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Recovering a ring from a space determined by its prime ideals

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RECOVERING A RING FROM A SPACE

DETERMINED BY ITS PRIME IDEALS

By

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TABLE OF CONTENTS

INTRODUCTION

For certain classes of rings, the entire structure of the ring is determined by the prime ideal structure and that of the quotient rings by the prime ideals. However, for many rings this information is not valuable. In fields, for example, the only prime ideals are the trivial ones, and the quotient rings are just $\{0\}$ and the field itself. In fact, for most rings the quotient rings will be no simpler in structure than the original ring.

To investigate the structure of such rings more closely, Harrison [1] has used the notion of primes in rings with identity. An advantage gained is shown by fields, which have a much less trivial structure of primes than of prime ideals. The theory there is closely related to the valuation theory for fields [2]. Moreover, a finite prime P in a ring determines a subring A_P of which it is a prime ideal, and the quotient rings A_P/P, being locally finite fields, are, in a sense, completely known.

For commutative nil semi-simple rings with identity (see the definitions), Manis [3] has shown to what extent a ring is determined by its prime structure. More precisely, a topological space Arith R is defined in terms of the primes of R. This space determines a new ring $\overline{R}$ consisting of a certain class of continuous functions within Arith R. It is seen that $\bar{R}$ is an integral extension of R with Arith $\bar{R}$ homeomorphic to Arith R. Thus $\bar{\bar{R}} \cong \bar{R}$, and the rings $\bar{R}$ are ones determined by their primes in this way. Last, necessary and sufficient conditions are found that a ring R is isomorphic with $\bar{R}$, so it is known just what rings are determined by their primes in this way.

The object of this paper is to show that it is precisely the same rings which are determined by their prime ideals in an analogous way. Here also are discussed only commutative nil semi-simple rings with an identity $1 \neq 0$. Throughout this paper "ring" will mean a ring of this type unless otherwise specified. First a topological space V(R) analogous to Arith R will be defined in terms of the prime ideals of a ring R. Then σ-functions, from a subspace of V(R) into V(R), will be defined and will be shown to constitute a ring $\hat{R}$. This ring is likewise an integral extension of R with $V(\hat{R})$ homeomorphic to V(R); in fact, each element of $\hat{R}$ is a root of a monic polynomial in R[x] which splits completely in R[x]. Finally, conditions will be given that $R \cong \hat{R}$. These conditions are precisely the same conditions that $R \cong \bar{R}$. Thus it is the same rings that are determined by their prime ideals as are determined by their primes in this way. In fact, we will see that $\hat{R} \cong \bar{R}$ for each ring R, and the procedure here leads to the same ring as the procedure in [3].

Of the three chapters in this paper, the first one is concerned with characterizing σ-functions, the elements of $\hat{R}$. The second concerns the properties of $\hat{R}$ as a ring and its relation to R and $\bar{R}$. Last, we discuss some examples of rings and ask whether or not they are rings with $R \cong \hat{R}$. The only knowledge assumed is elementary principles in ring theory and topology.

Definitions and Notation. Let R be a ring. If A is an ideal of R, the radical of A, denoted $\sqrt{A}$, is the set of all elements x of R such that $x^n \in A$ for some $n \geq 1$. The radical of an ideal is an ideal. The nil radical of R is $\sqrt{\{0\}}$. R is nil semi-simple when $\sqrt{\{0\}} = \{0\}$. A pre prime of R is a non-empty subset P which is closed under subtraction and multiplication and does not contain 1. A prime of R in the sense of [3] is a maximal pre prime. (This is a finite prime in the sense of [1].) For P a prime of R, $A_P = \{x \in R \mid xP \subset P\}$. Notice that A_P is a subring of R and P is an ideal of A_P. By an ideal of R we always mean a proper ideal.

If R is a ring, then V(R) is given by

$$V(R) = \{a + P \mid P \text{ is a prime ideal of } R, a \in R\}.$$

We consider the Stone topology on V(R). This is the topoogy with a basis for the open sets consisting of the collection of all sets of the form $N(E) = \{X \in V(R) \mid X \cap E = \emptyset\}$, where E is a finite subset of R. It is clear that this collection does form a basis for the open sets of a topology, since it is closed under finite intersections.

We will also be concerned with the subset $V_o(R) = \{P \mid P \text{ is a prime ideal of } R\} = \{O + P \mid P \text{ is a prime ideal of } R\}$, with the relative topology induced by that on $V(R)$. The assumption that we have $1 \neq 0$ in our rings guarantees that $V_o(R)$ is non-empty.

CHAPTER I

σ-FUNCTIONS ON THE SPACE V(R)

Before we define the set $\hat{R}$ of σ-functions from $V_o(R)$ into V(R) for a ring R, we need to know more about the topology on $V_o(R)$. Notice that a basic open set in $V_o(R)$ is of the form $N(E) \cap V_o(R) = \{P \in V_o(R) | P \cap E = \emptyset\}$. That is, the relative topology on $V_o(R)$ is just the Stone topology which is given directly for $V_o(R)$. We will also be concerned with sets of the following form:

<u>Definition 1</u>. Let $a \in R$. Then $\mathcal{P}(a) = \{X \in V(R) | a \in X\}$ is called the <u>path</u> through V(R) determined by a. Notice that for $a \in R$, $\mathcal{P}(a) = \{a + P | P \in V_o(R)\}$.

Let R be a ring and let E and F be two finite subsets of R. Notice that $N(E) \cap N(F) = N(E \cup F)$. In particular, if $E = \{e_1, e_2, \ldots, e_n\}$ is a finite subset of R, then $N(E) = N(\bigcup_{i=1}^{n}\{e_i\}) = \bigcap_{i=1}^{n} N(\{e_i\})$. Thus the basis of sets N(E) is generated by the subbasis of sets $N(\{a\})$ with $a \in R$. We will always denote $N(\{a\})$ by N(a). Notice that $N(a) = \{X \in V(R) | a \notin X\} = V(R) \setminus \mathcal{P}(a)$ so that the paths form a subbasis for the closed subsets of V(R). In particular, since $a + P = 0 + P$ if and only if $0 \in a + P$ for any $P \in V_o(R)$ and $a \in R$, it follows that

$$V_o(R) = \{0 + P | P \text{ is a prime ideal of } R\}$$
$$= \{a + P | a \in R, P \text{ is a prime ideal of } R, \text{ and } 0 \in a + P\}$$

$$= \{X \in V(R) \mid 0 \in X\} = \mathcal{P}(0),$$

and so $V_0(R)$ is closed in $V(R)$.

For purposes of computation we will need the following well-known result due to Krull (see [4, p. 73]), which we state without proof. This result holds for more general rings than we are considering. In particular, it holds for commutative rings.

Lemma 1.1. Let R be a ring and A be an ideal of R. If M is a multiplicative system in R with $A \cap M = \emptyset$, then there is a prime ideal P of R with $A \subset P$ and $P \cap M = \emptyset$.

We will also use another result that is an easy consequence of the first. It also holds for any commutative ring.

Lemma 1.2. Let A be an ideal of a ring R. If $E = \{e_1, e_2, \ldots, e_n\}$ is a finite subset of R with $\prod_{i=1}^{n} e_i \notin \sqrt{A}$ then there is a prime ideal P of R with $A \subset P$ such that $E \cap P = \emptyset$.

Proof. Let $M = \{\prod_{i=1}^{n} e_i^{k_i} \mid k_i$ is a non-negative integer for $i = 1, 2, \ldots, n\}$. Suppose that $\prod_{i=1}^{n} e_i^{k_i} \in M \cap A$. Then $(\prod_{i=1}^{n} e_i)^m \in A$ whenever $m \geq \max\{k_1, k_2, \ldots, k_n\}$ since R is commutative and A is an ideal. But then $\prod_{i=1}^{n} e_i \in \sqrt{A}$, contradicting the hypothesis. Thus $M \cap A = \emptyset$. But M is a multiplicative system in R, since R is commutative. Hence by lemma 1.1 there is a prime ideal P of R with $A \subset P$ and $P \cap M = \emptyset$. But since $E \subset M$ we also get that $E \cap P = \emptyset$.

Another consequence of lemma 1.1 which we have for all commutative rings is

<u>Lemma 1.3</u>. Let R be a ring. If A is any ideal of R, then $\sqrt{A} = \cap\{P \in V_o(R) \mid A \subset P\}$. In particular, the nil radical of R is the intersection of all prime ideals of R.

<u>Proof</u>. Let $a \in \sqrt{A}$. Then $a^n \in A$ for some integer $n \geq 1$. Let P be any prime ideal of R with $A \subset P$. Then $a^n \in P$ so $a \in P$. Thus $a \in \cap\{P \in V_o(R) \mid A \subset P\}$, and $\sqrt{A} \subset \cap\{P \in V_o(R) \mid A \subset P\}$.

Conversely, let $a \in \cap\{P \in V_o(R) \mid A \subset P\}$. Suppose $a \notin \sqrt{A}$. Then $\{a^n \mid n \text{ is a positive integer }\} = M$ is a multiplicative system in R, and $M \cap A = \emptyset$. Thus there is a prime ideal $P \supset A$ of R with $M \cap P = \emptyset$. Further, $a \in M$ so that $a \notin P$, a contradiction. Hence $a \in \sqrt{A}$ so that $\sqrt{A} = \cap\{P \in V_o(R) \mid A \subset P\}$.

In what follows, we will want to distinguish elements of R by their paths in V(R). If R were not nil semi-simple, this would not be possible. But in a ring of the type we are considering we have

<u>Theorem 1.4</u>. Let $a \in R$, $b \in R$, and $a \neq b$. Then there is a prime ideal P of R such that $a + P \neq b + P$.

<u>Proof</u>. If $a \neq b$, then $a - b \neq 0$, and $a - b \notin \sqrt{\{0\}}$; so there is a prime ideal P of R with $a - b \notin P$. Hence $a + P \neq b + P$, and we are done.

Now we are ready to define σ-functions. For the remainder of the first two chapters let R be a fixed ring, let $V = V(R)$, and let $V_o = V_o(R)$.

Definition 2. A function $f\colon V_o \longrightarrow V$ is called a σ-function if it satisfies:

1) $f(P) \in R/P$ for all $P \in V_o$.

2) f is continuous and closed.

Thus a σ-function f is a homeomorphism of V_o and $f(V_o)$; it is one-to-one because $(R/P) \cap (R/Q) = \emptyset$ if $P \neq Q$. The set of all σ-functions is called $\hat{R}$.

The most obvious candidate for a σ-function is a map $P \longrightarrow a + P$ for some fixed $a \in R$. We will first check that such maps are indeed σ-functions.

Definition 3. For each $a \in R$ define $f_a\colon V_o \longrightarrow V$ by $f_a(P) = a + P$ for each $P \in V_o$. f_a is called a path function.

Theorem 1.5. Let $a \in R$. Then f_a is a σ-function.

Proof. Clearly f_a satisfies condition 1). For condition 2), let $N(b)$ be a subbasic open set in V. Then $(f_a)^{-1}(N(b)) = \{P \in V_o \mid a + P \in N(b)\} = \{P \in V_o \mid b \notin a + P\}$ $= \{P \in V_o \mid b - a \notin P\} = N(b-a) \cap V_o$ is open in V_o, so f_a is continuous. Let $\mathcal{P}(b) \cap V_o$ be a subbasic closed set in V_o. Then $f_a(\mathcal{P}(b) \cap V_o) = \{a + P \mid P$ is a prime ideal of R, $b \in P\} = \{a + P \mid P$ is a prime ideal of R and $a + b \in a + P\}$ $= \mathcal{P}(a) \cap \mathcal{P}(a+b)$ is closed in V. Thus f_a is closed. In fact, f_a is clearly a homeomorphism of $\mathcal{P}(0)$ and $\mathcal{P}(a)$.

To make $\hat{R}$ into a ring which contains R isomorphically as a subring, addition and multiplication will be defined pointwise in terms of the operations on the quotient rings R/P. We first check that this is possible for path functions.

Theorem 1.6. Let $a \in R$ and $b \in R$. Then

1) $f_a = f_b$ if and only if $a = b$.

2) The path function $f_{a+b} = f$ has the property that $f(P) = f_a(P) + f_b(P)$ for all $P \in V_o$.

3) The path function $g = f_{ab}$ has the property that $g(P) = f_a(P)f_b(P)$ for all $P \in V_o$.

