta) = (\eta - \eta_0)(\lambda_\eta - \omega) - (\lambda_\eta - \omega)^2 \phi_\omega(\eta) \geq 0 \iff 1 \geq \frac{(\lambda_\eta - \omega)\phi_\omega(\eta)}{(\eta - \eta_0)}. \tag{5.3.1}$$

Hence, on $(\eta_0, +\infty)$, we get the factorization

$$\begin{aligned}
A_\omega(\eta) &= (\eta - \eta_0)(\lambda_\eta - \omega) - (\lambda_\eta - \omega)^2 \phi_\omega \\
&= (\eta - \eta_0)(\lambda_\eta - \omega)\left[1 - (\lambda_\eta - \omega)\frac{\phi_\omega(\eta)}{(\eta - \eta_0)}\right] \\
&=: (\eta - \eta_0) F_\omega(\eta).
\end{aligned}$$

By Lemma 4.3.5 and inequality (5.3.1), F_ω is continuous from the right and bounded on each finite, non-empty interval (η_0, N). This gives us the desired result. □

In our final chapter, we will use this result to show divergence of our main integral on a specified domain. But for now, we move on to acquire some additional estimates needed.

Chapter 6

Estimates for $N(\eta, \tau)$

When we consider the question of absolute convergence for the Szegö kernel, the integral to be analyzed is

$$S[(x,0,0),(r,0,0)] = c\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\tau e^{-\tau[b(x)+b(r)-\eta(x+r)]}}{\int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda]}\,d\lambda}\,d\tau\,d\eta. \tag{6.0.1}$$

In Chapter 3, we established asymptotic estimates for the largest location $\lambda(\eta)$ of the global minimum of $B_\eta(\lambda)$ as well as for the global minimum $B_\eta(\lambda(\eta))$ as $|\eta| \to \infty$. In order to obtain size estimates for the integrand, more needs to be done. We turn our attention to the denominator integral,

$$N(\eta,\tau) := \int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda]}\,d\lambda.$$

For each $\eta \in \mathbb{R}$ and $\tau > 0$, the integral N is of the form

$$\int_{-\infty}^{\infty} e^{-\rho(\lambda)}\,d\lambda,$$

where ρ satisfies $\lim_{|\lambda|\to\infty}\rho(\lambda) = \infty$. The heuristic principle that guides the analysis of such integrals is that the main contribution comes from a neighborhood of the point(s) at which the exponent attains its global maximum. Using the Legendre transform of b, we can describe the main contribution to N.

$$
\begin{aligned}
N(\eta, \tau) &= \int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda]}\, d\lambda \\
&= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-2\tau[b(\lambda)-\eta\lambda+b^*(\eta)]}\, d\lambda \\
&= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-2\tau[b(\lambda+\lambda_\eta)-\eta(\lambda+\lambda_\eta)+b^*(\eta)]}\, d\lambda \\
&=: e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-p(\lambda)}\, d\lambda,
\end{aligned}
$$

where $p(\lambda) := 2\tau[b(\lambda + \lambda_\eta) - \eta(\lambda + \lambda_\eta) + b^*(\eta)]$. The polynomial p is a non-negative, even-degree polynomial in λ, which vanishes to even order at the origin. Sharp estimates have been obtained for integrals of the form $\int_{\mathbb{R}} e^{-p(\lambda)}\, d\lambda$ whenever p is a non-negative, convex polynomial which vanishes to even order at the origin. Hence it is natural to ask if similar results follow when p is *non-convex*. The answer is in the affirmative only if the degree of p is four. Because of this, we are forced take a different approach when the degree of p, or equivalently b, is greater than four.

This chapter consists of three sections. In Section 6.1, we use an excerpt of our earlier paper [2] to obtain uniform estimates of N in the fourth-degree setting. This section surveys integrals of the form $\int_{\mathbb{R}} e^{-p(\lambda)}\, d\lambda$, where p satisfies "nice" conditions. The techniques used in obtaining these results will help the reader appreciate the complexity of these estimates. Starting in Sections 6.2, we generalize to even-degree polynomials of the form (2.1.1). We develop a canonical way of getting a uniform lower bound for N, which will allow us to establish the convergence of (6.0.1) on a particular open set. Section 6.3 is dedicated to finding a local, uniform upper bound for N, which in turn will be sufficient to show the divergence of the Szegö kernel at specific points.

6.1 Fourth-degree estimates, [[2],§4]

6.1.1 Definitions and notation

Let p be a real polynomial of even-degree with positive leading coefficient. We are interested in estimates for

$$\int_{-\infty}^{\infty} e^{-p(x)}\, dx \tag{6.1.1}$$

which are uniform in the coefficients of p.

Definition 6.1.1. *We say that A and B are* comparable*, denoted $A \approx B$, if for some positive constant c, $cB \leq A \leq c^{-1}B$. It will be understood whenever this notation is used that the underlying constant c is independent of all important parameters.*

If p is convex (i.e., if $p''(x) \geq 0$ for all x) with $p(0) = p'(0) = 0$, we know that

$$\int_{-\infty}^{\infty} e^{-p(x)}\, dx \approx |\{\, x : p(x) \leq 1 \,\}|, \tag{6.1.2}$$

where this comparability is independent of the coefficients of the polynomial p and depends only, perhaps, on the degree of p.

Our goal is to extend the estimate (6.1.2) to the situation in which p is a fourth-degree polynomial with positive leading coefficient. By translating, shifting, and reflecting about the y-axis if necessary, we can arrange it so that the (not necessarily unique) global minimum of the polynomial is zero and occurs at $x = 0$, and so that p is convex for all $x \leq 0$. Since p'' has degree 2, if p fails to be convex on all of $\mathbb{R}$, there is a single interval on which p'' is negative.

Thus suppose p'' has zeros at $x = A$ and $x = A + C$ where $A, C > 0$. Then there exists $B > 0$ so that

$$p''(x) = B(x - A)(x - (A + C)). \tag{6.1.3}$$

Remark 6.1.2. *One checks easily that if $A = 0$, p can not have its global minimum at 0 unless $C = 0$ as well. In this case, we would have $p(x) = Bx^4$, which is convex. Furthermore, if $A > 0$ but $C = 0$, p'' is never negative, hence p is convex.*

If we anti-differentiate (6.1.3) twice, using the assumption that $p(0) = p'(0) = 0$, we find

$$p(x) = \frac{B}{12}x^2[x^2 - 2(2A + C)x + 6A(A + C)],$$

which is of the form (2.1.1). In the analysis that follows, it will be essential to know what relationship, if any, exists between A and C. Thus write $C = \alpha A$ for $\alpha > 0$. Then

$$p(x) = \frac{B}{12}x^2[x^2 - 2A(2 + \alpha)x + 6A^2(1 + \alpha)]. \tag{6.1.4}$$

Proposition 6.1.3. *Let p be as in (6.1.4), with $A, B, \alpha > 0$. p is non-negative if and only if*

$$0 < \alpha \leq 1 + \sqrt{3}.$$

Proof. p is non-negative if and only if the expression $x^2 - 2A(2 + \alpha)x + 6A^2(1 + \alpha)$ is non-negative for all x. The conclusion follows by finding those positive α for which this quadratic has non-positive discriminant. □

Next, we prove an inequality concerning the value of p at its inflection points:

Proposition 6.1.4. *If $p(x) = \frac{B}{12}[x^4 - 2A(2+\alpha)x^3 + 6A^2(1+\alpha)x^2]$ and $0 < \alpha \leq 1 + \sqrt{3}$, then there exists $c > 0$ independent of A and B so that $p((1 + \alpha)A) \geq p(A) \geq cBA^4$.*

Proof. We have

$$\begin{aligned} p((1+\alpha)A) &= \frac{BA^4}{12}(1+\alpha)^3(3-\alpha) \\ &= \frac{BA^4}{12}(3+8\alpha+6\alpha^2-\alpha^4) \\ p(A) &= \frac{BA^4}{12}(3+4\alpha) \end{aligned}$$

The lower bound on $p(A)$ follows immediately since for $0 < \alpha \leq 1+\sqrt{3}$, $3+4\alpha$ is bounded below by a positive constant.

Observe,

$$p((1+\alpha)A) - p(A) = \frac{B}{12}A^4(4\alpha+6\alpha^2-\alpha^4).$$

One confirms easily that $\alpha = -2$, $\alpha = 0$, and $\alpha = 1+\sqrt{3}$ are roots. One also verifies that the fourth root is in $(-1, -.5)$. Finally, since at $\alpha = 1$,

$$p((1+\alpha)A) - p(A) = \frac{B}{12}A^4(4+6-1) > 0$$

we conclude that this difference is positive for all $0 < \alpha < 1+\sqrt{3}$. This proves the proposition.

$\square$

A convex polynomial clearly has only one local extremum, which is necessarily the location of the global minimum. For non-convex p, however, it is possible that p has other extrema. More specifically, if p is a fourth-degree polynomial, p' is a polynomial of degree three, hence it has either a single real root or three real roots. We have the following:

Proposition 6.1.5. *Let* $p(x) = \frac{B}{12}[x^4 - 2A(2+\alpha)x^3 + 6A^2(1+\alpha)x^2]$, *with* $A, B > 0$ *and* $0 < \alpha \leq 1+\sqrt{3}$. *Then* p' *has three real roots if and only if* $2 \leq \alpha \leq 1+\sqrt{3}$.

Proof. We compute:

$$p'(x) = \frac{B}{12}x[4x^2 - 6A(2+\alpha)x + 12A^2(1+\alpha)].$$

This has three real roots if and only if

$$9A^2(2+\alpha)^2 - 48A^2(1+\alpha) = 3A^2(3\alpha^2 - 4\alpha - 4) \geq 0.$$

This occurs if and only if

$$\alpha \leq -\frac{2}{3} \quad \text{or} \quad \alpha \geq 2.$$

Since we have assumed $0 < \alpha \leq 1 + \sqrt{3}$, the conclusion follows. ☐

To analyze the integral (6.1.1), we begin by writing it as a sum:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-p(x)}\,dx &= \int_{-\infty}^{0} e^{-p(x)}\,dx + \int_{0}^{A} e^{-p(x)}\,dx \\ &\quad + \int_{A}^{(1+\alpha)A} e^{-p(x)}\,dx + \int_{(1+\alpha)A}^{\infty} e^{-p(x)}\,dx \\ &= I + II + III + IV. \end{aligned} \tag{6.1.5}$$

Observe that p is convex on the intervals of integration for I, II, and IV. Obtaining sharp estimates for these integrals requires the results of the next subsection.

6.1.2 Some estimates for functions on intervals of convexity

We will use the following results repeatedly. The first gives the size of the integral of e^{-p} over any interval on which p is convex. It uses a modification of an argument in Halfpap, Nagel, and Wainger [18] proving an analogous estimate if p is convex on all of $\mathbb{R}$.

Lemma 6.1.6. *Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\lim_{|x|\to\infty} p(x) = \infty$.*

1. *Suppose* p' *is positive and increasing on an interval* (x_0, x_f)*, where* x_f *may equal* ∞*. Suppose further that in the case in which* $x_f < \infty$*,* $p(x_f) \geq p(x_0) + 1$*. Then*

$$\int_{x_0}^{x_f} e^{-p(x)}\, dx \approx e^{-p(x_0)} |\{x \in (x_0, x_f) : p(x_0) < p(x) < p(x_0) + 1\}|.$$

2. *Suppose* p' *is negative and increasing on an interval* (x_f, x_0)*, where* x_f *may equal* $-\infty$*. Suppose further that in the case in which* $x_f > -\infty$*,* $p(x_f) \geq p(x_0) + 1$*. Then*

$$\int_{x_f}^{x_0} e^{-p(x)}\, dx \approx e^{-p(x_0)} |\{x \in (x_f, x_0) : p(x_0) < p(x) < p(x_0) + 1\}|.$$

3. *Suppose* p' *is (i) positive and increasing on* $I = (x_0, x_f)$ *with* $x_f < \infty$ *or (ii) negative and increasing on* $I = (x_f, x_0)$ *with* $x_f > -\infty$*. Suppose further that* $p(x_0) < p(x_f) < p(x_0) + 1$*. Then*

$$\int_I e^{-p(x)}\, dx \approx e^{-p(x_0)} |x_f - x_0|.$$

Proof. We sketch the proof of (1). The remaining parts follow in a similar manner.

