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Greedoid invariants and the greedoid Tutte polynomial

by

Chris Anne Clouse

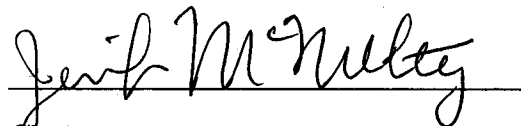
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the requirements for the degree of

Doctor of Philosophy

The University of Montana

May 2004

Approved by:



Chairperson



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Clouse, Chris May 2004

Mathematical Sciences

Greedoids and Invariant Theory

Advisor: Jennifer McNulty



An extension of invariant theory for greedoids is developed, one that parallels matroid invariant theory. We provide a summary of existing greedoid invariants including the characteristic polynomial and beta invariant. We define a new greedoid Tutte polynomial, g -invariant, generalized g -invariant, group invariant and Tutte invariant. These invariants are shown to be evaluations of the greedoid Tutte polynomial. Examples and applications of each class are given including applications specific to antimatroids.

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1 Overview

During a course on matroid theory, we were introduced to the close relationship between matroids and the greedy algorithm. The question arose: “If, for matroids, the greedy algorithm always produces an optimal solution, then why are there greedoids”? (The answer to this question is found in Appendix E.) In the course of researching this question and exploring combinatorial optimization, we were exposed to other areas of matroid and greedoid theory; in particular, invariant theory and the Tutte polynomial.

Matroid invariant theory is well-developed and unified. A fundamental result characterizes much of the theory: many key classes of matroid invariants are evaluations of a single, universal polynomial. In contrast, greedoid invariant theory was piecemeal and less developed. Several authors had developed greedoid counterparts to matroid invariants. But, specific classes of greedoid invariants were not identified nor were they characterized in terms of a single invariant. In this dissertation, we develop a framework for greedoid invariant theory that is modelled after the matroid case. We introduce a greedoid Tutte polynomial fashioned after its matroid counterpart. Two key theorems, Theorem 6.2.2 and Theorem 6.3.1, characterize two major classes of greedoid invariants in terms of the greedoid Tutte polynomial.

We begin Chapter 2 with a synopsis of basic matroid invariant theory, then, in Chapter 3, introduce the reader to greedoid facts and structures necessary to support greedoid invariant theory. We include contrasts between matroids and greedoids and

highlight a special class of greedoids called antimatroids.

Our development of greedoid invariant theory begins in Chapter 5. Here, we define three fundamental types of greedoid invariants. The following chapter characterizes each of these in terms of the greedoid Tutte polynomial. In Chapters 7–9, we turn our attention to applications of greedoid invariants moving from basic invariants (e.g., the number of feasible sets) to more advanced applications (e.g., broken circuits in antimatroids).

We conclude with some areas for future research. In addition to those topics mentioned earlier, the five appendices contain summaries of key features of greedoid subclasses and a summary of the relationship between the various characterizations of greedoids.

2 Overview of Matroid Theory

Greedoids were originally proposed as a generalization of matroids and thus a study of greedoids requires some familiarity with basic matroid theory. Furthermore, in this dissertation we focus on invariant theory of greedoids that parallels that for matroids. Thus, this introductory chapter has two goals: (1) to provide a brief summary of basic matroid definitions and (2) to provide some details about matroid invariant theory. Throughout this chapter, we use the terminology in [19] to which the reader may refer for more information. The material on matroid invariant theory is based on [4] and [6].

2.1 Definition of a matroid

Matroids can be defined in several equivalent ways. We use the independent set definition because it is closely related to the definition of a greedoid.

Definition 2.1.1. *A **matroid**, M , is an ordered pair, (E, \mathcal{I}) , in which E is a finite set and \mathcal{I} is a collection of subsets of E that satisfies the following three axioms:*

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $X \in \mathcal{I}$, and $Y \subseteq X$, then $Y \in \mathcal{I}$,

(I3) if X and Y are in \mathcal{I} with $|Y| < |X|$, then there is an element x of $X - Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

Remark 2.1.2. We refer to (I2) to as the *subclusive axiom*, and to (I3) as the *independence augmentation axiom*.

The sets in \mathcal{I} are called **independent** sets. If the ground set, E , consists of a single element, e , the matroid is denoted by $M(e)$.

2.2 Examples

Matroids were introduced by Whitney in 1935 as an abstraction of linear independence and the cycle structure of graphs. Examples of matroids are found throughout graph theory and combinatorics; we provide two examples of matroids that highlight the close relationship between matrices and graphs.

Whitney introduced the name *matroid* because an important prototype follows from matrices in a natural way.

Example 2.2.1. Let E be the set of column labels of an $n \times m$ matrix A over a field F , and let \mathcal{I} be the set of subsets, X , of E for which the multiset of columns labelled by X is linearly independent in the vector space $V(m, F)$. Then (E, \mathcal{I}) is a matroid.

The matroid in Example 2.2.1 is called a **vector matroid** or a **linear matroid**.

A second key class of matroids is the class of graphic matroids.

Example 2.2.2. Let $G = (V, E)$ be a graph and let \mathcal{I} be the collection of subsets of E that contain no cycles of G . Then (E, \mathcal{I}) is a matroid.

The matroid in Example 2.2.2 is a **cycle matroid** or a **polygon matroid**. Any matroid that is isomorphic to the cycle matroid of a graph is called **graphic**.

2.3 Basic matroid concepts

Let $M = (E, \mathcal{I})$ be a matroid. A **basis** of M is a maximal independent set. As a consequence of (I3), all bases, $B \subseteq E$, have the same cardinality. This cardinality is called the **rank** of B and is denoted by $r(B)$ or $r(M)$. Similarly, for all $X \subseteq E$, $r(X)$ is defined as the largest independent set that is contained in X .

There are three fundamental matroid constructions: restriction, deletion and contraction.

Definition 2.3.1. Let $M = (E, \mathcal{I})$ be a matroid and let $X \subseteq E$.

- Let $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$. Then, $(X, \mathcal{I}|X)$ is a matroid called the **restriction of M to X** , denoted $M|X$.
- Let $\mathcal{I} - X = \{I \subseteq E - X : I \in \mathcal{I}\}$. Then, $(E - X, \mathcal{I} - X)$ is a matroid called the **deletion of X from M** , denoted $M - X$.
- Suppose B_T is a basis for $M|T$. Let $\mathcal{I}/X = \{I \subseteq E - X : I \cup B_T \in \mathcal{I}\}$. Then, $(E - X, \mathcal{I}/X)$ is a matroid called the **contraction of X from M** , denoted M/X .

We can perform a sequence of deletions and contractions on a matroid, written in the form $M - X/Y$, for some pair of disjoint sets $X, Y \subseteq E$. The order in which deletion and contraction are performed does not matter. Matroids formed this way are called **minors** of M .

For a matroid, $M = (E, \mathcal{I})$, there are two types of *special* elements that may be contained in E . An **isthmus** is an element that is contained in every basis of M and a **loop** is an element that is in no basis of M . Loops and isthmuses play an important role in matroid invariant theory.

2.4 Matroid invariants

We now focus our attention on matroid invariants. We say two matroids M and N are **isomorphic**, written $N \cong M$, if there is a bijection ψ from $E(N)$ to $E(M)$ such that, for all $X \subseteq E(N)$, $\psi(X) \in \mathcal{I}(M)$ if and only if $X \in \mathcal{I}(N)$. A commutative ring-valued function f on the class of all matroids is an **isomorphism invariant** if $f(M) = f(N)$ whenever $M \cong N$. From here, we build up other invariants.

Definition 2.4.1. *Let \mathcal{K} be the class of matroids closed under isomorphism and the taking of minors. Let f be a function on \mathcal{K} that satisfies the following three axioms:*

- (1) *f is an isomorphism invariant,*
- (2) *if e is neither a loop nor an isthmus, then $f(M) = f(M - e) + f(M/e)$,*
- (3) *if e is a loop or an isthmus, then $f(M) = f(M(e))f(M - e)$.*

*Then, f is a **Tutte–Grothendieck** or **T–G invariant** for matroids.*

In [6], it is shown that the number of independent sets and the number of bases of a matroid are examples of T–G invariants. There are other invariants that do not satisfy axiom (2) but do satisfy a generalized deletion–contraction formula.

Definition 2.4.2. Let \mathcal{K} be a class of matroids closed under isomorphism and the taking of minors. Let f be a function on \mathcal{K} that satisfies the following three axioms:

(1) f is an isomorphism invariant,

(2') for some non-zero constants (in the commutative ring) σ and τ , $f(M) = \sigma f(M - e) + \tau f(M/e)$ if e is neither a loop nor an isthmus,

(3) if e is a loop or an isthmus, then $f(M) = f(M(e))f(M - e)$.

Then, f is a **generalized T-G invariant** for matroids.

The Möbius function, $\mu(M)$, and characteristic polynomial, $p(M; \lambda)$, of a matroid are generalized T-G invariants.

2.5 The Tutte polynomial

The fundamental result from matroid invariant theory can be stated as follows: there is a unique, universal T-G invariant that characterizes all other T-G invariants. It is known as the Tutte polynomial.

Definition 2.5.1. Let M be a matroid on ground set E . The **Tutte polynomial** of M , $t(M; x, y)$, is the integer coefficient, two variable polynomial defined by

$$t(M; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

Remark 2.5.2. The **corank-nullity** or **rank generating polynomial**, $S(M; x, y)$, developed by Whitney, can be recovered from the Tutte polynomial in the following

way:

$$S(M; x, y) = t(M; x + 1, y + 1).$$

If we write $S(M; x, y) = \sum_i \sum_j a_{ij} x^i y^j$, then a_{ij} is the number of submatroids having corank, i , and nullity, j . This accounts for the name corank-nullity polynomial. The Tutte polynomial, $t(M; x, y)$, is the foundation of matroid invariants. It can be calculated directly from the definition, or, it can be determined recursively.

Proposition 2.5.3 ([6], Theorem 6.2.2). *There is a unique function t from the set of isomorphism classes of matroids into the polynomial ring $\mathbb{Z}[x, y]$ having the following properties:*

(i) $t(I; x, y) = x$ and $t(L; x, y) = y$, where I is an isthmus and L is a loop,

(ii) (Deletion-contraction) if e is an element of the matroid, M , and e is neither a loop nor an isthmus, then

$$t(M; x, y) = t(M - e; x, y) + t(M/e; x, y),$$

(iii) if e is a loop or an isthmus of the matroid M , then

$$t(M; x, y) = t(M(e); x, y)t(M - e; x, y).$$

Furthermore, let R be a commutative ring and suppose that f is any function from the set of isomorphism classes of non-empty matroids into R . If f is a T - G invariant, then for all matroids, M ,

$$f(M) = t(M; x, y) \big|_{x=f(I), y=f(L)} = t(M; f(I), f(L)).$$

Remark 2.5.4. *The special element, I , is an element of the isthmus isomorphism class of matroids and L is an element of the loop isomorphism class of matroids. For the sake of brevity, we refer to I as an isthmus and L as a loop.*

There are many T-G invariants. As mentioned earlier, two important examples are the number of independent sets and the number of bases of a matroid. For a matroid, M , let $b(M)$ be the number of bases of M and $i(M)$ be the number of independent sets. It is shown in Proposition 6.2.11 of [6] that $b(M) = t(M; 1, 1)$ and $i(M) = t(M; 2, 1)$. Generalized T-G invariants can be characterized in terms of the Tutte polynomial too.

Corollary 2.5.5 ([6], Corollary 6.2.6). *Let σ and τ be non-zero elements of a field, F . There is a unique function, t' , from the set of isomorphism classes of matroids into the polynomial ring $\mathbb{F}[x, y]$ having the following properties:*

- (i) $t'(I; x, y) = x$ and $t'(L; x, y) = y$, for I an isthmus and L a loop,
- (ii) (Deletion-contraction) if e is an element of the matroid, M , and e is neither a loop nor an isthmus, then

$$t'(M; x, y) = \sigma t'(M - e; x, y) + \tau t'(M/e; x, y),$$

- (iii) if e is a loop or an isthmus of the matroid M , then

$$t'(M; x, y) = t'(M(e); x, y)t'(M - e; x, y).$$

Furthermore, this function t' is given by

$$t'(M) = \sigma^{|E|-r(E)} \tau^{r(E)} t\left(M; \frac{x}{\tau}, \frac{y}{\sigma}\right).$$

The Möbius invariant, $\mu(M)$, and the characteristic polynomial, $p(M; \lambda)$, of a matroid are discussed in detail in [22]. The Möbius invariant is related to the combinatorial Möbius function and the characteristic polynomial is the matroid counterpart of the chromatic polynomial of a graph. As an application of Corollary 2.5.5, it can be shown that each of these is a generalized T–G invariant given by $\mu(M) = (-1)^{r(M)} t(M; 1, 0)$ and $p(M; \lambda) = (-1)^{r(M)} t(M; 1 - \lambda, 0)$.

There are still other matroid invariants that are neither T–G nor generalized T–G invariants. The **beta invariant**, $\beta(M)$, can be used to determine if a matroid is connected. The beta invariant is an example of an invariant that satisfies axiom (2) (additive recursion) but not axiom (3) (multiplicative recursion). However, such invariants, known as **(T–G) group invariants**, can still be characterized in terms of the Tutte polynomial.

Proposition 2.5.6 ([6], **Proposition 6.2.8**). *Let $M = (E, \mathcal{I})$ be a non-empty matroid. Let $U_{i,i}$ be the matroid consisting of i isthmuses and $U_{0,j}$ be the matroid consisting of j loops. If A is an Abelian group, then there is a unique function, g , from the set of isomorphism classes of non-empty matroids into A such that*

$$(i) \quad g(M) = g(M - e) + g(M/e) \text{ if } e \text{ is neither a loop nor an isthmus, and}$$

$$(ii) \quad g(U_{i,i} \oplus U_{0,j}) = \alpha_{ij} \text{ for all } i \text{ and } j \text{ such that } i + j > 0.$$

Moreover, if $t(M; x, y) = \sum_i \sum_j b_{ij} x^i y^j$, then $g(M) = \sum_i \sum_j b_{ij} \alpha_{ij}$.

By Proposition 6.2.12 of [6], $\beta(M)$ is a group invariant. Using the notation of Proposition 2.5.6, $\beta(M) = b_{10}$.

Finally, we state which matroid invariants can be characterized by the Tutte polynomial.

Definition 2.5.7. *Let M and N be two matroids and let f be a function from the isomorphism classes of matroids into a set Ω . Then, f is a **Tutte invariant** if $f(M) = f(N)$ whenever M and N have the same Tutte polynomial.*

All T-G, generalized T-G, and group invariants are Tutte invariants. Other examples of Tutte invariants include the rank and nullity (difference between the size and rank) of a matroid. There is an abundance of applications of Tutte invariants in graph theory, coding theory, and network theory. See [6] for a study that focuses on the application of matroid invariants. A detailed development of matroid invariant theory is found in [4].

3 Introduction to Greedoids

Next, our focus switches to greedoids. Greedoids come about by loosening the subclusive axiom of matroids. Because of this close relationship, many terms, such as rank, bases and contraction, are shared by greedoids and matroids. However, there are sometimes differences in the definitions of shared terms. In this chapter, we present the foundations of greedoid theory including key facts and examples as well as a comparison to matroids.

3.1 Basic definitions

Definition 3.1.1. A *greedoid*, G , is a pair, (E, \mathcal{F}) , in which E is a finite set and $\mathcal{F} \subseteq 2^E$ is a set system that satisfies three axioms:

(G1) $\emptyset \in \mathcal{F}$,

(G2) If $\emptyset \neq X \in \mathcal{F}$, then there is an $x \in X$ such that $X - \{x\} \in \mathcal{F}$,

(G3) For all $X, Y \in \mathcal{F}$ with $|X| > |Y|$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

The elements of \mathcal{F} are called **feasible** sets and sets not in \mathcal{F} are called **infeasible**. When $\{e\} \in \mathcal{F}$, we call e a **feasible element**. A greedoid is **full** if $E \in \mathcal{F}$. For $X \in \mathcal{F}$, the **set of continuations of X** , $\Gamma(X)$, is the set of those elements that can be added to X to form a larger feasible set. Thus,

$$\Gamma(X) = \{a \in E - X \mid X \cup \{a\} \in \mathcal{F}\}.$$

Maximal feasible sets are known as **bases**. Any $X \subseteq E$ that contains a basis is a **spanning** set. Definition 3.1.1 is often referred to as the *unordered* definition of a greedoid, so called because it is written in terms of sets. Appendix E contains a second definition, in terms of languages, which is equivalent.

The reader will notice the similarities between Definition 3.1.1 and the independent set definition of a matroid, Definition 2.1.1. (G3) is the usual independence augmentation axiom for matroids, (I3). Set systems that obey (G2) are known as **accessible**. Accessibility is a relaxation of the subclusive axiom (subset property) of matroids, (I2). It tells us that not all subsets of feasible sets need be feasible; only one such subset must exist. It is still a powerful requirement. It allows one to peel away elements successively from any feasible set until the empty set is reached. Thus, for every $X \in \mathcal{F}$, there exist X_i such that $\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq X_k = X$, where $|X_i| = i$. This chain, however, may not be unique.

One may wonder why this relaxation of matroids was developed. To answer this question, we will review how greedoids came about.

3.2 Greedoids and algorithms

Greedoids trace their roots to the construction of set systems using various greedy-type algorithms. Consider the problem of constructing a minimum spanning tree of an n edge-labelled connected edge-weighted graph. Two well-known techniques for solving this problem are Prim's and Kruskal's algorithms. Kruskal's algorithm

begins with an empty forest and adds edges by choosing the cheapest edge such that no cycles are formed. It stops when the forest is a spanning tree. Prim's algorithm begins at a designated starting node (root) and selects an edge adjacent to it that is as inexpensive as possible. An edge is added to this rooted tree so as to build a larger rooted tree as cheaply as possible, stopping when the tree is spanning.

Observe that, in each case, the collection of sequences of edges forms a set system that satisfy (G1), (G2), and (G3) and thus are greedoids. There is, however, a key difference between the two resulting set systems. We could permute the edges of the spanning forests generated by Kruskal's algorithm and the result would still be "feasible". In this system, every subset of a feasible set is again feasible (i.e., satisfies Axiom (M2)). Hence, this collection of sets forms a matroid. The order in which elements are added in executing Prim's algorithm *does* matter. Not every subgraph of the rooted trees are themselves rooted trees; the set system is not a matroid. Greedoids were invented to accommodate such set systems.

See Appendix E.2 for details about Prim's and Kruskal's algorithms, greedoids in terms of languages, and more information about greedoid optimization.

3.3 Branching greedoids: a concrete example

In order to demonstrate structural concepts of a greedoid, we define an important class of greedoids, branching greedoids. We refer to it throughout the remainder of this chapter.

A **rooted directed graph** or **rooted digraph**, $D = (V, E, r)$, is a finite non-empty set, V , of vertices with a distinguished vertex, r , called the **root**, and a multiset, E , of edges that are ordered pairs of elements of V . An edge, $e = (u, v)$, has u as its initial vertex and v as its terminal vertex. A **directed path** of D is an alternating sequence, $v_1 e_1 v_2 e_2 \dots v_{k-1} e_k v_k$, where $e_i \in E$, $v_i \in V$, $v_i \neq v_j$ for $i \neq j$, and $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq k$. A **rooted directed path** is a directed path in which $v_1 = r$. A **rooted branching**, F of D , is the edge set of a collection of rooted directed paths of D such that, if v is a vertex of F , then there is a unique directed path from r to v . A rooted branching is also called a **rooted arborescence**; it can be thought of as a rooted tree in which the edges are directed away from the root. A branching **covers** a vertex v if it contains an edge with endpoint v .

Proposition 3.3.1. *Given a rooted digraph $D = (V, E, r)$, let \mathcal{F} be the collection of rooted branchings of D . Then, (E, \mathcal{F}) is a greedoid called the **directed branching greedoid**.*

Proof. We will show that \mathcal{F} satisfies (G1), (G2), and (G3). The empty rooted branching is in \mathcal{F} ; therefore, (G1) is satisfied. Every non-empty rooted branching in \mathcal{F} has a terminal leaf that can be pruned. When it is removed, the resulting rooted branching is still a rooted branching. Thus, (G2) is satisfied. For $A \in \mathcal{F}$, $|A|$ is the number of vertices of $V - r$ that are covered by edges of A . Suppose X, Y are rooted branchings with $|X| > |Y|$. There is at least one vertex that is covered by the edges of X but not the edges of Y . Let e be an edge of X such that one endpoint is covered by X

but not by Y . Then, $Y \cup \{e\} \in \mathcal{F}$. \square

Remark 3.3.2. *There could be more than one edge of X with one endpoint covered by X but not by Y . If e_i and e_j are two such edges, then both $Y \cup \{e_i\} \in \mathcal{F}$ and $Y \cup \{e_j\} \in \mathcal{F}$.*

An undirected branching greedoid can be defined in a similar way. Next, we give a specific example of a directed branching greedoid. We will refer to this example throughout the chapter to help illustrate fundamental greedoid features and constructs.

Example 3.3.3 (A directed branching greedoid). *Given $D = (V, E, r)$ as shown in Figure 1, the set of feasible sets of the corresponding directed branching greedoid is*

$$\mathcal{F} = \{\emptyset, \{a\}, \{f\}, \{g\}, \{a, b\}, \{a, f\}, \{a, g\}, \{f, g\}, \\ \{a, b, e\}, \{a, b, f\}, \{a, b, g\}, \{a, f, g\}, \{a, b, e, g\}, \{a, b, f, g\}\}.$$

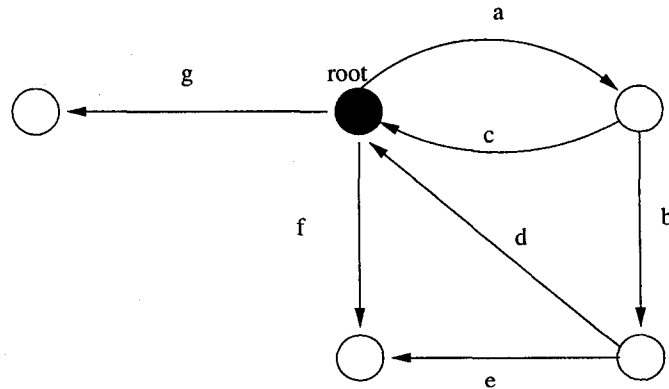


Figure 1: A directed branching greedoid

The feasible sets of a greedoid $G = (E, \mathcal{F})$, ordered by inclusion, form a poset $P(G) = (\mathcal{F}, \subseteq)$. The poset is an organized way to represent a greedoid and can assist when performing operations and calculations. When we present a poset, we denote the set $\{a, b\}$ by ab . Figure 2 shows the poset of feasible sets for the directed branching greedoid of Example 3.3.3.

