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On two-dimensional Lebesgue measure and rectangle functions

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ON TWO-DIMENSIONAL LEBESGUE

MEASURE AND RECTANGLE FUNCTIONS

by

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R.D.R.

TABLE OF CONTENTS

INTRODUCTION

Throughout the entire discussion, the underlying space being considered is R_2, the Euclidean plane. Any point p in this space may be represented by an ordered pair of real numbers (a,b). As in common practice, points will be located with reference to two coordinate, perpendicular axes, the x (horizontal) and y (vertical) axes.

Some of the notations and conventions encountered will be as follows. A <u>set</u> will be a collection of objects called <u>points</u>. A collection of sets will be called a <u>class</u>. Lower case English letters will denote points; upper case English letters will denote sets; and script capital English letters will denote classes. The following symbols with definitions indicated will be extensively used.

Symbol	Definition
$\in$	"is a member of" or "belongs to"
$\notin$	"is not a member of" or "does not belong to"
$\subset$	"is contained in" or "is a subset of"
$\not\subset$	"is not contained in" or "is not a subset of"
$\supset$	"contains"
$\not\supset$	"does not contain"
$\therefore$	"therefore"
$d(p_1, p_2)$	"the distance from p_1 to p_2"
$N(p,\epsilon)$	"the neighborhood of p of radius ϵ"

The distance between points will be defined in the ordinary sense. That is, if $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, then $d(p_1,p_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

A neighborhood of a point p of radius ϵ is the set of all points q such that $d(p,q)<\epsilon$. Thus, it will consist of the interior of a circle having p as center and radius ϵ .

If E and F are two sets, then $E + F$ will denote the set of all points p such that either $p \in E$ or $p \in F$. If $E_1, E_2, ---, E_n$ are sets, then $\sum_{i=1}^{n} E_i$ will denote the set of points p such that $p \in E_i$ for some $i = 1,2,---,n$. If $E_1, E_2, ---$ are sets, then $\sum_{i=1}^{\infty} E_i$ will denote the set of points p such that $p \in E_i$ for some $i = 1,2,---$. If $\mathfrak{Y}$ is any class of sets, then $\sum_{E \in \mathfrak{Y}} E$ will denote the set of points p such that $p \in E$ for some set $E \in \mathfrak{Y}$.

If E and F are two sets, then $E \cdot F$ will denote the set of all points p such that p is in both E and F. If $E_1, E_2, ---, E_n$ are sets, then $\prod_{i=1}^{n} E_i$ will denote the set of points p such that $p \in E_i$ for $i = 1,2,---,n$. If $E_1, E_2, ---$ are sets, then $\prod_{i=1}^{\infty} E_i$ denotes the set of points p such that $p \in E_i$ for each $i = 1,2,---$. If $\mathfrak{Y}$ is any class of sets, then $\prod_{E \in \mathfrak{Y}} E$ denotes the set of points p such that $p \in E$ for each set $E \in \mathfrak{Y}$.

The empty set or set consisting of no points will be denoted by ϕ.

$\mathcal{C}(E)$, the complement of E will denote the set of all points p such that $p \notin E$.

$E - F$ will denote the set of points p such that $p \in E$ and $p \notin F$. i.e. $E - F = E \cdot \mathcal{C} F$.

Sometimes a set of points in the plane will be explicitly denoted. For example $E_{x,y}\left[a \leqq x < b; \;\; c \leqq y < d\right]$ will denote the set of points p whose x and y coordinates fulfill the restrictions indicated inside the brackets.

An open set is a set G such that if $p \in G$, then there exists an $\epsilon > 0$ such that $N(p, \epsilon) \subset G$.

A point p is a limit point of a set E if for every $\epsilon > 0$, there exists $q \neq p$ such that $q \in E$ and $q \in N(p, \epsilon)$.

A closed set is a set F such that if p is a limit point of F, then $p \in F$.

If E is any set, then $\bar{E}$ will denote the closure of E and will be defined as the set of all points p such that either $p \in E$ or p is a limit point of E.

If E is any set, then E^o will denote the interior of E and will be defined as the set of points p such that $N(p,\epsilon) \subset E$ for some $\epsilon > 0$.

If $\{a_n\}$ is a sequence of real numbers, then we say the limit of $\{a_n\}$ as n approaches infinity is L, if for any $\epsilon > 0$ there exists an integer M such that if $n > M$, then $|a_n - L| < \epsilon$. We write $\lim_{n \to \infty} a_n = L$.

The limit inferior of a sequence of real numbers $\{a_n\}$ is abbreviated $\lim. \inf._{n \to \infty} a_n$ and is defined as follows. $\lim. \inf._{n \to \infty} a_n = c$ means that c is the smallest number for which there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k \to \infty} a_{n_k} = c$.

The limit superior of a sequence of real numbers $\{a_n\}$ is abbreviated $\lim. \sup._{n \to \infty} a_n$ and is defined as follows. $\lim. \sup._{n \to \infty} a_n = d$ means that d is the largest number for which there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k \to \infty} a_{n_k} = d$.

If E is any set of real numbers, then the least upper bound, abbreviated l.u.b., of E is defined as follows. M is the least upper bound of E if both these conditions are satisfied.

1. If $p \in E$, then $p \leq M$.
2. If L is such that $p \leq L$ for each $p \in E$, then $L \geq M$.

If E is any set of real numbers, then the greatest lower bound, abbreviated g.l.b., of E is defined as follows. m is the greatest lower bound of E if both these conditions are satisfied.

1. If $p \in E$, then $p \geqq m$.
2. If l is such that $p \geqq l$ for each $p \in E$, then $l \leqq m$.

If E is a set of real numbers, then we say that E is a bounded set if E has both a least upper bound and a greatest lower bound.

If $\{f_n(p)\}$ is a sequence of functions defined on a set E and if f(p) is a function defined on E, then we say $\{f_n(p)\}$ converges to f(p) on E, if for any $\epsilon > 0$, there exists an integer M depending upon both ϵ and p, such that if $n > M$, then $|f_n(p) - f(p)| < \epsilon$. We write $\lim_{n\to\infty} f_n(p) = f(p)$ on E or $f_n(p) \longrightarrow f(p)$ on E.

If $\{f_n(p)\}$ is a sequence of functions defined on a set E and if f(p) is a function defined on E, then we say $\{f_n(p)\}$ converges to f(p) uniformly on E, if for any $\epsilon > 0$, there exists an integer M, depending only upon ϵ and independent of the point $p \in E$, such that if $n > M$, then $|f_n(p) - f(p)| < \epsilon$. We write $\lim_{n\to\infty} f_n(p) = f(p)$ uniformly on E or $f_n(p) \Longrightarrow f(p)$ on E.

CHAPTER I

TWO-DIMENSIONAL LEBESGUE MEASURE

Let $\mathcal{P}$ be the collection of all oriented half-open rectangles of the form $R_{a,b;c,d} = E_{x,y}[a \leqq x < b;\ \ c \leqq y < d]$.

<u>1.1</u> $\emptyset$ (the empty set) $\in \mathcal{P}$ since $\emptyset = R_{a,a;c,c}$.

<u>1.2</u> If $R \in \mathcal{P}$ and if $S \in \mathcal{P}$ then $R \cdot S \in \mathcal{P}$. This is a conclusion which may be easily verified.

1.3 If $E \in \mathcal{P}$, $F \in \mathcal{P}$, then $F - E = R_1 + R_2 + R_3 + R_4$, where each $R_i \in \mathcal{P}$ and $R_i \cdot R_j = \emptyset$ if $i \neq j$. Note: one or more of the R_i's may be empty.

1.4 Definition. If $R \in \mathcal{P}$ and if $R = E_{x,y}\left[a \leqq x < b;\ \ c \leqq y < d\right]$, then $A(R) = (b-a)\ (d-c)$ (area of R).

1.5 $A(\emptyset) = (a-a)\ (c-c) = 0$

1.6 If $R \in \mathcal{P}$, then $A(R) \geqq 0$.

1.7 If $R = R_{a,b;c,d}$ and if $R_1, R_2, \ldots, R_n$ are such that $R_j = R_{a_j,b_j;c_j,d_j}$ for each j, $R = \sum_{j=1}^{n} Rj$, and $R_j \cdot R_k = \emptyset$, if $j \neq k$, then $\sum_{j=1}^{n} A(R_j) = A(R)$.

Proof: By induction. Conclusion true if $n = 1$. $A(R) = A(R)$

Suppose $n = 2$. We may without loss of generality assume that $(a,c) \in R_1$.

. Then $a_1 = a$, $c_1 = c$. There are two cases.

(1) Suppose $b_1 = a_2$. Then $b_2 = b_1$, $c_2 = c_1 = c$, and $d_2 = d_1 = d$.

$A(R_1) + A(R_2) = (b_1 - a_1)\ (d_1 - c_1) + (b_2 - a_2)\ (d_2 - c_2) = (a_2 - a)\ (d - c) + (b - a_2)\ (d - c) = (b - a)\ (d - c) = A(R)$.

(2) Suppose $d_1 = c_2$. Then $a = a_1 = a_2$, $b = b_1, = b_2$ and $d = d_2$.

$A(R_1) + A(R_2) = (b_1 - a_1)\ (d_1 - c_1) + (b_2 - a_2)\ (d_2 - c_2) = (b - a)\ (c_2 - c) + (b - a)\ (d - c_2) = (b - a)\ (d - c) = A(R)$.

In the general case we may assume without loss of generality that $(a,c) \in R_1$

.

Then $a_1 = a$, $c_1 = c$. $R_1 = R_{a,b1;c,d1}$.

Let $R' = R_{b_1,b;c,d_1}$, $R'' = R_{a,b;d_1,d}$.

$A(R) = (b-a)\ (d-c) = (b_1-a)\ (d_1-c) + (b-b_1)\ (d_1-c) + (b-a)\ (d-d_1) = A(R_1) + A(R') + A(R'')$.

Suppose conclusion is true for all $k < n$.

$R' \subset R - R_1$, $\quad \sum_{j=2}^{n} R_j = R - R_1$

$R' = R' \cdot \sum_{j=2}^{n} R_j = \sum_{j=2}^{n} R' \cdot R_j$.

Similarly, $R'' = \sum_{j=2}^{n} R'' \cdot R_j$

By inductive assumption,

$$A(R') = \sum_{j=2}^{n} A(R' \cdot R_j), \quad A(R'') = \sum_{j=2}^{n} A(R'' \cdot R_j)$$

$$\therefore \quad A(R) = A(R_1) + \sum_{j=2}^{n} \left[A(R' \cdot R_j) + A(R'' \cdot R_j)\right].$$

We must show $A(R' \cdot R_j) + A(R'' \cdot R_j) = A(R_j)$ for $j = 2, \ldots, n$.

Case 1: Either $R_j \subset R'$ or $R_j \subset R''$. R_1, R', R'', are disjoint.

Hence $A(R' \cdot R_j) + A(R'' \cdot R_j) = A(R_j)$

Case 2. Suppose $R_j \subset R' + R''$, $R_j \cdot R' \neq \emptyset$ and $R_j \cdot R'' \neq \emptyset$.

Then $R_j = R_j \cdot R' + R_j \cdot R''$.

$\therefore$ $A(R_j) = A(R_j \cdot R') + A(R_j \cdot R'')$, by the inductive assumption.

Thus, $\sum_{j=2}^{n} \left[A(R' \cdot R_j) + A(R'' \cdot R_j)\right] = \sum_{j=2}^{n} A(R_j)$. $A(R) = \sum_{j=1}^{n} A(R_j)$.

<u>1.8</u> If $R \in \mathcal{P}$ and if $R_i \in \mathcal{P}$, $i = 1,2,\ldots,n$, and if $R_j \cdot R_k = \emptyset$, if $j \neq k$, and if $\sum_{i=1}^{n} R_i \subset R$, then $\sum_{i=1}^{n} A(R_i) \leq A(R)$.

Proof: By induction.

$R = R_1 + \sum_{j=1}^{m} S_j$ where $S_j \in \mathcal{P}$ for each j, $R_1 \cdot S_j = \emptyset$, and $S_i \cdot S_j = \emptyset$ if $i \neq j$.

From the preceding conclusion, $A(R) = A(R_1) + \sum_{j=1}^{m} A(S_j)$,

$$\sum_{i=2}^{n} R_i \subset R - R_1, \quad \sum_{j=1}^{m} S_j = R - R_1$$

$$\left(\sum_{i=2}^{n} R_i\right)\left(\sum_{j=1}^{m} S_j\right) = \sum_{i=2}^{n} R_i = \sum_{i=2}^{n}\sum_{j=1}^{m} R_i \cdot S_j =$$

$$\sum_{j=1}^{m}\sum_{i=2}^{n} R_i \cdot S_j; \quad S_j \cdot \sum_{i=2}^{n} R_i \subset S_j.$$

$$\sum_{i=2}^{n} R_i \cdot S_j \subset S_j.$$

Assume conclusion is true for all $k < n$. It is true for $n = 1$.

$\therefore \sum_{i=2}^{n} A(R_i \cdot S_j) \leqq A(S_j)$

If $i = 2,3,\ldots,n$, $\sum_{j=1}^{m} R_i \cdot S_j = R_i \cdot \sum_{j=1}^{m} S_j = R_i(R-R_1) = R_i$

$\therefore A(R_i) = \sum_{j=1}^{m} A(R_i \cdot S_j)$ by 1.7.

$A(R) \geqq A(R_1) + \sum_{j=1}^{m}\sum_{i=2}^{n} A(R_i \cdot S_j) = A(R_1) + \sum_{i=2}^{n}\sum_{j=1}^{m} A(R_i \cdot S_j) =$

$A(R_1) + \sum_{i=2}^{n} A(R_i) = \sum_{i=1}^{n} A(R_i)$.

<u>1.9</u> If $\sum_{i=1}^{\infty} R_i \subset R$, where $R \in \mathcal{P}$, $R_i \in \mathcal{P}$ for $i = 1,\ldots,n,\ldots$, $R_i \cdot R_j = \emptyset$, if $i \neq j$, then $\sum_{i=1}^{\infty} A(R_i) \leqq A(R)$.

Proof: From the above, $\sum_{i=1}^{n} A(R_i) \leqq A(R)$ for each n.

$A(R_i) \geqq 0$ for each i. Thus the sequence of partial sums of $\sum_{i=1}^{\infty} A(R_i)$ is an increasing sequence bounded above by $A(R)$ and therefore converges to a limit less than or equal to $A(R)$.

i.e. $\sum_{i=1}^{\infty} A(R_i) \leqq A(R)$.

<u>1.10</u> Suppose $R \subset \sum_{i=1}^{n} R_i$, where $R = R_{a,b;c,d}$; $R_i = R_{a_i,\, b_i;\, c_i,\, d_i}$; $R \in \mathcal{P}$, $R_i \in \mathcal{P}$ for each i.
Then $A(R) \leqq \sum_{i=1}^{n} A(R_i)$.

Proof: Induction on the number of R_i.

1. When $n = 1$, $R \subset R_1$, $\therefore A(R) \leqq A(R_1)$.

2. Assume that the conclusion is true when $k < n$.

3. Let $p = (a,c)$. Without loss of generality we may assume $p \in R_1$.

Let $R' = R_{a,b_1;c,d_1} = R \cdot R_1$, $R'' = R_{b_1,b;c,d}$; $R''' = R_{a,b_1;d_1,d}$. $R'' \subset R''$. $\sum_{i=2}^{n} R_i = \sum_{i=2}^{n} R'' \cdot R_i$; $R''' \subset R'''$. $\sum_{i=2}^{n} R_i = \sum_{i=2}^{n} R''' \cdot R_i$.

$R = R' + R'' + R'''$; R', R'', R''' are all disjoint. $A(R) = A(R') + A(R'') + A(R''')$.

By inductive assumption,

$$A(R'') \leqq \sum_{i=2}^{n} A(R'' \cdot R_i); \quad A(R''') \leqq \sum_{i=2}^{n} A(R''' \cdot R_i); \quad A(R') \leqq A(R_1).$$

$$A(R) \leqq A(R') + \sum_{i=2}^{n} \left[A(R'' \cdot R_i) + A(R''' \cdot R_i)\right] \quad R'' \cdot R_i + R''' \cdot R_i \subset R_i.$$

$$A(R'' \cdot R_i) + A(R''' \cdot R_i) \leqq A(R_i) \text{ by 1.8.}$$

$$A(R) = A(R') + A(R'') + A(R''') \leqq A(R_1) + \sum_{i=2}^{n} A(R_i) = \sum_{i=1}^{n} A(R_i).$$

<u>1.11</u> Suppose $R \subset \sum_{i=1}^{\infty} R_i$, $R \in \mathcal{P}$, $R_i \in \mathcal{P}$ for each i. Then $A(R) \leqq \sum_{i=1}^{\infty} A(R_i)$.

Proof: Give $\epsilon > 0$. Suppose $R = R_{a,b;c,d}$, $R_i = R_{a_i,b_i;c_i,d_i}$.

Let $S \subset R$, $S = R_{a,\beta;c,\delta}$ so that $A(R) > A(S) > A(R) - \frac{\epsilon}{2}$.

Let $R_i \subset S_i$, $S_i = R_{\alpha_i,b_i;\gamma_i,d_i}$, so that

$A(R_i) < A(S_i) < A(R_i) + \frac{\epsilon}{2^{i+1}}$.

Let $\bar{S}$ be the closure of S. Let S_i° be the interior of S_i.

$$\bar{S} \subset R \subset \sum_{i=1}^{\infty} R_i \subset \sum_{i=1}^{\infty} S_i^{\circ}, \qquad R_i \subset S_i^{\circ} \text{ for each i.}$$

By the Heine-Borel Covering theorem,

$$\bar{S} \subset \sum_{i=1}^{n} S_i^{\circ}; \qquad S \subset \sum_{i=1}^{k} S_i$$

$$A(R) - \frac{\epsilon}{2} < A(S) \leqq \sum_{i=1}^{n} A(S_i) < \sum_{i=1}^{\infty} \left[A(R_i) + \frac{\epsilon}{2^{i+1}}\right] = \sum_{i=1}^{\infty} A(R_i) + \epsilon/2.$$

$$\therefore \quad A(R) - \epsilon < \sum_{i=1}^{\infty} A(R_i).$$

Since ϵ was arbitrary,

$$A(R) \leqq \sum_{i=1}^{\infty} A(R_i).$$

<u>1.12</u> If $R \in \mathcal{P}$, if $R_i \in \mathcal{P}$ for each i, if $R_i \cdot R_j = \emptyset$ for $i \neq j$, and if $R = \sum_{i=1}^{\infty} R_i$, then $A(R) = \sum_{i=1}^{\infty} A(R_i)$.

Proof: 1. $\sum_{i=1}^{\infty} R_i \subset R \therefore \sum_{i=1}^{\infty} A(R_i) \leqq A(R)$.

2. $R \subset \sum_{i=1}^{\infty} R_i, \therefore A(R) \leqq \sum_{i=1}^{\infty} A(R_i)$. Thus,

$A(R) = \sum_{i=1}^{\infty} A(R_i)$.

<u>1.13</u> If E is any set and if for every countable sequence of sets $\{R_i\}_{i=1}^{\infty}$ such that $R_i \in \mathcal{P}$ for each i and such that $E \subset \sum_{i=1}^{\infty} R_i$ we have $\sum_{i=1}^{\infty} A(R_i) = +\infty$, then we define $\mu^*(E) = +\infty$.

<u>1.14</u> Definition.

If E is any subset of R_2, the Euclidean plane, then $\mu^*(E)$, the exterior Lebesgue measure of E, is defined thus: $\mu^*(E) = \text{g.l.b.} \sum_{i=1}^{\infty} A(R_i)$ where g.l.b. is taken with respect to all possible countable coverings of E by means of sets $R_i \in \mathcal{P}$. i.e. where $E \subset \sum_{i=1}^{\infty} R_i$.

This means that if $\mu^*(E)$ is finite, then if $E \subset \sum_{i=1}^{\infty} R_i$, where $R_i \in \mathcal{P}$ for each i, then $\mu^*(E) \leqq \sum_{i=1}^{\infty} A(R_i)$. Also if $\epsilon > 0$, then there exists a collection of sets $\{R_i\}_{i=1}^{\infty}$, such that $R_i \in \mathcal{P}$ for each i, and such that $\mu^*(E) + \epsilon > \sum_{i=1}^{\infty} A(R_i)$.

<u>1.15</u> $\mu^*(R_2) = +\infty$

Proof: Deny. Suppose $\mu^*(R_2) < +\infty$. Then by 1.14 there exists a countable sequence of sets $\{R_i\}_{i=1}^{\infty}$ such that $R_i \in \mathcal{P}$ for each i and such that $R_2 \subset \sum_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} A(R_i) = a < +\infty$ and $\mu^*(R_2) \leqq a$. But there exists $R = R_{-\frac{\sqrt{2a}}{2}, \frac{\sqrt{2a}}{2}, -\frac{\sqrt{2a}}{2}, \frac{\sqrt{2a}}{2}} \in \mathcal{P}$, $R \subset R_2 \therefore A(R) \leqq \mu^*(R_2)$. But $A(R) = 2a$. This is a contradiction. We conclude that $\mu^*(R_2) = +\infty$.

1.16 If E is any set, $\mu^*(E) \geq 0$.

1.17 $\mu^*(\emptyset) = 0$

1.18 If E is a countable set, then $\mu^*(E) = 0$

Proof: Let $E = \{p_1, p_2, \ldots, p_n, \ldots\}$. Give $\epsilon > 0$.

Suppose $p_i = (a_i, c_i)$ for each i.

Let $R_1 = R_{a_1, a_1+\sqrt{\frac{\epsilon}{2}}; c_1, c_1+\sqrt{\frac{\epsilon}{2}}}$, $R_2 = R_{a_2, a_2+\sqrt{\frac{\epsilon}{4}}, c_2, c_2+\sqrt{\frac{\epsilon}{4}}}$.

$R_n = R_{a_n, a_n+\sqrt{\frac{\epsilon}{2^n}}; c_n, c_n+\sqrt{\frac{\epsilon}{2^n}}}, \ldots$, $E \subset \sum_{n=1}^{\infty} R_n$

$$\sum_{n=1}^{\infty} A(R_n) = \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots = \epsilon$$

$$\mu^*(E) \leq \sum_{n=1}^{\infty} A(R_n) = \epsilon .$$

Since ϵ was arbitrary and since $\mu^*(E) \geq 0$, we conclude that $\mu^*(E) = 0$.

1.19 Let $R \in \mathcal{P}$. Then $\mu^*(R) = A(R) = (b-a)(d-c)$, if $R = R_{a,b;c,d}$.

Proof:

1. $R \subset R$; $\mu^*(R) \leq A(R)$

2. Suppose $R \subset \sum_{i=1}^{\infty} R_i$, where $R_i \in \mathcal{P}$ for each i. $A(R) \leq \sum_{i=1}^{\infty} A(R_i)$ for all such coverings of R. But $\mu^*(R) = \text{g.l.b.} \sum_{i=1}^{\infty} A(R_i)$ for all such sums. $\therefore$ $A(R) \leq \mu^*(R)$.

We conclude that $\mu^*(R) = A(R)$.

1.20 Suppose $E \subset F$, then $\mu^*(E) \leq \mu^*(F)$.

Proof:

1. Suppose $\mu^*(F) = +\infty$. Then conclusion is true.

2. Suppose $\mu^*(F)$ is finite. Give $\epsilon > 0$. Then by 1.14 there is a covering $R_1, R_2, \ldots$, such that $F \subset \sum_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} A(R_i) < \mu^*(F) + \epsilon$.

$E \subset \sum_{i=1}^{\infty} R_i$, $\mu^*(E) \leq \sum_{i=1}^{\infty} A(R_i)$. $\mu^*(E) < \mu^*(F) + \epsilon$.

Since ϵ is arbitrary, we conclude that $\mu^*(E) \leq \mu^*(F)$.

1.21 Let $G = E_{x,y}\left[\, a<x<b,\ \ c<y<d \,\right]$, i.e. an oriented open rectangle. Then $\mu^*(G) = (b-a)\ (d-c)$.

Proof:

1. Let $R = E_{x,y}\left[\, a \leq x<b,\ \ c \leq y<d \,\right]$.

$\mu^*(R) = A(R) = (b-a)\ (d-c)$. $\quad G \subset R$.

$\therefore$ by 1.20 $\mu^*(G) \leq \mu^*(R) = (b-a)\ (d-c)$.

2. Give $\epsilon > 0$. Let $0 < \delta < \dfrac{\epsilon}{(d-c)+(b-a)}$ Let $S = R_{a+\delta,b;c+\delta,d}$.

$\mu^*(S) = A(S) = (b-a-\delta)\ \ (d-c-\delta) = (b-a)\ (d-c) - \delta\left((d-c)+(b-a)\right)+\delta^2 =$ $(b-a)\ (d-c) - \delta\left((d-c)+(b-a)-\delta\right)$; $S \subset G$, $\therefore$ by 1.20 $\mu^*(S) \leq \mu^*(G)$.

$(b-a)\ (d-c) - \epsilon < (b-a)\ (d-c) - \delta\ \left((d-c)\ \ (b-a)-\delta\right) \leq \mu^*(G)$.

Since ϵ is arbitrarily small, though positive, we conclude

$(b-a)\ (d-c) \leq \mu^*(G)$, $\therefore \mu^*(G) = (b-a)\ (d-c)$.

1.22 Let $F = E_{x,y}\left[\, a \leq x \leq b,\ c \leq y \leq d \,\right]$. Then $\mu^*(F) = (b-a)\ (d-c)$.

Proof:

1. Let $R = R_{a,b;c,d}$. $\quad R \subset F$

$\therefore$ by 1.20 $\mu^*(R) \leq \mu^*(F)$; $\mu^*(R) = (b-a)\ (d-c)$, $(b-a)\ (d-c) \leq \mu^*(F)$.

2. Give $\epsilon > 0$. Take $0<\delta<1$, such that $\delta < \dfrac{\epsilon}{(d-c)+(b-a)+1}$. Let $S = R_{a,b+\delta\,;\,c,d+\delta}$. $F \subset S$. $\mu^*(S) =$

$A(S) = (b+\delta-a)\ (d+\delta-c) = (b-a+\delta)\ (d-c+\delta)$. $= (b-a)\ (d-c) +$ $\delta\left((d-c)+(b-a)\right)+\delta^2 = (b-a)(d-c)+\delta\left((b-a)+(d-c)+\delta\right)$.