Suppose $x_f < \infty$, and let J be the largest positive integer such that $p(x_f) \geq p(x_0) + J$. Our hypotheses guarantee that such a J exists. For each positive integer $j \leq J$, define x_j to be the largest element of (x_0, x_f) for which $p(x_j) = p(x_0) + j$. (Of course, if p is a non-constant polynomial satisfying all hypotheses of (1), x_j is unique). Clearly,

$$e^{-p(x_0)}(x_1 - x_0) \leq \int_{x_0}^{x_f} e^{-p(x)}\, dx.$$

For the reverse inequality, observe that

$$\begin{aligned}\int_{x_0}^{x_f} e^{-p(x)}\,dx &= \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}} e^{-p(x)}\,dx + \int_{x_J}^{x_f} e^{-p(x)}\,dx \\ &\leq \sum_{j=0}^{J-1} e^{-p(x_0)-j}(x_{j+1}-x_j) + e^{-p(x_0)-J}(x_f - x_J) \\ &\leq e^{-p(x_0)}\left[(x_1-x_0) + \sum_{j=1}^{J-1} e^{-j}(x_{j+1}-x_1) + e^{-J}(x_f - x_1)\right].\end{aligned}$$

We now estimate $x_{j+1}-x_1$ in terms of $x_1 - x_0$.

$$\begin{aligned} j &= p(x_{j+1}) - p(x_1) \\ &= \int_{x_1}^{x_{j+1}} p'(x)\,dx \\ &\geq p'(x_1)(x_{j+1}-x_1).\end{aligned}$$

Since

$$p'(x_1)(x_1-x_0) \geq \int_{x_0}^{x_1} p'(x)\,dx = 1,$$

we have

$$x_{j+1} - x_1 \leq j(x_1 - x_0).$$

A similar estimate holds for $x_f - x_1$. It follows that

$$\int_{x_0}^{x_f} e^{-p(x)}\,dx \lesssim (x_1 - x_0)e^{-p(x_0)}.$$

□

Lemma 6.1.7 (Bruna, Nagel, Wainger [17])**.** *Let p be a polynomial of degree m satisfying $p(0) = p'(0) = 0$; i.e., $p(x) = \sum_{k=2}^{m} a_k x^k$. If p is convex on an interval $[0, A]$, then there exists* *a constant C_m, depending on m but independent of A, such that*

$$p(x) \geq C_m \sum_{k=2}^{m} |a_k| x^k \quad \textit{for all } x \in [0, A].$$

This lemma is useful to us because it allows us to prove the following:

Proposition 6.1.8. *Let p be as in Lemma 6.1.7. Suppose that $p(A) > 1$. Then $p(x) = 1$ has a unique solution μ in $[0, A]$ and*

$$\mu \approx \left[\sum_{k=2}^{m} |a_k|^{1/k} \right]^{-1}. \tag{6.1.6}$$

Proof. This is a standard argument, included here for completeness.

It follows from Lemma 6.1.7 that there exists C_m such that for all $x \in [0, A]$

$$C_m \sum_{k=2}^{m} |a_k| x^k \leq \sum_{k=2}^{m} a_k x^k \leq \sum_{k=2}^{m} |a_k| x^k.$$

Define $\tilde{p}(x) = \sum_{k=2}^{m} |a_k| x^k$. Then if y_1 is the positive solution to

$$\tilde{p}(x) = 1$$

and y_2 is the positive solution to

$$C_m \tilde{p}(x) = 1,$$

then

$$y_1 \leq \mu \leq y_2.$$

It therefore suffices to show that y_1 and y_2 are comparable to the expression on the right of (6.1.6). We show this for y_2.

By definition, y_2 satisfies

$$\sum_{k=2}^{m} C_m|a_k|y_2^k = 1.$$

Thus for every k, $2 \leq k \leq m$,

$$C_m|a_k|y_2^k \leq 1,$$

and hence

$$y_2 \leq [C_m^{1/k}|a_k|^{1/k}]^{-1}.$$

Since this is true for any k, it is true for the k_0 such that $C_m^{1/k_0}|a_{k_0}|^{1/k_0} = \max_{\{2\leq k\leq m\}} C_m^{1/k}|a_k|^{1/k}$. On the other hand,

$$C_m^{1/k_0}|a_{k_0}|^{1/k_0} \geq \frac{1}{m-1}\sum_{k=2}^{m} C_m^{1/k}|a_k|^{1/k}.$$

It follows that

$$y_2 \leq \left[\frac{1}{m-1}\sum_{k=2}^{m} C_m^{1/k}|a_k|^{1/k}\right]^{-1} \leq \frac{m-1}{C_m^{1/2}}\left[\sum_{k=2}^{m}|a_k|^{1/k}\right]^{-1}.$$

This gives the desired upper bound on y_2.

Next we obtain a lower bound. Let k_1 be such that $C_m|a_{k_1}|y_2^{k_1} = \max_{\{2\leq k\leq m\}} C_m|a_k|y_2^k$. Then

$$(m-1)C_m|a_{k_1}|y_2^{k_1} \geq \tilde{p}(y_2) = 1,$$

and so

$$\begin{aligned}
y_2 &\geq \left[(m-1)^{1/k_1}C_m^{1/k_1}|a_{k_1}|^{1/k_1}\right]^{-1} \\
&\geq \left(\frac{1}{m-1}\right)^{1/2}\left(\frac{1}{C_m}\right)^{1/m}\left[|a_{k_1}|^{1/k_1}\right]^{-1} \\
&\geq \left(\frac{1}{m-1}\right)^{1/2}\left(\frac{1}{C_m}\right)^{1/m}\left[\sum_{k=2}^{m}|a_k|^{1/k}\right]^{-1}.
\end{aligned}$$

We have now proved the desired estimates for y_2. The estimates on y_1 follow by setting

$C_m = 1$. $\square$

6.1.3 Estimates of the integral (6.1.1)

In this section we prove:

Lemma 6.1.9. *If $\beta > 0$ and $p(x) = \beta x^4 + \gamma x^3 + \delta x^2$ attains its global minimum at the origin, then*

$$\int_{-\infty}^{\infty} e^{-[\beta x^4+\gamma x^3+\delta x^2]}\,dx \approx [\beta^{\frac{1}{4}} + |\gamma|^{\frac{1}{3}} + \delta^{\frac{1}{2}}]^{-1}.$$

Since the result is already known for convex p, it suffice to establish it for non-convex p, taking $\beta = \dfrac{B}{12}$, $\gamma = -\dfrac{BA(2+\alpha)}{6}$, and $\delta = \dfrac{BA^2(1+\alpha)}{2}$. As in (6.1.5), we consider this as a sum of four integrals.

The integral I.

To estimate I, note that

$$q(x) = p(-x) = \frac{B}{12}[x^4 + 2A(2+\alpha)x^3 + 6A^2(1+\alpha)x^2]$$

is convex on $(0,\infty)$ with $q(0) = q'(0) = 0$. Thus by Lemma 6.1.6 and Proposition 6.1.8, I satisfies the estimate (6.1.9), i.e.,

$$\begin{aligned} I &\approx \left[\left(\frac{B}{12}\right)^{1/4} + \left(\frac{BA}{6}(2+\alpha)\right)^{1/3} + \left(\frac{BA^2}{2}(1+\alpha)\right)^{1/2}\right]^{-1} \\ &\approx \left[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\right]^{-1}. \end{aligned} \tag{6.1.7}$$

In (6.1.7), we have also used Proposition 6.1.3 to conclude that $2+\alpha$ and $1+\alpha$ are both comparable to 1.

Since clearly $I \leq I+II+III+IV$, the lemma will follow if we can show that $II, III, IV \lesssim I$.

The integral II.

We have two cases, depending on whether $p(A) \geq 1$ or $p(A) < 1$.

First, if $p(A) \geq 1$, then Lemma 6.1.6 and Proposition 6.1.8 imply, as they did in the case of integral I, that

$$II \approx \left[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\right]^{-1} \approx I,$$

as desired.

Suppose, then, that $p(A) < 1$. Then by Lemma 6.1.6,

$$II \approx A. \tag{6.1.8}$$

By Proposition 6.1.4, $cBA^4 \leq p(A)$, and so if $p(A) < 1$, $BA^4 \lesssim 1$. Thus to show that $II \lesssim I$, we must show that if $BA^4 \lesssim 1$, then $A[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A]$ is bounded. Indeed,

$$\begin{aligned} A[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A] &= B^{1/4}A + B^{1/3}A^{4/3} + B^{1/2}A^2 \\ &= (BA^4)^{1/4} + (BA^4)^{1/3} + (BA^4)^{1/2} \\ &\lesssim 1. \end{aligned}$$

The integral III.

This is the integral over the interval on which p'' is negative. This forces the minimum of p on this interval to be either $p((1+\alpha)A)$ or $p(A)$. By Proposition 6.1.4, both are bounded below by cBA^4 for some uniform positive constant c. Therefore

$$III \leq \alpha A e^{-cBA^4} \leq A e^{-cBA^4}.$$

This contribution is always less than that from the integral I. Indeed,

$$Ae^{-cBA^4}[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A]$$

is uniformly bounded since

$$Ae^{-cBA^4}[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A] = [(BA^4)^{1/4} + (BA^4)^{1/3} + (BA^4)^{1/2}]e^{-cBA^4}$$

and the function $f(x) = (x^{1/4} + x^{1/3} + x^{1/2})e^{-cx}$ is bounded on the positive real axis.

The integral IV.

As with integrals I and II, we are integrating over an interval on which p is convex. In order to use Lemma 6.1.6, we need to know the minimum value of p on this interval. We distinguish two cases.

First, suppose the minimum occurs at $x = (1+\alpha)A$. Note that this implies that $p'((1+\alpha)A) \geq 0$. We must find

$$|\{\, x > (1+\alpha)A : p[(1+\alpha)A] \leq p(x) \leq p[(1+\alpha)A] + 1 \,\}|.$$

If y is the unique solution to $p(y) = p[(1+\alpha)A] + 1$ in this interval, then the desired measure is $\nu = y - (1+\alpha)A$. Expanding p about $(1+\alpha)A$ and recalling that p has an inflection point at $x = (1+\alpha)A$, we find

$$p(x) = p[(1+\alpha)A] + p'((1+\alpha)A)(x - (1+\alpha)A) + \frac{\alpha BA}{6}(x - (1+\alpha)A)^3 + \frac{B}{12}(x - (1+\alpha)A)^4.$$

Thus ν is the solution to

$$p'((1+\alpha)A)\nu + \frac{\alpha BA}{6}\nu^3 + \frac{B}{12}\nu^4 = 1.$$

It follows that the solution to (6.1.3) is less than the $\tilde{\nu}$ satisfying

$$\frac{\alpha BA}{6}\tilde{\nu}^3 + \frac{B}{12}\tilde{\nu}^4 = 1.$$

Since $\alpha, A, B > 0$ and $\tilde{\nu}$ is non-negative,

$$\nu \le \tilde{\nu} \approx \left[B^{1/4} + B^{1/3}(\alpha A)^{1/3}\right]^{-1}.$$

Thus

$$IV \lesssim e^{-cBA^4}\left[B^{1/4} + B^{1/3}(\alpha A)^{1/3}\right]^{-1}.$$

We claim that $IV \lesssim I$. It suffices to show that

$$e^{-cBA^4}\frac{B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A}{B^{1/4} + B^{1/3}(\alpha A)^{1/3}}$$

is uniformly bounded.

Since $\alpha > 0$, the above is

$$\begin{aligned} &\le \quad e^{-cBA^4}\frac{B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A}{B^{1/4}} \\ &= \quad e^{-cBA^4}[1 + (BA^4)^{1/12} + (BA^4)^{1/4}]. \end{aligned}$$

Since $f(x) = [1 + x^{1/12} + x^{1/4}]e^{-cx}$ is a bounded function on the positive real axis, the conclusion follows.

Suppose, next, that the minimum of p on $[(1+\alpha)A, \infty)$ occurs at some point x_0 interior to the interval at which p' vanishes. In this case, p' has three real roots, and so by Proposition

6.1.5, $2 < \alpha \leq 1 + \sqrt{3}$. Precisely the same argument we used above to show that, regardless of the size of $p(A)$,

$$\int_{-\infty}^{A} e^{-p(x)}\,dx \approx \int_{-\infty}^{0} e^{-p(x)}\,dx \approx e^{-p(0)}|\{\, x < 0 : 0 < p(x) < 1 \,\}|$$

shows that, regardless of the size of $p[(1+\alpha)A]$,

$$IV = \int_{(1+\alpha)A}^{\infty} e^{-p(x)}\,dx \approx \int_{x_0}^{\infty} e^{-p(x)}\,dx \approx e^{-p(x_0)}|\{\, x > x_0 : p(x_0) < p(x) < p(x_0) + 1 \,\}|.$$

Thus we must estimate $p(x_0)$ and the positive number y satisfying

$$p(x_0) + 1 = p(x_0 + y).$$

Expanding p in powers of $y = x - x_0$ yields

$$\begin{aligned} p(x) &= p(x_0) + p'(x_0)y + \frac{1}{2}p''(x_0)y^2 + \frac{1}{6}p'''(x_0)y^3 + \frac{1}{24}p^{(4)}(x_0)y^4 \\ &= p(x_0) + \frac{B}{2}[x_0^2 - A(2+\alpha)x_0 + A^2(1+\alpha)]y^2 \\ &\quad + \frac{B}{6}[2x_0 - A(2+\alpha)]y^3 + \frac{B}{12}y^4 \end{aligned} \tag{6.1.9}$$