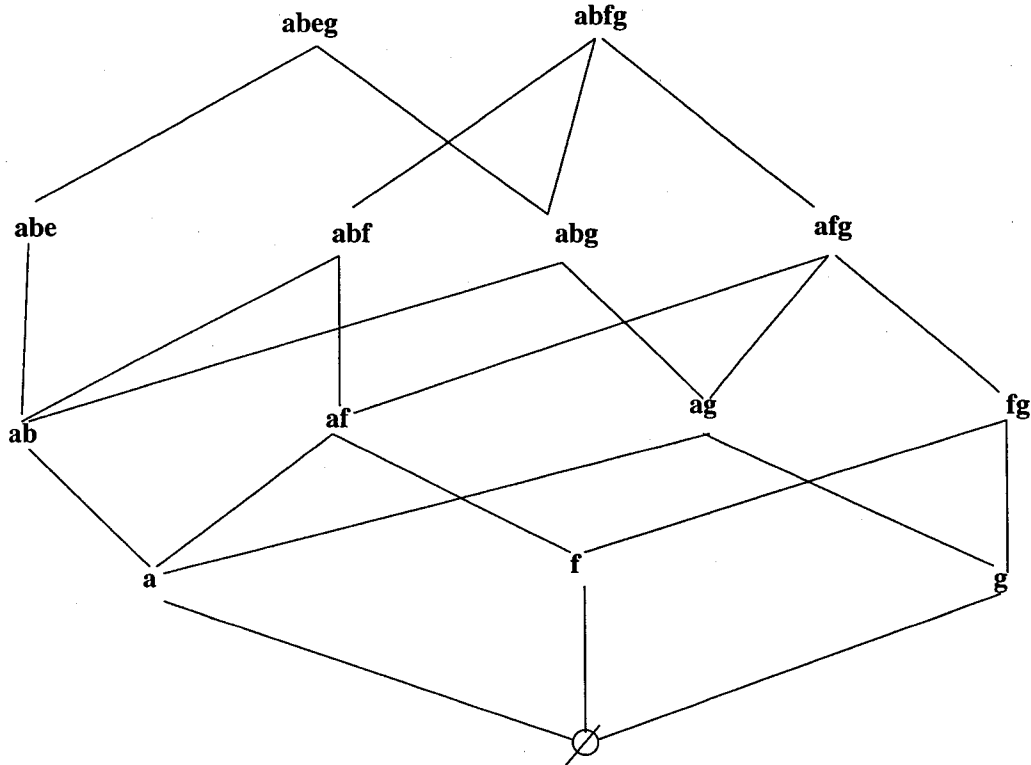


Figure 2: The poset of feasible sets of a directed branching greedoid

Up to this point, we have defined greedoids in terms of feasible sets and looked at a concrete example. Now, we wish to explore several equivalent ways to define greedoids. There are two key reasons for doing so: (1) different definitions lend themselves to better understanding different applications and, (2) we will better understand the

underlying theory and structure by examining the relationships among the definitions. The material in the next several sections is adapted from a variety of sources including [2], [11], and [18].

3.4 Rank

Let $G = (E, \mathcal{F})$ be a greedoid. The **rank** of any subset, $S \subseteq E$, is the size of the largest feasible set contained in S . That is, $r(S) = \max\{|F| : F \subseteq S, F \in \mathcal{F}\}$. Thus, a set is feasible if its rank is equal to its cardinality. A **basis** of G is a maximal feasible set of G . Axiom (G3) guarantees that all bases have the same cardinality. Thus, the rank, $r(E)$, of a greedoid (sometimes written as $r(G)$), is equal to the rank of a basis. For $S \subseteq E$, S is **spanning** if $r(S) = r(E)$. The directed branching greedoid in Example 3.3.3 has two bases: $\{a, b, f, g\}$, and $\{a, b, e, g\}$ each of rank four. A spanning set is any arc set containing $\{a, b, f, g\}$, or $\{a, b, e, g\}$.

We can characterize a greedoid using the rank function.

Proposition 3.4.1 ([18], Chapter V, Theorem 1.1). *Let E be a finite set. A function $r : E \rightarrow \mathbb{Z}$ is a rank function of a greedoid if and only if for $X, Y \subseteq E$, and $x, y \in E$,*

$$(R1) \quad r(\emptyset) = 0,$$

$$(R2) \quad r(X) \leq |X|,$$

$$(R3) \quad X \subseteq Y \text{ implies } r(X) \leq r(Y),$$

(R4) $r(X) = r(X \cup \{x\}) = r(X \cup \{y\})$ implies $r(X) = r(X \cup \{x\} \cup \{y\})$.

Axiom (R4) is known as local submodularity. A rank function satisfies (R4) and the unit increase property $r(X \cup \{x\}) \leq r(X) + 1$ if and only if it is submodular. The rank function of a matroid is submodular but, in general, that of a greedoid is not. This is an important difference between matroids and greedoids. Removing a single element from a feasible set of a greedoid can greatly effect its rank. Referring back to Example 3.3.3, notice that $r(\{a, b, e\}) = 3$ while $r(\{b, e\}) = 0$.

Axiom (G2) guarantees that every non-empty feasible set has at least one element that, when deleted, will cause the rank to decrease by exactly one. For the feasible set $\{a, b, e\}$ the only element that has the property is e . That is because e is a pendant arc that can be pruned from the rooted branching $\{a, b, e\}$.

3.5 Closure

We can use the rank function of a greedoid to define a (rank) closure operator σ for greedoids.

Definition 3.5.1. *Let E be a finite set. For $X \subseteq E$ define the **closure operator for greedoids**, $\sigma(X)$, by*

$$\sigma(X) = \{x \in E : r(X \cup \{x\}) = r(X)\}.$$

The usual definition for a closure operator is given in the next definition.

Definition 3.5.2. A *closure operator*, cl , is a function from 2^E to 2^E that satisfies three axioms:

$$(Cl1) \quad X \subseteq cl(X),$$

$$(Cl2) \quad X \subseteq Y \text{ implies } cl(X) \subseteq cl(Y),$$

$$(Cl3) \quad cl(cl(X)) = cl(X).$$

The second of these axioms, (Cl2), is called monotonicity. Unfortunately, just as the rank function of a greedoid does not necessarily satisfy the unit-increase property, the closure operator for greedoids is not necessarily monotone. Refer to Example 3.3.3 to see that $\sigma(\{a\}) = \{a, c, d, e\}$ and $\sigma(\{a, b\}) = \{a, b, c, d\}$. Thus, $\{a\} \subset \{a, b\}$ but $\sigma(\{a\}) \not\subseteq \sigma(\{a, b\})$.

Hence, the greedoid closure operator, σ , is not a closure operator in the usual topological sense. This is certainly an undesirable property and helps illustrate the fact that greedoids are not nearly as well behaved as matroids. Nevertheless, greedoids can be defined axiomatically in terms of the closure operator.

Proposition 3.5.3 ([2], Theorem 8.4.2). *Let E be a finite set. A function $\sigma : 2^E \rightarrow 2^E$ is a closure operator of a greedoid if and only if the following three properties are satisfied for all $X, Y \subseteq E$ and $v, w \in E$.*

$$(\sigma1) \quad X \subseteq \sigma(X).$$

$$(\sigma2) \quad \text{If } X \subseteq Y \subseteq \sigma(X) \text{ then } \sigma(X) = \sigma(Y).$$

($\sigma 3$) Suppose there is a $v \notin X$ such that, for all $z \in X \cup \{v\}$, the following holds:
 $z \notin \sigma(X \cup \{v\} - z)$. Then, $v \in \sigma(X \cup \{w\})$ implies $w \in \sigma(X \cup \{v\})$.

Using Proposition 3.5.3 we can characterize the feasible sets of a greedoids:

If $\sigma : 2^E \rightarrow 2^E$ is a greedoid closure operator, then

$$\mathcal{F} = \{X \subseteq E : x \notin \sigma(X - x) \text{ for all } x \in X\}.$$

For any closure operator, CL , we say a set X is **closed** if $CL(X) = X$. The closure operator for matroids satisfies (Cl1) - (Cl3) and thus is a closure operator in the usual topological sense. It also satisfies a fourth axiom known as *Steinitz–MacLane exchange*:

$$\text{if } X \subseteq E, x \in E, \text{ and } y \in CL(X \cup \{x\}) - CL(X), \text{ then } x \in CL(X \cup \{y\}). \quad (3.1)$$

Property ($\sigma 3$) is a relaxation of Steinitz–MacLane exchange.

In Section 4.1, we use the closure definition of greedoids to discuss an important class of greedoids known as antimatroids. The closure operator for antimatroids is an abstraction of the convex closure operator of \mathbb{R}^n and as such has special properties that give antimatroids added structure. A summary of the various axiomatizations for greedoids is found in Appendix A.

3.6 Greedoid constructions

We have seen that greedoids and matroids share several key definitions such as rank and basis. Next, we consider three more terms that are common to both greedoids and

matroids. As before, the definitions are slightly different for greedoids and matroids.

Recall that a loop of a matroid, M , is an element, x , that is in no basis of M , or equivalently, $r(\{x\}) = 0$. An isthmus, y , of M is an element that is in every basis, or equivalently, $r(A \cup \{y\}) = r(A) + 1$ for all $A \subseteq E - y$.

The definitions of loop, coloop and isthmus for greedoids vary among authors. We use the definitions established in [14].

Definition 3.6.1. *Let $G = (E, \mathcal{F})$ be a greedoid.*

- *A **loop**, L , is an element that is in no feasible set.*
- *A **coloop** is an element that is in every basis.*
- *An **isthmus**, I , is an element that can be added to or deleted from any feasible set with the resulting set being feasible.*

Thus, an isthmus is a special kind of coloop. A **normal** greedoid is one that contains no loops.

Example 3.6.2. *Referring to Example 3.3.3, there are two loops, c and d , three coloops, a , b , and g , and only one isthmus, g .*

Next, we define the greedoid operations of deletion and contraction.

Definition 3.6.3. *Let $G = (E, \mathcal{F})$ be a greedoid and let $A \subseteq E$. Define the **deletion of A from G** by*

$$G - A = (E - A, \mathcal{F} - A) \text{ where } \mathcal{F} - A = \{X \subseteq E - A \mid X \in \mathcal{F}\}.$$

*This operation is also known as the **restriction of G to $E - A$** .*

Definition 3.6.4. *Let $G = (E, \mathcal{F})$ and let $A \in \mathcal{F}$. The **contraction of A from G** is the greedoid*

$$G/A = (E - A, \mathcal{F}/A) \text{ where } \mathcal{F}/A = \{X \subseteq E - A \mid X \cup A \in \mathcal{F}\}.$$

Remark 3.6.5. *It can be easily checked, using Definition 3.1.1, that the deletion of any subset from G results in a greedoid and that the contraction of any feasible set from G results in a greedoid. Note that, to ensure that $\emptyset \in \mathcal{F}/A$, we only define contraction of feasible subsets. Unless \mathcal{F} is empty, axiom (G2) guarantees that there is always a feasible singleton that can be contracted.*

A **minor** of (E, \mathcal{F}) is any series of restrictions and contractions of the form $(E - (X \cup Y), (\mathcal{F}/X - Y))$, where $X \in \mathcal{F}$ and $Y \subseteq E - X$. Because the operations of deletion and contraction commute, the order in which they are applied does not matter.

At this point, we would like to draw attention to how the rank can be affected by deletion and contraction. Figures 3 and 4 illustrate deletion and contraction by a coloop and a non-coloop. Notice that in Figure 4 deleting a non-coloop did not change the rank of G . However, in Figure 3, deleting a coloop decreases the rank by two. When the coloop is an isthmus, deletion and contraction are identical. The following proposition summarizes these and other properties.

Proposition 3.6.6 ([2], [14], [18]). *Let $G = (E, \mathcal{F})$ be a greedoid with $x \in E$.*

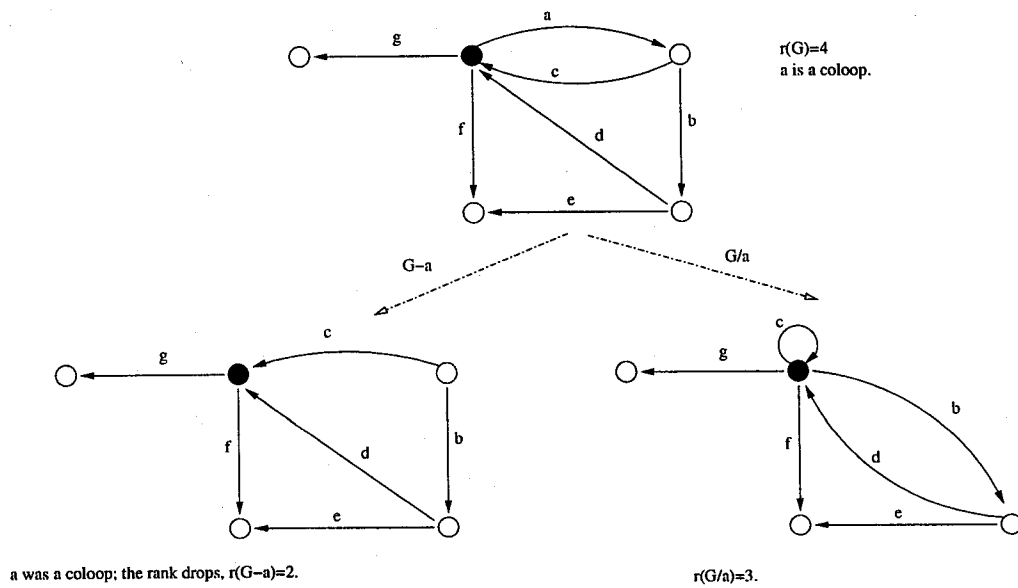


Figure 3: Deletion and contraction by a coloop

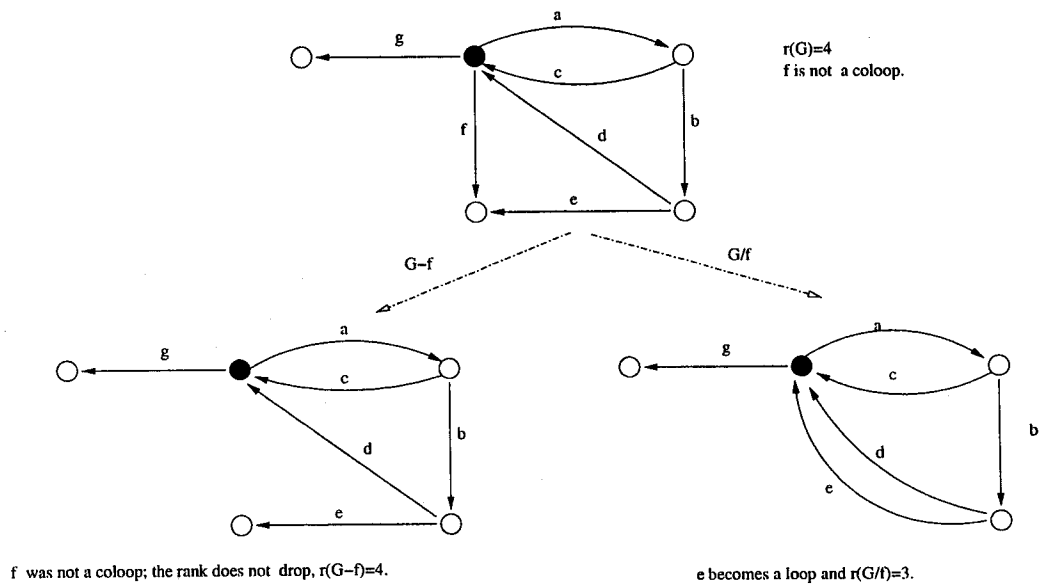


Figure 4: Deletion and contraction by a non-coloop

- If x is not a coloop, then $r(E - x) = r(E)$.
- If x is a coloop, then $r(E - x) < r(E)$.
- When a feasible singleton, x , is deleted and contracted from G , the feasible sets of G are partitioned into two sets: $\mathcal{F} - x$ and \mathcal{F}/x where

$$\mathcal{F} - x = \{X - x \mid x \in X, X \in \mathcal{F}\} \text{ and } \mathcal{F}/x = \{X \in \mathcal{F} \mid x \notin X\}.$$

- If x is an isthmus, then $G - x = G/x$. Therefore, if G contains an isthmus, $|\mathcal{F}(G)|$ must be even.
- Let r_G , r_{G-A} , and $r_{G/A}$ be the rank functions on G , $G-A$, and G/A , respectively. For $X \subseteq E - A$,

$$r_{G-A}(X) = r_G(X),$$

$$r_{G/A}(X) = r_G(X \cup A) - r_G(A).$$

We use these facts about the rank function, deletion and contraction extensively when we discuss the Tutte polynomial for greedoids and greedoid invariant theory.

Given two arbitrary greedoids on disjoint ground sets, we define the direct sum operation.

Definition 3.6.7. Let $G_1 = (E_1, \mathcal{F}_1)$ and $G_2 = (E_2, \mathcal{F}_2)$ be two greedoids defined on finite non-empty disjoint sets E_1 and E_2 . Let $E = (E_1 \cup E_2)$ and $\mathcal{F} = \{F_1 \cup F_2 \mid F_1 \in \mathcal{F}(G_1), F_2 \in \mathcal{F}(G_2)\}$. Then, $G = (E, \mathcal{F})$ is a greedoid called the **direct sum** of G_1 and G_2 and is denoted $G_1 \oplus G_2$.

Example 3.6.8. *Example 3.3.3 could be thought of as the direct sum of G_1 and G_2 where $E_1 = \{g\}$ and $E_2 = \{a, b, c, d, e, f\}$.*

If the two ground sets are not disjoint, the result is, in general, not a greedoid. When $|E_2| = 1$, direct sum is the addition of a loop or an isthmus. The direct sum operation can be extended recursively to n greedoids, G_1, G_2, \dots, G_n with disjoint ground sets.

3.7 External activity

In sections 8.2 and 9.4, we will use external activity in our study of invariants. We introduce the notion here because it is developed using deletion and contraction. The material in this section is adapted from two main sources, [16] and [17].

Let $G = (E, \mathcal{F})$ be a greedoid. If $r(E) > 0$, we can always find a feasible singleton, x , to delete and contract. If G/x and $G-x$ have rank greater than zero, we can repeat the process until we are left with only loops or the empty set. This process can be diagramed using a computation tree.

Definition 3.7.1. *For a greedoid, $G = (E, \mathcal{F})$, a **computation tree**, T_G , is a rooted binary tree in which each vertex is labelled by a minor of G . The computation tree is created recursively.*

- (a) *If $\mathcal{F} = \emptyset$, then T_G is the trivial tree consisting of a single vertex labelled G .*
- (b) *If $\{x\} \in \mathcal{F}$, then form two branches emanating from the root, G , by deleting and contracting x . Label their endpoints as $G-x$ and G/x .*

(c) Apply (b) to each branch until a rank zero minor is reached. Label each endpoint with the corresponding minor of G .

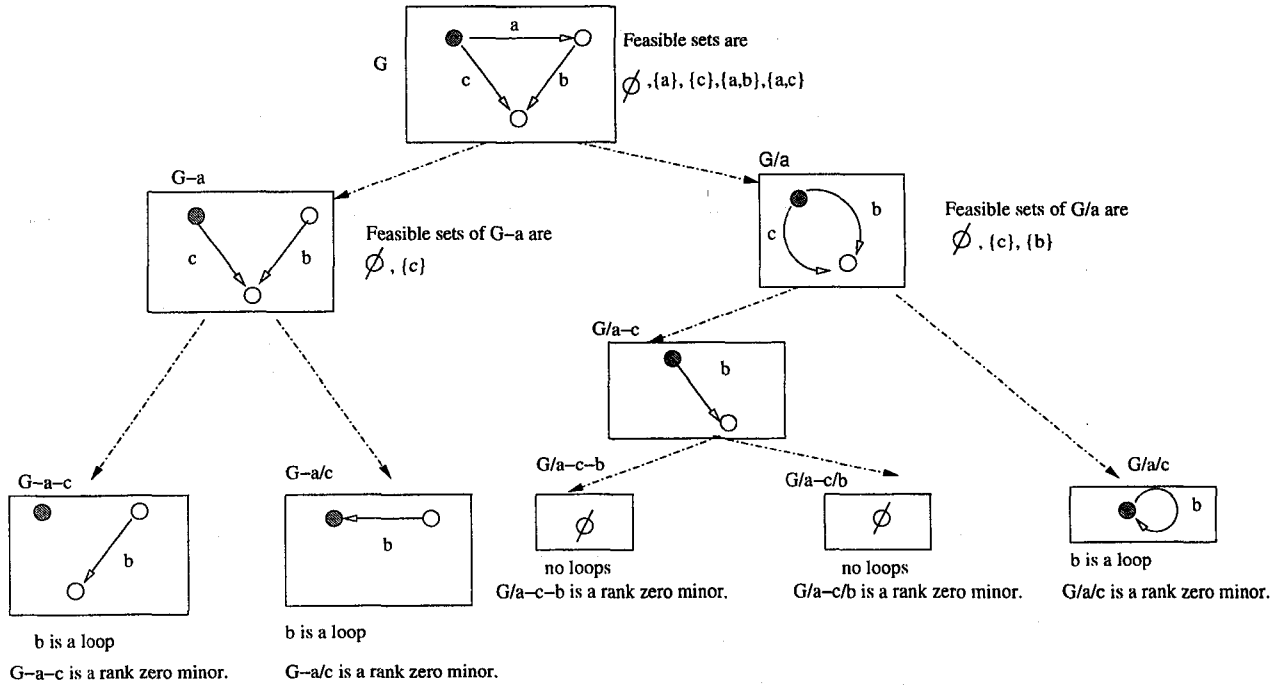


Figure 5: A computation tree for a directed branching greedoid.

Figure 5 contains an example of a computation tree. T_G is not necessarily unique. We use the computation tree, T_G , to define the external activity for greedoids in terms of a given computation tree.

Definition 3.7.2 (External activity in terms of sets). Let $G = (E, \mathcal{F})$ be a greedoid and let T_G be a computation tree of G . Further, let F_k be the elements of E that were contracted in the unique path from the root, labelled G , to the leaves, labelled G_k . For $F_k \subseteq E$, the **external activity of F_k with respect to T_G** , denoted $ext_T(F_k)$, is the collection of loops of G_k .

Thus, the external activity of the set F_k , is the set of elements of G that were neither deleted nor contracted along the corresponding path in T_G . The following proposition relates the feasible sets of G to its computation tree, T_G .

Proposition 3.7.3 ([16], Proposition 2.3). *Let $G = (E, \mathcal{F})$ be a greedoid and let T_G be a computation tree of G with m leaves. Let F_k be the elements of E that were contracted along the way in the unique path from the root to G_k . Then, $\{F_k, 1 \leq k \leq m\} = \mathcal{F}$. Further,*

$$\text{ext}_T(F_k) \subseteq \sigma(F_k) - F_k,$$

where σ is the greedoid closure operator of Definition 3.5.1.

For a given branch, there may be elements that were neither deleted nor contracted along the way because they were never feasible. These elements will remain at the terminal end of the branch as elements of a rank zero minor. The set of elements that are contracted along the paths to the leaves comprise the feasible sets of G . That is, we can “read” the feasible sets of G from the branches. Since G_k is a rank zero minor, G_k is a collection of loops or is empty. For some greedoids, a branch of T_G will end with \emptyset in which case the corresponding feasible set has no external activity.

