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Operations on graphs and matroids

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OPERATIONS ON GRAPHS AND MATROIDS

by

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Two graph operations, edge slides and K_3-moves, are defined and investigated. Under certain circumstances, it is possible to transform one graph into another by the repeated application of these operations. Investigating edge slides leads to a natural metric on the space of graphs. This distance measure is considered within the context of combinatorial Gray codes. Gray code enumerations of some small classes of trees are given. The K_3-move leads to a partial ordering, $\preceq$, of graphs and later to a partial ordering of matroids when the operation is suitably generalized. The resulting posets are very diverse in the sense that any poset may be embedded in one of them. A fundamental question for two matroids G and H is whether $G \preceq H$. This question is considered from a number of different perspectives, and is shown to have some implications for the matroid dual operator.

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List of notation

P_n the path on n vertices

S_n the star on n vertices

C_n the cycle on n vertices

K_n the complete graph on n vertices

$K_{m,n}$ the complete bipartite graph with parts of size m and n

W_{n-1} the wheel on n vertices ($n-1$ spokes)

$N_G(x)$ the vertex neighborhood of $x \in V(G)$, the subscript omitted under clear context

$A_{n,e}$ the isomorphism classes of connected graphs on n vertices and e edges

$\mathcal{G}_n$ the isomorphism classes of connected graphs on n vertices

$M(G)$ the cycle matroid of the graph G

M^* the dual of a matroid M

cl_M the closure operator of a matroid M

$PG(r-1, q)$ the projective geometry of rank r and order q

$U_{m,n}$ the rank-m uniform matroid on n elements

$\binom{n}{m}$ for integers n and m, the number of m element subsets of an n element set

$\binom{A}{m}$ the collection of subsets of size m from a set A

Contents

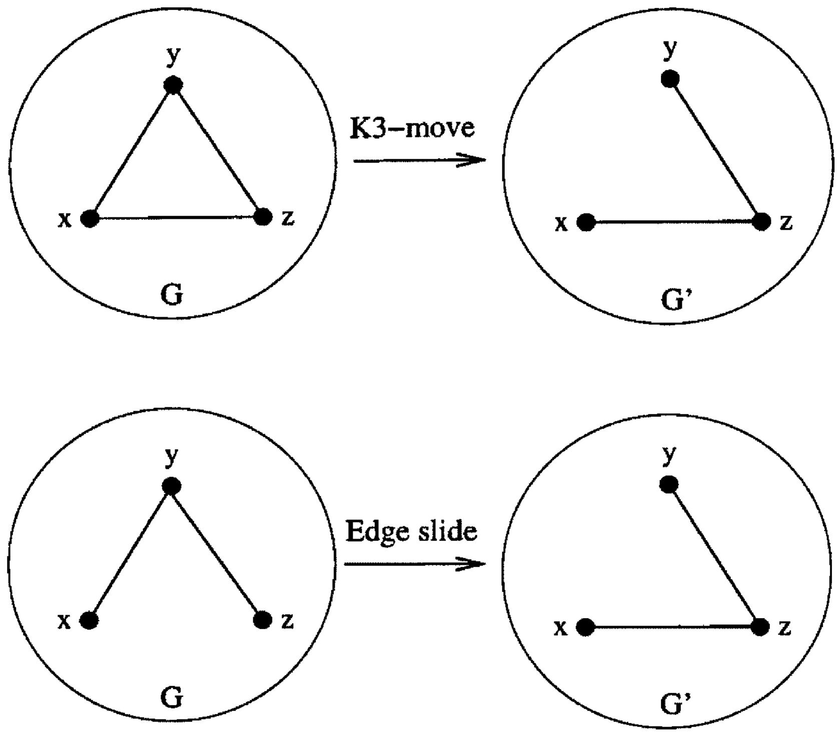
Chapter 1

Introduction

Combinatorialists count discrete objects and investigate their structure. In this thesis, we do some counting, but mostly investigate structure. The objects of interest are graphs and matroids. Their structure is investigated to the extent that it sheds light on the relationship between the two. Questions like "how 'far apart' are two graphs?" or "does this matroid 'come from' this other matroid?" are investigated by defining meaningful metrics and meaningful partial orders on the objects.

The tools for building metrics and partial orders in this thesis are operations on graphs and matroids. Two operations in particular, the "K_3-move" and "edge slide", are refined versions of the common graph operations, edge deletion and edge transfer. Since operations allow the creation of new objects from old, we can indeed ask questions like the ones above. If one object is realized by an operation on some other object, then there must be something important and interesting about the structure of the two.

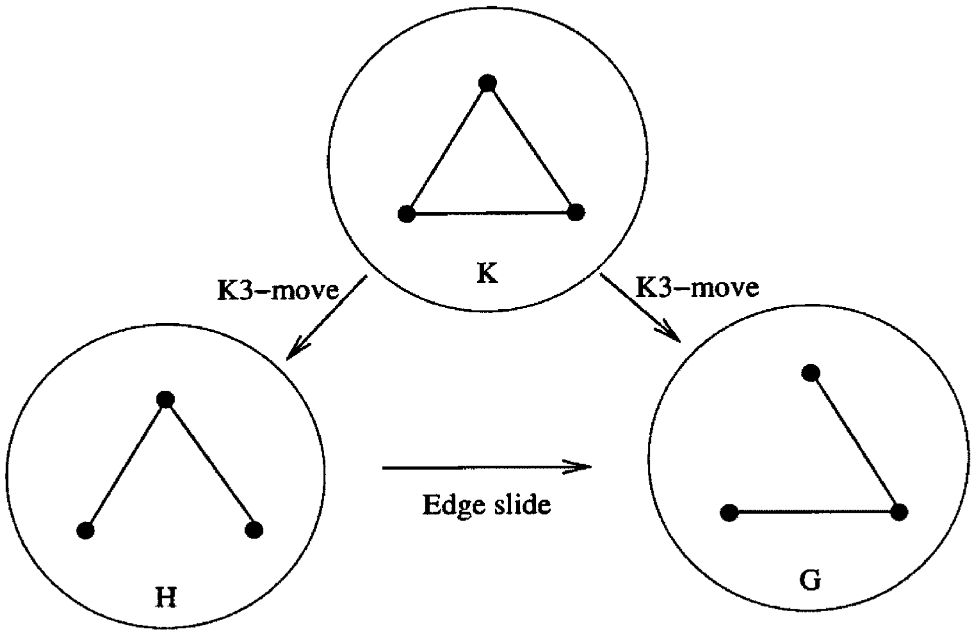
Consider the following graph operation, whereby a graph G is transformed into another graph G': given edges xy and yz of G, put $E' = (E(G) \setminus xy) \cup xz$ and put $G' = (V(G), E')$.



Outcomes of the operation: K_3-moves and edge slides

Clearly, the outcome of such an operation depends on whether the edge xz is in G. If $xz \in E(G)$, then the operation is referred to as a *K_3-move.* Otherwise, it is referred to as an *edge slide.* Edge slides fall under a general category of operations known as edge transfers, whereas K_3-moves are a type of edge deletion. As such, both operations are highly restrictive. Edge slides were introduced by Johnson in [11], and have been investigated by Zelinka [20], Jarrett [10], and Chartrand, et al. [4].

