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Development of tests for the convergence and divergence of infinite series

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THE DEVELOPMENT OF TESTS FOR THE

CONVERGENCE AND DIVERGENCE OF INFINITE SERIES.

by

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THE DEVELOPMENT OF TESTS FOR THE CONVERGENCE OF INFINITE SERIES.

1. Introduction.

It is the purpose of this paper: first, to trace briefly the development of attention to infinite series and to the subject of convergence; second, to list the various tests for convergence, with the proof of each; and, finally, to give brief account of recent developments in connection with infinite series.

<u>Definitions</u>.

If a set of numbers $u_0,\ u_1,\ u_2,$------ $u_n,$---- are so arranged as to correspond to the set of positive integers 1, 2, 3, -----n,--- , the set is defined as an <u>infinite sequence</u>. The word infinite simply means that every term in the sequence is followed by another term. It is often convenient to write u_n to represent the general term of the sequence. The rule defining the sequence may be expressed either by some formula giving u_n as an explicit function of <u>n</u>, or by some statement which indicates how each term can be determined either directly or from the preceding term.

The most important sequences in the applications of analysis are those that tend to a limit.

The limit of a sequence (u_n) is said to be L, if an index m can be found to correspond to every positive number ϵ, however small, such that

$$|L - u_n| < \epsilon$$

provided only that $n > m$.

This property may be expressed by the following notation

$$L = \lim_{n \to \infty} u_n$$

When the sequence has a finite limit L, it is said to be <u>convergent</u>. Otherwise it is <u>divergent</u>.

If an infinite sequence of numbers

$u_0, u_1, u_2, \cdots u_n, \cdots$ is given, then the expression

(A). $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$

is called an <u>infinite series</u>.

Series (A) is said to be convergent if the sequence of the successive sums

$$S_0 = u_0, \quad S_1 = u_0 + u_1, \cdots \quad S_n = u_0 + u_1 + \cdots + u_n + \cdots$$

is convergent. Let S be the limit of the latter sequence, i.e. the limit which the sum S_n approaches as n increases indefinitely:

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_0^{\infty} u_n$$

Then S is called <u>the sum of the preceding series</u>, and this relation is indicated by writing the symbolic equation

$$S = u_0 + u_1 + \cdots + u_n + \cdots = \sum_0^{\infty} u_n.$$

A series which is not convergent is said to <u>divergent</u>.

Absolutely convergent series are infinite series in which the series formed by the absolute value of the terms is also convergent. Absolute value means the value of the term without regard to its sign. In this paper absolute value be indicated in this way -- $|u_N|$.

Conditionally convergent series are infinite series in which the series are themselves convergent, but the series formed by moduli (or absolute value of terms) is divergent.

11. HISTORY

The Eleatic Schools, established by Xenophanes in Sicily and centered at Elea in Italy, were famous for the difficulties they raised in connection with questions which required the use of infinite series. The most famous of these was the well known paradox of Achilles and the tortoise. This paradox was enunciated by Zeno, one of their most prominent members.

Achilles runs a race with a tortoise. He runs ten times as fast as the tortoise. The tortoise has 100 yards start. Now, says Zeno, Achilles runs this 100 yards and reaches the place where the tortoise started. Meantime the tortoise has gone a tenth as far as Achilles and is, therefore, 10 yards ahead of Achilles. Achilles runs this ten yards. Meantime the tortoise has gone one-tenth as far and is, therefore, one-tenth of a yard ahead of Achilles, and so on.

So, argued Zeno, Achilles is always getting nearer the tortoise, but can never quite catch him.

The difficulty here was, of course, that the Greek mathematicians had no definite way of expressing a limit.

To us with our convenient fractional notation

$$10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\cdots,$$

or the even more convenient decimal notation

$$10 + 1 + 0.1 + 0.01 + 0.001 + \cdots\cdots,$$

the problem presents no difficuly. For it is clear that the distance the tortoise can move until there is no distance between himself and Achilles, or the total of all the yards the tortoise moves, is 11.1111111111-------
or 11 $\frac{1}{9}$ yards, and no more.

These paradoxes led the Greeks to look with suspicion upon the use of infinite series, and ultimately led to the "Method of exhaustion" of Eudoxus (408-355 BC)[1].
"If from the greatest of two unequal magnitudes there be taken more than its half, and so on, there at length remains a magnitude less than the least of the proposed magnitudes" By the use of this theorem the ancient geometers were able to avoid the use of infinitesimals.

About 1323-1382, Nichole Oresme, a bishop in Normany, in an unplublished manuscript[2] found the sum of the infinite series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + $
Such recurrent infinite series were formerly supposed to have made their first appearance in the 18 Century. The use of infinite series is explained also in the Liber de Triplici Motu, by the Portuguese mathematician Alvarus Thomas in 1509.

1. Ball, W. W."A Short History of Mathematics" MacMillan Co. 1893.

2. Cajori, F."A History of Mathematics" MacMillan Co. 1924.

He gave the division of a line segment into parts representing the terms of a convergent geometric series, that is, a segment AB is divided into parts such that

$$AB : P_1B = P_1B : P_2B = \text{---------} = P_iB : P_{i+1}B.$$

Such a division of a line segment occurs in Napiers kinematical discussion of logarithms.

Infinite series, which sprang into prominence at the time of the invention of the differential and integral calculus, were used by a few writers before that time. Pietro Mengole (1626-1686) in Bologna, treats them in a book, Novae Quadraturas Arithmeticas, of 1650. He proves the divergence of the harmonic series by dividing its terms into an infinite number of groups, such that the sum of the terms in each group is greater than one. Mengoli also showed the convergence of the reciprocals of the triangular numbers, and reached creditable results on the summation of infinite series.[3]

However, in spite of the fact that mathematicians met infinite series in various connections, very little in a definite way was accomplished until the beginning of the 17 Century. The interest in the infinitesimal as an element in analysis carried with it the notation of an infinite number of elements. At this time the study of series with an infinite ~~number~~ of terms was suggested.

3 F. Cajori op. cit.

James Gregory (1638-1667) showed in his "Vera Circuli et Hyperbolae Quadrature" that the areas of the circle and the hyperbola could be obtained in the form of convergent infinite series. Here for the first time, a distinction is made between convergent and divergent series, although Gregory does not use the term "divergent". It was Nicolaus Bernoulli who in 1713 first used "divergens" and " divergentia series".[4]

In 1671 , Gregory established the theorem that

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots ,$$

the result being true only if θ lies between $-\frac{1}{4}\pi$, and $\frac{1}{4}\pi$.

This is the theorem on which most of the subsequent calculations of the approximations of the numerical value of π have been based.[5]

4. Cajori, F. "The Name 'Divergent Series' " Bulletin of the American Mathematical Society Volume 29. page 55

5. Ball, W. W. "A Short History of Mathematics" MacMillan Co. New York. 1893.

There are three general periods in the later development of infinite series:

1. the period of Newton and Leibnitz -- the introduction.
2. the period of Euler-- the formal stage.
3 the modern period starting with Gauss and Cauchy-- ------ the scientific investigation of the validity of infinite series.[6]

1. The Introduction.

Newton and Leibnitz felt the necessity for enquiring into the convergence of infinite series. Newton in the first part of De Quadratura Curvarum--the second appendix to the Optics--deals with effecting the quadrature and rectification of curves by means of infinite series. The main object is to give rules for developing a function of x, so as to effect the quadrature of any curve whose ordinate can be expressed as the sum of an infinite number of such terms. In this way he effects the quadrature of the curves

$$y = (a^2 + x)^{1/2}, \quad y = (x - x^2)^{1/2}, \quad y = \left(\frac{1 + ax^2}{1 + bx^2}\right)^{1/2}.$$ [7,8]

6 Reiff, R. " Geschichte der Undichen Richen" Tiibinger 1889.

7. Smith, D.E. History of Mathematics. Topical Survey Ginn Co. 1925

8. Ball, W.W. op. cit.

but the results are as infinite series. He then proceeds to curves whose ordinant is given as implicit function of the abscissa, and he gives an method by which y can be expressed as an infinite series in descending powers of x, but the application demands such complicated calculus as to be of little value. Newton points out the importance of determining whether the infinite series are convergent ---an observation far in advance of his time, but he apparently knew of no general test for this purpose.

Leibnitz (1673) also pointed out the importance of determining whether a series was convergent or divergent. He proposed a test to distinguish series whose terms are alternately positive and negative.

Colin Maclaurin (1698-1746) gave proof of the theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

in his "Treatise of Fluxion" published in 174?, but he did not investigate the convergence of the series. He also enumerated the important theorem that, if $\phi(x)$ be positive and decrease from $x = a$ to $x = \infty$, then the series

$$\phi(a) + \phi(a+1) + \phi(a+2) + \cdots$$

is convergent or divergent as

$$\int_a^\infty \phi(x)\,dx \quad \text{is finite or infinite.}$$

Jacques Bernoulli (1654-1705), the first of the Bernoulli family of mathematicians, wrote one of the earlieast treatises on probability-"The Ars Conjectandi". This was published post-humously in 1713. At the close of a section entitled "Tractatus de Seriebus Infinitis Earumque Summa Finita et Usa in Quadraturis Spaticrum et Rectificationibus Curvarium " following Pars Quantis the following six verses appear. They represent one of the clearest of the early statements relating to the limit of an infinite series.

"Ut non-finitam seriem finita coercet
Summula, et in nullo limite limes adest.
Sic modico immensi vestigia numminis haerent
Corpore, et augusto limite limes abest.
Cernere in immenso parvum, dic, quanta voluptas!
In parvo immensum cernere, quanta, Deum!"

"Even as the finite encloses an infinite series
And in the unlimited limits appear,
So the soul of immensity dwells in minitia.
And in narrowest limits no limits inhere.
What joy to discern the minute in infinity!
The vast to perceive in the small,
what divinity![9]

9 Smith, D. E. Source Book of Mathematics page 271

2. The Second Period.

In the second period (1740-1830) the leading mathematicians were Euler, Lagrange, La Place, and Legendre.

Euler extended, summed up, and completed the work of his predecessors. He created analysis, and revised almost all of the branches of pure mathematics. He paid particular attention to the expansion of various functions in series, and pointed out explicitly that an infinite series cannot be safely employed unless it is convergent. He did not, however, develop any general tests to determine whether or not a given series is convergent.

Mathematicians at the time of Euler, d'Alembert, and Bernoulli had difficulty in regard to the generality of a function represented by a trigonometric series in connection with the problem of vibrating strings. They failed to see that the several curves might be represented by one trigonometric series. Fourier in his Memoir on the Theory of Heat (1807) laid down the proposition that an arbitrary function given graphically by means of a curve, which may be broken or discontinuous, is capable of representation by means of a single trigonometric series. This theory was received with astonishment and incredulity.

Fourier set for himself a problem to expand a given function of x in terms of the sines and cosines of multiples of x, a problem which he embodied in his "Theorie

Analytique de la Chaleur" (1822). Although Fourier attained the correct views as to the nature of the convergence of the infinite series he employed, he did not give any complete general proof that the series in the general case actually converges to the value of the function.

3. The Third Period.

The third or critical period began with the publication of Gauss' Memoir in connection with the hypergeometric series in 1812. This particular series is not so important as is the standard of critcism which Gauss (1777-1855) set up, embodying the simpler criteria of convergence. Essentially his rule is

If it is possible to express the quotient a_N / a_{N+1} in the form $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right)$ (where $p > 1$),

the series $\sum a_N$ is divergent if $\mu \leqq 1$, convergent if $\mu > 1$.