Proof. If $a = b$, then clearly $f_a = f_b$. Conversely, if $a \neq b$, then by theorem 1.4 there is some $P \in V_o$ such that $a + P \neq b + P$. Thus $f_a(P) \neq f_b(P)$ and $f_a \neq f_b$. To prove 2) note that $f_{a+b}(P) = a + b + P = (a + P) + (b + P) = f_a(P) + f_b(P)$ for each $P \in V_o$. Similar considerations hold for 3).

Thus we can define addition of path functions pointwise by saying $(f_a + f_b)(P) = f_a(P) + f_b(P)$ for each $P \in V_o$, and by theorem 1.6 the result is another path function (f_{a+b}). The same holds for multiplication, and it is clear that with these operations the subset of path functions of $\hat{R}$ constitutes a ring isomorphic with R via the correspondence $a \longleftrightarrow f_a$. It will be shown in chapter II that these pointwise definitions yield a σ-function if one starts with arbitrary σ-functions. To prove this, it will first be useful to characterize arbitrary σ-functions in such a way as to associate a finite subset of R with each σ-function f in a generalization of the way an element $a \in R$ is associated with f_a. This procedure requires the notion of irreducible subsets of V_o.

Definition 4. Let X be any topological space. A non-empty subset $T \subset X$ is said to be irreducible if whenever G_1 and G_2 are open subsets of X with $T \cap G_1 \neq \emptyset$ and $T \cap G_2 \neq \emptyset$, then $T \cap G_1 \cap G_2 \neq \emptyset$. An equivalent formulation is to say that the above holds for any two sets in some basis for the open sets of X.

An example of an irreducible subset of V_o is $\{P\}$, for any $P \in V_o$. We now show some elementary properties of irreducible sets and characterize a special type of irreducible subset of V_o.

Lemma 1.7. Let X and Y be topological spaces, and let T be irreducible in X. Then

1) $\overline{T}$ is also irreducible.

2) If $T \subset \bigcup_{i=1}^{n} F_i$, and each F_i is closed in X, then $T \subset F_i$ for some i.

3) If $f: X \longrightarrow Y$ is continuous, then f(T) is irreducible in Y.

4) If A is dense in T, then A is also irreducible.

Proof. Let G_1 and G_2 be open in X with $\overline{T} \cap G_1 \neq \emptyset$ and $\overline{T} \cap G_2 \neq \emptyset$. If $T \cap G_1 = \emptyset$ then $T \subset \backslash G_1$ which is closed in X. But then $\overline{T} \subset \backslash G_1$ so that $\overline{T} \cap G_1 = \emptyset$, contradicting the above. Hence $T \cap G_1 \neq \emptyset$, $T \cap G_2 \neq \emptyset$, and $T \cap G_1 \cap G_2 \neq \emptyset$. Thus also $\overline{T} \cap G_1 \cap G_2 \neq \emptyset$, and $\overline{T}$ is irreducible.

To prove 2), suppose that $T \subset \bigcup_{i=1}^{n} F_i$ with each F_i closed, but for each i, $T \not\subset F_i$. Then $T \cap (\setminus F_i) \neq \emptyset$ for each i, and by induction $T \cap (\bigcap_{i=1}^{n} \setminus F_i) \neq \emptyset$, since T is irreducible and each $\setminus F_i$ is open. Hence $T \not\subset \setminus(\bigcap_{i=1}^{n} \setminus F_i) = \bigcup_{i=1}^{n} F_i$, a contradiction. Therefore, 2) must be true.

3) and 4) likewise follow readily from the definition.

<u>Theorem 1.8</u>. For each $\alpha \in V_o$ set $T_\alpha = \{P \in V_o | \alpha \subset P\}$. Then a set $T \subset V_o$ is closed and irreducible if and only if $T = T_\alpha$ for some $\alpha \in V_o$. In this case, $\alpha = \cap T$. More generally, if T is any irreducible subset of V_o, then $\alpha = \cap T$ is a prime ideal of R and $T_\alpha = \overline{T}$.

<u>Proof</u>. Let T be an irreducible subset of V_o, and let $\alpha = \cap T = \cap\{P \in V_o | P \in T\}$. We claim that α is a prime ideal of R. The intersection of any non-empty collection of ideals of R is again an ideal of R, so α is clearly an ideal. Suppose $a \not\in \alpha$ and $b \not\in \alpha$. Then there are prime ideals P and Q of R with $P \in T$ and $Q \in T$ such that $a \not\in P$ and $b \not\in Q$. Hence $P \in N(a)$ and $Q \in N(b)$. But T is irreducible, $T \cap (N(a) \cap V_o) \neq \emptyset$, and $T \cap (N(b) \cap V_o) \neq \emptyset$; so $T \cap N(a) \cap N(b) \neq \emptyset$. Let $P' \in T \cap N(a) \cap N(b)$. Then $a \not\in P'$ and $b \not\in P'$, so $ab \not\in P'$. Hence $ab \not\in \alpha$, since $P' \in T$. Therefore, α is a prime ideal of R.

Let $P \in \overline{T}$. If $\alpha \not\subset P$, then let $a \in \alpha \setminus P$. Then $a \not\in P$ so $P \in N(a)$. But $P \in \overline{T}$, so $T \cap N(a) \neq \emptyset$. Let $Q \in$

$T \cap N(a)$. Then $a \notin Q$ and $Q \in T$. Hence $a \notin \alpha = \cap T$, a contradiction. Thus $\alpha \subset P$ and $P \in T_\alpha$. Conversely, let $P \in T_\alpha$. Suppose that $P \in N(E)$ for some set $E = \{e_1, e_2, \ldots, e_n\}$. Then $P \cap E = \emptyset$ so $\alpha \cap E = \emptyset$ also. If $T \cap N(E) = \emptyset$, then $T \subset \bigcup_{i=1}^{n} \mathcal{P}(e_i)$, so by lemma 1.7 $T \subset \mathcal{P}(e_i)$ for some i. Hence $e_i \in \alpha$ so that $\alpha \cap E \neq \emptyset$, a contradiction. So $T \cap (N(E) \cap V_o) = T \cap N(E) \neq \emptyset$, and every basic neighborhood of P intersects T. Hence $P \in \overline{T}$, and $T_\alpha = \overline{T}$.

For the first part, suppose $T \subset V_o$ is closed and irreducible. Then $\alpha = \cap T$ is a prime ideal of R and $T = \overline{T} = T_\alpha$. Notice that the closure of T in V_o is the same as the closure of T in V, since $V_o = \mathcal{P}(0)$ is closed in V.

Conversely, let $\alpha \in V_o$. We need to show that T_α is closed and irreducible in V_o. Let $G_1 = N(E) \cap V_o$ and $G_2 = N(F) \cap V_o$ be basic open subsets of V_o with $T_\alpha \cap G_1 \neq \emptyset$ and $T_\alpha \cap G_2 \neq \emptyset$. Then $T_\alpha \cap N(E) \neq \emptyset$ and $T_\alpha \cap N(F) \neq \emptyset$, and there is some $P_1 \in V_o$ and $P_2 \in V_o$ with $\alpha \subset P_1$, $\alpha \subset P_2$, $E \cap P_1 = \emptyset$, and $F \cap P_2 = \emptyset$. Consequently, $\alpha \in N(E) \cap N(F)$ and $T_\alpha \cap G_1 \cap G_2 \neq \emptyset$. Thus T_α is irreducible. Further, suppose that $P \in V_o \setminus T_\alpha$. Then $\alpha \not\subset P$, so let $a \in \alpha \setminus P$. Now $a \notin P$, so $P \in N(a)$. Furthermore, $N(a) \cap T_\alpha = \emptyset$, for if $Q \in N(a)$, then $a \notin Q$, $\alpha \not\subset Q$, and $Q \notin T_\alpha$. Hence $P \notin \overline{T}_\alpha$. Thus $\overline{T}_\alpha \subset T_\alpha$ and T_α is closed.

Finally, if T is irreducible and $T = T_\alpha$ for some $\alpha \in V_o$, then $\alpha = \cap T_\alpha = \cap T$. This completes the proof of the theorem.

Theorem 1.9. The maximal irreducible subsets of V_o are the sets T_α, where α is a minimal prime ideal of R. The map $\alpha \longrightarrow T_\alpha$ is a one-to-one correspondence between the minimal prime ideals of R and the maximal irreducible subsets of V_o.

Proof. Let T be a maximal irreducible subset of V_o. Then $\overline{T}$ is irreducible and $T \subset \overline{T}$; so $T = \overline{T}$ and T is closed. Thus $T = T_\alpha$ for some $\alpha \in V_o$. Suppose there is some $\beta \in V_o$ with $\beta \subset \alpha$. Then $T_\alpha \subset T_\beta$, so $T_\alpha = T_\beta$ and $\beta \in T_\alpha$. Hence $\alpha \subset \beta$ and $\alpha = \beta$. Therefore, α is a minimal prime ideal of R.

Conversely, let α be a minimal prime ideal of R. Let T be an irreducible subset of V_o with $T_\alpha \subset T$. Then $T_\alpha \subset \overline{T} = T_\beta$, where $\beta = \cap T$. Hence $\alpha \in T_\beta$ and $\beta \subset \alpha$. But α is minimal so that $\alpha = \beta$. Thus $T \subset T_\beta = T_\alpha$ and $T = T_\alpha$, whence T_α is maximal.

Let Δ denote the collection of minimal prime ideals of R. For each $\alpha \in \Delta$, T_α is a maximal irreducible subset of V_o. Moreover, if $T_\alpha = T_\beta$ then $\alpha \subset \beta$ and $\beta \subset \alpha$ so that $\alpha = \beta$. Thus the map $\alpha \longrightarrow T_\alpha$ is injective. But if T is any maximal irreducible subset of V_o, then $T = T_\alpha$ for some $\alpha \in \Delta$; so the map is also surjective. Hence it is a one-to-one correspondence between Δ and the collection of maximal irreducible subsets of V_o, and the theorem is proved.

Notice that the collections above are non-empty; each $P \in V_o$ is seen to contain some minimal prime ideal by use of Zorn's lemma. With the aid of maximal irreducible subsets of V_o and minimal prime ideals of R we will begin to characterize arbitrary σ-functions.

Let f be any fixed σ-function, and let S = range f $= f(V_o)$. If $P \varepsilon V_o$, then $1 \notin P$ so $1 + P \neq P$. Therefore, $\{P, 1 + P\} \not\subset S$, since $f(Q) \varepsilon R/P$ only if $Q = P$; so we have that $S \neq V$. Pick an arbitrary element $X \varepsilon V \setminus S$. Since S is closed, there is a finite subset $E = \{a_1, a_2, \ldots, a_n\}$ of R such that $X \varepsilon N(E)$ and $N(E) \cap S = \emptyset$. Assume that E is a minimal set with these properties. Since $N(\emptyset) \cap S = S \neq \emptyset$, clearly $E \neq \emptyset$.and $n \geq 1$. If $Y \varepsilon S$, then $Y \notin N(E)$ so that $E \cap Y \neq \emptyset$. Thus $a_i \varepsilon Y$ for some i. But $Y = f(P)$ for some $P \varepsilon V_o$ so $Y = a_i + P$ for some $P \varepsilon V_o$, and so $Y \varepsilon \mathcal{P}(a_i)$. In other words, $S \subset \bigcup_{i=1}^{n} \mathcal{P}(a_i)$. Notice that $S \cap \mathcal{P}(a_i) \neq \emptyset$ for each i, since we could otherwise eliminate each a_i from E that gives $S \cap \mathcal{P}(a_i) = \emptyset$, and E would still have all of the properties above, contradicting the minimality of E.

Let $\alpha \varepsilon \Delta$. Then T_α is a maximal irreducible subset of V_o. But $f(T_\alpha) \subset S \subset \bigcup_{i=1}^{n} \mathcal{P}(a_i)$, and $f(T_\alpha)$ is irreducible in V by lemma 1.7. Therefore, $f(T_\alpha) \subset \mathcal{P}(a_i)$ for some i by the same lemma. Hence $f(P) = a_i + P$ whenever $\alpha \subset P$. Since each $P \varepsilon V_o$ contains a minimal prime ideal, we have that for each $P \varepsilon V_o$, $f(P) = a_i + P$ for some $a_i \varepsilon E$.