The definition of the edge slide operation may be recast solely in terms of K_3-moves and their inverses. If G is obtained from H by an edge slide, then there exists a graph K such that G and H can be obtained from K by a K_3-move. We can say that K is obtained from G and H by an *inverse K_3-move*, where this inverse is defined in the natural way.



Edge slides recast in terms of K_3-moves

In Chapter 2, the relationship between edge slides and K_3-moves is thoroughly investigated. Under certain circumstances, it is possible to transform one graph into another by means of edge slides and K_3-moves.

The remainder of the thesis is divided into two general lines of investigation, one exclusively in edge slides (Chapters 3 and 4) and one exclusively in K_3-moves (Chapter 5).

The discussion of edge slides focuses primarily on an associated metric for graphs. Here, we investigate distance between graphs and are led into the realm of combinatorial Gray codes. We exhibit lists of some small classes of trees, in which consecutive members differ by an edge slide.

The edge rotation is a generalization of the edge slide. Given a collection X of graphs, the edge rotation distance graph $D_{ER}(X)$ of X is the graph with vertex set X, wherein two graphs are adjacent in $D_{ER}(X)$ if and only if they differ by an edge rotation. We show that every graph is homeomorphic to an edge rotation distance graph, giving a partial answer to a conjecture of Chartrand, et al.

The consideration of K_3-moves leads to a partial ordering, $\preceq$, of graphs. The set $\mathcal{G}_n$ of connected graphs on n vertices, equipped with the partial order $\preceq$, is an interesting poset, and we take up several natural and attractive questions about its structure. It is shown that an arbitrary poset is isomorphic to a subposet of $(\mathcal{G}_n, \preceq)$, for n sufficiently large. Bounds on the minimum n needed to do so are given.

An interesting problem is the identification of a graph G of minimum size such that $T \preceq G$ for every tree T on n vertices. These graphs are completely determined for small values of n.

In Section 5.5, the K_3-move operation is extended to an operation on matroids. Using this operation, the partial order previously defined for graphs is extended to matroids. For each matroid M, a property characterizing the subsets $I \subseteq E(M)$ for which $M \setminus I \preceq M$ is given. It is conjectured that the collection of subsets of a matroid M having this property are the independent sets of a matroid on $E(M)$. This conjecture is verified for several classes of matroid and is shown to have a connection to the matroid dual operator.

Relevant background on graphs can be found in "Modern Graph Theory", (Bollobás, [2]). For background on matroids, consult "Matroid Theory", (Oxley, [14]). Stanley's "Enumerative Combinatorics" provides a useful introduction to partially ordered sets (posets), (Stanley, [18]). All graphs are finite, simple, and connected, and all matroids are finite and simple, unless otherwise stated.

Chapter 2

Preliminary results

The purpose of this chapter is to investigate the close relationship between edge slides and K_3-moves. Provided certain conditions are met, it is possible to transform one graph into another using these two operations. It should be worthwhile to exploit these properties as a general proof technique in graph theory.

2.1 Edge slides and K_3-moves: a numerical link

We have seen that edge slides and K_3-moves are both special cases of a more general operation on the edge set of a graph. Moreover, we have observed that while one is a restricted edge transfer, the other is a restricted edge deletion.

The two operations enjoy another, numerical relationship. An edge slide operates on a vertex induced subgraph isomorphic to P_3, and a K_3-move operates on a vertex induced subgraph isomorphic to K_3. Had the edge slide operation not already been proposed and named, we would have chosen to refer to it as a "P_3-move".

In an arbitrary graph G, we will count vertex induced subgraphs isomorphic to P_3, and those isomorphic to K_3. For each vertex $x \in V(G)$, let $d(x)$ denote its degree and $N(x)$ denote its vertex neighborhood.

Let $P_3(G)$ and $K_3(G)$ denote the vertex induced subgraphs of G isomorphic to P_3

and K_3, respectively. The following identities hold:

$$(1)\ \ |P_3(G)| = \sum_{xy\in E(G)} (|[N(x) \cup N(y)] \setminus [N(x) \cap N(y)]| - 2)/2, \text{ and}$$

$$(2)\ \ |K_3(G)| = \sum_{xy\in E(G)} |N(x) \cap N(y)|/3.$$

Technical note: As edges in a graph, xy and yx are indistinguishable.

The scale factor of 1/2 in item (1) adjusts for double counting contributed by the two edges in each P_3 of G. The factor of 1/3 in item (2) serves a similar purpose.

Although the computing formulas given for $|P_3(G)|$ and $|K_3(G)|$ may not be very enlightening, they lead to a rather pleasant identity. Defining $d^{(2)}(G) = \sum_{x\in V(G)} d(x)^2$, the sum of squared degrees of G, we obtain the following identity.

Theorem 2.1 *In a graph G,*

$$|P_3(G)| + 3|K_3(G)| = d^{(2)}(G)/2 - |E(G)|.$$

Proof.

$$\begin{aligned}
|P_3(G)| &= \sum_{xy\in E(G)} \frac{1}{2}|(N(x) \cup N(y)) \setminus (N(x) \cap N(y))| - 1 \\
&= \sum_{xy\in E(G)} \frac{1}{2}(|N(x)| + |N(y)| - 2|N(x) \cap N(y)|) - 1 \\
&= \sum_{xy\in E(G)} \frac{1}{2}(|N(x)| + |N(y)|) - \sum_{xy\in E(G)} |N(x) \cap N(y)| - |E(G)| \\
&= \sum_{xy\in E(G)} \frac{d(x) + d(y)}{2} - 3|K_3(G)| - |E(G)| \\
&= \sum_{x\in V(G)} \frac{d(x)^2}{2} - 3|K_3(G)| - |E(G)|
\end{aligned}$$

□

Theorem 2.1 gives an alternate proof that the number of odd vertices in a graph is even.

2.2 Transforming one graph into another

We desire to characterize the circumstances in which one graph may be transformed, by means of edge slides and K_3-moves, into another graph. After introducing the edge slide in [11], Johnson answered a similar question. At the time, he was dealing with only edge slides; thus, his proof applies to the case where the graphs in question are of the same order and size. With the addition of K_3-moves to the toolbox, we can answer a broader question. As in Johnson's first proof, it turns out that only the most obvious necessary conditions are also sufficient for one graph to be transformable into another.

The results that follow re-prove Johnsons result in slightly more general terms and suggest an algorithm for constructing the desired sequence of operations. The original proof has a different flavor and may be of independent interest to the reader (see [11]).

The most useful property of edge slides and K_3-moves is expressed in terms of connectivity.

Lemma 2.2 *1-connectivity is preserved by edge slides and K_3-moves.*

Proof. Let $E = E(G)$ and let u and v be vertices connected by a path P in G.
Case 1: An edge slide is performed using edges xy and yz so that $E' = (E \setminus xy) \cup xz$. If $xy \notin P$, then P is a path in G' connecting u and v. If $xy \in P$, then either $(P \setminus xy) \cup \{xz, yz\}$ or $(P \setminus \{xy, yz\}) \cup xz$ is a path in G' connecting u and v.
Case 2: A K_3-move is perfomed so that $E' = E \setminus e_1$ where e_1, e_2 , and e_3 form a triangle in G. If $e_1 \notin P$, then P is a path in G' connecting u and v. If $e_1 \in P$, then one of $(P \setminus e_1) \cup \{e_2, e_3\}$, $(P \setminus \{e_1, e_2\}) \cup e_3$, and $(P \setminus \{e_1, e_3\}) \cup e_2$ is a path in G' connecting u and v. □

In contrast, 2-connectivity is not preserved under edge slides and K_3-moves. For example, perform any edge slide on C_4 and any K_3-move on K_3. In general, if G

is k-connected, then the graph G' obtained from G by any edge slide or K_3-move is $\max(1, k-1)$-connected.