By 1.20 $\mu^*(F) \leq \mu^*(S) = A(S) = (b-a)\ (d-c) + \delta\left((b-a)+(d-c)+\delta\right) <$ $(b-a)(d-c)+\epsilon$.

Since ϵ is arbitrarily small but positive we conclude

$\mu^*(F) \leq (b-a)\ (d-c)$. $\therefore \mu^*(F) = (b-a)\ (d-c)$.

1.23 Suppose $R_{a,b;c,d} \in \mathcal{P}$. Let R^o denote the interior of R and $\bar{R}$ denote the closure of R. If S is such that $R^o \subset S \subset \bar{R}$, then $\mu^*(S) =$ $(b-a)\ (d-c)$.

Proof: $\mu^*(R^\circ) = (b-a)\ (d-c)$.

$\mu^*(\bar{R}) = (b-a)\ (d-c)$, By 1.20 $\mu^*(R^\circ) \leqq \mu^*(S) \leqq \mu^*(\bar{R})$

$\therefore \mu^*(S) = (b-a)\ (d-c)$.

1.24 If E and F are any two sets, then $\mu^*(E + F) \leqq \mu^*(E) + \mu^*(F)$.

Proof: Case 1. Suppose either $\mu^*(E)$ or $\mu^*(F)$ is $+\infty$. Then the conclusion is immediate.

Case 2. Suppose both $\mu^*(E)$ and $\mu^*(F)$ are finite. Give $\epsilon > 0$.

From 1.14 there exists $\{S_i\}$ such that $S_i \in \mathcal{P}$ for each i and such that $E \subset \sum_{i=1}^{\infty} S_i$ and $\mu^*(E) > \sum_{i=1}^{\infty} A(S_i) - \frac{\epsilon}{2}$.

There exists $\{T_i\}$ such that $T_i \in \mathcal{P}$ for each i and such that $F \subset \sum_{i=1}^{\infty} T_i$ and $\mu^*(F) > \sum_{i=1}^{\infty} A(T_i) - \frac{\epsilon}{2}$.

$E + F \subset \sum_{i=1}^{\infty} S_i + \sum_{i=1}^{\infty} T_i;\ \mu^*(E + F) \leqq \sum_{i=1}^{\infty} A(S_i) + \sum_{i=1}^{\infty} A(T_i)$

$\mu^*(E) + \mu^*(F) > \sum_{i=1}^{\infty} A(S_i) + \sum_{i=1}^{\infty} A(T_i) - \epsilon \geqq \mu^*(E + F) - \epsilon$.

Since $\epsilon > 0$ is arbitrary, we conclude

$$\mu^*(E) + \mu^*(F) \geqq \mu^*(E + F)$$

1.25 If $A = \sum_{i=1}^{n} A_i$, then $\mu^*(A) \leqq \sum_{i=1}^{n} \mu^*(A_i)$

Proof: Case 1. Suppose $\mu^*(A_i) = +\infty$ for some i. Then the conclusion is obvious.

Case 2. Suppose $\mu^*(A_i)$ is finite for each i. Proof by induction on the number of A_i.

a. The theorem is true if n = 1. $\mu^*(A_1) \leqq \mu^*(A_1)$.

By 1.24, $\mu^*(A_1 + A_2) \leqq \mu^*(A_1) + \mu^*(A_2)$

b. Suppose conclusion is true for n = k. Then

$\mu^*(\sum_{i=1}^{k} A_i) \leqq \sum_{i=1}^{k} \mu^*(A_i)$. Add $\mu^*(A_{k+1})$ to both sides.

Consider $\sum_{i=1}^{k} A_i$ as a set and using the case $n = 2$, we obtain

$$\mu^*\left(\sum_{i=1}^{k+1} A_i\right) \leq \mu^*\left(\sum_{i=1}^{k} A_i\right) + \mu^*(A_{k+1}) \leq \sum_{i=1}^{k+1} \mu^*(A_i)$$

Since the truth of the conclusion in any case implies its truth in the next, we conclude

$$\mu^*(A) = \mu^*\left(\sum_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu^*(A_i).$$

<u>1.26</u> If $B \subset \sum_{i=1}^{n} A_i$, then $\mu^*(B) \leq \sum_{i=1}^{n} \mu^*(A_i)$

Proof: By 1.20 $\mu^*(B) \leq \mu^*\left(\sum_{i=1}^{n} A_i\right)$

But by the preceding theorem $\mu^*\left(\sum_{i=1} A_i\right) \leq \sum_{i=1}^{n} \mu^*(A_i)$.

$$\therefore \mu^*(B) \leq \sum_{i=1}^{n} \mu^*(A_i).$$

<u>1.27</u> If $B \subset \sum_{i=1}^{\infty} A_i$, then $\mu^*(B) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Proof: Case 1. Suppose $\mu^*(A_i) = +\infty$ for some i. Then the conclusion is obvious.

Case 2. Suppose $\mu^*(A_i)$ is finite for each i. Give $\epsilon > 0$. By 1.14 there are sets $R_{1,1}; R_{1,2}; R_{1,3}; \ldots \in \mathcal{P}$ such that $A_1 \subset \sum_{j=1}^{\infty} R_{1,j}$ and

$$\sum_{j=1}^{\infty} A(R_{1,j}) < \mu^*(A_1) + \frac{\epsilon}{2}.$$

There are sets $R_{2,1}; R_{2,2}; \ldots \in \mathcal{P}$, such that $A_2 \subset \sum_{j=1}^{\infty} R_{2,j}$; $\sum_{j=1}^{\infty} A(R_{2,j}) < \mu^*(A_2) + \frac{\epsilon}{4}$. There are sets $R_{i,1}; R_{i,2}; \ldots \in \mathcal{P}$, such that $A_i \subset \sum_{j=1}^{\infty} R_{i,j}$ and $\sum_{j=1}^{\infty} A(R_{i,j}) < \mu^*(A_i) + \frac{\epsilon}{2^i}$.

$$B \subset \sum_{i=1}^{\infty} A_i \subset \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} R_{i,j}$$

$$\therefore \mu^*(B) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A(R_{i,j}) < \sum_{i=1}^{\infty} \left(\mu^*(A_i) + \frac{\epsilon}{2^i}\right) =$$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon .$$

$$\mu^*(B) < \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \quad \therefore \mu^*(B) \leqq \sum_{i=1}^{\infty} \mu^*(A_i).$$

<u>1.28</u> If E is the x-axis, then $\mu^*(E) = 0$

Proof: Give $\epsilon > 0$. Let E_+ be the non-negative x-axis. Let E_- be the negative x-axis. Let $R_1 = R_{0,1;-\frac{\epsilon}{8},\frac{\epsilon}{8}}$; $R_2 = R_{1,2;-\frac{\epsilon}{16},\frac{\epsilon}{16}}$; $R_3 = R_{2,3;-\frac{\epsilon}{32},\frac{\epsilon}{32}}$; ...; $R_n = R_{n-1,n;-\frac{\epsilon}{2^{n+2}},\frac{\epsilon}{2^{n+2}}}$; ...

$A(R_n) = \frac{\epsilon}{2^{n+1}}$ for each n; $E_+ \subset \sum_{n=1}^{\infty} R_n$; $\sum_{n=1}^{\infty} A(R_n) = \frac{\epsilon}{2}$.

$\therefore \mu^*(E_+) \leqq \frac{\epsilon}{2}$.

Similarly, it can be seen that $\mu^*(E_-) \leqq \frac{\epsilon}{2}$. $E = E_+ + E_-$.

$\mu^*(E) \leqq \mu^*(E_+) + \mu^*(E_-) \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Since $0 \leqq \mu^*(E) \leqq \epsilon$ and since ϵ is arbitrary, we conclude that $\mu^*(E) = 0$.

<u>1.29</u> If $E = y$ -axis, then $\mu^*(E) = 0$.

Proof: Give $\epsilon > 0$. Let E_+ be the non-negative y -axis. Let E_- be the negative y -axis. Let $R_1 = R_{-\frac{\epsilon}{8},\frac{\epsilon}{8};0,1}$; $R_2 = R_{-\frac{\epsilon}{16},\frac{\epsilon}{16};1,2}$; ...; $R_n = R_{-\frac{\epsilon}{2^{n+2}},\frac{\epsilon}{2^{n+2}};n-1,n}$; ... $A(R_n) = \frac{\epsilon}{2^{n+1}}$, $E_+ \subset \sum_{n=1}^{\infty} R_n$; $R_n \in \mathcal{P}$ for each n; $\sum_{n=1}^{\infty} A(R_n) = \frac{\epsilon}{2}$ $\therefore \mu^*(E_+) \leqq \frac{\epsilon}{2}$.

Similarly it can be seen that $\mu^*(E_-) \leqq \frac{\epsilon}{2}$.

$E = E_+ + E_-$. $\mu^*(E) \leqq \mu^*(E_+) + \mu^*(E_-) \leqq \epsilon$.

Since $0 \leqq \mu^*(E) \leqq \epsilon$ and ϵ is arbitrary, we conclude that $\mu^*(E) = 0$.

<u>1.30</u> If L is a line parallel to either the x or y axis, then $\mu^*(L) = 0$

Proof: By a translation of axes, L can be transformed into an axis and can thus be seen to have exterior measure 0.

<u>1.31</u> If L is a line segment, then $\mu^*(L) = 0$.

Proof: Case 1. If L has slope equal to either 0 or $\pm\infty$, then it is a subset of a line M parallel to an axis. $\mu^*(M) = 0$, $\mu^*(L) \leq \mu^*(M)$. $\therefore$ $\mu^*(L) = 0$.

Case 2. The slope of L is positive but finite. Let $p = (a,c)$ and $q = (b,d)$ be the endpoints of L, where $a < b$, $c < d$. (Note: This will exclude degenerate line segments consisting of either no points or a single point. An empty segment of course has exterior measure 0 and a single point segment may be included in Case 1 above).

Consider $R_1 = R_{a,b;c,d}$. $L-q \subset R_1$. $L = (L-q) + q$. $\mu^*(L) \leq \mu^*(L-q) + \mu^*(q) = \mu^*(q)$. But $L-q \subset L$, $\therefore$ $\mu^*(L-q) \leq \mu^*(L)$. $\mu^*(L) = \mu^*(L-q)$. $A(R_1) = (b-a)\,(d-c)$.

Consider $R_{21} = R_{a,\frac{b+a}{2};c,\frac{c+d}{2}}$ and $R_{22} = R_{\frac{b+a}{2},b;\frac{c+d}{2},d}$.

$R_{21} \cdot R_{22} = \emptyset$. $L-q \subset R_{21} + R_{22}$.

$A(R_{21} + R_{22}) = A(R_{21}) + A(R_{22}) = \frac{A(R_1)}{2}$.

Consider $R_{31} = R_{a,\frac{3a+b}{4};c,\frac{3c+d}{4}}$; $R_{32} = R_{\frac{3a+b}{4},\frac{a+b}{2};\frac{3c+d}{4},d}$;

$R_{33} = R_{\frac{a+b}{2},\frac{a+3b}{4};\frac{c+d}{2},\frac{c+3d}{4}}$; $R_{34} = R_{\frac{a+3b}{4},b;\frac{c+3d}{4},d}$.

$L-q \subset R_{31} + R_{32} + R_{33} + R_{34}$

$R_{3i} \cdot R_{3j} = \emptyset$ if $i \neq j$

$A(R_{31} + R_{32} + R_{33} + R_{34}) = A(R_{31}) + A(R_{32}) + A(R_{33}) + A(R_{34}) = A(R_1)/4$

Continuing this process indefinitely, we find that we can cover $L-q$ with a sequence of oriented half-open rectangles of arbitrarily small total area. We conclude, therefore, that $\mu^*(L-q) = 0 = \mu^*(L)$.

Case 3. The slope of L is negative but finite. Let $p = (a,d)$, $q = (b,c)$ be the endpoints of L, where $a < b$, $c < d$.

Again let $R_1 = R_{a,b;c,d}$, $A(R_1) = (b-a)\,(d-c)$.

$L-(p+q) \subset R_1$. $\mu^*(p+q) = 0$.

$\mu^*(L) \leqq \mu^*(L-(p+q)) + \mu^*(p+q) = \mu^*(L-(p+q))$

$L-(p+q) \subset L$. $\therefore \mu^*(L-(p+q)) \leqq \mu^*(L)$

$\mu^*(L) = \mu^*(L-(p+q))$

Let $R_{21} = R_{a, \frac{a+b}{2}; \frac{c+d}{2}, d}$; $R_{22} = R_{\frac{a+b}{2}, b; c, \frac{c+d}{2}}$.

$R_{21} \cdot R_{22} = \emptyset$. $L-(p+q) \subset R_{21} + R_{22}$

$A(R_{21} + R_{22}) = A(R_{21}) + A(R_{22}) = A(R_1)/2$

Again, as before, we can by continuing this process cover $L-(p+q)$ with a sequence of oriented half-open rectangles of arbitrarily small total area. We conclude that $\mu^*(L) = 0$.

1.32 If L is any line, then $\mu^*(L) = 0$

Proof: $L = \sum_{i=1}^{\infty} l_i$, where each l_i is a half-open line segment of unit of length and $l_i \cdot l_j = \emptyset$ if $i \neq j$.

$\mu^*(L) = \mu^*(\sum_{i=1}^{\infty} l_i) = \sum_{i=1}^{\infty} \mu^*(l_i) = 0$

1.33 Definition. A set E is said to be a Legesgue measurable set if, for every set A we have

$\mu^*(A) = \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

Henceforth, the word "measurable" will be understood to mean "Lebesgue measurable."

1.34 For any two sets A and E, we have

$\mu^*(A) \leqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

Proof: $A = A \cdot E + A \cdot \complement E$

$\therefore$ from 1.24 $\mu^*(A) \leqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

1.35 E is a measurable set if and only if, for every set A, we have

$\mu^*(A) \geqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

Proof: 1. If E is a measurable set, then for every set A,

$*(A) = \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$, hence $\mu^*(A) \geqq \mu^*(A \cdot E) +$
$\mu^*(A \cdot \complement E)$.

2. Suppose for every set A, $\mu^*(A) \geqq \mu^*(A \cdot E) +$
$\mu^*(A \cdot \complement E)$.

Then from 1.34 $\mu^*(A) \leqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

$\therefore \mu^*(A) = \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

$\therefore$ E is a measurable set.

<u>1.36</u> $\emptyset$ is a measurable set.

Proof: Let A be any set. We must show that

$\mu^*(A) \geqq \mu^*(A \cdot \emptyset) + \mu^*(A \cdot \complement \emptyset)$.

$\mu^*(A \cdot \emptyset) = \mu^*(\emptyset) = 0$. $\mu^*(A \cdot \complement \emptyset) = \mu^*(A)$.

$\mu^*(A) = \mu(A \cdot \emptyset) + \mu^*(A \cdot \complement \emptyset) = \mu^*(A)$.

<u>1.37</u> If E is such that $\mu^*(E) = 0$, then E is a measurable set.

Proof: Let A be any set. We must show that

$\mu^*(A) \geqq \mu^*(A \cdot E) + \mu(A \cdot \complement E)$

$\mu^*(A \cdot E) = 0$, since $A \cdot E \subset E$, and $\mu^*(A \cdot E) \leqq \mu^*(E) = 0$.

$A \cdot \complement E \subset A$, $\therefore \mu^*(A \cdot \complement E) \leqq \mu^*(A)$.

Hence, it follows that

$$\mu^*(A) \geqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E).$$

<u>1.38</u> If E is a measurable set, then $\complement E$ is a measurable set.

Proof: Let A be any set. We must show that $\mu^*(A) \geqq \mu^*(A \cdot \complement E)$
$\mu^*(A \cdot \complement \complement E)$. But E is a measurable set.

So, $\mu^*(A) \geqq \mu^*(A \cdot E) + \mu^*(A \cdot \complement E)$.

$E = \complement \complement E$. $\therefore \mu^*(A) \geqq \mu^*(A \cdot \complement \complement E) + \mu^*(A \cdot \complement E)$.

<u>1.39</u> R_2 is a measurable set

Proof: $\emptyset$ is measurable. $\therefore \complement \emptyset = R_2$ is measurable.

<u>1.40</u> If E and F are measurable sets, then $E + F$ is a measurable set.

Proof: Let A be any set. We shall show that

$$\mu^*(A) = \mu^*(A \cdot (E + F)) + \mu^*(A \cdot \mathcal{C}(E + F)).$$

Since E is measurable, $\mu^*(A) = \mu^*(A \cdot E) + \mu^*(A \cdot \mathcal{C}E)$

Since F is measurable,

$$\mu^*(A \cdot E) = \mu^*(A \cdot E \cdot F) + \mu^*(A \cdot E \cdot \mathcal{C}F)$$

$$\mu^*(A \cdot \mathcal{C}E) = \mu^*(A \cdot \mathcal{C}E \cdot F) + \mu^*(A \cdot \mathcal{C}E \cdot \mathcal{C}F)$$

$$\therefore \mu^*(A) = \mu^*(A \cdot E \cdot F) + \mu^*(A \cdot E \cdot \mathcal{C}F) + \mu^*(A \cdot \mathcal{C}E \cdot F) + \mu^*(A \cdot \mathcal{C}E \cdot \mathcal{C}F).$$

Since E is measurable,

$$\mu^*(A \cdot (E + F)) = \mu^*(A(E + F) \cdot E) + \mu^*(A(E + F) \cdot \mathcal{C}E)$$

Since F is measurable,

$$\mu^*(A \cdot (E + F) \cdot E) = \mu^*(A(E + F) \cdot E \cdot F) + \mu^*(A(E + F) \cdot E \cdot \mathcal{C}F);$$

$$\mu^*(A(E + F) \cdot \mathcal{C}E) = \mu^*(A(E + F)\mathcal{C}E \cdot F) + \mu^*(A(E + F)\mathcal{C}E \cdot \mathcal{C}F);$$

$$\mu^*(A(E + F)) = \mu^*(A(E + F) \cdot E \cdot F) + \mu^*(A(E + F) \cdot E \cdot \mathcal{C}F) + \mu^*(A(E + F) \cdot \mathcal{C}E \cdot F) + \mu^*(A(E + F) \cdot \mathcal{C}E \cdot \mathcal{C}F).$$

$A(E + F) \cdot E \cdot F = A \cdot E \cdot F$

$A(E + F) \cdot E \cdot \mathcal{C}F = A \cdot E \cdot \mathcal{C}F$; $A(E + F) \cdot \mathcal{C}E \cdot F = A \cdot \mathcal{C}E \cdot F$

$A(E + F) \cdot \mathcal{C}E \cdot \mathcal{C}F = \emptyset$

$$\therefore \mu^*(A(E + F)) = \mu^*(A \cdot E \cdot F) + \mu^*(A \cdot E \cdot \mathcal{C}F) + \mu^*(A \cdot \mathcal{C}E \cdot F)$$

$$\therefore \mu^*(A) = \mu^*(A(E + F)) + \mu^*(A \cdot \mathcal{C}E \cdot \mathcal{C}F) = \mu^*(A(E + F)) + \mu^*(A \cdot \mathcal{C}(E + F)).$$

<u>1.41</u> If $E_1, E_2, \ldots, E_n$ are measurable sets, then $\sum_{i=1}^{n} E_i$ is a measurable set.

Proof: By induction on n.

1. The conclusion is trivial of $n = 1$. By the preceding conclusion, it is true for $n = 2$.

2. Assume the conclusion is true for $n = k$. Then if $E_1, E_2, \ldots, E_k$ are measurable $\sum_{i=1}^{k} E_i$ is measurable.

If E_{k+1} is measurable, then the truth of the assertion for $n = 2$ implies that $\sum_{i=1}^{k} E_i + E_{k+1}$ is measurable, i.e. $\sum_{i=1}^{k+1} E_i$ is measurable. $\therefore$ by induction the conclusion is true for all values of n.

<u>1.42</u> If E and F are measurable sets, then $E \cdot F$ is a measurable set.

Proof: $\mathcal{C}(E \cdot F) = \mathcal{C}E + \mathcal{C}F$. $\mathcal{C}E$ and $\mathcal{C}F$ are measurable by 1.38. $\therefore$ $\mathcal{C}E + \mathcal{C}F$ is measurable. $\therefore$ $\mathcal{C}(E \cdot F)$ is measurable. This implies $\mathcal{C}\mathcal{C}(E \cdot F) = E \cdot F$ is measurable.

<u>1.43</u> If $E_1, E_2, \ldots, E_n$ are measurable sets, then $\prod_{i=1}^{n} E_i$ is a measurable set.

Proof: Induction on n.

1. Trivial for $n = 1$. True for $n = 2$ by 1.42.

2. Assume true for $n = k$. Then, if $E_1, E_2, \ldots, E_k$ are measurable, $\prod_{i=1}^{k} E_i$ is measurable. If E_{k+1} is measurable, $\prod_{i=1}^{k} E_i \cdot E_{k+1}$ is measurable, i.e. $\prod_{i=1}^{k+1} E_i$ is measurable. Thus, the conclusion is true for all values of n.

<u>1.44</u> If E and F are measurable sets, then $E-F$ is a measurable set.

Proof: $E-F = E \cdot \mathcal{C}F$ which is measurable.

<u>1.45</u> If $\{E_n\}$ is a sequence of measurable sets, such that $E_m \cdot E_n = \emptyset$ if $m \neq n$, then $\sum_{n=1}^{\infty} E_n$ is a measurable set.

Proof: We must show that if A is any set, then $\mu^*(A) \geq \mu^*(A \cdot \sum_{n=1}^{\infty} E_n) + \mu^*(A \cdot \mathcal{C}\sum_{n=1}^{\infty} E_n)$, i.e. $\mu^*(A) \geq \mu^*(A \cdot Q) + \mu^*(A \cdot \mathcal{C}Q)$, where $Q = \sum_{n=1}^{\infty} E_n$.

If E_1 and E_2 are measurable sets, then for every set A,

$\mu^*(A(E_1 + E_2)) = \mu^*(A \cdot E_1 \cdot E_2) + \mu^*(A \cdot E_1 \cdot \mathcal{C} E_2) +$

$\mu^*(A \cdot \mathcal{C} E_1 \cdot E_2)$, an equation was developed as part of the proof of 1.40. But $E_1 \cdot E_2 = \emptyset, \therefore A \cdot E_1 \cdot E_2 = \emptyset$.

Hence $\mu^*(A(E_1 + E_2)) = \mu^*(A \cdot E_1) + \mu^*(A \cdot E_2)$

We assert next that

$$\mu^*(A(E_1 + E_2 + \ldots + E_n)) = \mu^*(A \cdot E_1) + \mu^*(A \cdot E_2) + \ldots + \mu^*(A \cdot E_n)$$

This statement is true for n = 1 and n = 2.

Suppose it is true for n = k. Then

$$\mu^*(A(E_1 + E_2 + \ldots + E_k)) = \mu^*(A \cdot E_1) + \mu^*(A \cdot E_2) + \ldots + \mu^*(A \cdot E_k)$$

$$\mu^*(A(E_1 + E_2 + \ldots + E_k + E_{k+1})) = \mu^*(A(E_1 + E_2 + \ldots + E_k)) + \mu^*(A \cdot E_{k+1}) =$$

$$\mu^*(A \cdot E_1) + \mu^*(A \cdot E_2) + \ldots + \mu^*(A \cdot E_k) + \mu^*(A \cdot E_{k+1})$$

Thus, the assertion is true.

$$\mu^*(A) \geqq \mu^*(A(E_1 + E_2 + \ldots + E_m)) + \mu^*(A \cdot \mathcal{C}(E_1 + E_2 + \ldots + E_m)) =$$

$$\sum_{n=1}^{m} \mu^*(A \cdot E_n) + \mu^*(A \cdot \mathcal{C}(E_1 + E_2 + \ldots + E_m)) \geqq$$

$$\sum_{n=1}^{m} \mu^*(A \cdot E_n) + \mu^*(A \cdot \mathcal{C}(\sum_{n=1}^{\infty} E_n)), \text{ since } \sum_{n=1}^{m} E_n \subset \sum_{n=1}^{\infty} E_n$$

$$\therefore \mathcal{C}(\sum_{n=1}^{m} E_n) \supset \mathcal{C}(\sum_{n=1}^{\infty} E_n)$$

But $\sum_{n=1}^{\infty} \mu^*(A \cdot E_n) = \lim_{m \to \infty} \sum_{n=1}^{m} \mu^*(A \cdot E_n)$

$$\therefore \mu^*(A) \geqq \sum_{n=1}^{\infty} \mu^*(A \cdot E_n) + \mu^*(A \cdot \mathcal{C}(\sum_{n=1}^{\infty} E_n))$$

$$\mu^*(A \cdot \sum_{n=1}^{\infty} E_n) \leqq \mu^*(A \cdot E_1) + \mu^*(A \cdot E_2) + \ldots$$

By 1.27 $\mu^*(A \cdot \sum_{n=1}^{\infty} E_n) \leqq \sum_{n=1}^{\infty} \mu^*(A \cdot E_n)$

$$\mu^*(A) \geqq \mu^*(A \cdot \sum_{n=1}^{\infty} E_n) + \mu(A \cdot \mathcal{C} \sum_{n=1}^{\infty} E_n).$$

1.46 If $\{E_n\}$ is a sequence of measurable sets, then $\sum_{n=1}^{\infty} E_n$ is a measurable set.

Proof:

$$\sum_{n=1}^{\infty} E_n = E_1 + (E_2 - E_1) + (E_3 - (E_1 + E_2)) + (E_4 - (E_1 + E_2 + E_3)) + \dots + (E_n - (E_1 + E_2 + \dots + E_{n-1})) + \dots$$

Each of the sets in the right-hand member of the above equation is measurable. Furthermore, each of the sets in the sum is disjoint with the other sets.

From the preceding conclusion, we see that $\sum_{n=1}^{\infty} E_n$ is a measurable set.

1.47 If $\{E_n\}$ is a sequence of measurable sets, then $\prod_{n=1}^{\infty} E_n$ is a measurable set.

Proof: $\mathcal{C} E_n$ is measurable for each n.

$\therefore \sum_{n=1}^{\infty} \mathcal{C} E_n$ is measurable by 1.46. $\sum_{n=1}^{\infty} \mathcal{C} E_n = \mathcal{C} \prod_{n=1}^{\infty} E_n$.

$\therefore \mathcal{C}\mathcal{C} \prod_{n=1}^{\infty} E_n = \prod_{n=1}^{\infty} E_n$ is measurable.