We find that

$$p'(x) = \frac{B}{6}x[2x^2 - 3A(2+\alpha)x + 6A^2(1+\alpha)].$$

Set

$$\varepsilon = 9\alpha^2 - 12\alpha - 12.$$

This is positive by hypothesis. (See Proposition 6.1.5.) Then

$$x_0 = \frac{A}{4}[3(2+\alpha) + \sqrt{\varepsilon}] \tag{6.1.10}$$

and

$$x_0^2 = \frac{A^2}{8}[9\alpha^2 + 12\alpha + 12 + 3(2+\alpha)\sqrt{\varepsilon}]. \tag{6.1.11}$$

Substituting (6.1.10) and (6.1.11) into (6.1.9) yields

$$\begin{aligned} & p(x_0 + y) \\ = \ & p(x_0) + \frac{BA^2}{48}\left[\varepsilon + 3(2+\alpha)\sqrt{\varepsilon}\right] y^2 + \frac{BA}{12}\left[2 + \alpha + \sqrt{\varepsilon}\right] y^3 + \frac{B}{12} y^4. \end{aligned}$$

Thus

$$1 = \frac{BA^2}{48}\left[\varepsilon + 3(2+\alpha)\sqrt{\varepsilon}\right] y^2 + \frac{BA}{12}\left[2 + \alpha + \sqrt{\varepsilon}\right] y^3 + \frac{B}{12} y^4,$$

and so

$$y \approx \left[B^{1/4} + B^{1/3}A^{1/3}(2 + \alpha + \sqrt{\varepsilon})^{1/3} + B^{1/2}A\varepsilon^{1/4}(\sqrt{\varepsilon} + 3(2+\alpha))^{1/2}\right]^{-1}.$$

Since $2 < \alpha \le 1 + \sqrt{3}$, for such α,

$$0 < \varepsilon = 3(3\alpha^2 - 4\alpha - 4) \lesssim 1.$$

Hence

$$y \approx \left[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\varepsilon^{1/4}\right]^{-1}.$$

Recall that we wish to show that $IV \lesssim I$, or, equivalently, that

$$e^{-p(x_0)}\left[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\varepsilon^{1/4}\right]^{-1} \lesssim \left[B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\right]^{-1}. \tag{6.1.12}$$

Since $e^{-p(x_0)} \le 1$ and $\varepsilon \lesssim 1$, this follows immediately in the case in which ε is also bounded below by an absolute constant β.

To prove (6.1.12) for all ε, therefore, it suffices to find an absolute constant β such that (6.1.12) holds for all $0 < \varepsilon \le \beta$. Since such an estimate is likely to rely upon the relative smallness of $e^{-p(x_0)}$ compared to BA^4, we need more information about the size of $p(x_0)$. A calculation shows

$$\begin{aligned} p(x_0) &= \frac{B}{12}x_0^2[x_0^2 - 2A(2+\alpha)x_0 + 6A^2(1+\alpha)] \\ &= \frac{BA^4}{(12)(64)}(9\alpha^2 + 12\alpha + 12 + 3(2+\alpha)\sqrt{\varepsilon})(-3\alpha^2 + 12\alpha + 12 - (2+\alpha)\sqrt{\varepsilon}) \\ &\approx BA^4(-3\alpha^2 + 12\alpha + 12 - (2+\alpha)\sqrt{\varepsilon}). \end{aligned}$$

We claim that there exist positive constants β and d such that for all $\alpha \in (2, 1+\sqrt{3}]$, if $\varepsilon \leq \beta$,

$$-3\alpha^2 + 12\alpha + 12 - (2+\alpha)\sqrt{\varepsilon} \geq d,$$

from which it will follow that $p(x_0) \geq dBA^4$.

Indeed, one shows that

$$-3\alpha^2 + 12\alpha + 12 - (2+\alpha)\sqrt{\varepsilon} \geq 6(1 - \sqrt{\varepsilon}).$$

This is bounded below by 3 if $\varepsilon \leq \frac{1}{4}$. The claim follows.

Return to the proof of (6.1.12) when $\varepsilon \leq \frac{1}{4}$. The inequality will follow if we show that

$$e^{-dBA^4}\frac{B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A}{B^{1/4} + B^{1/3}A^{1/3} + B^{1/2}A\varepsilon^{1/4}} = e^{-dBA^4}\frac{1 + (BA^4)^{1/12} + (BA^4)^{1/4}}{1 + (BA^4)^{1/12} + (BA^4)^{1/4}(\varepsilon)^{1/4}} \qquad (6.1.13)$$

is bounded. This is indeed the case since

$$0 \leq f(x) = e^{-dx}\frac{1 + x^{1/12} + x^{1/4}}{1 + x^{1/12} + x^{1/4}\varepsilon^{1/4}} \leq e^{-dx}(1 + x^{1/12} + x^{1/4})$$

and the latter is bounded above on the positive real axis.

Another interpretation.

In the case in which p is convex, (6.1.2) holds, i.e., if $p(0) = p'(0) = 0$,

$$\int_{-\infty}^{\infty} e^{-p(x)}\,dx \approx |\{\, x : p(x) \leq 1 \,\}|.$$

We claim that our estimates show that the same is true in the case of any fourth-degree polynomial with positive leading coefficient and global minimum at the origin. Indeed, set

$$\begin{aligned} \mu &= \{\, x : p(x) \leq 1 \,\} \\ \mu^+ &= \{\, x > 0 : p(x) \leq 1 \,\} \\ \mu^- &= \{\, x < 0 : p(x) \leq 1 \,\}. \end{aligned}$$

Clearly

$$e^{-1}\mu \leq \int_{\{x:p(x)\leq 1\}} e^{-p(x)}\,dx \leq \int_{-\infty}^{\infty} e^{-p(x)}\,dx.$$

On the other hand, the estimates of the previous section imply the existence of a constant $C > 0$ such that

$$\int_{-\infty}^{\infty} e^{-p(x)}\,dx \leq C\mu^-.$$

Since $\mu^- \leq \mu$, it follows that

$$\int_{-\infty}^{\infty} e^{-p(x)}\,dx \approx \mu,$$

as claimed.

6.1.4 Remarks on polynomials of higher degree.

We now show that Lemma 6.1.9 does not extend to polynomials of degree greater than four. As a consequence, the results in this section can not easily be extended to tube domains (2.0.1) defined by higher-degree non-convex polynomials with positive leading coefficients. It is not clear what *uniform* estimates should replace Lemma 6.1.9 to obtain sharp estimates. Fortunately, we do not need sharp estimates to prove the results of this thesis. In the following sections, we obtain less precise estimates for N, which are sufficient for proving the main theorems.

For the moment, consider the analogue of Lemma 6.1.9 for convex polynomials:

Lemma 6.1.10. *Let n be a positive integer and define*

$$p(x) = \sum_{j=2}^{2n} \beta_j x^j.$$

Suppose p is convex on $\mathbb{R}$. Then

$$I := \int_{-\infty}^{\infty} e^{-p(x)}\, dx \approx \left[\sum_{j=2}^{2n} |\beta_j|^{\frac{1}{j}}\right]^{-1}. \tag{6.1.14}$$

This lemma is not new; it follows easily from the results of Bruna, Nagel, and Wainger discussed above. We saw in Lemma 6.1.9 that this same result holds if $n = 2$ even if we replace the hypothesis that p is convex with the weaker hypotheses that p attains its global minimum at 0 and $\beta_{2n} > 0$. We claim that such a result does *not* hold if $n = 3$.

Indeed, consider

$$\begin{aligned} p(x) &= x^2(x-a)^4 \\ &= x^6 - 4ax^5 + 6a^2x^4 - 4a^3x^3 + a^4x^2, \end{aligned}$$

with $a > 1$. Clearly p is non-negative, attains its global minimum at the origin, and is convex for $x \le 0$. If the estimate of Lemma 6.1.10 were true, we would have

$$\begin{aligned}
\frac{1}{a^2} &\approx [1 + a^{\frac{1}{5}} + a^{\frac{1}{2}} + a + a^2]^{-1} \\
&\approx I \\
&\ge \int_a^\infty e^{-x^2(x-a)^4}\,dx \\
&= \int_0^\infty e^{-(y+a)^2y^4}\,dy \\
&\approx [1 + a^{\frac{1}{5}} + a^{\frac{1}{2}}]^{-1} \\
&\approx \frac{1}{a^{\frac{1}{2}}}.
\end{aligned}$$

(We have used in the above the observation that $q(y) = (y+a)^2y^4$ is convex on the positive real axis with global minimum at the origin.) Since there is no positive C independent of $a > 1$ such that $\frac{1}{a^2} \ge \frac{C}{a^{\frac{1}{2}}}$, our claim is established. See Section ( ) of the appendix for a general argument.

It is not hard to see what is going on; in the case of a non-convex fourth-degree polynomial, if there are two competing global minima, they are both points at which the polynomial vanishes to order two. A higher-degree polynomial can have different orders of vanishing at different competing global minima. Thus order of vanishing must be taken into account in the higher-degree case.

6.2 Higher degree: uniform lower bound for $N(\eta, \tau)$

In this section, we fix a polynomial b of the form (2.1.1) and estimate N. Recall that we have written N as

$$N(\eta,\tau) = e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-2\tau p(\lambda)}\,d\lambda,$$

where now $p(\lambda) := b(\lambda+\lambda_\eta) - \eta(\lambda+\lambda_\eta) + b^*(\eta)$. The polynomial p is a non-negative, even-degree polynomial in λ which has a monic derivative and vanishes to even order at the origin. Moreover, p is non-convex precisely when b is non-convex. It was shown in the previous section that the uniform estimate (6.1.14) does not hold when p is non-convex; hence we take a different approach.

Consider the Taylor expansion of p, $p(\lambda) = \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda_\eta)\lambda^j}{j!}$, about the origin. By the triangle inequality, we can get an upper bound for p that is a convex function,

$$p(\lambda) = \left| \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda_\eta)\lambda^j}{j!} \right| \le \sum_{j=2}^{2n} \frac{|b^{(j)}(\lambda_\eta)||\lambda|^j}{j!}.$$

Thus, $p(\lambda) \le \sum_{j=2}^{2n} \frac{|b^{(j)}(\lambda_\eta)|\lambda^j}{j!}$ when $\lambda \ge 0$. Since p is non-negative, this upper bounds gives a lower bound on the integral N,

$$\begin{aligned} N(\eta,\tau) &= e^{2\tau b^*(\eta)} \left[\int_0^\infty e^{-2\tau \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda_\eta)\lambda^j}{j!}}\, d\lambda + \int_{-\infty}^0 e^{-2\tau \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda_\eta)\lambda^j}{j!}}\, d\lambda \right] \\ &\ge e^{2\tau b^*(\eta)} \int_0^\infty e^{-2\tau \sum_{j=2}^{2n} \frac{|b^{(j)}(\lambda_\eta)|\lambda^j}{j!}}\, d\lambda. \end{aligned} \tag{6.2.1}$$

For each $\eta \in \mathbb{R}$ and $\tau > 0$, $\rho(\lambda) := 2\tau \sum_{j=2}^{2n} \frac{|b^{(j)}(\lambda_\eta)|\lambda^j}{j!}$ is convex on the positive real line. By the estimate obtained at (6.1.7),

$$\int_0^\infty e^{-\rho(\lambda)}\, d\lambda \approx \left[\sum_{j=2}^{2n} \tau^{1/j} \left| b^{(j)}(\lambda_\eta) \right|^{1/j} \right]^{-1} \tag{6.2.2}$$

independent of the positive coefficients of ρ.

After applying (6.2.2) to inequality (6.2.1), we have the lower bound

$$e^{2\tau b^*(\eta)} \left[\sum_{j=2}^{2n} \tau^{1/j} \left| b^{(j)}(\lambda_\eta) \right|^{1/j} \right]^{-1} \lesssim N(\eta,\tau)$$

uniformly in η and $\tau > 0$.

6.3 Higher degree: upper bound for $N(\eta,\tau)$

In a later chapter, we will use the uniform estimate from the prior section to show that the η-integral is always absolutely convergent at infinity. Thus if the η-integral were to diverge, it would do so because of the contribution from some interval of the form $[\eta_0, \eta_0 + \epsilon]$. Guided by that reasoning, we only need to find an upper bound for N which is uniform on the set $[\eta_0, \eta_0 + \epsilon] \times \mathbb{R}_+$.

Completely factoring p might be a rather difficult task, maybe impossible because $\mathbb{R}$ is not a splitting field. But, in theory, every even-degree real polynomial factors as a product of real quadratics. Since p is a non-negative, even-degree polynomial satisfying $p(0) = 0$, its roots are either real with even order or they occur as complex conjugate pairs. Thus

$$p(\lambda) = c_n \lambda^2 \prod_{i=1}^{n-1} \left[(\lambda - h_i)^2 + k_i \right], \tag{6.3.1}$$

for some $h_i = h_i(\eta) \in \mathbb{R}$, $k_i = k_i(\eta) \geq 0$, $n \geq 1$, $i = 1, \ldots, n-1$, and $c_n = (2n+2)^{-1}$. For each $1 \leq n \in \mathbb{N}$, it will suffice to consider the family of polynomials

$$\lambda^2 \prod_{i=1}^{n-1} [(\lambda - h_i)^2 + k_i],$$

where $h_i \in \mathbb{R}$, and $k_i \geq 0$.