Owing to the strangeness of treatment and rigor of proof, this memoir caused very little interest among mathematicians. More fortunate in reaching the public was Cauchy whose Analyse Algebrique of 1821 contains a rigorous treatment of series. In this book all series which do not approach a fixed limit as the number of terms increase without limit are called divergent. Thus establishing for the first time effective criteria for divergent series, although the term was used before.

To Cauchy and Gauss we owe the scientific treatment of series which have an infinite number of terms. One of the chief merits of Cauchy's work was that he placed the treatment of series and the fundamental conceptions of the calculus on a logical foundation. Like Gauss, he instituted comparisons with geometric series, and found that series with positive terms are convergent or not accordingly as the n^{th} root of the n^{th} term, or the ratio of the $(n+1)^{th}$ and n^{th} term, is ultimately less or greater than unity. To reach some of the cases when these expressions become unity ultimately and the tests fail, Cauchy established two other tests. He showed that series with negative terms converge when the absolute values of the terms converge, and he also deduced Leibnitz' test for alternating series.

The most outspoken critic of the methods in series was Abel (1802-1829). He established the fact that in any convergent series it is necessary for the general term u to approach zero. He also proved the ratio test for convergent and divergent positive term series. He established the theorem that if two series and the product series are all convergent, then the product series will converge toward the product of the sums of the given series. He was emphatic against the reckless use of series, and showed the necessity of considering the subject of continuity in questions of

First in order of time in the evolution of more delicate criteria of convergence and divergence come the researches of Raabe (1801-1859), then those of De Morgan (1806-1871) In his Calculus De Morgan established the logarithmic criteria which was later discovered in part independently by Bertrand (1842). The form used by Bertrand and also by Bonnet (1843) is more convenient than that used by De Morgan. Discussion of these tests will be given in the actual tests. Although Bonnet believed that the logarithmic criteria never failed, Du Bois Reymond (1831-1889) and Pringsheim in (1889) have discovered series which are known to be convergent but in which the logarithmic criteria fails to determine their convergency. However, since these series converge very slowly, they occur very seldom in practical problems.

Among the first to suggest general criteria and to consider the subject from a wider point of view culminating in a regular mathematical theory was E. E. Kummer (1810 - 1893). He established a test in two parts, the first part of which was afterwards proved by Dini (1845-1918) to be superfluous. This test is a ratio test. Kummor gave the test in the form:

If $\lim \phi(n)\, a_n = 0$, in a series of positive terms the condition for convergence is

$$\lim\left\{\phi(n) - \phi(n+1)\frac{a_{n+1}}{a_n}\right\} > 0$$

The form given by Dini, which will be given in the tests.

is the most convenient to use in practice.

Du Bois-Reymond(1831-1889) divides criteria into classes: criteria of the first kind, and criteria of the second kind, accordingly as the general n^{th} term, or the ratio of the $(n+1)^{th}$ term and the n^{th} term is made the basis. Pringsheim goes further and makes use of "generalized criteria of the second kind" in which he makes use of the ratio of two terms which do not have to be consecutive.

At the close of the 19^{th} Century, interest was revived in divergent series. In 1886 T.J. Stieltjes and H. Poincare showed the importance of the asymptotic series, at that time employed in astronomy only. In more recent work, other writers have developed other uses for the divergent series.

Difficult questions arose in the study of Fourier series in connection with its convergence. Cauchy (1826) was the first to inquire into its convergence. Dirichlet (1829) made the most thorough researches on this subject. He decided that whenever the function does not have infinite number of discontinuites, and does not possess an infinite number of maxima and minima, then Fourier's series converges toward the value of that function at all places except points of discontinuity and then it converges to the mean of the two bounding values. These conditions of Dirichlet's are sufficient but not necessary. Rudolf Lipschitz (1832-1903) of Bonn, proved that Fourier's series still represents the function when the number of dis-continuities is infinite, and he establishes a condition on which it represents a function having an infinite number of maxima and minima. In 1893, de la Vallée Poussin gave a proof requiring merely that the function and its square be integrable. Several others have worked with Fourier's series more recently.

Uniform convergence was first investigated in 1847 by Stokes of Cambridge. Uniform convergence assumed great importance in Weierstrass theory of functions. It was necessary to prove that a trigonometric series representing a continuous function converges uniformly. Doubts on some of the conclusions about Fourier's series were made by the observation made by Weierstrass that the integral of an infinite series can be shown to be equal to the sum of the integrals of the separate terms only when the series converges uniformly within the region in question.

Fourier found the series

$$F(t) = \frac{6\theta}{\pi^2} \sum \frac{1}{n^2} e^{-\lambda n^2 t}$$

to represent the mean temperature at time t of a sphere originally heated to temperature θ and cooling with its surface kept at zero temperature. Here λ is a certain positive constant depending on the size, mass, and thermal properties of the sphere. Weierstrass' test shows at once that the series converges uniformly if $t \geq 0$; and so $\lim_{t \to \infty} F(t) = \theta$

"The corresponding formula for the temperature at any point is

$$f(t) = \sum a_N e^{-N\lambda^2 t}$$

where a is of the form $(-1)^{N-1}(2\theta \sin n\omega)/(n\omega)$, and ω/π is equal to the quotient of the distance from the center by the radius of the circle of the sphere."[10]

As a conclusion of this historical account, the following quotation from Emile Picard will be illuminating.

"Under the impulse of the problems suggested by geometry, mechanics, and physics, we see almost all the great divisions of analysis develop or originate. In the first place, we met equations with a single independent variable. Soon equations with partial derivatives were to appear in connection with vibrating strings, mechanics of fluids, and the infinitesimal geometry of surfaces. But of all the applications of analysis, no one was more brilliant than that suggested by knowledge of the laws of gravitation, with which are associated the names of the greatest mathematicians. In all this period, especially in the second half of the 18 Century, what strikes us with admiration, and at the same time introducing some confusion, is the extreme

10. Bromwich Theory of Infinite Series. Second Edition. page 126
MacMillan and Co. Limited
St. Martin's St. London, England 1926.

importance of the application realized, while the pure theory appears still so insecure.

One of the most important examples of the influence of physics on the development of analysis is Fourier's theory of heat. Fourier begins by forming the partial differential equations which govern temperature. He applied his theory of the cooling of a sphere to the terrestrial globe, and investigated the law governing the variations of temperature in the sun, endeavoring to obtain numerical applications. In the face of so many beautiful results, we can understand Fourier's enthusiasm when he says, speaking of mathematical analysis, "There cannot be a langauge more universal and more simple, more devoid of errors and of obscurities, that is to say, more suitable for the expression of the invariable relations of natural objects. Considered from this point of view, it is as wide as nature itself; it defines all perceptible relations, measures time and space, forces, and temperatures; this difficult science is formed slowly, but it maintains all the principles that it has once acquired. It increases and strengthtens without ceasing in the midst of so many errors of the human intellect."

....

As for series, they occur, to begin with in the very existance proof of the integrals; for instance, Cauchy's first method gives convergent developments as long as the integrals and their differential coefficients are continuous. when any circumstance allows us to predict that this is always the case, we obtain developments always convergent"[11]

Hence it is clear that as need for knowledge of the nature of the convergence of infinite series became necessary through physical problems of wave mechanics, and celestial mechanics, and similar problems, mathematical analysis has kept step, and has provided the necessary logical foundation for theory as well as for actual practical applications.[12]

11. Emile Picard (Translated by M. L. Haskell). "On the Development of Mathematical Analysis, and Its Relation to Certain Other Sciences." Bulletin of The American Mathematical Society Vol. 11. 1904.

12. Historical references:

Ball, W. W. "A Short History of Mathematics" MacMillan Co. New York. 1893.

Cajori, F. "A History of Mathematics." Revised. MacMillan Co. 1924. New York.

Smith, D. E. "History of Modern Mathematics". Ginn and Co. New York.

III. TESTS FOR CONVERGENCE OF INFINITE SERIES.

1. Fundamental Tests For Positive Term Series.

Introduction.

The expression

(A). $u_0 + u_1 + u_2 + \text{------} u_n + \text{-----}$,

where $u_0, u_1, u_2, \text{----} u_n, \text{--}$ is an unlimited sequence of numbers is called an *infinite series*.

Let S_n denote the sum of the first *n* terms of the series:

$S_n = u_0 + u_1 + u_2 \text{---------} u_n$.

If as *n* increases without limit, S_n approaches a limit S, this limit is called the *sum* of the series, and the series (A) is said to be *convergent*.

The infinite series (A) is said to be *divergent*, if, as *n* increases without limit, S_n does not approach a limit.

A necessary but not sufficient condition for convergency of the series (A) is $\lim_{n \to \infty} u_n = 0$.[13]

13. For example, it will be shown later in this paper that the harmonic series
$1 + \frac{1}{2} + \frac{1}{3} + \text{--------} + \frac{1}{n} + \text{---}$
is divergent, although the general term, u_n, does approach zero as *n* increases without limit.

The necessary and sufficient condition for the convergence of any series is that it may be possible to find an index m, $m = m(\epsilon)$, corresponding to any positive number ϵ, however small, such that

$$|u_N - u_m| < \epsilon,$$

for all values of n greater than m.

It is evident that the problem of determining whether a series is convergent or divergent or divergent is equivalent to the problem of determining whether the sequence of the terms is convergent or divergent.

The test for the convergence of any infinite sequence, applied to a series, gives Cauchy's general test for convergence. This test is absolutely general but in practice is often difficult to apply.

In order that a series be convergent it is necessary and sufficient that corresponding to any preassigned positive number ϵ, an integer n should exist, such that the sum of any number of terms whatever, starting with u, is less than ϵ in absolute value. For

$$|S_{N+p} - S_N| = u_{N+1} + u_{N+2} + u_{N+3} + \cdots + u_{N+p}.$$

In particular, the general term $u_{N+1} = S_{N+1} - S_N$ must approach zero as n becomes infinite.

In a series of positive terms S_N increases with n. Hence in order that the series converge it is sufficient that the sum S_N remain less than some fixed number for all values of n. The most general test for convergence in such series is based upon comparison of the given series with other series known to be divergent or convergent.

A. COMPARISON TESTS

1\.

In a series of positive terms if each of the terms is less than or at most equal to the corresponding term of a known convergent series of positive terms, the given series is convergent also.

Proof:

For the sum S_N of the first n terms of the given series is evidently less than the sum S' of the second series. Hence S_N approaches a limit S which is less than S'. Therefore the given series must converge.

2\.

If each of the terms of a given series of positive terms is greater than or equal to the corresponding term of a known divergent series, the given series must diverge.

Proof:

For the sum of the first n terms of the given series is not less than the sum of the first n terms of the second series. Hence it increases with n indefinitely and the series is divergent.

3\. We may compare two series also by means of the following lemma: let

$$(U) = u_0 + u_1 + u_2 + \cdots + u_N + \cdots$$
$$(V) = v_0 + v_1 + v_2 + \cdots + v_N + \cdots$$

(1). If series (U) converges, and if after a certain term we always have

$$\frac{v_{N+1}}{v_N} \leq \frac{u_{N+1}}{u_N}$$

series (V) converges also.

Proof:

Suppose $\dfrac{v_{N+1}}{v_N} \leq \dfrac{u_{N+1}}{u_N}$ whenever $n \geq p$.

Since the convergence of a series is not affected by multiplying each term by the same constant, and since the ratio of two consecutive terms also remains unchanged, we may suppose that $v_p \leq u_p$, and it is evident that we should have $v_{p+1} \leq u_{p+1}$, $v_{p+2} \leq u_{p+2}$, ------.

Hence series (V) must converge.

(2) If series (U) diverges, and if after a certain term we always have

$$\frac{v_{N+1}}{v_N} \geq \frac{u_{N+1}}{u_N}$$

the series (V) diverges also.