The following result shows that the girth of a graph can be easily manipulated by edge slides.

Lemma 2.3 *If G has girth $g \geq 4$, then there is a edge slide on G which will decrease the girth to $g-1$.*

Proof. Let C be a cycle in G with $|C| = g$. If xy and yz are adjacent edges in C, then $x \neq z$ and $xz \notin E(G)$ (since G is triangle-free). Now the subgraph of G induced by vertices x, y, and z is isomorphic to P_3. The edge slide resulting in the addition of xz and the deletion of xy gives a graph G' with a cycle C' satisfying $E(C') = (E(C) \setminus \{xy, yz\}) \cup xz$. Clearly, $|C'| = |C| - 1$ and G' has girth at most $g-1$. Moreover, this bound is tight, since a cycle in G' of size less than $g-1$ implies that there is a cycle of size less than g in G. □

Suppose that $G_1, G_2, \ldots, G_k$ is a sequence of graphs so that for each $i = 2, 3, \ldots, k$, the graph G_i is obtained from G_{i-1} by an edge slide or a K_3-move. We say that G_k is *obtained* from G_1 by a *sequence of moves*, and that the sequence of moves associated with the graph sequence *starts* with G_1 and *ends* with G_k. The main theorem may now be addressed. The result is stated and proven for connected labelled graphs, but the reader will find it easy to extend it to unlabelled graphs or graphs with more than one component.

Theorem 2.4 *Given two connected labelled graphs, G and H, on the same vertex set, there is a sequence of moves starting with G and ending with H if and only if $|E(G)| \geq |E(H)|$.*

To complete the proof, we need three lemmas.

Lemma 2.5 *If G contains a cycle, then there is a sequence of moves starting with G and ending with a graph G' such that $|E(G')| = |E(G)| - 1$.*

Proof. Let g be the girth of G. If $g = 3$, then G contains a triangle, which can be broken by a K_3-move. Suppose that $g > 3$. By invoking Lemma 2.3 $g - 3$ times, the girth of G may be reduced to 3 by edge slides. A well placed K_3-move does the trick. □

We need a tool which is an easy generalization of the edge slide.

Lemma 2.6 *Let $P = xy_1y_2 \cdots y_q$ be a path in a graph G such that for every $i \in \{2, 3, \ldots, q\}$, $xy_i \notin E(G)$. Then there is a sequence of $q - 1$ edge slides starting with G and ending with a graph G' such that $E(G') = (E(G) \setminus xy_1) \cup xy_q$.* □

Lemma 2.7 *Let G be a connected graph with vertices x, y, u, v such that $xy \in E(G)$ and $uv \notin E(G)$. If*

(1) xy cuts G into two components g_1 and g_2, each of which contains an endpoint of uv, or

(2) xy is not a cut edge,

then there is a finite sequence of edge slides starting with G and ending with the graph H such that $E(H) = (E(G) \setminus xy) \cup uv$.

Proof. There are two cases.

Case 1: (1) holds. We may assume that $x, u \in V(g_1)$ and $y, v \in V(g_2)$. By construction, $g_1 \cup xy$ and $g_2 \cup uy$ are connected. Hence, there is a path $P_{(1)}$ in $g_1 \cup xy$ with endpoints u and y. Since xy is a cut of G, it is necessary that $P_{(1)}$ uses xy. The application of Lemma 2.6 yields a sequence of edge slides starting with G and ending with the graph G' where $E(G') = (E(G) \setminus xy) \cup uy$ (perhaps $x = u$, in which case the null sequence suffices). Now $g_2 \cup uy$ is a connected subgraph of G'. If $y = v$, then there is nothing more to show. Otherwise, there is a path $P_{(2)}$ in $g_2 \cup uy$ with endpoints u and v. Once again, $P_{(2)}$ is required to use uy. By Lemma 2.6, there is a sequence of edge slides starting with G' and ending with the graph G'' where $E(G'') = (E(G) \setminus uy) \cup uv$. Clearly, G'' is H.

Case 2: xy is not a cut edge.

Subcase 1: xy and uv are adjacent, in which case we may assume that $x = u$. Since xy is not a cut, there is a path $yz_1z_2 \cdots z_qv$ which does not use the edge xy. Let P be the path $xyz_1z_2 \cdots z_qv$ and let $S = \{z_{s_1}, z_{s_2}, \ldots, z_{s_n}\}$ be the collection of vertices in $V(P) \setminus y$ adjacent to x. If $n = 0$, then the application of Lemma 2.6 gives a sequence of edge slides starting with G and ending with H. Assume $n > 0$ and consider the paths $P_{(0)} = xyz_1z_2 \cdots z_{s_1}$ and $P_{(n)} = xz_{s_n}z_{(s_n+1)} \cdots z_qv$ together with the paths $P_{(i)} = xz_{s_i}z_{(s_i+1)} \cdots z_{s_{(i+1)}}$ for each $1 \leq i < n$. All such paths are in G. We may repeatedly apply Lemma 2.6 to the paths, provided we progress in order of decreasing subscript, to procure by means of edge slides a graph G' such that $E(G') = E(G) \setminus xz_{s_n} \cup xv \setminus xz_{s_{(n-1)}} \cup xz_{s_n} \cdots \setminus xz_{s_1} \cup xz_{s_2} \setminus xy \cup xz_{s_1} = (E(G) \setminus xy) \cup xv = E(H)$.

Subcase 2: xy and uv are vertex disjoint. Since xy is not a cut, there is a path P with endpoints u and y which containts xy. Assume that P is of minimal length with respect to these constraints. Evidently, $yi \notin E(G)$ for all $i \in V(P) \setminus x$. By Lemma 2.6, there is a sequence of edge slides starting with G and ending with a graph G' where $E(G') = (E(G) \setminus xy) \cup uy$. Setting $x' = u$, the graph G' and the choice of vertices x', y, u, v satisfy the hypothesis whose conclusion is assured in Case 2 Subcase 1. Appending the sequence of edge slides needed to pass from G' to H to the sequence used to pass from G to G' gives the desired sequence. □

Properties (1) and (2) from Lemma 2.7 characterize the situation in which we may delete an edge and add a new edge through a sequence of edge slides. Indeed, the invariance of 1-connectivity makes it impossible to remove a cut edge without adding a cut edge.

We are now sufficiently equipped to prove the main result of this section.