Definition 3.7.4. *Let G be a greedoid and let T_G be a computation tree of G . Then, the set of feasible sets with no external activity, $\mathcal{F}_{\emptyset_T}(G)$, are given by*

$$\mathcal{F}_{\emptyset_T}(G) = \{F \in \mathcal{F} : \text{ext}_T(F) = \emptyset\}.$$

Referring again to Figure 5, we list the external activity of each feasible set in Table

1. For the greedoid whose computation tree is shown in Figure 5, $\mathcal{F}_{\emptyset_T} = \{\{a\}, \{a, b\}\}$.

Table 1: The external activity for the feasible sets in Figure 5.

Terminal rank 0 minors from T_G	Feasible Set, F	External Activity, $Ext(F)$
$G - \{a\} - \{c\}$	\emptyset	$\{b\}$
$G - \{a\}/\{c\}$	$\{c\}$	$\{b\}$
$G/\{a\} - \{c\} - \{b\}$	$\{a\}$	\emptyset
$G/\{a\} - \{c\}/\{b\}$	$\{a, b\}$	\emptyset
$G/\{a\}/\{c\}$	$\{a, c\}$	$\{b\}$

For an arbitrary greedoid the external activity of a feasible set may depend on T_G and thus not be unique. However, it is shown in [13] that the number of feasible sets with no external activity is the same for all computation trees. Thus, we refer to $|\mathcal{F}_{\emptyset}|$ without reference to T_G . In the next chapter, we revisit external activity when we discuss antimatroids. We will see that, for antimatroids, the external activity of a feasible set is unique.

4 Important Subclasses of Greedoids

In this chapter we provide a few examples of greedoids. Our purpose is to highlight a few important subclasses on which we focus in our discussion of greedoid invariants. We begin with interval greedoids - a class that contains both matroids and antimatroids.

Definition 4.0.5. *Let $G = (E, \mathcal{F})$ be a greedoid with $A, B, C \in \mathcal{F}$ such that $A \subseteq B \subseteq C$, $x \in E - C$. Then, G has the **interval property** if $A \cup \{x\}$, $C \cup \{x\} \in \mathcal{F}$ implies $B \cup \{x\} \in \mathcal{F}$.*

Remark 4.0.6. *We note that the interval property derives its name from the fact that, when the feasible sets are ordered by inclusion, A and C are the boundaries of an interval and the property applies to all B in that interval.*

Interval greedoids are those greedoids that satisfy the interval property. Every greedoid of rank less than three necessarily has the interval property.

Proposition 4.0.7. *Directed branching greedoids have the interval property.*

Proof. Let A, B and C be branchings with $A \subseteq B \subseteq C$, $x \in E - C$. If $A \cup \{x\} \in \mathcal{F}$, then x is an edge that joins two vertices, v and w , such that v is covered by A but w is not. If $C \cup \{x\} \in \mathcal{F}$, then w is still uncovered by C since $x \in E - C$ and $A \subseteq C$. Since $B \subseteq C$, w is uncovered by B and since $A \subseteq B$, v is covered by B . Then, $B \cup \{x\} \in \mathcal{F}$. □

The next example shows that not all greedoids have the interval property.

Example 4.0.8. Let $E = \{a, b, x\}$ and

$$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{x\}, \{a, b\}, \{b, x\}, \{a, b, x\}\}.$$

Let $A = \emptyset$, $B = \{a\}$ and $C = \{a, b\}$. Then, $A \subseteq B \subseteq C$, $\{x\} \in \mathcal{F}$ and $\{a, b\} \cup \{x\} = \{a, b, x\} \in \mathcal{F}$. But, we cannot augment $\{a\}$ with x because $\{a, x\} \notin \mathcal{F}$.

Matroids and antimatroids are two important classes of interval greedoids.

Definition 4.0.9. A greedoid $G = (E, \mathcal{F})$ has the *interval property without lower bound* if, for all $B, C \in \mathcal{F}$ with $B \subseteq C$, if $C \cup \{x\} \in \mathcal{F}$, then $B \cup \{x\} \in \mathcal{F}$.

Remark 4.0.10. Note that the bottom restriction of the interval $[A, C]$ (from Definition 4.0.5) is removed to form an interval without lower bound.

As noted in Section 2, this is the subclusive axiom for matroids. See Appendix B for details. Thus, we can characterize matroids as a special type of interval greedoids.

Proposition 4.0.11 ([18], Chapter IV, Corollary 1.6). *Matroids are precisely those greedoids that have the interval property without lower bound.*

4.1 Antimatroids

Antimatroids make up one of the most important classes of greedoids. They can be characterized in a number of different ways and for this reason have been discovered and rediscovered in a variety of settings over the years. In this section, we describe antimatroids in several ways.

Definition 4.1.1. A greedoid, $G = (E, \mathcal{F})$, is an *antimatroid* if $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$.

This definition amounts to saying that the feasible sets of an antimatroid are closed under union. This is in direct contrast to the feasible (independent) sets of a matroid that are closed under intersection. Thus, we have one reason for the name *antimatroid*.

4.1.1 Antimatroids as special interval greedoids

We used a loosening of the interval property to characterize matroids as special interval greedoids. Next, we modify the interval property to characterize antimatroids.

Definition 4.1.2. Let $G = (E, \mathcal{F})$ be a greedoid. For all $A, B \in \mathcal{F}$ such that $A \subseteq B$, if $A \cup \{x\} \in \mathcal{F}$ implies $B \cup \{x\} \in \mathcal{F}$, then G has the *interval property without upper bound*.

Antimatroids can be thought of as interval greedoids without upper bound.

Proposition 4.1.3. A greedoid $G = (E, \mathcal{F})$ is an *antimatroid* if it satisfies the *interval property without upper bound*.

4.1.2 Antimatroids in terms of convexity

Convexity is the geometric aspect of antimatroids. Just as matroids are an abstraction of independence, antimatroids serve as an abstraction of convexity. To illustrate this, consider the convex closure operator of Euclidean space. For E , a finite subset of

\mathbb{R}^n , this convex closure operator, $conv : 2^E \rightarrow 2^E$, is a topological closure operator as given by Definition 3.5.2. It also satisfies the following property.

For $X \subseteq E$ and $x, y \in E$ with $x, y \notin conv(X)$, if $x \in conv(X \cup \{y\})$, then $y \notin conv(X \cup \{x\})$.

(4.1)

Figure 6 illustrates this property.

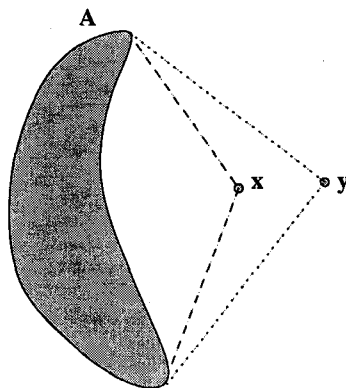


Figure 6: The anti-exchange property of the convex closure operator

The following **anti-exchange** property generalizes (4.1). Let τ be a topological closure operator on finite set, E .

For $X \subseteq E$ and $x, y \in E$ with $x, y \notin \tau(X)$, if $x \in \tau(X \cup \{y\})$, then $y \notin \tau(X \cup \{x\})$.

(4.2)

The anti-exchange property was named to indicate the close relationship to the Steinitz–MacLane exchange property, Equation (3.1), that characterizes matroid closure operators. This is another reason why antimatroids are so called.

A **convex geometry** is a pair, (E, τ) , such that E is a finite set and τ is a topological closure operator that satisfies the anti-exchange property, (4.2). If $X =$

$\tau(X)$, then we say X is **convex**. The collection of convex sets of a convex geometry is denoted \mathcal{C} . Because τ determines \mathcal{C} , convex geometries are often denoted (E, \mathcal{C}) . See Appendix B for details. Convex geometries and antimatroids are dual in the following sense.

Proposition 4.1.4 ([2], **Proposition 8.7.3**). *Let $G = (E, \mathcal{F})$ be a greedoid and let $\mathcal{F}^c = \{X \subseteq E \mid E - X \in \mathcal{F}\}$. Then, (E, \mathcal{F}) is an antimatroid if and only if (E, \mathcal{F}^c) is a convex geometry.*

At this point, we will provide three classes of antimatroids and their corresponding convex geometries.

Example 4.1.5. *Let $P = (E, \leq)$ be a poset and let \mathcal{F} be the order ideals of P . That is, $\mathcal{F} = \{X \subseteq E \mid \text{for all } x \in X, \text{ if } y \leq x, \text{ then } y \in X\}$. Then, (E, \mathcal{F}) is an antimatroid known as a **poset antimatroid**. The convex sets of the convex geometry, (E, \mathcal{C}) , are the order filters (dual ideals) of P .*

Example 4.1.6. *Let E be the edge set of a tree, T . Define \mathcal{F} to be the collection of complements of subtrees of T . Then, (E, \mathcal{F}) is the **edge pruning antimatroid** of T . The convex sets of the dual convex geometry are the subtrees of T .*

The third example requires the following definition.

Definition 4.1.7. *Let τ be a closure operator as defined in Definition 3.5.2. A point $x \in X$ is an **extreme point** of X if $x \notin \tau(X - x)$.*

Example 4.1.8. *Let E be a finite set of points in Euclidean space. Let $\mathcal{F} = \{X \subseteq E \mid E - X \text{ is convex}\}$. Then, $\mathcal{A} = (E, \mathcal{F})$ is a **convex pruning antimatroid**. If $\tau(X) = E \cap \text{conv}(X)$, then (E, τ) is the dual convex geometry. We now describe how to determine \mathcal{F} given C . Let A be a non-empty convex set. Then, A contains at least one extreme point, x , such that $A - x$ is convex. Hence, $E - A - x$ is feasible. Thus, the convex pruning antimatroid results from the process of repeatedly removing the extreme points from convex sets.*

Remark 4.1.9. *The antimatroids in these three examples are normal; i.e., they have no loops. In fact, many “naturally occurring” antimatroids have no loops and hence, the ground set, E , is feasible. Throughout the literature, antimatroids are often assumed to be normal and we follow this convention throughout this dissertation.*

4.1.3 Antimatroids in terms of lattices

Recall that the feasible sets of a greedoid, $G = (E, \mathcal{F})$, ordered by inclusion form a poset $P = (\mathcal{F}, \subseteq)$. Let G be an antimatroid with $X, Y \in \mathcal{F}$. Because the feasible sets of an antimatroid are closed under union, $X \vee Y$ equals $X \cup Y$ in P and $X \wedge Y$ equals $X \cap Y$ in P . For normal antimatroids, $E = \hat{1}$. Thus, the poset of feasible sets of an antimatroid form a lattice.

Proposition 4.1.10 ([2], Proposition 8.7.5). *Let $G = (E, \mathcal{F})$ be a greedoid and let $P = (\mathcal{F}, \subseteq)$ be the poset of feasible sets of G . Then, the following are equivalent.*

- (i) (E, \mathcal{F}) is an antimatroid.

(ii) P is a join-distributive lattice.

(iii) P is a semimodular lattice.

Remark 4.1.11. *A lattice L is join-distributive if, for every $x \in L - \{\hat{1}\}$, the interval $[x, j(x)]$ is Boolean, where $j(x)$ is the join of all the elements that cover x .*

Poset antimatroids were introduced in Example 4.1.5. The lattice of feasible sets of a poset antimatroid has a special structure due to the fact that the feasible sets (order ideals) are closed under both union and intersection. This means that the lattice distributive laws will be satisfied. Recall the Fundamental Theorem of Finite Distributive Lattices (FTFDL).

Theorem 4.1.12 ([21], Theorem 3.4.1). *(FTFDL) Let $J(P)$ denote the set of all order ideals of poset P ordered by inclusion, and let L be a finite distributive lattice. Then, there is a unique (up to isomorphism) finite poset P for which $L \cong J(P)$.*

The next proposition follows from the FTFDL.

Proposition 4.1.13. *An antimatroid $\mathcal{A} = (E, \mathcal{F})$ is a poset antimatroid if and only if $P = (\mathcal{F}, \subseteq)$ is a distributive lattice.*

These lattice characterizations allow us to apply lattice theory to our investigation of antimatroid invariants.

Poset antimatroids are quite special because they generalize all other antimatroids as the following proposition states.

Proposition 4.1.14 ([2], Proposition 8.7.8). *Let $f : P \rightarrow E$ be a function from a poset, P , to a set, E , and let $\mathcal{F} = \{f(A) \subseteq E \mid A \text{ is an ideal of } P\}$. Then, (E, \mathcal{F}) is an antimatroid and all antimatroids can be generated this way.*

4.1.4 Antimatroid and external activity

In Section, 3.7 we noted that, for an arbitrary greedoid, the external activity of a feasible set depends on its computation tree and therefore may not be unique. However the added structure of antimatroids guarantees the uniqueness of the external activity of feasible sets. We can characterize antimatroids in terms of external activity.

Proposition 4.1.15 ([16], Proposition 2.4). *Let G be a greedoid and let T_G be any computation tree of G . Then, $\text{ext}_{T_G}(F) = \sigma(F) - F$ for all feasible sets F if and only if G is an antimatroid.*

Remark 4.1.16. *For an antimatroid, $\mathcal{A} = (E, \mathcal{F})$, the external activity of a feasible set, $F_k \in \mathcal{F}$, does not depend on the computation tree, T_G , and is thus denoted $\text{ext}(F_k)$.*

Recall that, for an arbitrary greedoid, $\text{ext}_{T_G}(F) \subseteq \sigma(F) - F$. The strengthening to equality is due to the interval property without upper bound that characterizes antimatroids. Refer to [16] for a detailed study of external activity and interval partitions. In Chapter 9.2, we focus on an invariant for antimatroids and use the external activity of feasible sets extensively.

We wrap up this discussion of classes of greedoids with a few examples of greedoids that are not antimatroids.

- A greedoid, $G = (E, \mathcal{F})$, has the **transposition property** if it satisfies the following:

$$\begin{aligned} &\text{if } A, A \cup \{x\}, A \cup \{y\} \in \mathcal{F}, \text{ and } A \cup \{x\} \cup \{y\} \notin \mathcal{F}, \text{ then } A \cup \{x\} \cup B \in \mathcal{F} \\ &\text{implies } A \cup \{y\} \cup B \in \mathcal{F}, \text{ for all } B \subseteq E - (A \cup \{x\} \cup \{y\}). \end{aligned} \quad (4.3)$$

A **transposition greedoid**, is a greedoid that satisfies the transposition property, (4.3). The interval property is a strengthening of the transposition property. So, all interval greedoids are transposition greedoids. Not all transposition greedoids are interval greedoids. For example, series-parallel decomposition is a well-studied graph decomposition that can be used to characterize a transposition greedoid. See [18] for details.

- Let A be a matrix. Use Gaussian elimination to reduce A to an upper triangular matrix. Let $\{j_1, j_2, \dots, j_k\}$ be the column indices of the pivots of A . Let E be the set of column indices of A and let

$$\mathcal{F} = \{\{j_1, j_2, \dots, j_k\} \mid (j_1, j_2, \dots, j_k) \text{ is a subsequence of } (j_1, j_2, \dots, j_m)\}.$$

Then, (E, \mathcal{F}) is a greedoid called the **Gaussian elimination greedoid**.

Figure 7 illustrates the relationship between some of the classes of greedoids including those mentioned here.

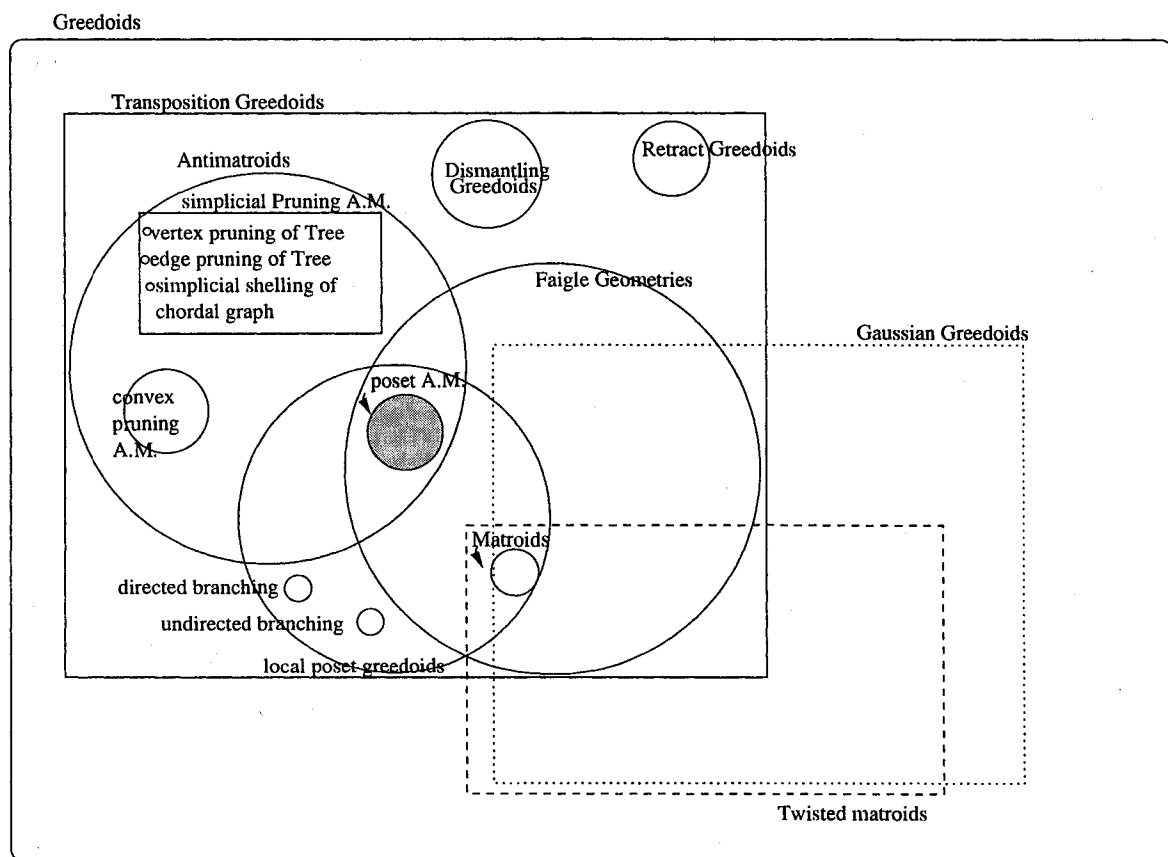


Figure 7: Relationship among some classes of greedoids

5 Greedoid Invariants

In Chapter 2, we reviewed basic matroid concepts and discussed the fundamental results of matroid invariant theory: many key classes of invariants are actually evaluations of one universal invariant, the Tutte polynomial, $t(M; x, y)$. We now embark on a study of greedoid invariant theory. Our goal is to develop greedoid counterparts to the Tutte polynomial and key classes of matroid invariants. We begin by stating that an **isomorphism invariant** on a class of objects is a \mathbb{Z} -valued function, f , such that

$$f(A) = f(B) \text{ whenever } A \cong B.$$

5.1 G-invariants

Let $G = (E, \mathcal{F})$ be a greedoid. From Definition 3.6.3, the restriction of G to $E - A$ is a greedoid with ground set $E - A$ and collection of feasible sets $\mathcal{F} - A = \{X \subseteq E - A \mid X \in \mathcal{F}\}$. For $e \in E$, we denote the greedoid of the restriction of G to $\{e\}$ by $G(e)$.

Definition 5.1.1. A *g-invariant*, π , is any isomorphism invariant on the class of greedoids that satisfies the following axioms:

$$(g1) \text{ if } e \text{ is a loop or an isthmus, then } \pi(G) = \pi(G(e)) \cdot \pi(G - e),$$

$$(g2) \text{ if } \{e\} \text{ is feasible, and if } \pi(I) = t \text{ for an isthmus, } I, \text{ then}$$

$$\pi(G) = \pi(G/e) + (t - 1)^{r(E) - r(E - e)} \pi(G - e).$$

A g-invariant is the greedoid counterpart to the matroid T-G invariant. Axiom

(g1) can be extended to the direct sum of two greedoids using an inductive argument. Thus, axiom (g1) can be replaced by axiom (g1'): If $G = G_1 \oplus G_2$, then $\pi(G) = \pi(G_1) \cdot \pi(G_2)$.

5.2 Generalized g-invariants

As was the case with matroids, some invariants do not satisfy axiom (g2) but do satisfy a generalization of axiom (g2). Such functions are known as generalized g-invariants.

Definition 5.2.1. *Let a, b be non-zero constants. A **generalized g-invariant**, h' , is a greedoid isomorphism invariant that satisfies*

$$(g1) \quad \text{if } e \text{ is a loop or an isthmus, then } h'(G) = h'(G(e)) \cdot h'(G - e),$$

$$(g2') \quad \text{if } \{e\} \text{ is feasible, and if } \pi(I) = t \text{ for an isthmus, } I, \text{ then}$$

$$h'(G) = b \cdot h'(G/e) + a \left(\frac{t-b}{a}\right)^{r(E)-r(E-e)} h'(G - e).$$

5.3 Greedoid group invariants

Thirdly, we define a counterpart to a matroid group invariant.

Definition 5.3.1. *A **greedoid group invariant**, π , is a greedoid isomorphism invariant that satisfies axiom (g2), the additive recursion, but not necessarily (g1), the multiplicative recursion.*

6 A Universal Greedoid Invariant

Next, we introduce a universal greedoid invariant and prove fundamental results that characterizes all g -invariants, generalized g -invariants and group invariants.

6.1 The greedoid Tutte polynomial

We begin with a counterpart to the matroid Tutte polynomial, $t(M; x, y)$.

Definition 6.1.1. *Let $G = (E, \mathcal{F})$ and define the **greedoid Tutte polynomial**, $h(G; t, z)$, as follows:*

$$h(G; t, z) = \sum_{A \subseteq E} (t - 1)^{r(E) - r(A)} (z - 1)^{|A| - r(A)}.$$

It is often convenient to write $h(G)$ for $h(G; t, z)$; we do so when there is no chance of confusion. Notice that if I is an isthmus and L is a loop, then $h(I) = t$ and $h(L) = z$. For these and other reasons, the choice of variables $t - 1$ and $z - 1$ will become apparent.

6.2 A characterization of g -invariants

The greedoid Tutte polynomial, $h(G; t, z)$ serves as the universal g -invariant. A precise statement of this result is given in Lemma 6.2.1 and Theorem 6.2.2.

Lemma 6.2.1. *The greedoid Tutte polynomial, $h(G; t, z)$, is a g -invariant.*

Proof. We first show that axiom (g2) holds, so let e be a feasible element of G . Then, for all $X \subseteq E$ either $e \in X$ or $e \notin X$. Thus,

$$h(G) = \sum_{e \in X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} + \sum_{e \notin X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)}.$$

We will show that these two sums become the two terms in axiom (g2).