Even though it is clear that the coefficients of a polynomial vary continuously with the parameters h_i, k_i, it is not evident that the parameters vary continuously with the coefficients of p. In order for our approach to be valid, it must be shown that the h_i, $i = 1, \ldots, n-1$, are locally bounded in η.

Lemma 6.3.1. *Suppose p is of the form (6.3.1). For each $\eta_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists $C > 0$ such that*

$$|h_i(\eta)| < C$$

for all $i = 1, \ldots, n-1$ and all $\eta \in [\eta_0, \eta_0 + \epsilon]$.

Proof. Suppose that the conclusion did not hold for a polynomial p of degree $2n$. In particular, assume that for some i, say $i = 1$, there is an η_0 and $\epsilon > 0$ such that $|h_1(\eta_j)| \longrightarrow \infty$ for some sequence $\{\eta_j\}_j \subset [\eta_0, \eta_0 + \epsilon]$. Bolzano-Weiestrass guarantees a subsequence will converge to a point in $[\eta_0, \eta_0 + \epsilon]$, so we may assume the original sequence has the limit $\hat{\eta} \in [\eta_0, \eta_0 + \epsilon]$. Since we can find another subsequence so that $\{h_1(\eta_{j_k})\}_k$ is monotone, we may assume $\{h_1(\eta_j)\}_j$ and $\{\eta_j\}_j$ are both monotone.

Define $p(\eta, x) := b(x + \lambda_\eta) - \eta(x + \lambda_\eta) + b^*(\eta)$. For each $x \in \mathbb{R}$, $p(\eta, x)$ is bounded whenever η is bounded because b^* is continuous and λ_η is bounded on each compact subset of $\mathbb{R}$. For each $m \in \mathbb{N}$, $p(\eta_j, m)$ is bounded for all $j \in \mathbb{N}$. On the other hand, the factor

$$[(m - h_1(\eta_j))^2 + k_1(\eta_j)] \longrightarrow \infty \quad \text{as } j \to \infty. \tag{6.3.2}$$

This implies that there is a subsequence, $\{\eta_{j_\ell}\}_\ell$ and one other factor, say $i = 2$, that converges to zero, i.e.,

$$[(m - h_2(\eta_{j_\ell}))^2 + k_2(\eta_{j_\ell})] \longrightarrow 0 \quad \text{as } \ell \to \infty. \tag{6.3.3}$$

Since $k_i(\eta_{j_\ell}) \geq 0$, $h_2(\eta_{j_\ell}) \to m$ as $\ell \to \infty$.

If we replace m with $m + 1$, the boundedness of $p(\eta_{j_\ell}, m + 1)$ and divergence (6.3.2) still hold as $\ell \to \infty$. Also, the factor associated with $i = 2$ will be bounded away from zero because $|m - h_2(\eta_{j_\ell})|$ converges to zero. Thus there must be some subsequence of $\{\eta_{j_\ell}\}_\ell$, and a term, say $i = 3$, such that convergence (6.3.3) occurs after m is replaced by $m + 1$. We may suppose the subsequence is $\{\eta_{j_\ell}\}_\ell$.

If we iterate the process above $n-2$ times, each factor $[(m+n-2-h_i(\eta_{j_\ell}))^2+k_i(\eta_{j_\ell})]$, with $i=2,\ldots,n-1$, is bounded away from zero as $\ell\to\infty$. Moreover, $[(m+n-2-h_1(\eta_{j_\ell}))^2+k_1(\eta_{j_\ell})]\longrightarrow\infty$ as $\ell\to\infty$. Thus $p(\eta_{j_\ell},m+n-2)$ cannot possibly be bounded, which is a contradiction.

It follows that there is no sequence $\{\eta_n\}_n$ in $[\eta_0,\eta_0+\epsilon]$ such that $|h_1(\eta_n)|\longrightarrow\infty$. Hence each $h_i(\eta)$, $i=1,\ldots,n-1$, is bounded on $[\eta_0,\eta_0+\epsilon]$. $\square$

Fix $\eta_0\in\mathbb{R}$ and $\epsilon>0$. From the lemma above, there exists some $C>1$, depending on ϵ and η_0, such that

$$|h_i(\eta)|<C-1$$

for all $i=1,\ldots,n-1$ and all $\eta\in[\eta_0,\eta_0+\epsilon]$. We can make the following estimates:

$$\begin{aligned}
e^{-2\tau b^*(\eta)}N(\eta,\tau) &= \int_{-\infty}^{\infty} e^{-2\tau[b(\lambda-\lambda_\eta)-\eta(\lambda-\lambda_\eta)+b^*(\eta)]}\,d\lambda\\
&= \int_{-\infty}^{\infty} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}\left[(\lambda-h_i)^2+k_i\right]}\,d\lambda\\
&\le \int_{-\infty}^{\infty} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(\lambda-h_i)^2}\,d\lambda\\
&= \int_{C}^{\infty} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(\lambda-h_i)^2}\,d\lambda+\int_{-C}^{C} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(\lambda-h_i)^2}\,d\lambda\\
&\qquad+\int_{-\infty}^{-C} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(\lambda-h_i)^2}\,d\lambda\\
&=: I+II+III.
\end{aligned}$$

Consider I.

$$\begin{aligned} I &\leq \int_C^\infty e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(C-h_i)^2}\,d\lambda \\ &\leq \int_{-\infty}^\infty e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(C-h_i)^2}\,d\lambda \\ &\approx \left[\tau^{1/2}\prod_{i=1}^{n-1}|C-h_i|\right]^{-1} \\ &\leq \tau^{-1/2}. \end{aligned}$$

The last inequality follows from the fact that $|h_i(\eta)| < C-1$, for all $\eta \in [\eta_0, \eta_0+\epsilon]$, implies $1 < C - |h_i| \leq |C-h_i|$; hence $|C-h_i|^{-1} < 1$ for all $i = 1, \dots, n-1$.

Consider II.

$$II = \int_{-C}^C e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(\lambda-h_i)^2}\,d\lambda \leq 2C \lesssim 1.$$

Consider III.

$$\begin{aligned} III &\leq \int_{-\infty}^{-C} e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(-C-h_i)^2}\,d\lambda \\ &\leq \int_{-\infty}^\infty e^{-2c_n\tau\lambda^2\prod_{i=1}^{n-1}(C+h_i)^2}\,d\lambda \\ &\approx \left[\tau^{1/2}\prod_{i=1}^{n-1}|C+h_i|\right]^{-1} \\ &\leq \tau^{-1/2}, \end{aligned}$$

where the last inequality follows from the estimate of I.

After collecting the estimates for I, II, and III, we have the upper bound

$$N(\eta,\tau) \lesssim e^{2\tau b^*(\eta)}\left(\frac{1+\tau^{1/2}}{\tau^{1/2}}\right)$$

uniformly in $\tau > 0$ and $\eta \in [\eta_0, \eta_0+\epsilon]$.

Chapter 7

Non-Convex Quartic Results

In the first half of Chapter 6, we restricted our analysis to fourth-degree polynomials and established uniform estimates for $N(\eta, \tau)$. In this chapter, we focus on fourth-degree polynomials in order to analyze $\lambda(\eta)$, which is defined as the largest location of the global minimum of $B_\eta(\lambda) = b(\lambda) - \eta\lambda$ as a function of λ. This chapter is essentially Chapter 4 redux and will consist almost entirely of work done in early 2010. See [2] for a full exposition.

In Section 1.2, we summarized Carracino's detailed analysis of the Szegö kernel on the non-pseudoconvex model domain

$$\Omega = \{\, (z_1 = x + iy, z_2 = t + i\xi) : \xi > b(x) \,\},$$

for a non-convex, piecewise quadratic function b.

Domains of the form Ω are non-pseudoconvex precisely when b fails to be convex. Recall that very little research has been done on such domains. Thus Carracino's findings are novel. Building on her work, the results that follow extend the comprehensive study [16] from convex to non-convex polynomial domains.

We start by making an appeal to Section 2.4. It suffices to consider quartic polynomials of the form

$$b(x) = \frac{1}{4}x^4 + \frac{1}{2}px^2 + qx, \quad p < 0, q \in \mathbb{R}. \tag{7.0.1}$$

These are the precise conditions needed for b to be non-convex since $b''(x) = 3x^2 + p \geq 0$ for all x only if $p \geq 0$. Hence Ω is a non-pseudoconvex domain. Recall that the Szegö kernel associated with this domain takes the form (1.2.3) and that our goal is to identify subsets of $\mathbb{C}^2 \times \mathbb{C}^2$ on which the Szegö kernel and its derivatives converge absolutely.

7.1 Statement of the results, [[2],§2]

Let b be as in (7.0.1) and let

$$z = (z_1, z_2) = (x + iy, t + ib(x) + ih)$$
$$w = (w_1, w_2) = (r + is, u + ib(r) + ik).$$

Define

$$\Sigma = \{\, (z, w) : x = r,\ |x| > \sqrt{-p}\,\} \cup \{\, (z, w) : |x| = |r| = \sqrt{-p}\,\}. \tag{7.1.1}$$

Theorem 7.1.1. *The integral defining $S(z, w)$ is absolutely convergent in the region in which*

$$h + k + b(x) + b(r) - 2b^{**}\left(\frac{x + r}{2}\right) > 0.$$

This is an open neighborhood of $(\overline{\Omega} \times \overline{\Omega}) \setminus \Sigma$. More generally, if i_1, j_1, i_2, and j_2 are non-negative integers, then

$$\partial^{i_1}_{z_1}\partial^{j_1}_{\bar{w}_1}\partial^{i_2}_{z_2}\partial^{j_2}_{\bar{w}_2} S(z, w) = c\int_{-\infty}^{\infty}\int_0^{\infty} e^{\eta\tau[z_1+\bar{w}_1]+i\tau[z_2-\bar{w}_2]}\,\frac{\eta^{i_1+j_1}\tau^{i_1+j_1+i_2+j_2+1}}{N(\eta, \tau)}\,d\eta\,d\tau$$

is absolutely convergent in the same region.

Remark 7.1.2. *Compare this with Theorem 3.2 in* [?HNW:09]. *In that theorem, the domain is of the form* (**??**) *for b convex, and the region in which the integrals converge absolutely is defined by the inequality*

$$h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

These two theorems are, in fact, analogous since the Legendre transform is an involution on the set of convex functions.

Theorem 7.1.3. *If $(x + iy, t + ib(x)), (r + is, u + ib(r)) \in \Sigma$, $S[(x, 0, 0), (r, 0, 0)]$ is infinite. Also, if $\delta = h + k > 0$,*

$$\lim_{\delta \to 0^+} S[(x, i(b(x) + h)), (r, i(b(r) + k))] = \infty.$$

These two theorems identify the subset of $\mathbb{C}^2 \times \mathbb{C}^2$ on which the Szegö kernel converges absolutely. In order to prove these results, we move directly to the study of the function $\lambda(\cdot)$.

In [[2],§3], there is one subtle difference in the development of the notation. First, the local minima of B_η are denoted as λ_- and λ_+. Then the function $\lambda(\cdot)$, as defined in Chapter 3, is defined piecewise. Unlike in the higher-degree setting, simple calculus will allow us to explicitly describe the set on which the Szegö kernel is absolutely convergent in terms of a coefficient of b. It will become evident how the structure theorem in Section 4.3 generalizes this simpler result.

7.2 Behavior of $b(\lambda) - \eta\lambda$, [[2],§3]

Given a polynomial b of the form (7.0.1), $B_\eta(\lambda) = b(\lambda) - \eta\lambda$ is a fourth-degree polynomial in λ with positive leading coefficient. Hence it tends to infinity as $|\lambda| \to \infty$. It follows that its infimum is achieved at some λ for which B'_η vanishes; i.e., at some λ satisfying

$$\lambda^3 + p\lambda - (\eta - q) = 0.$$

Note that this is a depressed cubic equation. Therefore, by considering its discriminant, one finds:

Proposition 7.2.1. *1. If $4(-p)^3 \leq 27(\eta - q)^2$, there is a single λ at which B'_η changes sign, hence B_η has a single local extremum, which is necessarily the location of the global minimum.*

2. If $4(-p)^3 > 27(\eta - q)^2$, $B'_\eta(\lambda) = 0$ has three distinct solutions. Two correspond to local minima of B_η. We label them $\lambda_-(\eta)$ and $\lambda_+(\eta)$, with $\lambda_-(\eta) < \lambda_+(\eta)$.

The next propositions contain more specific information about the location(s) of the global minima of B_η.

Proposition 7.2.2. *Let $g(\lambda) = \lambda^3 + p\lambda$. Then*

1. *g is negative on $(-\infty, -\sqrt{-p})$ and $(0, \sqrt{-p})$, and g is positive on $(-\sqrt{-p}, 0)$ and $(\sqrt{-p}, \infty)$.*
2. *g is decreasing on $\left(-\sqrt{-p/3}, \sqrt{-p/3}\right)$ and increasing on $(-\infty, -\sqrt{-p/3})$ and $(\sqrt{-p/3}, \infty)$.*

Proof. (i): Notice that $g(\lambda) < 0$ only if one of the following two conditions are satisfied: $\lambda < 0$ and $\lambda^2 + p > 0$, or $\lambda > 0$ and $\lambda^2 + p < 0$. The first occurs whenever $\lambda < 0$ and $\lambda^2 > -p$. The second occurs whenever $\lambda > 0$ and $\lambda^2 < -p$. By putting these together, it follows that g is negative on $(-\infty, -\sqrt{-p})$ and $(0, \sqrt{-p})$.