Proof: A similar proof holds for the second part. 14.

14\. Goursat-Hedrick "Mathematical Analysis" Vol. 1. Ginn Co. New York 1904.

There are three basic series which may be most conveniently used in the comparison tests.

(a)

The simpliest of these is the geometric progression

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots, \text{ where the ratio is } \underline{r}.$$

This series is convergent if $r < 1$, $r > -1$;

and is divergent if $r \geq 1$, or $r \leq -1$.

Proof:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{ar^n - a}{r - 1} = \frac{a}{1-r} - \frac{ar^n}{1-r} \text{ (from algebra).}$$

If $-1 < r < 1$ $\lim_{n \to \infty} r^n = 0$ and hence $\lim \frac{ar^n}{1-r} = 0$

Therefore $\lim_{n \to \infty} (a + ar + ar^2 + \cdots + ar^{n-1}) = \frac{a}{1-r}$, and the series converges.

For $r = 1$ and $r = -1$, the series become

$a + a + a + a + \cdots$, and $a - a + a - a + \cdots$, respectively.

Both series are divergent obviously.

For r greater than 1 the series is divergent since the terms increase indefinitely in value as n increases indefinitely.

(b)

A second useful series is the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

This series is divergent, since the total sum can be made to exceed any fixed number.

(c)

The third series of value in the comparison tests is "p-series"

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$$

If $p > 1$ the series is convergent.

Proof: $\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$; $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$

Similarly the sum of the next 8 terms is $< \frac{1}{8^{p-1}}$, the sum of the next 16 terms is $< \frac{1}{16^{p-1}}$, ---------. That is the sum of the given series is less than the sum of the series

$$\frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots$$

which is a convergent geometric series when $p > 1$. Hence by comparison the series given is convergent also.

If $p = 1$ the series is the same as the harmonic series, and is therefore divergent.

If $p < 1$ the series is divergent also, since diminishing p tends to increase the respective terms of the series.

(4) Cauchy's Radical Test.

If the series $\sum a_n$ is compared with the geometric series we can infer Cauchy's important radical test.

If the n^{th} root $\sqrt[n]{u_n}$ of the general term u_n of a series of positive terms, after a certain term, is constantly less than a fixed number less than unity, the series converges. If $\sqrt[n]{u_n}$, after a certain term, is constantly greater than unity, the series diverges.

Proof:
In the first case, $\sqrt[n]{u_n} < k < 1$ or $u_n < k^n$. Hence, each term of the series after a certain one is less than the

corresponding term of a certain geometric progression whose ratio is less than unity. Therefore the series is convergent by comparison.

In the second case, u_N 1 or u_N 1 or u_N 1.
Here the general term does not approach zero, therefore the series diverges.

B. RATIO TEST.

Since in practice it is often difficult to apply the comparison tests, another type of test making use of the n and the (n 1) terms has been developed. This type has been called "Tests of the Second Kind", or ratio tests.

The ratio tests depend on the quotient $\frac{a_{N+1}}{a_N}$ obtained by the division of two consecutive terms of the series.

(1)
If in the series $\sum a_i$, $\lim_{n \to \infty} \frac{a_{N+1}}{a_N} = p$, then the series is convergent if $p < 1$, and is divergent if $p > 1$. If $p = \pm 1$, the series may be either convergent or divergent.

Proof:
if p 1 , after a certain term a_k all terms are positive and are numerically less than a_k. Let r be any number greater than p but less than 1 . Consider the series

$$a_1 + a_2 + a_3 \text{ --- } a_k + a_k r + a_k r^2 + \text{---}.$$

Since after a_k this is a geometric series, it is convergent. Moreover, after the term a_K the terms of the second series are greater than the corresponding terms of the given series, the given series must converge by means of the comparison test.

When $p > 1$, the series is divergent since the general term does not approach zero as n increases without limit.

When $p \leq 1$, further investigation is necessary.

(2) Kummer Ratio Test.

This test due to Kummer was arranged in this form by Dini.

If $\sum D_\nu^{-1}$ is a divergent series and if

$$T_\nu = D_\nu \frac{a_\nu}{a_{\nu+1}} - D_{\nu+1}$$

then

(C) $\sum a_\nu$ is convergent if $\lim\limits_{n \to \infty} T_\nu > 0$.

(D) $\sum a_\nu$ is divergent if $\lim\limits_{n \to \infty} T_\nu < 0$.

In particular, if T_ν tends to a definite limit L, then

(C) $\sum a_\nu$ is convergent if $L > 0$,

(D) $\sum a_\nu$ is divergent if $L < 0$.

Proof:

If the minimum limit g is positive, and if h is any positive number less than g, an integer m can be found such that

$$T_\nu = D_\nu \frac{a_\nu}{a_{\nu+1}} - D_{\nu+1} > h, \quad \text{if } n \geq m.$$

Thus $a_\nu D_\nu - a_{\nu+1} D_{\nu+1} > h a_{\nu+1}$, if $n \geq m$

or adding, we have

$$a_m D_m - aD > h (a_{m+1} a_{m+2} --- a)$$

Hence $a_{m+1} + a_{m+2} + \cdots a_n < a_m I_m / h$, and the last expression on the right does not involve n ; so that $\sum_1^n a$ remains always less than a fixed number, and therefore $\sum a_n$ is convergent.

On the other hand, if the maximum limit is negative, all the expressions I_n must be negative after a certain stage, and thus we find m , so that

$$D_n \frac{a_n}{a_{n+1}} - D_{n+1} < 0, \qquad \text{if } n \geqq m,$$

or $a_n D_n < a_{n+1} D_{n+1}$, if $n \geqq m$, Hence $a_n D_n > a_m D_m$, if $n > m$, and so the terms of $\sum a_n$ are, after the m^{th}, greater than those of the divergent series $(a_m D_m) \sum D_n^{-1}$ Thus $\sum a_n$ is also divergent by means of the comparison test.[15]

(2) Special Ratio Tests of Importance

(a). d'Alembert's Test.

This test is essentially the same as the first ratio test given in this paper.

Let $D_n = D_{n+1} = 1$; then the conditions are

(C) Convergent if $\lim_{n \to \infty} (a_n / a_{n+1}) > 1$;

(D) Divergent if $\lim_{n \to \infty} (a_n / a_{n+1}) < 1$.

15 Bromwich, T. J. I. "The Theory of Infinite Series" MacMillan Co., London, Second Edition. page 37.

(b) Raabe's Test

This test should be tried when $\lim (a_n / a_{n+1}) = 1$

Let $D_n = n$, then the conditions are

(C) Convergent if $\lim \left\{ n \ (a_n / a_{n+1} - 1) \right\} > 1;$

(D) Divergent if $\lim \left\{ n \ (a_n / a_{n+1} - 1) \right\} < 1.$ " 16

In case the limits used in Raabe's test are both equal to 1, it is necessary to apply more delicate tests.

C. LOGARITHMIC TESTS

1. Cauchy's Logarithmic Criteria.

Taking the general harmonic series

$$\frac{1}{1^{\mu}} + \frac{1}{2^{\mu}} + \frac{1}{3^{\mu}} + \cdots \frac{1}{N^{\mu}}$$

as a comparison series, Cauchy deduced another test for convergence:

If after a certain term the expression $\dfrac{\log \frac{1}{u_n}}{\log n}$ is always greater than a fixed number greater than unity, the series converges. If after a certain term the expression $\dfrac{\log \frac{1}{u_n}}{\log n}$ is always less than unity the series diverges.

If $\dfrac{\log \frac{1}{u_n}}{\log n}$ approaches a limit L as n increases indefinitely, the series converges if $L > 1$, and diverges if $L < 1$. The case in which $L = 1$ remains in doubt.

16 Bromwich, op. cit. page 39.

Proof:

Suppose $\frac{\log \frac{1}{u_n}}{\log n} > k$, where $k > 1$.

This may be written $\log \frac{1}{u_n} > k \log n$; or this is equivalent to $\frac{1}{u_n} > n^k$; or $u_n < \frac{1}{n^k}$

Therefore since $k > 1$ the series must converge.

Likewise, suppose $\frac{\log \frac{1}{u_n}}{\log n} < 1$, or $\log \frac{1}{u_n} < \log n$.

Here $\frac{1}{u_n} < n$, or, $u_n > \frac{1}{n}$ [17]

Hence the series must diverge.

Goursat[17] has this to say of the logarithmic tests in general: "This test enables us to determine whether a series converges or diverges whenever the terms of the series, after a certain one, are each respectively less than the corresponding term of the series

$$\frac{A}{1^{\mu}} + \frac{A}{2^{\mu}} + \cdots \frac{A}{n^{\mu}} +$$

where A is a constant factor and $\mu > 1$. For, if $u_n < \frac{A}{n^{\mu}}$, we shall have $\log u_n + \mu \log n < \log A$, or

$$\frac{\log \frac{1}{u_n}}{\log n} > \mu - \frac{\log A}{\log n},$$ and the right-hand side approaches the limit μ as $\underline{n}$ increases indefinitely.

If K denotes a number between unity and μ, we shall

17 Goursat-Hedrick "Mathematical Analysis" Vol. 1 Ginn and Co. New York. page 339.

have, after a certain term

$$\frac{\log \frac{1}{u_n}}{\log n} > K.$$

Similarly, taking the series $\sum \frac{1}{n\,(\log n)^{\mu}}$, $\sum \frac{1}{n \log n (\log n)^{\mu}}$ as comparison series, we obtain an infinite number of tests for convergence which may be obtained mechanically from the preceding by replacing the expression $\log\,(1/u_n)\ /\log n$ by $\log\,(1/nu_n)\ /\log n$; and then by $\dfrac{\log\left(\dfrac{1}{n\,u_n \log n}\right)}{\log^3 n}$.

and so on.

> These tests apply in more or less general cases. Indeed, it is easy to show that if the convergence or divergence of a series can be established by means of any one of them the same will be true of any which follow. It may happen that no matter how far we proceed with these trial tests, no one of them will enable us to determine whether the series converges or diverges. This result is of great theoretical importance, but convergent series of this type evidently converge very slowly, and it scarsely seems possible that they should ever have any practical applicati n whatever[18] in problems which involve numerical calculations."

2. Raabe's or Duhamel's Test.

Taking the series

$$\frac{1}{1^{\mu}} + \frac{1}{2^{\mu}} + \frac{1}{3^{\mu}} + \cdots \frac{1}{n^{\mu}} + \cdot$$

18. Goursat-Hedrick op. cit. page339-340.

as a comparison series and comparing the ratio of two consecutive terms instead of the terms themselves, gives the test discovered first by Raabe[19], and then by Duhamel[20].

Consider the series

$$u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

in which the ratio u_{n+1}/u_n approaches unity, remaining constantly less than unity. Then we may write

$$\frac{u_{n+1}}{u_n} = \frac{1}{1+\alpha_n},$$ where α_n approaches zero as n becomes infinite. The comparison of this ratio with $\left(\frac{n}{n+1}\right)^{\mu}$ gives the Raabe rule:

<u>If after a certain term the product $n\alpha_n$ is always greater than a fixed number which is greater than unity, the series converges. If after a certain term the same product is always less than unity, the series diverges.</u>

Proof:

The second part of the theorem follows immediately. For since $n\alpha_n < 1$ after a certain term, it follows that

$$\frac{1}{1+\alpha_n} > \frac{n}{n+1},$$ and the ratio $\frac{u_{n+1}}{u_n}$ is greater than the ratio of two consecutive terms of the harmonic series. Hence by comparison the series diverges.