In fact, there is a finite set $\{A_1, A_2, \ldots, A_n\}$ of ideals of R with $f(P) = a_i + P$ whenever $P \supset A_i$ and such that each $P \in V_o$ contains some A_i. To see this, for each $i \in [1, n]$, where $[1, n]$ denotes $\{1, 2, \ldots, n\}$, let $\Delta_i = \{\alpha \in \Delta \mid f(T_\alpha) \subset \mathcal{P}(a_i)\}$, $D_i = f^{-1}(\mathcal{P}(a_i))$, and $A_i = \cap\Delta_i = \cap\{\alpha \in \Delta \mid \alpha \in \Delta_i\}$. Notice that for each $\alpha \in \Delta$, $f(T_\alpha) \subset \mathcal{P}(a_i)$ for some i as mentioned above; so $\Delta = \bigcup_{i=1}^{n}\Delta_i$ and each $\alpha \in \Delta$ is in some set Δ_i.

Furthermore, each A_i is a proper ideal of R. For this it is sufficient that each collection Δ_i be non-empty. Suppose that for some i, $f(T_\alpha) \subset \mathcal{P}(a_i)$ for no $\alpha \in \Delta$. Let $E' = E \setminus \{a_i\}$. Then $N(E) \subset N(E')$ so $X \in N(E')$. Also $N(E') \cap S = \emptyset$, for if $f(P) \in N(E') \cap S$, then let $\alpha \in \Delta$ with $\alpha \subset P$. This gives that $E' \cap f(P) = \emptyset$ so $f(P) = a_j + P$ only for $j = i$. But $f(T_\alpha) \subset \mathcal{P}(a_j)$ for some j and then $f(P) = a_j + P$ for this j. Hence $j = i$ and $f(T_\alpha) \subset \mathcal{P}(a_i)$, a contradiction. So in fact $N(E') \cap S = \emptyset$, contradicting the minimality of E. Thus for each $i \in [1, n]$, $f(T_\alpha) \subset \mathcal{P}(a_i)$ for some $\alpha \in \Delta$, and so each $\Delta_i \neq \emptyset$, whence A_i is a proper ideal of R for each i. We now find some additional information about the sets Δ_i, D_i, and A_i.

<u>Theorem 1.10</u>. For any $i \in [1, n]$, $\{P \in V_o \mid A_i \subset P\}$ $= \cap\{\bigcup_{k=1}^{m}\mathcal{P}(b_k) \mid m \geq 1,\ b_k \in R \text{ for each } k, \text{ and } \prod_{k=1}^{m} b_k \in A_i\}$ $\subset D_i$.

<u>Proof</u>. Since f is continuous, $D_i = f^{-1}(\mathcal{P}(a_i))$ is closed in both V_o and V. Thus D_i is the intersection of all basic closed sets which contain it; i.e., $D_i = \cap\{\bigcup_{k=1}^{m}\mathcal{P}(b_k) \mid D_i \subset \bigcup_{k=1}^{m}\mathcal{P}(b_k)\}$.

Let $F = \{b_1, b_2, \ldots, b_m\}$ be a finite subset of R such that $b = \prod_{k=1}^{m} b_k \notin A_i$. Then there is some $\alpha \in \Delta_i$ with $b \notin \alpha$. Thus $b_k \notin \alpha$ for all k, and $\alpha \cap F = \emptyset$ so that $\alpha \in N(F)$. But $\alpha \in \Delta_i$ so that $f(T_\alpha) \subset \mathcal{P}(a_i)$; hence $T_\alpha \subset f^{-1}(f(T_\alpha)) \subset f^{-1}(\mathcal{P}(a_i)) = D_i$, and $\alpha \in D_i \cap N(F) \neq \emptyset$. Therefore, $D_i \not\subset \setminus N(F) = \bigcup_{k=1}^{m} \mathcal{P}(b_k)$, and so $D_i \subset \bigcup_{k=1}^{m} \mathcal{P}(b_k)$ implies that $\prod_{k=1}^{m} b_k \in A_i$. We get then that $\{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid D_i \subset \bigcup_{k=1}^{m} \mathcal{P}(b_k)\} \subset \{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid \prod_{k=1}^{m} b_k \in A_i\}$. Hence $\bigcap\{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid \prod_{k=1}^{m} b_k \in A_i\} \subset \bigcap\{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid D_i \subset \bigcup_{k=1}^{m} \mathcal{P}(b_k)\} = D_i$.

To get the stated equality, let $P \in V_o$ with $A_i \subset P$. Let $F = \{b_1, b_2, \ldots, b_m\} \subset R$ such that $\prod_{k=1}^{m} b_k \in A_i$. Then $\prod_{k=1}^{m} b_k \in P$, so $b_k \in P$ for some k. This gives that $P \cap F \neq \emptyset$ and $P \notin N(F)$. Hence $P \in \setminus N(F) = \bigcup_{k=1}^{m} \mathcal{P}(b_k)$, and so $P \in \bigcap\{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid \prod_{k=1}^{m} b_k \in A_i\}$. Conversely, let $P \in \bigcap\{\bigcup_{k=1}^{m} \mathcal{P}(b_k) \mid \prod_{k=1}^{m} b_k \in A_i\}$. Then $P \in D_i$ so $P \in V_o$. For any $b = b_1 \in A_i$ we have that $P \in \bigcup_{k=1}^{1} \mathcal{P}(b_k) = \mathcal{P}(b_1)$, and $b \in P$. Therefore, $A_i \subset P$ and $P \in \{P \in V_o \mid A_i \subset P\}$.

<u>Theorem 1.11</u>. $\Delta_i = \{\alpha \in \Delta \mid A_i \subset \alpha\}$, and $\prod_{i=1}^{n} A_i = \{0\}$.

<u>Proof</u>. Let $\alpha \in \Delta$ with $A_i \subset \alpha$. Then $\alpha \in D_i$ by the last theorem. In fact, $A_i \subset P$ for all $P \in T_\alpha$, and so $T_\alpha \subset D_i$. Thus $f(T_\alpha) \subset f(D_i) = f(f^{-1}(\mathcal{P}(a_i))) \subset \mathcal{P}(a_i)$, and $\alpha \in \Delta_i$. Conversely, let $\alpha \in \Delta_i$. Then $A_i = \bigcap \Delta_i \subset \alpha$. Therefore, $\Delta_i = \{\alpha \in \Delta \mid A_i \subset \alpha\}$ as asserted.

For the second part, suppose $\prod_{i=1}^{n} A_i \neq \{0\}$. Then let

$b_i \in A_i$ for each $i \in [1, n]$ such that $\prod_{i=1}^{n} b_i \neq 0$. By lemma 1.3 there is some $P \in V_o$ such that $\prod_{i=1}^{n} b_i \notin P$. Thus we get $P \cap \{b_1, b_2, \ldots, b_n\} = \emptyset$, since P is an ideal. But $P \in T_\alpha$ for some $\alpha \in \Delta$, and $f(T_\alpha) \subset \mathcal{P}(a_i)$ for some i; so $\alpha \in \Delta_i$ for some i. It follows that $A_i \subset \alpha$ by the first part of this theorem, and so $A_i \subset P$. Hence $b_i \in P$, a contradiction. Therefore, $\prod_{i=1}^{n} A_i = \{0\}$.

Theorem 1.12. For each $(i,j) \in [1, n] \times [1, n]$, $a_i - a_j \in \sqrt{A_i + A_j}$.

Proof. If this were not so, then by lemma 1.2 there would be a prime ideal P with $\sqrt{A_i + A_j} \subset P$ and $a_i - a_j \notin P$ for some pair (i,j). But then $A_i \subset P$ so that $P \in D_i$. Likewise $P \in D_j$. Hence we would have $f(P) = a_i + P$ and $f(P) = a_j + P$ also, giving $a_i - a_j \in P$. But this is a contradiction.

With the σ-function f has thus been associated a finite set of elements $\{a_i \mid i \in [1, n]\}$ and a corresponding set $\{A_i \mid i \in [1, n]\}$ of proper ideals of R such that $\prod_{i=1}^{n} A_i = \{0\}$ and $a_i - a_j \in \sqrt{A_i + A_j}$ for each pair (i,j). Notice that this set of ideals has the property that was promised. If P is arbitrary in V_o, then $\alpha \subset P$ for some $\alpha \in \Delta$, and $\Delta = \bigcup_{i=1}^{n} \Delta_i$ so that $\alpha \in \Delta_i$ for some i. Hence by theorem 1.11, $A_i \subset \alpha \subset P$ and $A_i \subset P$ for some i. Moreover, for any i such that $A_i \subset P$ we get that $P \in D_i$ by theorem 1.10; hence $f(P) = a_i + P$. It will now be shown that these properties characterize σ-functions completely.

Theorem 1.13. For each σ-function f, there are finite sequences $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ of ideals and elements of R such that

1) $\prod_{i=1}^{n} A_i = \{0\}$.

2) $a_i - a_j \in \sqrt{A_i + A_j}$ for each (i,j).

3) $f(P) = a_i + P$ whenever $P \in V_o$ and $A_i \subset P$.

Conversely, whenever sequences $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ of ideals and elements of R satisfy conditions 1) and 2) above, then 3) defines a σ-function f.

Proof. As remarked above, the last three theorems yield the first part of this theorem.

Conversely, suppose sequences $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ satisfy 1) and 2) above. Then:

f given by 3) is a well-defined function. To see this, suppose $A_i \subset P$ and $A_j \subset P$ for some pair (i,j). We must show that $a_i + P = a_j + P$. But $A_i + A_j \subset P$ and P is a prime ideal, so $\sqrt{A_i + A_j} \subset \sqrt{P} = P$ and $a_i - a_j \in P$. Therefore, $a_i + P = a_j + P$ and f is well-defined.

Domain $f = V_o$. Let $P \in V_o$. If $A_i \not\subset P$ for each i, then let $b_i \in A_i \setminus P$ for each i. Then $\prod_{i=1}^{n} b_i \in \prod_{i=1}^{n} A_i = \{0\}$, so $\prod_{i=1}^{n} b_i = 0 \in P$. Hence $b_i \in P$ for some i, a contradiction. Thus $A_i \subset P$ for some $i \in [1, n]$ so that f is defined at P. We have then that $f\colon V_o \longrightarrow V$.

$f(P) \in R/P$ for each $P \in V_o$. This is clear from 3).

f is continuous. For this, it is sufficient to show that $f^{-1}(\mathcal{P}(b))$ is closed in V_o for every subbasic closed set $\mathcal{P}(b)$ in V. Let $\mathcal{P}(b)$ be an arbitrary subbasic closed set in V. Let $P \in f^{-1}(\mathcal{P}(b))$; i.e., $f(P) = b + P$. Let $A_i \subset P$. Then $f(P) = a_i + P$ also, so $b - a_i \in P$. Let $a \in A_i$. Then $a \in P$ so that $P \in \mathcal{P}(a)$. Thus $P \in \mathcal{P}(a)$ for each $a \in A_i$, and $P \in \mathcal{P}(b-a_i) \cap (\bigcap_{a \in A_i} \mathcal{P}(a)) = S_i$. So $P \in \bigcup_{i=1}^{n} S_i$.

Conversely, let $P \in S = \bigcup_{i=1}^{n} S_i$. Then for some i we have $P \in S_i$. If $a \in A_i$, then $P \in \mathcal{P}(a)$ so that $a \in P$; thus $A_i \subset P$ and $f(P) = a_i + P$. But $P \in \mathcal{P}(b-a_i)$ also, so $a_i + P = b + P$ and $f(P) = b + P$. Hence $P \in f^{-1}(\mathcal{P}(b))$. Therefore, $f^{-1}(\mathcal{P}(b)) = S$. Clearly, S is closed in V and $S \subset V_o$; thus S is closed in V_o, whence $f^{-1}(\mathcal{P}(b))$ is closed in V_o. Hence f is continuous.

<u>f is closed</u>. Since f is injective, to show f is closed it is sufficient to show that $f(V_o \cap \mathcal{P}(b))$ is closed in V for every subbasic closed subset $V_o \cap \mathcal{P}(b)$ of V_o. Let $V_o \cap \mathcal{P}(b)$ be any such set. For each $i \in [1, n]$, let $S_i = \mathcal{P}(a_i) \cap (\bigcap_{a \in A_i} \mathcal{P}(a_i+b+a))$ and let $S = \bigcup_{i=1}^{n} S_i$. S is clearly closed in V.