Since g vanishes to first order at $0, \pm\sqrt{-p}$, g is positive on $(-\sqrt{-p}, 0)$ and $(\sqrt{-p}, \infty)$.

(ii): Consider $g'(\lambda) = 3\lambda^2 + p$. This is a quadratic polynomial with positive leading coefficient, which has zeros at $\pm\sqrt{-p/3}$. Thus g' is negative on $\left(-\sqrt{-p/3}, \sqrt{-p/3}\right)$ and positive on $(-\infty, -\sqrt{-p/3})$ and $(\sqrt{-p/3}, \infty)$. From this, the result follows.

$\square$

Proposition 7.2.3. *Let $B_\eta(\lambda) = b(\lambda) - \eta\lambda$, with b as in* (7.0.1).

(i) If $\eta - q = 0$, $\lambda_-(\eta) = -\sqrt{-p}$, $\lambda_+(\eta) = \sqrt{-p}$, and $B_\eta(\lambda_-(\eta)) = B_\eta(\lambda_+(\eta))$. In other words, the global minimum of B_η is achieved at two distinct points $\lambda_-(\eta)$ and $\lambda_+(\eta)$.

(ii) If $0 < \eta - q < \left(\frac{4(-p)^3}{27}\right)^{\frac{1}{2}}$,

$$-\sqrt{-p} < \lambda_-(\eta) < 0 < \sqrt{-p} < \lambda_+(\eta)$$

and $B_\eta(\lambda_+) < B_\eta(\lambda_-)$.

(iii) If $\left(\frac{4(-p)^3}{27}\right)^{\frac{1}{2}} \leq \eta - q$, B_η has a single local (hence global) minimum at $\lambda = \lambda_+(\eta) > \sqrt{-p}$, and $\lambda_+(\eta) \sim \eta^{\frac{1}{3}}$ as $\eta \to \infty$.

(iv) If $-\left(\frac{4(-p)^3}{27}\right)^{\frac{1}{2}} < \eta - q < 0$,

$$\lambda_-(\eta) < -\sqrt{-p} < 0 < \lambda_+(\eta) < \sqrt{-p}$$

and $B_\eta(\lambda_-) < B_\eta(\lambda_+)$.

(v) If $\eta - q < -\left(\frac{4(-p)^3}{27}\right)^{\frac{1}{2}} < 0$, B_η has a single local (hence global) minimum at $\lambda = \lambda_-(\eta) < -\sqrt{-p}$, and $\lambda_-(\eta) \sim \eta^{\frac{1}{3}}$ as $\eta \to -\infty$.

Proof. (i): If $\eta = q$, then the local extrema of B_η occur at solutions to $g(\lambda) = 0$. The three solutions are $\lambda = -\sqrt{-p}$, 0, $\sqrt{-p}$, and the local minimum is attained at $\lambda = \pm\sqrt{-p}$. Since in this case $B_\eta(\lambda) = \frac{1}{4}\lambda^4 + \frac{1}{2}p\lambda^2$ is even, the conclusion follows.

(ii): By Proposition 7.2.1, the upper bound on η guarantees that B_η in fact has two local minima. Since $\eta - q > 0$, for $\lambda \in [0, \sqrt{-p}]$,

$$B_\eta'(\lambda) = (\lambda^3 + p\lambda) - (\eta - q) = g(\lambda) - (\eta - q) < 0.$$

Since g is increasing for $\lambda > \sqrt{\frac{-p}{3}}$, $g(\lambda) - (\eta - q) = 0$ has precisely one solution in $(\sqrt{-p}, \infty)$, and it is the location of a local minimum for B_η. We have named this point $\lambda_+(\eta)$. On the other hand, since $g(\lambda) - (\eta - q)$ is also negative on $(-\infty, -\sqrt{-p}]$, the second local minimum $\lambda_-(\eta)$ is in $(-\sqrt{-p}, 0)$.

Now, $\eta - q > 0$ and $\lambda_- < 0$ imply $(\eta - q)\lambda_- < (\eta - q)(-\lambda_-)$. Since λ_+ is the location of the global minimum of the restriction of B_η to the positive real axis,

$$\begin{aligned}
B_\eta(\lambda_-) &= \left(\frac{1}{4}\lambda_-^4 + \frac{1}{2}p\lambda_-^2\right) - (\eta - q)\lambda_- \\
&> \left(\frac{1}{4}(-\lambda_-)^4 + \frac{1}{2}p(-\lambda_-)^2\right) - (\eta - q)(-\lambda_-) \\
&= B_\eta(-\lambda_-) \\
&> B_\eta(\lambda_+).
\end{aligned}$$

This proves (ii).

(iii): By Proposition 7.2.1, we are in the situation in which $B'_\eta(\lambda) = 0$ has a single solution. An identical argument to the one used to prove (ii) shows that the solution, which we call $\lambda_+(\eta)$, satisfies $\sqrt{-p} < \lambda_+(\eta)$.

We now prove the statement about the asymptotic behavior of $\lambda_+(\eta)$. Since $\lambda_+^3 > \lambda_+^3 + p\lambda_+ = \eta - q$, $\lambda_+(\eta) \to \infty$ as $\eta \to \infty$. Also, since

$$\lambda_+^3 = \eta - q - p\lambda_+,$$

we have

$$1 = \frac{\eta}{\lambda_+^3} + o(1).$$

Thus $\lambda_+^3 \sim \eta$, i.e., $\lambda_+^3 = \eta[1 + o(1)]$ as $\eta \to \infty$. It follows that $\lambda_+(\eta) \sim \eta^{\frac{1}{3}}$ as $\eta \to \infty$.

The proofs of (iv) and (v) are almost identical to the proofs of (ii) and (iii) and are omitted.

$\square$

Define the function

$$\lambda(\eta) = \begin{cases} \lambda_-(\eta) & \eta < q \\ \sqrt{-p} & \eta = q \\ \lambda_+(\eta) & \eta > q. \end{cases}$$

Thus for $\eta \neq q$, $\lambda(\eta)$ is the location of the global minimum of B_η. For $\eta = q$, the global minimum is achieved at two points, $\pm\sqrt{-p}$. Which of these we choose for the value of $\lambda(q)$ is arbitrary. Notice that by our choice of $\lambda(q)$, it follow that $\sigma(q) = -\sqrt{-p}$, which is the smallest location of the global minimum.

Proposition 7.2.4. *The function $\eta \mapsto \lambda(\eta)$ maps $\mathbb{R}$ onto $\mathbb{R} \setminus [-\sqrt{-p}, \sqrt{-p})$. Furthermore, it is*

(a) differentiable on $\mathbb{R} \setminus \{q\}$,

(b) continuous from the right at $\eta = q$, and

(c) increasing and injective on $\mathbb{R}$.

Proof. The equation

$$\eta = q + \lambda^3 + p\lambda$$

clearly expresses η as a function of λ. Furthermore, the restriction of this function to $(-\infty, -\sqrt{-p}) \cup [\sqrt{-p}, \infty)$ is easily seen to be one-to-one with image $\mathbb{R}$. Thus its inverse function is well-defined on $\mathbb{R}$ and maps this set to $(-\infty, -\sqrt{-p}) \cup [\sqrt{-p}, \infty)$. Since λ restricted to $\mathbb{R} \setminus \{q\}$ is the inverse of a function which is smooth with non-vanishing derivative on its (restricted) domain, λ is itself continuous and differentiable there, with derivative

$$\lambda'(\eta) = \frac{1}{3[\lambda(\eta)]^2 + p} > 0.$$

The proposition is established. $\square$

Remark 7.2.5. *The first statement of the proposition can be reformulated using the notation established in Chapter 3. This gives an explicit version of the structure theorem, Theorem 4.3.6. In particular,*

$$\Lambda = \mathbb{R} \setminus [-\sqrt{-p}, \sqrt{-p})$$

with $\Lambda_q = \{-\sqrt{-p}, \sqrt{-p}\}$ *being the only non-singleton. Also, we can rewrite* Σ*, given in* (7.1.1)*, as* $\Sigma = \{\, (z, w) : x, r \in \Lambda_\eta \text{ for some } \eta \in \mathbb{R}, \text{and } \delta = 0 \,\}$.

Corollary 7.2.6. *The functions* $\eta \mapsto |\lambda(\eta)|$ *and* $\eta \mapsto b^*(\eta)$ *are continuous on* $\mathbb{R}$.

Proof. Since λ is continuous on $\mathbb{R} \setminus \{q\}$, we need only consider the behavior at q. The first statement then follows since $\lim_{\eta \to q^-} \lambda(\eta) = -\sqrt{-p}$ and $\lim_{\eta \to q^+} \lambda(\eta) = \sqrt{-p} = |\lambda(q)|$.

For the second statement, note

$$b^*(\eta) = (\eta - q)\lambda(\eta) - \frac{1}{4}[\lambda(\eta)]^4 - \frac{p}{2}[\lambda(\eta)]^2.$$

Since $\eta - q$ and $\lambda(\eta)$ both switch from negative to positive at $\eta = q$,

$$b^*(\eta) = |\eta - q||\lambda(\eta)| - \frac{1}{4}|\lambda(\eta)|^4 - \frac{p}{2}|\lambda(\eta)|^2.$$

The continuity of b^* follows. $\square$

Chapter 8

Proofs for Main Theorems

In this chapter, we prove the main results of this thesis: Theorem 1.3.1, Theorem 1.3.2, and Theorem 1.3.3. Together with Corollary 4.3.10, these theorems give the necessary and sufficient conditions on b to guarantee the smoothness of the Szegö kernel and its derivatives off of the diagonal of the boundary: Corollary 1.3.4. These novel results are an incremental step toward extending [16] to all even-degree polynomials with positive leading coefficients.

8.1 Fourth degree

Let us start by proving the main results for the special case in which b is a fourth-degree polynomial of the form (7.0.1). We begin by proving Theorems 7.1.1 and 7.1.3 in order to shed light on the issues that surface while trying to prove more general results. Although the higher-degree analogues will follow in a similar fashion, it is worth seeing how we are able to express certain singularities of the Szegö kernel off of the diagonal in terms of a coefficient of b. Aside from minor changes in notation and a few remarks, this section is from our earlier paper.

8.1.1 Fourth degree: proof of Theorem 7.1.1, [[2],§5]

We now return to the analysis of the integral N.

$$\begin{aligned}
&N(\eta,\tau)\\
&= e^{2\tau b^*(\eta)}\int_{-\infty}^{\infty} e^{2\tau[\eta\lambda-b(\lambda)+B_\eta(\lambda(\eta))]}\,d\lambda\\
&= e^{2\tau b^*(\eta)}\int_{-\infty}^{\infty} e^{2\tau[\eta\lambda-b(\lambda)-\eta\lambda(\eta)+b(\lambda(\eta))]}\,d\lambda\\
&= e^{2\tau b^*(\eta)}\int_{-\infty}^{\infty} e^{2\tau[-b''(\lambda(\eta))\frac{(\lambda-\lambda(\eta))^2}{2}-b'''(\lambda(\eta))\frac{(\lambda-\lambda(\eta))^3}{6}-\frac{(\lambda-\lambda(\eta))^4}{4}]}\,d\lambda\\
&= e^{2\tau b^*(\eta)}\int_{-\infty}^{\infty} e^{-[2\tau b''(\lambda(\eta))\frac{y^2}{2}+2\tau b'''(\lambda(\eta))\frac{y^3}{6}+2\tau\frac{y^4}{4}]}\,dy\\
&\approx e^{2\tau b^*(\eta)}\left[\left(\frac{\tau}{2}\right)^{\frac{1}{4}}+\left|\frac{\tau b'''(\lambda(\eta))}{3}\right|^{\frac{1}{3}}+\left(\tau b''(\lambda(\eta))\right)^{\frac{1}{2}}\right]^{-1}\\
&\approx e^{2\tau b^*(\eta)}\left[\tau^{\frac{1}{4}}+\tau^{\frac{1}{3}}|\lambda(\eta)|^{\frac{1}{3}}+\tau^{\frac{1}{2}}(3\lambda(\eta)^2+p)^{\frac{1}{2}}\right]^{-1},
\end{aligned}$$

where we have used Lemma 6.1.9 in the second-to-last line.

We set $z=(z_1,z_2)=(x+iy,t+ib(x)+ih)$, $w=(w_1,w_2)=(r+is,t+ib(r)+ik)$, and $\delta=h+k$. Also, for non-negative integers i_1,j_1,i_2, and j_2, define $s=i_1+j_1$ and $m=i_1+j_1+i_2+j_2$ (so that $m\geq s$).