19. Raabe, "Zeitschrift fur Mathematik und Physik", 1832.

20 Duhamel, "Journal de Liouville". 1838.

In order to prove the first part, suppose that after a certain term we always have $n\alpha_N > k > 1$. Let be a number which lies between 1 and k, $1 < \mu < k$. Then the series surely converges if after a certain term the ratio u_{N+1} / u_N is less than the ratio $[n/(n+1)]$ of two consecutive terms of the series whose general term is $n^{-\mu}$. The necessary condition that this should be true is that

$$\frac{1}{1+\alpha_N} < \frac{1}{(1+\frac{1}{N})^{\mu}}$$

or by developing $(1 + 1/n)^{\mu}$ by Taylor's theorem limited to the term $1/n^2$,

$$1 + \frac{\mu}{N} + \frac{\lambda_N}{N^2} < 1 + \alpha_N$$

, where λ_N always remains less than a fixed number as n becomes infinite. Simplifying this inequality we may write it in the form

$$\mu + \frac{\lambda}{N} < N\alpha_N$$

The left-hand side of this inequality approaches μ as its limit as n becomes infinite. Hence, after a sufficiently large value of n, the left-hand side will be less than $n\alpha_N$, which proves the inequality $\frac{1}{1+\alpha_N} < \frac{1}{(1+\frac{1}{N})^{\mu}}$.

It follows that the series is convergent. If the product n approaches a limit L as n becomes infinite, we may apply the preceding rule. The series is convergent if $L > 1$, and divergent if $L < 1$. A doubt exists if $L = 1$, except when $n\alpha_N$ approaches unity remaining constantly less than unity, in that case the series diverges.

(3) In the test above, if the product $n\,\alpha_n$ approaches unity as its limit, we may compare the ratio u_n / u_{n+1} with the ratio of two consecutive terms of the series

$$\frac{1}{2(\log 2)^{\mu}} + \cdots + \frac{1}{n(\log n)^{\mu}} + \cdots$$

which converges if $\mu > 1$, and diverges if $\mu \leq 1$. The ratio of two consecutive terms of the given series may be written in the form $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{\beta_n}{n}$, where β_n approaches zero as n becomes infinite.

<u>If after a certain term the product $\beta_n \log n$ is always greater than a fixed number which is greater than unity, the series converges. If after a certain term the same product is always less than unity, the series diverges.</u>

Proof:

In order to prove the first part of the theorem, suppose that $\beta_n \log n > k > 1$. Let μ be a number between 1 and k. Then the series surely will converge if after a certain term we have

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)[\log(n+1)]^{\mu}}{n(\log n)^{\mu}}$$

which may be written in the form

$$1 + \frac{1}{n} + \frac{\beta_n}{n} > \left(1 + \frac{1}{n}\right)\left[1 + \frac{\log\left(1 + \frac{1}{n}\right)}{\log n}\right]^{\mu}$$

or applying Taylor's theorem to the right-hand side,

$$1 + \frac{1}{n} + \frac{\beta_n}{n} > \left(1 + \frac{1}{n}\right)\left[1 + \frac{\mu}{n \log n} + \frac{\epsilon_n}{n^2 \log n}\right],$$

where ϵ_n always remains less than a fixed number as n becomes infinite. Simplifying this inequality, it becomes

$$\beta_n \log n > \mu (n+1) \log (1+1/n) + \frac{\lambda_n (n+1) \log(1+1/n)}{\log n}$$

The product $(n+1) \log (1+1/n)$ approaches unity as n becomes infinite, for it may be written, by Taylor's theorem

$$(n+1) \log (1+1/n) = 1 + (1/2n)(1+\epsilon),$$

where ϵ approaches zero. The right-hand side of the above inequality therefore approaches μ as its limit, and the truth of the inequality is established for sufficiently large values of n, since the left-hand side is greater than k, which is itself greater than μ.

The second part of the theorem may be proved by comparing the ratio u_{n+1}/u_n with the ratio of two consecutive terms of the series whose general term is $1/(n \log n)$. For the inequality

$$\frac{u_{n+1}}{u_n} > \frac{n}{n+1} \frac{\log n}{\log (n+1)}$$

which is to be proved, may be written in the form

$$1 + \frac{1}{n} + \frac{\lambda_n}{n} < \left(1+\frac{1}{n}\right)\left[1 + \frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right].$$

or $\beta_n \log n < (n+1) \log \left(1+1/n\right)$.

The right-hand side approaches unity through values which are greater than unity, as is seen from the equation

$$(n+1) \log (1+1/n) = 1 + 1/2n \; (1+\epsilon)$$

The truth of the inequality is therefore established for sufficiently large values of $\underline{n}$, for the left-hand side cannot exceed unity.[21]

4. In Raabe's Ratio Test"

"Let $D_n = n$

Convergent if $\lim_{n\to\infty} \{ n(a_n/a_{n+1} - 1) \} > 1$;

Divergent if $\lim_{n\to\infty} \{ n (a_n / a_{n+1} - 1) \} < 1$." .

If the limits are both equal to 1, we may use more delicate tests, found by writing

$D_n = n \log n, \quad n \log n \log(\log n), \text{-----}$

These functions are of the form $f(n)$, where $f(x)$ is continuous and $f''(x)$ tends to zero as $\underline{x}$ tends to infinity. Then the test is :

Convergent if $\lim p_n > 0$;

Divergent if $\lim p_n < 0$.

where $a_n / a_{n+1} = 1 + f'(n) / f(n) + p_n / f'(n)$.

Proof:

$$f(n+1) - f(n) - f'(n) = \int_0^1 \left[f'(n+x) - f'(n) \right] dx.$$
$$= \int_0^1 dx \int_0^x f''(n+t)\, dt.$$

Now we can find $\underline{v}$ so that $|f''(E)| < \delta$, if $E > V$, and so the last integral is easily seen to be less than $\frac{1}{2}\delta$, if $n > v$.

Thus $\quad f(n+1) - f(n) - f'(n) \to 0$, as $n \to \infty$

Writing $f(n+1)$ and $f(n)$ for D_{n+1} and D_n in Kummer's test, gives the form above.[22]

21. Goursat-Hedrick op. cit. page 342-343.

5. de Morgan's and Bertrand's First Test.

In the test (4) above, if $f(x) = x \log x$, we get the test of de Morgan and Bertrand:

Convergent if $\lim p > 1$;

Divergent if $\lim p < 1$.

where $a_N / a_{N+1} = 1 + 1/n + p/(n \log n)$.

Their further tests are of less importance.[22]

6. Ermakoff's Tests--Logarithmic Scale Tests.

The series $\sum f(n)$, in which $f(n)$ is subject to the condition that $f(x)$ is definite for values of x which are not integers, and that $f(x)$ never increases with x, is:

Convergent if $\lim\limits_{n\to\infty} \frac{e^x f(e^x)}{f(x)} < 1$

Divergent if $\lim\limits_{n\to\infty} \frac{e\, f(e)}{f(x)} > 1$.

Proof:

In the first place, in the first case, if p is any number between the maximum limit and unity, we can find E so that

$$e^x f(e^x) < p\, f(x), \quad \text{if } x > E$$

Thus $\int_E^X e^x f(e^x)\, dx < p \int_E^X f(x)\, dx$, if $X > E$.

or, changing the independent variable to e^x in the left-hand integral, we have $\int_G^Z f(x)\,dx < p \int_E^X f(x)\,dx$, where $Z = e^X$, $G = e^E$

22. Bromwich, op. cit. page 39-40

That is, $(1 - p)\int_{\zeta}^{X} f(x)\,dx < p\left\{\int_{\xi}^{X} f(x)\,dx - \int_{\zeta}^{Z} f(x)\,dx\right\}$

or $< p\left\{\int_{\xi}^{\zeta} f(x)\,dx - \int_{X}^{Z} f(x)\,dx\right\}$.

Or, again, since the last term in the bracket is positive (because $Z = e^X$ is greater than X), we have $(1 - p)\int_{\zeta}^{Z} f(x)\,dx < p\int_{\xi}^{\zeta} f(x)$.
As this inequality is true for any value of X greater than E, it is clear that the infinite integral $\int_{\zeta}^{\infty} f(x)\,dx$ must converge; and, therefore, so does the series $\sum f(n)$.[23] (As will be shown in the Cauchy Integral Test following.)

23. Bromwich, op. cit. page43-44.

D. INTEGRAL TESTS.

1. Cauchy's Integral Test.

This is a test of the first kind.[24]

Let $\phi(x)$ be a function which is positive for values of x greater than a certain number a, and which constantly decreases as x increases past $x = a$, approaching zero as x increases without limit. Then, the x-axis is an asymptote to the curve $y = \phi(x)$, and the definite integral $\int_a^L \phi(x)\,dx$ may or may not approach a limit as L increases without limit. The series $\phi(a) + \phi(a+1) + \text{----} + \phi(a+n) + \text{--}$ converges if the preceding integral approaches a limit, and diverges if it does not.

Proof:

Consider the set of rectangles whose bases are each unity and whose altitudes are $\phi(a)$, $\phi(a+1)$, ---- $\phi(a+n)$, -- respectively. Since each of these rectangles extends beyond the curve $y = \phi(x)$, the sum of their areas is evidently greater than the area between the x-axis, the curve $y = \phi(x)$, and the two ordinates $x = a$, $x = a+n$. That is

$$\phi(a) + \phi(a+1) + \text{------} + \phi(a+n) > \int_a^{a+n} \phi(x)\,dx.$$

24. In testing a series of the form $u_1 + u_2 + \text{---} + u_n + \text{---}$ duBois-Reymond called those tests which make use of the ratio of two consecutive terms "tests of the second kind", while those tests which make use of the general term u_n are called tests of the "first kind". The same notation is used here to designate the two kinds of integral tests.

On the other hand, if we consider the rectangles inside the curve, with a common base equal to unity and with altitudes

$$\phi(a+1),\ \phi(a+2), \text{----}\ \phi(a+n),$$

respectively, the sum of the areas of these rectangles is evidently less than the area under the curve $Y = \phi(x)$, and we may write

$$\phi(a) + \phi(a+1) + \text{--} + \phi(a+n) < \phi(a) + \int_a^{a+n} \phi(x)\, dx.$$

Hence, if the integral $\int_a^L \phi(x)\,dx$ approaches a limit I' as L increases indefinitely, the sum

$\phi(a)+ ---+\phi(a+n)$ always remains less than $\phi(a)+I'$.

It follows that the sum in question approaches a limit; hence the series $\phi(a)+\phi(a+1)+---+\phi(a+n)$ is convergent. On the other hand, if the integral $\int_a^{a+n}\phi(x)\,dx$ increases beyond all limit as n increases without limit, the same is true of the sum $\phi(a)+\phi(a+1)+---+\phi(a+n)$--, as is seen in the first of the above inequalities; hence the series diverges.[25]

Bromwich calls this test the "Maclaurin Logarithmic Scale Test", and tells us that although this test is commonly attributed to Cauchy it occurs in Maclaurin's "Fluxions" 1742, article 350. He states and establishes the theorem in a different manner, as follows:

"The series $\sum^{\infty} a_n$ converges or diverges with the integral $\int_1^{\infty} f(x)\,dx$; if convergent the sum of the series differs from the integral by less than a_1; if divergent the limit of $(S_n - I)$ nevertheless exists and lies between 0 and a_1.