1.48 If $R \in \mathcal{P}$, then R is a measurable set.

Proof: Let E be any set. We must show that $\mu^*(E) \geqq \mu^*(E \cdot R) + \mu^*(E \cdot \mathcal{C} R)$.

Case 1. If $\mu^*(E) = +\infty$, the conclusion is immediate.

Case 2. Suppose $\mu^*(E)$ is finite. Give $\epsilon > 0$. There is a covering $\{S_j\}$, such that $E \subset \sum_{j=1}^{\infty} S_j$, $S_j \in \mathcal{P}$ for each j and $\sum_{j=1}^{\infty} A(S_j) < \mu^*(E) + \epsilon$ by 1.14.

$E \cdot R \subset \sum_{j=1}^{\infty} S_j \cdot R.$ $S_j \cdot R \in \rho$ for each j from 1.2.

$E \cdot \mathcal{C} R \subset \sum_{j=1}^{\infty} S_j \cdot \mathcal{C} R.$ From 1.3 $S_j \cdot \mathcal{C} R = S_j - R = T_j + U_j + V_j + W_j$,

where $T_j, U_j, V_j, W_j \in \rho$ and T_j, U_j, V_j, W_j are all disjoint.

$$E \cdot \mathcal{C} R \subset \sum_{j=1}^{\infty} (T_j + U_j + V_j + W_j) = \sum_{j=1}^{\infty} T_j + \sum_{j=1}^{\infty} U_j + \sum_{j=1}^{\infty} V_j + \sum_{j=1}^{\infty} W_j.$$

$S_j = S_j \cdot R + S_j \cdot \mathcal{C} R = S_j \cdot R + T_j + U_j + V_j + W_j$.

The sets in the sum on the right of the above equation are disjoint.

$\therefore$ by 1.7, $A(S_j) = A(S_j \cdot R) + A(T_j) + A(U_j) + A(V_j) + A(W_j)$.

$\mu^*(E \cdot R) \leq \sum_{j=1}^{\infty} A(S_j \cdot R)$ by 1.19 and 1.20.

$$\mu^*(E \cdot \mathcal{C} R) \leq \sum_{j=1}^{\infty} A(T_j) + \sum_{j=1}^{\infty} A(U_j) + \sum_{j=1}^{\infty} A(V_j) + \sum_{j=1}^{\infty} A(W_j).$$

$$\mu^*(E \cdot R) + \mu^*(E \cdot \mathcal{C} R) \leq \sum_{j=1}^{\infty} A(S_j \cdot R) + \sum_{j=1}^{\infty} A(T_j) + \sum_{j=1}^{\infty} A(U_j) + \sum_{j=1}^{\infty} A(V_j) + \sum_{j=1}^{\infty} A(W_j) = \sum_{j=1}^{\infty} (A(S_j \cdot R) + A(T_j) + A(U_j) + A(V_j) + A(W_j)) = \sum_{j=1}^{\infty} A(S_j) < \mu^*(E) + \epsilon.$$

$\therefore$ We conclude that

$$\mu^*(E) \geq \mu^*(E \cdot R) + \mu^*(E \cdot \mathcal{C} R).$$

<u>1.49</u> If $R \in \rho$ and if S is such that $R^\circ \subset S \subset \bar{R}$, then S is a measurable set and $\mu^*(R^\circ) = \mu^*(\bar{R}) = \mu^*(R) = \mu^*(S)$.

Proof: $\bar{R}$ is a closed oriented rectangle.

Let s_1 = left side of $\bar{R}$, $\mu^*(s_1) = 0$ by 1.31.

Let s_2 = bottom side of $\bar{R}$, $\mu^*(s_2) = 0$.

Let s_3 = right side of $\bar{R}$, $\mu^*(s_3) = 0$.

Let s_4 = top side of $\bar{R}$, $\mu^*(s_4) = 0$.

$R^\circ + s_1 + s_2 = R$. $\therefore$ R° is measurable.

$\mu^*(R) \leq \mu^*(R^o) + \mu^*(s_1) + \mu^*(s_2) = \mu^*(R^o)$

$R^o \subset R$, $\therefore \mu^*(R^o) \leq \mu^*(R)$ $\therefore \mu^*(R^o) = \mu^*(R)$.

$\bar{R} = R + s_3 + s_4$ $\therefore$ $\bar{R}$ is measurable.

$\mu^*(\bar{R}) \leq \mu^*(R) + \mu^*(s_3) + \mu^*(s_4) = \mu^*(R)$. $R \subset \bar{R}$

$\therefore \mu^*(\bar{R}) \geq \mu^*(R)$.

$\mu^*(\bar{R}) = \mu^*(R)$

$\therefore \mu^*(R^o) = \mu^*(\bar{R}) = \mu^*(R)$

$R^o \subset S \subset \bar{R}$

$S = R^o + B$, where $\mu^*(B) = 0$. $\therefore$ S is measurable.

$\mu^*(S) \leq \mu^*(R^o) + \mu^*(B) = \mu^*(R^o)$

But $\mu^*(R^o) \leq \mu^*(S)$. $\mu^*(R^o) = \mu^*(S) = \mu^*(R)$

<u>1.50</u> If G is any open set, then there is a countable sequence of open squares, $\{S_n\}$, such that $G = \sum_{n=1}^{\infty} S_n$.

Proof: Let $\mathcal{V}$ be the collection of all open squares having centers with both coordinates rational and half-side length equal to $\frac{1}{n}$ where n is a positive integer. $\mathcal{V}$ is a countable collection.

We shall show $G = \sum_{\substack{S \in \mathcal{V} \\ S \subset G}} S$

1. Suppose $p \in \sum_{\substack{S \in \mathcal{V} \\ S \subset G}} S$. Then $p \in S_o$ for some set S_o, where $S_o \subset G$, and $S_o \in \mathcal{V}$.

Hence, $p \in G$. $\therefore$ $G \supset \sum_{\substack{S \in \mathcal{V} \\ S \subset G}} S$

2. Suppose $p \in G$. There exists $1 > \epsilon > 0$ such that $N(p,\epsilon) \subset G$. Let q be a point having rational coordinates such that $d(p,q) < \frac{\epsilon}{4}$.

Let n be such that $\frac{\epsilon}{4} < \frac{1}{n} < \frac{\epsilon}{2}$.

Let S be the square having q as center and $\frac{1}{n}$ as half-side length. $S \in \mathcal{V}$, $d(p,q) < \frac{1}{n}$.

$p \in S$. Let $r \in S$. Then $d(q,r) < \frac{\sqrt{2}}{n} < \frac{\sqrt{2}\,\epsilon}{2}$.

$d(p,q) < \frac{\epsilon}{4}$. $d(p,r) < \frac{\epsilon}{4} + \frac{\sqrt{2}\,\epsilon}{2} < \epsilon$. $r \in N(p, \epsilon)$

$r \in G$. $\therefore$ $S \subset G$. $G \subset \sum_{\substack{S \in \mathcal{V} \\ S \subset G}} S$

Hence, $G = \sum_{\substack{S \in \mathcal{V} \\ S \subset G}} S$

<u>1.51</u> In view of the preceding conclusion, we immediately conclude that every open set is measurable.

<u>1.52</u> Every closed set is measurable.

<u>1.53</u> Definition. The class of Borel sets in the plane is the smallest class of sets containing the open sets and closed under countable sums and countable products. Let $\mathcal{B}$ denote this class.

<u>1.54</u> If $E \in \mathcal{B}$, then E is a measurable set.

To summarize then,

<u>1.55</u> Definition. Let $\mathcal{L}$ denote the collection of all Lebesgue measurable sets.

<u>1.56</u> If $E \in \mathcal{L}$, then $\complement E \in \mathcal{L}$.

<u>1.57</u> If $E_n \in \mathcal{L}$ for each n, then $\sum_{n=1}^{\infty} E_n \in \mathcal{L}$ and $\prod_{n=1}^{\infty} E_n \in \mathcal{L}$.

1.58 If E is open or if E is closed, then $E \in \mathcal{L}$.

1.59 If $\mu^*(E) = 0$, then $E \in \mathcal{L}$. Also if $\mu^*(E) = 0$, and $F \subset E$, then $F \in \mathcal{L}$.

1.60 Definition. If $E \in \mathcal{L}$, then we define $\mu(E) = \mu^*(E)$ and $\mu(E)$ is called the Lebesgue measure of E.

1.61 If $E \in \mathcal{L}$, then $\mu(E) \geqq 0$, and $\mu(E) \leqq +\infty$.

1.62 If $E \in \mathcal{L}$ and if $F \in \mathcal{L}$, and if $E \subset F$, then $\mu(E) \leqq \mu(F)$.

1.63 If $\{E_n\}$ is a sequence of disjoint sets, such that $E_n \in \mathcal{L}$ for each n, then $\mu(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

Proof: From the proof of 1.45, if A is any set

$$\mu^*(A) \geqq \sum_{n=1}^{\infty} \mu^*(A \cdot E_n) + \mu^*(A \cdot \mathcal{C}(\sum_{n=1}^{\infty} E_n)).$$

Let $A = \sum_{n=1}^{\infty} E_n$. $\quad E_n \cdot \sum_{n=1}^{\infty} E_n = E_n$

$$\mu^*(\sum_{n=1}^{\infty} E_n) \geqq \sum_{n=1}^{\infty} \mu^*(E_n) + \mu^*(\emptyset) = \sum_{n=1}^{\infty} \mu^*(E_n).$$

But, we always have $\mu^*(\sum_{n=1}^{\infty} E_n) \leqq \sum_{n=1}^{\infty} \mu^*(E_n)$. (1.27).

$$\therefore \mu^*(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n) \text{ and } \mu(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

1.64 Definition. A sequence of sets $\{A_n\}$ is called an increasing sequence if, for each n, $A_n \subset A_{n+1}$.

1.65 Definition. A sequence of sets $\{A_n\}$ is called a decreasing sequence

$+ 1.$

<u>1.66</u> If $\{A_n\}$ is an increasing sequence of measurable sets, then $\mu(\sum_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$

Proof: Let $B_1 = A_1$; $B_2 = A_2 - A_1$; $B_3 = A_3 - (A_1 + A_2)$; ...;
$B_n = A_n - (A_1 + A_2 + \dots + A_{n-1})$; ...

$B_n \subset A_n$ for each n. B_n is a measurable set for each n from 1.41 and 1.44.

$B_n \cdot B_m = \emptyset$, if $m \neq n$. $\sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} A_n$.

From 1.63 $\sum_{n=1}^{\infty} \mu(B_n) = \mu(\sum_{n=1}^{\infty} B_n) = \mu(\sum_{n=1}^{\infty} A_n)$; $\sum_{n=1}^{\infty} \mu(B_n) = \lim_{k\to\infty} \sum_{n=1}^{\infty} \mu(B_n)$.

$\sum_{n=1}^{k} \mu(B_n) = \mu(\sum_{n=1}^{k} B_n)$.

We shall show that $\sum_{n=1}^{k} B_n = A_k$

1. Suppose $x_o \in \sum_{n=1}^{k} B_n$

$x_o \in B_n$, $n \leq k$; $x_o \in A_n$, $n \leq k$, $A_n \subset A_k$

$\therefore$ $x_o \in A_k$ and $\sum_{n=1}^{k} B_n \subset A_k$.

2. Suppose $x_o \in A_k$. Let n be the smallest integer such that $x_o \in A_n$, $n \leq k$.

a. If $n = 1$, then $x_o \in A_1 = B_1$, $x_o \in B_1$, $x_o \in \sum_{n=1}^{k} B_n$ and $A_k \subset \sum_{n=1}^{k} B_n$.

b. If $n > 1$, then $x_o \in A_n$, $x_o \notin A_m$ if $m < n$
$x_o \in B_n$; $x_o \in \sum_{n=1}^{k} B_n$ and $A_k \subset \sum_{n=1}^{k} B_n$.

$\therefore \mu(\sum_{n=1}^{k} B_n) = \mu(A_k)$. $\lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n) = \lim_{k\to\infty} \mu(A_k)$,
$\mu(\sum_{n=1}^{\infty} A_n) = \lim_{k\to\infty} \mu(A_k)$.

<u>1.67</u> If $\{A_n\}$ is a decreasing sequence of measurable sets, and if $\mu(A_1) < +\infty$, then $\mu(\prod_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Proof: Let $B_n = A_1 \cdot \complement A_n = A_1 - A_n$ for each n.

B_n is a measurable set, for each n.

$A_n \supset A_{n+1}$. $\complement A_n \subset \complement A_{n+1}$ $B_n = A_1 \cdot \complement A_n \subset A_1 \cdot \complement A_{n+1} = B_{n+1}$.

$\{B_n\}$ is an increasing sequence of measurable sets.

From 1.66 $\mu(\sum_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$. $A_1 = A_1 \cdot \complement A_n + A_1 \cdot A_n = B_n + A_n$.

$\mu(A_1) = \mu(B_n) + \mu(A_n)$ from 1.63 $\mu(A_1) - \mu(B_n) = \mu(A_n)$.

$A_1 = A_1 \cdot \complement \prod_{n=1}^{\infty} A_n + A_1 \cdot \prod_{n=1}^{\infty} A_n = A_1 \cdot \sum_{n=1}^{\infty} \complement A_n + \prod_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} A_1 \cdot \complement A_n + \prod_{n=1}^{\infty} A_n =$
$\sum_{n=1}^{\infty} B_n + \prod_{n=1}^{\infty} A_n$. $\therefore \mu(A_1) = \mu(\sum_{n=1}^{\infty} B_n) + \mu(\prod_{n=1}^{\infty} A_n)$ from 1.63.

$\mu(A_1) - \mu(\sum_{n=1}^{\infty} B_n) = \mu(\prod_{n=1}^{\infty} A_n)$.

$\lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \mu(\sum_{n=1}^{\infty} B_n) = \mu(\prod_{n=1}^{\infty} A_n)$.

<u>1.68</u> Definition. If $\{E_n\}$ is a sequence of sets, we define the limit inferior (lim inf) of $\{E_n\}$ as follows:

Let $C_k = \prod_{n=k}^{\infty} E_n$. Then $\liminf_{n\to\infty} E_n = \sum_{k=1}^{\infty} C_k$.

It may be noticed that the limit inferior of $\{E_n\}$ is the set of all points which belong to all but a finite number of the sets E_n.

<u>1.69</u> Definition. If $\{E_n\}$ is a sequence of sets, we define the limit superior (lim sup) of $\{E_n\}$ as follows:

Let $B_k = \sum_{n=k}^{\infty} E_n$. Then $\limsup_{n\to\infty} E_n = \prod_{k=1}^{\infty} B_k$.

It may be noticed that the limit superior of $\{E_n\}$ is the set of all points which belong to E_n for infinitely many values of n.

<u>1.70</u> If $\{E_n\}$ is a sequence of measurable sets, then $\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n)$.

Proof: $\liminf_{n\to\infty} E_n = \sum_{k=1}^{\infty} C_k$, where $C_k = \prod_{n=k} E_n$

$C_k \subset C_{k+1} \subset \ldots.$

$\lim_{k\to\infty} \mu(C_k) = \mu(\sum_{k=1}^{\infty} C_k) = \mu(\liminf_{n\to\infty} E_n)$ by 1.66.

$\mu(E_k) \geq \mu(C_k)$ by 1.20. $\therefore$ $\liminf_{k\to\infty} \mu(E_k) \geq \liminf_{k\to\infty} \mu(C_k)$.

$\liminf_{n\to\infty} \mu(E_n) \geq \lim_{k\to\infty} \mu(C_k) = \mu(\liminf_{n\to\infty} E_n)$.

<u>1.71</u> If $\{E_n\}$ is a sequence of measurable sets such that $\mu(\sum_{n=1}^{\infty} E_n) < +\infty$, then $\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n)$.

Proof: From 1.20 $\mu(E_k) \leq \mu(B_k)$, where $B_k = \sum_{n=k}^{\infty} E_n$.

$\limsup_{k\to\infty} \mu(E_k) \leq \limsup_{k\to\infty} \mu(B_k) =$

$\lim_{k\to\infty} \mu(B_k) = \mu(\prod_{k=1}^{\infty} B_k) = \mu(\limsup_{n\to\infty} E_n)$ from 1.67.

<u>1.72</u> If E is measurable, $\mu(E) < +\infty$, and if $\epsilon > 0$, then there exists an open set G such that $G \supset E$ and such that $\mu(G) < \mu(E) + \epsilon$.

Proof: $\mu(E) = \mu^*(E)$.

There exists $\{R_n\}$ such that $R_n \in \mathcal{P}$ for each n, $E \subset \sum_{n=1}^{\infty} R_n$

and such that $\sum_{n=1}^{\infty} \mu(R_n) = \sum_{n=1}^{\infty} A(R_n) < \mu^*(E) + \frac{\epsilon}{2} = \mu(E) + \frac{\epsilon}{2}$ from 1.14.

Let $\{S_n\}$ be a sequence of open rectangle such that $R_n \subset S_n$ for each n and such that $\mu(S_n) < \mu(R_n) + \frac{\epsilon}{2^{n+1}}$

Let $G = \sum_{n=1}^{\infty} S_n$. $E \subset G$, $\mu(G) = \mu(\sum_{n=1}^{\infty} S_n) \leq$

∞

CHAPTER II

THE LEBESGUE INTEGRAL AND LEBESGUE MEASURABLE AND SUMMABLE FUNCTIONS

Suppose that $f(p)$ is a real-valued function defined on a measurable set E of finite measure. Suppose further that there exist numbers m and M such that $p \in E$ implies $m \leq f(p) \leq M$.

2.1 Definition. A measurable partition P of E means a finite collection of disjoint measurable sets $E_1, E_2, \ldots, E_n$ such that $E = E_1 + E_2 + \ldots + E_n$. Such a partition will be denoted by $P[E_1, E_2, \ldots, E_n]$.

2.2 Definition. If $P[E_1, E_2, \ldots, E_n]$ is a measurable partition of E, let $M_1 = \underset{p \in E_1}{\text{l.u.b.}}\, f(p)$, Let $M_2 = \underset{p \in E_2}{\text{l.u.b.}}\, f(p), \ldots,$ $M_n = \underset{p \in E_n}{\text{l.u.b.}}\, f(p)$.

Let $S(P) = M_1 \mu(E_1) + M_2 \mu(E_2) + \ldots + M_n \mu(E_n) = \sum_{i=1}^{n} M_i \mu(E_i)$.

$S(P)$ is called the upper sum for the partition P.

Let $m_1 = \underset{p \in E_1}{\text{g.l.b.}}\, f(p)$, let $m_2 = \underset{p \in E_2}{\text{g.l.b.}}\, f(p), \ldots,$ let $m_n = \underset{p \in E_n}{\text{g.l.b.}}\, f(p)$

Let $s(P) = \sum_{i=1}^{n} m_i \mu(E_i)$. $s(P)$ is called the lower sum for the partition P.

2.3 If $P[E_1, E_2, \ldots, E_n]$ is a measurable partition of E, if $S(P) = \sum_{i=1}^{n} M_i \mu(E_i)$, $s(P) = \sum_{i=1}^{n} m_i \mu(E_i)$, then

$$m\mu(E) \leq s(P) \leq S(P) \leq M\mu(E).$$

Proof: $m \leq m_i \leq M_i \leq M$ for each i.

$M_i = \underset{p \in E_i}{\text{l.u.b.}}\, f(p)$ and $m_i = \underset{p \in E_i}{\text{g.l.b.}}\, f(p)$. For each i, $M_i \geq m_i$,

$$\therefore \sum_{i=1}^{n} M_i \mu(E_i) \geqq \sum_{i=1}^{n} m_i \mu(E_i).$$

But for each i, $\sum_{i=1}^{n} M_i \mu(E_i) \leqq \sum_{i=1}^{n} M \mu(E_i) = M\mu(E)$

and $\sum_{i=1}^{n} m_i \mu(E_i) \geqq \sum_{i=1}^{n} m\mu(E_i) = m\mu(E)$.

$$\therefore \quad m\mu(E) \leqq s(P) \leqq S(P) \leqq M\mu(E).$$

2.4 Definition. The lower Lebesgue integral of f(p) on E is denoted by $\underline{\int_E} f(p)\, d\mu$. It is defined as follows.

$\underline{\int_E} f(p)d\mu$ = l.u.b.s(P) where l.u.b. is taken with respect to all measurable partitions P of E. $m\mu(E) \leqq \underline{\int_E} f(p)d\mu \leqq M\mu(E)$.

2.5 Definition. The upper Lebesgue integral of f(p) on E is denoted by $\overline{\int_E} f(p)d\mu$. It is defined as follows.

$\overline{\int_E} f(p)d\mu$ = g.l.b.S(P) where g.l.b. is taken with respect to all measurable partitions P of E. $m\mu(E) \leqq \overline{\int_E} f(p)d\mu \leqq M\mu(E)$.

2.6 Suppose that P $[E_1, E_2, \ldots, E_n]$ and Q $[F_1, F_2, \ldots, F_m]$ are measurable partitions of E. Then Q is a refinement of P if each F_i is a subset of some E_j.

2.7 If Q is a refinement of P, then $S(Q) \leqq S(P)$ and $s(Q) \geqq s(P)$.

Proof: $E_j = \sum_{F_i \subset E_j} F_i$ for each j.

$\mu(E_j) = \sum_{F_i \subset E_j} \mu(F_i)$ for each j.

If $F_i \subset E_j$, then $\overline{M}_i = \text{l.u.b.}_{p \in F_i} f(p) \leqq \text{l.u.b.}_{p \in E_j} f(p)$.

$$\underset{p \in F_i}{\text{g.l.b.}} f(p) \geq \underset{p \in E_j}{\text{g.l.b.}} f(p)$$

$$S(P) = \sum_{j=1}^{n} M_j \mu(E_j), \text{ where } M_j = \underset{p \in E_j}{\text{l.u.b.}} f(p)$$

$$S(Q) = \sum_{i=1}^{m} \bar{M}_i \mu(F_i), \text{ where } \bar{M}_i = \underset{p \in F_i}{\text{l.u.b.}} f(p)$$

$$\sum_{F_i \subset E_j} \bar{M}_i \mu(F_i) \leq \sum_{F_i \subset E_j} M_j \mu(F_i) = M_j \sum_{F_i \subset E_j} \mu(F_i) = M_j \mu(E_j) \text{ for each } j.$$

$$S(Q) = \sum_{i=1}^{m} \bar{M}_i \mu(F_i) = \sum_{j=1}^{n} \sum_{F_i \subset E_j} \bar{M}_i \mu(F_i) \leq \sum_{j=1}^{n} M_j \mu(E_j) = S(P)$$

$$s(P) = \sum_{j=1}^{n} m_j \mu(E_j), \text{ where } m_j = \underset{p \in E_j}{\text{g.l.b.}} f(p)$$

$$s(Q) = \sum_{i=1}^{m} \bar{m}_i \mu(F_i), \text{ where } \bar{m}_i = \underset{p \in F_i}{\text{g.l.b.}} f(p)$$

$$\sum_{F_i \subset E_j} \bar{m}_i \mu(F_i) \geq \sum_{F_i \subset E_j} m_j \mu(F_i) = m_j \sum_{F_i \subset E_j} \mu(F_i) = m_j \mu(E_j) \text{ for each } j.$$

$$s(Q) = \sum_{i=1}^{m} \bar{m}_i \mu(F_i) = \sum_{j=1}^{n} \sum_{F_i \subset E_j} \bar{m}_i \mu(F_i) \geq \sum_{j=1}^{n} m_j \mu(E_j) = s(P)$$

<u>2.8</u> Suppose P $[E_1, E_2, \ldots, E_n]$ and Q $[F_1, F_2, \ldots, F_m]$ are measurable partitions of E. Then there is a partition R of E such that R is a refinement of P and a refinement of Q.

Proof: Let R be the collection of sets

$E_j \cdot F_i, j = 1,2,\ldots,n,\ i = 1,2,\ldots,m.$ $E_j \cdot F_i \subset E_j$, $E_j \cdot F_i \subset F_i$.
Each set $E_j \cdot F_i$ is measurable since both E_j and F_i are measurable. From the disjointness of the sets F_i and the sets E_j, we see that $(E_j \cdot F_i) \cdot (E_k \cdot F_l) = \emptyset$, unless $j = k$ and $i = l$.

$$\sum_{i=1}^{m} E_j \cdot F_i = E_j \cdot \sum_{i=1}^{m} F_i = E_j \cdot E = E_j. \quad \sum_{j=1}^{n} \sum_{i=1}^{m} E_j \cdot F_i = \sum_{j=1}^{n} E_j = E$$

Thus we see that R is a measurable partition of E and is a refinement of both P and Q.

<u>2.9</u> For every measurable partition P of E, $\overline{\int}_E f(p)d\mu \leqq S(P)$ and $\underline{\int}_E f(p)d\mu \geqq s(P)$. The proof of this assertion is immediate from the definitions of the upper and lower Lebesgue integrals, respectively.

<u>2.10</u> If $\epsilon > 0$, there is a measurable partition P_1 of E such that $S(P_1) < \overline{\int}_E f(p)d\mu + \epsilon$. Also, if $\epsilon > 0$, there is a measurable partition P_2 such that $s(P_2) > \underline{\int}_E f(p)d\mu - \epsilon$. Both these conclusions follow directly from definition.

<u>2.11</u> $\underline{\int}_E f(p)d\mu \leqq \overline{\int}_E f(p)d\mu$.

Proof: Deny the conclusion. Suppose

$\underline{\int}_E f(p)d\mu = \overline{\int}_E f(p)d\mu + \epsilon$, where $\epsilon > 0$. There is a measurable partition P_1 such that $s(P_1) < \overline{\int}_E f(p)d\mu + \frac{\epsilon}{2}$. Also, there is a measurable partition P_2 such that $s(P_2) > \underline{\int}_E f(p)d\mu - \frac{\epsilon}{2}$. Let R be a common refinement of P_1 and P_2. Then $S(R) \leqq S(P_1)$ and $s(R) \geqq s(P_2)$. But we notice that $S(P_1) < s(P_2)$. $\therefore$ $S(R) < s(R)$.

This, of course, is a contradiction and we conclude that

$\underline{\int}_E f(p)d\mu \leqq \overline{\int}_E f(p)d\mu$.

<u>2.12</u> Definition. With the above restrictions on f(p) and E, if

$\underline{\int}_E f(p)d\mu = \overline{\int}_E f(p)d\mu$, then we say that f(p) is Lebesgue integrable on E, and $\int_E f(p)d\mu$ denotes the common value of $\overline{\int_E} f(p)d\mu$ and $\underline{\int_E} f(p)d\mu$ and is called the Lebesgue integral of f(p) on E. We note that $m\mu(E) \leq \int_E f(p)d\mu \leq M\mu(E)$.