We now prove Theorem 7.1.1. If we show that each integral

$$\int_{-\infty}^{\infty}\int_0^{\infty} e^{\eta\tau[z_1+\bar{w_1}]+i\tau[z_2-\bar{w_2}]}\frac{\eta^{i_1+j_1}\tau^{i_1+j_1+i_2+j_2}}{N(\eta,\tau)}\,d\eta\,d\tau$$

is absolutely convergent in the region

$$h+k+b(x)+b(r)-2b^{**}\left(\frac{x+r}{2}\right)>0,$$

it will follow that the integral in fact equals $\partial_{z_1}^{i_1}\partial_{\bar{w}_1}^{j_1}\partial_{z_2}^{i_2}\partial_{\bar{w}_2}^{j_2}S(z,w)$. The integral becomes

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{\eta\tau[x+r+i(y-s)]+i\tau[t-u+i(b(x)+b(r)+\delta)]}\frac{\eta^s\tau^{m+1}}{N(\eta,\tau)}\,d\eta\,d\tau,$$

which we will call $S^{s,m,\delta}$.

$S^{s,m,\delta}$ converges absolutely if and only if

$$\widetilde{S}^{s,m,\delta} = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)]}\frac{|\eta|^s\tau^{m+1}}{N(\eta,\tau)}\,d\tau\,d\eta < \infty. \tag{8.1.1}$$

From above,

$$\begin{aligned}
\widetilde{S}^{s,m,\delta} \quad &\approx \quad \int_{-\infty}^{\infty}\int_{0}^{\infty} \tau^{\frac{1}{4}}|\eta|^s\tau^{m+1}e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]}\,d\tau\,d\eta \\
&\qquad + \int_{-\infty}^{\infty}\int_{0}^{\infty} \tau^{\frac{1}{3}}|\lambda(\eta)|^{\frac{1}{3}}|\eta|^s\tau^{m+1}e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]}\,d\tau\,d\eta \\
&\qquad + \int_{-\infty}^{\infty}\int_{0}^{\infty} \tau^{\frac{1}{2}}(3\lambda(\eta)^2+p)^{\frac{1}{2}}|\eta|^s\tau^{m+1}e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]}\,d\tau\,d\eta. \\
&=: \quad \mathcal{I}_1^{s,m,\delta} + \mathcal{I}_2^{s,m,\delta} + \mathcal{I}_3^{s,m,\delta}.
\end{aligned}$$

Furthermore, let $\mathcal{I}_j^{s,m,\delta}(\eta)$ denote the integrand of the η-integral defining $\mathcal{I}_j^{s,m,\delta}$, so that

$$\mathcal{I}_j^{s,m,\delta} = \int_{-\infty}^{\infty}\mathcal{I}_j^{s,m,\delta}(\eta)\,d\eta.$$

Set

$$A(x,r,\eta) = b(x)+b(r)-\eta(x+r)+2b^*(\eta).$$

Since

$$A(x,r,\eta) = \sup_{\lambda}[\eta\lambda-b(\lambda)]-[\eta x-b(x)]+\sup_{\lambda}[\eta\lambda-b(\lambda)]-[\eta r-b(r)],$$

A is non-negative.

Each $\mathcal{I}_j^{s,m,\delta}(\eta)$ involves an integral in τ of the form

$$\int_0^\infty e^{-\tau[\delta + A(x,r,\eta)]}\tau^\alpha \, d\tau,$$

which yields

$$c_\alpha \frac{1}{[\delta + A(x,r,\eta)]^{\alpha+1}} \quad \text{if } \delta + A(x,r,\eta) > 0. \tag{8.1.2}$$

It is now clear that there are two potential barriers to the convergence of the full integrals $\mathcal{I}_j^{s,m,\delta}$:

1. insufficient growth of A in η at infinity, and
2. vanishing of $\delta + A(x,r,\eta)$ for some finite η for certain choices of x, r, and δ.

The next subsections explore these issues and in so doing establish the theorem.

8.1.2 Behavior of $A(x,r,\eta)$ for large $|\eta|$.

Lemma 8.1.1. *Fix $x, r \in \mathbb{R}$. Then*

$$A(x,r,\eta) \sim \frac{3}{2}\eta^{\frac{4}{3}}, \quad |\eta| \to \infty.$$

Proof. Recall from Proposition 7.2.3 that $\lambda(\eta) \sim \eta^{\frac{1}{3}}$ as $|\eta| \longrightarrow \infty$, i.e., $\lambda(\eta) = \eta^{\frac{1}{3}}(1 + o(1))$

as $|\eta| \to \infty$. Thus as $|\eta| \to \infty$,

$$\begin{aligned}
&A(x,r,\eta) \\
&= b(x)+b(r)-\eta(x+r)+2b^*(\eta) \\
&= b(x)+b(r)-\eta(x+r)+2\left[\eta\lambda(\eta)-b\left(\lambda(\eta)\right)\right] \\
&= b(x)+b(r)-\eta(x+r) \\
&\quad +2\left[\eta^{\frac{4}{3}}(1+o(1))-\frac{1}{4}\eta^{\frac{4}{3}}(1+o(1))^4-\frac{1}{2}p\eta^{\frac{2}{3}}(1+o(1))^2-q\eta^{\frac{1}{3}}(1+o(1))\right] \\
&= b(x)+b(r)-\eta(x+r) \\
&\quad +2\left[\eta^{\frac{4}{3}}(1+o(1))-\frac{1}{4}\eta^{\frac{4}{3}}(1+o(1))\left(1-\frac{1}{2}p\eta^{-\frac{2}{3}}(1+o(1))-q\eta^{-1}\right)\right] \\
&= \frac{3}{2}\eta^{\frac{4}{3}}+\eta^{\frac{4}{3}}o(1)+O(|\eta|) \\
&= \frac{3}{2}\eta^{\frac{4}{3}}(1+o(1)).
\end{aligned}$$

□

Remark 8.1.2. *Our arguments can be extended to obtain a generalized asymptotic expansions for $\lambda(\cdot)$ and $A(x,r,\cdot)$. See Olver* [8]*, Section 1.5 for a detailed discussion of such techniques.*

This lemma, equation (8.1.2), and parts (iii) and (v) of Proposition 7.2.3, allow us to conclude the following:

1. $\mathcal{I}_1^{s,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{5}{4}+m+1)}|\eta|^s = c|\eta|^{-3-\frac{4}{3}m+s}$. Since $m \geq s \geq 0$, $-3-\frac{4}{3}m+s \leq -3$, and so for any fixed s, m, and δ, $\mathcal{I}_1^{s,m,\delta}$ is convergent at infinity.

2. $\mathcal{I}_2^{s,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{4}{3}+m+1)} \cdot |\eta^{\frac{1}{3}}|^{\frac{1}{3}} \cdot |\eta|^s = c|\eta|^{-3-\frac{4}{3}m+s}$, and so each $\mathcal{I}_2^{s,m,\delta}$ is convergent at infinity.

3. $\mathcal{I}_3^{s,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{3}{2}+m+1)} \cdot |\eta|^{\frac{1}{3}} \cdot |\eta|^s = c|\eta|^{-3-\frac{4}{3}m+s}$, and so each $\mathcal{I}_3^{s,m,\delta}$ is convergent at infinity.

8.1.3 Vanishing of $\delta + A(x, r, \eta)$

The estimates of the previous sections show that whether or not the integrals $\mathcal{I}_j^{s,m,\delta}$ converge depends upon whether or not for some fixed x, r, and δ the function

$$\eta \mapsto \delta + A(x, r, \eta)$$

vanishes for some finite η_0 and, if so, the behavior of this function near such a point. In particular, we have proved:

Proposition 8.1.3. *If for some x, r, and δ fixed*

$$\inf_{\eta \in \mathbb{R}} [\delta + A(x, r, \eta)] > 0,$$

then each $\mathcal{I}_j^{s,m,\delta}$ is finite.

Note, moreover, that

$$\begin{aligned} \inf_{\eta \in \mathbb{R}} [\delta + A(x, r, \eta)] &= \inf_{\eta \in \mathbb{R}} [\delta + b(x) + b(r) - \eta(x + r) + 2b^*(\eta)] \\ &= \delta + b(x) + b(r) - 2 \sup_{\eta \in \mathbb{R}} \left[\eta \left(\frac{x + r}{2} \right) - b^*(\eta) \right] \\ &= \delta + b(x) + b(r) - 2b^{**} \left(\frac{x + r}{2} \right). \end{aligned}$$

(By Lemma 3.3.1, b^* has super-linear growth. This guarantees the finiteness of the supremum in the second-to-last line.) We have thus proved that the integrals defining the Szegö kernel and all its derivatives converge absolutely in the region

$$\delta + b(x) + b(r) - 2b^{**} \left(\frac{x + r}{2} \right) > 0. \tag{8.1.3}$$

We do not yet know which x, r, and δ are in this set. We claim first that if $z = (z_1, z_2) = (x + iy, t + ib(x) + ih) \in \Omega$ and $w = (w_1, w_2) = (r + is, t + ib(r) + ik) \in \Omega$, (z, w) is in the region in $\mathbb{C}^2$ defined by (8.1.3). Indeed, $(z,w) \in \left(\overline{\Omega} \times \Omega\right) \cup \left(\Omega \times \overline{\Omega}\right)$ implies $\delta = h + k > 0$ by the definition of Ω. It follows that $\delta + A(x,r,\eta) \geq \delta$, and hence its infimum over η is bounded below by δ as well. Thus inequality (8.1.3) is satisfied whenever $(z,w) \in \left(\overline{\Omega} \times \Omega\right) \cup \left(\Omega \times \overline{\Omega}\right)$.

To prove the remainder of Theorem 7.1.1, we must determine which $(z,w) \in \partial\Omega \times \partial\Omega$ are in the region (8.1.3). For such (z,w), $\delta = 0$. We thus need to know for which x and r fixed $A(x,r,\eta)$ is bounded away from zero independently of η.

Since

$$A(x,r,\eta) = \left[\sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta x - b(x))\right] + \left[\sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta r - b(r))\right],$$

A is a sum of two non-negative functions

$$A_x(\eta) := \sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta x - b(x)) = b^*(\eta) - (\eta x - b(x)) \tag{8.1.4}$$

$$A_r(\eta) := \sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta r - b(r)) = b^*(\eta) - (\eta r - b(r)). \tag{8.1.5}$$

Thus for fixed x and r, A vanishes at some η_0 if and only if both $A_x(\eta_0)$ and $A_r(\eta_0)$ vanish. Furthermore, by Corollary 7.2.6, $\eta \mapsto A(x,r,\eta)$ is continuous, and by Lemma 8.1.1, $A_x(\eta) \sim c\eta^{\frac{4}{3}}$ as $|\eta| \to \infty$. Thus if for some fixed x and r, $A(x,r,\cdot)$ never vanishes, it is bounded below by a positive constant for all η. We thus identify (z,w) in the region (8.1.3) by identifying pairs x and r for which $A(x,r,\cdot)$ never vanishes.

Case 1: $|x| < \sqrt{-p}$ or $|r| < \sqrt{-p}$. For definiteness, suppose $|x| < \sqrt{-p}$. A_x could only vanish if x were such that, for some value of η, the infimum of $B_\eta(\lambda) = b(\lambda) - \eta\lambda$ were achieved at x. But Proposition 7.2.3 shows that the infimum of B_η is always achieved at one or more points outside of $(-\sqrt{-p}, \sqrt{-p})$. This completes the proof in this case.

Case 2: $|x|, |r| > \sqrt{-p}$, and $x \neq r$. Since the map $\eta \mapsto \lambda(\eta)$ maps $\mathbb{R}\backslash\{q\}$ onto $\mathbb{R}\backslash[-\sqrt{-p}, \sqrt{-p}]$ and is injective, there exists a unique $\eta_1 \neq q$ and a unique $\eta_2 \neq q$ such that $\lambda(\eta_1) = x$ and

$\lambda(\eta_2) = r$. Since $x \neq r$, $\eta_1 \neq \eta_2$. By the comment in the prior paragraph, it follows that $A(x, r, \cdot)$ never vanishes in this case.

Case 3: $|x| = \sqrt{-p}$ *but* $|r| > \sqrt{-p}$. (A symmetric argument covers the case $|r| = \sqrt{-p}$ but $|x| > \sqrt{-p}$.) Then $A_x(\eta) = 0$ only at $\eta = q$, where one easily computes that $A_r(q) = \frac{1}{4}(r^2 + p)^2 > 0$. Thus $A(x, r, \cdot)$ does not vanish.

This completes the proof of Theorem 7.1.1. $\square$

Remark 8.1.4. *Let us state these three cases using the notation established in Chapter 3. It was just shown that A does not vanish when one of the following holds:*

Case 1: For all $\eta \in \mathbb{R}$, $x \notin \Lambda_\eta$ or $r \notin \Lambda_\eta$.

Case 2: $x \in \Lambda_{\eta_0}$ and $r \in \Lambda_{\eta_1}$ for some $\eta_0, \eta_1 \in \mathbb{R} \setminus \{q\}$ with $\eta_0 \neq \eta_1$.

Case 3: $x \in \Lambda_q$ and $r \in \Lambda_{\eta_0}$ for some $\eta_0 \in \mathbb{R} \setminus \{q\}$.