Let r' be the rank function of G/e . If $e \in X$, then $r'(X-e) = r(X) - 1$, $r'(E/e) = r(E) - 1$, $|X-e| = |X| - 1$, and $|E-e| = |E| - 1$.

So, we can rewrite this first sum as

$$\begin{aligned} & \sum_{e \in X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} \\ &= \sum_{e \in X \subseteq E} (t-1)^{[r(E)-1-(r(X)-1)]} (z-1)^{[|X|-1]-[r(X)-1]} \\ &= \sum_{e \in X \subseteq E} (t-1)^{r'(E/e)-r'(X-e)} (z-1)^{|X-e|-r'(X-e)} \\ &= \sum_{X \subseteq E-e} (t-1)^{r'(E/e)-r'(X)} (z-1)^{|X|-r'(X)} \\ &= h(G/e). \end{aligned}$$

Next, we address the second sum, $\sum_{e \notin X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)}$. Let r'' be the rank function of $G-e$. For, $X \subseteq E-e$, $r''(X) = r(X)$. With this fact, we express

this second sum in terms of $G - e$.

$$\sum_{e \notin X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} = \quad (6.1)$$

$$= \sum_{e \notin X \subseteq E} (t-1)^{r(E)-r''(X)} (z-1)^{|X|-r''(X)} \quad (6.2)$$

$$= (t-1)^{r(E)-r(E-e)}. \quad (6.3)$$

$$\sum_{X \subseteq E-e} (t-1)^{r''(E-e)-r''(X)} (z-1)^{|X|-r''(X)} \quad (6.4)$$

$$= (t-1)^{r(E)-r(E-e)} \cdot h(G-e). \quad (6.5)$$

Thus, axiom (g2) is satisfied. To prove axiom (g1), consider the special cases in which e is an isthmus or a loop.

If e is an isthmus, then $G - e = G/e$ and $r(E) - r(E - e) = 1$. Since e is feasible, the argument above holds, and so, $h(G) = h(G/e) + (t-1)^{r(E)-r(E-e)} \cdot h(G - e)$ reduces to $h(G) = h(G - e) + (t-1)h(G - e) = t \cdot h(G - e)$. Since $h(G(e)) = t$, we conclude that $h(G) = h(G(e)) \cdot h(G - e)$.

If e is a loop, contraction is not defined. However, we can still break the sum into the two parts:

$$h(G) = \sum_{e \in X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} + \sum_{e \notin X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)}.$$

If $e \notin X$, since $r(E) = r(E - e)$, by Equation 6.1, we have

$$\sum_{e \notin X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} = h(G - e). \quad (6.6)$$

If $e \in X$, then $|X - e| = |X| - 1$ and $r(X) = r''(X) = r(X - e)$, where r'' is the

rank function of $G - e$. Also, $r''(E) = r(E)$; thus,

$$\begin{aligned}
 & \sum_{e \in X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} \\
 &= \sum_{e \in X \subseteq E} (t-1)^{r''(E)-r''(X)} (z-1)^{|X-e|+1-r''(X)} \\
 &= \sum_{X \subseteq E-e} (t-1)^{r''(E)-r''(X)} (z-1)^{|X|-r''(X)} \\
 &= (z-1) \cdot h(G-e).
 \end{aligned}$$

Putting these two sums together we have $h(G) = h(G-e) + (z-1) \cdot h(G-e) = z \cdot h(G-e)$. Since $h(G(e)) = z$, we obtain $h(G) = h(G(e)) \cdot h(G-e)$.

□

Not only is the greedoid Tutte polynomial a g -invariant, but like its matroid counterpart, all other g -invariants can be realized as evaluations of $h(G)$.

Theorem 6.2.2. *There is a unique function, $h(G; t, z)$, from the set of isomorphism classes of greedoids into the polynomial ring $\mathbb{Z}[t, z]$ that has the following properties:*

(i) $h(I; t, z) = t$ and $h(L; t, z) = z$ where I is an isthmus and L is a loop,

(ii) if $\{e\}$ is feasible, then

$$h(G) = h(G/e) + (t-1)^{r(E)-r(E-e)} \cdot h(G-e),$$

(iii) if e is a loop or an isthmus, then

$$h(G) = h(G(e)) \cdot h(G-e).$$

Further, if π is any g -invariant, then

$$\pi(G) = h(G; t, z) \Big|_{t=\pi(I), z=\pi(L)} = h(G; \pi(I), \pi(L)).$$

Proof. Parts (i), (ii), and (iii) follow from the previous lemma. To show uniqueness, suppose there is a second function, $j(G; t, z)$, that satisfies properties (i)-(iii). We will show that $j(G; t, z) = h(G; t, z)$, for $G = (E, \mathcal{F})$, by induction on $|E|$.

If $|E| = 1$, then G is a loop or an isthmus. If G is a loop, property (i) implies that $h(L; t, z) = z = j(L; t, z)$. If G is an isthmus, we have $h(I; t, z) = t = j(I; t, z)$.

Assume $h(G; t, z) = j(G; t, z)$ if $|E| = n - 1 \geq 1$. Suppose $G = (E, \mathcal{F})$ is a greedoid such that $|E| = n$. Either G consists of n loops or G has a feasible element. If G consists of n loops, then property (iii) implies $j(G; t, z) = z^n = h(G; t, z)$. If G has a feasible element, e , it can be deleted and contracted from G . Then, by property (ii),

$$j(G) = j(G/e) + (t - 1)^{r(E) - r(E-e)} \cdot j(G - e).$$

Since $|E(G/e)| = |E(G - e)| = n - 1$, we can apply the inductive hypothesis. Thus,

$$j(G) = h(G/e) + (t - 1)^{r(E) - r(E-e)} \cdot h(G - e) = h(G).$$

Therefore, $j(G; t, z) = h(G; t, z)$.

Let π be any g -invariant. We show that $\pi(G) = h(G; \pi(I), \pi(L))$ for I , an isthmus, and L , a loop. This follows from another inductive argument on the size of E .

Suppose $|E| = 1$. Then, G is either a loop, L , or an isthmus, I . Hence, $h(I; \pi(I), \pi(L)) = \pi(I)$ and $h(L; \pi(I), \pi(L)) = \pi(L)$. Thus, $h(G; \pi(I), \pi(L)) = \pi(G)$.

Next, suppose that $|E| = n$. Then, either G consists of n loops or G has at least one feasible element. If G consists of all loops, by (g1) of Definition 5.1.1, $\pi(G) = (\pi(L))^n$ and $h(G) = z^n$. Thus, $h(G; \pi(I), \pi(L)) = (\pi(L))^n = \pi(G)$. If G has a feasible element, e , it can be deleted and contracted to get two greedoids of size $n - 1$. Then, by axiom (g2) of Definition 5.1.1,

$$\pi(G) = \pi(G/e) + (\pi(I) - 1)^{r(E) - r(E-e)} \pi(G - e).$$

By the inductive hypothesis,

$$\pi(G) = h(G/e; \pi(I), \pi(L)) + (\pi(I) - 1)^{r(E) - r(E-e)} h(G - e; \pi(I), \pi(L)).$$

Since h is a g -invariant, axiom (g1) of Definition 5.1.1 implies

$$\pi(G) = h(G; \pi(I), \pi(L)).$$

□

Remark 6.2.3. *If e is not a coloop, then $r(E) = r(E - e)$ and property (ii) reduces to the usual matroid recursion $h(G) = h(G/e) + h(G - e)$. However, if e is a coloop, we only know that $r(E) > r(E - e)$ and so the $t - 1$ term is necessary.*

Property (iii) can be generalized to

$$(iii)' \quad \text{If } G = G_1 \oplus G_2, \text{ then } h(G) = h(G_1) \cdot h(G_2).$$

Then, $h(G; t, z)$ can be defined by properties (i), (ii), and (iii)'. See Appendix B.

Figure 8 shows a calculation of $h(G; t, z)$ for this the directed branching greedoid of Example 3.3.3 using the recursive properties. The result is $h(G; t, z) = tz^2[(t - 1)^2tz^2 + (t - 1)tz + t + z]$.

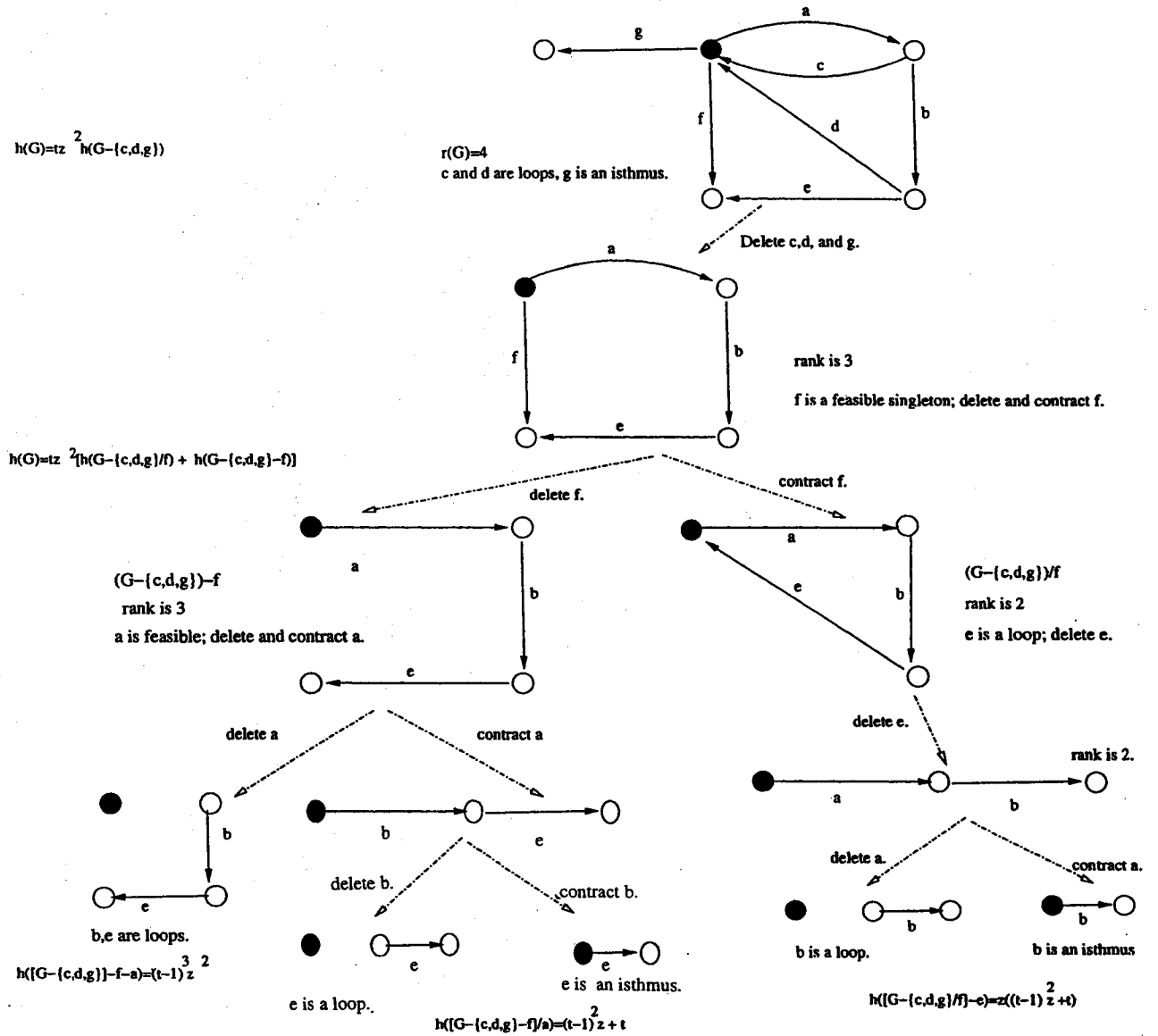


Figure 8: Calculating the greedoid Tutte polynomial

6.3 A characterization of generalized g -invariants

In the same manner used in Theorem 6.2.2 to characterize g -invariants, we characterize generalized g -invariants in terms of the greedoid Tutte polynomial.

Theorem 6.3.1. *Let a, b be non-zero elements of a field, F . Then, there is a unique function, h' , from the class of greedoids into $F[t, z]$ with the following properties:*

(i) $h'(I; t, z) = t$ and $h'(L; t, z) = z$, where I is an isthmus and L is a loop,

(ii') if $\{e\} \in \mathcal{F}$, then

$$h'(G; t, z) = b \cdot h'(G/e; t, z) + a \cdot \left(\frac{t-b}{a}\right)^{r(E)-r(E-e)} \cdot h'(G-e; t, z),$$

(iii) if e is a loop or an isthmus, then $h'(G) = h'(G(e)) \cdot h'(G-e)$.

Furthermore, the function h' is given by

$$h'(G; t, z) = a^{|E|-r(E)} \cdot b^{r(E)} \cdot h\left(G; \frac{t}{b}, \frac{z}{a}\right).$$

Proof. Let

$$h'(G; t, z) = a^{|E|-r(E)} \cdot b^{r(E)} \cdot h\left(G; \frac{t}{b}, \frac{z}{a}\right).$$

Then,

$$h'(L) = (a)^{1-0} \cdot b^0 \cdot h\left(L; \frac{t}{b}, \frac{z}{a}\right) = a \cdot \left(\frac{z}{a}\right) = z,$$

and

$$h'(I) = (a)^0 \cdot b^1 \cdot h\left(I; \frac{t}{b}, \frac{z}{a}\right) = b \cdot \left(\frac{t}{b}\right) = t.$$

Thus, (i) is satisfied. Next, we will show that (ii) is satisfied for all feasible elements,

e . By definition,

$$\begin{aligned} h'(G; t, z) &= a^{|E|-r(E)} \cdot b^{r(E)} \cdot h\left(G; \frac{t}{b}, \frac{z}{a}\right) \\ &= a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{X \subseteq E} \left(\frac{t}{b} - 1\right)^{r(E)-r(X)} \left(\frac{z}{a} - 1\right)^{|X|-r(X)}. \end{aligned}$$

As in the proof of Lemma 6.2.1, we break the sum into two parts, subsets that contain e and those that do not.

First, consider subsets, $X \subseteq E$, such that $e \in X$. Let r' be the rank function of G/e . Then $r'(X - e) = r(X) - 1$, $r'(E/e) = r(E) - 1$, $|X - e| = |X| - 1$, and $|E/e| = |E| - 1$.

$$\begin{aligned} &a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{e \in X \subseteq E} \left(\frac{t}{b} - 1\right)^{r(E)-r(X)} \left(\frac{z}{a} - 1\right)^{|X|-r(X)} = \\ &= a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{e \in X \subseteq E} \left(\frac{t}{b} - 1\right)^{[r(E)-1]-[r(X)-1]} \left(\frac{z}{a} - 1\right)^{[|X|-1]-[r(X)-1]} \\ &= a^{|E|-1-(r(E)-1)} \cdot b^{r(E)-1} \cdot b \cdot \sum_{e \in X \subseteq E} \left(\frac{t}{b} - 1\right)^{r'(E/e)-r'(X-e)} \left(\frac{z}{a} - 1\right)^{|X|-1-r'(X-e)} \\ &= b \cdot [a^{|E/e|-r'(E/e)} \cdot b^{r'(E/e)} \cdot \sum_{X \subseteq E-e} \left(\frac{t}{b} - 1\right)^{r'(E/e)-r'(X)} \left(\frac{z}{a} - 1\right)^{|X|-r'(X)}] \\ &= b \cdot h'[(G/e; t, z)]. \end{aligned}$$

Next, consider subsets X , such that $e \notin X$. Let r'' be the rank function of $G - e$.

For, $X \subseteq E - e$, $r''(X) = r(X)$. Use this to rewrite the second sum.

$$\begin{aligned}
& a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{e \notin X \subseteq E} \left(\frac{t}{b} - 1\right)^{r(E)-r(X)} \left(\frac{z}{a} - 1\right)^{|X|-r(X)} = \\
& = a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{e \notin X \subseteq E} \left(\frac{t}{b} - 1\right)^{r(E)-r''(X)} \left(\frac{z}{a} - 1\right)^{|X|-r''(X)} \\
& = a^{|E|-r(E)} \cdot b^{r(E)} \cdot \sum_{e \notin X \subseteq E} \left(\frac{t}{b} - 1\right)^{r(E)-r(E-e)+r(E-e)-r''(X)} \left(\frac{z}{a} - 1\right)^{|X|-r''(X)} \\
& = a^{|E|-r(E)} \cdot b^{r(E)} \cdot \left(\frac{t}{b} - 1\right)^{r(E)-r(E-e)} \sum_{X \subseteq E-e} \left(\frac{t}{b} - 1\right)^{r''(E-e)-r''(X)} \left(\frac{z}{a} - 1\right)^{|X|-r''(X)} \\
& = a \cdot a^{|E|-1-(r(E-e))-(r(E)-r(E-e))} \cdot b^{r(E-e)} (t-b)^{r(E)-r(E-e)} \cdot \\
& \quad \sum_{X \subseteq E-e} \left(\frac{t}{b} - 1\right)^{r(E-e)-r''(X)} \left(\frac{z}{a} - 1\right)^{|X|-r''(X)} \\
& = a \left[\frac{t-b}{a} \right]^{r(E)-r(E-e)} \cdot \\
& \quad \left[a^{|E-e|-r''(E-e)} b^{r''(E-e)} \cdot \sum_{X \subseteq E-e} \left(\frac{t}{b} - 1\right)^{r''(E-e)-r''(X)} \left(\frac{z}{a} - 1\right)^{|X|-r''(X)} \right] \\
& = a \cdot \left[\frac{t-b}{a} \right]^{r(E)-r(E-e)} \cdot h'(G-e; t, z)
\end{aligned}$$

□

6.4 Greedoid corank-nullity polynomial

Through a change of variables, we can form another polynomial for greedoids, $h(G; t, z)$, that is analogous to the matroid corank-nullity polynomial, $S(M; x, y)$.

Definition 6.4.1. *Let $G = (E, \mathcal{F})$ be a greedoid. The **greedoid corank-nullity polynomial** of G , $f(G; t, z)$, is defined by*

$$f(G; t, z) = h(G; t+1, z+1).$$

This polynomial was introduced by Gordon and McMahon to distinguish rooted arborescences. See [14] for details.

6.5 Greedoid group invariants and the greedoid Tutte polynomial

Matroid group invariants are characterized in terms of the matroid Tutte polynomial. Similarly, we characterize greedoid group invariants in terms of the greedoid Tutte polynomial.

Proposition 6.5.1. *Let \mathcal{A} be an Abelian group. There is a unique function, f , from the isomorphism class of non-empty greedoids to \mathcal{A} , such that the following two axioms are satisfied:*

(i) $f(G) = f(G/e) + (t - 1)^{\tau(E) - \tau(E-e)} f(G - e)$ where $\{e\}$ is feasible and, for an isthmus, I , $f(I) = t$,

(ii) If I_i denotes the greedoid that consists of i isthmuses and L_j denotes the greedoid that consists of j loops, then $f(I_i \oplus L_j) = \alpha_{ij}$ for all i and j where $i + j > 0$.

Furthermore, if $h(G; t, z) = \sum_i \sum_j b_{ij} t^i z^j$ is the greedoid Tutte polynomial, then

$$f(G) = \sum_i \sum_j b_{ij} \alpha_{ij}.$$

Proof. Let $h(G; t, z) = \sum_i \sum_j b_{ij} t^i z^j$ and let $f(G) = \sum_i \sum_j b_{ij} \alpha_{ij}$. Because $h(G; t, z)$ is a g -invariant, it satisfies (i). Let $\alpha_{ij} = t^i z^j$. Then, $f(G)$ satisfies (i). Also, $h(I_i \oplus L_j; t, z) = t^i z^j$, so $f(I_i \oplus L_j) = \alpha_{ij}$, and (ii) is satisfied. Then, at least one

function, $f(G) = \sum_i \sum_j b_{ij} \alpha_{ij}$ has the required properties. Uniqueness is shown using an inductive argument similar to that of Theorem 6.2.2. \square

6.6 Greedoid Tutte invariants

Finally, we wish to discuss those invariants that are not g , generalized g or group invariants but that can still be determined from the greedoid Tutte polynomial.

Definition 6.6.1. *A function, f , from the class of greedoids into a set, ζ , is a **greedoid Tutte invariant** if $f(G) = f(H)$ whenever G and H have the same greedoid Tutte polynomial.*

7 Examples of g -invariants

Having characterized g -invariants and their relationship to the greedoid Tutte polynomial, we next examine some applications.

7.1 The number of feasible sets, bases, spanning sets and subsets

Theorem 7.1.1. *Let $F(G)$, $B(G)$, $S(G)$, and $n(G)$ denote the number of feasible sets, bases, spanning sets, and subsets respectively, of greedoid, G . Each of these is a g -invariant given by,*

$$(i) \quad F(G) = h(G; 2, 1),$$

$$(ii) \quad B(G) = h(G; 1, 1),$$

$$(iii) \quad S(G) = h(G; 1, 2),$$

$$(iv) \quad N(G) = 2^{|E|} = h(G; 2, 2).$$

Proof. We first prove (i) in some detail. As the other proofs are similar, we give only an outline.

Proof of (i): To prove $F(G) = h(G; 2, 1)$, we show that $F(G)$ is a g -invariant. That is, it satisfies axiom (g1') and axiom (g2). To that end, we note that if two greedoids are isomorphic they have the same number of feasible sets. Hence, $F(G)$ is an isomorphism invariant. Secondly, suppose $G = G_1 \oplus G_2$ with feasible sets \mathcal{F} ,

\mathcal{F}_1 , and \mathcal{F}_2 , respectively. By Definition 3.6.7, $\mathcal{F} = \{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. It readily follows that $F(G) = F(G_1) \cdot F(G_2)$. So, axiom (g1') is easily satisfied. Next, we show axiom (g2) is satisfied.

If e is a feasible singleton, then there are two types of feasible sets of G : those that contain e and those that do not. Those that contain e are denoted \mathcal{F}' . The collection of feasible sets of G/e is given by

$$\mathcal{F}(G/e) = \{X \subseteq E - e \mid X \cup \{e\} \in \mathcal{F}\}.$$

Thus, $|\mathcal{F}(G/e)| = |\mathcal{F}'|$. Those feasible sets that do not contain e are denoted \mathcal{F}'' and are given by

$$\mathcal{F}'' = \{X \subseteq E - e \mid X \in \mathcal{F}\}.$$

These are the feasible sets of $G - e$. Thus, we have $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, where the union is disjoint, which implies $F(G) = F(G/e) + F(G - e)$. Since $t = F(I) = 2$, axiom (g2) is satisfied. Hence, F is a g -invariant with $F(I) = 2$ and $F(L) = 1$. Theorem 6.2.2 gives $F(G) = h(G; 2, 1)$.

Proof of (ii): As with feasible sets, it is easy to see that $B(G)$ is an isomorphism invariant and that the direct sum rule applies. Also, $B(I) = 1$ and $B(L) = 1$.

Now, we focus on axiom (g2). We consider two cases.