<u>2.13</u> If $m \leq f(p) \leq M$ and if $E = E_{x,y}\left[a \leq x \leq b,\ c \leq y \leq d\right]$, i.e. E is a closed rectangle, and if f(p) is Riemann integrable on E, then f(p) is Lebesgue integrable on E and $(R)\int_E f(p)dA = (L)\int_E f(p)d\mu$, where $(R)\int_E f(p)dA$ denotes the Riemann integral of f(p) on E and $(L)\int_E f(p)d\mu$ denotes the Lebesgue integral of f(p) on E.

Proof: Suppose f(p) is Riemann integrable on E. Then $(R)\underline{\int_E} f(p)dA = (R)\overline{\int_E} f(p)dA$. Give $\epsilon > 0$.

There is a Riemann partition P_1 of E (i.e. P_1 is a partition of E into closed rectangles two of which may have a side in common) such that $s(P_1) > (R)\int_E f(p)dA - \epsilon$. To form the corresponding Lebesgue measurable partition Q_1, we remove from any closed rectangle in P_1 its upper and/or right sides, depending upon whether the rectangle is bordered above or on the right by another rectangle. This will give a disjoint measurable partition of E. If $P_1 = P_1\left[R_1, R_2, \ldots, R_n\right]$ and if $Q_1 = Q_1\left[S_1, S_2, \ldots, S_n\right]$, then $R_i \supset S_i$ for each i and $s(P_1) = \sum_{i=1}^{n} m_i A(R_i)$,

$m_i = \underset{p\in R_i}{\text{g.l.b.}} f(p)$, $s(Q_1) = \sum_{i=1}^{n} l_i \mu(S_i)$, $l_i = \underset{p\in S_i}{\text{g.l.b.}} f(p)$

But $A(R_i) = \mu(R_i) = \mu(S_i)$ and $m_i \leq l_i$ for each i. (1.19, 1.23)

$\therefore\ s(P_1) \leq s(Q_1) \leq (L)\int_E f(p)d\mu$.

$(R)\underline{\underline{\int_E}} f(p)dA - \epsilon < (L)\underline{\underline{\int_E}} f(p)d\mu$. We conclude that

$(R)\underline{\int_E} f(p)d\mu \leq (L)\underline{\underline{\int_E}} f(p)d\mu$.

As before we can find a Riemann partition $P_2(T_1,T_2,\ldots,T_n)$ of E

such that $S(P_2) < \overline{\int_E} f(p)dA + \epsilon$. There exists a corresponding Lebesgue

measurable partition $Q_2(U_1,U_2,\ldots,U_n)$ of E formed as before. $T_i \supset U_i$ for

each i. $S(P_2) = \sum_{i=1}^{n} M_i A(T_i)$, $M_i = \underset{p\in T_i}{\text{l.u.b.}} f(p)$.

$S(Q_2) = \sum_{i=1}^{n} L_i \mu(U_i)$, $L_i = \underset{p\in U_i}{\text{l.u.b.}} f(p)$, $A(T_i) = \mu(T_i) = \mu(U_i)$

and $L_i \leq M_i$ for each i. Hence, $(L)\overline{\int_E} f(p)d\mu \leq S(Q_2) \leq S(P_2)$

$(L)\overline{\int_E} f(p)d\mu < (R)\overline{\int_E} f(p)dA$. We conclude that

$(L)\overline{\int_E} f(p)d\mu \leq (R)\overline{\int_E} f(p)dA$.

Combining the above inequalities $(R)\underline{\underline{\int_E}} f(p)dA \leq$

$(L)\underline{\underline{\int_E}} f(p)d\mu \leq (L)\overline{\int_E} f(p)d\mu \leq (R)\overline{\int_E} f(p)dA$.

But $(R)\underline{\underline{\int_E}} f(p)dA = (R)\overline{\int_E} f(p)dA$. $\therefore$ $(L)\underline{\underline{\int_E}} f(p)d\mu = (L)\overline{\int_E} f(p)d\mu$.

We conclude that f(p) is Lebesgue integrable on E and

$(L)\int_E f(p)d\mu = (R)\int_E f(p)dA$.

<u>2.14</u> **Definition.** Let E be a measurable set, and let f(p) be a function defined on E. f(p) is said to be a measurable function on E, if for every real number a, the set of points p of E for which $f(p) > a$ is a measurable set.

2.15 Definition. Suppose $f(p)$ is defined on E. If $p_o \in E$, then we say that $f(p)$ is continuous at p_o if, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $d(p,p_o) < \delta$, and if $p \in E$ then $|f(p) - f(p_o)| < \epsilon$.

2.16 If $f(p)$ is a continuous function on a measurable set E, then $f(p)$ is a measurable function on E.

Proof: Let a be a real number. Let E_a be the set of points p in E for which $f(p) > a$. Suppose $p_o \in E_a$. Then $p_o \in E$ and $f(p_o) > a$. Let $f(p_o)-a = \epsilon > 0$. There is a $\delta > 0$ such that if $d(p,p_o) < \delta$ and $p \in E$, then $|f(p)-f(p_o)| < \epsilon$, i.e. $f(p_o)-\epsilon < f(p) < f(p_o)+\epsilon$. But $f(p_o)-\epsilon = a$.

Hence if $d(p,p_o) < \delta$ and $p \in E$, then $f(p) > a$. Let $G_{p_o} = N(p_o, \delta)$. G_{p_o} is an open set and $p_o \in G_{p_o}$.

$G_{p_o} \cdot E \subset E_a$. $p_o \in G_{p_o} \cdot E \subset E_a$. $\sum_{p_o \in E_a} p_o \subset \sum_{p_o \in E_a} G_{p_o} \cdot E \subset E_a$,

$E_a \subset \sum_{p_o \in E_a} G_{p_o} \cdot E \subset E_a$. $\therefore$ $E_a = \sum_{p_o \in E_a} G_{p_o} \cdot E = E \cdot \sum_{p_o \in E_a} G_{p_o}$.

But the set on the right is a measurable set. (1.42, 1.51). We conclude that E_a is measurable, i.e. that $f(p)$ is a measurable function.

2.17 Given $f(p)$ on a measurable set E. Let N be the set of points of E where $f(p)$ is discontinuous. Suppose $\mu(N) = 0$. Then $f(p)$ is a measurable function on E.

Proof: Let E_a be the set of points $p \in E$ for which $f(p) > a$. Consider $E-N$. Let $N_a = N \cdot E_a$. Let $H_a = E_a - N_a$. $\therefore$ $E_a - H_a = N_a \subset N$. Let $p_o \in H_a$. Then $p_o \in E_a - N_a$. Hence $p_o \in E$, $f(p_o) > a$. $p_o \notin N$. $\therefore$ $f(p)$ is continuous at p_o. Let $f(p_o)-a = \epsilon > 0$. There is a $\delta > 0$ such that if $d(p,p_o) < \delta$ and if $p \in E$, then $|f(p)-f(p_o)| < \epsilon$, i.e. $f(p) > a$. Let $G_{p_o} = N(p_o, \delta)$. $p_o \in G_{p_o} \cdot E \subset E_a$.

Let $M = \sum_{p_o \in H_a} G_{p_o} \cdot E$. $H_a = \sum_{p_o \in H_a} p_o \subset \sum_{p_o \in H_a} G_{p_o} \cdot E = M \subset E_a$.

$H_a \subset M \subset E_a$. $E_a - M \subset E_a - H_a = N_a \subset N$.

$\mu(N) = 0$. $\mu^*(N) = 0$. $\mu^*(E_a - M) = 0$. $E_a - M$ is measurable.

(1.37, 1.44). M is measurable. $E_a = M + (E_a - M)$. $\therefore$ E_a is measurable.

2.18 Definition.

Let $E_p\left[p \in E, f(p) > a\right]$ denote the set of points p in E for which $f(p) > a$.

Let $E_p\left[p \in E, f(p) \geqq a\right]$ denote the set of points p in E for which $f(p) \geqq a$.

Let $E_p\left[p \in E, f(p) < a\right]$ denote the set of points p in E for which $f(p) < a$.

Let $E_p\left[p \in E, f(p) \leqq a\right]$ denote the set of points p in E for which $f(p) \leqq a$.

2.19 If f(p) is a measurable function on a measurable set E, then for every a, the set $E_p\left[p \in E, f(p) \geqq a\right]$ is a measurable set.

Proof: Let m be a positive integer. We shall show that

$\prod_{m=1}^{\infty} E_p\left[p \in E, f(p) > a - \frac{1}{m}\right] = E_p\left[p \in E, f(p) \geqq a\right]$. The set on the left is a countable product of measurable sets and hence is measurable. (1.47)

Suppose $p_o \in E_p\left[p \in E, f(p) \geqq a\right]$, i.e. $p_o \in E$, $f(p_o) \geqq a$.

For every m, $f(p_o) > a - \frac{1}{m}$. $\therefore$ $p_o \in E_p\left[p \in E, f(p) > a - \frac{1}{m}\right]$ for each m;

or $p_o \in \prod_{m=1}^{\infty} E_p\left[p \in E, f(p) > a - \frac{1}{m}\right]$. $E_p\left[p \in E, f(p) \geqq a\right] \subset \prod_{m=1}^{\infty} E_p\left[p \in E, f(p) > a - \frac{1}{m}\right]$.

Suppose $p_o \in \prod_{m=1}^{\infty} E_p\left[p \in E, f(p) > a - \frac{1}{m}\right]$ for each m.

Then $p_o \in E$, $f(p_o) > a - \frac{1}{m}$ for each m. $\therefore$ $f(p_o) \geqq a$. $\therefore$ $p_o \in E_p\left[p \in E, f(p) \geqq a\right]$.

$\prod E_p\left[p \in E, f(p) > a - \frac{1}{m}\right] \subset E_p\left[p \in E, f(p) \geqq a\right]$.

$$\prod_{m=1}^{\infty} E_p\left[p \in E,\ f(p) > a-\frac{1}{m}\right] = E_p\left[p \in E,\ f(p) \geqq a\right] .$$

This implies that $E_p\left[p \in E,\ f(p) \geqq a\right]$ is a measurable set.

2.20 If f(p) is a measurable function on a measurable set E, then for every a, the set $E_p\left[p \in E,\ f(p) \leqq a\right]$ is a measurable set.

Proof: We shall show that

$$E_p\left[p \in E,\ f(p) \leqq a\right] = E \cdot C\ E_p\left[p \in E,\ f(p) > a\right] .$$

The set on the right is the product of a measurable set and the complement of a measurable set (2.14) and hence is measurable.

Suppose $p_o \in E_p\left[p \in E,\ f(p) \leqq a\right]$, $p_o \in E$, $f(p_o) \leqq a$,

$p_o \notin E_p\left[p \in E,\ f(p) > a\right]$. $p_o \in C\ E_p\left[p \in E,\ f(p) > a\right]$.

$\therefore\ p_o \in E \cdot C\ E_p\left[p \in E,\ f(p) > a\right]$.

$E_p\left[p \in E,\ f(p) \leqq a\right] \subset E \cdot C\ E_p\left[p \in E,\ f(p) > a\right]$.

Suppose $p_o \in E \cdot C\ E_p\left[p \in E,\ f(p) > a\right]$.

$p_o \in E$, $p_o \notin E_p\left[p \in E,\ f(p) > a\right]$. $f(p_o) \leqq a$.

$p_o \in E_p\left[p \in E,\ f(p) \leqq a\right]$.

$E \cdot C\ E_p\left[p \in E,\ f(p) > a\right] \subset E_p\left[p \in E,\ f(p) \leqq a\right]$.

$\therefore\ E \cdot C\ E_p\left[p \in E,\ f(p) > a\right] = E_p\left[p \in E,\ f(p) \leqq a\right]$.

This implies that $E_p\left[p \in E,\ f(p) \leqq a\right]$ is a measurable set.

2.21 If f(p) is a measurable function on a measurable set E, then for every real number a the set $E_p\left[p \in E,\ f(p) < a\right]$ is measurable.

Proof: In an argument similar to that used in the preceding conclusion we can show that $E_p\left[p \in E,\ f(p) < a\right] = E \cdot C\ E_p\left[p \in E,\ f(p) \geqq a\right]$.

The set on the right is again seen to be measurable. (2.19)

2.22 If $f(p)$ is a measurable function on a measurable set E, then $E_p\left[p \in E,\ a \leq f(p) < b\right]$ is a measurable set.

Proof: We notice that

$$E_p\left[p \in E,\ a \leq f(p) < b\right] = E_p\left[p \in E,\ f(p) \geq a\right] \cdot E_p\left[p \in E,\ f(p) < b\right].$$

The set on the right is measurable. (2.19, 2.21)

2.23 If $f(p)$ is a measurable function on a measurable set E, $\mu(E) < +\infty$, and if $m \leq f(p) < M$, then $f(p)$ is Lebesgue integrable on E.

Proof: We must show that $\underline{\int_E} f(p)\,d\mu = \overline{\int_E} f(p)\,d\mu$. Give $\epsilon > 0$. Choose an integer N such that $\frac{\mu(E)}{N} < \epsilon$. We may suppose that M and m are integers.

Let $z_0 = m$, $z_1 = m + \frac{1}{N}$, $z_2 = m + \frac{2}{N}$, $z_3 = m + \frac{3}{N}, \ldots, z_k = m + \frac{k}{N}, \ldots,$ $z_{(M-m)N} = m + \frac{(M-m)N}{N} = M$. Let $E_i = E_p\left[p \in E,\ z_{i-1} \leq f(p) < z_i\right]$, $i = 1, \ldots, (M-m)N$. E_i is a measurable set for each i. (2.22).

$E_i \cdot E_j = \emptyset$ if $i \neq j$. $E = \sum_{i=1}^{(M-m)N} E_i$. Thus, we have a measurable partition $P(E_1, \ldots, E_{(M-m)N})$ of E. $S(P) = \sum_{i=1}^{(M-m)N} M_i\,\mu(E_i)$, where $M_i = \underset{p \in E_i}{\text{l.u.b.}}\, f(p)$.

$s(P) = \sum_{i=1}^{(M-m)N} m_i\,\mu(E_i)$, where $m_i = \underset{p \in E_i}{\text{g.l.b.}}\, f(p)$.

$m_i \geq z_{i-1}$ $\therefore$ $s(P) \geq \sum_{i=1}^{(M-m)N} z_{i-1}\,\mu(E_i)$. $M_i \leq z_i$

$\therefore$ $S(P) \leq \sum_{i=1}^{(M-m)N} z_i\,\mu(E_i)$, $S(P) - s(P) \leq \sum_{i=1}^{(M-m)N} (z_i - z_{i-1})\,\mu(E_i) =$ $\sum_{i=1}^{(M-m)N} \frac{1}{N}\mu(E_i) = \frac{1}{N}\sum_{i=1}^{(M-m)N} \mu(E_i) = \frac{\mu(E)}{N} < \epsilon$. $S(P) < s(P) + \epsilon$.

$$\overline{\int_E} f(P)d\mu \leqq S(P) < s(P) + \epsilon \leqq \underline{\int_E} f(p)d\mu + \epsilon.$$

Since ϵ is arbitrary and since we always have

$\underline{\int_E} f(p)d\mu \leqq \overline{\int_E} f(p)d\mu$, we conclude $\underline{\int_E} f(p)d\mu = \overline{\int_E} f(p)d\mu$,

and that $\therefore$ f(p) is Lebesgue integrable on E.

2.24 Definition. A condition is said to hold almost everywhere on a set E, if the subset F of E on which it does not hold is such that $\mu(F) = 0$.

2.25 Suppose f(p) is measurable on a measurable set E, $\mu(E) < +\infty$, $0 \leqq f(p) \leqq M$. Then $\int_E f(p)d\mu = 0$ if and only if $f(p) = 0$ almost everywhere on E.

Proof: 1. Suppose $f(p) = 0$ almost everywhere on E. Let N be the set of points of E for which $f(p) \neq 0$, that is $N = E_p\left[p \in E,\ f(p) > 0\right]$. $\mu(N) = 0$. N is a measurable set. E-N is also measurable. $N + (E-N) = E$. N and E-N form a measurable partition P of E. $S(P) \leqq M\cdot 0 \quad 0\cdot\mu(E-N) = 0$

$$\overline{\int_E} f(p)d\mu \leqq 0 \leqq \underline{\int_E} f(p)d\mu \quad \therefore \int_E f(p)d\mu = 0. \quad (2.4,\ 2.11)$$

2. Define N as above. Suppose

$\mu(N) > 0$, i.e. that it is not true that $f(p) = 0$ almost everywhere on E. We shall show that the following identity holds.

$$N = E_p\left[p \in E,\ f(p) > 0\right] = E_p\left[p \in E,\ f(p) > 1\right] + \sum_{n=1}^{\infty} E_p\left[p \in E, \frac{1}{n+1} < f(p) \leqq \frac{1}{n}\right].$$

Suppose $p_o \in E_p\left[p \in E,\ f(p) > 0\right]$.

Case 1. If $f(p_o) > 1$, then $p_o \in E_p\left[p \in E,\ f(p) > 1\right]$.

Case 2. If $0 < f(p) \leqq 1$, then there is an integer n such that $\frac{1}{n+1} < f(p) \leqq \frac{1}{n}$.

Suppose $p_o \in E_p\left[p \in E, f(p) > 1\right] + \sum_{n=1}^{\infty} E_p\left[p \in E, \frac{1}{n+1} < f(p) \leq \frac{1}{n}\right]$.

Case 1. Suppose $p_o \in E_p\left[p \in E, f(p) > 1\right]$. Then $p_o \in E_p\left[p \in E, f(p) > 0\right]$.

Case 2. Suppose $p_o \in E_p\left[p \in E, \frac{1}{n+1} < f(p) \leq \frac{1}{n}\right]$ for some n. Then $p_o \in E_p\left[p \in E, f(p) > 0\right]$. This verifies the above identity.

$$E_p\left[p \in E, f(p) > 1\right] \cdot \sum_{n=1}^{\infty} E_p\left[p \in E, \frac{1}{n+1} < f(p) \leq \frac{1}{n}\right] = \emptyset.$$

Let $F_o = E_p\left[p \in E, f(p) > 1\right]$, $F_n = E_p\left[p \in E, \frac{1}{n+1} < f(p) \leq \frac{1}{n}\right]$ for each n. Then $N = \sum_{n=0}^{\infty} F_n$; $0 < \mu(N) = \sum_{n=0}^{\infty} \mu(F_n)$.

$\therefore$ There exists an integer j such that $\mu(F_j) > 0$.

F_j is a measurable set. $E-F_j$ is also a measurable set.

F_j and $E-F_j$ form a measurable partition P of E.

$$s(P) = (\underset{p \in F_j}{\text{g.l.b.}}\ f(p))\mu(F_j) + (\underset{p \in E-F_j}{\text{g.l.b.}}\ f(p))\mu(E-F_j)$$

$$s(P) \geq \frac{1}{j+1}\mu(F_j) + 0 \cdot 0 = \frac{\mu(F_j)}{j+1} > 0.$$

$\therefore \underline{\int_E} f(p)d\mu > 0$ and $\int_E f(p)d\mu > 0$

$\therefore$ We conclude that if $\int_E f(p)d\mu = 0$, then $\mu(N) = 0$.

<u>2.26</u> Suppose we have $\{f_n(p)\}$ defined on a measurable set E and $f_n(p)$ is measurable for each n. Suppose $\lim_{n \to \infty} f_n(p) = f(p)$ on E. Then f(p) is measurable on E.

Proof: Let a be any real number. We must show that $E_p\left[p \in E, f(p) > a\right]$ is a measurable set. If we can establish the following identity the proof will be complete, since the set on the right is measurable. (2.14, 1.46, 1.47).

$$E_p\left[p \in E, f(p) > a\right] = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \prod_{n=k}^{\infty} E_p\left[p \in E, f_n(p) > a + \frac{1}{m}\right].$$

Suppose $p_o \in \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right]$. Then there is an m such that $p_o \in \sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right]$.

There is an m and a k such that $p_o \in \prod_{n=k}^{\infty} E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right]$.

$\therefore$ If $n \geqq k$, then $p_o \in E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right]$. If $n \geqq k$,

$f_n(p_o) > a + \frac{1}{m}$. $\lim_{n \to \infty} f_n(p_o) = f(p_o)$

$\therefore$ $f(p_o) \geqq a + \frac{1}{m} > a$ and $p_o \in E_p\left[p \in E,\ f(p) > a\right]$.

$$\therefore \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right] \subset E_p\left[p \in E,\ f(p) > a\right].$$

Suppose $p_o \in E_p\left[p \in E,\ f(p) > a\right]$. $f(p_o) > a$. There is an integer m such that $f(p_o) > a + \frac{1}{m}$. $\lim_{n \to \infty} f_n(p_o) = f(p_o)$. There is an integer k such that if $n \geqq k$, then $f_n(p_o) > a + \frac{1}{m}$. There is an integer m and an integer k such that if $n \geqq k$, then $p_o \in E_p\left[p \in E,\ f_n(p) > a + \frac{1}{m}\right]$.

$$\therefore \quad p_o \in \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p\left[p \in E,\ f(p) > a + \frac{1}{m}\right].$$

$$E_p\left[p \in E,\ f(p) > a\right] \subset \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p\left[p \in E,\ f(p) > a + \frac{1}{m}\right].$$

$$\therefore E_p\left[p \in E,\ f(p) > a\right] = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{n=k}^{\infty} E_p \quad p \quad E,\ f(p) \quad a \quad .$$

<u>2.27</u> If f(p) is a measurable function on a measurable set E, and if g(p) = -f(p), then g(p) is a measurable function on E.

Proof: Let a be any real number. We must show that $E_p\left[p \in E,\ g(p) > a\right]$ is a measurable set. We shall verify the following identity.

$E_p\left[p \in E,\ g(p) > a\right] = E_p\left[p \in E,\ f(p) < -a\right]$. The set on the right is measurable (2.21); therefore, this will establish the conclusion.

Suppose $p_o \in E_p\left[p \in E,\ g(p) > a\right]$.

$p_o \in E; g(p_o) > a; -f(p_o) > a; f(p_o) < -a.$

$\therefore \quad p_o \in E_p\left[p \in E,\ f(p) < -a\right].$

Suppose $p_o \in E_p\left[p \in E,\ f(p) < -a\right].$

$p_o \in E$; $f(p) < -a$, $-f(p_o) > a$, $g(p_o) > a$. $\therefore p_o \in E_p\left[p \in E,\ g(p) > a\right].$

Thus, the conclusion is established.

<u>2.28</u> If $f(p)$ and $g(p)$ are measurable functions on a measurable set E and if $h(p) = f(p) + g(p)$, then $h(p)$ is a measurable function on E.

Proof: Let $\{r_n\}$ be a sequence containing all of the rational numbers. Let a be any real number. We must show that $E_p\left[p \in E,\ h(p) > a\right]$ is a measurable set. We shall establish the following identity.

$$E_p\left[p \in E,\ h(p) > a\right] = \sum_{n=1}^{\infty} E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right].$$

The set on the right is obviously measurable and this will establish the conclusion.

Suppose $p_o \in \sum_{n=1}^{\infty} E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right].$

There is an integer n such that

$p \in E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right]$, $p_o \in E$, $f(p_o) > r_n$;

$g(p_o) > a - r_n$, $h(p_o) = f(p_o) + g(p_o) > r_n + a - r_n = a$. $p_o \in E_p\left[p \in E,\ h(p) > a\right].$

$E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right] \subset E_p\left[p \in E,\ h(p) > a\right].$

Suppose $p_o \in E_p\left[p \in E,\ h(p) > a\right].$

$p_o \in E$, $h(p_o) > a$, $f(p_o) + g(p_o) > a$, $f(p_o) > a - g(p_o)$,

$f(p_o) + g(p_o) = a + \epsilon$, $\epsilon > 0$, $f(p_o) > f(p_o) - \epsilon$. There is an integer n such that

$f(p_o) > r_n > f(p_o) - \epsilon$, $\epsilon - f(p_o) > -r_n$, $g(p_o) = a + \epsilon - f(p_o) > a - r_n$,

$g(p_o) > a - r_n$, $p_o \in E_p\left[p \in E,\ f(p) > r_n\right]$, $p_o \in E_p\left[p \in E,\ g(p) > a - r_n\right].$

$\therefore \quad p_o \in E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right].$

$E_p\left[p \in E,\ h(p) > a\right] \subset E_p\left[p \in E,\ f(p) > r_n\right] \cdot E_p\left[p \in E,\ g(p) > a - r_n\right].$

This establishes the identity.

<u>2.29</u> If $f(p)$ and $g(p)$ are measurable functions on a measurable set E, and if $k(p) = f(p) - g(p)$, then $k(p)$ is a measurable function on E.

Proof: $k(p) = f(p) + (-g(p))$. $-g(p)$ is measurable by an earlier conclusion (2.27) and the sum of two measurable functions is a measurable function (2.28).

<u>2.30</u> If $f(p)$ is a measurable function on a measurable set E, and if c is a constant, and if $\emptyset(p) = c f(p)$, then $\emptyset(p)$ is measurable on E.

Proof: 1. Suppose $c = 0$. Then $\emptyset(p) = 0$ on E. $\emptyset(p)$ is measurable on E.

2. Suppose $c > 0$. Let a be any real number. Consider the following identity, which we shall establish: $E_p\left[p \in E, \emptyset(p) > a\right] = E_p\left[p \in E, f(p) > \frac{a}{c}\right]$.

Suppose $p_o \in E_p\left[p \in E, \emptyset(p) > a\right]$,
$p_o \in E$, $\emptyset(p_o) > a$, $\emptyset(p_o) = cf(p_o) > a$, $f(p_o) > \frac{a}{c}$,
$\therefore p_o \in E_p\left[p \in E, f(p) > \frac{a}{c}\right]$. Thus $E_p\left[p \in E, \emptyset(p) > a\right] \subset E_p\left[p \in E, f(p) > \frac{a}{c}\right]$
The opposite relationship may be shown by reversing the steps above. Since the set on the right is measurable, the conclusion is established.

3. Suppose $c < 0$. Then $\emptyset(p) = -|c| f(p)$.
But $g(p) = |c| f(p)$ is a measurable function by Case 2. and
$-g(p) = -|c| f(p) = \emptyset(p)$ is measurable by 2.27.

<u>2.31</u> If $f(p)$ is a measurable function on a measurable set E and if $g(p) = (f(p))^2$, then $g(p)$ is a measurable function.