Proof:

If we write $f(x)= a_n$ it may happen that the

25 Goursat-Hedrick, op. cit. page 335-337.

function $f(x)$ is also definite for values of <u>x</u> which are not integers, and that $f(x)$ never increases with <u>x</u>. Then, if <u>x</u> lies between $(n-1)$ and n, it is plain that

$$a_{n-1} \geqq f(x) \geqq a_n > 0.$$

Thus from the definition of an integral, we have:

$$\int_{n-1}^{n} a_{n-1}\,dx \geqq \int_{n-1}^{n} f(x)\,dx \geqq \int_{n-1}^{n} a_n\,dx.$$

or, $$a_{n-1} \geqq \int_{n-1}^{n} f(x)\,dx \geqq a_n$$

Write now
$$I_n = \int_{1}^{n} f(x)\,dx$$

and we find on addition for $n = 1, 2,$ ------

$$a_1 + a_2 + \text{---}\ a_{n-1} \geqq I_n \geqq a_2 + a_3\ \text{------}\ a_n$$

or, $$S_n - a_n \geqq I_n \geqq S_n - a_1.$$

Hence $$a_1 \geqq S_n - I_n \geqq a_n > 0.$$

Further, $$(S_n - I_n) - (S_{n-1} - I_{n-1}) = a_n - \int_{n-1}^{n} f(x)\,dx \leqq 0,$$

and therefore the sequence whose n^{th} term is $S_n - I_n$ <u>never increases</u>; and since its terms are contained between 0 and a_1, the sequence must have a limit and[26]

$$a_1 \geqq \lim_{n \to \infty} (S_n - I_n) \geqq 0."$$

26. Bromwich, op. cit. page 22.

2. Integral Tests of the Second Kind.

Integral tests of the second kind apply to series for which a function is known that for successive values of the ratio of one term to the preceding term. Such a series can be written in the following normal form:

$$c + ca + ca_0 a_1 + ca_0 a_1 a_2 + ca_0 a_1 a_2 a_3 ---------$$

where $a_n = a(n)$, a (x) being a known function.

Brink gives several integral tests for testing the convergence or divergence of infinite series. His first test is a fundamental test of the second kind; then he gives three other tests which are merely variations of this test. His fifth test is a generally useful integral test of the second kind in a simple form. Here the fundamental test and its proof are quoted in detail, the tests only are quoted for the second, third, and fourth, tests. and the fifth test with its proof is quoted in full.

a. Fundamental Integral Test of the Second Kind

Theorem 1.

" Given the series $u_0 + u_1 + u_2 + -----$ $\quad (u_n > 0, \quad n \geq \mu)$

Let $r_n = u_{n+1} / u_n$ and suppose that from a certain point $r(x)$ is a continuous funtion such that $r(n) = r_n$, and suppose that a constant $\underline{m}$ exists, positive or zero, such that $r(x') \geqq r(x)$ when $x' \geqq x + m$.

" Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral

$$\int_{\mu}^{\infty} e^{\int^{x} \log r(x)dx} dx.$$

Proof:

Under the condition, from a certain point on, either $r(x) > 1$, or $r(x) \leqq 1$. Suppose that $r(x) \leqq 1$, $\mu < x$. We take μ to be an integer. Then

(1)
$$\log r_{\nu+m} \leqq \int_{\nu}^{\nu+1} \log r(x)\, dx \leqq \log r.$$

we write

(2)
$$\int_{N}^{N+1} e^{\int_{\mu}^{x} \log r(x)dx} dx = \int_{N}^{N+1} e^{\overline{\int_{\mu}^{\mu+1} + \int_{\mu+1}^{\mu+2} + \cdots + \int_{N-1}^{N} + \int_{N}^{x}} \log r(x)\, dx} dx$$

(the bar indicates that the integrals under it have the same integrand.)

(3) Therefore by (1)

$$\int_{N}^{N+1} e^{\int_{\mu}^{x} \log r(x)\, dx} dx \leqq \int_{N}^{N+1} e^{\log r_{\mu+1+m} \log r_{\mu+2+m} \cdots\cdots \log r_{N+m}} dx.$$

$$r_{\mu+1+m} \cdot r_{\mu+2+m} \cdots\cdots r_{N+m} \geqq \frac{1}{u_{\mu+1+m}} \cdot u_{N+1+m}$$

Likewise,

(4)
$$\int_{N}^{N+1} e^{\int_{\mu}^{x} \log r(x)\, dx} dx \geqq r_{\mu+m} \cdot r_{\mu+1+m} \cdots\cdots r_{N+m} \geqq \frac{1}{u_{\mu-m}} \cdot u_{N+1-m}$$

Similar inequalities hold if $r(x) > 1$, $\mu < x$. Since the integral $\int e^{\int \log r(x)\, dx} dx$ cannot oscillate, the theorem follows at once from a comparison of the two series

$$\sum_{\nu}^{\infty} u_{\nu} \quad \text{and} \quad \sum \int_{\nu}^{\nu+1} e^{\int^{x} \log r(x)\, dx} dx, \; ,$$

by means of (3) and (4).

b. Theorem II.

"Given the series $u_1 + u_2 + u_3 + \cdots$, $(u_n > 0,\ n \geq 1)$

Let $r(x)$ be a function with a continuous derivative $r'(x)$ $x \geq 1$, such that $r(n) = r_n = u_n / u_{n-1}$.

Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral

$$\int_{1}^{\infty} e^{\int^{x} \log r(x)\, dx} dx."$$

(Theorem II is included in the following theorem).

c. Theorem III.

"Given the series $u_1 + u_2 + u_3 + \cdots$ $(u_n > 0,\ n \geq 1)$.

Let $r(x)$ be a continuous function such that, for $x \geq 1$

(1) $r(n) = r_n = u_n / u_{n-1}$.

(2) $0 < A \leq r(x) \leq B$.

(3) $|r(x') - r(x)| \leq f(x)$, whenever $0 \leq (x' - x) \leq 1$

The series $\sum_{1}^{\infty} f(n)$ being a convergent series.

Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral

$$\int_{m}^{\infty} e^{\int_{m}^{x} \log r(x)\, dx}\, dx.$$

----

d. Theorem IV.

"Given the series

$$u_0 + u_1 + u_2 + \cdots\cdots \quad (u_n > 0, \quad n \geq 0).$$

Let $r_n = u_{n+1} / u_n$ and suppose that $r(x)$ is a function satisfying the preliminary condition in any one of the theorems I, II, or III, and that $r(x) \leq 1$, then the series converges if

$$\frac{\left[r\,(e^x) \right]^{e^x}}{r(x)} < \cdots < \frac{1}{e^x},$$

$m \leq x$.

and diverges if

$$\frac{\left[r\,(e^x) \right]^{e^x}}{r(x)} > \cdots > \frac{1}{e^x}, \quad m \leq x."$$

e. <u>Theorem V</u> -- <u>An Integral Test Involving $r(x) - 1$.</u>

"Given the series

$$u_1 + u_2 + u_3 + \cdots \qquad (u_n > 0, \; n \to \infty)$$

Let $r_n = u_n / u_{n+1}$, and suppose that $r(x)$ is a positive integral function satisfying the preliminary conditions in one of the theorems I, II, or III, and the further condition that $\left| r(x) - 1 \right| < a < 1$, x.

Then a sufficient condition for the convergence of the series is the convergence of the integral

$$\int^{\infty} e^{\int [r(x) - 1]\, dx}\, dx.$$

This condition is also necessary for the convergence of the series if from a certain point on $\left| r(x) - 1 \right| < \frac{k}{x}$, where <u>k</u> is a constant.

Proof:

(1) $\log r(x) = [r(x) - 1] - \frac{1}{2}[r(x) - 1]^2 + \frac{1}{3}[r(x) - 1]^3 - \cdots$

Then $\log r(x) \leq [r(x) - 1]$, so that

$$\int e^{\int \log r(x)\, dx}\, dx \leq \int e^{\int [r(x) - 1]\, dx}\, dx.$$

Therefore since the preliminary conditions of one of the theorems I, II, or III, are satisfied, so that the given series and the integral

$$\int e^{\int r(x)\, dx}\, dx,$$

converge or diverge together, the convergence of the integral

$$\int^{\infty} e^{\int [r(x) - 1]\, dx}\, dx$$

is sufficient for the convergence of the given series.

The expansion (1) converges uniformly for $|r(x)-1| < \varepsilon < 1$, that is $\mu_1 \leq x$. We can therefore integrate it term by term over any interval $\mu_1 \leq x \leq A$. Then

$$\int_{x_0}^{x} \log r(x)\,dx = \int_{x_0}^{x} [r(x)-1]\,dx - \frac{1}{2}\int_{x_0}^{x} [r(x)-1]^2\,dx \text{ --}$$

$\mu_1 < x_0 < x$. and

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)\,dx}\,dx = \int_{x_0}^{x} e^{\int_{x_0}^{x} [r(x)-1]\,dx - \frac{1}{2}\int_{x_0}^{x} [r(x)-1]^2\,dx} \,dx$$

------, $\mu_1 < x_0 < x$. We have

$$|r(x)-1| < \frac{k}{x}, \text{ so that } [r(x)-1] > -\frac{k}{x},$$

$$-[r(x)-1]^2 > \frac{-k^2}{x^2}, \quad |r(x)-1|^3 > \frac{-k^3}{x^3}, \text{------}, \quad x > \mu_1 .$$

Therefore

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)\,dx}\,dx > \int_{x_0}^{x} e^{\int_{x_0}^{x} [r(x)-1]\,dx - \frac{1}{2}\int_{x_0}^{x}\frac{k^2}{x^2}dx - \frac{1}{3}\int_{x_0}^{x}\frac{k^3}{x^3}\,dx}\,dx$$

$$> \int_{x_0}^{x} e^{\int_{x_0}^{x} [r(x)-1]\,dx - \frac{1}{2}\int_{x_0}^{x}\frac{k^2}{x^2}dx - \frac{1}{3}\int_{x_0}^{x}\frac{k^3}{x^3}dx \text{------}}\,dx =$$

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} [r(x)-1]\,dx - \left(\frac{k}{2x} + \frac{k^2}{3.2\,x^2} + \frac{k^3}{4.3\,x^3} + \text{---}\right)}\,dx.$$

$\mu_1 < x_0 < x$.

If x_0 is taken greater than k, the series in parenthesis converges to a value C. Consequently

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)\, dx}\, dx > C \int_{x_0}^{x} e^{\int_{x_0}^{x} [r(x) - 1]\, dx}\, dx.$$

Therefore when x increases indefinitely, if the second of these two integrals diverges, the first one also diverges, and the given series diverges. "[27]

Brink also gives two other tests which are extensions of the first ones to double and complex series. His test for complex series is given later in this paper. The other one is not included as it is a simple application of the other tests.

It is evident that integral tests of the second kind are closely related to the more familiar ratio tests.

27 Brink, R. W. "A New Integral Test For Convergence and Divergence of Infinite Series. Transactions of the American Mathematical Society. Vol. 19. 1918. page 186-204.

2. ALTERNATING SERIES.

If the terms of a series are alternately positive and negative the series is called an alternating series.

(a)

An important test for the convergence of an alternating series is :

If in an alternating series the terms approach zero as $n \to \infty$, and if after a certain term each term is smaller than the one preceding it, then the series converges.

Proof·

Let a be a term after which the terms grow smaller numerically, and let S_N be the sum of the first n terms. Then it is obvious that all sums after S_N lie between S_N and S_{N+1}. Since by taking n sufficiently large the difference between S_N and S_{N+1} may be made as small as we please it follows at once that the series is convergent.

(b)

Another test similar to Raabe's ratio test is:

If v_N / v_{N+1} can be expressed in the form

$$\frac{v_N}{v_{N+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^\rho}\right) \qquad \rho > 1.$$

The series $(-1)^{N-1} v_N$ is convergent if $\mu > 0$, and is divergent if $\mu \leq 0$.