Proof of Theorem 2.4. The necessity follows from the fact that edge slides and K_3-moves certainly to not increase the size of a graph. In proving the sufficiency, we assume $|E(G)| \geq |E(H)|$ and show that there is a sequence of moves starting at G and ending at H. By repeatedly applying Lemma 2.5, we may find a sequence of moves starting with G and ending with a graph G' such that $|E(G')| = |E(H)|$. We claim that we may now pass from G' to H; using induction on $n = |E(G') \setminus E(H)|$, we show that there is a sequence of edge slides starting with G' and ending with H. Assuming, for the moment, that this claim is true, then appending the sequence used to pass from G' to H to the sequence used to pass from G to G' yeilds a sequence starting with G and ending with H, thus proving the theorem. Clearly, we are done if $n = 0$ for then $E(G') = E(H)$. Let $n > 0$. Then there is an edge $e_g \in E(G') \setminus E(H)$ and an edge $e_h \in E(H) \setminus E(G')$. If e_g is a cut edge in G', then it follows from the connectivity of H that e_h can be chosen so that it has an endpoint in each of the components of $G' \setminus e_g$. By Lemma 2.7 there is a sequence of edge slides starting with G' and ending with the graph $G'' = (G' \setminus e_g) \cup e_h$. If e_g is not a cut edge in G', then the same conclusion follows from Lemma 2.7. Now $|E(G'')| = |E(H)|$ since only edge slides were used and $|E(G'') \setminus E(H)| = n - 1$. By Lemma 2.2, G'' is connected. The result follows by induction. □

An examination of the proof technique shows that it is no more difficult to proceed from G to H in the critical case where $|E(G)| = |E(H)|$ than when the size of G is much larger than the size of H. Moreover, the process of deleting edges can be done haphazardly and the process of "edge exchange" in the critical case needs to be performed with only slight judiciousness.

We believe that K_3-moves and edge slides may provide a valuable tool for proving graph theorems. An application may look like the following: if Π is a property of K_n for all n and Π is preserved under edge slides and K_3-moves, then every connected graph has property Π.

Chapter 3

Edge slide distance

What does it mean for two graphs to be "close" together? Several metrics for spaces of graphs have been proposed, including the greatest common subgraph metric, edge jump distance, and edge rotation distance (resp. [19], [5], and [3]). The edge slide distance was introduced by Johnson in [11]. Consider two connected graphs G and H in $A_{n,e}$, and write $d_{ES}(G, H)$ for the minimum number of edge slides needed to transform G into H. The analogous distance function can be defined for other types of edge transfers.

The "edge rotation", formulated by Chartrand, et al. is a closely related operation, one which will play an important part in our discussions. We say that H is obtained from G by an *edge rotation* if there exist distinct vertices $x, y, z \in V(G)$ such that $xy \in E(G)$, $xz \notin E(G)$, and H is isomorphic to $(G \setminus xy) \cup xz$. We write $d_{ER}(G, H)$ for the minimum number of edge rotations needed to transform G into H. Both functions, d_{ES} and d_{ER}, are metrics.

An edge slide is an edge rotation, but not every edge rotation is an edge slide. Hence,

$$d_{ER}(G, H) \leq d_{ES}(G, H)$$

for all graphs $G, H \in A_{n,e}$.

For arbitrary graphs, g and h, the decision problem, "Is $d_{ES}(g, h) \leq k$?," is not easy. The corresponding decision problem for the edge rotation metric d_{ER} was shown in [12] to be NP-hard. Note that the $k = 0$ cases of both problems are equivalent to a classic NP-complete decision problem: Is g isomorphic to h? For more on complexity theory, consult [7].

The quest for a closed form expression for the edge slide distance between two arbitrary graphs is quixotic. Researchers have, instead, settled for crude bounds on these distances for general graphs; when seeking more refined results, however, the space of trees has attracted the most attention. The following bounds are immediate but very useful.

Lemma 3.1 (Degree bound) *(Goddard and Swart, [8]) Let G and H be graphs in $A_{n,e}$ with degree sequences $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$, respectively. Then*

$$d_{ES}(G,H) \geq \frac{1}{2}\sum_{i=1}^{n} |a_i - b_i|.$$

Lemma 3.2 (Diameter bound) *(Zelinka, [20]) Let G and H be graphs in $A_{n,e}$ with diameters d_G and d_H, respectively. Then $d_{ES}(G,H) \geq |d_G - d_H|$.*

Lemma 3.3 (Girth and circumference bound) *Let G and H be graphs in $A_{n,e}$, each of which contains a cycle. If g_G and g_H are the respective girths of G and H and if c_G and c_H are the respective circumferences of G and H, then*

$$d_{ES}(G,H) \geq \max(|g_G - g_H|, |c_G - c_H|).$$

Proof. An edge slide increases the size of a cycle by one, decreases it by one, or leaves it the same. □

The numerical results for the space of trees are summarized by the following theorem.

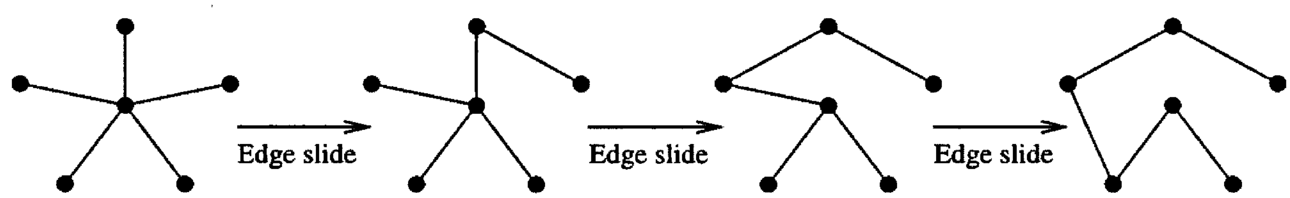
Theorem 3.4 *(Zelinka, [20]) For every $n \geq 3$, $d_{ES}(S_n, P_n) = n - 3$. If T is a tree on $n \geq 3$ vertices with diameter d_T and maximum degree δ_T, then*

- $d_{ES}(T, P_n) = n - d_T - 1$, *and*

- $d_{ES}(T, S_n) = n - \delta_T - 1$.

Proof. The necessary lower bounds are implied by the degree bound and the diameter bound (Lemmas 3.1 and 3.2). If P is a maximal path in a tree T and there is an edge e of T not contained in P but nevertheless meeting P at a common vertex, then T can be transformed by an edge slide into a new tree having a maximal path with size exactly one larger than the size of P. It follows that $d_{ES}(T, P_n) \leq n - d_T - 1$. If x is a vertex of a tree T that is not adjacent to every other vertex in T, then T can be transformed by an edge slide into a new tree, in which the degree of x is increased by one. It follows that $d_{ES}(T, S_n) \leq n - \delta_T - 1$. The value of $d_{ES}(S_n, P_n)$ now follows from either of these results. □

A sequence of three edge slides realizing the transformation of S_6 into P_6 is given below.



The edge slide distance between S_6 and P_6 is three

How "far apart" can two trees on n vertices be? A simple application of the triangle inequality for metrics gives $2n - 6$ as an upper bound. A significantly tighter bound has so far been illusive to researchers. Based on anecdotal evidence, however, we are confident in the following significant improvement.

Conjecture 3.5 *If T_1 and T_2 are trees on $n \geq 3$ vertices, then $d_{ES}(T_1, T_2) \leq n - 3$. Equality holds if and only if T_1 and T_2 are the star and the path.*

This conjecture has been verified for all trees on up to 8 vertices. The reader may observe in Section 4.2 the girth of certain "distance graphs". These computations verify Conjecture 3.5 for all trees on up to 8 vertices.

Goddard and Swart showed in [8] that $d_{ER}(T_1, T_2) \leq n - 3$ holds under the same hypothesis. In this light, Conjecture 3.5 suggests that the restriction of edge rotations to edge slides does not dramatically inhibit transformability of trees.