Proof: Let a be a real number.

1. Suppose $a < 0$. $E_p\left[p \in E, g(p) > a\right] = E$, since $g(p) = (f(p))^2 \geq 0$ on E. E is a measurable set.

2. Suppose $a \geq 0$.

$E_p[p \in E, g(p) > a] = E_p[p \in E, f(p) > \sqrt{a}] + E_p[p \in E, f(p) < -\sqrt{a}]$,

Since suppose $p_o \in E_p[p \in E,\ g(p) > a]$; $p_o \in E$, $g(p_o) > a$, $(f(p_o))^2 > a$, then either $f(p_o) > \sqrt{a}$ or $f(p_o) < -\sqrt{a}$. Thus

$E_p \quad p \quad E,\ g(p) \quad a \quad E_p \quad p \quad E,\ f(p) \quad a \quad E_p \quad p \quad E,\ f(p) \quad - \quad a$.

A reversal of steps gives the opposite relationship. Since the set on the right is the sum of two measurable sets (2.14, 2.21), it is measurable and the conclusion is established.

2.32 If $f(p)$ and $g(p)$ are measurable functions on a measurable set E, and if $\Theta(p) = f(p)g(p)$, then $\Theta(p)$ is measurable on E.

Proof: $\Theta(p) = f(p)g(p) = \frac{1}{4}(f(p)+g(p))^2 - \frac{1}{4}(f(p)-g(p))^2$.

The function on the right is measurable from preceding conclusions (2.27-2.31); therefore, the conclusion is established.

2.33 If $f(p)$ is a measurable function on a measurable set E, then $|f(p)|$ is a measurable function.

Proof: Case 1. If $a < 0$, then $E_p[|f(p)| > a] = E$.

Case 2. If $a \geqq 0$, then $E_p[|f(p)| > a] = E_p[f(p) > a] + E_p[f(p) < -a]$.

This identity is readily established, and since the sets on the right are measurable, the conclusion follows.

2.34 If $f(p)$ and $g(p)$ are measurable functions on a measurable set E, $\mu(E) < +\infty$ and if $m \leqq f(p) \leqq M$, $l \leqq g(p) \leqq N$, then

$$\int_E (f(p)+g(p))d\mu = \int_E f(p)d\mu + \int_E g(p)d\mu .$$

Proof: Give $\epsilon > 0$. There is a measurable partition P_1 of E such that $s^f(P_1) > \int_E f(p)d\mu - \epsilon$, where $s^f(P_1)$ denotes the lower sum of the partition P_1 with respect to the function $f(p)$. (2.4, 2.12.) There is a measurable partition P_2 of E, such that $S^f(P_2) < \int_E f(p)d\mu + \epsilon$,

where $S^f(P_2)$ denotes the upper sum of the partition P_2 with respect to the function f(p). (2.5, 2.12.) Let P be a measurable partition of E which is a refinement of both P_1 and P_2. Then, in similar notation

$s^f(P) > \int_E f(p)d\mu - \epsilon$, $S^f(P) < \int_E f(p)d\mu + \epsilon$. (2.7) There is a partition Q_1 of E such that $s^g(Q_1) > \int_E g(p)d\mu - \epsilon$, where again $s^g(Q_1)$ denotes the lower sum of the partition Q_1 with respect to the function g(p). There is a partition Q_2 of E such that $S^g(Q_2) < \int_E g(p)d\mu + \epsilon$. $S^g(Q_2)$ is the upper sum of the partition Q_2 with respect to the function g(p). Let Q be a measurable partition of E which is a refinement of Q_1 and Q_2.

Then $s^g(Q) > \int_E g(p)d\mu - \epsilon$ and $S^g(Q) < \int_E g(p)d\mu + \epsilon$.

Let R be a partition which is a refinement of both P and Q. Then the following relationships hold. (2.7)

$$s^f(R) > \int_E f(p)d\mu - \epsilon, \quad S^f(R) < \int_E f(p)d\mu + \epsilon,$$

$$s^g(R) > \int_E g(p)d\mu - \epsilon, \quad S^g(R) < \int_E g(p)d\mu + \epsilon.$$

Let $R = R\left[E_1, E_2, \ldots, E_n\right]$.

$$S^f(R) = \sum_{i=1}^{n} M_i^f \mu(E_i), \quad M_i^f = \underset{p \in E_i}{\text{l.u.b.}}\ f(p).$$

$$S^g(R) = \sum_{i=1}^{n} M_i^g \mu(E_i), \quad M_i^g = \underset{p \in E_i}{\text{l.u.b.}}\ g(p).$$

$$S^{f+g}(R) = \sum_{i=1}^{n} M_i^{f+g} \mu(E_i), \quad M_i^{f+g} = \underset{p \in E_i}{\text{l.u.b.}}\ f(p) + g(p).$$

$$S^f(R) + S^g(R) = \sum_{i=1}^{n} (M_i^f + M_i^g)\mu(E_i).$$

Give $\delta > 0$. There is a $p_i \in E_i$ such that

$M_i^{f+g} - \delta < f(p_i) + g(p_i) \leqq M_i^f + M_i^g$. Since δ is arbitrary we conclude $M_i^{f+g} \leqq M_i^f + M_i^g$ for each i. $\therefore$ $S^{f+g}(R) \leqq S^f(R) + S^g(R)$.

$$s^f(R) = \sum_{i=1}^{n} m_i^f \mu(E_i),\ m_i^f = \underset{p\in E_i}{\text{g.l.b.}}\ f(p).$$

$$s^g(R) = \sum_{i=1}^{n} m_i^g \mu(E_i),\ m_i^g = \underset{p\in E_i}{\text{g.l.b.}}\ g(p).$$

$$s^{f+g}(R) = \sum_{i=1}^{n} m_i^{f+g} \mu(E_i),\ m_i^{f+g} = \underset{p\in E_i}{\text{g.l.b.}}\ f(p)+g(p).$$

$$s^f(R) + s^g(R) = \sum_{i=1}^{n} (m_i^f + m_i^g)\mu(E_i).$$

Give $\eta > 0$. There is a $p_i \in E_i$ such that $m_i^f + m_i^g \leqq f(p_i) + g(p_i) < m_i^{f+g} + \eta$. Since η is arbitrary, we conclude that $m_i^f + m_i^g \leqq m_i^{f+g}$ for each i. $\therefore$ $s^{f+g}(R) \geqq s^f(R) + s^g(R)$.

$$\int_E (f(p)+g(p))d\mu \leqq S^{f+g}(R) \leqq S^f(R) + S^g(R) < \int_E f(p)d\mu + \int_E g(p)d\mu + 2\epsilon.$$

$$\int_E (f(p)+g(p))d\mu \geqq s^{f+g}(R) \geqq s^f(R) + s^g(R) > \int_E f(p)d\mu + \int_E g(p)d\mu - 2\epsilon.$$

$$\int_E f(p)d\mu + \int_E g(p)d\mu - 2\epsilon < \int_E (f(p)+g(p))d\mu < \int_E f(p)d\mu + \int_E g(p)d\mu + 2\epsilon,$$

$$\therefore \int_E (f(p)+g(p))d\mu = \int_E f(p)d\mu + \int_E g(p)d\mu.$$

2.35 If $m \leqq f(p) \leqq M$ and if $l \leqq g(p) \leqq M$ are functions defined on a measurable set E of finite measure, and if f(p) and g(p) are Lebesgue integrable on E, and if $f(p) \leqq g(p)$ for all p in E, then $\int_E f(p)d\mu \leqq \int_E g(p)d\mu$.

Proof: Let $P[E_1, E_2, \ldots, E_n]$ be any measurable partition of E.
$s^f(P) = \sum_{i=1}^{n} m_i \mu(E_i)$, $m_i = \underset{p\in E_i}{\text{g.l.b.}}\ f(p)$; $s^g(p) = \sum_{i=1}^{n} l_i \mu(E_i)$,
$l_i = \underset{p\in E_i}{\text{g.l.b.}}\ g(p)$. $m_i \leqq l_i$ for each i. $\therefore$ $s^f(P) \leqq s^g(P)$. Give $\epsilon > 0$. There is a measurable partition Q of E such that

$s^f(Q) > \int_E f(p)d\mu - \epsilon$. (2.4). $s^f(Q) \leqq s^g(Q)$.

$\int_E g(p)d\mu \geqq s^g(Q) > \int_E f(p)d\mu - \epsilon$.

Since ϵ is arbitrary, $\int_E g(p)d\mu \geqq \int_E f(p)d\mu$.

2.36 Let c be any real number. If f(p) is a bounded measurable function on a measurable set E of finite measure, then cf(p) is Lebesgue integrable on E and $\int_E cf(p)d\mu = c\int_E f(p)d\mu$.

Proof: Case 1. Suppose c = 0; then the conclusion is obvious.

Case 2. Suppose c > 0. f(p) is integrable on E. (2.23).

$c\int_E f(p)d\mu$ = cg.l.b.S(P), = g.l.b.cS(P), where g.l.b. is taken with respect to all measurable partitions P of E. Let $P(E_1,E_2,\ldots,E_n)$ be any measurable partition of E.

$$S(P) = \sum_{i=1}^{n} M_i\mu(E_i),\ M_i = \underset{p\in E_i}{\text{l.u.b.}}\ f(p).$$

$$cS(P) = c\sum_{i=1}^{n} M_i\mu(E_i) = \sum_{i=1}^{n} cM_i\mu(E_i),\ cM_i = \underset{p\in E_i}{\text{l.u.b.}}\, cf(p)$$

If g(p) = cf(p), then $cS(P) = S^g(P)$, since $cM_i = \underset{p\in E_i}{\text{l.u.b.}}\ g(p)$, where $S^g(P)$ denotes the upper sum of the partition P with respect to g(p).

$\therefore\ c\int_E f(p)d\mu = \overline{\int_E} g(p)d\mu = \overline{\int_E} cf(p)d\mu$.

Similarly, $c\int_E f(p)d\mu = \underline{\int_E} g(p)d\mu = \underline{\int_E} cf(p)d\mu$.

Case 3. Suppose c < 0. Leg g(p) = cf(p). Let $P[E_1,E_2,\ldots,E_n]$ be any measurable partition of E. If E_i is any set in P, and if M_i and m_i denote, respectively, the l.u.b. f(p) on E_i and g.l.b. f(p) on E_i, then cM_i and cm_i are respectively, the g.l.b. g(p) on E_i and l.u.b. g(p) on E_i.

$$S(P) = \sum_{i=1}^{n} M_i\mu(E_i);\ cS(P) = c\sum_{i=1}^{n} M_i\mu(E_i) = \sum_{i=1}^{n} cM_i\mu(E_i) = s^g(P).$$

$s(P) = \sum_{i=1}^{n} m_i \mu(E_i)$; $cs(P) = c \sum_{i=1}^{n} m_i \mu(E_i) = \sum_{i=1}^{n} cm_i \mu(E_i) = S^g(P)$.

Since P is arbitrary, we conclude that

$$c \overline{\int_E} f(p)d\mu = \underline{\int_E} g(p)d\mu = \underline{\int_E} cf(p)d\mu \quad \text{and}$$

$$c \underline{\int_E} f(p)d\mu = \overline{\int_E} g(p)d\mu = \overline{\int_E} cf(p)d\mu .$$

But $\overline{\int_E} f(p)d\mu = \underline{\int_E} f(p)d\mu = \int_E f(p)d\mu$.

$$\therefore \underline{\int_E} cf(p)d\mu = \overline{\int_E} cf(p)d\mu .$$

We conclude that f(p) is integrable and

$$c \int_E f(p)d\mu = \int_E cf(p)d\mu .$$

<u>2.37</u> If $m \leq f(p) \leq M$ and $l \leq g(p) \leq N$ are functions defined on a measurable set E, $\mu(E) < +\infty$, then $f(p) - g(p)$ is Lebesgue integrable on E and $\int_E (f(p) - g(p))d\mu = \int_E f(p)d\mu - \int_E g(p)d\mu$.

Proof: From 2.36 we see by letting c = -1 that

$$-\int_E g(p)d\mu = \int_E -g(p)d\mu .$$

$$\int_E (f(p) - g(p))d\mu = \int_E (f(p) + (-g(p)))d\mu = \int_E f(p)d\mu + \int_E -g(p)d\mu =$$

$$\int_E f(p)d\mu - \int_E g(p)d\mu . \quad (2.34)$$

<u>2.38</u> If f(p) is a measurable function on a measurable set E of finite measure and if f(p) = g(p) almost everywhere on E, then g(p) is measurable on E.

Proof: Let a be any real number. We must show that $E_p[p \in E, g(p) > a]$ is a measurable set. The following identity will be established.

(1). $E_p[p \in E,\ g(p) > a] = E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a] +$ $E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) > a]$. $E_p[p \in E,\ f(p) > a]$ is a measurable set. $E_p[p \in E,\ f(p) \neq g(p)]$ is by hypothesis a measurable set of measure 0.

$E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a] \subset E_p[p \in E,\ f(p) \neq g(p)]$ $\therefore$ The set on the left is measurable. (1.16, 1.20, 1.37).

$E_p[p \in E,\ f(p) = g(p)] = E - E_p[p \in E,\ f(p) \neq g(p)]$ $\therefore$ the set on the left of this relationship is measurable (1.37, 1.44). These statements imply that the set on the right of the identity (1) is measurable. (1.40, 1.42)

Suppose $p_o \in E_p[p \in E,\ g(p) > a]$. There are two cases here.

Case 1. $f(p_o) \neq g(p_o)$, $p_o \in E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a]$.

Case 2. $f(p_o) = g(p_o)$, $\therefore p_o \in E_p[p \in E,\ f(p) = g(p)]$, $f(p_o) > a$, $\therefore p_o \in E_p[p \in E,\ f(p) > a]$. This shows that

$E_p[p \in E,\ g(p) > a] \subset E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a] +$

$E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) > a]$.

Suppose $p_o \in E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a] +$

$E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) > a]$.

There are two cases here also.

Case 1. $p_o \in E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a]$

$p_o \in E,\ f(p_o) \neq g(p_o),\ g(p_o) > a$. $\therefore p_o \in E_p[p \in E,\ g(p) > a]$.

Case 2. $p_o \in E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) > a]$.

$p_o \in E,\ f(p_o) = g(p_o),\ f(p_o) > a$. $\therefore g(p_o) > a$, $p_o \in E_p[p \in E,\ g(p) > a]$.

$E_p[p \in E,\ g(p) > a] \supset E_p[p \in E,\ f(p) \neq g(p),\ g(p) > a] +$

$E_p[p \in E,\ f(p) = g(p)] \cdot E_p[p \in E,\ f(p) > a]$.

This establishes the identity, and we conclude that $E_p[p \in E,\ g(p) > a]$ is a measurable set, and hence that $g(p)$ is a measurable function.

<u>2.39</u> If $f(p)$ is a bounded function on a measurable set E of finite

measure, and if $f(p)$ is Lebesgue integrable on E, then $f(p)$ is measurable on E.

Proof: There is a measurable partition $P_1[E_1', E_2', \ldots, E_{n_1}']$ of E such that $s(P_1) > \int_E f(p)d\mu - 1$, and such that $S(P_1) < \int_E f(p)d\mu + 1$. (2.4, 2.5). If $p \in E_k'$, let $f_1(p) = \text{g.l.b.}_{p \in E_k'} f(p) = m_k'$;
$g_1(p) = \text{l.u.b.}_{p \in E_k'} f(p) = M_k'$.

$s^f(P_1) = \sum_{k=1}^{n_1} m_k' \mu(E_k')$, $S^f(P_1) = \sum_{k=1}^{n_1} M_k' \mu(E_k')$. $f_1(p)$ is a measurable function, since if a is any real number, $E_p[p \in E, f_1(p) > a] = \sum E_k'$, summation extended over those integers k for which $m_k' > a$ and each set E_k' is measurable. $f_1(p) \leqq f(p)$ for each p from the definition of $f_1(p)$.

$\therefore \int_E f_1(p)d\mu \leqq \int_E f(p)d\mu$. (2.35).

$s^{f_1}(P_1) = m_1'\mu(E_1') + m_2'\mu(E_2') + \ldots + m_{n_1}'\mu(E_{n_1}') =$
$\sum_{k=1}^{n_1} m_k'\mu(E_k') = s^f(P_1)$

$S^{f_1}(P_1) = m_1'\mu(E_1') + m_2\mu(E_2') + \ldots + m_{n_1}'\mu(E_{n_1}') =$
$\sum_{k=1}^{n_1} m_k'\mu(E_k') = s^{f_1}(P_1)$.

$\therefore \int_E f_1(p)d\mu = s^f(P_1) > \int_E f(p)d\mu - 1$.

There is a measurable partition Q_2 of E such that

$s^f(Q_2) > \int_E f(p)d\mu - \frac{1}{2}$, $S^f(Q_2) < \int_E f(p)d\mu + \frac{1}{2}$.

Let $P_2[E_1^2, E_2^2, \ldots, E_{n_2}^2]$ be a measurable partition of E which is a refinement of both P_1 and Q_2.

$s^f(P_2) > \int_E f(p)d\mu - \frac{1}{2}$, $S^f(P_2) < \int_E f(p)d + \frac{1}{2}$.

If $p \in E_k^2$, let $f_2(p) = \text{g.l.b.}_{p \in E_k^2} f(p)$. By the same reasoning as for

$f_1(p)$, we see that $f_2(p)$ is a measurable function on E, and further $f_2(p) \leq f(p)$, $f_1(p) \leq f_2(p)$. As before we observe that $s^{f_2}(P_2) = s^f(P_2)$. and $S^{f_2}(P_2) = s^f(P_2)$

$$\int_E f(p)d\mu - \frac{1}{2} < s^f(P_2) = \int_E f_2(p)d\mu \leq \int_E f(p)d\mu .$$

Construct in a similar manner a measurable function $f_3(p)$ such that $f_2(p) \leq f_3(p) \leq f(p)$. and such that

$$\int_E f(p)d\mu - \frac{1}{3} < \int_E f_3(p)d\mu \leq \int_E f(p)d\mu .$$

Continuing this process we obtain a sequence of functions $\{f_n(p)\}$ where $f_n(p)$ is a measurable function for each n, and such that $f_1(p) \leq f_2(p) \leq f_3(p) \leq \ldots \leq f_n(p) \leq f_{n+1}(p) \leq \ldots$ where $f_n(p) \leq f(p)$ for each n.

$$\int_E f(p)d\mu - \frac{1}{n} < \int_E f_n(p)d\mu \leq \int_E f(p)d\mu .$$

$\{f_n(p)\}$ converges, since if $p_o \in E$, we have $\{f_n(p_o)\}$, where $f_1(p_o) \leq f_2(p_o) \leq \ldots \leq f_n(p_o) \leq \ldots \leq f(p_o)$

Let $g(p_o) = \lim_{n\to\infty} f_n(p_o)$. Let $g(p) = \lim_{n\to\infty} f_n(p)$. $g(p)$ is a measurable function since it is the limit of a sequence of measurable functions. (2.26)

$$f_n(p) \leq g(p) \leq f(p) \text{ for each } n. \therefore \int_E f_n(p)d\mu \leq \int_E g(p)d\mu \leq \int_E f(p)d\mu . \quad (2.35)$$

$$\int_E f(p)d\mu - \frac{1}{n} < \int_E f_n(p)d\mu \text{ for each } n.$$

$$\int_E f(p)d\mu \leq \int_E g(p)d\mu \;\therefore \int_E g(p)d\mu = \int_E f(p)d\mu .$$

By similar reasoning we can construct a decreasing sequence of measurable functions $\{g_n(p)\}$, i.e.

$g_1(p) \geq g_2(p) \geq \ldots \geq g_n(p) \geq \ldots \geq f(p)$, such that

$i\mu < \int_E f(p)d\mu + \frac{1}{n}$. This sequence will converge to some function h(p), where $f(p) \leq h(p) \leq g_n(p)$ and h(p) is measurable.

$$\int_E f(p)d\mu \leq \int_E h(p)d\mu \leq \int_E g_n(p)d\mu < \int_E f(p)d\mu + \frac{1}{n}.$$

$$\int_E h(p)d\mu \leq \int_E f(p)d\mu \quad \therefore \int_E h(p)d\mu = \int_E f(p)d\mu,$$

$g(p) \leq f(p) \leq h(p)$.

Since g(p) and h(p) are measurable functions and $g(p) \leq h(p)$, then

$$\int_E (h(p) - g(p))d\mu = \int_E h(p)d\mu - \int_E g(p)d\mu = 0.$$

We know $h(p) - g(p) \geq 0$, $\therefore$ $h(p) - g(p) = 0$ almost everywhere on E, or $h(p) = g(p)$ almost everywhere on E, $\therefore$ $f(p) = g(p)$ almost everywhere on E and since g(p) is measurable on E, we conclude, by 2.38, that f(p) is measurable on E.

<u>2.40</u> Definition. If f(p) is a non-negative measurable function on a measurable set E, let $f_N(p) = \begin{cases} f(p) \text{ if } 0 \leq f(p) < N \\ N \text{ if } f(p) \geq N, \end{cases}$

where N is a positive integer.

<u>2.41</u> Definition. If f(p) is a negative measurable function on a measurable set E, let $f_{-N}(p) = \begin{cases} f(p) \text{ if } 0 > f(p) > -N \\ -N \text{ if } f(p) \leq -N, \end{cases}$

where N is a positive integer.

<u>2.42</u> If f(p) is a non-negative, measurable function on a measurable set E, then for each N, $f_N(p)$ is a bounded, non-negative function on E. The proof of this assertion is immediate from the definition of $f_N(p)$.

<u>2.43</u> If f(p) is a negative, measurable function on a measurable set E,

then for each N, $f_{-N}(p)$ is a bounded negative function on E.

Again, the truth of this assertion follows directly from the definition of $f_{-N}(p)$.

2.44 If f(p) is a non-negative, measurable function on a measurable set E, then for each N, $f_N(p) \leq f(p)$.

Proof: The proof follows from the definition of $f_N(p)$.

2.45 If f(p) is a negative, measurable function on a measurable set E, then for each N, $f_{-N}(p) \geq f(p)$.

Proof: The proof follows immediately from the definition of $f_{-N}(p)$.

2.46 If f(p) is a non-negative, measurable function on a measurable set E, then for each N, $f_N(p)$ is a non-negative measurable function on E.

Proof: From a previous conclusion (2.42), we see that $f_N(p)$ is non-negative and bounded. Let a be any real number. We must show that for each N, $E_p[p \in E, f_N(p) > a]$ is a measurable set. Let N be any positive integer.

Case 1. If $a \geq N$, then let $E_p[p \in E, f_N(p) > a] = \emptyset$, which is a measurable set.

Case 2. If $a < N$, then $E_p[p \in E, f_N(p) > a] =$ $E_p[p \in E, f(p) > a]$.

We must establish this identity.

1. Suppose $p_o \in E_p[p \in E, f_N(p) > a]$, $p_o \in E$, $f_N(p_o) > a$, $f(p_o) > a$. $\therefore$ $p_o \in E_p[p \in E, f(p) > a]$.

2. Suppose $p_o \in E_p[p \in E, f(p) > a]$, $p_o \in E$, $f(p_o) > a$.

a. If $f(p_o) \geq N$, then $f_N(p_o) = N > a$, $p_o \in E_p[p \in E, f_N(p) > a]$.

b. If $f(p_o) < N$, then $f_N(p_o) = f(p_o) > a$, $p_o \in E_p[p \in E, f_N(p) > a]$.

Thus, the identity is established, and since $f(p)$ is a measurable function, it follows that $E_p\left[p \in E,\ f(p) > a\right]$ is a measurable set. (2.14). Hence, $E_p\left[p \in E,\ f_N(p) > a\right]$ is a measurable set and $f_N(p)$ is a measurable function on E.

2.47 If $f(p)$ is a negative, measurable function on a measurable set E, then for each N, $f_{-N}(p)$ is a negative, bounded, measurable function on E.

Proof: The proof to this conclusion is similar to that of 2.46.

2.48 If $f(p)$ is a non-negative, measurable function on a measurable set E, and if $N < M$, then $f_N(p) \leqq f_M(p)$.

Proof: If $f(p) < N$, then $f_N(p) = f_M(p) = f(p)$. (2.40). If $f(p) \geqq N$, then $f_N(p) = N$ and either $f_M(p) = f(p) \geqq f_N(p)$ or $f_M(p) = M > N = f_N(p)$. In each of these situations $f_N(p) \leqq f_M(p)$.

2.49 If $f(p)$ is a negative, measurable function on a measurable set E, and if $-M < -N$, then $f_{-M}(p) \leqq f_{-N}(p)$.

Proof: The proof of this theorem is similar to that of 2.48.

2.50 Definition. Let $f(p)$ be a non-negative, measurable function on a measurable set E, $\mu(E) < +\infty$. For each positive integer N, consider $f_N(p)$. $f_N(p)$ is a non-negative, bounded, measurable function on E. Therefore, $f_N(p)$ is Lebesgue integrable on E, for each N. If $N < M$, then $f_N(p) \leqq f_M(p)$ and hence $\int_E f_N(p)d\mu \leqq \int_E f_M(p)d\mu$.

Consider $\left\{\int_E f_N(p)d\mu\right\}$. This sequence is an increasing sequence of real numbers. If $\left\{\int_E f_N(p)d\mu\right\}$ is an unbounded sequence, we say that $f(p)$ is not a summable function on E.

If $\left\{ \int_E f_N(p)d\mu \right\}$ is a bounded sequence, then suppose $\lim_{N\to\infty} \int_E f_N(p)d\mu = a$. Then we say that f(p) is Lebesgue summable on E, and we write $\int_E f(p)\, d\mu = a = \lim_{N\to\infty} \int_E f_N(p)\, d\mu$.

<u>2.51</u> Definition. Let f(p) be a negative, measurable function on a measurable set E of finite measure. For each positive integer N, consider $f_{-N}(p)$. $f_{-N}(p)$ is a negative, bounded, measurable function on E. Therefore, $f_{-N}(p)$ is Lebesgue integrable on E, for each N. If $-M < -N$, then $f_{-M}(p) \leqq f_{-N}(p)$ and hence $\int_E f_{-M}(p)\, d\mu \leqq \int_E f_{-N}(p)\, d\mu$.

Consider $\left\{ \int_E f_{-N}(p)\, d\mu \right\}_{-N=-1}^{-\infty}$. This sequence is a decreasing sequence of real numbers. If $\left\{ \int_E f_{-N}(p)\, d\mu \right\}$ is an unbounded sequence, then we say that f(p) is not a summable function on E.