We claim that $S = f(V_o \cap \mathcal{P}(b))$. For let $X \in S$; then $X \in S_i$ for some i, so $X \in \mathcal{P}(a_i)$ and $X = a_i + P$ for some $P \in V_o$. Let $a \in A_i$. Then $X \in \mathcal{P}(a_i+b+a)$ so $a_i + b + a + P = a_i + P = X$ for all $a \in A_i$. Hence $b + a \in P$ for all $a \in A_i$. In particular, $0 \in A_i$ so that $b \in P$. Hence $X = a_i + a + P = a_i + P$ for all $a \in A_i$, and so $A_i \subset P$. Thus $f(P) = a_i + P = X$, and $X = f(P) \in f(V_o \cap \mathcal{P}(b))$, since

$b \varepsilon P$.

Conversely, let $X = f(P) \varepsilon f(V_o \cap \mathcal{P}(b))$, where $P \varepsilon V_o \cap \mathcal{P}(b)$. Let $A_i \subset P$. Then $X = f(P) = a_i + P$ and $X \varepsilon \mathcal{P}(a_i)$. Further, let $a \varepsilon A_i$. Then $a \varepsilon P$ and $b \varepsilon P$, so $X = a_i + P = a_i + b + a + P$ and $X \varepsilon \mathcal{P}(a_i+b+a)$. Hence $X \varepsilon S_i$ and $X \varepsilon \bigcup_{i=1}^{n} S_i = S$. Thus $f(V_o \cap \mathcal{P}(b)) = S$ and $f(V_o \cap \mathcal{P}(b))$ is closed in V. Hence f is closed. Therefore, f is a σ-function as asserted. This completes the proof of the theorem.

<u>Remark</u>. Notice that in showing that an arbitrary σ-function f has the form given in the last theorem, no use was made of the fact that f was closed other than to get the set E with $X \varepsilon N(E)$ and range $f \cap N(E) = \emptyset$. But this can be done if we only assume that range f is not dense in V. Therefore, any continuous function $f: V_o \longrightarrow V$ which has non-dense range and satisfies condition 1) of definition 2 satisfies also the conditions of theorem 1.13; by the second part of the theorem it must be a σ-function. In particular, if range f is closed, then it is non-dense and the above goes through. Conversely, if f is any σ-function, then f closed implies range f is closed and thus also non-dense. We thus have two equivalent formulations for σ-functions.

<u>Theorem 1.14</u>. Let $f: V_o \longrightarrow V$. Then f is a σ-function if and only if it satisfies conditions 1) and 2') below, and also if and only if it satisfies conditions 1)

and 2").

1) $f(P) \varepsilon R/P$ for all $P \varepsilon V_0$.

2') f is continuous and has a non-dense range.

2") f is continuous and has a closed range.

CHAPTER II

THE RING $\hat{R}$

To make $\hat{R}$ into a ring, it must first be shown that the pointwise addition and multiplication of two functions yield functions which are also in $\hat{R}$. This is done by using the characterization of σ-functions which was developed in the preceding chapter.

Theorem 2.1. Let $f \in R$ and $g \in R$. Define $f + g : V_o \longrightarrow V$ by $(f+g)(P) = f(P) + g(P)$ for each $P \in V_o$, where the addition on the right is that of R/P. Similarly, define $fg: V_o \longrightarrow V$ by $(fg)(P) = f(P)g(P)$. Then $f + g$ and fg are also σ-functions.

Proof. Let $\{A_i | i \in [1, n]\}$ and $\{a_i | i \in [1, n]\}$ define f according to theorem 1.13. Let $\{B_j | j \in [1, m]\}$ and $\{b_j | j \in [1, m]\}$ similarly define g. Set $C_{ij} = A_i + B_j$ and $c_{ij} = a_i + b_j$ for each $(i,j) \in [1, n] \times [1, m]$.

We first claim that $\pi C_{ij} = \{0\}$, where the product is taken over all $(i,j) \in [1, n] \times [1, m]$. To see this, let $P \in V_o$ and choose (i,j) so that $A_i \subset P$ and $B_j \subset P$. Then $C_{ij} \subset P$. But P is an ideal so $\pi C_{ij} \subset P$. Therefore, $\pi C_{ij} \subset \cap\{P | P \in V_o\} = \{0\}$.

Second, notice that $c_{ij} - c_{kr} = (a_i + b_j) - (a_k + b_r) = (a_i - a_k) + (b_j - b_r) \in \sqrt{A_i + A_k} + \sqrt{B_j + B_r} \subset \sqrt{A_i + A_k + B_j + B_r} = \sqrt{C_{ij} + C_{kr}}$ for each (i,j) and (k,r) in $[1, n] \times [1, m]$.

If some ideal $C_{ij} = R$, then it can be deleted from the collection $\{C_{ij} \mid (i, j) \in [1, n] \times [1, m]\}$, and we will still have $\pi C_{ij} = \{0\}$, where the product is now taken over the ideals which are not so deleted. Certainly not all of the ideals will be removed; for if $P \in V_0$ then $C_{ij} \subset P$ for some (i, j) so that $C_{ij} \neq R$ for this (i, j). Hence we may assume that $\{C_{ij} \mid (i, j) \in [1, n] \times [1, m]\}$ is a non-empty collection of proper ideals of R, and by theorem 1.13 a σ-function h is defined by setting $h(P) = c_{ij} + P$ whenever $C_{ij} \subset P$. It is readily verified that $h(P) = f(P) + g(P)$ for each $P \in V_0$. Therefore $f + g = h$, and $f + g$ is a σ-function.

For the product fg, set $C_{ij} = A_i + B_j$ as before, but this time set $c_{ij} = a_i b_j$ for each (i, j). Then $\pi C_{ij} = \{0\}$ as before. We also have that $a_i - a_k \in \sqrt{A_i + A_k}$ and $b_j - b_r \in \sqrt{B_j + B_r}$ for each (i, j) and (k, r) in $[1, n] \times [1, m]$. Hence $(a_i - a_k)b_j = a_i b_j - a_k b_j \in \sqrt{A_i + A_k}$, and $a_k(b_j - b_r) = a_k b_j - a_k b_r \in \sqrt{B_j + B_r}$. Therefore, $c_{ij} - c_{kr} = a_i b_j - a_k b_r = (a_i b_j - a_k b_j) + (a_k b_j - a_k b_r) \in \sqrt{A_i + A_k} + \sqrt{B_j + B_r} \subset \sqrt{A_i + B_j + A_k + B_r} = \sqrt{C_{ij} + C_{kr}}$ for each (i, j) and (k, r). So as above these sequences also define a σ-function h'. It is also easily verified that $h'(P) = f(P)g(P)$ for each $P \in V_0$. Therefore, $h' = fg$ is a σ-function also.

Now we can show that $\hat{R}$ actually constitutes a ring (in our original sense).

Theorem 2.2. The collection $\hat{R}$ of all σ-functions, with the operations that have been defined above, is a ring.

Proof. The commutative and associative laws for addition and multiplication and the distributive law hold in $\hat{R}$ because they hold in each ring R/P for each $P \varepsilon V_0$. It is easily seen that the path functions f_0 and f_1 are respectively additive and multiplicative identities.

For additive inverses, let f be a σ-function and let f be defined by sequences $\{A_i | i \varepsilon [1, n]\}$ and $\{a_i | i \varepsilon [1, n]\}$ according to theorem 1.13. For each $i \varepsilon [1, n]$ set $B_i = A_i$ and $b_i = -a_i$. These new sequences clearly satisfy the conditions of theorem 1.13 also, so a new σ-function -f is defined by setting $(-f)(P) = b_i + P = -a_i + P = -(a_i + P) = -(f(P))$ whenever $B_i = A_i \subset P$. Now -f is clearly an additive inverse for f. Hence $\hat{R}$ is at least a commutative ring with identity.

It remains to show that $\hat{R}$ is nil semi-simple. Suppose that $\hat{P} = \{f \varepsilon \hat{R} | f(P) = P\}$ is a prime ideal of $\hat{R}$ for each $P \varepsilon V_0$. Actually, we will prove a much more comprehensive result below in theorem 2.4. Thus if some $f \varepsilon \hat{R}$ is in every prime ideal of $\hat{R}$, then $f \varepsilon \hat{P}$ for each $P \varepsilon V_0$ in particular. But this says that $f(P) = P$ for all $P \varepsilon V_0$. It follows that $f = f_0$ and $\cap\{Q | Q$ is a prime ideal of $\hat{R}\} = \{f_0\}$, the zero ideal of $\hat{R}$, whence $\hat{R}$ is nil semi-simple by lemma 1.3.

Closure properties of $\hat{R}$. We will turn now to an investigation of $\hat{R}$. It will be shown that $\hat{R}$ is a "closure" of R in the sense that $\hat{R} \cong \hat{\hat{R}}$. Using this result, and the characterization of σ-functions developed in the first chapter, we will find conditions that R is itself "closed"; i.e., that $R \cong \hat{R}$. We begin with

Theorem 2.3. The map $\Psi: R \longrightarrow \hat{R}$ given by $\Psi(a) = f_a$ is an isomorphism of R into $\hat{R}$, and for each $f \in \hat{R}$ there is a finite set $\{a_i \mid i \in [1, n]\}$ such that $\prod_{i=1}^{n}(f-f_{a_i}) = f_0$. Hence defining $R' = \Psi(R) \cong R$, we have that $\hat{R}$ is an integral extension of R'.

Proof. The first statement is simply a reformulation of theorem 1.6. Let $f \in \hat{R}$ and let $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ be defining for f. Let $P \in V_o$ and $A_j \subset P$. Then $[\prod_{i=1}^{n}(f-f_{a_i})](P) = (f-f_{a_j})(P)[\prod_{i \neq j}(f-f_{a_i})(P)]$

$= [(a_j + P) + (-a_j + P)][\prod_{i \neq j} (f-f_{a_i})(P)]$

$= (0 + P)[\prod_{i \neq j} (f-f_{a_i})(P)] = 0 + P = f_0(P)$. Hence

$f_0 = \prod_{i=1}^{n}(f-f_{a_i})$.

Theorem 2.4. There is a one-to-one correspondence between $V_o(R)$ and $V_o(\hat{R})$. Specifically, $\hat{P} = \{f \in \hat{R} \mid f(P) = P\}$ is a prime ideal of $\hat{R}$ for each $P \in V_o(R)$, $Q' = \{a \in R \mid f_a \in Q\}$ is a prime ideal of R for each $Q \in V_o(\hat{R})$, and $\widehat{Q'} = Q$ and $\hat{P}' = P$ for each $Q \in V_o(\hat{R})$ and each $P \in V_o(R)$. Pairing the prime ideals in this way, we have $R/P \cong \hat{R}/\hat{P}$ for each $P \in V_o(R)$.

<u>Proof</u>. Let $P \in V_o(R)$ and let $f \in \hat{P}$, $g \in \hat{P}$, and $h \in \hat{R}$. Suppose that $h(P) = a + P$. Then $(f-g)(P) = f(P) - g(P) = P - P = P$, so $f - g \in \hat{P}$. Also $(fh)(P) = f(P)h(P) = (0 + P)(a + P) = 0 + P = P$, and so $fh \in \hat{P}$. Hence $\hat{P}$ is an ideal of $\hat{R}$. On the other hand, suppose that $f \notin \hat{P}$ and $g \notin \hat{P}$. Let $f(P) = a + P$ and $g(P) = b + P$. Then $a \notin P$ and $b \notin P$ so that $ab \notin P$. Hence $(fg)(P) = f(P)g(P) = (a + P)(b + P) = ab + P \neq P$. Therefore $fg \notin \hat{P}$ and $\hat{P}$ is a prime ideal of $\hat{R}$. (This is enough to complete the proof of theorem 2.2.)

Now let $Q \in V_o(\hat{R})$, and let $a \in Q'$, $b \in Q'$ and $c \in R$. Then $f_{a-b} = f_a - f_b \in Q - Q = Q$ so $a - b \in Q'$. Also, $f_{ac} = f_a f_c \in Q$ so $ac \in Q'$. Hence Q' is an ideal of R. But if $a \notin Q'$ and $b \notin Q'$, then $f_a \notin Q$ and $f_b \notin Q$ so that $f_{ab} = f_a f_b \notin Q$ and $ab \notin Q'$. Therefore, Q' is a prime ideal of R. It is clear that the operations $'$ and $\hat{}$ yield proper ideals from proper ideals; for if $a \notin P$ then $f_a(P) \neq P$, and if $f_1 \notin Q$ then $1 \notin Q'$.