Proof:

For if μ 0, after a certain term we shall have

$$\frac{\mu}{n} > O\left(\frac{1}{n^\rho}\right), \text{ so that } v_N > v_{N+1}; \text{ further, that } \lim_{N \to \infty} v_N = 0.$$

Therefore the series converges. But on the other hand, if $\mu = 0$, it is clear that $\lim v_n$ is not 0 as n increases without limit, therefore the series must be divergent; and if $\mu < 0$, we have $v_n < v_{n+1}$, so that $\lim v_n$ cannot be zero as n increase without limit, therefore, again, the series must diverge.

3. ABSOLUTELY CONVERGENT SERIES.

A series is said to be absolutely convergent if the series formed of the absolute values of its terms (or its moduli) form a convergent series also. The series formed of the absolute values of the terms of a series is called the <u>adjoint series</u>, also.

(a)

Any series whatever is convergent if the series formed of the absolute values of the terms of the given series converges.

Proof:

let

(1) $u_1 + u_2 + u_3 + \cdots + u_n$ be a series of positive and negative terms, and let

(2) $U_1 + U_2 + U_3 + \cdots + U_n + \cdots$ be the series of the moduli of the given series, where $U_n = |u_n|$

If series (2) converges, series (1) must likewise converge.

For we have $|u_N + u_{N+1} + u_{N+2} + \cdots + u_{N+p}| < U_N + U_{N+1} + U_{N+2} + \cdots + U_{N+p}$.

And the right-hand side may be made less than any preassigned number by choosing <u>n</u> sufficiently large, for any subsequent choice of p, hence the same is true of the left-hand side, and the series (1) must surely converge.

(b)

An absolutely convergent series remains convergent if each term a_ν is multiplied by a factor v_ν whose numerical value does not exceed a constant <u>k</u>.

Proof:

for, since $\sum a_\nu$ is absolutely convergent, the index <u>m</u> can be chosen so that $\sum_{m+1}^{m+p} a_\nu < \frac{\epsilon}{k}$, however small ϵ may be.

But

$$\left|\sum_{m+1}^{m+p} a_\nu v_\nu\right| \leq \sum_{m+1}^{m+p} |a_\nu v_\nu|, \text{ and } |a_\nu v_\nu| = |a_\nu||v_\nu| \leq a_\nu k,$$

Thus

$$\left|\sum_{m+1}^{m+p} a_\nu v_\nu\right| \leq k \sum_{m+1}^{m+p} a_\nu < \epsilon,$$ and therefore the series $\sum a_\nu v_\nu$ is convergent.

Any absolutely convergent series may be regarded as the difference of two convergent series of positive terms. In numerical calculations such series may be treated as if it were a sum of a finite number of terms.

(c)

The sum of an absolutely convergent series is the same no matter in what order its terms are arranged.

Proof:

(1) let $\sum a_i$ be an absolutely convergent series with sum S, (2) and let $\sum a_i'$, (3) $\sum a_i''$,

be the series formed from the positive and negative terms respectively of (1). Then (2) and (3) are both convergent since (1) is absolutely convergent. let the sums of (2) and (3) be S', S'' respectively, S' being positive, and S'' being negative. Then $S' + S'' = S$.

If the sum of the first n terms of (2) differ from S' by less than d, and the sum of the first n terms of (3) differ from S'' by less than d, also, then the sum of the collection of k terms from (2) and (3) which contain the first n terms of both series, and other terms, must differ from $S' + S'' = S$ by less than 2d. But d may be made as small as we please and hence the sum of the first k terms of the rearranged series can be made to differ from S by as little as we please. That is the sum of the rearranged series is S.

(d) Comparison theorem for absolutely convergent series.

A series

$$u_0 + u_1 + u_2 + \cdots$$

is absolutely convergent provided that $|u_n|$ is always less than $C|v_n|$ where C is some number independent of n, and v_n is the n^{th} term of another series known to be absolutely convergent.

Proof:

$$|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < C\left\{|v_{n+1}| + |v_{n+2}| + \cdots + |v_{n+p}|\right\}.$$

where n and p are any integers, but since the series $\sum v_n$ is absolutely convergent, the series $\sum |v_n|$ is convergent, and so, given ϵ, we can find n such that $|v_{n+1}| + |v_{n+2}| + \cdots + |v_{n+p}| < \frac{\epsilon}{C}$

for all values of p. It follows therefore that we can find n such that $|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \varepsilon$. for all values of p.

That is the series $\sum u_\nu$ is absolutely convergent.

(e) Corollary.

A series is absolutely convergent if the ratio of the n^{th} term to the n^{th} term of a series known to be absolutely convergent is less than some number independent of n.

4. CONDITIONALLY CONVERGENT INFINITE SERIES.

If a series is convergent, but the series formed of the moduli of its terms is not convergent, the series is said to be conditionally convergent

For example, the series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$$

is convergent, but the series formed by its moduli

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

or the harmonic is obviously divergent.

5. UNIFORMLY CONVERGENT SERIES.

A series of the form

$$U_0(x) + U_1(x) + \cdots + U_\nu(x) + \cdots$$

each of whose terms is a function of x which is definite in an interval (a, b) is said to be uniformly convergent in that interval if it converges for every value of x between a and b, and if, corresponding to any arbitrarily preassigned positive number ϵ, a positive integer N, independent of x can be found such that the absolute value of the remainder R of the given series

$$R = U_{n+1}(x) + U_{n+2}(x) + \ldots + U_{n+p}(x) + \text{----}$$

is less than for every value of $n \geqq N$, and for every value of x which lies in the interval (a, b).

The latter condition is essential in this definition. For any preassigned value of x for which the series converges it is apparent from the very definition of convergence that, corresponding to any positive number ϵ, a number N can be found which will satisfy the condition in question. But, in order that the series should converge uniformly, it is necessary further that the same number N should satisfy this condition, no matter what value of x be selected in that interval (a, b).

(a)

The importance of uniformly convergent series rests upon the following property:

<u>The sum of a series whose terms are continuous functions of a variable</u> x <u>in an interval</u> (a,b) <u>and which converges uniformly in that interval, is itself a continuous function of</u> x <u>in that interval.</u>

Let x_0 be a value of x between a and b, and let $x_0 + h$

be a value in the neighborhood of x, which also lies in the interval (a, b). Let n be chosen so large that the remainder

$$R_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots$$

is less than $\epsilon / 3$ in absolute value for all values of x in the interval (a, b), where ϵ is an arbitrarily preassigned positive number. Let $f(x)$ be the sum of the given convergent series. Then we may write

$$f(x) = \phi(x) + R_n(x).$$

Where $\phi(x)$ denotes the sum of the first $n+1$ terms of the series

$$\phi(x) = u_0(x) + u_1(x) + \cdots u_n(x).$$

Subtracting the two equalities

$$f(x) = \phi(x) + R_n(x), \quad \text{and} \quad f(x+h) = \phi(x+h) + R_n(x+h)$$

we find

$$f(x+h) - f(x) = \left[\phi(x+h) - \phi(x)\right] + R_n(x+h) - R_n(x).$$

The number $\underline{n}$ was so chosen that we have

$$\left| R_n(x) \right| < \frac{\epsilon}{3}, \quad \left| R_n(x+h) \right| < \frac{\epsilon}{3}.$$

On the other hand, since each of the terms of the series is a continuous function of x, $\phi(x)$ is itself a continuous function of x. Hence a positive number η may be found such that

$$\left| \phi(x+h) - \phi(x) \right| < \frac{\epsilon}{3}.$$

whenever $|h|$ is less than η. It follows that we shall

$$f(x+h) - f(x) < 3\,\frac{\epsilon}{3}, \quad \text{whenever } h \text{ is less than } \eta.$$

This shows that $f(x)$ is continuous for $x = x_0$.

(b)

The following theorem enables us to show in many cases that a given series converges uniformly.

Let

$$u_0(x) + u_1(x) + \cdots + u_N(x) \cdots$$

be a series each of whose terms is a continuous function of x in an interval (a, b) and let

$$M_0 + M_1 + \cdots + M_N + \cdots$$

be a convergent series whose terms are positive constants. Then if $|u_N| \leqq M_N$ for all values of x in the interval (a, b) and for all values of n, the first series converges uniformly in the interval considered.

Proof:

it is evident that we shall have

$$\left| u_{N+1} + u_{N+2} + \cdots \right| \leqq M_{N+1} + M_{N+2} + \cdots$$

for all values of x between a and b. If N is chosen so large that the remainder R_N of the second series is less than for all values of n greater than N, we shall have, also,

$$\left| u_{N+1} + u_{N+2} \cdots \right| < \epsilon.$$

whenever n is greater than N, for all values of x in the interval (a, b).

For example, the series

$$M_0 + M_1 \sin x + M_2 \sin 2x + \cdots + M_N \sin nx + \cdots,$$

where M_0, M_1, ---- have the same meaning as above, converges in any interval whatever.

The test just given is usually called <u>Weierstrass' M-Test for Uniform Convergence.</u> Series which satisfy the M-test have been called "normally convergent" series by Baire, since it can be applied to nearly all series in ordinary everday use.[28]

(c) <u>Abel's Test for Uniform Convergence.</u>

<u>The series $\sum a_n(x)\, v_n(x)$ is uniformly convergent in an interval (a, b) provided that $\sum a_n(x)$ is uniformly convergent in the same interval; that for any particular value of x in the interval $v_n(x)$ is positive and never increases with n; and that $v(x)$ remains less than a fixed number K for all values of x in the interval.</u>

Proof:
For in virtue of the uniform convergence of $\sum a_n(x)$, we can find <u>m</u> so that, whatever positive integer <u>p</u> may be,

$$a_{m+1},\ a_{m+2} + a_{m+3},\ \text{------}\ a_{m+1} + a_{m+2} + \text{------} + a_{m+p}.$$

are all numerically less than ϵ. Then by Abel's Lemma,

> " If the sequence (v) of positive terms never increase, the sum $\sum a\,v$ lies between Hv and hv, where H and h are the upper and lower limits of the sums[28]
> $$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \text{------},\ a_1 + a_2 + a_3 \text{----} a_p."$$

we see that

$$\left|\sum_{m+1}^{m+p} a_n(x)\, v_n(x)\right| < \epsilon\, v_{m+1}(x) < \epsilon K,$$ since by hypothesis, $\sum a(x)\, v_n(x)$ converges uniformly in the interval. $v_n(x) \leqq v_0(x) < K$

28. Brink, op. cit. page 124, 57,125.

The most important special cases are when :

(1) a_ν is independent of (x); and

(2) v_ν is independent of (x).

This is a more delicate test for uniform convergence.

(d) Dirichlet's Test for Uniform Convergence.

The series $\sum u(x)\, v_\nu(x)$ is uniformly convergent in an interval (a, b), provided that (1) the series $\sum u_\nu(x)$ oscillates so that the absolute values of its limits of oscillation remain less than a fixed number K;
(2) for any particular value of x in the interval $v_\nu(x)$ is positive and never increases with n; and (3) as n tend to ∞, $v_\nu(x)$ tends uniformly to zero for all values of x in the interval.

Proof:
for then throughout the interval, the expressions

$$|u_{n+1}|,\ |u_{n+1} + u_{n+2}|,\ \text{--------},\ |u_{n+1} + u_{n+2} + \cdots + u_{n+p}|,$$

are less than $2K$; and we can find an index m such that

$v_m(x) < \epsilon$ for all values of x in the interval.

Thus, using Abel's lemma as before, we see that

$$\left| \sum_{n+1}^{n+p} u_\nu(x)\, v_\nu(x) \right| < 2\epsilon K$$ 29

for all points in the interval.