If $\left\{ \int_E f_{-N}(p)\, d\mu \right\}$ is a bounded sequence, then suppose that $\lim_{-N\to-\infty} \int_E f_{-N}(p)\, d\mu = -a$. Then we say that f(p) is Lebesgue summable on E, and we write $\int_E f(p)\, d\mu = -a = \lim_{-N\to-\infty} \int_E f_{-N}(p)\, d\mu$.

<u>2.52</u> Definition. Let f(p) be a measurable function on a measurable set E of finite measure. Let $P = E_p\left[p \in E,\ f(p) \geqq 0 \right]$ and let $N = E_p\left[p \in E,\ f(p) < 0 \right]$. Then clearly $E = P + N$ and $P \cdot N = \emptyset$. If f(p) is a Lebesgue summable function on both P and N, and if $\int_P f(p)\, d\mu = a$ and $\int_N f(p)\, d\mu = -b$, then we say that f(p) is Lebesgue summable on E and we write

$$) \, d\mu + \int_N f(p)\, d\mu = a - b.$$

2.53 If $f(p)$ and $g(p)$ are non-negative, measurable functions on a measurable set E of finite measure, and if $f(p)$ and $g(p)$ are summable, and if $h(p) = f(p) + g(p)$, then $h(p)$ is summable on E, and

$$\int_E h(p)\, d\mu = \int_E f(p)\, d\mu + \int_E g(p)\, d\mu .$$

Proof: $h(p)$ is non-negative and measurable.

$$\text{Let } h_N(p) = \begin{cases} h(p) & \text{if } 0 \leq h(p) < N \\ N & \text{if } h(p) \geq N. \end{cases}$$

$$f_N(p) = \begin{cases} f(p) & \text{if } 0 \leq f(p) < N \\ N & \text{if } f(p) \geq N. \end{cases}$$

$$g_N(p) = \begin{cases} g(p) & \text{if } 0 \leq g(p) < N \\ N & \text{if } g(p) \geq N. \end{cases}$$

Since $f(p)$ and $g(p)$ are summable, $\lim_{N\to\infty} \int_E f_N(p)\, d\mu = \int_E f(p)\, d\mu$ and

$$\lim_{N\to\infty} \int_E g_N(p)\, d\mu = \int_E g(p)\, d\mu .$$

We shall show that for each N, $h_N(p) \leq f_N(p) + g_N(p)$.

Let N be any positive integer; suppose $p_0 \in E$.

Case 1. Suppose $0 \leq h(p_0) < N$. Then $h_N(p_0) = h(p_0)$. Then $0 \leq f(p_0) < N$. Then $f_N(p_0) = f(p_0)$. Then $0 \leq g(p_0) < N$. Then $g_N(p_0) = g(p_0)$.

$\therefore$ $h_N(p_0) = f_N(p_0) + g_N(p_0)$.

Case 2. Suppose $h(p_0) \geq N$ and

a. suppose $f(p_0) \geq N$. Then $h_N(p_0) = N$, $f_N(p_0) = N$ and $g_N(p_0) \geq 0$.

$\therefore$ $h_N(p_0) \leq f_N(p_0) + g_N(p_0)$. A similar argument gives the same result if $g(p_0) \geq N$.

b. Suppose $f(p_0) < N$ and $g(p_0) < N$.

Then $h_N(p_0) = N \leqq h(p_0)$, $f_N(p_0) = f(p_0)$, $g_N(p_0) = g(p_0)$.

We have $h_N(p_0) = N \leqq h(p_0) = f(p_0) + g(p_0) = f_N(p_0) + g_N(p_0)$.

Thus, in any possible case we see that $h_N(p) \leqq f_N(p) + g_N(p)$. This implies that for each N,

$$\int_E h_N(p)\, d\mu \leqq \int_E (f_N(p) + g_N(p))\, d\mu = \int_E f_N(p)\, d\mu \ +$$
$$\int_E g_N(p)\, d\mu \leqq \int_E f(p)\, d\mu + \int_E g(p)\, d\mu \quad (2.34,\ 2.35).$$ Therefore,

$h(p)$ is summable on E, since $\left\{\int_E h_N(p)\, d\mu\right\}$ is an increasing sequence bounded above by $\int_E f(p)\, d\mu + \int_E g(p)\, d\mu$ and furthermore

$$\int_E h(p)\, d\mu = \lim_{N\to\infty} \int_E h_N(p)\, d\mu \leqq \int_E f(p)\, d\mu + \int_E g(p)\, d\mu .$$

Hence this limit exists.

We shall next show that for each N, $h_{2N}(p) \geqq f_N(p) + g_N(p)$. Suppose N is any positive integer and $p_0 \in E$.

Case 1. Suppose $0 \leqq f(p_o) < N$ and $0 \leqq g(p_o) < N$,

Then $0 \leqq h(p_0) < 2N$, $f_N(p_0) = f(p_0)$ and $g_N(p_0) = g(p_0)$.

Hence, $h_{2N}(p_0) = h(p_0)$ and $h_{2N}(p_0) = f_N(p_o) + g_N(p_0)$.

Case 2. Suppose $f(p_0) \geqq N$ and $g(p_0) \geqq N$.

Then $h(p_o) = f(p_0) + g(p_0) \geqq 2N$.

$f_N(p_0) = N$ and $g_N(p_0) = N$, $h_{2N}(p_0) = 2N$.

$h_{2N}(p_0) = f_N(p_0) + g_N(p_0)$.

Case 3. Suppose $f(p_0) \geqq N$ and $g(p_0) < N$ and

a. suppose $h(p_0) \geqq 2N$. $f_N(p_0) = N$,

$g_N(p_0) = g(p_0) < N$, $h_{2N}(p_0) =$

$2N > f_N(p_0) + g_N(p_0)$.

b. suppose $h(p_0) < 2N$. $f_N(p_0) = N \leq f(p_0)$, $g_N(p_0) = g(p_0)$, $h_{2N}(p_0) = h(p_0)$. $f_N(p_0) + g_N(p_0) \leq f(p_0) + g(p_0) = h(p_0) = h_{2N}(p_0)$.

In this case similar results follow if we assume initially that $f(p_0) < N$ and $g(p_0) \geq N$. In each case we see that $h_{2N}(p) \geq f_N(p) + g_N(p)$.

$$\therefore \int_E h_{2N}(p)\, d\mu \geq \int_E (f_N(p) + g_N(p))\, d\mu = \int_E f_N(p)\, d\mu + \int_E g_N(p)\, d\mu \;.\; \int_E h(p)\, d\mu \geq \int_E h_{2N}(p)\, d\mu \;.$$

$$\therefore \int_E f_N(p)\, d\mu + \int_E g_N(p)\, d\mu \leq \int_E h(p)\, d\mu \;.$$

$$\lim_{N\to\infty} \int_E f_N(p)\, d\mu = \int_E f(p)\, d\mu \;;\; \lim_{N\to\infty} \int_E g_N(p)\, d\mu = \int_E g(p)\, d\mu \;.\; \lim_{N\to\infty} \left(\int_E f_N(p)\, d\mu + \int_E g_N(p)\, d\mu \right) = \int_E f(p)\, d\mu + \int_E g(p)\, d\mu \leq \int_E h(p)\, d\mu \;.$$

Therefore, since the reverse relationship has already been established, we conclude that

$$\int_E h(p)\, d\mu = \int_E f(p)\, d\mu + \int_E g(p)\, d\mu \;.$$

<u>2.54</u> Suppose $f(p)$ is a bounded, integrable function on a measurable set E of finite measure. Suppose that G is a measurable subset of E. Then $f(p)$ is integrable on G.

Proof: $f(p)$ is measurable on E. We shall first show that $f(p)$ is measurable on G.

To do this we shall establish the following identity.

Let a be any real number.

$E_p\left[p \in G,\ f(p) > a\right] = G \cdot E_p\left[p \in E,\ f(p) > a\right]$. The set on right is measurable since f(p) is a measurable function on the set E and since G is measurable by hypothesis.

Suppose $p_o \in E_p\left[p \in G,\ f(p) > a\right]$. Then $p_o \in G$, $f(p_o) > a$, $p_o \in E$, $\therefore$ $p_o \in G \cdot E_p\left[p \in E,\ f(p) > a\right]$.

Suppose $p_o \in G \cdot E_p\left[p \in E,\ f(p) > a\right]$, $p_o \in G$, $p_o \in E$, $f(p_o) > a$, $\therefore$ $p_o \quad E_p\left[p \in G,\ f(p) > a\right]$.

Thus the identity is established. We conclude that $E_p\left[p \in G,\ f(p) > a\right]$ is a measurable set and hence that f(p) is a measurable function on the set G. Since f(p) is bounded on E, it follows that it is bounded on the subset G. Therefore, f(p) is Lebesgue integrable on G. (2.23)

<u>2.55</u> If f(p) is a bounded, measurable function on a measurable set E of finite measure and if $E = E_1 + E_2$, $E_1 \cdot E_2 = \emptyset$ and E_1 and E_2 are measurable sets, then f(p) is Lebesgue integrable on E_1 and on E_2, and

$$\int_E f(p)\, d\mu = \int_{E_1} f(p)\, d\mu + \int_{E_2} f(p)\, d\mu .$$

Proof: The fact that f(p) is Lebesgue integrable on E_1 and on E_2 is immediate from the preceding conclusion.

Give $\epsilon > 0$. There is a measurable partition $P_1\left[F_2, \ldots, F_n\right]$ of E_1 such that $s(P_1) > \int_{E_1} f(p)\, d\mu - \frac{\epsilon}{2}$. (2.4) There is a measurable partition $P_2\left[G_1, G_2, \ldots, G_m\right]$ of E_2 such that $s(P_2) > \int_{E_2} f(p)\, d\mu - \frac{\epsilon}{2}$. Then $P\left[F_1, F_2, \ldots, F_n, G_1, G_2, \ldots, G_m\right]$ is a measurable partition of E.

$s(P_1) = \sum_{k=1}^{n} m_k' \mu(F_k)$; $\qquad m_k' = \text{g.l.b.}_{p \in F_k}\, f(p)$;

$s(P_2) = \sum_{k=1}^{m} m_k^2 \mu(G_k)$; $\qquad m_k^2 = \text{g.l.b.}_{p \in G_k}\, f(p)$;

$s(p) = s(P_1) + s(P_2)$;

$$\int_E f(p)d\mu \geq s(P) > \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu - \epsilon \quad . \quad (2.9)$$

Since ϵ is arbitrary, $\int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu \leq \int_E f(p)d\mu$.

There is a measurable partition $Q_1 \left[H_1, H_2, \ldots, H_r\right]$ of E, such that

$S(Q_1) < \int_{E_1} f(p)d\mu + \frac{\epsilon}{2}$. (2.5) There is a measurable partition

$Q_2 \left[J_1, J_2, \ldots, J_s\right]$ such that $S(Q_2) < \int_{E_2} f(p)d\mu + \frac{\epsilon}{2}$.

$Q\left[H_1, H_2, \ldots, H_r, J_1, J_2, \ldots, J_s\right]$ is a measurable partition of E.

$$S(Q_1) = \sum_{k=1}^{r} M_k^1 \mu(H_k), \quad M_k^1 = \underset{p \in H_k}{\text{l.u.b.}}\ f(p)$$

$$S(Q_2) = \sum_{k=1}^{s} M_k^2 \mu(J_k), \qquad M_k^2 = \underset{p \in J_k}{\text{l.u.b.}}\ f(p), \quad S(Q) = S(Q_1) + S(Q_2);$$

$$\int_E f(p)d\mu \leq S(Q) < \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu + \epsilon \quad . \ (2.9)$$

Since ϵ is arbitrary, $\int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu \geq \int_E f(p)d\mu$.

The opposite relationship having already been established, we conclude that

$$\int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu = \int_E f(p)d\mu \ .$$

<u>2.56</u> If $m \leq f(p) \leq M$ on E if E is a measurable set of finite measure, and if f(p) is measurable on E, then $m \cdot \mu(E) \leq \int_E f(p)d\mu = M \cdot \mu(E)$.

Proof: Consider the measurable partition P of E consisting of the set E alone.

$$\int_E f(p)d\mu \leq S(P) = (\underset{p \in E}{\text{l.u.b.}}\ f(p)) \cdot \mu(E) \leq M \cdot \mu(E). \quad (2.9)$$

$$\int_E f(p)d\mu \geq s(P) = (\underset{p \in E}{\text{g.l.b.}}\ f(p)) \cdot \mu(E) \geq m \cdot \mu(E). \quad (2.9)$$

<u>2.57</u> If f(p) is a non-negative, measurable and summable function on a measurable set E of finite measure, and if $E = E_1 + E_2$, $E_1 \cdot E_2 = \emptyset$ and E_1 and E_2 are measurable sets, then f(p) is summable on E_1 and E_2,

$$\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu, \text{ and}$$

$$\int_{E_1} f(p)d\mu \leqq \int_E f(p)d\mu \text{ and } \int_{E_2} f(p)d\mu \leqq \int_E f(p)d\mu.$$

Proof: Let $f_N(p)$ be defined as before.

We know that $\int_{E_1} f_N(p)d\mu \leqq \int_E f_N(p)d\mu \leqq \int_E f(p)d\mu$, since

$$\int_E f_N(p)d\mu = \int_{E_1} f_N(p)d\mu + \int_{E_2} f_N(p)d\mu \text{ and } \lim_{N\to\infty} \int_E f_N(p)d\mu = \int_E f(p)d\mu.$$

$\therefore \left\{\int_{E_1} f_N(p)d\mu\right\}$ is a bounded, increasing sequence and hence f(p) is summable on E_1.

$$\int_{E_1} f(p)d\mu = \lim_{N\to\infty} \int_{E_1} f_N(p)d\mu$$

$\int_{E_1} f(p)d\mu \leqq \int_E f(p)d\mu$. From symmetry in the definitions of E_1 and E_2 we see that f(p) is summable on E_2 and $\int_{E_2} f(p)d\mu \leqq \int_E f(p)d\mu$. We know that $\int_E f_N(p)d\mu = \int_{E_1} f_N(p)d\mu + \int_{E_2} f_N(p)d\mu$ for each N. Taking limits as N becomes infinite we obtain $\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu$.

<u>2.58</u> If f(p) is a negative, measurable and summable function on a measurable set E of finite measure and if $E = E_1 + E_2$, where $E_1 \cdot E_2 = \emptyset$ and E_1 and E_2 are measurable sets, then f(p) is summable on E_1 and E_2,

$$\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu, \text{ and}$$

$$)d\mu \text{ and } \int_{E_2} f(p)d\mu \geqq \int_E f(p)d\mu.$$

The proof of this theorem is similar to that of 2.57.

2.59 If $f(p)$ is a measurable and summable function on a measurable set E of finite measure, if $E = E_1 + E_2$, $E_1 \cdot E_2 = \emptyset$, and if E_1 and E_2 are measurable sets, then $\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu$.

Proof: Let $N = E_p\left[p \in E,\ f(p) < 0\right]$.
Let $P = E_p\left[p \in E,\ f(p) \geq 0\right]$. $E = N + P$. Since $f(p)$ is a measurable function, N and P are measurable sets. (2.19, 2.21)
$N \subset E = E_1 + E_2$. $\therefore N = N \cdot E_1 + N \cdot E_2$; $(NE_1)\cdot(N \cdot E_2) = \emptyset$.
Similarly $P = P \cdot E_1 + P \cdot E_2$; $(P \cdot E_1)\cdot(P \cdot E_2) = \emptyset$.

$\therefore \int_N f(p)d\mu = \int_{N \cdot E_1} f(p)d\mu + \int_{N \cdot E_2} f(p)d\mu$ (2.58) and
$\int_P f(p)d\mu = \int_{P \cdot E_1} f(p)d\mu + \int_{P \cdot E_2} f(p)d\mu$ (2.57)
$E_1 = E_1 \cdot P + E_1 \cdot N$, $E_2 = E_2 \cdot P + E_2 \cdot N$.

$$\int_E f(p)d\mu = \int_N f(p)d\mu + \int_P f(p)d\mu =$$
$$\int_{N \cdot E_1} f(p)d\mu + \int_{N \cdot E_2} f(p)d\mu + \int_{P \cdot E_1} f(p)d\mu + \int_{P \cdot E_2} f(p)d\mu =$$
$$\int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu. \quad (2.55)$$

2.60 If $f(p)$ is a bounded, measurable function on a measurable set E of finite measure and if $\epsilon > 0$, then there is a $\delta > 0$, such that if G is a measurable subset of E and if $\mu(G) < \delta$, then

$$\left| \int_G f(p)d\mu \right| < \epsilon .$$

Proof: Since $f(p)$ is bounded, we can find a positive real number M such that $-M \leq f(p) \leq M$ on E. If G is any subset of E, then certainly

$-M \leqq f(p) \leqq M$ on G. Let $\delta = \frac{\epsilon}{M}$. Then $\delta > 0$. Suppose that G is a measurable subset of E and that $\mu(G) < \delta$. Then

$$-\epsilon = -M \cdot \frac{\epsilon}{M} < -M \cdot \mu(G) \leqq \int_G f(p)d\mu \leqq M \cdot \mu(G) < \frac{M\epsilon}{M} = \epsilon \quad (2.4).$$

or in other words $\left| \int_G f(p)d\mu \right| < \epsilon$.

2.61 If f(p) is a non-negative, measurable and summable function on a measurable set E of finite measure, and if $\epsilon > 0$, then there is $\delta > 0$ such that if G is a measurable subset of E and if $\mu(G) < \delta$, then

$$\int_G f(p)d\mu < \epsilon .$$

Proof: $\int_E f(p)d\mu = \lim_{N\to\infty} \int_E f_N(p)d\mu$.

For each N, $\int_E f_N(p)d\mu \leqq \int_E f(p)d\mu$. (2.35),

$\therefore \int_E f(p)d\mu - \int_E f_N(p)d\mu \geqq 0$.

Choose an integer N such that $0 \leqq \int_E f(p)d\mu - \int_E f_N(p)d\mu < \frac{\epsilon}{2}$.

$f_N(p)$ is a bounded, non-negative, measurable function on E. There is a $\delta > 0$ such that if G is any measurable subset of E and if $\mu(G) < \delta$, then

$\left| \int_G f_N(p)d\mu \right| = \int_G f_N(p)d\mu < \frac{\epsilon}{2}$ (2.60). Let G be a measurable subset of E such that $\mu(G) < \delta$.

$$\int_E f(p)d\mu = \int_G f(p)d\mu + \int_{E-G} f(p)d\mu \quad (2.57) \text{ and}$$

$$\int_E f_N(p)d\mu = \int_G f_N(p)d\mu + \int_{E-G} f_N(p)d\mu \quad \text{for each } N \quad (2.55).$$

$$\int_E f(p)d\mu - \int_E f_N(p)d\mu = \int_G f(p)d\mu - \int_G f_N(p)d\mu + \int_{E-G} f(p)d\mu - \int_{E-G} f_N(p)d\mu .$$

$\int_{E-G} f(p)d\mu = \lim_{N\to\infty} \int_{E-G} f_N(p)d\mu$. By similar reasoning to that used above,

$$\int_{E-G} f(p)d\mu - \int_{E-G} f_N(p)d\mu \geq 0$$

$$0 \leq \int_G f(p)d\mu - \int_G f_N(p)d\mu \leq \int_E f(p)d\mu - \int_E f_N(p)d\mu < \frac{\epsilon}{2} \quad (2.57).$$

$$\int_G f(p)d\mu < \int_G f_N(p)d\mu + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

<u>2.62</u> If f(p) is a negative, measurable and summable function on a measurable set E of finite measure, and if $\epsilon > 0$, then there is a $\delta > 0$ such that if G is a measurable subset of E and if $\mu(G) < \delta$, then

$$\int_G f(p)d\mu > -\epsilon .$$

The proof of this theorem is similar to that of 2.61.

<u>2.63</u> If f(p) is a measurable and summable function on a measurable set E of finite measure, and if $\epsilon > 0$, then there is a $\delta > 0$ such that if G is a measurable subset of E and if $\mu(G) < \delta$, then $\left|\int_G f(p)d\mu\right| < \epsilon$.

Proof: Let $N = E_p[p \in E, f(p) < 0]$. Let $P = E_p[p \in E, f(p) \geq 0]$.

$$\int_E f(p)d\mu = \int_N f(p)d\mu + \int_P f(p)d\mu .$$

There is a $\delta_1 > 0$ such that $G \subset P$, G measurable, $\mu(G) < \delta$ implies

$$\left|\int_G f(p)d\mu\right| < \frac{\epsilon}{2} , \quad (2.61),$$

There is a $\delta_2 > 0$ such that $G \subset N$, G measurable $\mu(G) < \delta_2$ implies

$$\left|\int_G f(p)d\mu\right| < \frac{\epsilon}{2} . \quad (2.62),$$

Let $\delta = \min. \delta_1 , \delta_2$. Then if $G \subset E$, G is measurable, $\mu(G) < \delta$, it follows that

$$\int_G f(p)d\mu = \int_{G\cdot P} f(p)d\mu + \int_{G\cdot N} f(p)d\mu \text{ ; (2.57) and}$$

$$\left|\int_G f(p)d\mu\right| \leqq \left|\int_{G\cdot P} f(p)d\mu\right| + \left|\int_{G\cdot N} f(p)d\mu\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

<u>2.64</u> If f(p) is a measurable, summable function on a measurable set E of finite measure, and if B is any measurable subset of E, then f(p) is measurable and summable on B.

Proof: The fact that f(p) is measurable on B is obvious.

Let $P = E_p\left[p \in E,\ f(p) \geqq 0\right]$,

By 2.57 and 2.58

$$\int_{B\cdot P} f_N(p)d\mu \leqq \int_{E\cdot P} f_N(p)d\mu \leqq \int_E f(p)d\mu \text{ for each N. and}$$

$$\int_{B\cdot M} f_{-N}(p)d\mu \geqq \int_{E\cdot P} f_{-N}(p)d\mu \geqq \int_{E\cdot M} f_{-N}(p)d\mu \text{ for each N.}$$

$\left\{\int_{B\cdot P} f_N(p)d\mu\right\}$ is an increasing sequence bounded above, and hence

$$\lim_{N\to\infty} \int_{B\cdot P} f_N(p)d\mu = \int_{B\cdot P} f(p)d\mu \text{ exists,}$$

$\left\{\int_{B\cdot M} f_{-N}(p)d\mu\right\}$ is a decreasing sequence bounded below and hence

$$\lim_{-N\to-\infty} \int_{B\cdot M} f_{-N}(p)d\mu = \int_{B\cdot M} f(p)d\mu \text{ exists,}$$

Therefore, f(p) is summable on B.

<u>2.65</u> Let f(p) be a measurable, summable function on a measurable set E of finite measure. If A_1 and A_2 are disjoint, measurable subsets of E, then $\int_{A_1+A_2} f(p)d\mu = \int_{A_1} f(p)d\mu + \int_{A_2} f(p)d\mu$.

Proof: Let $B = A_1 + A_2$; $B \subset E$; B is a measurable set

$\therefore \mu(B) < +\infty$. f(p) is a measurable, summable function on b. (2.64)

$$\text{(p)}d\mu + \int_{A_2} f(p)d\mu = \int_{A_1+A_2} f(p)d\mu \text{ . (2.59)}$$

2.66 If f(p) is a measurable, summable function on a measurable set E of finite measure, and if $A_1, A_2, \ldots, A_n$ are disjoint, measurable subsets of E, then $\int_{\sum_{i=1}^{n}} f(p)d\mu = \sum_{i=1}^{n} \int_{A_i} f(p)d\mu$.

Proof: By induction on the number of sets A_n. The assertion is true if n = 1 or n = 2. (2.65)

Assume it is true when n = k. Suppose $A_1, A_2, \ldots, A_{k+1}$ are disjoint measurable subsets of E.

$$\int_{\sum_{i=1}^{k+1} A_i} f(p)d\mu = \int_{A_{k+1}} f(p)d\mu + \int_{\sum_{i=1}^{k}} f(p)d\mu = \int_{A_{k+1}} f(p)d\mu + \sum_{i=1}^{k} \int_{A_i} f(p)d\mu = \sum_{i=1}^{k+1} \int_{A_i} f(p)d\mu .$$

The first equality holds since the assertion is true when n = 2. Thus, the truth of the assertion for n = k implies it for n = k+1; hence it is true for all positive integral values of n.

2.67 Let f(p) be a measurable summable function on a measurable set E of finite measure. If $\{A_i\}$ is a sequence of disjoint measurable subsets of E, then $\int_{\sum_{i=1}^{\infty} A_i} f(p)d\mu = \sum_{i=1}^{\infty} \int_{A_i} f(p)d\mu$.

Proof: Let $A = \sum_{i=1}^{\infty} A_i$. Let $R_n = \sum_{i=n+1}^{\infty} A_i$ for each n.

$A = \sum_{i=1}^{n} A_i + R_n$. $\int_A f(p)d\mu = \sum_{i=1}^{n} \int_{A_i} f(p)d\mu + \int_{R_n} f(p)d\mu$. (2.59)

$\sum_{i=1}^{\infty} \int_{A_i} f(p)d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f(p)d\mu$, provided that this limit exists.

$\left| \int_A f(p)d\mu - \sum_{i=1}^{n} \int_{A_i} f(p)d\mu \right| = \left| \int_{R_n} f(p)d\mu \right|$; $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$.

Give $\epsilon > 0$. There is a $\delta > 0$ such that if G is any measurable subset of E, and if $\mu(G) < \delta$, then $\left| \int_G f(p)d\mu \right| < \epsilon$. (2.63)

There is an integer M such that if $n > M$, then

$\sum_{i=n+1}^{\infty} \mu(A_i) < \delta$. $\sum_{i=n+1}^{\infty} \mu(A_i) = \mu(R_n)$. If $n > M$, $\mu(R_n) < \delta$, and therefore, if $n > M$, $\left| \int_{R_n} f(p)d\mu \right| < \epsilon$. If $n > M$,

$\left| \int_A f(p)d\mu - \sum_{i=1}^{n} \int_{A_i} f(p)d\mu \right| < \epsilon$. Since ϵ is arbitrary,

$$\sum_{i=1}^{\infty} \int_{A_i} f(p)d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f(p)d\mu = \int_A f(p)d\mu = \int_{\sum_{i=1}^{\infty} A_i} f(p)d\mu .$$

2.68 If $g(p)$ is a bounded, Lebesgue integrable function on a measurable set E of finite measure, then $\left| \int_E g(p)d\mu \right| \leqq \int_E |g(p)| \, d\mu$.