McGuinness has proposed studying the degree sequences of trees, rather than the isomorphism classes of trees themselves [13]. It is well known that every degree sequence can be realized by a tree (see [1]); that is, for every non-decreasing sequence of positive integers $a_1 \leq a_2 \leq \cdots \leq a_p$ there is a tree with $\sum_{i=1}^{p} a_i/2 + 1$ vertices having degree sequence $a_1 \leq a_2 \leq \cdots \leq a_p$.

The next two conjectures are subsumed by Conjecture 3.5; yet, they possess a distinct flavor, and so are included here.

Conjecture 3.6 *If two trees, T_1 and T_2, on n vertices have the same degree sequence, then $d_{ES}(T_1, T_2) \leq n - o(n)$.*

Conjecture 3.7 *If T_1 is a tree on n vertices and D is a degree sequence of a tree on n vertices, then there exists another tree T_2 having degree sequence D such that $d_{ES}(T_1, T_2) \leq n - o(n)$.*

Perhaps these conjectures can be more easily proven if we restrict our attention to caterpillars. Indeed, every degree sequence can be realized by a caterpillar (see [1]).

What can be said for graphs that are not trees? Here is an accessible result.

Theorem 3.8 *Let $G \in A_{n,n}$ have girth g. Then $d_{ES}(G, C_n) = n - g$.*

Proof. By the girth and circumference bound (Lemma 3.3), we have $d_{ES}(G, C_n) \geq n - g$. The graph G has exactly one cycle, C. Suppose that e is an edge of G not contained in C, but nevertheless having a common vertex with C. There is an edge slide transformation of $C \cup e$ into C_{g+1}. Therefore, the girth of G can be increased by one, and $d_{ES}(G, C_n) \leq n - g$. □

This proof is very similar to the one verifying that $d_{ES}(T, P_n) = n - d_T - 1$ from Lemma 3.4. There is no coincidence; in some sense, cycles are to 2-connected graphs what paths are to 1-connected graphs. Similarly, the graphs $K_{2,n}$ serve as 2-connected analogues to stars. The investigation of $d_{ES}(G, K_{2,n})$ for arbitrary graphs $G \in A_{n+2,2n}$ is likely to be fruitful.

Chapter 4

Distance graphs

If (X, d) is a metric space and d is an integer valued metric, then it is sometimes possible to associate the metric space with a graph G having vertex set X such that for all $x, y \in X$, the length of the shortest path in G with endpoints x and y is $d(x, y)$. A graph can be viewed as a metric space, so in this sense G and (X, d) are isomorphic as metric spaces. The graph is referred to as a *distance graph* and it abstracts the essential structure of the metric space.

4.1 Hamiltonian distance graphs

In [6], Cummins, an electrical engineer, considered a metric on the set of spanning trees of a graph. Two spanning trees T_1 and T_2 of a graph G differ by *basis exchange* if there is an edge $e \in T_1$ and an edge $b \in T_2$ such that $T_2 = (T_1 \setminus e) \cup b$. A distance d_{BE} can be defined on the space of spanning trees of a graph. Here, $d_{BE}(T_1, T_2)$ is the minimum number of basis exchanges required to transform one spanning tree T_1 into another T_2. There is an associated distance graph for this metric space, called the *tree graph of* G and denoted $T(G)$. In his paper, Cummins proved an amazing fact about tree graphs.

Theorem 4.1 *The tree graph of any graph is Hamiltonian.*

This result was later generalized to matroids by Harary and Holzmann in [9]. The *tree graph of a matroid* is defined analogously on the bases of a matroid.

Theorem 4.2 *The tree graph of any matroid is Hamiltonian.*

Why are Hamiltonian distance graphs interesting? Suppose that we desire to generate the members of some combinatorial family – trees, permutations, partitions, and subsets, for example. If an operation on members of the family leads to a natural metric space, whose associated distance graph is Hamiltonian (or at least contains a Hamilton path), then we get a natural and efficient way to list the members of the family. Very few interesting combinatorial families have an obvious ordering. For example, there is no natural way to totally order graphs on the basis of common graph parameters, such as size, order, girth, circumference, diameter, and chromatic number.

When a combinatorial family is listed such that consecutive members of the list are in some sense "close together", we call the list a *Gray code* (see [17] for an excellent survey on the topic). Thus, Theorem 4.1 asserts the existence of a Gray code enumeration of spanning trees of a graph, in which consecutive trees differ by basis exchange. These types of results can have significant implications for data storage and combinatorial simulation.

Every edge slide transformation of a tree is a basis exchange. Thus, $d_{BE}(T_1, T_2) \leq d_{ES}(T_1, T_2)$. Can the edge slide operation lead to a new Gray code listing of trees? We take this up in the next section.

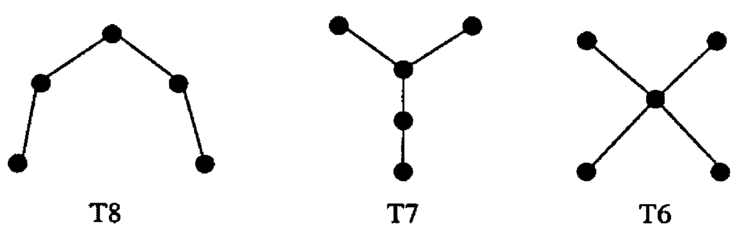
4.2 Edge slide distance graphs and Gray codes

Given a collection of graphs X, define the *edge slide distance graph* $D_{ES}(X)$ of X to be the graph with vertex set X, where g and h are adjacent as vertices in $D_{ES}(X)$ if and only if g can be transformed into h by an edge slide. Thus, for each set of graphs we may generate the associated edge slide distance graph. The following result can be found in [4].

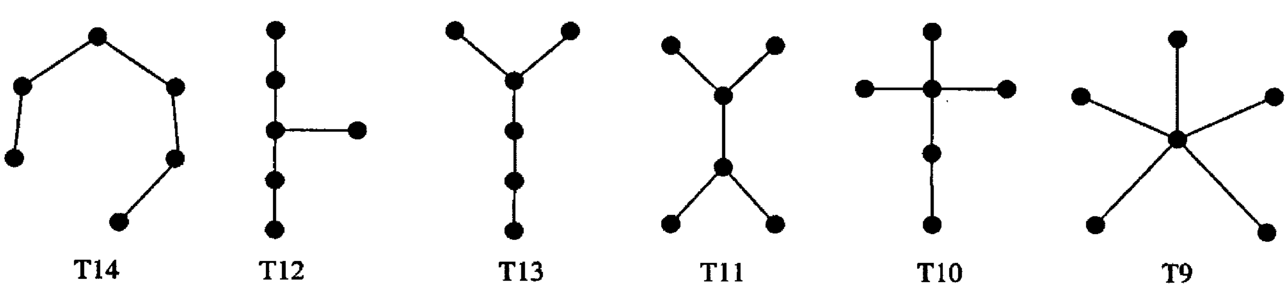
Theorem 4.3 *Every graph is an edge slide distance graph.*

Outline of proof. Let G be a graph on vertex set $\{v_1, v_2, \ldots, v_n\}$ and G' be the graph obtained from G by adding, for each $i = 1, 2, \ldots, n$, an additional $2i$ vertices, each adjacent only to v_i. For each i, define H_i to be the graph obtained from G' by adding an additional vertex, adjacent only to v_i. One can show that $D_{ES}(\{H_1, H_2, \ldots, H_n\})$ is isomorphic to G. $\square$