If $P \in V_o(R)$, then first notice that $f_a \in \hat{P}$ if and only if $f_a(P) = a + P = P$, and this is so if and only if $a \in P$. So we get $\hat{P}' = \{a \in R \mid f_a \in \hat{P}\} = \{a \in R \mid a \in P\} = P$.

Let $Q \in V_o(\hat{R})$. To show that $\widehat{Q'} = Q$ is somewhat more difficult. First we pick f arbitrary in Q and let $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ define f. For each i such that $A_i \not\subseteq Q'$, pick some element $t_i \in A_i \setminus Q'$. Then for each $i \in [1, n]$ we set:

$a_i' = a_i$ if $A_i \subset Q'$ or $a_i \notin Q'$.

$a_i' = a_i + t_i$ if $A_i \not\subset Q'$ and $a_i \in Q'$.

It is clear that the sequences $\{A_i | i \in [1, n]$ and $\{a_i' | i \in [1, n]$ also satisfy conditions 1) and 2) of theorem 1.13 because t_i is always chosen in A_i. Hence these sequences define a σ-function f'. But let $P \in V_0(R)$ and choose $A_i \subset P$. If $A_i \subset Q'$ or $a_i \notin Q'$, then $f'(P) = a_i' + P = a_i + P = f(P)$. Otherwise $A_i \not\subset Q'$ and $a_i \in Q'$, and in this case $f'(P) = a_i' + P = a_i + t_i + P = a_i + P = f(P)$ also, since $t_i \in A_i \subset P$. Therefore, $f = f'$ and f is defined also by the sequences $\{A_i | i \in [1, n]\}$ and $\{a_i' | i \in [1, n]\}$.

Now $\prod_{i=1}^{n}(f-f_{a_i'}) = f_0 \in Q$, and Q is a prime ideal, so $f - f_{a_i'} \in Q$ for some i. But $f \in Q$ also, so $f_{a_i'} \in Q$. Hence $a_i' \in Q'$. If $A_i \not\subset Q'$ and $a_i \in Q'$, then $t_i = a_i' - a_i \in Q'$, a contradiction. Thus $A_i \subset Q'$ or $a_i \notin Q'$. But if $a_i \notin Q'$, then $a_i' = a_i \notin Q'$, a contradiction. So $A_i \subset Q'$ and $f(Q') = a_i' + Q' = Q'$, whence $f \in \widehat{Q'}$. So $Q \subset \widehat{Q'}$.

Conversely, let f be arbitrary in $\widehat{Q'}$, and let sequences $\{A_i | i \in [1, n]\}$ and $\{a_i | i \in [1, n]\}$ define f. Choose a sequence $\{a_i' | i \in [1, n]\}$ with respect to Q' as above. Then the new sequences also define f as before. Now $\prod_{i=1}^{n}(f-f_{a_i'}) = f_0 \in Q$, and Q is a prime ideal of $\hat{R}$, so $f - f_{a_i'} \in Q$ for some i. Thus $f - f_{a_i'} \in \widehat{Q'}$ and $f_{a_i'} \in \widehat{Q'}$. So we see that $a_i' + Q' = f_{a_i'}(Q') = Q'$, and $a_i' \in Q'$.

Hence $f_{a_i'} \in Q$ and $f = (f - f_{a_i'}) + f_{a_i'} \in Q$. So $\widehat{Q'} \subset Q$, and $\widehat{Q'} = Q$.

To show that the quotient rings are isomorphic, let $P \in V_o(R)$ and map $\Psi: R/P \longrightarrow \hat{R}/\hat{P}$, where Ψ is given by

$$\Psi(a + P) = f_a + \hat{P} \text{ for each } a + P \in R/P.$$

Suppose $a + P = b + P$. Then $(a - b) + P = P$ so $f_{a-b}(P) = P$, and $f_{a-b} = f_a - f_b \in \hat{P}$. Thus $f_a + \hat{P} = f_b + \hat{P}$ whenever $a + P = b + P$, and we see that Ψ as given above is a genuine mapping.

If $\Psi(a + P) = \Psi(b + P)$, then $f_a + \hat{P} = f_b + \hat{P}$; so $f_{a-b} = f_a - f_b \in \hat{P}$. Hence $f_{a-b}(P) = (a - b) + P = P$ so that $a - b \in P$. Consequently, $a + P = b + P$, and Ψ is injective. On the other hand, if $f + \hat{P}$ is arbitrary in $\hat{R}/\hat{P}$, then let $\{A_i | i \in [1, n]\}$ and $\{a_i | i \in [1, n]\}$ define f. Then $A_i \subset P$ for some i, so $f(P) = a_i + P$. We claim that $f + \hat{P} = f_{a_i} + \hat{P}$. To see this notice that $(f - f_{a_i})(P) = f(P) - f_{a_i}(P) = (a_i + P) - (a_i + P) = P$ so that $f - f_{a_i} \in \hat{P}$. Hence $\Psi(a_i + P) = f_{a_i} + \hat{P} = f + \hat{P}$. Therefore, Ψ is a one-to-one correspondence of R/P and $\hat{R}/\hat{P}$.

Last, let $a + P$ and $b + P$ be arbitrary in R/P. Then $\Psi(a + P) + \Psi(b + P) = (f_a + \hat{P}) + (f_b + \hat{P}) = (f_a + f_b) + \hat{P} = f_{a+b} + \hat{P} = \Psi(a + b + P) = \Psi((a + P) + (b + P))$, and $\Psi(a + P)\Psi(b + P) = (f_a + \hat{P})(f_b + \hat{P}) = f_a f_b + \hat{P} = f_{ab} + \hat{P} = \Psi(ab + P) = \Psi((a + P)(b + P))$. Therefore, Ψ is an isomorphism of R/P and $\hat{R}/\hat{P}$ as asserted. The theorem has thus been proved.

Now we introduce a map from $V(\hat{R})$ to $V(R)$, which turns out to be the same as $Q \longrightarrow Q'$ restricted to $V_o(\hat{R})$ and the inverse of the map Ψ of the last theorem when restricted to $\hat{R}/\hat{P}$. The map is shown to be a homeomorphism, and it induces an isomorphism of $\hat{R}$ and $\hat{\hat{R}}$.

<u>Theorem 2.5</u>. $V(\hat{R})$ and $V(R)$ are homeomorphic via the mapping $\varphi: V(\hat{R}) \longrightarrow V(R)$ given by $\varphi(X) = \{a \in R \mid f_a \in X\}$ for each $X \in V(\hat{R})$. Since $\varphi(Q) = Q'$ for each $Q \in V_o(\hat{R})$, it follows that $\varphi_{|V_o(\hat{R})}$ is a homeomorphism of $V_o(\hat{R})$ and $V_o(R)$.

<u>Proof</u>. Surely φ is a well-defined function of some sort, but we are not yet assured that its range is in $V(R)$. To do this we will first find a more convenient formula for computing $\varphi(X)$. By theorem 2.4 an arbitrary $X \in V(\hat{R})$ has the form $X = f + \hat{P}$, where $f \in \hat{R}$ and $P \in V_o(R)$. Let $f(P) = c + P$ and we obtain

$$\begin{aligned} \varphi(X) &= \varphi(f + \hat{P}) = \{a \in R \mid f_a \in f + \hat{P}\} \\ &= \{a \in R \mid (f - f_a) \in \hat{P}\} \\ &= \{a \in R \mid a \in c + P\} = c + P = f(P). \end{aligned}$$

Clearly $\varphi(X) \in V(R)$ as we wish; moreover, $\varphi(f + \hat{P}) = f(P)$ is the natural way to compute φ. Even easier, for $f + Q$ arbitrary in $V(\hat{R})$ we find that $\varphi(f + Q) = \varphi(f + \widehat{Q'}) = f(Q')$. Notice that $\varphi(Q) = Q'$ for all $Q \in V_o(\hat{R})$.

To show φ is a bijection, suppose $\varphi(f + Q) = \varphi(g + H)$; i.e., $f(Q') = g(H')$. We must have $Q' = H'$, $Q = H$, and $(f - g)(Q') = Q'$. This gives that $f - g \in \widehat{Q'}$, whence $f + Q = g + Q = g + H$, and φ is injective.

Moreover, if $a + P$ is arbitrary in $V(R)$ we see that $\varphi(f_a + \hat{P}) = f_a(P) = a + P$ so that φ is a surjection.

Since φ is a bijection we need only consider subbasic closed sets to verify that φ is a homeomorphism. But

$$\begin{aligned}\varphi(\mathcal{P}(f)) &= \{\varphi(f + Q) \mid Q \text{ is a prime ideal of } \hat{R}\} \\ &= \{f(Q') \mid Q' \text{ is a prime ideal of } R\} \\ &= f(V_o)\end{aligned}$$

is closed in $V(R)$ for every subbasic closed set $\mathcal{P}(f)$ of $V(\hat{R})$, since each corresponding f is a closed map. Similarly, it is easily seen that $\varphi^{-1}(\mathcal{P}(a)) = \mathcal{P}(f_a)$ for every subbasic closed set $\mathcal{P}(a)$ of $V(R)$. Thus φ is a homeomorphism as claimed, and we are done.

Since the spaces $V(\hat{R})$ and $V(R)$ are essentially the same, it is not surprising that the ring $\hat{\hat{R}}$ determined by $V(\hat{R})$ is the same as the ring $\hat{R}$ determined by $V(R)$. We now demonstrate this.

<u>Theorem 2.6</u>. The rings $\hat{R}$ and $\hat{\hat{R}}$ are isomorphic.

<u>Proof</u>. We use the mapping φ above. If $g \in \hat{\hat{R}}$ (i.e., g is a σ-function for $\hat{R}$), then $g\colon V_o(\hat{R}) \longrightarrow V(\hat{R})$. So for any $P \in V_o(R)$, $\varphi^{-1}(P) \in V_o(\hat{R})$, $g(\varphi^{-1}(P)) \in V(\hat{R})$, and $\varphi(g(\varphi^{-1}(P))) \in V(R)$. The natural mapping from $\hat{\hat{R}}$ to $\hat{R}$ would seem to be obtained then by setting $gT = T(g) = \varphi \circ g \circ \varphi^{-1}$ for each $g \in R$. Certainly $gT\colon V_o(R) \longrightarrow V(R)$ for any $g \in \hat{\hat{R}}$.

Moreover, it is easily verified that for any

$P \in V_o(R)$, $gT(P) = f(P)$ for any $f \in \hat{R}$ such that $g(\hat{P}) = f + \hat{P}$, whence condition 1) of σ-functions for gT follows from the fact that the function f has this property. Further, since g is continuous and closed and φ and φ^{-1} are homeomorphisms, gT must be continuous and closed for each $g \in \hat{\hat{R}}$. Therefore, $T: \hat{\hat{R}} \longrightarrow \hat{R}$.

Suppose that $f \in \hat{R}$ and P is a prime ideal of R. This implies that $(\varphi^{-1} \circ f \circ \varphi)(\hat{P}) = (\varphi^{-1} \circ f)(P) = \varphi^{-1}(f(P)) = f + \hat{P} = g_f(\hat{P})$, since $\varphi(f + \hat{P}) = f(P)$. So if f is fixed while P varies, we see that $\varphi^{-1} \circ f \circ \varphi = g_f$; consequently $g_f T = \varphi \circ g_f \circ \varphi^{-1} = f$ for each $f \in \hat{R}$, and T maps $\hat{\hat{R}}$ onto $\hat{R}$.

Recall that the natural injection $f \longrightarrow g_f$ of $\hat{R}$ into $\hat{\hat{R}}$ must be an isomorphism. But if g is arbitrary in $\hat{\hat{R}}$ and $gT = f \in \hat{R}$, then $g = \varphi^{-1} \circ (\varphi \circ g \circ \varphi^{-1}) \circ \varphi = \varphi^{-1} \circ f \circ \varphi = g_f$ so that the only functions in $\hat{\hat{R}}$ are the path functions g_f. We conclude that this natural injection must be an isomorphism of $\hat{R}$ and all of $\hat{\hat{R}}$; thus T is also an isomorphism, since it is clearly the inverse of the injection $f \longrightarrow g_f$. So $\hat{R} \cong \hat{\hat{R}}$ as stated.