Proof: $g(p)$ is a measurable function. Let $E_1 = E_p[\, p \in E, \; g(p) \geqq 0 \,]$. Let $E_2 = E_p[\, p \in E, \; g(p) < 0 \,]$. E_1 and E_2 are measurable sets. $E_1 \cdot E_2 = \emptyset$, $E_1 + E_2 = E$.

$\therefore \int_E g(p)d\mu = \int_{E_1} g(p)d\mu + \int_{E_2} g(p)d\mu$, (2.55) $g(p) = |g(p)|$ if $p \in E_1$; $g(p) = -|g(p)|$ if $p \in E_2$.

$\int_E g(p)d\mu = \int_{E_1} |g(p)| \, d\mu + \int_{E_2} -|g(p)| \, d\mu =$
$\int_{E_1} g(p)d\mu - \int_{E_2} |g(p)| \, d\mu$ (2.36). $\int_E |g(p)| d\mu = \int_{E_1} |g(p)| \, d\mu$
$\int_{E_2} |g(p)| d\mu$. $\int_{E_2} |g(p)| d\mu \geqq 0$. $\therefore \int_E g(p)d\mu \leqq \int_E |g(p)| d\mu$.

$-\int_E g(p)d\mu = \int_{E_2} |g(p)| \, d\mu - \int_{E_1} |g(p)| \, d\mu$.

$-\int_E g(p)d\mu \leqq \int_E |g(p)| \, d\mu$. $\int_E g(p)d\mu \geqq -\int_E |g(p)| \, d\mu$.

$-\int_E |g(p)| \, d\mu \leqq \int_E g(p)d\mu \leqq \int_E |g(p)| \, d\mu$.

$\left| \int_E g(p)d\mu \right| \leqq \int_E |g(p)| \, d\mu$.

2.69 If E is a measurable set of finite measure, if $\{f_n(p)\}$ is a sequence of bounded, measurable functions on E, and if $\{f_n(p)\}$ converges uniformly to f(p) on E, and if f(p) . is bounded on E, then f(p) is integrable on E and $\lim_{n\to\infty} \int_E fn(p)d\mu = \int_E f(p)d\mu$.

Proof: f(p) is measurable and bounded on E. (2.26) $\therefore$ f(p) is Lebesgue integrable on E. Give $\epsilon > 0$.

$$\left| \int_E (f_n(p) - f(p))\, d\mu \right| = \left| \int_E f_n(p)d\mu - \int_E f(p)d\mu \right| \quad ; \quad (2.37)$$

$$\left| \int_E (f_n(p) - f(p))\, d\mu \right| \leq \int_E |f_n(p) - f(p)|\, d\mu \, . \quad (2.69)$$

There exists an integer M such that if $n > M$, then

$|f_n(p) - f(p)| < \frac{\epsilon}{\mu(E)}$ for all points p in E.

$$\int_E |f_n(p) - f(p)|\, d\mu < \int_E \frac{\epsilon}{\mu(E)}\, d\mu = \frac{\mu(E)\cdot\epsilon}{\mu(E)} = \epsilon \, .$$

2.70 If E is a measurable set of finite measure, if $f_n(p)$ is a bounded, measurable function on E for each positive integer n, if f(p) is a bounded, measurable function on E, if $\lim_{n\to\infty} f_n(p) = f(p)$ on E, and if $\epsilon > 0$, then there exists a measurable set F such that $F \subset E$, $\mu(F) < \epsilon$, and such that $\lim_{n\to\infty} f_n(p) = f(p)$ uniformly on E - F.

Proof: Let $E_{mn} = E_p\left[p \in E, |f_n(p) - f(p)| < \frac{1}{2^m}\right]$.

Let $G_{mk} = \prod_{n=k}^{\infty} E_{mn}$ for fixed m. Let $E_m = \sum_{k=1}^{\infty} G_{mk} = \sum_{k=1}^{\infty} \prod_{n=k}^{\infty} E_{mn}$.

Then $E_m = \liminf_{n\to\infty} E_{mn} = E$ since $f_n(p)$ converges to f(p) at every point of E. (1.68) $\mu(E) = \liminf_{n\to\infty} \mu(E_{mn})$. (1.70). $\{G_{mk}\}$ is an increasing sequence of sets for fixed m. $E = \sum_{k=1}^{\infty} G_{mk}$. $\lim_{k\to\infty} \mu(G_{mk}) = \mu(E)$. (1.66).

Choose an integer k_m such that $\mu(G_{mk_m}) > \mu(E) - \frac{\epsilon}{2^m}$.

Let $F_m = E - G_{mk_m}$. Then $F_m + G_{mk_m} = E$. $\mu(F_m) < \frac{\epsilon}{2^m}$. Let $F = \sum_{m=1}^{\infty} F_m$.

Then $\mu(F) < \epsilon$. F is a measurable set. $F \subset E$. Give $\delta > 0$. We must find an integer L such that if $n > L$, then $|f_n(p) - f(p)| < \delta$ if $p \in E - F$.

Choose m so that $\frac{1}{m} < \delta$. Then $E - F \subset E - F_m \subset G_{mk_m}$. Let $L = k_m$.

If $n \geqq L$ and if $p \in E - F$, then $p \in G_{mk_m}$.

$G_{mk_m} = \prod_{n=k_m}^{\infty} E_{mn}$, $p \in E_{mn}$. $|f_n(p) - f(p)| < \frac{1}{2^m} < \delta$.

<u>2.71</u> If E is a measurable set of finite measure, if $f_n(p)$ is a bounded, measurable function on E for each n, if f(p) is a bounded measurable function on E, if $\lim_{n\to\infty} f_n(p) = f(p)$, if $0 \leqq f_n(p) \leqq K$ on E for each N, then

$$\lim_{n\to\infty} \int_E f_n(p)d\mu = \int_E f(p)d\mu .$$

Proof: Give $\epsilon > 0$. We must find an integer L such that if $n > L$, then $\left|\int_E f_n(p)d\mu - \int_E f(p)d\mu\right| < \epsilon$. $0 \leqq f(p) \leqq K$ on E. Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2K}$. Choose F such that F is a measurable set, $F \subset E$, $\mu(F) < \delta$, and $\lim_{n\to\infty} f_n(p) = f(p)$ uniformly on $E - F$. (2.70)

$$\left|\int_E f_n(p)d\mu - \int_E f(p)d\mu\right| = \left|\int_E (f_n(p) - f(p))d\mu\right| \leqq$$

$$\int_E |f_n(p) - f(p)|\, d\mu = \int_{E-F} |f_n(p) - f(p)|\, d\mu + \int_F |f_n(p) - f(p)|\, d\mu$$

(2.37, 2.68). Choose L such that if $n > L$ and if $p \in E - F$, then $|f_n(p) - f(p)| < \frac{\epsilon}{2\mu(E)}$. If $n > L$, then $\left|\int_E f_n(p)d\mu - \int_E f(p)d\mu\right| < \int_{E-F} \frac{\epsilon}{2\mu(E)} d\mu$ ·

F) + $K\mu(F) < \frac{\epsilon}{2} + K\delta < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. (2.55, 2.68, 2.3

2.72 If E is a measurable set of finite measure, if f(p) is measurable on E, if $f_n(p)$ is non-negative, bounded and measurable on E for each n, if $\lim\limits_{n\to\infty} f_n(p) = f(p)$ on E, and if $\int_E f_n(p)d\mu \leq Q$ for each n, then f(p) is summable on E and $\int_E f(p)d\mu \leq Q$.

Proof: Let $f^N(p) = \begin{cases} f(p) & \text{if } f(p) \leq N \\ N & \text{if } f(p) > N \end{cases}$

Let $f_n^N(p) = \begin{cases} f_n(p) & \text{if } f_n(p) \leq N \\ N & \text{if } f_n(p) > N \end{cases}$. We must show that

$\lim\limits_{N\to\infty} \int_E f^N(p)d\mu$ exists. $f(p) \geq 0$ on E. Consider $f^N(p)$ and $\left\{ f_n^N(p) \right\}_{n=1}^{\infty}$ for fixed N, $\lim\limits_{n\to\infty} f_n^N(p) = f^N(p)$. Then $\lim\limits_{n\to\infty} \int_E f_n^N(p)d\mu = \int_E f^N(p)d\mu$ by 2.71, but $\int_E f_n(p)d\mu \leq Q$ for each n. $\therefore$ $\int_E f_n^N(p)d\mu \leq Q$ and $\int_E f^N(p)d\mu \leq Q$.

$\therefore$ $\lim\limits_{N\to\infty} \int_E f^N(p)d\mu \leq Q$ and hence f(p) is summable on E and $\int_E f(p)d\mu \leq Q$.

CHAPTER III

RECTANGLE FUNCTIONS AND DERIVATIVES

3.1 Definition: A rectangle function is a real-valued function whose domain of definition is $\mathcal{P}$, the class of all oriented half-open rectangles.

3.2 Definition: A rectangle function ϕ will be said to be finitely additive if $R_1, R_2, \ldots, R_n$ belonging to $\mathcal{P}$ and $R_i \cdot R_j = \emptyset$ if $i \neq j$ imply that $\phi\left(\sum_{i=1}^{n} R_i\right) = \sum_{i=1}^{n} \phi(R_i)$, provided of course that $\sum_{i=1}^{n} R_i \in \mathcal{P}$.

3.3 Definition: A rectangle function ϕ will be said to be countably additive if $R_1, R_2, \ldots$ belonging to $\mathcal{P}$ and $R_i \cdot R_j = \emptyset$ if $i \neq j$ imply that $\phi\left(\sum_{i=1}^{\infty} R_i\right) = \sum_{i=1}^{\infty} \phi(R_i)$, provided that $\sum_{i=1}^{\infty} R_i \in \mathcal{P}$.

3.4 Definition: A rectangle function ϕ is said to be of Type A if ϕ is non-negative and if

$\sum_{i=1}^{n} R_i \subset R$, $R_i \cdot R_j = \emptyset$ if $i = j$ imply that

$\sum_{i=1}^{n} \phi(R_i) \leqq \phi(R)$.

3.5 If ϕ is a finitely additive and non-negative rectangle function, then ϕ is of Type A. That is, if $\sum_{i=1}^{n} R_i \subset R$, $R_i \cdot R_j = \emptyset$, if $i \neq j$, then $\sum_{i=1}^{n} \phi(R_i) \leqq \phi(R)$.

Proof: If $\sum_{i=1}^{n} R_i = R$, then $\sum_{i=1}^{n} \phi(R_i) = \phi(R)$ and we are finished. Suppose $\sum_{i=1}^{n} R_i \neq R$. $R = R_1 + \sum_{j=1}^{k} S_j$ where $S_j \in \mathcal{P}$, $R_1 \cdot S_j = \emptyset$, $S_i \cdot S_j = \emptyset$, if $i \neq j$. $\phi(R) = \phi(R_1) + \sum_{j=1}^{k} \phi(S_j)$, since ϕ is finitely additive.

$\sum_{i=1}^{n} R_i \subset R - R_1$, $\sum_{j=1}^{k} S_j = R - R_1$.

$$\left(\sum_{i=2}^{n} R_i\right)\left(\sum_{j=1}^{k} S_j\right) = \sum_{i=2}^{n} R_i = \sum_{i=2}^{n}\sum_{j=1}^{k} R_i S_j = \sum_{j=1}^{k}\sum_{i=2}^{n} R_i S_j$$

$S_j \cdot \sum_{i=2}^{n} R_i \subset S_j$, $\sum_{i=2}^{n} R_i \cdot S_j \subset S_j$.

The conclusion will be proved by induction. It is trivial in case $n = 1$. We shall assume its truth for all integers less than n.

Then $\sum_{i=2}^{n} \phi(R_i \cdot S_j) \leq \phi(S_j)$.

$\sum_{j=1}^{k} R_i \cdot S_j = R_i \sum_{j=1}^{k} S_j = R_i (R - R_1) = R_i$, since $R_i \subset R - R_1$.

$\therefore$ by finite additivity $\phi(R_i) = \sum_{j=1}^{k} \phi(R_i \cdot S_j)$ for each i.

$$\phi(R) \geq \phi(R_1) + \sum_{j=1}^{k}\sum_{i=2}^{n} \phi(R_i \cdot S_j) = \phi(R_1) + \sum_{i=2}^{n}\sum_{j=1}^{k} \phi(R_i \cdot S_j) =$$

$$\phi(R_1) + \sum_{i=2}^{n} \phi(R_i) = \sum_{i=1}^{n} \phi(R_i).$$

3.6 Definition. Suppose ϕ is a rectangle function. Let $S \in \mathcal{P}$, where S is a square. Then $\lim_{\substack{p_o \in S^o \\ A(S) \to 0}} \frac{\phi(S)}{A(S)} = \phi'(p_o)$, provided this limit exists and is finite. $\phi'(p_o)$ us called the two-dimensional derivative of ϕ at p_o. This definition implies that given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $A(S) < \delta$ and if $p_o \in S^o$, then $\left| \frac{\phi(S)}{A(S)} - \phi'(p_o) \right| < \epsilon$.

3.7 Definition: Let $\overline{D}(\phi, p_o)$ be the largest number 1 such that there exists a sequence S_n of oriented half-open squares, such that $p_o \in S_n^{\,o}$ for each n, $\lim_{n\to\infty} A(S_n) = 0$ and $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)} = 1$. For the purpose of this discussion may be $\pm\infty$. $\overline{D}(\phi, p_o)$ is called the upper derivative of ϕ at p_o.

3.8 Definition: Let $\underline{D}(\emptyset, p_o)$ be the smallest number l such that there exists a sequence $\{S_n\}$ of oriented half-open squares, such that $p_o \in S_n^o$ for each n, $\lim_{n\to\infty} A(S_n) = 0$ and $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)} = l$. Again l may be $\pm\infty$. $\underline{D}(\emptyset, p_o)$ is called the lower derivative of $\emptyset$ at p_o.

3.9 $-\infty \leq \underline{D}(\emptyset, p_o) \leq \overline{D}(\emptyset, p_o) \leq +\infty$

Proof: The proof follows immediately from the preceding definitions.

3.10 If $\emptyset$ is of Type A, then $0 \leq \underline{D} \leq \overline{D} \leq +\infty$.

Proof: $\emptyset(S) \geq 0$ for all $S \in \mathcal{P}$. $A(S) \geq 0$.

$\therefore \frac{\phi(S)}{A(S)} \geq 0$ for all S. Thus, it follows that $\underline{D} \geq 0$.

3.11 $\emptyset$ has a derivative $\emptyset'(p_o)$ at p_o if and only if, for every sequence $\{S_n\}$ of squares such that $p_o \in S_n^o$ for each n, and $\lim_{n\to\infty} A(S_n) = 0$, then $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)} = \emptyset'(p_o)$.

Proof: 1. Suppose $\emptyset$ has a derivative $\emptyset'(p_o)$ at p_o. Suppose $\{S_n\}$ is a sequence of squares such that $p_o \in S_n^o$ for each n, and $\lim_{n\to\infty} A(S_n) = 0$.

$\lim_{\substack{p_o \in S^o \\ A(S)\to 0}} \frac{\phi(S)}{A(S)} = \emptyset'(p_o)$. Give $\epsilon > 0$. There exists $\delta > 0$ such that if $A(S) < \delta$, $p_o \in S^o$ then $\left|\frac{\phi(S)}{A(S)} - \emptyset'(p_o)\right| < \epsilon$. There exists an integer m such that if $n > m$ then $A(S_n) < \delta$, $p_o \in S_n^o$. Then

$\left|\frac{\phi(S_n)}{A(S_n)} - \emptyset'(p_o)\right| < \epsilon$. This implies that $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)}$ exists and equals $\emptyset'(p_o)$.

2. Suppose for every sequence $\{S_n\}$ of squares such that $p_o \in S_n^o$ for each n and $\lim_{n\to\infty} A(S_n) = 0$, then $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)} = L$. Suppose $\emptyset'(p_o) \neq$

There exists $\epsilon_0 > 0$ such that no $\delta > 0$ works. In particular $\frac{1}{n}$ does not work for each n.

There exists S_1 such that $A(S_1) < 1$, $p_o \in S_1^{\circ}$ and $\left|\frac{\phi(S_1)}{A(S_1)} - L\right| \geqq \epsilon_0$.

There exists S_2 such that $A(S_2) < \frac{1}{2}$, $p_o \in S_2^{\circ}$ and $\left|\frac{\phi(S_2)}{A(S_2)} - L\right| \geqq \epsilon_0$

Continue this process.

There exists S_m such that $A(S_m) < \frac{1}{m}$, $p_o \in S_m^{\circ}$ and $\left|\frac{\phi(S_m)}{A(S_m)} - L\right| \geqq \epsilon_0$.

Continue indefinitely. We obtain a sequence $\{S_m\}$ such that $p_o \in S_m^{\circ}$ for each m, $\lim_{m\to\infty} A(S_m) = 0$, but $\lim_{m\to\infty} \frac{\phi(S_m)}{A(S_m)} \neq L$. This contradicts the hypothesis and hence we conclude that $\phi'(p_o) = L$.

<u>3.12</u> $\phi'(p_o)$ exists if and only if $\overline{D}(\phi,p_o)$ and $\underline{D}(\phi,p_o)$ are finite and equal.

Proof: 1. Suppose $\phi'(p_o)$ exists. Then for every sequence of squares $\{S_n\}$ such that $p_o \in S_n^{\circ}$ for each n and $\lim_{n\to\infty} A(S_n) = 0$, $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)} = \phi'(p_o) < |\infty|$. Then by definition $\overline{D}(\phi,p_o) = \underline{D}(\phi,p_o) = \phi'(p_o)$ and is finite.

2. Suppose $\overline{D}(\phi,p_o)$ and $\underline{D}(\phi,p_o)$ are finite and equal. Let $\{S_n\}$ be such that $p_o \in S_n^{\circ}$ for each n and $\lim_{n\to\infty} A(S_n) = 0$. Suppose $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)}$ does not exist. Let $q_n = \frac{\phi(S_n)}{A(S_n)}$ for each n. There exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ such that $\lim_{k\to\infty} q_{nk} = r$. Since $\lim_{n\to\infty} q_n$ does not exist there exists $\delta > 0$ such that infinitely many terms of $\{q_n\}$ do not belong to $N(r, \delta)$. These terms form a subsequence $\{q_{m_k}\}$ of $\{q_n\}$. There exists a subsequence $\{q_{m_{k_l}}\}$ of $\{q_{m_k}\}$ such that $\lim_{l\to\infty} q_{m_{k_l}}$ exists but is different

from r. $\lim_{k\to\infty} q_{n_k} = r$. $\lim_{l\to\infty} q_{m_{kl}} = t$. $t \neq r$. Since $\overline{D}(\emptyset,p_o)$ and $\underline{D}(\emptyset,p_o)$ are finite and equal to say Q, we know that $r = t = Q$. This is a contradiction and we conclude that $\lim_{n\to\infty} \frac{\phi(S_n)}{A(S_n)}$ does exist.

$\underline{3.13}$ Suppose $\emptyset$ and λ are two rectangle functions. Let $K = \emptyset + \lambda$, and suppose $\emptyset'(p_o)$ and $\lambda'(p_o)$ exist, then $K'(p_o) = \emptyset'(p_o) + \lambda'(p_o)$.

Proof: Give $\epsilon > 0$. $\lim_{\substack{p_o \in S^o \\ A(S)\to 0}} \frac{\phi(S)}{A(S)}$ exists and equals $\emptyset'(p_o)$. There exists $\delta_\phi > 0$ such that if $A(S) < \delta_\phi$ and $p_o \in S^o$, then $\left|\frac{\phi(S)}{A(S)} - \emptyset'(p_o)\right| < \frac{\epsilon}{2}$. $\lim_{\substack{p_o \in S^o \\ A(S)\to 0}} \frac{\lambda(S)}{A(S)}$ exists and equals $\lambda'(p_o)$. There exists $\delta_\lambda > 0$ such that if $A(S) < \delta_\lambda$ and $p_o \in S^o$, then $\left|\frac{\lambda(S)}{A(S)} - \lambda'(p_o)\right| < \frac{\epsilon}{2}$. Let $\delta = \min.\ \delta_\phi$ and δ_λ. If $A(S) < \delta$, then $\left|\frac{K(S)}{A(S)} - K'(p_o)\right| = \left|\frac{\phi(S)}{A(S)} + \frac{\lambda(S)}{A(S)} - (\emptyset'(p_o) + \lambda'(p_o))\right| \leq$

$$\left|\frac{\phi(S)}{A(S)} - \emptyset'(p_o)\right| + \left|\frac{\lambda(S)}{A(S)} - \lambda'(p_o)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\underline{3.14}$ Suppose $\emptyset$ is a rectangle function. Let $\beta = a\emptyset$ where a is any real number, and suppose $\emptyset'(p_o)$ exists. Then $\beta'(p_o) = a\emptyset'(p_o)$.

Proof: Give $\epsilon > 0$. There exists $\delta > 0$ such that if $A(S) < \delta$, $p_o \in S^o$, then $\left|\frac{\phi(S)}{A(S)} - \emptyset'(p_o)\right| < \frac{\epsilon}{|a|}$. $\left|\frac{a\phi(S)}{A(S)} - a\emptyset'(p_o)\right| =$

$|a|\left|\frac{\phi(S)}{A(S)} - \emptyset'(p_o)\right| < \epsilon$. $\therefore$ since $\beta(S) = a\emptyset(S)$, $\beta'(p_o)$ exists and equals $a\emptyset'(p_o)$.

$\underline{3.15}$ If $\emptyset'(p_o)$ exists, then $\lim_{\substack{p_o \in S^o \\ A(S)\to 0}} \emptyset(S) = 0$.

Proof: Give $\epsilon > 0$. Suppose $\epsilon_1 = 1$. There exists $\delta_1 > 0$ such that if $A(S) < \delta_1$ and $p_o \in S^o$, then $\left| \frac{\phi(S)}{A(S)} - \phi'(p_o) \right| < 1$

i.e. $A(S)(\phi'(p_o) - 1) < \phi(S) < A(S)(\phi'(p_o) + 1)$.

Let $M = \max. \; |\phi'(p_o) - 1|, \; |\phi'(p_o) + 1|$. Let $\delta =$

$\min. \; \delta_1, \frac{\epsilon}{M}$; $\delta > 0$. Suppose $A(S) < \delta$, $p_o \in S^o$. $|\phi(S)| <$

$\max. \; A(S) \; |\phi'(p_o) + 1|, \; A(S) \; |\phi'(p_o) - 1| =$

$A(S) \max. \; |\phi'(p_o) + 1|, \; |\phi'(p_o) - 1| = A(S) \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$.

$\therefore \lim\limits_{\substack{p_o \in S^o \\ A(S) \to 0}} \phi(S) = 0$.

3.16 If $\phi'(p_o)$ and $\lambda'(p_o)$ exist and if $K = \phi \cdot \lambda$, then $K'(p_o)$ exists and $K'(p_o) = 0$.

Proof: $\dfrac{K(S)}{A(S)} = \dfrac{\phi(S)\lambda(S)}{A(S)} = \phi(S) \cdot \dfrac{\lambda(S)}{A(S)}$.

The existence of $\phi'(p_o)$ implies $\lim\limits_{\substack{p_o \in S^o \\ A(S) \to 0}} \phi(S) = 0$.

$$\lim_{\substack{p_o \in S^o \\ A(S) \to 0}} \frac{K(S)}{A(S)} = \lim_{\substack{p_o \in S^o \\ A(S) \to 0}} \frac{\phi(S)\lambda(S)}{A(S)} = \lim_{\substack{p_o \in S^o \\ A(S) \to 0}} \phi(S) \lim_{\substack{p_o \in S^o \\ A(S) \to 0}} \frac{\lambda(S)}{A(S)} = 0 \cdot \lambda'(p_o) = 0.$$

3.17 Let $\mathcal{B}$ denote the class of Borel sets in the plane. Let $\mathcal{L}$ denote the class of Lebesgue measurable sets in the plane. Then $\mathcal{B} \subset \mathcal{L}$.

Proof: By definition $\mathcal{B}$ is the smallest class of sets in the plane which contains the open sets and which is closed under the formation of countable unions (sums) and countable intersections (products). Since $\mathcal{L}$ contains the open sets and is also closed under the formation of countable unions and intersections, (1.46, 1.47, 1.51), it follows that $\mathcal{B} \subset \mathcal{L}$.

3.18 Definition. A function ϕ defined on a set E will be said to be Borel measurable on E if for every real number a the set of points $E_p\left[p \in E,\ \emptyset(p) > a\right]$ is a Borel set.

3.19 The upper and lower derivatives are Borel measurable functions.

Proof: The proof will be given for the upper derivative. A similar proof will give the conclusion for the lower derivative.

Let a be any real number. Let S be a generic notation for an oriented square. For every pair of positive integers m and n, let E_{amn} be defined as follows.

$E_{amn} = \sum S^o$, where the summation is extended over those squares S for which $A(S) < \frac{1}{n}$, and $\frac{\phi(S)}{A(S)} > a + \frac{1}{m}$.

Let E_a denote the set of points p such that $\overline{D}(\phi, p) > a$

We shall verify the following identity.

$E_a = \sum_{m=1}^{\infty} \prod_{n=1}^{\infty} E_{amn}$. E_{amn} is an open set, since it is a sum of open sets.

Thus E_a is a Borel set and the conclusion will follow.

Suppose $p \in E_a$. $\overline{D}(\emptyset, p) > a$. There exists a sequence of oriented half-open squares $\{S_i\}$ such that for each i, $p \in S_i^o$, $\lim_{i \to \infty} A(S_i) = 0$ and $\lim_{i \to \infty} \frac{\phi(S_i)}{A(S_i)} > a$. Choose an integer m so that $a + \frac{1}{m} < \overline{D}(\emptyset, p)$. Let n be any positive integer. Then there exists an integer k such that if $i > k$, then $\frac{\phi(S_i)}{A(S_i)} > a + \frac{1}{m}$ and such that $A(S_i) < \frac{1}{n}$. Therefore we see that $p \in E_{amn}$ for a fixed m and any n.

$\therefore\ E_a \subset \sum_{m=1}^{\infty} \prod_{n=1}^{\infty} E_{amn}$.