8.1.4 Proof of Theorem 7.1.3

We begin by observing that on $\mathbb{C}^2 \times \mathbb{C}^2$

$$\begin{aligned} S[(x, i(b(x) + h)), (r, i(b(r) + k))] &= c \int_{-\infty}^{\infty} \int_0^{\infty} \tau e^{\eta\tau(x+r) - \tau[b(x)+b(r)+h+k]} N(\eta, \tau)^{-1} \, d\eta \, d\tau \\ &= \widetilde{S}^{0,0,\delta}, \end{aligned}$$

and on $\mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{aligned} S[(x, 0, 0), (r, 0, 0)] &= c \int_{-\infty}^{\infty} \int_0^{\infty} \tau e^{\eta\tau(x+r) - \tau[b(x)+b(r)]} N(\eta, \tau)^{-1} \, d\eta \, d\tau \\ &= \widetilde{S}^{0,0,0}, \end{aligned}$$

where $\widetilde{S}^{n,m,\delta}$ is as defined in (8.1.1). We will shorten the notation for these integrals to $\widetilde{S}^{\delta}$ for $\delta \geq 0$. Thus to prove Theorem 7.1.3, we must show that whenever x and r satisfy the hypotheses stated there

(i) $\widetilde{S}^0$ is divergent, and

(ii) $\lim_{\delta\to 0^+} \widetilde{S}^{\delta} = \infty$.

It is immediately clear that (ii) will follow from (i) since the integrand of $\widetilde{S}^{\delta}$ is non-negative and converges pointwise and monotonically to the integrand of $\widetilde{S}^0$ as $\delta \to 0^+$. Furthermore, (i) will follow if the corresponding statement holds for any of the three integrals $\mathcal{I}_j^{0,0,0}$ (again, abbreviated $\mathcal{I}_j^0$). We will show that

(iii) $\mathcal{I}_1^0$ is divergent.

As we saw in the previous section, $\mathcal{I}_1^0$ converges if x and r are chosen in such a way that $A(x,r,\cdot)$ never vanishes. Thus in order to establish (iii), we need detailed information about the behavior of A near values η_0 for which $A(x,r,\eta_0)=0$.

Remark 8.1.5. *In the language of Chapter 3, Theorem 7.1.3 is equivalent to saying that the Szegö kernel is not absolutely convergent if $x, r \in \Lambda_{\eta_0}$ for some $\eta_0 \in \mathbb{R}$.*

Recall that if $A(x,r,\eta)\neq 0$, (8.1.2) shows that the integrand of $\mathcal{I}_1^0$ is comparable to

$$[A(x,r,\eta)]^{-\frac{9}{4}}. \tag{8.1.6}$$

We prove (iii) by considering the behavior of A in three subcases.

8.1.5 Case 1: $x = r$ and $|x| > \sqrt{-p}$.

In this case, there exists a unique $\eta_0 \neq q$ such that $x = r = \lambda(\eta_0)$.

Suppose $\eta \neq \eta_0$ and recall that

$$\eta = [\lambda(\eta)]^3 + p\lambda(\eta) + q \quad \text{and} \quad \eta_0 = x^3 + px + q$$

so that

$$\eta_0 - \eta = (x - \lambda(\eta))(x^2 + x\lambda(\eta) + [\lambda(\eta)]^2 + p). \tag{8.1.7}$$

Then (suppressing the dependence of λ on η)

$$\begin{aligned}
A(x, x, \eta) &= 2A_x(\eta) \\
&= 2[\eta\lambda - \frac{1}{4}\lambda^4 - \frac{p}{2}\lambda^2 - q\lambda - \eta x + \frac{1}{4}x^4 + \frac{p}{2}x^2 + qx] \\
&= 2(x - \lambda)\left[\frac{1}{4}(x + \lambda)(x^2 + \lambda^2) + \frac{p}{2}(x + \lambda) - (\eta - q)\right] \\
&= 2(x - \lambda)\left[\frac{1}{4}(x + \lambda)(x^2 + \lambda^2) + \frac{p}{2}(x + \lambda) - \lambda^3 - p\lambda\right] \\
&= \frac{1}{2}(x - \lambda)^2(x^2 + 2\lambda x + 3\lambda^2 + 2p).
\end{aligned}$$

We are concerned with how this function varies with η. We have the following proposition and corollary:

Proposition 8.1.6. *For $|x| > \sqrt{-p}$ fixed,*

(a)

$$x^2 + x\lambda(\eta) + [\lambda(\eta)]^2 + p \geq \begin{cases} -2p & |x| \geq 2\sqrt{-p} \\ |x|(|x| - \sqrt{-p}) & \sqrt{-p} < |x| < 2\sqrt{-p}. \end{cases}$$

(b)

$$x^2 + 2x\lambda(\eta) + 3[\lambda(\eta)]^2 + 2p \geq \begin{cases} -4p & |x| \geq 3\sqrt{-p} \\ (|x| - \sqrt{-p})^2 & \sqrt{-p} < |x| < 3\sqrt{-p}. \end{cases}$$

That is, both expressions are bounded below by a positive constant independent of η.

Proof. Recall that $|\lambda(\eta)| \geq \sqrt{-p}$. Our task in part (a) is thus to find the global minimum of

$$f(\lambda) = x^2 + x\lambda + \lambda^2 + p$$

on $\{\, \lambda : |\lambda| \geq \sqrt{-p}\,\}$. There are two cases to consider depending on whether f attains its minimum at a critical point or at $\lambda = \pm\sqrt{-p}$.

Observe, $f'(\lambda) = x + 2\lambda = 0$ when $\lambda = -\frac{1}{2}x$. If $\frac{1}{2}|x| \geq \sqrt{-p}$, this indeed is the location of the global minimum, which is then seen to be

$$\frac{3}{4}x^2 + p \geq -2p.$$

If $\frac{1}{2}|x| < \sqrt{-p}$, the global minimum is one of the two quantities $x^2 \pm x\sqrt{-p} - p + p$, which is in turn

$$\geq x^2 - |x|\sqrt{-p} = |x|(|x| - \sqrt{-p}).$$

This proves (a). The proof of (b) is similar and is omitted. $\square$

Corollary 8.1.7. *If $|x| > \sqrt{-p}$,*

$$A(x, x, \eta) \approx (\eta - \eta_0)^2(1 + |\eta|)^{-\frac{2}{3}}.$$

Proof. By Proposition 8.1.6, we may write

$$A(x,x,\eta) = (\eta-\eta_0)^2 \frac{x^2+2x\lambda(\eta)+3[\lambda(\eta)]^2+2p}{2(x^2+x\lambda(\eta)+[\lambda(\eta)]^2+p)^2} =: (\eta-\eta_0)^2 g(\eta).$$

The proposition shows that g is finite for all η and bounded away from zero. Thus for η on any fixed interval $[-K,K]$, $g(\eta)\approx 1$. On the other hand, Proposition 7.2.3 shows that $\lambda(\eta)\sim\eta^{\frac{1}{3}}$ as $|\eta|\to\infty$, and so

$$g(\eta)\approx|\eta^{\frac{1}{3}}|^{-2} \quad \text{for } |\eta|>K.$$

Thus for all η,

$$g(\eta)\approx(1+|\eta|)^{-\frac{2}{3}}.$$

□

By (8.1.6),

$$\mathcal{I}_1^0(\eta)\approx[(\eta-\eta_0)^2(1+|\eta|)^{-\frac{2}{3}}]^{-\frac{9}{4}},$$

and thus $\mathcal{I}_1^0$ is divergent, establishing (iii) in this case.

8.1.6 Case 2: $x=r$ and $|x|=\sqrt{-p}$.

Here, the analysis is slightly more delicate because the discontinuity of λ occurs at $\eta=q$, and one of $\pm x-\lambda(\eta)$ vanishes at $\eta=q$. In this situation, the analogue of (8.1.7) above is the relationship

$$\eta-q=\lambda(\eta)[\lambda(\eta)-x][\lambda(\eta)+x]. \tag{8.1.8}$$

Furthermore, since $x^2=-p$, in this case

$$A(x,x,\eta)=\frac{1}{2}(x-\lambda)^2(3\lambda-x)(\lambda+x)$$

Proposition 8.1.8. *Let A be as above.*

1. *If $x = \sqrt{-p}$, then for $\eta > q$,*

$$A(x, x, \eta) \approx (\eta - q)^2 (1 + |\eta|)^{-\frac{2}{3}}.$$

2. *If $x = -\sqrt{-p}$, then for $\eta < q$,*

$$A(x, x, \eta) \approx (\eta - q)^2 (1 + |\eta|)^{-\frac{2}{3}}.$$

Proof. In both cases, it is enough to observe that for the values of η indicated, both $|x+\lambda(\eta)| \geq 2|x| > 0$ and $|3\lambda(\eta) - x| \geq 2|x| > 0$. We may thus solve (8.1.8) for $x - \lambda$ and substitute into the expression for A. The estimate then follows, again using the fact (Proposition 7.2.3) that $\lambda(\eta) \sim \eta^{\frac{1}{3}}$ as $|\eta| \to \infty$. □

Thus in this case, as above, $\mathcal{I}_1^0$ is divergent.

8.1.7 Case 3: $|x| = |r| = \sqrt{-p}$ but $x = -r$.

The point here is that although $x \neq r$, there is an η_0 for which B_{η_0} achieves its global minimum at both x and r: when $\eta = q$ and $x = \pm\sqrt{-p}$ and $r = \mp\sqrt{-p}$ (See Proposition 7.2.3). In this case, $A_x(\eta)$ vanishes if and only if $\eta = q$. For $\eta \neq q$

$$A_{\pm\sqrt{-p}}(\eta) = (\eta - q)(\lambda(\eta) \mp \sqrt{-p}) - \frac{1}{4}([\lambda(\eta)]^2 + p)^2.$$

Proposition 8.1.9. *If $|x| = \sqrt{-p}$,*

$$A(x, -x, \eta) = (\eta - q)h(\eta),$$

where

$$|h(\eta)| \approx (1 + |\eta|)^{\frac{1}{3}}.$$

Proof. It follows from (8.1.8) that

$$[\lambda(\eta)]^2 + p = \frac{\eta - q}{\lambda(\eta)}$$

and so

$$\begin{aligned} A(\pm\sqrt{-p}, \mp\sqrt{-p}, \eta) &= 2(\eta - q)\left[\lambda(\eta) - \frac{[\lambda(\eta)]^2 + p}{4\lambda(\eta)}\right] \\ &= (\eta - q)\frac{3[\lambda(\eta)]^2 - p}{2\lambda(\eta)} \\ &=: (\eta - g)h(\eta). \end{aligned}$$

Since the numerator is bounded below by $-4p > 0$ and the denominator is bounded in absolute value away from zero, it follows that if we fix an interval $[-K, K]$, $|h(\eta)| \approx 1$ on the interval.

On the other hand, since $\lambda(\eta) \sim \eta^{\frac{1}{3}}$ as $|\eta| \to \infty$, for sufficiently large K,

$$|h(\eta)| \approx |\eta|^{\frac{1}{3}} \quad \text{for } |\eta| > K.$$

The proposition follows. $\square$

Since the integrand of $\mathcal{I}_1^0$ is

$$\approx [|\eta - q|(1 + |\eta|)^{\frac{1}{3}}]^{-\frac{9}{4}},$$

$\mathcal{I}_1^0$ diverges in this case as well. The proof of Theorem 7.1.3 is now complete.

8.2 Higher-degree polynomials

We now turn to the higher-degree setting. In this context, our polynomial b has the form (2.1.1). As before, we set $\delta = h + k$, $z = (z_1, z_2) = (x + iy, t + ib(x) + ih)$, and $w = (w_1, w_2) = (r + is, u + ib(r) + ik)$ on $\overline{\Omega}$. With $h, k \geq 0$, we have $\delta > 0$ if and only if $(z, w) \in \left(\overline{\Omega} \times \Omega\right) \cup \left(\Omega \times \overline{\Omega}\right)$.

8.2.1 Proof of Theorem 1.3.1

We prove Theorem 1.3.1, the analogue to Theorem 7.1.1, by applying the upper bound found in Section 6.2. We acquired the upper bound

$$N(\eta,\tau)^{-1} \lesssim e^{-2\tau b^*(\eta)} \sum_{j=2}^{2n} \tau^{1/j} \left|b^{(j)}(\lambda_\eta)\right|^{1/j}, \tag{8.2.1}$$

for all $\eta \in \mathbb{R}$ and $\tau > 0$. As stated earlier, we need to show that for all non-negative integers i_1, j_1, i_2, and j_2, each integral

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{\eta\tau[z_1+\bar{w_1}]+i\tau[z_2-\bar{w_2}]} \frac{\eta^{i_1+j_1}\tau^{i_1+j_1+i_2+j_2+1}}{N(\eta,\tau)}\, d\eta\, d\tau \tag{8.2.2}$$

is absolutely convergent in the region

$$\delta + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

If this integral does converge absolutely, it equals $\partial_{z_1}^{i_1}\partial_{\bar{w_1}}^{j_1}\partial_{z_2}^{i_2}\partial_{\bar{w_2}}^{j_2} S(z,w)$.