Case 1: G has a feasible coloop, e , then $r(G) > r(G - e)$, and axiom (g2) reduces to $B(G) = B(G/e)$. Since e is in every basis of G , the bases of G/e are

$$B(G/e) = \{B \subseteq E - e \mid B \cup \{e\} \in \mathcal{B}\} = \{B - e \mid B \in \mathcal{B}\} = B(G).$$

Case 2: Every feasible element of G is not a coloop. Let e be a feasible non-coloop.

Then, $r(E) = r(E - e)$ and axiom (g2) reduces to

$$B(G) = B(G/e) + B(G - e).$$

Since e is not a coloop, there are bases of G that do not contain e . These are precisely the bases of $G - e$, denoted $\mathcal{B}(G - e)$. We have

$$|\mathcal{B}| = |\mathcal{B}(G/e)| + |\mathcal{B}(G - e)|$$

or

$$B(G) = B(G/e) + B(G - e)$$

and axiom (g2) is satisfied. Hence, $B(G)$ is a g -invariant. Since $B(I) = 1$ and $B(L) = 1$, Theorem 6.2.2 gives $B(G) = h(G; 1, 1)$.

Proof of (iii): This proof is left to the reader, as it is similar to the proof of (ii).

Proof of (iv): If two greedoids are isomorphic, they have the same number of subsets of their respective ground sets. So $N(G)$ is an isomorphism invariant. Next, suppose that $G = G_1 \oplus G_2$ and that $|G| = n$, $|G_1| = k$, and $|G_2| = n - k$. Then, $\pi(G_1) \cdot \pi(G_2) = 2^k \cdot 2^{n-k} = 2^n = \pi(G)$, and π satisfies axiom (g1'). To show axiom (g2), note that $N(G/e) = 2^{n-1}$ and $N(G - e) = 2^{n-1}$. Hence, $N(G/e) + N(G - e) = 2^{n-1} + 2^{n-1} = 2^n = N(G)$. If G has one element, then this reduces to $N(I) = N(L) = 2$. Thus, $N(G) = 2^{|E|} = h(G; 2, 2)$.

□

7.2 A partitioning of spanning sets

The following proposition illustrates another g -invariant.

Proposition 7.2.1. *Let G be a greedoid of rank r and cardinality n , and let s_k be the number of spanning sets of size k , for $r \leq k \leq n$. Then,*

$$h(G; 1, 1 + w) = \sum_{j=0}^{n-r} s_{r+j} w^j.$$

Proof. Since $h(G; t, z) = \sum_{X \subseteq E} (t - 1)^{r(E) - r(X)} (z - 1)^{|X| - r(X)}$, setting $t = 1$ and $z = 1 + w$ gives

$$h(G; 1, 1 + w) = \sum_{\substack{X \subseteq E \\ r(E) = r(X)}} (w)^{|X| - r(E)}.$$

Let $j = |X| - r(E)$, $|X| = j + r$. Then,

$$h(G; 1, 1 + w) = \sum_{j=0}^{n-r} s_{r+j} w^j.$$

□

7.3 The greedoid invariant of Korte, Lovász, and Schrader

In [18], Korte, Lovász and Schrader, introduced a greedoid polynomial that partially extends the concept of the Tutte polynomial for matroids. It can be defined recursively.

Definition 7.3.1. *Let G be a greedoid. Define a **greedoid polynomial**, $\lambda_G(t)$, as follows: $\lambda_G(I) = 1$, $\lambda(L) = t$, and*

$$\lambda_G(t) = \begin{cases} \lambda_{G/e}(t) & \text{if } \{e\} \in \mathcal{F} \text{ and } e \text{ is coloop,} \\ \lambda_{G/e}(t) + \lambda_{G-e}(t) & \text{if } \{e\} \in \mathcal{F} \text{ and } e \text{ is not a coloop,} \\ \lambda_{G_1}(t)\lambda_{G_2}(t) & \text{if } G \text{ is the direct sum of } G_1 \text{ and } G_2. \end{cases}$$

The following theorem shows how $\lambda_G(t)$ fits into our framework of greedoid invariant theory.

Theorem 7.3.2. *The greedoid polynomial $\lambda_G(t)$ is a g -invariant where $\lambda_G(t) = h(G; 1, t)$.*

Proof. If G is an isthmus, I , $\lambda_I(t) = 1$. If G is a loop, L , $\lambda_L(t) = t$. If e is a feasible and not a coloop, then $r(E) = r(E - e)$ and (g2) reduces to $\lambda_G(t) = \lambda_{G/e}(t) + \lambda_{G-e}(t)$ which is true by Definition 7.3.1. If e is a feasible coloop, then $r(E) > r(E - e)$ and (g2) reduces to $\lambda_G(t) = \lambda_{G/e}(t)$ which also follows from Definition 7.3.1. Thus, $\lambda_G(t)$ is a g -invariant with $\lambda_G(t) = h(G; 1, t)$. \square

Korte, Lovász, and Schrader also introduced an invariant for greedoids based on $\lambda_G(t)$. We refer to it as a KLS invariant.

Definition 7.3.3. *Let ϕ be a function that assigns a complex value to every greedoid, $G = (E, \mathcal{F})$. We call ϕ a **KLS invariant** if it satisfies the following five axioms:*

- (KLS1) $\phi(G) = \phi(G/e)$ *if e is a feasible coloop,*
- (KLS2) $\phi(G) = \phi(G - e) + \phi(G/e)$ *if e is feasible and not a coloop,*
- (KLS3) $\phi(G) = \phi(G_1) \cdot \phi(G_2)$ *if $G = G_1 \oplus G_2$,*
- (KLS4) $\phi(G_1) = \phi(G_2)$ *if G_1 is isomorphic to G_2 ,*
- (KLS5) $\phi(G) \neq 0$ *for at least one greedoid.*

Every KLS invariant is an evaluation of $\lambda_G(t)$ which makes KLS invariants a subclass of g -invariants.

Corollary 7.3.4. *Every KLS invariant is a g -invariant.*

Proof. By Proposition 8.6.4 of [18], if $\phi(G)$ is a KLS invariant, then $\phi(G) = \lambda_G(z)$. By Theorem 7.3.2, $\phi(G)$ is a g -invariant given by $\phi(G) = h(G; 1, z)$. \square

Not all g -invariants are KLS invariants. If there exists an $e \in E$ such that e is in every feasible set of G , then the number of feasible sets of G is equal to the number of feasible sets of G/e . However, for an arbitrary greedoid, there may be no such e .

Proposition 7.3.5. *The number of feasible sets is not a KLS invariant.*

Proof. We will show that if G has a feasible coloop that is not in every feasible set, then axiom (KLS1) fails.

Let e be a feasible coloop of $G = (E, \mathcal{F})$ and let $F \in \mathcal{F}$. By Proposition 3.6.6,

$$\begin{aligned} \mathcal{F} &= \mathcal{F}/e \cup \mathcal{F} - e \\ &= \{X - e \mid e \in X, X \in \mathcal{F}\} \cup \{X \in \mathcal{F} \mid e \notin X\}. \end{aligned}$$

Choose $Y \in \mathcal{F}$ such that $e \notin Y$. Then, $Y \subseteq \mathcal{F} - e$ but $Y \not\subseteq \mathcal{F}/e$. Thus, $Y \in \mathcal{F}$ but $Y \notin \mathcal{F}/e$. Since $\mathcal{F}(G/e) \subseteq \mathcal{F}(G)$, then $|\mathcal{F}(G)| \neq |\mathcal{F}(G/e)|$. The number of feasible sets does not satisfy axiom (KLS1). Therefore, the number of feasible sets is not a KLS invariant. \square

8 Examples of Tutte, Group and Generalized g -invariants

Several key graph and poset functions have been generalized to matroids. These include the Möbius function, chromatic polynomial and beta invariant. In this chapter we generalize these functions and we classify each of them as a type of greedoid invariant. We first address the Möbius function.

8.1 The Möbius function and the Möbius invariant

Definition 8.1.1. *The Möbius function $\mu(x, z)$ of a finite poset, P , is defined recursively as follows:*

$$(i) \mu(x, x) = 1 \text{ for all } x \in P,$$

$$(ii) \mu(x, z) = - \sum_{x \leq z < y} \mu(x, y) \text{ for all } x < z \text{ in } P,$$

$$(iii) \mu(x, y) = 0 \text{ if } x \not\leq y.$$

Remark 8.1.2. *Sometimes, the Möbius function of P is written as μ_P to avoid confusion.*

As we mentioned in 4.1.3, the feasible sets of a greedoid, ordered by inclusion, form a poset. For an arbitrary greedoid $G = (E, \mathcal{F})$, the poset $P(G) = (\mathcal{F}, \subseteq)$ will be graded with height $r(E)$ and will have $\hat{0} = \emptyset$. The maximal elements of P will

be the bases of G . Thus, we can define a Möbius function for greedoid, μ_G , as $\mu_G(x, y) = \mu(x, y)$ for all $x, y \in P(G)$.

8.1.1 Computing μ_G

For small posets, the Möbius function can be computed using Definition 8.1.1. But, more sophisticated techniques for calculating μ have been developed. We refer the reader to Chapter 3 of [21] for a detailed study. Next we develop a method for calculating the Möbius function of an antimatroid. We begin with a lemma that describes the intervals of the lattice of feasible sets of an antimatroid. Recall that Proposition 4.1.3 states that the poset $P = (\mathcal{F}, \subseteq)$ of the antimatroid is a join-distributive lattice.

Lemma 8.1.3. *Let $P = (\mathcal{F}, \subseteq)$ be the poset of feasible sets of antimatroid, \mathcal{A} . Then, every interval of the form $[x, k(x)]$ is Boolean, where $k(x)$ is the join of any non-empty subset of elements covering x .*

Proof. First, recall that the lattice of feasible sets of an antimatroid is a join-distributive lattice L . By definition, every interval $[x, j(x)]$ in $L - \{\hat{1}\}$ is Boolean, where $j(x)$ is the join of all the elements covering x . We wish to show that this is also true for the join of some of the elements covering x . To that end, suppose that $k(x) \neq \hat{1}$ is the join of the elements $\{x_i\}_i$ all of which cover x . The elements of $P(\mathcal{A})$ are closed under union because \mathcal{A} is an antimatroid. So every possible union of the element of $\{x_i\}_i$ must also be in $P(\mathcal{A})$. Thus, $[x, k(x)]$ is Boolean. \square

We will also use the following result by Stanley ([21], Example 3.9.6).

Lemma 8.1.4. *Let P be a finite distributive lattice. For $X, Y \in P$,*

$$\mu(X, Y) = \begin{cases} (-1)^{|Y-X|} & \text{if } [X, Y] \text{ is Boolean,} \\ 0 & \text{otherwise.} \end{cases} \quad (8.1)$$

Now, we wish to characterize the Möbius function of an antimatroid. Recall that $\Gamma(X)$ is the set of continuations of $X \in \mathcal{F}$.

Proposition 8.1.5. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid and let $\gamma(X) \subseteq \Gamma(X)$. For $X, Y \in \mathcal{F}$,*

$$\mu(X, Y) = \begin{cases} (-1)^{r(Y)-r(X)} & \text{if } Y = X \cup \gamma(X) \\ 0 & \text{otherwise.} \end{cases} \quad (8.2)$$

Proof. If $Y = X \cup \gamma(X)$, then $[X, Y]$ is Boolean, by Lemma 8.1.3, and thus $\mu(X, Y) = (-1)^{r(Y)-r(X)}$ by Lemma 8.1.4. Now, suppose $Y \neq X \cup \gamma(X)$. If $X \not\subseteq Y$, then $\mu(X, Y) = 0$ by Definition 8.1.1, part (iii). If $X \subseteq Y$, and X contains some z such that there is an $x \not\subseteq z$, then $\{x\} \cup \Gamma(X) \subset Y$, in which case $\mu(X, Y) = 0$ by Definition 8.1.1, part (ii). \square

8.1.2 The Möbius invariant

An account of the several key matroid invariants is found in [22], including an invariant based on the Möbius function. The flats (closed sets) of a matroid ordered by inclusion form a geometric lattice, L . For $X, Y \subseteq E(M)$, the Möbius function μ_M of matroid

M is defined by:

$$\mu_M(X, Y) = \begin{cases} \mu_L(X, Y) & \text{if } X, Y \in L \\ 0 & \text{if } X \notin L, Y \in L \\ \text{undefined} & Y \notin L. \end{cases} \quad (8.3)$$

With the added structure of a lattice, one can define a **Möbius invariant**, μ_M , of M by $\mu_M = \mu(\hat{0}, \hat{1})$ where $\hat{0}$ and $\hat{1}$ are the bottom and top elements of the lattice of flats. In [22], several expansions of the Möbius invariant for matroids are developed and it is shown that $\mu(M)$ is a generalized TG-invariant. We next consider when a Möbius invariant for greedoids can be defined.

For an arbitrary greedoid, the poset of feasible sets may not have a $\hat{1}$ and $X \vee Y$ may not exist for all $X, Y \in P(G)$. Hence, $P(G)$ will not admit a lattice structure. However, by Proposition 4.1.10, the feasible sets of an antimatroid form a semimodular lattice, we can define a Möbius invariant for antimatroids.

Definition 8.1.6. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid with lattice of feasible sets, \mathcal{L} .*

*The **Möbius invariant of \mathcal{A}** is defined by:*

$$\mu(\mathcal{A}) = \mu(\hat{0}, \hat{1}).$$

We need the following lemma to characterize the Möbius invariant, $\mu(\mathcal{A})$.

Lemma 8.1.7 ([21], Corollary 3.9.5). *If L is a finite lattice for which $\hat{1}$ is not the join of atoms, then $\mu(\hat{0}, \hat{1}) = 0$.*

The following theorem characterizes the Möbius invariant for antimatroids.

Theorem 8.1.8. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid with lattice of feasible sets, $\mathcal{L}_{\mathcal{F}}$.*

Then,

$$\mu(\mathcal{A}) = \begin{cases} (-1)^{|E|} & \text{if } \mathcal{L}_{\mathcal{F}} \text{ is Boolean} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $\mathcal{L}_{\mathcal{F}}$ is Boolean. By Proposition 8.1.4, $\mu(\mathcal{A}) = \mu(\hat{0}, \hat{1}) = (-1)^{|E|}$. Now, suppose $\mathcal{L}_{\mathcal{F}}$ is not Boolean. We claim that in a semimodular lattice, P , ordered by inclusion, if P is not Boolean, then $\hat{1}$ is not the join of atoms. To see this, notice that for $\hat{1}$ to be the join of atoms, every singleton must be feasible. But, because the feasible sets are closed under union, P would necessarily be Boolean. Thus, $\hat{1}$ is not the join of atoms. Now, we invoke Lemma 8.1.7 and the theorem follows. \square

In [22], the Möbius invariant is shown to be an evaluation of the matroid characteristic polynomial (that we discuss in Section 8.2). This is what accounts for it being a generalized TG-invariant. Unlike its matroid counterpart, the Möbius function of an antimatroid is not a generalized g -invariant.

Example 8.1.9. *We show that $\mu(\mathcal{A})$ is not a generalized g -invariant because it does not satisfy axiom (g2), the deletion-contraction rule. Notice that if \mathcal{A} were an isthmus, then $\mu(I) = 1$ and axiom (g2) reduces to $\mu(\mathcal{A}) = \mu(\mathcal{A}/e)$. Now, let $E = \{a, b, c\}$ and $\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then, $\mathcal{A} = (E, \mathcal{F})$ is an antimatroid. The feasible sets of \mathcal{A}/b are $\{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, $\mu(\mathcal{A}) = 0$ while $\mu(\mathcal{A}/b) = 1$.*

The Möbius invariant of an antimatroid is an example of a greedoid Tutte invariant.

Corollary 8.1.10 (Corollary to Theorem 8.1.8). *The Möbius invariant, $\mu(\mathcal{A})$, of an antimatroid is a greedoid Tutte invariant.*

Proof. By Theorem 8.1.8, $\mu(\mathcal{A}) = (-1)^{|E|}$ if $\mathcal{L}_{\mathcal{F}}$ is Boolean and is zero otherwise. Now, $\mathcal{L}_{\mathcal{F}}$ is Boolean if and only if $h(\mathcal{A}) = t^{|E|}$, therefore $\mu(\mathcal{A})$ satisfies Definition 6.6.1. □

Next, we consider the characteristic polynomial.

8.2 The characteristic polynomial

Definition 8.2.1. *For a finite, graded poset P with rank n and rank function ρ , the **characteristic polynomial**, $\chi(P, q)$ is given by:*

$$\chi(P, q) = \sum_{x \in P} \mu(\hat{0}, x) \cdot q^{n-\rho(x)} = \sum_{k=0}^n w_k q^{n-k}$$

where $\mu(\hat{0}, x)$ is the Möbius function of P and

$$w_k = \sum_{\substack{x \in P \\ \rho(x)=k}} \mu(\hat{0}, x). \tag{8.4}$$

The flats, \mathcal{F} , (closed sets) of a matroid, M , form a geometric lattice, L . If we apply Definition 8.2.1 to L , then

$$\chi(M; q) = \sum_{F \in L} \mu(\hat{0}, F) \cdot q^{r(E)-r(F)}.$$

An alternative definition of the characteristic polynomial of a matroid is reviewed in [22]. Denoted $p(M; \lambda)$, it is the counterpart to the chromatic polynomial of a graph.

Definition 8.2.2. *The characteristic polynomial, $p(M; \lambda)$, of a matroid, M , is defined by $p(M; \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(E) - r(X)}$.*

The Möbius function of M has the following Boolean expansion.

Lemma 8.2.3 ([22], **Proposition 7.1.4**). *Let L be the lattice of flats of a matroid, M , with ground set, E . Let $W \subseteq E$ and $F \in L$. Then,*

$$\mu_M(W, F) = \sum_{\substack{W \subseteq X \subseteq F \\ clX = F}} (-1)^{|X - W|}. \quad (8.5)$$

We use the Boolean expansion of the Möbius function to relate $\chi(P, q)$ and $p(M; \lambda)$.

Proposition 8.2.4. *If P is a lattice of flats of matroid, M , then $\chi(P, q)$ and $p(M; \lambda)$ are equivalent.*

Proof. Let L be the lattice of flats of matroid, M , with ground set E . From Lemma 8.2.3,

$$\mu_M(\emptyset, F) = \sum_{\substack{X \subseteq F \\ clX = F}} (-1)^{|X|}.$$

For $X \in E$, $clX = F$ and $r(X) = r(F)$. Thus,

$$\chi(M; \lambda) = \sum_{F \in L} \sum_{\substack{X \subseteq F \\ clX = F}} (-1)^{|X|} \lambda^{r(E) - r(X)} = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(E) - r(X)} = p(M; \lambda).$$

□

8.3 A greedoid characteristic polynomial

In [15], Gordon and McMahon define a greedoid characteristic polynomial, $p(G; \lambda)$ in several equivalent ways. We use the following definition to show similarity of $p(G; \lambda)$ to the matroid characteristic polynomial, $p(M; \lambda)$.

Definition 8.3.1. *Let $G = (E, \mathcal{F})$ be a greedoid with rank function, r . The **characteristic polynomial of G** is defined by*

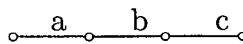
$$p(G, \lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{r(E) - r(S)}.$$

In Proposition 8.2.4, we showed that, for matroids, Definition 8.2.1 and (8.2.2) are equivalent. For an arbitrary greedoid, $p(G; \lambda)$ is not equivalent to $\chi(P(G); q)$.

Example 8.3.2. *Let G be a greedoid on ground set $E = \{a, b, c\}$ and with feasible sets $\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then, $\chi(P; q) = q^3 - 2q^2 + q$ but $p(G; \lambda) = -\lambda^2 + 2\lambda - 1$.*

Next, we point out another difference between $p(M; \lambda)$ and $p(G; \lambda)$. If G is a matroid, and \emptyset is not closed, then $p(G, \lambda) \equiv 0$. However, this is not the case for a general greedoid as the next example shows.

Figure 9: \emptyset is not closed; $p(G; \lambda) \neq 0$



Example 8.3.3. Let G be the edge pruning greedoid of the tree in Figure 8.3. Then, the closure of \emptyset is $\{b, c\}$, while the characteristic polynomial of G is $p(G, \lambda) = \lambda - 1$.

It is possible to formulate the characteristic polynomial in terms of the feasible sets of a greedoid. Recall from Definition 3.7.2, that the external activity of a feasible set, $ext_T(F)$, is the collection of greedoid loops that remain as the terminal vertex at the end of the branch of the computation tree, T_G , of the greedoid.

Proposition 8.3.4 ([15] Proposition 2). (*Feasible set expansion*)

Let $G = (E, \mathcal{F})$ be a greedoid and let T_G be any computation tree of G . Let $\mathcal{F}_{\emptyset_T}$ denote the set of feasible sets of G with no external activity. Then, the characteristic polynomial of G is given by:

$$p(G; \lambda) = \sum_{F \in \mathcal{F}_{\emptyset_T}} (-1)^{|F|} \lambda^{r(E) - |F|}.$$

Corollary 8.3.5. Let I be an isthmus and L a loop; then I , $p(I) = \lambda - 1$, and $p(L; \lambda) = 0$.

Now that we are familiar with the greedoid characteristic polynomial we categorize it as a generalized g -invariants.

Theorem 8.3.6. The greedoid characteristic polynomial, $p(G; \lambda)$, is a generalized g -invariant and $p(G; \lambda) = (-1)^{r(E)} h(G; 1 - \lambda, 0)$.

Proof. For $G = (E, \mathcal{F})$, we show that $p(G; \lambda)$ is a generalized g -invariant and invoke Theorem 6.3.1 to show that $p(G; \lambda)$ is an evaluation of the greedoid Tutte polynomial, $h(G; t, z)$. Axiom (g1) is satisfied as a result of Corollary 8.3.5.

By Proposition 3 of [15], for $\{e\} \in \mathcal{F}$, $p(G; \lambda)$ satisfies the following recursion:

$$p(G; \lambda) = -p(G/e; \lambda) + \lambda^{r(E)-r(E-e)} \cdot p(G - e; \lambda). \quad (8.6)$$

Thus, axiom (g2) is satisfied, with $a = 1$, $b = -1$, and $t = \lambda - 1$.

To show axiom (g1), let G be a greedoid that contains a loop, e . Then, e is in the external activity of every feasible set. Hence, $\mathcal{F}_\emptyset = \emptyset$. By Proposition 8.3.4, $p(G; \lambda) = 0$. Thus, $p(G) = p(G(e); \lambda)p(G - e; \lambda)$.

Let G be a greedoid that contains an isthmus, e . We know that if $|E| = 1$, $p(G; \lambda) = \lambda - 1$. If $|E| > 1$, then $G - e = G/e$ and equation (8.6) reduces to

$$p(G; \lambda) = (\lambda - 1)p(G - e; \lambda).$$

Hence, $p(G; \lambda) = p(G(e); \lambda)p(G - e; \lambda)$.