Notice that the proofs of theorems 2.3 to 2.6, with minor changes, remain valid when $\hat{R}$ is replaced by any subring S of $\hat{R}$ such that $S \supset R' = \{f_a \mid a \in R\}$. Thus we actually have the stronger closure property that $\hat{R} \cong \hat{S}$ for any ring S with $R' \subset S \subset \hat{R}$.

We pause here to ask in what sense $\hat{R}$ is the ring determined by the prime ideal structure of R. This is true of course in the sense that it is determined from the space V(R), which in turn is determined (partially) by the prime ideal structure of R. Moreover, by theorem 2.4 it has the same prime ideal structure as R in such a comprehensive sense that it yields the same space as R and thus gives us back an identical third-order ring in theorems 2.5 and 2.6. The hitch, however, is that the space V(R) seems also to depend partially on the elements of R. This gives us the natural correspondences (i.e., $a \longrightarrow f_a$) which have greatly aided us so far. We would like to get some results in a more general setting.

Consider, for example, the

Conjecture. Let R and S be two rings. Then there is an order-preserving bijection Ψ: $V_o(R) \longrightarrow V_o(S)$ such that $R/P \cong S/\Psi(P)$ for each $P \in V_o(R)$ if and only if $\hat{R} \cong \hat{S}$.

Notice first that we have already shown the "if" part of this; for if $\hat{R} \cong \hat{S}$ then there is such a bijection for $\hat{R}$ and $\hat{S}$, and there are also such correspondences for R and $\hat{R}$ and for S and $\hat{S}$ by theorem 2.4. It is easily shown that the correspondence of 2.4 is order-preserving. So by composition we can get such a correspondence for R and S.

Secondly, notice that we cannot put $R \cong S$ in this conjecture. It will be shown in the third chapter that there is a ring R not isomorphic with $\hat{R}$. But these two rings do have the same prime ideal structure as formulated in the conjecture. However, since $\hat{R} \cong \hat{\hat{R}}$, such rings do not offer a counterexample to the above.

Notice, thirdly, that the "only if" part is true for integral domains, since $\{0\}$ is a prime ideal in this case. Thus by the order-preserving condition we are asuming that $R \cong R/\{0\} \cong S/\{0\} \cong S$. In fact, in this case we can substitute the conclusion $R \cong S$. Now it would seem that we constructed the ring $\hat{R}$ precisely to take care of the case that the rings are not integral domains; i.e., that we have found the ring $\hat{R}$ to generalize what is a theorem for integral domains to a wider theorem, replacing rings with their "^-closures", which holds for the wider class of commutative nil semi-simple rings with identity.

However, it is not known whether this conjecture is true. The major difficulty is in showing that the spaces $V(R)$ and $V(S)$ are homeomorphic. For the case of R and $\hat{R}$ we have that the isomorphisms $R/P \rightarrow \hat{R}/\hat{P}$ are "compatible" as we range over $P \in V(R)$ in the sense that if $a + P \longrightarrow f_a + \hat{P}$ for some prime ideal P, then $a + Q \longrightarrow f_a + \hat{Q}$ for every prime ideal Q. We certainly would not want to assume such a strong compatibility as this for R and S above since we would then have, to begin with, a map from R to S or vice versa analogous to $a \longrightarrow f_a$, and this is getting away from the prime ideal structure.

However, a more likely conjecture would include some sort of compatibility condition. Merle Manis has suggested adding the condition that there is some given set $\{\Psi_P\colon R/P \longrightarrow S/\Psi(P) \mid P \in V_o(R)\}$ of isomorphisms such that whenever $P \subset Q$, then the diagram

$$\begin{array}{ccc} R/P & \xrightarrow{\Psi_P} & S/\Psi(P) \\ \downarrow \text{nat.} & & \downarrow \text{nat.} \\ R/Q & \xrightarrow{\Psi_Q} & S/\Psi(Q) \end{array}$$

commutes. The truth of this conjecture is likewise unknown. He has also pointed out that this modified conjecture is not true in the case of primes. The prime structure of the integers and rationals is the same with respect to all of these conditions, but $\overline{Z} \cong Z \not\cong Q \cong \overline{Q}$. This clearly does not offer a counterexample to our conjecture, however, since these systems are both integral domains. What happens is that $\{0\}$ is a prime ideal but not a prime.

Failing in the preceding problem, we can yet go back and find more information about $\hat{R}$ for some fixed ring R. Conditions will now be found that $R \cong \hat{R}$. We start with

<u>Theorem 2.7</u>. Let f be a σ-function in $\hat{R}$. Then $f \notin \{f_a \mid a \in R\}$ if and only if f is defined by sequences $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ of ideals and elements of R satisfying

1) $\prod_{i=1}^{n} A_i = \{0\}$.
2) $a_i - a_j \in \sqrt{A_i + A_j}$ for all (i, j).
3) $a_i - a_j \notin \sqrt{A_i} + \sqrt{A_j}$ for some (i, j).

<u>Proof</u>. Suppose that f is defined by some sequences satisfying the above conditions. We want to show that f

is not a path function. Suppose, to the contrary, that $f = f_a$ for some $a \varepsilon R$. We have then that $f(P) = a + P = a_i + P$ and $a_i - a \varepsilon P$ whenever $A_i \subset P$. It follows that $a_i - a \varepsilon \cap\{P \varepsilon V_o(R) | A_i \subset P\}$ for each $i \varepsilon [1, n]$. But if $a_i - a \notin \sqrt{A_i}$ for some i, lemma 1.2 tells us that there is a prime ideal P of R with $A_i \subset P$ and $a_i - a \notin P$. Consequently, $a_i - a \varepsilon \sqrt{A_i}$ for each $i \varepsilon [1, n]$. Thus $a_i - a_j = (a_i - a) - (a_j - a) \varepsilon \sqrt{A_i} + \sqrt{A_j}$ for each (i, j), contradicting condition 3) above. Therefore, f cannot be a path function.

Conversely, suppose that f is not a path function. Theorem 1.13 guarantees that f is defined by sequences satisfying at least the first two conditions. Choose sequences $\{A_i | i \varepsilon [1, n]\}$ and $\{a_i | i \varepsilon [1, n]\}$ which define f such that n is minimal. It is clear that $n \geq 2$, since $n = 1$ obviously imples that f is a path function; namely, $f = f_{a_1}$.

Now notice that we may assume that the ideals A_i are radical ideals. For if not, then $\prod_{i=1}^{n} A_i = \{0\}$ if and only if $\prod_{i=1}^{n} \sqrt{A_i} = \{0\}$, $\sqrt{A_i + A_j} = \sqrt{\sqrt{A_i} + \sqrt{A_j}}$ for each (i, j), and $A_i \subset P$ if and only if $\sqrt{A_i} \subset P$ for any prime ideal P of R and any $i \varepsilon [1, n]$. That is, the sequence of radicals defines the same σ-function as the original sequence.

We claim that condition 3) must hold for such a minimal sequence defining f. In particular, suppose that

$a_1 - a_2 \in \sqrt{A_1} + \sqrt{A_2} = A_1 + A_2$. Let $a_1 - a_2 = b_1 + b_2$, where $b_1 \in A_1$ and $b_2 \in A_2$. Then let $b' = a_1 - b_1 = a_2 + b_2$. Consider the sequence $B' = A_1A_2$, $B_i = A_i$ for $i = 3, 4, \ldots, n$ of ideals of R and the sequence b', $b_i = a_i$ for $i = 3, 4, \ldots, n$ of elements of R. Using the defining property of prime ideals, it is clear that $A_1A_2 \subset P$ if and only if $A_1 \subset P$ or $A_2 \subset P$ for any $P \in V_o(R)$.

We show that these new sequences must define a σ-function. Certainly $B'(\underset{i\geq 3}{\pi} B_i) = \overset{n}{\underset{i=1}{\pi}} A_i = \{0\}$ at least. Moreover, $b_i - b_j = a_i - a_j \in \sqrt{A_i + A_j} = \sqrt{B_i + B_j}$ at least for $i \geq 3$ and $j \geq 3$. Since $b' - b' = 0 \in \sqrt{B' + B'}$, and $b' - b_i \in \sqrt{B' + B_i}$ for $i \geq 3$ if and only if $b_i - b' \in \sqrt{B_i + B'}$, we need only show that $b_i - b' \in \sqrt{B_i + B'}$ for each $i \geq 3$. So suppose that $b_i - b' \notin \sqrt{A_i + A_1A_2}$ for some $i \geq 3$. Then by lemma 1.2 there is some $P \in V_o(R)$ with $\sqrt{A_i + A_1A_2} \subset P$ and $b_i - b' \notin P$. Now $A_i \subset P$, and without loss of generality we can conclude from the remark above that $A_1 \subset P$. Hence $f(P) = a_i + P = a_1 + P$, and $a_i - a_1 \in P$. But $b_i - b' = a_i - (a_1 - b_1) = (a_i - a_1) + b_1$ and $b_1 \in A_1 \subset P$. Consequently, $b_i - b' \in P$, a contradiction. Note that we could similarly have used a_2 and b_2 if we had assumed that $A_2 \subset P$.

Theorem 1.13 now tells us that a σ-function f' is defined by these new sequences. Further, let $P \in V_o(R)$. If $A_i \subset P$ for some $i \geq 3$, then $f'(P) = b_i + P = a_i + P = f(P)$. Otherwise $A_1 \subset P$ or $A_2 \subset P$ so that $B' = A_1A_2 \subset P$.

Without loss of generality we may suppose that $A_1 \subset P$, whence it follows that $f(P) = a_1 + P$ and $f'(P) = b' + P = a_1 - b_1 + P$. However, $b_1 \in A_1 \subset P$ so that $a_1 + P = a_1 - b_1 + P$ and $f(P) = f'(P)$ in this case also. Therefore $f = f'$ and f is defined by sequences of $n - 1$ ideals and elements, but this contradicts the minimality of n. Thus $a_1 - a_2 \notin \sqrt{A_1} + \sqrt{A_2}$ after all so that condition 3) holds in particular for $i = 1$ and $j = 2$ with any minimal sequences using radical ideals which define f. This completes the proof.

Notice that when a minimal choice is made for any σ-function and the radical ideals are substituted as above, then we will have $a_i - a_j \notin A_i + A_j$ for all $(i, j) \in [1, n] \times [1, n]$ with $i \neq j$ by repeating the above argument for an arbitrary such pair (i, j). (For a path function minimality obtains when $n = 1$, and we never have $i \neq j$.) Thus we have actually proved the stronger result:

<u>Theorem 2.8.</u> $\hat{R} \neq \{f_a | a \in R\}$ if and only if there is a sequence of radical ideals of R and a sequence of elements of R with length $n \geq 2$ such that

1) $\prod_{i=1}^{n} A_i = \{0\}$.

2) $a_i - a_j \in \sqrt{A_i + A_j} \setminus (A_i + A_j)$ for all $(i, j) \in [1, n] \times [1, n]$ such that $i \neq j$.

Moreover, in this case each σ-function which is not a path function is defined by such a sequence, but no path function is so defined.

Notice further that we can now state precisely what rings have $R \cong \hat{R}$. They are just those for which the above does not happen. We demonstrate this fact now as the final theorem involving only the rings R and $\hat{R}$.

Theorem 2.9. $R \cong \hat{R}$ if and only if $\hat{R} = \{f_a \mid a \in R\}$; i.e., if and only if the natural injection $a \rightarrow f_a$ maps R onto $\hat{R}$.

Proof. The "if" part here is just theorem 2.3. For the converse, notice that any isomorphism of two rings will preserve conditions 1) and 2) of the last theorem. Suppose then that $R \cong \hat{R}$. If $\hat{R} \neq \{f_a \mid a \in R\}$, then R has sequences as in 2.8. Hence $\hat{R}$ also has such sequences. Applying the theorem again we conclude that $\hat{\hat{R}} \neq \{g_f \mid f \in \hat{R}\}$. However, it was shown in theorem 2.6 that the injection $f \longrightarrow g_f$ was an isomorphism of $\hat{R}$ and $\hat{\hat{R}}$ so that $\hat{\hat{R}} = \{g_f \mid f \in \hat{R}\}$, a contradiction. Thus $\hat{R} = \{f_a \mid a \in R\}$ whenever $R \cong \hat{R}$, and the theorem is proved.