Suppose $p \in \sum_{m=1}^{\infty} \prod_{n=1}^{\infty} E_{amn}$. There exists an integer m such that

$p \in \prod_{n=1}^{\infty} E_{amn}$. $p \in E_{am1}$ implies that there exists S_1 such that $A(S_1) < 1$, $\frac{\phi(S_1)}{A(S_1)} > a + \frac{1}{m}$, and $p \in S_1^{\circ}$. Continue this process.

$p \in E_{ami}$ implies that there exists S_i such that $A(S_i) < \frac{1}{i}$,

$\frac{\phi(S_i)}{A(S_i)} > a + \frac{1}{m}$, and $p \in S_i^{\circ}$.

Continue this process indefinitely.

We obtain a sequence $\{S_i\}$ such that $p \in S_i^{\circ}$ for each i, $\lim_{i\to\infty} A(S_i) = 0$ and $\frac{\phi(S_i)}{A(S_i)} > a + \frac{1}{m}$ for each i. There exists a subsequence $\{S_{i_k}\}$ of $\{S_i\}$ such that $\lim_{k\to\infty} \frac{\phi(S_{i_k})}{A(S_{i_k})} \geqq a + \frac{1}{m} > a$, $p \in S_{i_k}^{\circ}$, and $\lim_{k\to\infty} A(S_{i_k}) = 0$.

$\therefore$ $\overline{D}(\emptyset,p) > a$ and $p \in E_a$. $\sum_{m=1}^{\infty}\prod_{n=1}^{\infty} E_{amn} \subset E_a$ and hence $E_a = \sum_{m=1}^{\infty}\prod_{n=1}^{\infty} E_{amn}$.

3.20 Let R_o be a fixed, oriented half-open rectangle. $E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) = \underline{D}(\emptyset,p)\right]$ is a Borel set.

Proof: The following identity is easily verified.

$$E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) = \underline{D}(\emptyset,p)\right] = \prod_{n=1}^{\infty} E_p\left[p \in R_o^{\circ}, \underline{D}(\emptyset,p) \geqq \overline{D}(\emptyset,p) - \frac{1}{n}\right].$$

If we can show that $E_p\left[p \in R_o^{\circ}, \underline{D}(\emptyset,p) \geqq \overline{D}(\emptyset,p) - \frac{1}{n}\right]$ is a Borel set it will follow that $\prod_{n=1}^{\infty} E_p\left[p \in R_o^{\circ}, \underline{D}(\emptyset,p) \geqq \overline{D}(\emptyset,p) - \frac{1}{n}\right]$ is a Borel set.

$$E_p\left[p \in R_o^{\circ}, \underline{D}(\emptyset,p) \geqq \overline{D}(\emptyset,p) - \frac{1}{n}\right] = E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) - \underline{D}(\emptyset,p) \leqq \frac{1}{n}\right]$$

Let $\{r_k\}$ denote the sequence of rational numbers. Let a be any real number. If $E_p\left[\overline{D}(\emptyset,p) - \underline{D}(\emptyset,p) > a\right]$ is a Borel set, it is easy to show that $\complement E_p\left[\overline{D}(\emptyset,p) - \underline{D}(\emptyset,p) > a\right]$ is also a Borel set.

We shall verify that $E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) - \underline{D}(\emptyset,p) > a\right] =$

$$\sum_{k=1}^{\infty} E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) > r_k\right] \cdot E_p\left[\underline{D}(\emptyset,p) < r_k - a\right].$$

From the preceding theorem we know that for each k. $E_p\left[p \in R_o^{\circ}, \overline{D}(\emptyset,p) > r_k\right]$

and $E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k - a\right]$ are Borel sets and hence that

$\sum_{k=1}^{\infty} E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]\cdot E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k - a\right]$ is a Borel set.

Suppose $p_o\in \sum_{k=1}^{\infty} E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]\cdot$

$E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k -a\right]$. Then for some k, $p_o\in E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]$

and $p_o\in E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k -a\right]$. $\therefore$ $-\underline{D}(\emptyset,p_o)> a-r_k$, $\overline{D}(\emptyset,p_o)> r_k$.

$\overline{D}(\emptyset,p_o) - \underline{D}(\emptyset,p_o)> a$; $p_o\in E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p) - \underline{D}(\emptyset,p)> a\right]$.

$\sum_{k=1}^{\infty} E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]\cdot E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k -a\right]\subset$

$E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p) - \underline{D}(\emptyset,p)> a\right]$.

Suppose $p_o\in E_p\left[\overline{D}(\emptyset,p) - \underline{D}(\emptyset,p)> a\right]$, $\overline{D}(\emptyset,p_o)> \underline{D}(\emptyset,p_o)+ a$

There exists a rational number r_k such that

$\overline{D}(\emptyset,p_o) > r_k > a + \underline{D}(\emptyset,p_o)$. $\overline{D}(\emptyset,p_o) > r_k$, $\underline{D}(\emptyset,p_o)< r_k -a$.

$p_o\in E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]\cdot E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p_o)< r_k -a\right]$ for some k.

$\therefore$ $E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p) - \underline{D}(\emptyset,p)> a\right]\subset$

$\sum_{k=1}^{\infty} E_p\left[p\in R_o^o,\ \overline{D}(\emptyset,p)> r_k\right]\cdot E_p\left[p\in R_o^o,\ \underline{D}(\emptyset,p)< r_k -a\right]$

Thus the identity is established.

3.21 If R_o is a fixed, oriented half-open rectangle, and if E is the set of points p of R_o^o for which the derivative $\emptyset'(p)$ is defined, then E is a Borel set.

Proof: The set E is by definition the set of points p of R_o^o for which the following three conditions hold simultaneously.

1. $-\infty < \overline{D}(\emptyset,p) < +\infty$
2. $-\infty < \underline{D}(\emptyset,p) < +\infty$
3. $\overline{D}(\emptyset,p) = \underline{D}(\emptyset,p)$

Each of these three sets is a Borel set, hence E is the intersection of three Borel sets and is itself a Borel set. The set E may of course be empty, but $\emptyset$ is a Borel set (an open set).

3.22 Definition. A family $\mathcal{F}$ of closed oriented squares is said to be a Vitali covering of a set E, if $E \subset \sum_{G \in \mathcal{F}} G$, and if $p \in E$, there exists a sequence $\{S_n\}$ of squares of $\mathcal{F}$ such that $p \in S_n$ for each n and $\lim_{n\to\infty} A(S_n) = 0$.

3.23 If E is a bounded measurable set and if $\mathcal{F}$ is a Vitali covering of E, then there exists a countable sequence $\{S_n\}$ of disjoint squares of $\mathcal{F}$ such that $\mu(E - \sum_{n=1}^{\infty} S_n) = 0$.

Proof: Let U be a bounded open set containing E. Discard from $\mathcal{F}$ all sets not contained in U. Define $e(S) = \frac{1}{2}$ side of S for each set S in $\mathcal{F}$.

The sequence $\{S_n\}$ will be defined inductively. Choose S_1 arbitrarily. After having chosen the sets $S_1,\ldots,S_p$, it is possible that $\sum_{n=1}^{p} S_n$ contains all of E. In this case the proof is complete.

Otherwise, there will exist a point x_o of E not in $\sum_{n=1}^{p} S_n$ which is a closed set, being a finite sum of closed sets. $x_o \in \mathcal{C}(\sum_{n=1}^{p} S_n)$ which is open. $\therefore$ There exists $\delta > 0$ such that $N(x_o, \delta) \subset \mathcal{C}(\sum_{n=1}^{p} S_n)$. There exists $\{S_n'\}$ where $S_n' \in \mathcal{F}$ for each i such that $\lim_{n\to\infty} A(S_n') = 0$ and $x_o \in S_n'$ for each n. $\therefore$ all but a finite number of the squares of this sequence are contained in $N(x_o, \delta)$. Thus there exist infinitely many squares S_n' such that $S_n' \cdot \sum_{n=1}^{\infty} S_n = \emptyset$. Let ϵ_{p+1} be l.u.b. $e(S_n')$ for S_n'

fulfilling this condition. Choose S_{p+1} to be a set of $\mathcal{F}$ having no points in common with $\sum_{n=1}^{p} S_n$ and such that $e(S_{p+1}) > \frac{\epsilon_{p+1}}{2}$. This inductively exhibits a countable sequence of sets $\{S_n\}$. We must show that this is the sequence which satisfies the conditions of the theorem.

$\sum_{n=1}^{\infty} S_n \subset U$. $S_i \cdot S_j = \emptyset$ if $i \neq j$, from the method of selection of the sets of $\{S_n\}$. We must show that $\mu(E - \sum_{n=1}^{\infty} S_n) = 0$.

Deny this. Suppose $\mu(E - \sum_{n=1}^{\infty} S_n) > 0$. Let x_n be the center point of the square S_n for each n. Consider the square S_n^* having center x_n and such that $e(S_n^*) = 5e(S_n)$. $\mu(S_n^*) = 5^2 \mu(S_n)$.

The series $\sum_{n=1}^{\infty} \mu(S_n)$ converges, since $\{S_n\}$ is a sequence of disjoint closed sets all contained in a set U of finite measure.

$\therefore \sum_{n=1}^{\infty} \mu(S_n^*)$ also converges. Since $\mu(E - \sum_{n=1}^{\infty} S_n) > 0$, there exists an integer N such that $\sum_{n=N+1}^{\infty} \mu(S_n^*) < \mu(E - \sum_{n=1}^{\infty} S_n)$

$\mu(\sum_{n=N+1}^{\infty} S_n^*) \leq \sum_{n=N+1}^{\infty} \mu(S_n^*) < \mu(E - \sum_{n=1}^{\infty} S_n)$. (1.26).

$\therefore$ $E - \sum_{n=1}^{\infty} S_n \not\subset \sum_{n=N+1}^{\infty} S_n$. (1.20). There exists x_o such that $x_o \in E - \sum_{n=1}^{\infty} S_n$ and $x_o \notin \sum_{n=N+1}^{\infty} S_n^*$. $x_o \notin \sum_{n=1}^{\infty} S_n$, $x_o \in E$.

As previously there exists $\delta > 0$ such that $N(x_o, \delta) \cdot \sum_{n=1}^{N} S_n = \emptyset$.

Again we choose a set $S \in \mathcal{F}$, such that $x_o \in S$ and such that

$S \cdot \sum_{n=1}^{N} S_n = \emptyset$.

This leaves two cases; either the set S has a point in common with some S_n, $n > N$, or it has not.

Case 1. Suppose the set S has no point in common with any S_n. For each integer p, $S \cdot \sum_{n=1}^{p} S_n = \emptyset$. Let ϵ_{p+1} be the l.u.b. of $e(S')$ for all $S' \in \mathcal{F}$ and such that $S' \cdot \sum_{n=1}^{p} S_n = \emptyset$. $\epsilon_{p+1} \geqq e(S)$. By the law of formation of $\{S_n\}$, $e(S_{p+1}) > \frac{e(S)}{2}$.

$e(S_{p+1}^*) = 5e(S_{p+1}) > \frac{5e(S)}{2}$. $\therefore$ the side of S_{p+1}^* is greater than $5e(S)$.

$\mu(S_{p+1}^*) > (5e(S))^2$. $(5e(S))^2$ is a positive number independent of p.

This is a contradiction since the series $\sum_{n=1}^{\infty} \mu(S_n^*)$ converges.

$\therefore \mu(E - \sum_{n=1}^{\infty} S_n) = 0$.

Case 2. Suppose there is an n such that S_n has a point in common with S.

Let $p+1$ be the least integer such that S_{p+1} and S have a point in common, let $\bar{x} \in S \cdot S_{p+1}$. From the above $p+1$ cannot be any integer $1, 2, \ldots, N$, i.e. $p \geqq N$.

Since $S \in \mathcal{F}$ and $S \cdot \sum_{n=1}^{p} S_n = \emptyset$, $\epsilon_{p+1} \geqq e(S)$.

$\therefore e(S_{p+1}) > \frac{e(S)}{2}$. $\bar{x}$ and x_o both belong to S.

Let $\bar{x} = (\bar{a}, \bar{b})$ and $x_o = (a_o, b_o)$. Then $|a_o - \bar{a}| \leqq 2e(S)$ and $|b_o - \bar{b}| \leqq 2e(S)$. $\bar{x} \in S_{p+1}$. If x_{p+1} is the center of S_{p+1} and $x_{p+1} = (a_{p+1}, b_{p+1})$, $|\bar{a} - a_{p+1}| \leqq e(S_{p+1})$ and $|\bar{b} - b_{p+1}| \leqq e(S_{p+1})$.

$$|a_o - a_{p+1}| \leqq |a_o - \bar{a}| + |\bar{a} - a_{p+1}| \leqq 2e(S) + e(S_{p+1}) < 5e(S_{p+1})$$

$$|b_o-b_{p+1}| \leq |b_o-\bar{b}|+|\bar{b}-b_{p+1}| \leq 2e(S) + e(S_{p+1}) < 5e(S_{p+1})$$

The last two inequalities imply that $x_o \in S_{p+1}^*$, but $p+1 > N$ and this contradicts a previous condition on x_o. $\therefore$ again $\mu(E-\sum_{n=1}^{\infty} S_n) = 0$.

<u>3.24</u> If R_o is an oriented half-open rectangle, and if ϕ is of type A in R_o, then its derivative $\phi'(p)$ exists almost everywhere in R_o and is summable in R_o.

Furthermore, for every oriented rectangle $R \subset R_o$ we have the inequality $\int_R \phi'(p)d\mu \leq \phi(R)$.

Proof: The proof will be based on several preliminary statements.

(a) Let α be a positive number, and let E_α be the subset of R_o^o where $\bar{D}(\phi,p) > \alpha$. Then $\alpha\mu(E_\alpha) \leq \phi(R_o)$.

Proof: Let $\mathcal{F}$ be the family of those oriented closed squares S that satisfy the following conditions: $S \subset R_o^o$, $\frac{\phi(S)}{A(S)} > \alpha$. It is clear that the squares of $\mathcal{F}$ form a Vitali covering for E_α. (3.22) Hence there are a countable number of squares of $\mathcal{F}$, $\{S_n\}$ such that $S_i \cdot S_j = \emptyset$ if $i \neq j$ and $\mu(E - \sum_{n=1}^{\infty} S_n) = 0$. (3.23) Since ϕ is of type A, it follows that for every positive integer k the inequality $\phi(R_o) \geq \phi(S_1)+\phi(S_2)+ \dots + \phi(S_k) > \alpha(\mu(S_1)+\mu(S_2)+\dots+\mu(S_k))$ holds. (3.4)

Since $\sum_{n=1}^{\infty} S_n$ and E_α are measurable sets, it follows that

$$\mu(E_\alpha) = \mu(E_\alpha \cdot \sum_{n=1}^{\infty} S_n)+\mu(E \cdot \mathcal{C}\sum_{n=1}^{\infty} S_n) =$$

$$\mu(E_\alpha \cdot \sum S_n)+\mu(E - \sum_{n=1}^{\infty} S_n) = \mu(E_\alpha \cdot \sum_{n=1}^{\infty} S_n) \quad (1.33).$$

$\cdot S_n)$

$$\sum_{n=1}^{\infty}\mu(S_n) \geqq \sum_{n=1}^{\infty}\mu(E_\alpha \cdot S_n) = \mu(E_\alpha)$$

$\therefore\ \phi(R_o) \geqq \alpha\sum_{n=1}^{\infty}\mu(S_n) \geqq \alpha\mu(E_\alpha)$, which is obtained from the above by letting k tend to infinity.

(b) Since ϕ is of type A in every oriented rectangle, $R \subset R_o$ also, (a) implies the inequality $\alpha\mu(E_\alpha \cdot R) \leqq \phi(R)$ for all such rectangles R.

(c) Let E^* be the subset of R_o^o where $\overline{D}(\phi,p) = +\infty$. Then $\mu(E^*) = 0$. That is $\overline{D}(\phi,p) < +\infty$ almost everywhere in R_o.

Proof: $E^* \subset E_\alpha$ for all $\alpha > 0$. $\mu(E_\alpha) \leqq \dfrac{\phi(R_o)}{\alpha}$ from (a). Give $\epsilon > 0$. Choose α so that $\alpha > \dfrac{\phi(R_o)}{\epsilon}$. $\mu(E^*) \leqq \mu(E_\alpha) \leqq \dfrac{\phi(R_o)}{\alpha} < \epsilon$.

$\therefore\ \mu(E^*) = 0$.

(d) The subset E_* of R_o^o where $\underline{D}(\phi,p) < \overline{D}(\phi,p)$ is of measure zero.

Proof: Deny. Suppose $\mu(E_*) > 0$. Then there exist rational numbers $0 < x < y$ such that the subset E_{xy} of R_o where $\underline{D}(\phi,p) < x < y < \overline{D}(\phi,p)$ is of positive measure. Give $\epsilon > 0$. There exists an open set G such that $E_{xy} \subset G \subset R_o^o$ and $\mu(G) < \mu(E_{xy}) + \epsilon$. (1.72). Let $\mathcal{F}$ denote the family of oriented closed squares S in G such that $\phi(S)/A(S) < x$. Clearly, the squares constitute a Vitali covering of E_{xy}. (3.22) Hence $\mathcal{F}$ contains a countable sequence $\{S_n\}$ of disjoint squares such that $\mu(E_{xy} - \sum_{n=1}^{\infty} S_n) = 0$. (3.23). We obtain the following inequalities.

$$\sum_{n=1}^{\infty}\phi(S_n) < x\sum_{n=1}^{\infty}\mu(S_n) \leqq x\mu(G) < x(\mu(E_{xy}) + \epsilon).$$

From (b) we have $\displaystyle\sum_{n=1}^{\infty}\phi(S_n) \geqq y\sum_{n=1}^{\infty}\mu(E_y \cdot S_n) \geqq$ $\displaystyle y\sum_{n=1}^{\infty}\mu(E_{xy} \cdot S_n) = y\mu(E_{xy})$. We notice that while each square S_n was originally taken to be closed we may replace it by its corresponding half-open square in the above inequalities, since this merely entails deleting in each case a set of measure 0. Since ϵ was arbitrary it follows that $x\mu(E_{xy}) \geq y\mu(E_{xy})$. Since $\mu(E_{xy})$ was assumed to be positive, we have $x \geq y$ which is a contradiction. Therefore, we conclude that $\mu(E_{xy}) = 0$, and hence that $\mu(E_*) = 0$.

(c) and (d) together imply that $\phi'(p)$ exists almost everywhere in R_o^o, and this proves the first part of the theorem.

Let us denote, for each positive integer n and each point $p = (u,v)$ in R_o, by $\mathcal{K}_n$ the collection of all squares $S \subset R_o$ of the form $(i-1)/n \leq u \leq i/n$, $(j-1)/n \leq v \leq j/n$ where i,j are integers (positive, negative, or zero). For given n, the collection $\mathcal{K}_n$ is finite, since R_o is bounded. Let us replace each square $S_n \in \mathcal{K}_n$ by a somewhat smaller oriented square S_{-n} having the same center, such that $\sum_{S_n \in \mathcal{K}_n} \mu(S_n - S_{-n}) < \frac{1}{n}$.

Let G_n denote the set of interior points of all the squares S_{-n} for given n. G_n is an open set and $\lim_{n\to\infty} \mu(R_o - G_n) = 0$. We have a subsequence $\{G_{n_k}\}$ of $\{G_n\}$ such that $\sum_{k=1}^{\infty} \mu(R_o - G_{n_k}) < +\infty$. Let $F_m = \prod_{k=m}^{\infty} G_{n_k}$.

Then $\lim_{m\to\infty} \mu(R_o - F_m) = 0$. Let us define for each positive integer k, a function $g_k(p)$ in R_o as follows. If p is an interior point of some square S_{-n_k}, then $g_k(p) = \phi(S_{-n_k})/A(S_{-n_k})$. Otherwise $g_k(p) = 0$. Clearly, since ϕ is of type A, $\int_{R_o} g_k(p)d\mu \leq \phi(R_o)$.

Let m be a positive integer and let p be a point of F_m such that $\phi'(p)$ exists. Then $p \in G_{n_k}$ for $k \geq m$ and hence $g_k(p)$ is equal to a quotient of the form $\phi(S)/A(S)$ where S is one of the squares S_{-n_k} and p is an interior point of S. Hence $\lim_{k\to\infty} g_k(p) = \phi'(p)$. Since $\phi'(p)$

exists almost everywhere in R_o, it follows that $\lim_{k\to\infty} g_k(p) = \phi'(p)$ almost everywhere on F_m, $m = 1,2,\ldots$ Since $\lim_{m\to\infty} \mu(R_o - F_m) = 0$, it follows that $\lim_{k\to\infty} g_k(p) = \phi'(p)$ almost everywhere in R_o. Since $g_k(p)$ is a non-negative measurable function in R_o, from 2.72 we conclude that

$$\int_{R_o} \phi'(p)\,d\mu \leqq \phi(R_o).$$

Since ϕ is of type A in every oriented half-open rectangle $R \subset R_o$, we can replace R_o by any such rectangle R and the proof is complete.

The theory presented in this chapter does not depend upon the dimensionality involved. Whereas it has been presented in the two-dimensional case, it generalizes immediately to the one-dimensional case.

In this case we should consider interval functions, i.e. functions whose domain of definition is the class of half-open intervals of the form $E_x[a \leqq x < b]$, indicated $[a,b)$.

We would define the one-dimensional derivative as follows. If I is a half-open interval, then $\phi'(x) = \lim_{\substack{x \in I^o \\ l(I)\to 0}} \frac{\phi(I)}{l(I)}$ provided that this limit exists, where ϕ is an interval function and $l(I)$ denotes the length of I.

If $f(x)$ is an increasing function of a real variable, and if $I = [a,b)$ then we can define a function $\phi(I) = f(b)-f(a)$. It is easily seen that an interval function thus defined is of type A. We may apply 3.24 to conclude that if I_o is a fixed half-open interval, then $\phi'(x)$ exists at almost every point x of I_o.

$\phi'(x)$ thus defined has a direct application to the ordinary derivative of differential calculus. $\phi'(x_o)$ is called the straddling derivative of

$f(x)$ at x_o. We shall explicitly define the straddling derivative and then prove two theorems which will show its relationship to the ordinary derivative of calculus.

3.25 Definition. $f_s'(x_o)$, the straddling derivative of $f(x)$ at x_o is defined as $\lim\limits_{\substack{x_1 \to x_o \\ x_2 \to x_o \\ x_2 > x_o \\ x_1 < x_o}} \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$, provided that this limit exists. $f(x)$ is not here assumed increasing. It is easily seen that this definition is equivalent to that given above.

3.26 If $f(x)$ has a derivative at x_o, then $f(x)$ has a straddling derivative at x_o, and the two derivatives are equal.

Proof: Give $\epsilon > 0$. Let $f'(x_o)$ denote the derivative of $f(x)$ at x_o. The derivative is independent of the manner in which x approaches x_o.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{x_2-x_o}{x_2-x_1} \cdot \frac{f(x_2)-f(x_o)}{x_2-x_o} + \frac{x_o-x_1}{x_2-x_1} \cdot \frac{f(x_1)-f(x_o)}{x_1-x_o}$$

Choose $\delta > 0$ so that $0 < |x-x_o| < \delta$ implies $\left|\dfrac{f(x)-f(x_o)}{x-x_o} - f'(x_o)\right| < \dfrac{\epsilon}{2}$.

Then, if $x_o < x_2 < x_o + \delta$, and if $x_o - \delta < x_1 < x_o$, we have

$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1} - f'(x_o)\right| = \left|\left(\frac{f(x_2)-f(x_o)}{x_2-x_o} - f'(x_o)\right)\left(\frac{x_2-x_o}{x_2-x_1}\right) + \left(\frac{f(x_1)-f(x_o)}{x_1-x_o} - f'(x_o)\right)\left(\frac{x_o-x_1}{x_2-x_1}\right)\right|$$

$$\leq \left|\frac{f(x_2)-f(x_o)}{x_2-x_o} - f'(x_o)\right|\left|\frac{x_2-x_o}{x_2-x_1}\right| + \left|\frac{f(x_1)-f(x_o)}{x_1-x_o} - f'(x_o)\right|\left|\frac{x_o-x_1}{x_2-x_1}\right| < \epsilon .$$

Note that $\dfrac{x_2-x_o}{x_2-x_1} + \dfrac{x_o-x_1}{x_2-x_1} = 1$, $\left|\dfrac{x_2-x_o}{x_2-x_1}\right| < 1$, and $\left|\dfrac{x_o-x_1}{x_2-x_1}\right| < 1$.

3.27 If f(x) has a straddling derivative at x_o and is continuous at x_o, then f(x) has a derivative at x_o and the derivatives are equal.

Proof: Give $\epsilon > 0$. There exists $\delta > 0$ such that if $x_o < x_2 < x_o + \delta$ and $x_o - \delta < x_1 < x_o$ then

$$\left| \frac{f(x_2)-f(x_1)}{x_2-x_1} - f_s'(x_o) \right| < \epsilon .$$

Let $x = x_1$. Then $f_s'(x_o) - \epsilon < \dfrac{f(x_2)-f(x)}{x_2-x} < f_s'(x_o) + \epsilon$

$$f_s'(x_o) - \epsilon \leqq \lim_{\substack{x \to x_o \\ x < x_o}} \frac{f(x_2)-f(x)}{x_2-x} \leqq f_s'(x_o) + \epsilon$$

$$f_s'(x_o) - \epsilon \leqq \frac{f(x_2)-f(x_o)}{x_2-x_o} \leqq f_s'(x_o) + \epsilon$$

Similarly, let $x = x_2$.

$$f_s'(x_o) - \epsilon < \frac{f(x)-f(x_1)}{x-x_1} < f_s'(x_o) + \epsilon$$

$$f_s'(x_o) - \epsilon \leqq \frac{f(x_o)-f(x_1)}{x_o-x_1} \leqq f_s'(x_o) + \epsilon$$

$\therefore$ if $q \neq x_o$ and if $|q - x_o| < \delta$, then $\left| \dfrac{f(q)-f(x_o)}{q-x_o} - f_s'(x_o) \right| < \epsilon$

and we see that this implies that

$$\lim_{x \to x_o} \frac{f(x)-f(x_o)}{x-x_o} = f'(x_o) \text{ exists and is equal to } f_s'(x_o).$$

If we restrict f(x) to be an increasing function and define ϕ as before, we can obtain a final conclusion. It is known that if f(x) is defined on $[a,b]$, then f(x) is continuous at all but perhaps a countable set of points.[1] Since the straddling derivative exists almost everywhere on $[a,b)$ and since the set of discontinuities is a set of measure 0, it follows that f(x) is differentiable at almost every point of $[a,b]$.

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