29. Brink, op. cit. page 125-126.

This is another test more delicate than the M-Test.

(e)

The condition of uniform convergence makes possible the use of the operations associated with the calculus, differentiation and integration. These are of especial value in connection with the convergence of power series, Fourier's series, and with multiple series.

(1) Any series of continuous function which converges uniformly in an interval (a, b) may be integrated term by term, provided that the limits of integration are finite, and lie in the interval (a, b).

Proof:

let x_0 and x_1 be any two values of x which lie between a and b, and let N be a positive integer such that $|R_n(x)| < \delta$ for all values of x in the interval (a, b) whenever $n > N$. Let f(x) be the sum of the series.

$$f(x) = u_0(x) + u_1(x) + \cdots\cdots + u_n(x) + \cdots$$

and let us set $D_n = \int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} u_0\,dx - \int_{x_0}^{x_1} u_1\,dx - \cdots - \int_{x_0}^{x_1} u_n\,dx = \int_{x_0}^{x_1} R_n\,dx$.
The absolute value of D_n approaches zero as n increases indefinitely, and we have the equation

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} u_0(x)\,dx + \int_{x_0}^{x_1} u_1(x)\,dx + \cdots + \int_{x_0}^{x_1} u_n(x)\,dx + \cdots\cdots$$

Considering x_0 as fixed and x_1 as variable, we obtain a series $\int_{x_0}^{x} u_0(x)\,dx + \cdots + \int_{x_0}^{x} u_n(x)\,dx + \cdots\cdots$, which converges uniformly in the interval (a,b) and represents a continuous function whose derivative is $f(x)$.

(P)

Conversely, any convergent series may be differentiated term by term if the resulting series converges uniformly.

Proof: for let

$$f(x) = u_0(x) + u_1(x) + \text{---} + u_n(x) + \text{----}$$

be a series which converges in the interval (a, b). Let us suppose that the series whose terms are the derivatives of the terms of the given series, respectively, converge uniformly in the same interval, and let $\phi(x)$ denote the sum of the new series

$$\phi(x) = \frac{du_0}{dx} + \frac{du_1}{dx} + \text{----} + \frac{du_n}{dx}.$$

Integrating this series term by term between the two limits x_0 and x, each of which lies in the interval (a, b), we find

$$\int_{x_0}^{x} \phi(x)\,dx = \Big[u_0(x) - u_0(x_0)\Big] + \Big[u_1(x) - u_1(x_0)\Big] + \text{----}$$

or

$$\int_{x_0}^{x} \phi(x)\,dx = f(x) - f(x_0). \qquad 30.$$

This shows that $\phi(x)$ is the derivative of $f(x)$.

30. Goursat-Hedrick, op. cit. page 364-366.

6. COMPLEX INFINITE SERIES.

An infinite series of complex quantities

$$z_0 + z_1 + z_2 + z_3 + \cdots\cdots\cdots z_n + \cdots$$

is called convergent and the complex quantity S is called the sum of the series when

$$\lim_{n \to \infty} (z_0 + z_1 + \cdots\cdots\cdots z_n + \cdots)$$

exists and is equal to S.

A necessary and sufficient condition for the convergence of an infinite series of complex quantities is that the series of real parts and the series of imaginary parts respectively converge.

A series of complex quantities is called absolutely convergent when the series formed by the absolute values of its terms converge.

If a series of complex quantities converge absolutely then the series formed from its real parts, and from its imaginary parts converge absolutely.

If a series of complex quantities is absolutely convergent its sum is independent of the arrangement of the terms.

"Let $u_0 + u_1 + u_2 + \cdots u_n \cdots\cdots$

be a series whose terms are imaginary quantities:

$$u_0 = a_0 + b_0 i, \quad u_1 = a_1 + b_1 i, \quad \cdots\cdots \quad u_n = a_n + b_n i, \quad \cdots\cdots$$

(1) $a_0 + a_1 + a_2 + \cdots\cdots a_n = S'$, (2) $b_0 + b_1 + b_2 + \cdots\cdots b_n = S''$

S' and S'' are the sums of series (1) and (2) respectively. Then the quantity $S = S' + iS''$ is called the sum of the complex series. It is evident that S, is, as before, the limit of the sum S_n of the first n terms of the series as n becomes infinite. It is evident that a complex series is essentially only a combination of two series of real terms. When the series of absolute values of the terms of the series of complex terms

$$(3) \quad \sqrt{a_0^2+b_0^2} + \sqrt{a_1^2+b_1^2} + \cdots\cdots \sqrt{a_n^2+b_n^2} + \cdots$$

converges, each of the series (1) and (2) evidently converge absolutely, for $|a_n| \leq \sqrt{a_n^2+b_n^2}$ and $|b_n| \leq \sqrt{a_n^2+b_n^2}$. In this case the complex series is said to be absolutely convergent. The sum of such a series is not altered by a change in the order of the terms, nor by grouping the terms together in any way.

Conversely, if each of the series (1) and (2) converge absolutely, the series (3) converges absolutely, for [31]

$$\sqrt{a_n^2+b_n^2} \leq |a_n| + |b_n|.$$

It is easy to see that the necessary and sufficient test for convergence is that, corresponding to any real positive number ϵ, we can find m such that $|a_{m+1} + a_{m+2} + \cdots a_{m+p}| < \epsilon$ no matter how large p is. Since the series to be tested is $\sum |a_n|$ which consists of positive terms only, we can apply at once any of the common tests for positive term series.

31 Goursat-Hedrick Op. cit page 250-251.

" Cauchy's integral test may be extended to test series of positive and negative, or complex terms.

Given a series

$$u_0 + u_1 + u_2 + \text{-------}$$

Let $u(x)$ be an integral function of the real variable x such that

$$(1) \quad u(n) = u_N$$

$$(2) \quad \lim_{x \to \infty} u(x) = 0$$

$$(3) \quad \left| u(x) - u_N \right| \leq v_N. \qquad 0 \leq x - n \leq 1.$$

the series $\sum_{N=\mu}^{\infty} v_N$ being a convergent series. A necessary and sufficient condition for the convergence of the series $\sum_{N=0}^{\infty} u_N$ is the convergence of the integral $\int_0^{\infty} u(x)dx$

Proof:

If we write [illegible]

we have [illegible]

[illegible]

since $\lim u(x) = 0$. $\lim \int u(x)\, dx = 0$. if $n \leq x \leq n+1$

Therefore

$$\lim \left| \int u(x)\, dx - S \right| \leq \sum v_N \text{, where } n \leq x \leq n + 1;$$

This inequality establishes the theorem, since by increasing [illegible] we can make $\sum v_N$ as small as we wish. Moreover the inequality provides limits on the value of the series if the series converges, or on its divergence, if it diverges." 32

32. Brink, R. W. "A New Sequence of Integral Tests" Annals of Mathematics 1919.

IV. RECENT DEVELOPMENTS.

In the study of infinite series, it is perhaps natural that studies were made first of convergent series and that little was done about divergent series until the close of the nineteenth century. Since that time interest in divergent series, the circle of convergence, the different types of summability of infinite series, convergence in mean, and Lebesgue integration, has developed rapidly. While most of these topics are beyond the scope of this paper, a brief discussion will be of interest.

The difficult problems in connection with the theory of convergence of Fourier's series cover many pages in mathematical books and magazines. The apparent hopelessness of obtaining simple necessary and sufficient conditions on f(x) for f(x) → f(x) is implied in the following theorem attributed to Hardy:

" A necessary and sufficient condition that f(x) → f(x) is that f(x) → f(x).".

R. P. Agnew of Cornell University has an interesting article "Convergence in Mean and Lebesgue Integration" in which he explains these two terms, tells of their merits, and explains their application to Fourier analysis. Quoting briefly from this article: "Let

$$u_1 + u_2 + u_3 + - - - -$$

be a series of constants or functions(real or complex) of a real variable x. It has been recognized for many years that life is too short to add up all the u's, and that some other methos of evaluating the series must be used. The classic method of convergence involves the sequence of partial sums defined by

$$f_n = u_1 + u_2 + \text{--------} u_n.$$

If there is an f such that $|f_n - f| \to 0$, as $n \to \infty$ then the series $u_1 + u_2 \text{----}$, and the sequence $\{f_n\}$ are said to converge to f, and we write $\lim f_N = f$. If there is no f such that the $\lim f_N = f$, the series is called divergent; this does not mean that the series has no value, but means rather that the method of convergence fails to assign a value to the series.

Methods of evaluation of series and of sequences (of which the method of convergence is one example) have come to be called methods of summability; --------

Another method of evaluating series and sequences of functions, differing from the methods of summability considered in these expository papers, is one known as the method of convergence in mean or mean convergence. We illustrate this method by an example.

Let

$I_1, I_2, I_3, \text{-----}$

denote in order the closed subintervals of the interval $(0, 1)$: $(0,\frac{1}{2}), (\frac{1}{2}, 1), (0,\frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3},1), (0,\frac{1}{4}),\text{------}$.

For each $n = 1, 2, 3, \cdots$ let the function $f_n(x)$ be defined by the formula

$$f_n(x) = 1 \quad \text{if } x \text{ lies in the interval } I_n,$$
$$\quad = 0 \text{ otherwise.}$$

This sequence $\{f_n(x)\}$ is the sequence of partial sums of the series $u_1 + u_2 + u_3 + \cdots$, where $u_1 = f_1$, $u_2 = f_2 - f_1$, $\cdots$. When x is fixed in the interval $0 \leq x \leq 1$, there is an infinite set of N's for which x lies in the interval I_n and $f_n(x) = 1$; this implies that $f_n(x) \to 1$ as n becomes infinite over a set containing some but not all positive integers. On the other hand there is an infinite set of n's for which x does not lie in the interval I_n and $f_n(x) = 0$; this implies that $f_n(x) \to 0$ as $n \to \infty$ over a set containing some but not all positive integers. Thus, for every x in the interval $0 \leq x \leq 1$, $\lim f_n(x)$ fails to exist; the series and sequence are therefore divergent, the method of convergence being inadequate to evalute the series.

We observe, even though $\{f(x)\}$ diverges, that when n is large, $f_n(x)$ is near 0 most of the time, that is, for all x except those belonging to the small interval I_n; and that as n increases the length of I_n tends to 0 and the graph of $f_n(x)$ tends in a sense to approximate more and more closely the graph of $f(x) = 0$. Accordingly we define $f(x)$ to be the function vanishing identically and investigate $|f_n(x) - f(x)|$, this being the absolute value of the difference between $f_n(x)$ and what seems to play the role of a limit of the sequence (in a sense which we have yet to make precise). We observe that

$$|f_n(x) - f(x)| = 1 \quad \text{when } x \text{ belongs to } I_N$$
$$= 0 \quad \text{otherwise.}$$

The question now arises how we can make precise the vague idea that when n is large $|f_n(x) - f(x)|$ is not very much different from 0 very much of the time; in other words, how can we assign a single number which tells in a sense how much the functions $f(x)$ and $f(x)$ differ over the interval $0 \leq x \leq 1$. The answer with which this lecture concerns itself lies in integration. It follows from $|f_N(x) - f(x)| = 1$ when x belongs to I, equal to 0 otherwise, that

$$\int |f_N(x) - f(x)| \, dx = |I_N|, \text{ where we use } |I_N|$$

to denote the length of the interval I. Since the integrand in this integral is always 0 or 1, its value is unchanged if we square it. Hence, if we define $\| f_N - f \|$ by the formula

$$\| f_n - f \| = \left(\int |f(x) - f(x)|^2 \, dx \right)^{1/2}$$

we have $\| f_n - f \| = |I_n|^{1/2}$.

and it follows that

$$\| f_n - f \| \to 0 \quad \text{as } n \to \infty.$$

We now introduce terminology which is associated with the computation of the preceding example.