A Comparison of $\hat{R}$ and $\bar{R}$. We consider finally the relation between $\hat{R}$ and the ring $\bar{R}$ constructed from R in [3]. There, an analogous space Arith R was defined in terms of the primes instead of the prime ideals. π-functions were defined much as our σ-functions, and the ring $\bar{R}$ consists of these, with similar operations. The π-functions of $\bar{R}$ are characterized [3] by

Theorem 2.10. f is a π-function if and only if there are ideals $A_1, A_2, \ldots, A_n$ of R and elements $a_1, a_2, \ldots, a_n$ of R satisfying

1) $\prod_{i=1}^{n} A_i = \{0\}$.

2) $a_i - a_j \in \sqrt{A_i + A_j}$ for all (i, j).

3) $f(P) = a_i + P$ whenever P is a prime of R with $A_i \subset P$ and $a_i \in A_P$.

The characterization of rings R for which $R \cong \bar{R}$ corresponds with what we have done here even more closely than the characterization of π-functions does. Specifically, our theorems 2.7 and 2.9 are valid for $\bar{R}$ [3] if $\hat{R}$ is replaced by $\bar{R}$ and σ-functions and path functions are replaced by π-functions and corresponding path functions for Arith R. This leads us to suspect that the rings $\hat{R}$ and $\bar{R}$ are the same. To show this is so, we introduce some new notation to facilitate setting up a correspondence between $\hat{R}$ and $\bar{R}$.

<u>Definition 5</u>. Let S be the collection of all pairs of sequences $(\{A_i \mid i \in [1, n]\}, \{a_i \mid i \in [1, n]\})$, $n \geq 1$, of ideals of R and elements of R satisfying conditions 1) and 2) of theorems 1.13 and 2.10. For each $X \in S$, let $f_\sigma(X)$ be the σ-function defined by X and let $f_\pi(X)$ be the π-function defined by X according to theorem 2.10. Since more than one sequence may define a single σ-function, we put them into equivalence classes. If $X \in S$ and $Y \in S$, we say $X \sim_\sigma Y$ when $f_\sigma(X) = f_\sigma(Y)$ and $X \sim_\pi Y$ when $f_\pi(X) = f_\pi(Y)$. Clearly $\sim_\sigma$ and $\sim_\pi$ are equivalence relations, and we denote the respective equivalence classes by $[X]_\sigma$ and $[X]_\pi$ for each $X \in S$.

Now the correspondence $[X]_\sigma \rightarrow f_\sigma(X)$ is a one-to-one correspondence between the equivalence classes of S under $\sim_\sigma$ and the ring $\hat{R}$. Thus a ring structure is induced on the set of equivalence classes which makes it a ring isomorphic with $\hat{R}$. Note that in showing $\hat{R}$ is a ring, we showed that the addition or multiplication of two σ-functions f and g can be defined in terms of certain operations on their respective defining sequences which yield other sequences defining their sum and product as previously defined pointwise. If we start with sequences in S for f and g, then the resultant sequences are in S, as is seen in the proof of theorem 2.1. Thus the equivalence classes of S under $\sim_\sigma$ are actually a ring under these operations definable purely in terms of the sequences, since these operations are clearly the same as those induced by the correspondence mentioned above. It did not matter which specific sequences we had used to define f and g to get that the resultant ones were defining for f + g and fg.

Hence the operation

$$[(\{A_i\}_{i=1}^{n}, \{a_i\}_{i=1}^{n})]_\sigma + [(\{B_j\}_{j=1}^{m}, \{b_j\}_{j=1}^{m})]_\sigma =$$
$$[(\{A_i + B_j \mid (i, j) \in [1, n] \times [1, m]\},$$
$$\{a_i + b_j \mid (i, j) \in [1, n] \times [1, m]\})]_\sigma$$

is well-defined, and similarly for multiplication. For convenience, then, we may identify $\hat{R}$ with the collection of equivalence classes and perform operations on $\hat{R}$ in accordance with the formulas in theorem 2.1 as restated for the equivalence classes.

Analogous considerations hold for $\overline{R}$ (see [3]); identical formulae may be used for combining two equivalence classes $[X]_\pi$ and $[Y]_\pi$. Consequently, we will regard both $\hat{R} = \{[X]_\sigma | X \in S\}$ and $\overline{R} = \{[X]_\pi | X \in S\}$. We want to show that these two rings are isomorphic. This is clearly so if the correspondence $[X]_\sigma \longleftrightarrow [X]_\pi$ is one-to-one between $\hat{R}$ and $\overline{R}$, since the computation formulae are identical. Assuming familiarity with [3], we will demonstrate that this indeed happens.

Theorem 2.11. $F\colon \hat{R} \cong \overline{R}$, where F is given by $F([X]_\sigma) = [X]_\pi$ for each $X \in S$.

Proof. We first verify that F is well-defined. Suppose that $[X]_\sigma = [Y]_\sigma$ for some X and Y in S, specifically, for

$$X = (\{A_i | i \in [1, n]\}, \{a_i | i \in [1, n]\}),$$
$$Y = (\{B_j | j \in [1, m]\}, \{b_j | j \in [1, m]\}).$$

Then $X \sim_\sigma Y$, $f_\sigma(X) = f_\sigma(Y)$, and theorem 1.13 tells us that $a_i + P = b_j + P$, i.e., $a_i - b_j \in P$, whenever we find $A_i \subset P$ and $B_j \subset P$ for any given $P \in V_o(R)$.

Turning to $\overline{R}$, suppose that P is any prime in the domain of $f_\pi(X)$. This means that for some i we have both $A_i \subset P$ and $a_i \in A_P$. (See theorem 2.10.) Then consider any B_j such that $B_j \subset P$, as must happen for some j. We get that $A_i + B_j \subset P$ so that $A_i + B_j \neq R$, whence there is a proper prime ideal P' of R such that $A_i + B_j \subset P'$. Now suppose that $P' \subset P$. By the first paragraph it follows that $a_i - b_j \in P' \subset P \subset A_P$; hence also $b_j \in A_P$. So some $B_j \subset P$ with $b_j \in A_P$ and P is also in the domain of $f_\pi(Y)$. Furthermore, the P' above may always be chosen so that $P' \subset P$; for primes share with prime ideals the property that their complements are multiplicative systems in R (see [1, Proposition 2.1]), whence we may obtain such a P' by use of lemma 1.1. It follows that domain($f_\pi(X)$) $\subset$ domain($f_\pi(Y)$), and by reversing the argument we get that domain($f_\pi(X)$) = domain($f_\pi(Y)$).

Now consider any prime P in this common domain. Then choose any (i, j) such that $A_i \subset P$, $a_i \in A_P$, $B_j \subset P$, and $b_j \in A_P$. Again, there is a prime ideal P' of R with $A_i + B_j \subset P' \subset P$. Thus $A_i \subset P'$ and $B_j \subset P'$ so that $a_i - b_j \in P'$ and $a_i - b_j \in P$. Hence $(f_\pi(X))(P) = a_i + P = b_j + P = (f_\pi(Y))(P)$. We have shown then that $f_\pi(X) = f_\pi(Y)$, since P was arbitrary in their common domains. Therefore, $X \sim_\pi Y$ and $[X]_\pi = [Y]_\pi$. So F is in fact well-defined.

By the definition, F certainly maps $\hat{R}$ onto $\overline{R}$, so by the remarks preceding the theorem we know that it is at least an epimorphism of the two rings. Thus to show that F is one-to-one and an isomorphism of the rings we need only show that the kernel of the mapping is $\{f_O\} = \{[(\{\{0\}\}, \{0\})]_\sigma\}$.

Suppose, then, that $F([X]_\sigma) = [X]_\pi$ is the zero element of $\overline{R}$ (which is the corresponding path function

$f_0 = [(\{\{0\}\}, \{0\})]_\pi$ in $\overline{R}$) for some $X \varepsilon S$. If $[X]_\sigma$ is not a path function, then it may be defined by ideals and elements as in theorem 2.7 so that $f_\sigma(X) = f_\sigma(X')$, with X' satisfying the conditions of theorem 2.7. But then, by the corresponding result in [3], $f_\pi(X) = f_\pi(X')$ is also not a path function in $\overline{R}$, a contradiction since it is f_0. So for some $a \varepsilon R$ we have that $[X]_\sigma = [(\{\{0\}\}, \{a\})]_\sigma$, and so $F([X]_\sigma) = [X]_\pi = [(\{\{0\}\}, \{a\})]_\pi = f_a = f_0$ in $\overline{R}$. However, just as in $\hat{R}$, the nil semi-simple condition implies that $f_a = f_b$ if and only if $a = b$ for any path functions f_a and f_b of $\overline{R}$. We conclude that $a = 0$ and $[X]_\sigma = f_0$; hence $\ker(F) = \{f_0\}$ and F is an isomorphism of $\hat{R}$ and $\overline{R}$. This completes the proof.

Summary. We have seen in this chapter, then, that the object $\hat{R}$ is a ring with the "same prime ideal structure as R"; that it is a ring from which our construction process yields the same ring back; that, conversely, any ring which yields an isomorphic ring by this process is of the form $\hat{R}$ for some ring R (in the sense of the natural injection); and that our process yields the same ring as that in [3]. Further, we have found necessary and sufficient conditions that a ring R be of the $\hat{}$-type.

Also, the operations $\hat{}$ and $\bar{}$ are closure operations in the sense that $\hat{R} \cong \hat{\hat{R}}$ for each ring R. Regarding $R = R' = \{f_a \mid a \varepsilon R\}$ as a subring of $\hat{R}$, we also have the closure property that $\hat{R} \cong \hat{S}$ for any ring S such that $R \subset S \subset \hat{R}$.

To conclude, we might consider a possible generalization of these ideas. We have seen that the spaces both of primes and of prime ideals in a ring R determine the same ring. Merle Manis has suggested that some other classes of structures in R might likewise give spaces which also lead to the same ring. In particular, in characterizing both π-functions and σ-functions much use was made of irreducible subsets and of minimal prime ideals of R. Notice that a σ-function f is determined by its values for minimal prime ideals; each prime ideal contains a minimal prime ideal, and if α is any minimal prime ideal then f is "constant" on the maximal irreducible set T_α. For if $\{A_i \mid i \in [1, n]\}$ and $\{a_i \mid i \in [1, n]\}$ define f, choose $A_i \subset \alpha$; then $A_i \subset P$ for all $P \in T_\alpha$. So $f(P) = a_i + P$ for all $P \in T_\alpha$. Thus it seems that the space of cosets of minimal prime ideals alone might very well determine the same ring as V(R) and Arith R.

CHAPTER III

EXAMPLES

For convenience of expression we introduce some new notation. Let R be a ring, commutative and nil semi-simple with identity as usual. We say that R is an <u>H-ring</u> if $R \cong \hat{R}$. One set of conditions has already been found that a ring R is an H-ring. It might be asked whether there are other, more natural conditions. At least we would like to know whether all, some, or no rings are H-rings and whether some well-known classes of rings are H-rings. This chapter gives a partial answer to such questions.

First, notice that the analysis of the first two chapters is somewhat trivial for integral domains; $\{0\}$ is the unique minimal prime ideal in any integral domain, and all domains (with identity) are H-rings. For by our earlier remarks, any σ-function must be constant on all prime ideals in $T_{\{0\}} = V_o(R)$ for an integral domain R, which is to say that each σ-function must be a path function.

However, it is not true that all rings are H-rings, as we see by an example due to Merle Manis [3]. Let F be a field, and consider the polynomial ring in two indeterminates $F[x, y] = R_1$. A well-known result is that $R[x_1, x_2, \ldots, x_n]$ is a unique factorization domain for any unique factorization domain R [5, p. 126]. Thus R_1 is a u.f.d.

and, consequently, an H-ring. But the ideal $K = (x^2 + y)yR_1$ is not a prime ideal of R_1, so R_1/K is not an integral domain. It is certainly a commutative ring with identity, however.

We will verify that $R = R_1/K$ is a ring in our sense; i.e., that it is nil semi-simple. Let $a \varepsilon R$ with $a^n = 0$ for some $n \geq 1$. Then a has the form $a = p(x, y) + K$ for some $p(x, y) \varepsilon F[x, y]$, and we are saying that $a^n = (p(x, y) + K)^n = p(x, y)^n + K = 0 + K = K$. In other words, $p(x, y)^n \varepsilon K = (x^2 + y)yR_1$ and $(x^2 + y)y$ divides $p(x, y)^n$. But y is irreducible, or a prime, in R_1 so y divides $p(x, y)$, since R_1 is a u.f.d. Similarly $x^2 + y$ must divide $p(x, y)$, but $x^2 + y$ and y are relatively prime so that $(x^2 + y)y$ must divide $p(x, y)$. Hence $p(x, y) \varepsilon K$ and $a = 0 + K$. Thus $\sqrt{\{0\}} = \{0\}$ in R, and R is nil semi-simple.