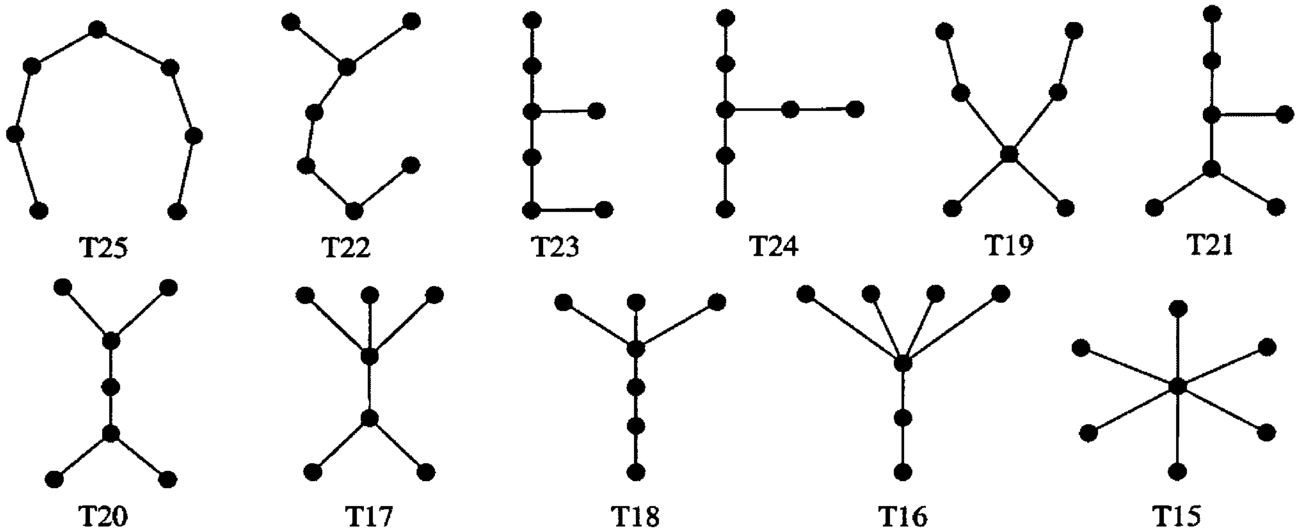
Recall that $A_{n,n-1}$ is the collection of trees on n vertices. A natural question is whether $D_{ES}(A_{n,n-1})$ has a Hamilton path. This is equivalent to asking if there is a Gray code of trees on n vertices, where consecutive trees in the code differ by an edge slide. Hamilton paths (Gray codes) have been found for all $n \leq 8$. The only interesting cases are for $n \geq 5$ and are listed below. Trees are identified using the system in [15].



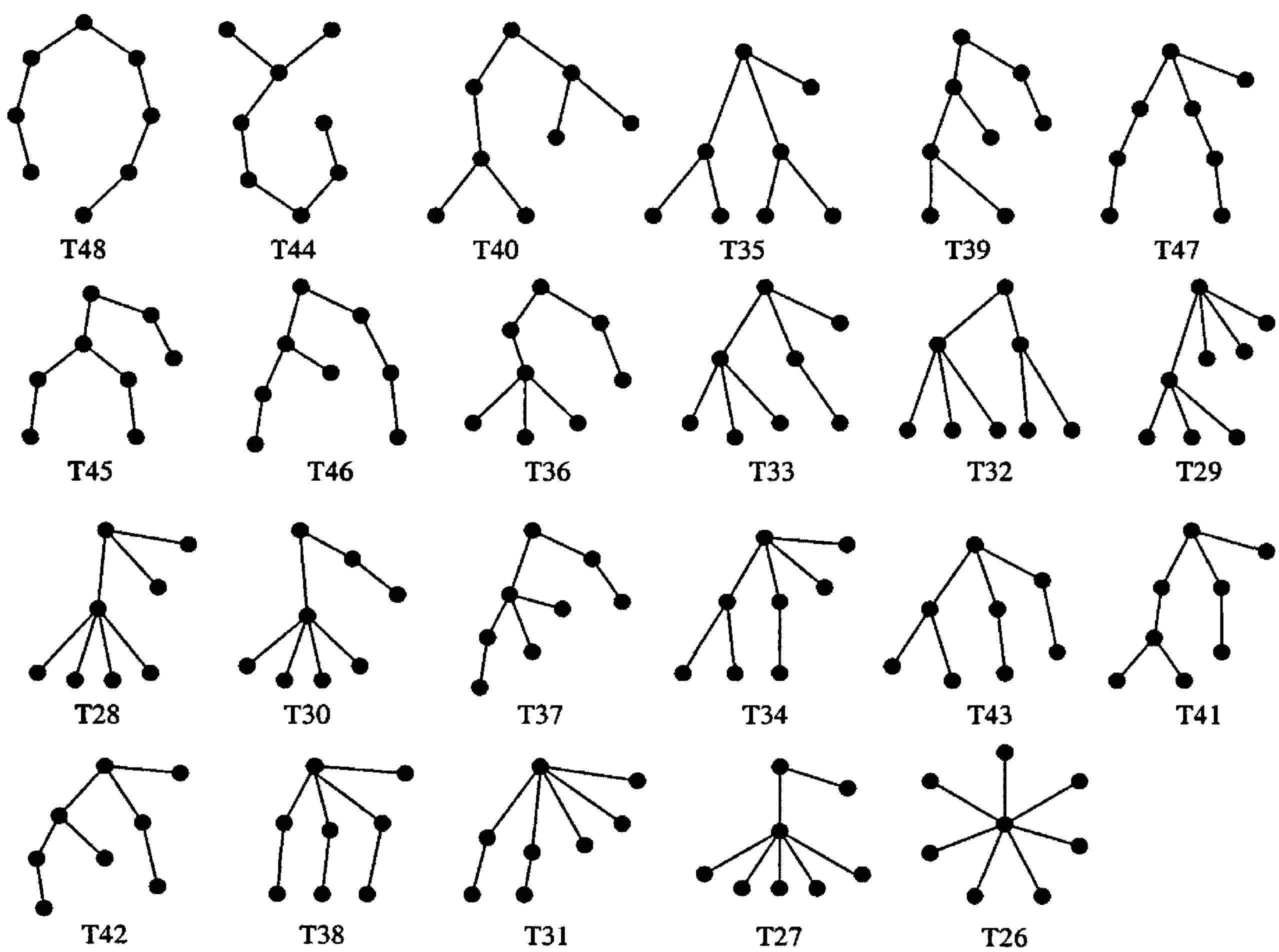
Trees with 5 *vertices (listed as a Gray code)*



Trees with 6 *vertices (listed as a Gray code)*



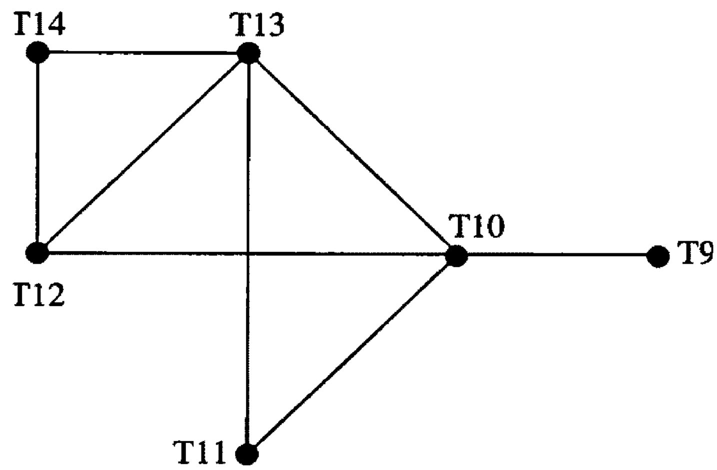
Trees with 7 vertices (listed as a Gray code)