Set $s = i_1 + j_1$, and $m = i_1 + j_1 + i_2 + j_2$ (so that $m \geq s$). As before, the integral becomes

$$S^{s,m,\delta} := \iint_{\tau>0} e^{\eta\tau[x+r+i(y-s)]+i\tau[t-u+i(b(x)+b(r)+\delta)]} \frac{\eta^s\tau^{m+1}}{N(\eta,\tau)}\, d\eta\, d\tau,$$

and it converges absolutely if and only if

$$\widetilde{S}^{s,m,\delta} = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)]}\frac{|\eta|^s\tau^{m+1}}{N(\eta,\tau)}\,d\tau\,d\eta < \infty.$$

From estimate (8.2.1),

$$\begin{aligned}\widetilde{S}^{s,m,\delta} &\lesssim \sum_{j=2}^{2n}\int_{-\infty}^{\infty}\int_0^{\infty}\tau^{1/j}|\eta|^s\left|b^{(j)}(\lambda_\eta)\right|^{1/j}\tau^{m+1}e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]}\,d\tau\,d\eta\\ &=: \sum_{j=2}^{2n}\mathcal{I}_j^{s,m,\delta},\end{aligned}$$

where $\mathcal{I}_j^{s,m,\delta} =: \int_{-\infty}^{\infty}\mathcal{I}_j^{s,m,\delta}(\eta)\,d\eta.$

Set $A(x,r,\eta) = b(x)+b(r)-\eta(x+r)+2b^*(\eta)$. With the added hypothesis $\delta+A(x,r,\eta)>0$,

$$\begin{aligned}\mathcal{I}_j^{s,m,\delta}(\eta) &= c_j|\eta|^s\left|b^{(j)}(\lambda_\eta)\right|^{1/j}\int_0^{\infty} e^{-\tau[\delta+A(x,r,\eta)]}\tau^{m+1+\frac{1}{j}}\,d\tau\\ &\approx \frac{|\eta|^s\left|b^{(j)}(\lambda_\eta)\right|^{1/j}}{[\delta+A(x,r,\eta)]^{m+2+\frac{1}{j}}}\left(\int_0^{\infty} e^{-\tau}\tau^{m+1+\frac{1}{j}}\,d\tau\right)\\ &\approx \frac{|\eta|^s\left|b^{(j)}(\lambda_\eta)\right|^{1/j}}{[\delta+A(x,r,\eta)]^{m+2+\frac{1}{j}}}.\end{aligned}\tag{8.2.3}$$

The same two issues that were addressed in the fourth-degree setting may inhibit the convergence of the integrals $\mathcal{I}_j^{s,m,\delta}$:

1. insufficient growth of A in η at infinity, and
2. vanishing of $\delta+A(x,r,\eta)$ for some finite η for certain choices of x, r, and δ.

Let us explore these issues.

8.2.2 Higher degree: behavior of $A(x, r, \eta)$ for large $|\eta|$.

Corollary 8.2.1. *Fix $x, r \in \mathbb{R}$. Then*

$$A(x,r,\eta) \sim \left(\frac{2n-1}{n}\right) \eta^{\frac{2n}{2n-1}}, \quad |\eta| \to \infty.$$

Proof. Since $A(x, r, \eta) = b(x) + b(r) - \eta(x + r) + 2b^*(\eta)$, the result follows directly from the asymptotic approximation given in (3.3.3). $\square$

Applying this lemma and estimate (3.4.2) to (8.2.3) yields

$$\begin{aligned}
\mathcal{I}_j^{s,m,\delta}(\eta) &\sim \frac{|\eta|^s}{\eta^{\frac{2n(m+2)}{2n-1}}} \frac{|\eta|^{\frac{2n-j}{j(2n-1)}}}{\eta^{\frac{2n}{j(2n-1)}}} \\
&= \frac{|\eta|^s \; |\eta|^{\frac{-1}{2n-1}}}{\eta^{\frac{2n(m+2)}{2n-1}}} \\
&= \left(|\eta|^{(2n-1)s-1-2n(m+2)}\right)^{\frac{1}{2n-1}} \\
&= |\eta|^{-2-(m-s)-\frac{m+3}{2n-1}}
\end{aligned}$$

as $|\eta| \to \infty$. Since $m \geq s \geq 0$,

$$-2 - (m-s) - \frac{m+3}{2n-1} < -2.$$

Hence for any fixed s, m, j, and δ, $\mathcal{I}_j^{s,m,\delta}$ is convergent at infinity.

8.2.3 Higher degree: vanishing of $\delta + A(x,r,\eta)$

The estimates from the previous section show that if the integral $\mathcal{I}_j^{s,m,\delta}$ fails to converge, then for some fixed x, r, and δ the function

$$\eta \mapsto \delta + A(x,r,\eta)$$

vanishes for some finite η_0. Therefore if for some x, r and δ fixed

$$\inf_{\eta}[\delta + A(x,r,\eta)] = \delta + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0, \tag{8.2.4}$$

then each $\mathcal{I}_j^{s,m,\delta}$ is finite. Hence the integrals defining the Szegö kernel and all its derivatives converge absolutely in the region defined by (8.2.4).

Using the argument directly following inequality (8.1.3), Theorem 1.3.1 will be established if we can identify for which x and r fixed the function $A(x,r,\cdot)$ is bounded away from zero. This is done by proving the higher-degree analogue of Remark (8.1.4).

Lemma 8.2.2. *If x and r are not in the same Λ_η for all η, then $A(x,r,\cdot)$ is bounded away from zero.*

Proof. We consider three cases.

Case 1: Assume $x \in \Lambda_{\eta_1}$ and $r \in \Lambda_{\eta_2}$ with $\eta_2 \neq \eta_1$. Without loss of generality, assume $\eta_1 < \eta_2$. Fix any $\eta_0 \in (\eta_1, \eta_2)$. By definition $A(x,r,\eta) = A_x(\eta) + A_r(\eta)$. Since b^* is continuous, both A_x and A_r are continuous, non-negative functions of η vanishing only at η_1 and η_2, respectively. Since both have the asymptotic approximation $c\eta^{2n/(2n-1)}$ as $|\eta| \to \infty$, $A(x,r,\cdot)$ is bounded away from zero.

Case 2: For all $\eta \in \mathbb{R}$, $x \notin \Lambda_\eta$ or $r \notin \Lambda_\eta$. Assume $r \notin \Lambda_\eta$ for all $\eta \in \mathbb{R}$. (A symmetric argument covers the case $x \notin \Lambda_\eta$ for all $\eta \in \mathbb{R}$.) Since $r \notin \Lambda_\eta$ for all η, A_r is positive and continuous in η with the asymptotic approximation $c\eta^{2n/(2n-1)}$ as $|\eta| \to \infty$. As a consequence, $A(x, r, \cdot)$ is bounded away from zero.

The lemma holds. $\square$

Earlier, we wrote $A(x, r, \eta) = A_x(\eta) + A_r(\eta)$, where A_x, A_r are the non-negative functions defined in (8.1.4) and (8.1.5). If we fix x and r, $A(x, r, \cdot)$ vanishes at η_0 if and only if A_x and A_r both vanish at η_0, which happens precisely when $x, r \in \Lambda_{\eta_0}$ by definition. Using this identification and Lemma 8.2.2, we have just identified all pairs x and r for which $A(x, r, \cdot)$ never vanishes. These pairs identify all points $(z, w) \in \partial\Omega \times \partial\Omega$ for which inequality (8.2.4) is satisfied. This proves Theorem 1.3.1. Since each integral $\mathcal{I}_j^{s,m,\delta}$ is convergent whenever $A(x, r, \cdot)$ is bounded away from zero, the Szegö kernel (1.2.4) is absolutely convergent whenever x and r are not in the same Λ_η for all η. This proves Theorem 1.3.2.

8.2.4 Proof of Theorem 1.3.3

As in Section 8.1.4, we set

$$\widetilde{S}^\delta[(x, i(b(x) + h)), (r, i(b(r) + k))] = c \int_{-\infty}^{\infty} \int_0^{\infty} \tau e^{-\tau[\delta + b(x) + b(r) - \eta(x+r)]} N(\eta, \tau)^{-1}\, d\tau\, d\eta,$$

for $\delta = h + k \geq 0$. To prove Theorem 1.3.3, the analogue to Theorem 7.1.3, we show that

(i) $\widetilde{S}^0$ is divergent, and

(ii) $\lim_{\delta \to 0^+} \widetilde{S}^\delta = \infty$,

whenever we choose x and r so that $A(x, r, \cdot)$ vanishes at a finite η_0. Notice that (i) implies (ii) since the integrand of $\widetilde{S}^\delta$ is non-negative and converges pointwise and monotonically to the integrand of $\widetilde{S}^0$ as $\delta \to 0^+$.

We begin by fixing $x, r \in \Lambda_{\eta_0}$, for any $\eta_0 \in \mathbb{R}$. Then recall the local, lower bound given in Section 6.3. We found that for fixed $\eta_0 \in \mathbb{R}$ and $\epsilon > 0$, we have

$$\frac{\tau^{1/2}}{1+\tau^{1/2}} \lesssim e^{2\tau b^*(\eta)} N(\eta,\tau)^{-1}, \tag{8.2.5}$$

for all $\tau > 0$ and $\eta \in (\eta_0, \eta_0 + \epsilon)$. With this, we appeal to Lemma 5.3.1. More specifically, for $\eta_0 \in \mathbb{R}$ and $\epsilon > 0$, there exist locally-bounded, positive functions in η, F_x and H_r, such that

$$A(x,r,\eta) = (\eta - \eta_0)\left(F_x(\eta) + H_r(\eta)\right),$$

for all $\eta \in (\eta_0, \eta_0 + \epsilon)$. For brevity, set $T = F_x + H_r$.

We now apply estimate (8.2.5) and Lemma 5.3.1 to get a lower bound on $\widetilde{S}^\delta$. By setting $A = A(x,r,\eta)$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\infty} \tau e^{-\tau A} e^{2\tau b^*(\eta)} N(\eta,\tau)^{-1}\, d\tau\, d\eta \\
&\quad > \int_{\eta_0}^{\eta_0+\epsilon}\int_0^{\infty} \tau e^{-\tau A} e^{2\tau b^*(\eta)} N(\eta,\tau)^{-1}\, d\tau\, d\eta \\
&\quad \gtrsim \int_{\eta_0}^{\eta_0+\epsilon}\int_0^{\infty} \frac{\tau^{3/2} e^{-\tau A}}{(1+\tau^{1/2})}\, d\tau\, d\eta \\
&\quad = \int_{\eta_0}^{\eta_0+\epsilon}\int_0^{\infty} \frac{e^{-\tau}\frac{\tau^{3/2}}{A^{3/2}}}{1+\frac{\tau^{1/2}}{A^{1/2}}}\, \frac{d\tau}{A}\, d\eta \\
&\quad = \int_{\eta_0}^{\eta_0+\epsilon}\int_0^{\infty} \frac{e^{-\tau}\frac{\tau^{3/2}}{A^2}}{A^{1/2}+\tau^{1/2}}\, d\tau\ d\eta \\
&\quad \geq \int_{\eta_0}^{\eta_0+\epsilon}\int_0^{1} \frac{e^{-\tau}\frac{\tau^{3/2}}{(\eta-\eta_0)^2 T(\eta)^2}}{(\eta-\eta_0)^{1/2} T(\eta)^{1/2}+1}\, d\tau\ d\eta \\
&\quad = \left(\int_0^1 \tau^{3/2} e^{-\tau}\ d\tau\right) \int_{\eta_0}^{\eta_0+\epsilon} \frac{\frac{1}{(\eta-\eta_0)^2 T(\eta)^2}}{(\eta-\eta_0)^{1/2} T(\eta)^{1/2}+1}\ d\eta \\
&\quad \approx \int_{\eta_0}^{\eta_0+\epsilon} \frac{G(\eta)}{(\eta-\eta_0)^2 T(\eta)^2}\, d\eta,
\end{aligned}$$

where $G(\eta)$ is continuous from the right and bounded away from zero. Since T is also a locally-bounded positive function, divergence follows. Since this happens whenever $x, r \in \Lambda_{\eta_0}$ for some η_0, Theorem 1.3.3 has been proved.

8.2.5 Proof of Corollary 1.3.4

In order to prove Corollary 1.3.4, we must recall Corollary 4.3.10. For a polynomial b of the form (2.1.1), it states the following: b is non-convex if and only if there exists an $\eta_0 \in \mathbb{R}$ such that $|\Lambda_{\eta_0}| > 1$. Thus for a non-convex b, we can pick $x, r \in \Lambda_{\eta_0}$ with $x \neq r$. By Theorem 1.3.3, the Szegö kernel is divergent, hence not smooth, at each point off of the diagonal $((x, y, t), (r, y, t))$, for each $y, t \in \mathbb{R}$. For the converse, we turn to [16]. They showed that if b is convex, then the Szegö kernel is smooth off of the diagonal. Together, these prove Corollary 1.3.4.

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