We conclude $p(G; \lambda)$ is a generalized g -invariant, and by Theorem 6.3.1, is given by $p(G; \lambda) = (-1)^{r(E)}h(G; 1 - \lambda, 0)$. \square

Remark 8.3.7. *It is worth noting that, in the case of matroids, $\mu(M)$ and $p(M; \lambda)$ are related by $\mu(M) = \mu_M(\emptyset, E) = p(M; 0)$. However, this is not true in the case of antimatroids. For example, the antimatroid in Example 8.3.2 has $\mu(\mathcal{A}) = 0$ but $\lambda(\mathcal{A}; 0) = -1$.*

8.4 Greedoid Tutte invariants

Greedoid Tutte invariants can be thought of as functions of the coefficients of $h(G; t, z)$.

Here are a few properties of a greedoid that are greedoid Tutte invariants.

Proposition 8.4.1. *Let $G = (E, \mathcal{F})$ be a greedoid with rank, r . Let $F_c(G)$ denote the number of feasible sets of G having $c = r - k$ elements, for $k = 0, 1, \dots, r$. If $h(G; t, z) = \sum_i \sum_j b_{ij} t^i z^j$, then $F_c(G)$ is a Tutte invariant, given by*

$$F_c(G) = \sum_i \sum_j b_{ij} \binom{i}{r-c}.$$

Proof. Let e be a feasible element of G that is not an isthmus. Let \mathcal{F}_c denote the feasible sets of G with c elements. There are two types of feasible sets in \mathcal{F}_c ; those that contain e , denoted \mathcal{F}'_c , and those that do not, denoted \mathcal{F}''_c .

First, we consider \mathcal{F}'_c . There is a one-to-one correspondence between \mathcal{F}'_c and the sets of $\mathcal{F}(G/e)$ having $(r-1) - k$ elements, where $X \in \mathcal{F}(G)$ corresponds to $X - e \in \mathcal{F}(G/e)$. Thus, $|\mathcal{F}'_c| = F_{c-1}(G/e)$. Next, we consider \mathcal{F}''_c . If $X \in \mathcal{F}''_c$, then $X \in \mathcal{F}(G - e)$, and $|\mathcal{F}''_c| = F_c(G - e)$. Thus,

$$F_c(G) = F_{c-1}(G/e) + F_c(G - e).$$

Next, consider the value of F_c on the greedoid $(I_i \oplus L_j)$. Every one of the $\binom{i}{i-k}$ possible combination of elements of I_i will be feasible while only one subset of L_j , the empty set, is feasible. Thus, $F_c(I_i \oplus L_j) = \binom{i}{i-k} = \binom{i}{k} = \binom{i}{r-c} = t^i z^j$. \square

We will use the following lemma in the proof of Proposition 8.4.3.

Lemma 8.4.2. *[[16], Theorem 2.5] Let $G = (E, \mathcal{F})$ be a greedoid with $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$. If T_G is any computation tree of G , let $\text{ext}_T(F_k)$ denote the external activity of $F_k \in \mathcal{F}$, $1 \leq k \leq m$. Then, the intervals of the form $[F_k, F_k \cup \text{ext}_T(F)]$, partition 2^E .*

Proposition 8.4.3. *Suppose that $h(G; t, z) = \sum_i \sum_j b_{ij} t^i z^j$ and that the term $b_{nm} t^n z^m$ has the highest degree of t for which $n + m$ is maximal. Then,*

- $n = r(E)$,
- G has m non-feasible singletons,
- If G is an antimatroid without loops, then $n - m$ is the number of feasible singletons.

Proof. Let $G = (E, \mathcal{F})$ be a greedoid and let T_G be any computation tree of G . For $F \in \mathcal{F}$, let $I(F)$ denote the interval $[F, F \cup \text{ext}_T(F)]$. The interval partition of Lemma 8.4.2 has the property that every subset $X \in I(F)$ has $r(X) = |F|$. We use this fact to rewrite $h(G; t, z)$.

$$\begin{aligned}
 h(G; t, z) &= \sum_{X \subseteq E} (t-1)^{r(E)-r(X)} (z-1)^{|X|-r(X)} \\
 &= \sum_{F \in \mathcal{F}} \sum_{X \in I(F)} (t-1)^{r(E)-|F|} (z-1)^{|X|-|F|} \\
 &= \sum_{F \in \mathcal{F}} (t-1)^{r(E)-|F|} \sum_{X \in I(F)} (z-1)^{|X|-|F|} \\
 &= \sum_{F \in \mathcal{F}} (t-1)^{r(E)-|F|} (z)^{|\text{ext}_T(F)|}.
 \end{aligned}$$

Using this feasible set expansion, the term corresponding to $F = \emptyset$ in $h(G; t, z)$ is $(t-1)^{r(E)} z^{|\text{ext}_T(\emptyset)|}$. Every non-feasible singleton, including loops, of E will be in the external activity of \emptyset . The first term in the expansion of $(t-1)^{r(E)} z^{|\text{ext}_T(\emptyset)|}$ will be of the form $b_{nm} t^n z^m$, where $n = r(E)$ and m is the number of non-feasible singletons. Finally, if G is an antimatroid without loops, then $r(E) = |E|$ and

$|ext_T(F)| = |\sigma(F)| - |F|$, where σ is the closure operator of the antimatroid. Then, $\sigma(\emptyset)$ contains all non-feasible singletons and thus $n - m$ is the number of feasible singletons. \square

Next, we discuss a greedoid β invariant and show that it is a greedoid Tutte invariant.

8.5 The beta invariant

In [9], Crapo introduced the β invariant for matroids. The invariant is an indicator of the separability (connectedness) of M .

Definition 8.5.1. *Let M be a matroid. The β invariant of M can be defined as*

$$\beta(M) = (-1)^{r(M)} \left. \frac{dp(M; \lambda)}{d\lambda} \right|_{\lambda=1}.$$

It follows from the fact that $\chi(M) = p(M) = (-1)^{r(M)} \cdot t(M; 1 - \lambda, 0)$ that $\beta(M) = \frac{\partial t(M; 0, 0)}{\partial t}$.

In [13] Gordon defines a β invariant for greedoids that is analogous to $\beta(M)$.

Definition 8.5.2. *Let G be a greedoid with characteristic polynomial $p(G; \lambda)$. Then,*

$$\beta(G) = (-1)^{r(E)} \cdot \left. \frac{dp(G; \lambda)}{d\lambda} \right|_{\lambda=1}.$$

The next proposition shows the relationship between β and the greedoid Tutte polynomial.

Proposition 8.5.3. *The β invariant can be expressed by:*

$$\beta(G) = \left. \frac{\partial h(G; t, 0)}{\partial t} \right|_{t=0}.$$

Proof. The result follows directly from Theorem 8.3.6 that states that $p(G; \lambda) = (-1)^{r(E)} \cdot h(G; 1 - \lambda, 0)$. □

As a direct consequence, we can classify $\beta(G)$ as a greedoid Tutte invariant.

Corollary 8.5.4. *If $h(G; t, z) = \sum_{ij} b_{ij} t^i z^j$, then $\beta(G)$ is a greedoid Tutte invariant with $\beta(G) = b_{10}$.*

Proof. Let $h(G; t, z) = \sum_{ij} b_{ij} t^i z^j$. Then, $\beta(G) = \left. \frac{\partial h(G; t, 0)}{\partial t} \right|_{t=0} = b_{10}$. □

Corollary 8.5.5. *Let $G = (E, \mathcal{F})$. If $G = G_1 \oplus G_2$, then $\beta(G) = 0$.*

Proof. Recall that $h(G; t, z) = h(G_1; t, z) \cdot h(G_2; t, z)$. Then,

$$\frac{\partial h(G; t, z)}{\partial t} = \frac{\partial h(G_1; t, z)}{\partial t} \cdot h(G_2; t, z) + \frac{\partial h(G_2; t, z)}{\partial t} \cdot h(G_1; t, z).$$

The result follows from $h(G; 0, 0) = 0$. □

9 Antimatroid Invariants

In this chapter, we focus on antimatroids and circuits. As a result, we obtain several invariants for antimatroids only. We begin by considering the number of feasible sets of an antimatroid that have no external activity.

9.1 The cardinality of the collection of feasible sets with no external activity

Proposition 9.1.1. *Let $\mathcal{F}_\emptyset(\mathcal{A})$ denote the collection of feasible sets of \mathcal{A} with no external activity. Then, $|\mathcal{F}_\emptyset(\mathcal{A})|$ is a g -invariant and $|\mathcal{F}_\emptyset(\mathcal{A})| = h(\mathcal{A}; 2, 0)$.*

Proof. If \mathcal{A} is an isthmus, I , then neither of the two feasible sets have external activity and $t = |\mathcal{F}_\emptyset(I)| = 2$. If \mathcal{A} is a loop, L , then L is in the external activity of the only feasible set, the empty set, and $z = |\mathcal{F}_\emptyset(L)| = 0$.

Let $\{e\}$ be feasible. Since $t = 2$, axiom (g2) reduces to

$$|\mathcal{F}_\emptyset(\mathcal{A})| = |\mathcal{F}_\emptyset(\mathcal{A}/e)| + |\mathcal{F}_\emptyset(\mathcal{A} - e)|.$$

There are two types of feasible sets with no external activity: those that contain e and those that do not. The former has a size equal to $|\mathcal{F}_\emptyset(\mathcal{A}/e)|$ and the latter's size is $|\mathcal{F}_\emptyset(\mathcal{A} - e)|$. Thus, axiom (g2) is satisfied and $|\mathcal{F}_\emptyset(\mathcal{A})|$ is a g -invariant given by $h(\mathcal{A}; 2, 0)$. \square

Proposition 9.1.1 is used in the discussion of rooted circuits found in Chapter 9.2.

In Section 6.6A. of [6], the authors show that the number of subsets of a matroid that contain no broken circuits is a T-G invariant. Next, we develop the greedoid analog.

9.2 Rooted circuits

Matroids serve as an abstraction of independence. Hence, the dependent sets of the matroid are of interest. In fact, the minimally dependent sets of the matroid, the circuits, completely characterize the matroid. For antimatroids, rooted circuits can likewise determine the antimatroid. To show this, we must build up some key concepts.

Definition 9.2.1. Let $G = (E, \mathcal{F})$ be a greedoid. For $X \subseteq E$, the *trace of X on G* , $\mathcal{F} : X$, is given by

$$\mathcal{F} : X = \{X \cap F : F \in \mathcal{F}\}.$$

For an arbitrary greedoid, the trace does not produce a greedoid. However, because the feasible sets of an antimatroid are closed under union, for all $X \subseteq E$, the trace $(E, \mathcal{F} : X)$ of an antimatroid is an antimatroid. Hence, we focus on antimatroids. For matroids, where feasible sets are independent sets, the trace, $\mathcal{F} : X$, reduced to the restriction on X . In this case, if X is free, then X is independent. There is a similar concept with antimatroids.

Definition 9.2.2. Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid and let $X \subseteq E$. If $\mathcal{F} : X = 2^X$, then X is called a *free set*.

There are several equivalent definitions of free.

Proposition 9.2.3 ([2] Lemma 8.7.9). *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid. The following are equivalent.*

1. X is free.
2. $X = \Gamma(A)$ for some $A \in \mathcal{F}$.
3. $X = \text{ex}(K)$ for some convex set, K .
4. $x \notin \tau(X - x)$ for all $x \in X$, where τ is the closure operator of the corresponding convex geometry.

For a matroid, a circuit is a minimally non-independent set. This idea can be extended to antimatroids. Namely, a circuit is a minimally non-free set.

Definition 9.2.4. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid. A subset $X \subseteq E$ is a **circuit** if $\mathcal{F} : X = 2^X - \{a\}$ for some $a \in E$.*

The element a is called the **root** of the circuit.

Proposition 9.2.5. *The root of a circuit is unique.*

Proof. Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid and let C be a circuit of \mathcal{A} . We will show that there is a unique element a such that $a \in \tau(C - a)$. By Proposition 9.2.3, there is some $a \in C$ such that $a \in \tau(C - a)$. Choose some $x \in C - a$ and set $B = C - \{a, x\}$. C is minimally non-free, so both $B \cup \{a\}$ and $B \cup \{x\}$ are free. Then, $a \in \tau(B \cup \{x\})$

implies $a, x \notin \tau(B)$. By Property 4.1, the anti-exchange property, $x \notin \tau(B \cup \{a\})$.

The fact that $\tau(B \cup \{a\}) = \tau(C - x)$ completes the proof. \square

To emphasize the importance of the root, we use (C, a) to denote the circuit C with root a . The set of **rooted circuits** of an antimatroid is denoted \mathcal{R} .

There is another way of defining the concept of free that sheds some light on the nature of the rooted circuits. If we consider the dual convex geometry of the antimatroid, we have an alternative definition of free.

Proposition 9.2.6 (Björner and Ziegler, Lemma 8.7.9). *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid with dual convex geometry (E, \mathcal{C}) . Let $ex(C)$ be the extreme points of the convex set C . If $X = ex(C)$ for some $C \in \mathcal{C}$, then X is free.*

9.3 Broken circuits

Let $M = (E, \mathcal{I})$ be a matroid. Suppose $|E| = n$ and impose a linear order on E by relabelling its elements as $1, 2, \dots, n$. A **broken circuit** is an independent set of M that results when the least element (with respect to the linear order) of a circuit is deleted. We now define a counterpart for antimatroids.

Definition 9.3.1. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid and let (C, a) be a rooted circuit of \mathcal{A} . A **broken circuit**, B of \mathcal{A} , is defined as $B = (C, a) - \{a\}$.*

9.4 Broken circuits and the greedoid Tutte polynomial

We now show that, for an antimatroid \mathcal{A} , the number of subsets that do not contain a broken circuit is an evaluation of the greedoid Tutte polynomial, $h(\mathcal{A}; t, z)$.

In Section 3.7 we defined the external activity of a feasible set and denoted those feasible sets with no external activity by $\mathcal{F}_\emptyset(\mathcal{A})$. Later, in Proposition 9.1.1, we showed that $|\mathcal{F}_\emptyset(G)|$ is a greedoid group invariant equal to $h(G; 2, 0)$. Next, we will relate external activity and broken circuits.

Lemma 9.4.1. *A feasible set F of an antimatroid, $\mathcal{A} = (E, \mathcal{F})$, has no external activity if and only if $E - F$ contains no broken circuits.*

Proof. Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid and let $\mathcal{C} = (E, \mathcal{F}^c)$ be its dual convex geometry. If $F \in \mathcal{F}$, then $K = E - F$ is convex. In [16] it is shown that a set $F \in \mathcal{F}_\emptyset$ if and only if $\sigma(F) = F$, where σ is the closure operator of the antimatroid. If $F \notin \mathcal{F}_\emptyset(\mathcal{A})$, then there is an $x \in \sigma(F) - F$. This implies that x is not an extreme point of K and thus K is not free. K must contain a minimally non-free set, i.e, a circuit. Thus $K - \{x\}$ is a broken circuit.

If $F \in \mathcal{F}_\emptyset$, then $\sigma(F) = F$ and $K = E - F$ will contain only extreme points. Hence, K is free. That is, K contains no non-free sets and thus no circuits. Finally, if K contained a broken circuit $(C, a) - a$, then $\{a\}$ is externally active in F which contradicts the assumption.

□

Let w_i be the coefficient of λ^{r-i} in the characteristic polynomial $p(\mathcal{A}; \lambda)$. Then,

$$h(\mathcal{A}; \lambda + 1, 0) = \sum_{i=0}^r |w_i| \lambda^{r-i}.$$

For $0 \leq i \leq r$, w_i are called the **Whitney numbers of the first kind**. The value w_i is the number of subsets of size i and rank $r - i$.

We are now ready to show the relationship between subsets with no broken circuits and the greedoid Tutte polynomial.

Theorem 9.4.2. *The number of subsets of E that do not contain a broken circuit is given by $h(\mathcal{A}; 2, 0) = \sum_i |w_i|$.*

Proof. By Proposition 9.1.1, $|\mathcal{F}_\emptyset| = h(\mathcal{A}; 2, 0)$. Also, $h(\mathcal{A}; 2, 0) = \sum_i |w_i|$. Thus, $\sum_i |w_i|$ is equal to the number of feasible sets with no external activity which, by Proposition 9.4.1, equals to the number of subsets with no broken circuits. \square

9.5 Critical circuits

Let \mathcal{A} be an antimatroid with rooted circuits, \mathcal{R} . There is a subset of \mathcal{R} , called the set of critical circuits, that is sufficient to determine the feasible sets of the antimatroid.

Definition 9.5.1. *Let (C, a) be a rooted circuit of antimatroid $\mathcal{A} = (E, \mathcal{F})$. Let B be a basis of $\mathcal{A} - C$. Then, (C, a) is a **critical circuit** if $B \cup \{a\} \notin \mathcal{F}$ but $B \cup \{a\} \cup \{c\} \in \mathcal{F}$ for all $c \in C - \{a\}$, for some B .*

Here are a few examples of types of antimatroids, their rooted circuits and their critical rooted circuits.

Example 9.5.2. • *Let A be the vertex pruning greedoid of tree T . The circuits are the triples of vertices $\{x, r, y\}$ with root r for which there is a path from x to y through r . The rooted circuit $(\{x, r, y\}, r)$ is critical if $\{x, r\}$ and $\{r, y\}$ are edges of T .*

- *If A is a poset antimatroid, then the circuits are the pairs $\{x, y\}$ in which $x < y$. The root of the circuit is y . The circuit is critical if there is no $z \in P$ such that $x < z < y$.*
- *Let A be a convex pruning (shelling) antimatroid in \mathbb{R}^n , and let E be a finite subset of \mathbb{R}^n . A subset $A \subseteq E$ is free if every point of A is an extreme point of the convex hull of A . The set (C, a) is a rooted circuit if $C - a$ is the set of vertices of a simplex and a is a point in the relative interior of the simplex. These circuits have size of at least three and at most $n+2$. The circuit is critical if and only if C is convex.*

Let \mathcal{R}_0 denote the set of critical circuits of an antimatroid.

Proposition 9.5.3 ([11], Section 2, Lemmas 2, 3). *The feasible sets of an antimatroid can be determined by \mathcal{R}_0 as follows:*

$$\mathcal{F} = \{X \subseteq E : X \cap (C, a) \neq \{a\}, \text{ for all } (C, a) \in \mathcal{R}_0.\}$$

Using Proposition 9.5.3 we can establish \mathcal{F} given \mathcal{R}_0 .

- (1) An element $e \in E$ is feasible if it is not the root of any circuit.

- (2) Let \mathcal{F}_1 denote the collection of all feasible singletons establish in (1). Then, X is feasible if $X \subseteq \mathcal{F}_1$.
- (3) Let $X \subset E$, but $X \notin \mathcal{F}_1$ and let $\{x\} \in \mathcal{F}_1$. Then, $X \cup \{e\}$ is feasible if and only if $X \cup e \cap C \neq \{a\}$ for any $(C, a) \in \mathcal{R}_0$.

Next, we consider an upper bound for the number of rooted circuits for some classes of antimatroids.

Proposition 9.5.4. *Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid with $h(\mathcal{A}) = \sum_{ij} b_{ij} t^i z^j$. Suppose that the term $b_{mk} t^m z^k$ has the highest degree of t among those terms for which $m + k$ is greatest. If $R(\mathcal{A})$ is the number of rooted circuits of \mathcal{A} then:*

- (1) *if \mathcal{A} is a poset antimatroid, then $R(\mathcal{A}) \leq k(m - k)$,*
- (2) *if \mathcal{A} is a vertex pruning greedoid, then $R(\mathcal{A}) \leq k \binom{m-1}{2}$,*
- (3) *if \mathcal{A} is a convex shelling antimatroid in \mathbb{R}^n , then $R(\mathcal{A}) \leq k \binom{m-1}{n+1}$.*

Proof. (1) By Corollary 3.10 of [18], an antimatroid is a poset antimatroid if and only if all of its circuits have cardinality 2. The feasible singletons of the antimatroid are the least elements in the poset. The circuits of the poset antimatroid are of the form $(\{x, y\}, y)$ in which $x < y$ so every non-feasible singleton y is a root of at least one circuit, and there are k such singletons. Since there are $m - k$ feasible singletons, by Proposition 8.4.3, there are at most $k(m - k)$ rooted circuits.

- (2) The rooted circuits of the vertex pruning greedoid have cardinality 3. The k non-feasible singletons act as the roots and can possibly be groups with pairs of any of the remaining $m - 1$ singletons to form circuits. Thus, each of the k roots can be in $\binom{m-1}{2}$ circuits.
- (3) As in part (2), this follows from the fact that the circuits of the convex shelling antimatroid can have as many as $n + 2$ elements and there are k possible roots. Thus, each of the k roots can be in $\binom{m-1}{n+1}$ circuits.

□

10 Areas of Future Research

There are some shortcomings to greedoid invariant theory that are worth pointing out.

One useful feature of the Tutte polynomial $t(M; x, y)$ is the following relationship between a matroid and its dual:

$$t(M; x, y) = t(M^*; y, x) \quad (10.1)$$

where M^* is the dual of M . The key reason for this result is the fact that for matroids, if $X \subseteq E$, then the rank function, r_{M^*} satisfies

$$r_{M^*}(X) = |X| - r_M(E) + r_M(E - X). \quad (10.2)$$

When (10.2) is applied to the definition of the Tutte polynomial, $t(M; x, y)$, (10.1) follows.

There are two notions of duality for greedoids but neither is a satisfactory counterpart to the matroid dual. First, one can define duality in terms of the *complement* of a feasible set. That is, given a greedoid $G = (E, \mathcal{F})$, form $\mathcal{F}^c = \{X \subseteq E : E - X \in \mathcal{F}\}$. As was discussed in Section 4.1.2, if $G = (E, \mathcal{F})$ is an antimatroid, then (E, \mathcal{F}^c) is a convex geometry. However, (E, \mathcal{F}^c) is a greedoid if and only if G is a poset antimatroid. The complement is obtained by inverting the poset. It is easy to see that a relationship similar to (10.2) does not hold for poset antimatroids. For example, in Figure 10, $P2$ results from inverting $P1$. Let A_{P1} be the antimatroid of

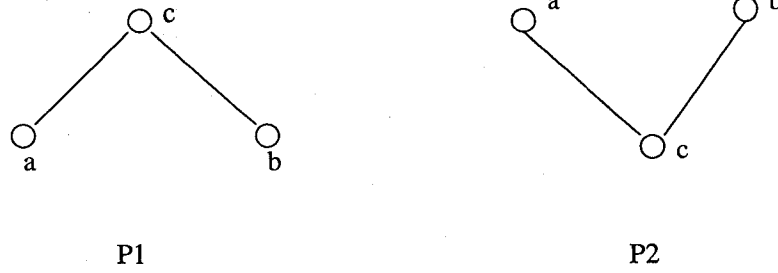


Figure 10: A poset and its complement

$P1$ and let A_{P2} be the antimatroid of $P2$. The greedoid Tutte polynomial of A_{P1} is $h(A_{P1}) = t^3z - t^2z - tz - z + t$ and the greedoid Tutte polynomial of A_{P2} is $h(A_{P2}) = t^3z^2 - 3t^2z^2 + 3tz^2 - z^2 + t^2$. Hence, there is no greedoid counterpart to (10.1) for greedoids with this notion of duality.