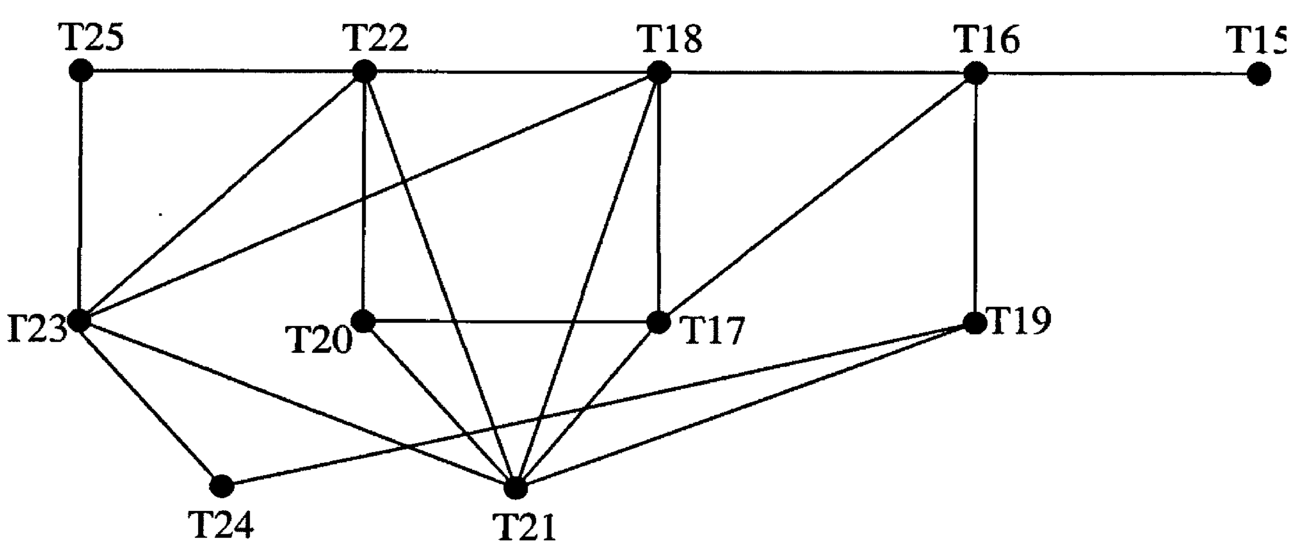
Trees with 8 vertices (listed as a Gray code)

Each Gray code above corresponds to a Hamilton path in the corresponding edge slide distance graph. The graphs $D_{ES}(A_{n,n-1})$ (for $n = 5, 6, 7, 8$) are exhibited below.