Now look at the principal ideals $A_1 = ((x^2 + y) + K)R$ and $A_2 = (y + K)R$. Let $a_1 = ((x^2 + y) + K)(p_1(x, y) + K) = (x^2 + y)p_1(x, y) + K \varepsilon A_1$ and $a_2 = (y + K)(p_2(x, y) + K) = yp_2(x, y) + K \varepsilon A_2$. Then $a_1a_2 = (x^2 - y)yp_1(x, y)p_2(x, y) + K = 0 + K$. Thus $A_1A_2 = \{0\}$ (where 0 here is $0 + K$, the zero of R).

Consider also the element $z = x + K \varepsilon R$. Then $z^2 = (x + K)^2 = (x^2 + K) = ((x^2 + y) + K) - (y + K) \varepsilon A_1 + A_2$ so that $z \varepsilon \sqrt{A_1 + A_2}$.

Now recall that in any quotient ring S/A, where A is any ideal of a ring S, there is a one-to-one correspondence between the ideals of S/A and the ideals of S which contain A, and that this correspondence preserves prime ideals. In our case the ideal A_1 derives from the ideal $A_1' = (x^2 + y)R_1$ of R_1 which contains K, and the ideal A_2 comes from the ideal $A_2' = yR_1$ of R_1 which also contains K. But these two ideals are prime ideals of R_1, since they are the principal ideals generated by the two irreducible elements $x^2 + y$ and y of R_1. Therefore, A_1 and A_2 are prime ideals of R; in particular they are radical ideals.

Suppose, then, that $z \in A_1 + A_2$. This means that there is some $p'(x, y)$ and some $q'(x, y)$ in $F[x, y]$ such that $x + K = ((x^2 + y)p'(x, y) + K) + (yq'(x, y) + K) = ((x^2 + y)p'(x, y) + yq'(x, y)) + K \in A_1 + A_2$. But then $(x^2 + y)p'(x, y) + yq'(x, y) - x \in K = (x^2 + y)yR_1$. Consequently, y divides $x^2p'(x, y) - x = x(xp'(x, y) - 1)$, and y divides $xp'(x, y) - 1$. But $xp'(x, y)$ has no constant term so that $xp'(x, y) - 1$ has the constant term -1, whereas y cannot divide any polynomial with a non-zero constant term. This contradiction shows that $x + K = z \notin A_1 + A_2 = \sqrt{A_1} + \sqrt{A_2}$ after all.

Now we can let $a_1 = z$, $a_2 = 0$ and obtain $a_i - a_j \in \sqrt{A_i + A_j}$ for all $(i, j) \in \{1, 2\} \times \{1, 2\}$, but $a_i - a_j \notin \sqrt{A_i} + \sqrt{A_j}$ for $i = 1$ and $j = 2$. Therefore, using theorem 2.7 we get that the σ-function determined by A_1, A_2, a_1, and a_2 is not a path function, and theorem 2.9 says then that R is not an H-ring.

We are at least assured, then, that some but not all rings are H-rings. It turns out that rings other than integral domains may be H-rings. Failing to put them all in some commonly recognized class, we might at least investigate whether some commonly considered rings are H-rings. A complete investigation, however, would be too lengthy to include here, and this paper will now be concluded with an examination of product rings. If Δ is an index set and for each $\alpha \in \Delta$ we have a ring R_α, recall that $R = \prod_{\alpha\in\Delta} R_\alpha = \{a\colon \Delta \longrightarrow \bigcup_{\alpha\in\Delta} R_\alpha \mid a(\alpha) = a_\alpha \in R_\alpha$ for each $\alpha \in \Delta\}$, called the <u>complete direct product</u> of the rings R_α, is a ring (see [6, pp. 172-178]) under the coordinatewise addition and multiplication. The <u>weak direct product</u> $R' = \{a \in \prod_{\alpha\in\Delta} R_\alpha \mid a_\alpha = 0_\alpha$, the zero of R_α, for all but finitely many $\alpha \in \Delta\}$ is a subring of R which we will also consider. For the rest of the paper we will use the term "ring" in the ordinary sense, not assuming any of the special properties that were previously assumed unless an H-ring is specified. Likewise, an ideal may not be proper.

We recall without proof some properties of such products. The weak direct product is actually an ideal of the complete direct product, and every ideal of R' is an ideal of R. R and R' are each commutative if and only if each R_α is commutative. The same is true of the nil semisimple property. For multiplicative identities, R has one if and only if each R_α does, but R' cannot have one unless Δ is finite.

For each $\alpha \varepsilon \Delta$ we have the projection epimorphism $p_\alpha\colon R \longrightarrow R_\alpha$ given by $p_\alpha(a) = a_\alpha$ for each $a \varepsilon R$ and the monomorphism $i_\alpha\colon R_\alpha \longrightarrow R$ given by $(i_\alpha(b))_\delta = 0_\delta$ if $\delta \neq \alpha$, $(i_\alpha(b))_\alpha = b$ for each $b \varepsilon R_\alpha$. If A is any ideal of R, then $A_\alpha = \{a_\alpha \mid a \varepsilon A\} = p_\alpha(A)$ is an ideal of R_α for each $\alpha \varepsilon \Delta$, and $A \subset \pi_{\alpha\varepsilon\Delta} A_\alpha$. If each ring R_α has identity 1_α, then $\pi'_{\alpha\varepsilon\Delta} A_\alpha \subset A \subset \pi_{\alpha\varepsilon\Delta} A_\alpha$, where π' denotes the weak direct product, and A_α is a prime ideal of R_α for each $\alpha \varepsilon \Delta$ if A is a prime ideal. If, in addition, Δ is finite, then the three ideals above are equal. Conversely, if $A(\alpha)$ is an ideal of R_α for each $\alpha \varepsilon \Delta$, then $A = \pi_{\alpha\varepsilon\Delta} A(\alpha)$ is an ideal of R and $A_\alpha = A(\alpha)$ for each α. Notice that if A and B are any subsets of R, then $(A + B)_\alpha = A_\alpha + B_\alpha$ for all $\alpha \varepsilon \Delta$.

Last, if A is an ideal of R, then $(\sqrt{A})_\alpha = \sqrt{A_\alpha}$ for each $\alpha \varepsilon \Delta$, and $\sqrt{A} \subset \sqrt{\pi_{\alpha\varepsilon\Delta} A_\alpha} \subset \pi_{\alpha\varepsilon\Delta} \sqrt{A_\alpha} = \pi_{\alpha\varepsilon\Delta} (\sqrt{A})_\alpha$. Now for a discussion of H-rings an identity is required, so only the complete direct product is considered in the theorem we now prove with the aid of the ideas above.

Theorem 3.1. Let $R = \pi_{\alpha\varepsilon\Delta} R_\alpha$ be any complete direct product of rings. R can be an H-ring only if R_α is an H-ring for each $\alpha \varepsilon \Delta$; if each R_α is an H-ring and Δ is finite, then R is an H-ring.

Proof. Since R is commutative and nil semi-simple with identity if and only if each ring R_α is, we may assume that they all are such and concern ourselves with the conditions of theorems 2.7 to 2.9 in the last chapter.

Suppose that R is not an H-ring. Then we choose ideals $\{A_i \mid i \in [1, n]\}$ and elements $\{a_i \mid i \in [1, n]\}$ of R according to theorem 2.8. Clearly, we have that $\prod_{i=1}^{n} (A_i)_\alpha = \{0_\alpha\}$ for each $\alpha \in \Delta$, and if $(i, j) \in [1, n] \times [1, n]$, then $(a_i)_\alpha - (a_j)_\alpha = (a_i - a_j)_\alpha \in (\sqrt{A_i + A_j})_\alpha = \sqrt{(A_i + A_j)_\alpha} = \sqrt{(A_i)_\alpha + (A_j)_\alpha}$ for each $\alpha \in \Delta$. Thus by deleting those indices i for which $(A_i)_\alpha = R_\alpha$, and not all will be such for any given α, we may assume that the sequences $\{(A_i)_\alpha \mid i \in [1, n]\}$ and $\{(a_i)_\alpha \mid i \in [1, n]\}$ are non-empty sequences with proper ideals which define a σ-function in R_α for each α.

We claim that one of these σ-functions is not a path function if Δ is finite. For if $(a_i)_\alpha - (a_j)_\alpha \in (A_i)_\alpha + (A_j)_\alpha$ for all $\alpha \in \Delta$ and all $(i, j) \in [1, n] \times [1, n]$, the finiteness condition gives $A_i + A_j = \prod_{\alpha \in \Delta} (A_i + A_j)_\alpha = \prod_{\alpha \in \Delta} [(A_i)_\alpha + (A_j)_\alpha]$ so that since $(a_i - a_j)_\alpha = (a_i)_\alpha - (a_j)_\alpha \in (A_i)_\alpha + (A_j)_\alpha$ for all $\alpha \in \Delta$, thus $(a_i - a_j) \in \prod_{\alpha \in \Delta} [(A_i)_\alpha + (A_j)_\alpha] = A_i + A_j$ for all $(i, j) \in [1, n] \times [1, n]$. This, however, contradicts our original choice. Hence for some $\delta \in \Delta$ we have that $(a_i)_\delta - (a_j)_\delta \in \sqrt{(A_i)_\delta + (A_j)_\delta} \setminus [(A_i)_\delta + (A_j)_\delta]$ for some (i, j). In this case $(A_i)_\delta$ and $(A_j)_\delta$ are not R_δ, and these ideals were not deleted to get defining sequences for this particular δ. Further, note that $(A_i)_\delta = \sqrt{(A_i)_\delta}$ and $(A_j)_\delta = \sqrt{(A_j)_\delta}$ by the original choice of ideals. Hence theorem 2.7 applies, and the σ-function for R_δ is not a path function; thus R_δ is not an H-ring. Therefore, if each R_α is an H-ring and Δ is finite, then R is an H-ring.

Conversely, suppose that R_δ is not an H-ring for some $\delta \in \Delta$. Choose ideals $\{B_j \mid j \in [1, n]\}$ and elements $\{b_j \mid j \in [1, n]\}$ according to theorem 2.8. For $j \in [1, n]$ let $A_j = \{i_\delta(b) \mid b \in B_j\}$. Then A_j is an ideal of R for each j because $A_j = \pi_{\alpha\in\Delta}(C_j)(\alpha)$, where $(C_j)(\alpha) = \{0_\alpha\}$ for $\alpha \neq \delta$ and $(C_j)(\delta) = B_j$. It is clear that $\pi_{j=1}^{n} A_j = \{0\}$, and each A_j is proper. Similarly let $a_j = i_\delta(b_j)$ for each j. It is easily checked that for $(j, k) \in [1, n] \times [1, n]$, $a_j - a_k \in i_\delta((\sqrt{A_j + A_k})_\delta)$. But since $(A_j + A_k)_\alpha = \{0_\alpha\}$ for each $\alpha \neq \delta$ and R_α is nil semi-simple for each α, we get that $(\sqrt{A_j + A_k})_\alpha = \{0_\alpha\}$ for each $\alpha \neq \delta$, and it follows that $i_\delta((\sqrt{A_j + A_k})_\delta) = \sqrt{A_j + A_k}$ and $a_j - a_k \in \sqrt{A_j + A_k}$ for all (j, k). By the same reasoning, we see that $\sqrt{A_j} = i_\delta(\sqrt{B_j}) = i_\delta(B_j) = A_j$ for each $j \in [1, n]$.

Now $n \geq 2$ by the original choice, so choose some $(i, j) \in [1, n] \times [1, n]$ with $i \neq j$. If $a_i - a_j \in \sqrt{A_i} + \sqrt{A_j}$, then $b_i - b_j = (a_i)_\delta - (a_j)_\delta = (a_i - a_j)_\delta \in (A_i + A_j)_\delta = (A_i)_\delta + (A_j)_\delta = B_i + B_j$, a contradiction. Thus $a_i - a_j \notin \sqrt{A_i} + \sqrt{A_j}$ for some $i \neq j$. By theorem 2.7 we conclude that R has a σ-function other than a path function; thus R cannot be an H-ring. Therefore, R can be an H-ring only if each ring R_α is an H-ring, and the theorem is proved.

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