A second notion of duality can be expressed in terms of the bases of $G = (E, \mathcal{F})$. Let $\mathcal{F}^* = \{A : A \subseteq E - B \text{ for some basis } B \in \mathcal{B}(G)\}$. Then, $G^* = (E, \mathcal{F}^*)$ is a greedoid but G cannot be determined from G^* . However, the possible dual nature of $h(G)$ may hold some promise for this type of duality and is open for future research.

Up to this point, we have not been able to characterize the number of non-free, rooted circuits or critical circuits of an antimatroid in terms of the greedoid Tutte polynomial. Nor have we been able to compute $h(G; t, z)$ using the rooted circuits for any subclass of antimatroids. However, these are open questions and results may arise from future research. We have only developed an upper bound on the number of rooted circuits for some subclasses of antimatroids. Future research could tighten these bounds or determine the number of rooted circuits for some other subclasses.

A Axioms for Greedoids and Antimatroids

A.1 Cryptomorphisms

The following is a summary of the common ways to describe greedoids and antimatroids. First, we provide an axiomatization for each item. Then, a cryptomorphic table is given. All greedoids, G , are defined on a finite ground set, E .

(1) *Feasible sets, \mathcal{F} .*

\mathcal{F} is a collection of subsets of E with the following properties:

F1. $\emptyset \in \mathcal{F}$,

F2. if $X \in \mathcal{F}$, then there is an $x \in X$ such that $X - x \in \mathcal{F}$,

F3. for all $X, Y \in \mathcal{F}$ with $|X| > |Y|$, there exists an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

(2) *Language, \mathcal{L} .*

We call the elements of a finite set, E , letters; \mathcal{L} is a collection of words that are comprised of letters such that

L1. $\emptyset \in \mathcal{L}$,

L2. if $\alpha \in \mathcal{L}$ and $\alpha = \beta x$, then $\beta \in \mathcal{L}$,

L3. for all $\alpha, \beta \in \mathcal{L}$, $|\alpha| > |\beta|$, there is a letter $x \in \alpha$ such that $\beta x \in \mathcal{L}$.

(3) *Rank function, $r : 2^E \rightarrow \mathbb{N}$*

r is a natural-number valued function from the set of all subsets of ground set E satisfying three axioms:

- r1. for all $A \subseteq E$, $r(A) \leq |A|$,
- r2. for all $B \subseteq A \subseteq E$, $r(B) \leq r(A)$,
- r3. for all $A \subseteq E$, $x, y \in E$, $r(A \cup \{x\}) = r(A \cup \{y\})$ implies $r(A) = r(A \cup \{x\} \cup \{y\})$.

(4) *Closure operator*, $\sigma : 2^E \longrightarrow 2^E$

σ is a function, defined for all $X \subseteq E$, such that

- σ 1. For all $A \subseteq E$, $A \subseteq \sigma(A)$.
- σ 2. For all $A, B \subseteq E$, $A \subseteq B \subseteq \sigma(A)$ implies $\sigma(B) = \sigma(A)$.
- σ 3. For all $A \subseteq E$, $x, y \in E$, $z \in A \cup \{x\}$, if $x \notin A$ and $z \notin \sigma(A \cup \{x\} - \{z\})$, then $x \in \sigma(A \cup \{y\})$ implies $y \in \sigma(A \cup \{x\})$.

For antimatroids only, we have an axiomatization in terms of lattices.

(5) *Lattice*, L , in which the set of atoms of L are denoted A .

- La1. There is a function, $f : E' \longrightarrow A$, that maps the subset, E' , consisting of the nonloops of E , onto the atoms, A .
- La2. Every maximal chain of L has the same length.
- La3. The lattice, L , has a function, $\rho : L \longrightarrow \mathbb{N}$, called the rank function, that satisfies $\rho(\hat{0}) = 0$ and $\rho(x) = \rho(y) + 1$ whenever y covers x . Furthermore, ρ satisfies $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$.

Next, we summarize these axiomatizations and show their relationships in Table 2. This table includes references to greedoid languages; see Section E.

A.2 Characterizations of convex geometries

Next, we review a characterization of a convex geometry of an antimatroid.

Definition A.2.1. *A function $\tau : 2^E \rightarrow 2^E$ is a **closure operator of an antimatroid** if, for all $A, B \subseteq E$,*

$$(\tau 1) \quad A \subseteq \tau(A),$$

$$(\tau 2) \quad \text{If } A \subseteq B \text{ then } \tau(A) \subseteq \tau(B),$$

$$(\tau 3) \quad \tau(\tau(A)) = \tau(A),$$

$$(\tau 4) \quad \text{For } x, y \notin A, \text{ if } x \in \tau(A \cup \{y\}), \text{ then } y \notin \tau(A \cup \{x\}).$$

Property $(\tau 4)$ is the anti-exchange property. We use τ to define a convex geometry.

Definition A.2.2. *A **convex geometry** is a pair, (E, τ) , where E is a finite set and τ is a closure operator of an antimatroid.*

A set, $A \subseteq E$ is **closed** if $\tau(A) = A$. The closed sets of a convex geometry are called **convex sets**. Let \mathcal{C} be a collection of convex sets. Then, we can denote the convex geometry, (E, τ) in terms of \mathcal{C} . That is, we say (E, \mathcal{C}) is a convex geometry if \mathcal{C} is a collection of convex sets of τ .

There is a one-to-one correspondence between antimatroids and convex geometries.

Table 2: Cryptomorphisms for greedoids

Define \rightarrow in terms of \downarrow	Closure, σ For $A \subseteq E$, $\sigma(A) =$	Rank, τ $\tau(A) =$	Spanning Sets, S $A \in S$ if	Language, \mathcal{L} For $A = \tilde{\alpha}$, $\alpha \in \mathcal{L}$ if	Feasible Sets, \mathcal{F} $A \in \mathcal{F}$ if
Closure, σ		$\left\{ \begin{array}{l} \tau(X) : x \in \sigma(X) \\ \tau(X)_{c>0} : x \notin \sigma(X) \\ \text{where } A = X \cup \{x\}, \\ X \subseteq E. \end{array} \right.$	$\sigma(A) = E.$	$x \notin \sigma(A-x)$ for all $x \in A.$	$x \notin \sigma(A-x)$ for all $x \in A.$
Rank, τ	$\{x: \tau(A) = \tau(A \cup \{x\})\}.$		$\tau(A) = \tau(E).$	$\tau(A) = A .$	$\tau(A) = A .$
Language, \mathcal{L} where $\tilde{\alpha} = \{\text{letters of } \alpha\}$	$E - \Gamma(A),$ where $A = \tilde{\alpha}$ and $\Gamma(A) =$ {continuations of A }.	$\max\{ \beta : \beta \in \mathcal{L}, \tilde{\beta} \subseteq A\},$ where $A = \tilde{\alpha}.$	maximal words in α are maximal in $E,$ where $A = \tilde{\alpha}.$		$\alpha \in \mathcal{L},$ where $A = \tilde{\alpha}.$
Feasible sets, \mathcal{F}	$E - \Gamma(A)$ where $\Gamma(A) = \{x A \cup \{x\} \in \mathcal{F}\}.$	$\max\{ F : F \in \mathcal{F}, F \subseteq A\}.$	maximal feasible sets in A are maximal in $E.$	$A \in \mathcal{F},$ where $\mathcal{F} = \tilde{\mathcal{L}}(\mathcal{F}).$	

Table 3: Lattice cryptomorphisms for antimatroids

Define \rightarrow in terms of \downarrow	Closure, σ For $A \subseteq E$, $\sigma(A) =$	Rank, r $r(A) =$	Spanning Sets, S $A \in S$ if	Language, \mathcal{L} For $A = \tilde{\alpha}$, $\alpha \in \mathcal{L}$ if	Feasible Sets, \mathcal{F} $A \in \mathcal{F}$ if
Lattice, L with atoms, x , such that $\{x\} \in \mathcal{F}$	$A \cup \{x \in E : \rho(\{x\} \cup A) \leq \rho(A)\}$, where ρ is the rank function of L .	$\rho(\{x : x \in A\})$ where ρ is the rank function of L .	$\hat{I} = \vee \{x : x \in A\}$,	A is an element in L .	A is an element in L .

Proposition A.2.3 ([2], **Proposition 8.7.3**). *Let E be a finite set and $\mathcal{F} \subseteq 2^E$. Then, (E, \mathcal{F}) is an antimatroid if and only if $\mathcal{F}^c = \{E - A \mid A \in \mathcal{F}\}$ is the collection of convex sets of a convex geometry.*

Proof. Let τ be the closure operator an antimatroid, \mathcal{A} . The collection of closed (convex) sets of τ are preserved under intersection. [Therefore, the collection of complements of closed sets is preserved under union.] Also, from Proposition 8.7.2 of [2], (E, τ) is a convex geometry if and only if, for every closed set $A \subseteq E$, there is an $x \in E - A$ such that $A \cup \{x\}$ is closed. Thus, the collection of complements of closed sets of τ satisfies:

if $X \in \mathcal{F}$, then there exists an $x \in E - X$ such that $X - x \in \mathcal{F}$.

Hence, \mathcal{F} is the collection of feasible sets of an antimatroid.

Let \mathcal{A} be an antimatroid with closure operator, τ , and collection of feasible sets, \mathcal{F} . The collection \mathcal{F} is closed under union; thus \mathcal{F}^c is closed under intersection. By Proposition 8.7.2 of [2], the collection \mathcal{F}^c forms a convex geometry. \square

A.3 Interval property without lower bound and the subclusive axiom

To complete this section, we show that the interval property without lower bounds (ipwlb) is equivalent to the subclusive axiom for matroids. The (ipwlb) is as follows: if $B, C \in \mathcal{F}$ and $B \subseteq C$, then $C \cup \{x\} \in \mathcal{F}$ implies $B \cup \{x\} \in \mathcal{F}$. We show that the

(ipwlb) implies the subclusive axiom for matroids, which states that if $C \in \mathcal{F}$ and $B \subseteq C$, then $B \in \mathcal{F}$. Choose $C \in \mathcal{F}$ and $B \subseteq C$, and apply induction to $|B|$. If B is such that $|C| = |B| + 1$, the (ipwlb) implies $B \in \mathcal{F}$. If $|C| = |B| + n$, then, B is contained in some subset D such that $D \subseteq C$, $|C| = |D| + (n - 1)$. The inductive hypothesis implies $D \in \mathcal{F}$. Then, the (ipwlb) implies $B \in \mathcal{F}$.

B Matroids and Antimatroids

B.1 Linear dependence and convexity

This section summarizes some facts about and similarities and differences between matroids and antimatroids that were not discussed in Chapter 4. First, in Table 4, we compare the underlying principles of matroids and antimatroids: linear dependence and convexity, respectively.

B.2 Circuits of matroids and antimatroids

Let $M = (E, \mathcal{I})$ be a matroid. A set $D \in E$ can be linearly dependent while every proper subset of D is independent. Then, D is a minimally dependent set called a **circuit**. The collection of circuits determines a matroid.

Proposition B.2.1 ([5], **Axiomatization 6**). *Let E be a finite set. Then, the collection $\mathcal{C} \subseteq 2^E$, is the collection of circuits of a matroid if and only if the following three properties are satisfied.*

Table 4: Linear dependence versus convexity

Let S be a subset of vector space, V , over a field, \mathbb{F} , such that $S = \{v_1, v_2, \dots, v_n\}$ for $v_i \in V$, $1 \leq i \leq n$.	
Matroid, $M = (E, \mathcal{I})$	Antimatroid, $\mathcal{A} = (E, \mathcal{F})$
A vector x is a linear combination of vectors in S if there exist $a_i \in \mathbb{F}$, such that $x = \sum_{i=1}^n a_i v_i$.	A vector x is a convex combination of vectors in S if $x = \sum_{i=1}^n a_i v_i$, $\sum a_i = 1$, $a_i \geq 0$.
S is linearly dependent if there exist $a_i \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^n a_i v_i = 0$. A set that is not linearly dependent is called linearly independent .	A subset S of V is convex if any convex combination of its elements is again in S . An extreme point of a closed convex set, K , is a point $z \in K$ that can be written as a convex combination of points in K in only a trivial way. That is, for $0 < a_1 < 1$, $z = a_1 x_1 + (1 - a_1)x_2$, $x_1, x_2 \in K$ implies that $z = x_1 = x_2$.
The span of S is the set of all vectors that can be written as a linear combination of elements of S .	The convex hull of subset S of V is the set of all vectors that can be written as a convex combination of members of S .
Any subset of a linearly independent set is itself linearly independent.	Not all subsets of convex sets are convex. But, if S is convex, then there exists an extreme point $x \in S$ such that $S - \{x\}$ is convex.

C1. $\emptyset \notin \mathcal{C}$.

C2. For $C_1, C_2 \in \mathcal{C}$, if $C_1 \subseteq C_2$, then $C_1 = C_2$.

C3. For $C_1, C_2 \in \mathcal{C}$, with $C_1 \neq C_2$, if $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that

$$C_3 \subseteq (C_1 \cup C_2) - e.$$

Axiom *C3*. is known as the weak circuit elimination axiom. An alternative to Axiom *C3*, called the strong circuit elimination axiom, can be used in its place.

C3'. For $C_1, C_2 \in \mathcal{C}$, with $C_1 \neq C_2$, if $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$, then there is a member $C_3 \in \mathcal{C}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$.

Thus, we can completely characterize a matroid in terms of circuits. There is an analogous characterization of antimatroids.

Let $\mathcal{A} = (E, \mathcal{F})$ be an antimatroid. For $S \subseteq E$, the **trace** of (E, \mathcal{F}) on S is $(S, \mathcal{F} : S)$ where

$$\mathcal{F} : S = \{A \cap S \mid A \in \mathcal{F}\}.$$

The set S is **free** in $\mathcal{A} = (E, \mathcal{F})$ if $\mathcal{F} : S = 2^S$. A **circuit** of \mathcal{A} is a minimally non-free set.

The collection of circuits of an antimatroid are not sufficient to determine the antimatroid, as the next example illustrates.

Example B.2.2. Let $E = \{a, b, c\}$ and define two collections of feasible sets:

$$\mathcal{F}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$$

and

$$\mathcal{F}_2 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

For antimatroid $\mathcal{A}_1 = (E, \mathcal{F}_1)$,

$$(\{a, b, c\}, \mathcal{F}_1 : \{a, b, c\}) = c$$

while for $\mathcal{A}_2 = (E, \mathcal{F}_2)$,

$$(\{a, b, c\}, \mathcal{F}_2 : \{a, b, c\}) = b.$$

Hence, $\{a, b, c\}$ is a circuit of both \mathcal{A}_1 and \mathcal{A}_2 . Thus, $(E, \mathcal{F}_1) \not\cong (E, \mathcal{F}_2)$ but they each have the same collection of circuits, $\mathcal{R} = \{\{a, b, c\}\}$.

Let $C \subseteq E$ be a circuit. The **root**, a of C , is the unique element satisfying $\mathcal{F} : C = 2^C - \{a\}$. We call the set (C, a) a **rooted circuit** of \mathcal{A} . In Example B.2.2, the rooted circuit of \mathcal{F}_1 is $(\{a, b, c\}, c)$ while the rooted circuit of \mathcal{F}_2 is $(\{a, b, c\}, b)$.

The collection of rooted circuits of antimatroid, \mathcal{A} , is denoted \mathcal{R} . We can use \mathcal{R} to determine an antimatroid.

Proposition B.2.3 ([10], Theorem 7). *Let E be a finite set. Then, $\mathcal{R} \subseteq 2^E$ is a set of rooted circuit of an antimatroid if and only if the following three properties are satisfied:*

R1. $\emptyset \notin \mathcal{R}$,

R2. for $(C_1, x_1), (C_2, x_2) \in \mathcal{R}$, if $C_1 \subseteq C_2$, then $C_1 = C_2$ and $x_1 = x_2$,

R3. for $(C_1, x_1), (C_2, x_2) \in \mathcal{R}$, with $x_1 \neq x_2$, if $x_1 \in (C_1 \cap C_2)$, then there exists $(C_3, x_2) \in \mathcal{R}$ such that $C_3 \subseteq (C_1 \cup C_2) - x_1$.

If the rooted circuits are used to determine an antimatroid, \mathcal{A} , we denote \mathcal{A} by (E, \mathcal{R}) . In section 9.5, we discuss a special subset of rooted circuits called critical circuits.

There is no antimatroid analog to the strong circuit elimination axiom of matroids.

Example B.2.4 illustrates this.

Example B.2.4. Let $\mathcal{A} = (E, \mathcal{R})$ be an antimatroid with ground set $E = \{a, b, c, d, e\}$ and let $\mathcal{R} = \{C_1, C_2, C_3\}$, where $C_1 = (\{a, b, c\}, c)$, $C_2 = (\{c, d, e\}, e)$, and $C_3 = (\{a, e\}, e)$. Then, $C_1 \cap C_2 = \{c\}$, $C_1 - C_2 = \{a, b\}$ and $(C_1 \cup C_2) - e = \{a, b, c, d\}$. But, $C_3 \not\subseteq (C_1 \cup C_2) - e$.

C Matroid Invariants and Greedoid Invariants

Theorem 6.2.2 shows that the greedoid Tutte polynomial, $h(G; t, z)$ is the universal g -invariant by showing it is the unique function satisfying the following three properties:

- (i) $h(I; t, z) = t$ and $h(L; t, z) = z$ where I is an isthmus and L is a loop,
- (ii) if $\{e\}$ is feasible, then

$$h(G) = h(G/e) + (t - 1)^{r(E) - r(E - e)} \cdot h(G - e),$$

(iii) if e is a loop or an isthmus, then

$$h(G) = h(G(e)) \cdot h(G - e).$$

Next, we define the direct sum of two greedoids.

Definition C.0.5. Let $G_1 = (E_1, \mathcal{F}_1)$ and $G_2 = (E_2, \mathcal{F}_2)$ be two greedoids defined on finite non-empty disjoint sets E_1 and E_2 . Let $E = (E_1 \cup E_2)$ and $\mathcal{F} = \{F_1 \cup F_2 | F_1 \in \mathcal{F}(G_1), F_2 \in \mathcal{F}(G_2)\}$. Then, $G = (E, \mathcal{F})$ is the **direct sum** of G_1 and G_2 and is denoted $G_1 \oplus G_2$.

Remark C.0.6. If $\{e\} \in \mathcal{F}_1$, then $\{e\} \in \mathcal{F}$ since $\{e\} = \{e\} \cup \emptyset$

Property (iii) can be extended to the direct sum, $G_1 \oplus G_2$ of two greedoids.

Proposition C.0.7. If $G = G_1 \oplus G_2$, then $h(G) = h(G_1) \cdot h(G_2)$.

Proof. We use induction of the size of E_1 . If $|E_1| = 1$, then the proposition reduces to Property (iii) and thus is true. Suppose $h(G) = h(G_1) \cdot h(G_2)$ for $|E_1| = n - 1$. Consider the case when $|E_1| = n$. If E_1 consists of n loops, then delete a loop, e , from G_1 to get $h(G) = h(G(e)) \cdot h(G - e) = h(G(e)) \cdot h(G_1 - e \oplus G_2)$. If there is a feasible element, e , in E_1 , then delete and contract e . Then, $h(G) = h(G/e) + (t - 1)^{r(G) - r(G - e)} \cdot h(G - e)$ and, by the inductive hypothesis, $h(G) = h(G_1/e) \cdot h(G_2) + (t - 1)^{r(E) - r(E - e)} h(G_1 - e) \cdot h(G_2) = h(G_2) \cdot (h(G_1/e) + (t - 1)^{r(E) - r(E - e)} h(G_1 - e))$. Therefore, $h(G) = h(G_1) \cdot h(G_2)$. \square

Tables 5 and 6 contain summaries of the Tutte and corank-nullity polynomials for matroid and greedoid.

Table 5: The matroid Tutte and co-rank nullity polynomials

Matroids	Tutte polynomial	Corank-nullity polynomial
Definition	$t(M; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{ X -r(X)}$	$S(M; x, y) = \sum_{X \subseteq E} (x)^{r(E)-r(X)} (y)^{ X -r(X)}$
For isthmus, I , and loop, L	$t(I) = x, t(L) = y$	$S(I) = x + 1, S(L) = y + 1$
Recursion, for e , non-special element	$t(M) = t(M - e) + t(M/e)$	$S(M) = S(M - e) + S(M/e)$
Direct sum, $M = M_1 \oplus M_2$	$t(M) = t(M_1) \cdot t(M_2)$	$S(M) = S(M_1) \cdot S(M_2)$

Table 6: The greedoid Tutte and co-rank nullity polynomials

Greedoids	Tutte polynomial	Corank-nullity polynomial
Definition	$h(G; t, z) = \sum_{X \subseteq E} (t-1)^{r(E)-r(X)} (x-1)^{ X -r(X)}$	$f(G; t, z) = \sum_{X \subseteq E} (t)^{r(E)-r(X)} (x)^{ X -r(X)}$
For isthmus, I , and loop, L	$h(I) = t, h(L) = z$	$f(I) = t + 1, f(L) = z + 1$
Recursion, for feasible element, e	$h(G) = h(G/e) + (t-1)^{r(E)-r(E-e)} \cdot h(G-e)$	$f(G) = f(G/e) + t^{r(E)-r(E-e)} \cdot f(G-e)$
Direct sum, $G = G_1 \oplus G_2$	$h(G) = h(G_1) \cdot h(G_2)$	$f(G) = f(G_1) \cdot f(G_2)$

D Classes of Antimatroids

D.1 Poset antimatroids

To discuss poset antimatroids, we provide some basic facts about posets. Let $P = (E, \preceq)$ be a finite partially ordered set (poset). An **antichain** of P is a subset, A

of P , such that for any two elements $x, y \in A$, x and y are incomparable. A subset, $I \subseteq E$, is an **ideal** of P if, for $x \in I$, $y \preceq x$ implies $y \in I$. A subset, $B \subseteq E$, is an **order filter** of P if, for $x \in B$, $y \succeq x$ implies $y \in B$. An ideal is **strict** if we replace \preceq with \prec . A strict order filter is likewise defined when we replace \succeq by \succ . The order ideals, I , and antichains, A , of a poset are in a one-to-one correspondence. More precisely, the order ideal generated by A is $I(A) = \{x \in P : x \preceq y \text{ for some } y \in A\}$, and A consists of the maximal elements of I . See [21] for more details about posets.

Let \mathcal{F} be the set of ideals of E . That is, $\mathcal{F} = \{I(A) : A \text{ is an antichain of } P\}$. Then, (E, \mathcal{F}) is the **poset antimatroid** of P . The set of ideals of a poset are closed under both union and intersection. Hence, (\mathcal{F}, \subseteq) forms a distributive lattice. Here are a few facts about invariants for poset antimatroids. See [12] for more details and development.

- Let $I^*(A)$ denote the set of order filters of P and let $\overline{I^*(A)}$ denote the set of strict order filters.

$$h(G; t; z) = \sum_{A \in A(P)} (t-1)^{|I^*(A)|} (z)^{|\overline{I^*(A)}|},$$

where $A(P)$ is the set of antichains of P .