If $\{f(x)\}$ is the sequence of partial sums of a series of functions

$$u_1(x) + u_2(x) + \cdots$$

defined over an interval (or more general set) A, and if f(x) is such that

$$\| f_n - f \| = \left\{ \int |f_n(x) - f(x)|^2 dx \right\}^{1/2} \to 0 \text{ as } n \to \infty$$

(the integral being taken over the set A), then the sequence and the series are said to converge in mean (over the set A) to f(x), and we write

$$\text{l. i. m. } f_n = f \qquad \text{and} \qquad f_n \xrightarrow{m} f .$$

As is the case for the sequence $f_n(x) = 1$ if x lies in the interval I_n, equal to 0 otherwise, it can happen that

$$f_n \xrightarrow{m} f$$

when there is no x for which $f_n(x) \to f(x)$. This shows that the method convergence in mean can serve to evaluate series and sequences when the method of convergence fails".[33]

Mr. Agnew then proceeds to define Fourier's coefficients, and Fourier's series of f(x). He shows clearly that in many cases convergence in mean of Fourier's series is much more complete and easier to apply than convergence. In fact

33. Agnew, R. P. "Convergence in Mean and Lebesgue Integration The American Mathematical Monthly Jan. 1937 Vol.44, page 4.

he shows that convergence in mean justifies calculations and processes which convergence cannot justify.

Dunham Jackson, University of Minnesota, has an article on "The Convergence of Fourier's Series" in which he outlines his plan for presenting an introductory course on Fourier's series to students who have had no further mathematics beyond the first course in calculus. His reason for teaching the course is the student interest in certain applications of Fourier's series as well as in its mathematical content. He gives the class " genuine appreciation of some of the properties of convergence, even the most elementary of which are so characteristic of the type of series in question, and have had so profound an influence on the course of modern mathematical defelopment."

In this article the theorems of convergence are clearly explained and applied to series of the Fourier or trigonometric type. He considers series which are continuous functions, or which are continuous except for a finite number of finite jumps in a period. The theorem on uniform convergence in this article is of particular interest. Quoting briefly:

"Let f(t) be continuous everywhere, and let it be supposed that any period interval can be divided into a finite number of subintervals throughout each of which f(t) has continuous first and second derivatives, but that that the derivatives may not be continuous in passing froone subinterval to the next. The graph of f(t) over period is then made up of a finite number of pieces, each having continuous curvature, but there may be corners (or, as an admissible alternative, abrupt changes of curvature without change of direction) at the points where two pieces come together. Let the successive points of division marking the subintervals of the period from $-\pi$ to π be x_1, x_2, ---- x_{p-1}, and for uniformity of notation let $x_0 = -\pi$, and $x_p = \pi$. The derivatives may have different values from the right and from the left at these points, but the function f(t) itself has a determinate value at each of them. For each value of i from 0 to p - 1,

$$\int_{x_i}^{x_{i+1}} f(t) \cos kt \, dt = \left| \frac{1}{k} f(t) \sin kt \right|_{x_i}^{x_{i+1}} - \frac{1}{k} \int_{x_i}^{x_{i+1}} f'(t) \sin kt \, dt$$

when equations of this form are written for all p subintervals and added, the terms (1/k f(x) sin kx) cancel, each occuring once with a plus sign and once with a minus sign, and

$$\int_{-\pi}^{\pi} f(t) \cos kt \, dt = -\frac{1}{k} \sum \int_{x_i}^{x_{i+1}} f'(t) \sin kt \, dt.$$

Another integration by parts gives

$$\int_{x_i}^{x_{i+1}} f'(t) \sin kt \, dt = \left| -\frac{1}{k} f'(t) \cos kt \right|_{x_i}^{x_{i+1}} + \frac{1}{k} \int_{x_i}^{x_{i+1}} f''(t) \cos kt \, dt$$

when these expressions are added for the various intervals the terms outside the signs of integration do not cancel, since f'(x) at the left-hand end of one interval does not in general mean the same thing as $f'(x)$ at the right-hand end of the preceding interval. But under the hypotheses $f'(t)$ and $f''(t)$ remain finite everywhere, in spite of their discontinuities; and if M and M_1 are numbers such that $|f''(t)| \leq M$, $|f'(t)| \leq M_1$ for all values of t, then

$$\left| f'(x_{i}) \cos kx_{i} - f'(x_{i+1}) \cos kx_{i+1} \right| \leq 2M_1$$

in each case, and

$$\left| \int_{x_i}^{x_{i+1}} f''(t) \cos kt \, dt \right| \leq M (x_{i+1} - x_i).$$

Hence

$$\left| \pi a_k \right| = \left| \int_{-\pi}^{\pi} f(t) \cos kt \, dt \right| \leq 2pM_1 / k^2 + 2\pi M / k^2.$$

A similar calculation applies to b_k.

This result may be stated as follows:

Theorem 1. If $f(x)$ is a function which has a continuous second derivative except for a finite number of corners in a period, and if a_k, b_k are the coefficients in its Fourier's series, there is a number C, independent of k, such that $|a_k| \leq C/k^2$, $|b_k| \leq C/k^2$. 35.

The convergence of the series is an immediate corollary.

35. Note: Fourier series for a given function has the form
$a_0/2 + a_1 \cos x + a_2 \cos 2x + \cdots$
$+ b_1 \sin x + b_2 \sin 2x + \cdots$
in which the coefficients are given by the formulas
$a_k = 1/\pi \int_{-\pi}^{\pi} f(t) \cos kt \, dt$; $b_k = 1/\pi \int_{-\pi}^{\pi} f(t) \sin kt \, dt$.

"-------

Let f(x) be a function for which the conditions of Theorem 1 are satisfied. By the conclusion of that theorem, together with the fact that the sum of the series is f(x)

$$\left| f(x) - S_n(x) \right| = \left| \sum_{k=n+1}^{\infty} a_k \cos kx + b_k \sin kx \right| \leq 2C/k^2.$$

It is clear that

$$1/k^2 = \int_{k-1}^{k} du/k^2 < \int_{k-1}^{k} du/u^2 \quad \text{since } u < k$$

throughout the interior of the interval of integration and hence

$$\sum_{k=n+1}^{\infty} 1/k^2 < \sum_{k=n+1}^{\infty} \int_{k-1}^{k} du/u^2 = \int_{n}^{\infty} du/u^2 = 1/n \quad .$$

So that

$$\left| f(x) - S_n(x) \right| \leq 2C/n \qquad \text{for all values of } x.$$

The fact that the remainder does not exceed a quantity which is independent of x and which approaches 0 as n becomes infinite is expressed by saying that the series is uniformly convergent . This conclusion may be recorded as Theorem IV.

If f(x) is a function which has a continuous second derivative except for a finite number of corners in a period, its Fourier's series converges uniformly to the value f(x) for all values of x."[25]

25. Jackson, Dunham "Convergence of Fourier Series" The American Mathematical Monthly Feb. 1934 page 67-84.

BIBLIOGRAPHY.

Agnew R. P. "Convergence in Mean and Lebesgue Integration"
The American Mathematical Monthly Jan. 1937, page 4.

Agnew, R. P. "Comparison of Products of Methods Of Summability"
Transactions of the American Mathematical Society Vol. 41, May 1939, page 327.

Ball, W. W. "A Short History of Mathematics"
MacMillan Co. New York. 1893.

Brink, R. W. " A New Integral Test for the Convergence and Divergence of Infinite Series"
Transactions of the American Mathematical Society Vol. 19. 1918. page 186-204.

Brink, R. W. " A New Sequence of Integral Tests for the Convergence and Divergence of Infinite Series."
Annals of Mathematics. Vol. 21. 1919 page 39-60.

Bromwich, T. J. I. "An Introduction to the Theory of Infinite Series"
MacMillan Co. Limited, London, England.
Second Edition 1926.

Burkhart, Heinrich "Theory of Functions of a Complex Variable"
D. C. Heath Co. Boston, Mass. 1913.

Cajori, F. "A History of Mathematics" Revised.
MacMillan Co. 1924. page 38, 373-376.

"The Name Divergent Series".
Bulletin of the American Mathematical Society.
Vol. 29. 1923. page 55.

Chaundy, T. A., and Phillips, E. "The Convergence of Sequences Defined by Quadratic Recurrence Formulae".
Quarterly Journal of Mathematics, Oxford Series.
Vol. 70. No. 25. March 1936. page 74-80.

Chow, H. C. "Note on the Absolute Cesaro Summability of Power Series.
Proceedings of the London Mathematical Society.
Dec. 1937. page 484.

Davenport. "On Some Infinite Series Involving Arithmetic Functions"
Quarterly Journal of Mathematics. Oxford Series.
Vol. 8. No. 29. March 1937. page 8-13.
Vol. 8 No. 32. Dec. 1937. page 313-320.

Bicke, [illegible] "Note on the Integration of Uniformly Convergent Series through an Infinite Integral"
The Bulletin of the American Mathematical Society. Vol. 3. 1896-1897

Ford, W. B. "Dini's Method of Showing the Convergence of Fourier's Series".
The Bulletin of the American Mathematical Society
Vol. 7 1900-1901. page 227.

Goursat-Hedrick, "Mathematical Analysis" Vol. 1.
Ginn and Co. Cambridge, Mass. 1904.

Hartman, P. "On a Class of Arithmetic Fourier's Series".
The American Mathematical Journal. Vol. 4. No. 1
Jan. 1938. page 6[illegible]-74.

Hille, E. "On the Absolute Convergence of Polynomial Series"
The American Mathematical Monthly. April 1938.
Vol. 45. No. 4. page 220-226.

Hobson, E. W. "On a General Convergence Theorem and the Theory of the Representation of a Function by a Series of Normal Functions."
Cambridge University Press. 1907. page 453-511.

Jackson, Dunham "The Convergence of Fourier Series".
The American Mathematical Monthly. Feb. 1934
page 67-84.

James, Glenn "On the Theory of Summability"
The Annals of Mathematics. Vol. 21. 1919-1920
page 120-127.

Lennes, N. J. "Differential and Integral Calculus"
Harper and Brothers. New York. 1921.
page 254-279, 451, 303-319.

March and Wolff, "Calculus"
McGraw-Hill Book Co. New York 1917.

Moore, C. N. "Summable Series and Convergent Factors".
The American Mathematical Society. 1938.

Picard, Emile "On the Development of Mathematical Analysis and Its Relation to Certain Other Sciences."
Bulletin of the American Mathematical Society
Vol. 11 1907.

Pierpont, James "The Theory of Functions of Real Variables"
Ginn Co. New York. Vol 11. 1918.

Rajagopal, C. T. "On an Integral Test of R. W. Brink for Convergence of Series."
Bulletin of the American Mathematical Society.
June 1937. Vol. 43. No. 6.

Randels, W. C. "On the Summability of Fourier Series."
Transactions of the American Mathematical Society
Jan. 1937. page 24.

Roman, Irwin "An Euler Summation Formula"
American Mathematical Society Monthly. Jan. 1936
page 9-21.

Smith, D. E. "History of Modern Mathematics"
Ginn Co. New York.

"History of Mathematics" Topical Survey.
Ginn Co. 1925. page [illegible]7.

"Source Book in Mathematics"
Ginn Co. page [illegible], [illegible]-[illegible]1.
Whittaker and Watson "Modern Analysis" Third Edition.
Cambridge University Press, Cambridge, Mass. 1920.

Young, W. H. " On a Test for Non-Uniform Convergence
Bulletin of the American Mathematical Society.
Vol. 1[illegible]. [illegible]