- Let $M(P)$ be the collection of maximal elements of P . Then, the characteristic polynomial, can be written as

$$p(\mathcal{A}, \lambda) = (-1)^{|E|} (1 - \lambda)^{|M(P)|}.$$

- The antimatroid is full, thus the greedoid polynomial $\lambda_G(\mathcal{A}) \equiv 1$.

- The β invariant can be expressed by

$$\beta = \begin{cases} 1 & \text{if there is only one maximal element in } P; \\ 0 & \text{otherwise.} \end{cases}$$

D.2 Unrooted trees

A class of greedoids is defined for an unrooted tree. A graph G is **connected** if every two nodes are joined by a path. An **unrooted tree**, $T = (V, E)$, is a connected graph without cycles. Let $\mathcal{F} = \{\text{complements of edge sets of unrooted trees}\}$. Then, (E, \mathcal{F}) is an antimatroid called the **edge pruning antimatroid of T** .

Let $R(T)$ denote the set of subtrees of T . For $S \in R(T)$, let $L_S = \{\text{leaves (as edges) of } S\}$.

Then,

- the Tutte polynomial of T is $h(T; t, z) = \sum_{S \in R(T)} (t - 1)^{|E(S)|} (z)^{|E(S)| - |L_S|}$.
- The characteristic polynomial of T is

$$p(T; \lambda) = (-1)^{|E|} \left(\sum_{v \in V} (1 - \lambda)^{\deg(v)} - |E| \cdot (1 - \lambda) \right).$$

- The β invariant satisfies the following:

$$|\beta| = \begin{cases} 1 & \text{if there is only one edge in } T \\ 0 & \text{if there are exactly two edges in } T. \end{cases}$$

Further,

$$|\beta| = (\text{number of interior points of } T) - 1$$

if T has more than two edges, where an interior point is a vertex that is incident to two or more edges.

D.3 Chordal graphs

A **simple** graph $G = (V, E)$ is one with no loops or parallel edges. A simple graph is **chordal** or **triangulated** if every cycle of length 4 or more has an edge joining two nonadjacent vertices. Such an edge is called a **chord**. A simple graph can be made into a chordal graph by adding chords. This process, shown in Figure 11, is called **triangulation**.

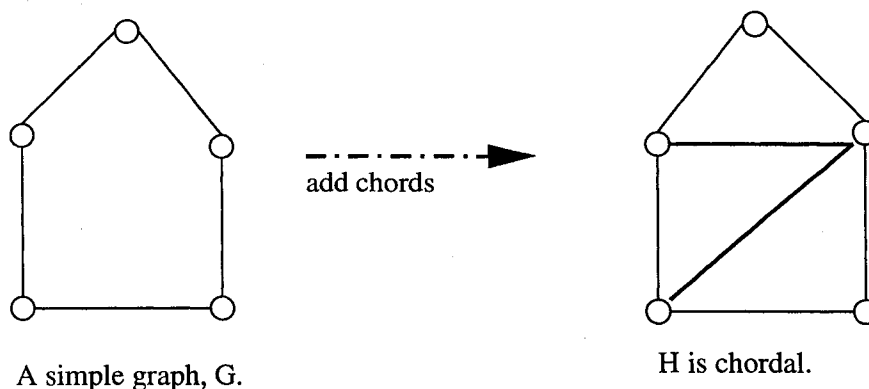
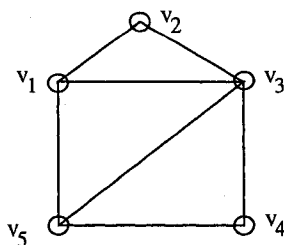


Figure 11: H is a triangulation of G .

For a vertex $v \in V$, let $N(v)$ denote the neighborhood of v i.e., the set of vertices u adjacent to v . For $A \subseteq G$, we let $G(A)$ denote the subgraph of G induced by A . A complete induced subgraph is called a **clique** of G . Thus, each pair of vertices in a clique are adjacent in G . A vertex, v , is **simplicial** if and only if the subgraph of $N(v)$ is a clique. See Figure 12 for an illustration.

Every chordal graph has at least two simplicial vertices. In Figure 12, vertices v_2 and v_4 are simplicial.

Let $|V| = n$ and $f : \{1, 2, \dots, n\} \rightarrow V$ be an **ordering** of G . Then, $f^{-1}(v)$ is the



v_2 is simplicial because its neighbors v_1 and v_3 are adjacent.

v_5 is not simplicial because v_4 is not adjacent to v_1 .

Figure 12: Simplicial and nonsimplicial vertices

position in the ordering assigned to the vertex, v . Define the set $N_f(v)$ to be the set of neighbors, u of v , such that $f^{-1}(u) > f^{-1}(v)$. Then, $N_f(v)$ consists of all neighbors of v that have a higher position in the ordering f . With this notation, we define a **perfect elimination ordering** of a graph, G , as an ordering, f , such that, for each $v \in V$, $N_f(v)$ is a clique. The following theorem characterizes graphs that have a perfect elimination ordering.

Theorem D.3.1 ([7], Theorem 2). *A graph G is chordal if and only if it has a perfect elimination ordering.*

If G is chordal, then the perfect elimination ordering is used to form an antimatroid.

Definition D.3.2. *Let G be a chordal graph with vertex set V and let $E = V(G)$.*

Let

$$\mathcal{F} = \{v : v \text{ occurs in a prefix of a perfect elimination ordering}\}.$$

Then, $\mathcal{A} = (E, \mathcal{F})$ is an antimatroid called the **simplicial vertex pruning antimatroid**.

For any graph, G , a **cut vertex** of G is a vertex that, when deleted, increases the number of components of G . A **block** of a graph, G , is a maximal subgroup of G with no cut vertex.

Proposition D.3.3 ([13], Theorem 5.1). *Let G be a connected chordal graph and let $b(G)$ be the number of the blocks of graph, G . Then, the β invariant is given by $\beta(G) = 1 - b(G)$.*

Let $\overline{N}(v) = N(V) \cup \{v\}$. Because the $v_i \in \mathcal{F}$ are simplicial, $\overline{N}(v_1) \cup \overline{N}(v_2) \cup \dots \cup \overline{N}(v_{i-1})$ are the cliques. Next, we relate the cliques of a graph to the dual convex geometry $\mathcal{C} = (E, \mathcal{F}^c)$ of \mathcal{A} .

Proposition D.3.4. *Let (E, \mathcal{F}) be a simplicial vertex pruning antimatroid of graph, G . A subset $K \subseteq E$ is a clique of G if and only if K is a free convex set of \mathcal{C} .*

Proof. Let K be a clique of G . Then, K is of the form $\overline{N}(v_1) \cup \overline{N}(v_2) \cup \dots \cup \overline{N}(v_{i-1})$. By Definition D.3.2, $K \in \{E - F \mid F \in \mathcal{F}\}$ and thus $K \in \mathcal{F}^c$. Hence, K is a convex set in \mathcal{C} . Also, any ordered subset of $K = \{\{\overline{N}(v_1) \cup \overline{N}(v_2) \cup \dots \cup \overline{N}(v_{i-1})\} \mid \{v_1, v_2, \dots, v_k\}, k \leq i\}$ is convex (because its elements are pairwise adjacent in G). Therefore, K is free.

Let K be a free convex set. Then, $F = E - C$ is feasible and so is every subset of F . Thus, $v \in F$ is a simplicial vertex of the subgraph induced by $K = \{v_k, v_{k+1}, \dots, v_n\}$ and, as such, K is a clique. □

E Greedoids as Exchange Languages

When we introduced greedoids in Section 3.1, we noted that they arose in the context of algorithms. Sometimes it is easier to define a greedoid as a language, as it is easier to depict the ordering of elements using a language. Here are some basic facts about languages. The information here is derived from [2] and [18].

Given a finite ground set, E , we can form sequences of elements of E . We call E the **alphabet** and form $E^* = \{x_1x_2x_3\dots x_n \mid x_i \in E, 1 \leq i \leq n\}$. The sequences in E^* are **words**; a collection of words $\mathcal{L} \subseteq E^*$ is a **language** over E . Letters from the alphabet are usually denoted by lower case Latin letters such as x, y and z , while lower case Greek letters such as α, β , and γ are reserved for words in the language. If $\alpha \in \mathcal{L}$, then α is a **feasible word**.

- If $\alpha = x_1x_2x_3\dots x_k$, then $\alpha y = x_1x_2x_3\dots x_k y$. This operation is known as **concatenation**.
- The **support** $\tilde{\alpha}$ of a word α is defined by $\tilde{\alpha} = \{\text{letters that comprise } \alpha\}$.
- For $\alpha \in \mathcal{L}$, such that $\alpha = x_1x_2x_3\dots x_n$, $|\alpha| = n$
- The support $\tilde{\mathcal{L}}$ of the language \mathcal{L} is the set system $\tilde{\mathcal{L}} = \{\tilde{\alpha} \mid \alpha \in \mathcal{L}\}$.
- A word is **simple** if none of its letters is repeated. That is, if $|\alpha| = |\tilde{\alpha}|$.
- A language is simple if all of its words are simple. We use E_s^* to denote the collection of simple words of a language.

- A language is **hereditary** if $\alpha = \beta y$ and $\alpha \in \mathcal{L}$ implies that $\beta \in \mathcal{L}$.
- The set of all letters that can be used to augment a word α to a larger feasible word is called the **set of continuations of α** and is denoted $\Gamma(\alpha)$.
- If $\alpha \in \mathcal{L}$, then $\tilde{\alpha} = A \subseteq E$. For $A \subseteq E$, a word $\alpha \in \mathcal{L}$ with $\tilde{\alpha} \subseteq A$ is a **basic word of A** if $\alpha x \notin \mathcal{L}$ for all $x \in A$.
- Recall that the exchange axiom (G3) for a greedoid $G = (E, \mathcal{F})$ reads as follows: for all $X, Y \in \mathcal{F}$ with $|X| > |Y|$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$. This axiom can be expressed in terms of languages.

If $\alpha, \beta \in \mathcal{L}$, with $|\alpha| > |\beta|$, then α contains a letter x such that $\beta x \in \mathcal{L}$.

Now, we can define a greedoid language.

Definition E.0.5. A *greedoid language* over a finite set E is a pair (E, \mathcal{L}) in which \mathcal{L} is a simple, hereditary language that obeys the exchange axiom, (G3).

The greedoid definitions of rank, bases and feasible sets all apply to greedoid languages.

Proposition E.0.6 ([2], Proposition 8.2.3). *Greedoids and greedoid languages are equivalent. More precisely,*

- (1) if (E, \mathcal{L}) is a greedoid language and $\mathcal{F} = \tilde{\mathcal{L}}$, then (E, \mathcal{F}) is a greedoid;
- (2) Let $\mathcal{L}(\mathcal{F}) = \{x_1, x_2, \dots, x_k \in E_s^* \mid \{x_1, x_2, \dots, x_k\} \in \mathcal{F}, 1 \leq i \leq k\}$. If (E, \mathcal{F}) is a greedoid, then $(E, \mathcal{L}(\mathcal{F}))$ is a greedoid language.

We now have a fourth, very useful, characterization of greedoids in terms of languages. This definition is known as the *ordered* version because the order in which letters are chosen is explicit and important. Because greedoids and greedoid languages are equivalent, it is common in the literature to use them interchangeably and write $G = (E, \mathcal{L})$ for $G = (E, \mathcal{L}(\mathcal{F}))$.

E.1 Computing the Möbius function using languages

For an arbitrary greedoid, the poset of feasible sets may not have much structure so the shortcuts detailed in [21] may not apply. However, the following theorem can be applied to any poset.

Theorem E.1.1 ([21], Corollary 3.8.5). *Let P be a poset with $x, y \in P$. If $\lambda(x, y; k)$ denotes the number of chains of length k that can be interpolated between x and y , then $\mu(x, y) = 1 + \sum_k (-1)^k \cdot \lambda(x, y; k)$.*

Remark E.1.2. *This theorem is due to Philip Hall and is sometimes known as Hall's First Theorem. We can use this theorem to calculate μ_G . To do so, we use the corresponding greedoid language, \mathcal{L} . For a greedoid language, \mathcal{L} , $\lambda(x, y; k)$ equals the number of words of a given length that begin with x and end with y . This gives an alternative way of calculating μ .*

E.2 Greedoids and the greedy algorithm

As mentioned previously, greedoids were developed by Korte and Lovász to describe set systems that arose from the application of various greedy algorithms such as Prim’s algorithm for finding a minimum spanning tree. The very name greedoid is a melding of the words greedy and matroid. Korte and Lovász report that a colleague from North America declared, “Only a Hungarian and a German could misuse the English language that much to come up with such a name”. In this section, we look at how the greedy algorithm can be used to characterize greedoids.

There is a natural optimization problem that arises in many areas of combinatorics: Find the shortest, fastest, cheapest or largest option. More formally, the problem can be stated as follows: given a set system S of subsets of set E , and a weight function $\omega : E \rightarrow \mathbb{R}^+$, find a subset of maximum weight.

One strategy for solving this optimization is a greedy strategy that can be boiled down to this: don’t look back, don’t look ahead, just take the best you can get now. When formalized, this strategy is known as the **Greedy Algorithm**. We present it here in terms of greedoids.

Definition E.2.1 (The Greedy Algorithm). *Let $G = (E, \mathcal{F})$ and let $\omega : \mathcal{L}(\mathcal{F}) \rightarrow \mathbb{R}$ be an objective function to be maximized. The problem is to find α_0 such that:*

$$\omega(\alpha_0) = \max\{\omega(\alpha) : \alpha \text{ is a basic word of } \mathcal{L}(\mathcal{F})\}.$$

The greedy strategy can then be written as:

1. Set $\alpha = \emptyset$.
2. If $\Gamma(\alpha) = \emptyset$, then stop.
3. Choose $x \in \Gamma(\alpha)$ so that $\omega(\alpha x) \geq \omega(\alpha y)$ for all $y \in \Gamma(\alpha)$.
4. Set $\alpha = \alpha x$ and go to 2.

This is a simple but elegant strategy. The question is, “Does it always work?”; the answer is, “It depends on ω and $\mathcal{L}(\mathcal{F})$ ”. In fact, their structures must be agreeable in the sense that ω can’t punish us later for a valid choice made from $\mathcal{L}(\mathcal{F})$ earlier. This relationship is formally called \mathcal{L} -admissibility.

Definition E.2.2. *Given a greedoid $G = (E, \mathcal{L})$ and a function $\omega : \mathcal{L} \rightarrow \mathbb{R}$. We call ω \mathcal{L} -admissible (or compatible with \mathcal{L}) if for every $\alpha \in \mathcal{L}$ and $x \in \Gamma(\alpha)$ such that when $\omega(\alpha x) \geq \omega(\alpha y)$ for all $y \in \Gamma(\alpha)$, two conditions are met:*

(LA1) $\omega(\alpha\beta x\gamma) \geq \omega(\alpha\beta z\gamma)$ for all $z \in E$, $\beta, \gamma \in E^*$ such that $\alpha\beta x\gamma, \alpha\beta z\gamma \in \mathcal{L}$.

(LA2) $\omega(\alpha x\beta z\gamma) \geq \omega(\alpha z\beta x\gamma)$.

In words, \mathcal{L} -admissibility requires that if x is the best choice after α , then

(LA1) x is the best choice at every later iteration,

(LA2) it is always better to choose x first and z later than the other way around.

For the sake of clarity and completeness, we next discuss admissibility in terms of greedoids and feasible sets. To do so, we first note that some objective functions

depend only on the support of a word and not on the order the letters are chosen. Such objective functions are called **stable**. Given $\alpha \in \mathcal{L}$, there is a subset, $A \subseteq E$, such that A consists of the letters of α . If a function is stable, we define $\omega(A) = \omega(\alpha)$. If ω is stable, then (LA2) is trivially true. Furthermore, if ω is stable, we can rewrite (LA1) in terms of subsets: For $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{F}$, suppose $A \subseteq B, x \in E - B$. If $\omega(A \cup x) \geq \omega(A \cup \{y\})$ for all $y \in \Gamma(A)$, then $\omega(B \cup \{x\}) \geq \omega(B \cup \{z\}) \in \Gamma(B)$.

\mathcal{L} -admissible functions are the key to determining optimal solutions using greedy methods. The next proposition states this relationship explicitly.

Proposition E.2.3 ([18] Chapter XI, Theorem 1.3). *If (E, \mathcal{L}) is a greedoid, then the Greedy Algorithm gives an optimal solution for any \mathcal{L} -admissible objective function.*

To understand when a greedy strategy will work, we must find compatible objective functions and set systems. What follows are two key theorems about functions that are compatible with matroids and with greedoids. They help explain how Kruskal's and Prim's algorithms both work with the same objective function while producing differing set systems.

Definition E.2.4. *An objective function $f : \mathcal{L} \rightarrow \mathbb{R}$ is called **linear** if*

$$f(x_1 x_2 x_3 \dots x_k) = \sum_{i=1}^k \omega(x_i)$$

for some weight function, ω .

In Example 8.5.3 of [2], Björner and Ziegler state that linear functions are \mathcal{L} -admissible for matroids. We state this result here as a proposition and supply a supporting proof.

Proposition E.2.5. ([2], Example 8.5.3) *If (E, \mathcal{L}) is a matroid, then linear functions are \mathcal{L} -admissible.*

Proof. The feasible sets of a matroid are the independent sets. Because linear functions are stable, (LA2) holds.

Recall that (LA1) can be written in terms of sets: For $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{F}$, suppose $A \subseteq B, x \in E - B$. If $\omega(A \cup x) \geq \omega(A \cup \{y\})$ for all $y \in \Gamma(A)$, then $\omega(B \cup \{x\}) \geq \omega(B \cup \{z\}) \in \Gamma(B)$.

To show (LA1) holds, let f be a linear function and suppose $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{I}$, such that $A \subseteq B$, and $x \in E - B$. Also suppose that for all $y \in \Gamma(A)$ we have $f(A \cup \{x\}) \geq f(A \cup \{y\})$. If we choose $z \in \Gamma(B)$ we want to show that $f(B \cup \{x\}) \geq f(B \cup \{z\})$. Since $A \subseteq B$, $z \in \Gamma(B)$ implies $z \in \Gamma(A)$. Then, $f(A \cup \{x\}) \geq f(A \cup \{z\})$. For $A = \{a_1, a_2, a_3, \dots, a_k\}$, $\sum \omega(a_1 a_2 a_3 \dots a_k) + \omega(x) \geq \sum \omega(a_1 a_2 a_3 \dots a_k) + \omega(z)$. For $b \in B - A$, $\sum \omega(a_1 a_2 a_3 \dots a_k) + \omega(x) + \omega(b) \geq \sum \omega(a_1 a_2 a_3 \dots a_k) + \omega(z) + \omega(b)$. Thus, $\sum \omega(a_1 a_2 a_3 \dots a_k b x) \geq \sum \omega(a_1 a_2 a_3 \dots a_k b z)$, which implies $f(B \cup \{x\}) \geq f(B \cup \{z\})$.

□

In section 3.2 we discuss two algorithms for finding a minimum spanning tree on a connected graph. More formally, given a connected graph $G = (V, E)$ with a weight function $u : E \rightarrow \mathbb{R}$, find a spanning tree, T of minimum weight. If we express

Kruskal's algorithm in terms of Theorem E.2.5, we have $f(T) = \sum_{e \in T} -u(e)$ and (E, \mathcal{L}) is the cycle matroid on G . Then, Kruskal's algorithm will find an optimal solution according to Proposition E.2.3.

Proposition E.2.5 states that if (E, \mathcal{L}) is a matroid, then any linear function is \mathcal{L} -admissible. However, there are greedoids with this property that are not matroids.

Definition E.2.6. *Given $G = (E, \mathcal{F})$, we can define the **hereditary closure** of \mathcal{F} as follows:*

$$H(\mathcal{F}) = \{Y \mid Y \subseteq X \in \mathcal{F}\}.$$

If $H(\mathcal{F}) = \mathcal{F}$, then G will be a matroid. Also, $H(\mathcal{F})$ may or may not be the set of feasible sets of a greedoid. The hereditary closure allows us to find precisely those greedoids for which linear functions are \mathcal{L} -admissible.

Theorem E.2.7 ([2], **Proposition 8.5.6**). *Let (E, \mathcal{F}) be a greedoid. The Greedy Algorithm will optimize any linear objective function on (E, \mathcal{F}) if and only if the hereditary closure $(E, H(\mathcal{F}))$ is a matroid and every set that is closed in (E, \mathcal{F}) is closed in $(E, H(\mathcal{F}))$.*

The hereditary closure of the undirected branching greedoid on the connected rooted graph $G = (V, E, r)$ is the cycle matroid of G . Given $G = (V, E)$, designate a node in V as the root, r . Then Prim's algorithm is equivalent to the greedy optimization of the linear function $f(T) = \sum_{e \in T} -u(e)$ over the undirected branching greedoid of $G = (V, E, r)$.

E.3 More on greedoid optimization

For a general greedoid, linear functions are not always \mathcal{L} -admissible; a generalized bottleneck function is always \mathcal{L} -admissible.

Definition E.3.1. *Given a set of elements, E , let $f : E \times \mathbb{N} \rightarrow \mathbb{R}$ satisfy $f(x, k) \leq f(x, k + 1)$ for all $x \in E, k \in \mathbb{N}$. Given $\alpha = x_1x_2x_3\dots x_k$, in which $x_j \in E, 1 \leq j \leq k$, let $\omega(\alpha) = \min\{f(x_i, i) \mid 1 \leq i \leq k\}$. Then, f is called a **generalized bottleneck function**.*

Proposition E.3.2 ([18], Theorem 1.3). *The generalized bottleneck function is \mathcal{L} -admissible for any greedoid $G = (E, \mathcal{L})$.*

Proposition E.3.3 ([18] Theorem 1.4). *If (E, \mathcal{L}) is a greedoid, then the Greedy Algorithm gives an optimal solution for any generalized bottleneck function.*

The shortest path problem can be defined as follows: let $D = (V, E, r)$ be a connected rooted digraph with length function $d : E \rightarrow \mathbb{R}^+$ on the arc set of D . For each $v \in V$ find a directed path from r to v of least cost, if one exists.

Dijkstra's algorithm is one technique for solving the shortest path problem. It greedily optimizes a generalized bottleneck function. To find an optimal solution using Dijkstra's algorithm, let $G = (E, \mathcal{F})$ be the directed branching greedoid of D and let $\mathcal{L} = \mathcal{L}(\mathcal{F})$ as defined Proposition E.0.6. Define an objective function $\omega : \mathcal{L} \rightarrow \mathbb{R}$ by $\omega(x_1x_2x_3\dots x_k) = -\sum_{i=1}^k d(r, v_i)$, in which v_i is the head node of the arc x_i reached by the branching $x_1x_2x_3\dots x_n$ and $d(r, v_i)$ is the sum of the arc lengths

on the unique path from r to v_i in this branching. See [8] for details about approaches to solving this and other bottleneck problems.

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