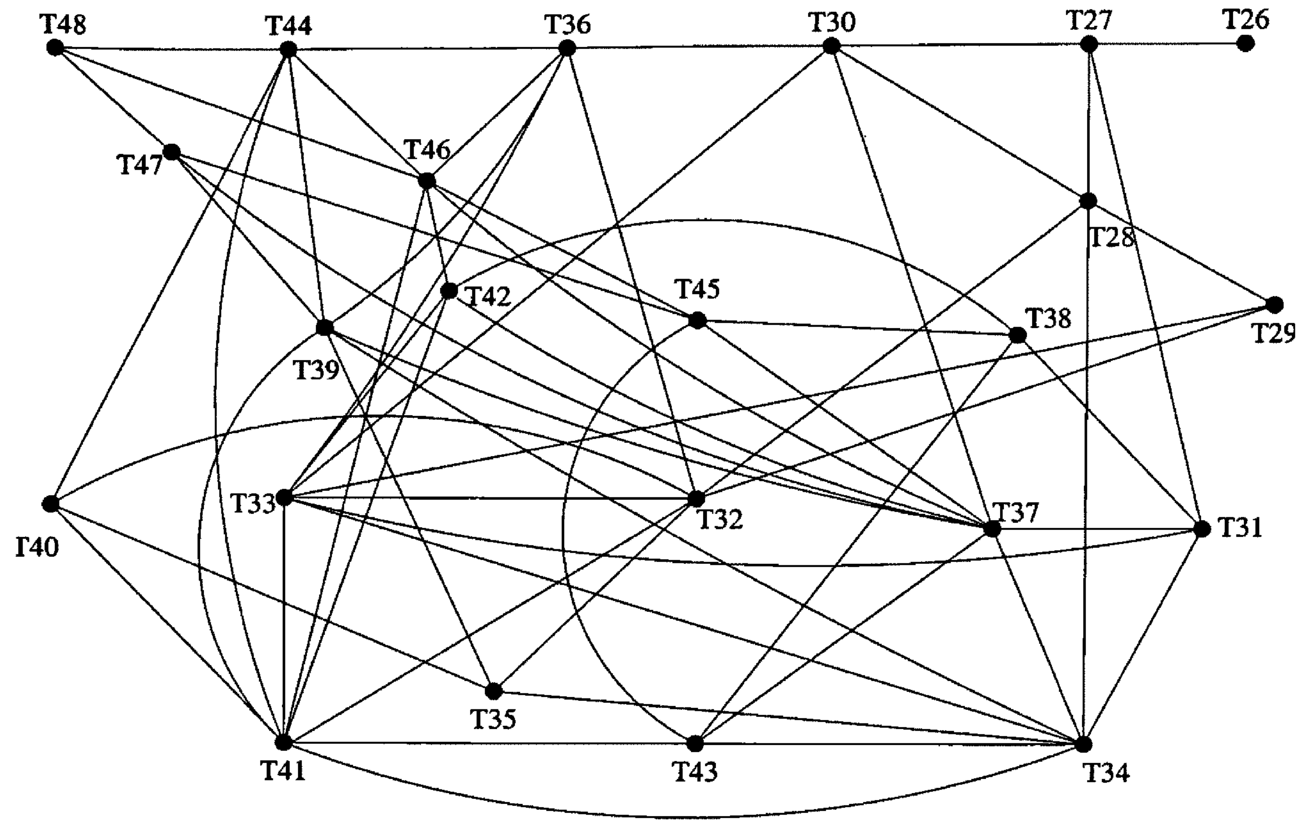
$D_{ES}(A_{5,4})$



$D_{ES}(A_{6,5})$



$D_{ES}(A_{7,6})$

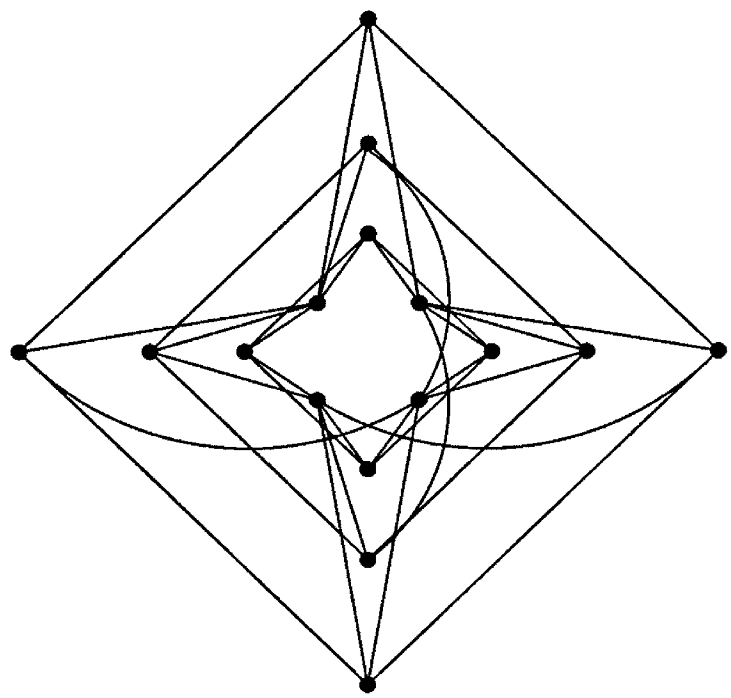


$D_{ES}(A_{8,6})$

In each graph above, there is a Hamilton path starting at P_n and ending at S_n. We believe that this can always be done; that is, we are confident in the following conjecture.

Conjecture 4.4 *$D_{ES}(A_{n,n-1})$ has a Hamilton path with endpoints P_n and S_n.*

Since Cummins was considering the spanning trees of a graph, he was essentially looking at labelled trees. Thus, if we consider the edge slide distance graph of *labelled* trees on n vertices, we are essentially looking at a spanning subgraph of $T(K_n)$, the tree graph of K_n. Indeed, the edge slide is a restricted basis exchange. This subgraph is depicted below for $n = 4$.



The edge slide distance graph of labelled trees on 4 vertices

The graph is not only Hamiltonian but is also *Hamiltonian laceable*, meaning that for every pair of vertices the graph has a Hamilton path having the pair as endpoints. We are naturally led to ask, for which values of n is the edge slide distance graph of labelled trees on n vertices Hamiltonian? The edge slide is a highly restictive basis exchange, and so an affirmative answer for all n would be surprising indeed. Whatever comes of these questions, the graph above will provide a fertile base case.

4.3 Edge rotation distance graphs

Recall that for any distinct vertices x, y, z of a graph G such that $xy \in E(G)$ and $xz \notin E(G)$, the graph H obtained from G by deleting xy and adding xz is said to have been obtained from G by an edge rotation. We require that $yz \in E(G)$ in order for H to be obtained from G by an edge slide; edge slides are restricted edge rotations.

For any collection of graphs $\mathcal{G}$, we define the *edge rotation distance graph*, $D_{ER}(\mathcal{G})$, of $\mathcal{G}$ by taking the vertices of $D_{ER}(\mathcal{G})$ to be the members of $\mathcal{G}$, with two vertices being adjacent if and only if they differ by an edge rotation. In other words,

$gh \in E(D_{ER}(\mathcal{G}))$ if and only if $d_{ER}(g,h) = 1$. The graph $D_{ES}(\mathcal{G})$ is a subgraph of $D_{ER}(\mathcal{G})$ saturating every vertex. Moreover, it follows from Theorem 2.4 that $D_{ES}(\mathcal{G})$ is connected.

In [4], Chartrand et al. conjectured that every graph is an edge rotation distance graph. They observed the following:

- K_n, C_n, and P_n are edge rotation distance graphs,
- If G and H are edge rotation distance graphs, then so are $G \cup H$ and $G \times H$, and
- Every tree is an edge rotation distance graph; in particular, if G and H are edge rotation distance graphs and $u \in V(G)$ and $v \in V(H)$, then the graph obtained from G and H by identifying u and v is an edge rotation distance graph.

In [10], Jarrett added the following:

- Wheels are edge rotation distance graphs, and
- Complete bipartite graphs are edge rotation distance graphs.

Every graph is an edge slide distance graph, and the necessary construction is given in the previous section. The refined nature of edge slides is readily exploited to prove this result. The flexibility of edge rotations, however, stands in the way of our efforts to prove Chartrand's conjecture. We settle for a weaker version, which asserts that the conjecture is true in a topological sense.

Theorem 4.5 *Every graph is homeomorphic to an edge rotation distance graph.*

The proof requires the construction of a collection of graphs $\mathcal{G}$, whose edge rotation distance graph has the desired properties. The remainder of this section contains the body of this involved proof. It is somewhat of a departure from the main thrust of the thesis and may be omitted without loss of continuity.

Let G be a graph. It is sufficient to prove the result for connected graphs, and so G is assumed to be connected. We will exhibit a collection of graphs, $\mathcal{G}$, and a homeomorphic mapping of G into $D_{ER}(\mathcal{G})$.

For simplicity of notation, $D_{ER}(*)$ will be denoted $D(*)$. Recall that for each graph G, the edge set $E(G)$ is a subset of $\binom{V(G)}{2}$.

The proof is by induction and is motivated by the following result about connected graphs:

Lemma 4.6 *If G is a connected graph, then there is a permutation of its vertex set $(v_1, v_2, \ldots, v_n)$ such that for every $i = 1, 2, \ldots, n$, the vertex v_i is adjacent in G to a member of $\{v_1, v_2, \ldots, v_{i-1}\}$.*

Proof. Let T be a spanning tree of G and let $(v_1, v_2, \ldots, v_n)$ be the visiting order of some depth first search of T beginning at v_1. This permutation satisfies the conclusion of the lemma. □

Theorem 4.5 holds when G is a single vertex, in which case G is isomorphic to the edge rotation distance graph of the collection containing the single graph, P_2. Let G have $n > 1$ vertices. Our induction hypothesis is as follows: *Every connected graph with $n-1$ vertices is homeomorphic to an edge rotation distance graph, whose vertices are connected graphs with at least one edge.*

In light of Lemma 4.6, there is a vertex $v_0 \in V(G)$ that is adjacent to vertices in the vertex-induced subgraph H of G, induced by $V(G) \setminus v_0$. By the induction hypothesis, there is a collection $\mathcal{H}$ of connected graphs, all with at least one edge, and a homeomorphic mapping,

$$\psi : V(H) \rightarrow V(D(\mathcal{H})).$$

Consider the collection of graphs,

$$\{h_1, h_2, \ldots, h_k\} = \psi(N(v_0)),$$

and let $m = |E(h_1)|$. The induction hypothesis allows us to assume that $m > 0$.

We now outline the construction of an edge rotation distance graph homeomorphic to G. We do so by augmenting each member of $\mathcal{H}$ and adding new graphs to the collection. This new collection will be denoted by $\mathcal{G}$. The mapping ψ is then extended to a homeomorphic mapping from G to $D(\mathcal{G})$.

For each graph $h \in \mathcal{H}$, we will construct a graph $g_h^{(0)}$ such that there is an isomorphism from $D(\mathcal{H})$ to $D(\{g_h^{(0)} : h \in \mathcal{H}\})$ which maps h to $g_h^{(0)}$, for every $h \in \mathcal{H}$. For each $i = 1, 2, \ldots, k$ and each $j = 1, 2, \ldots, 2m+1$ we will define graphs $g_i^{(j)}$, along with a graph $g^{(2m+2)}$. The collection

$$\mathcal{G} = \{g_h^{(0)} : h \in \mathcal{H}\} \cup \{g_i^{(j)} : i \in [k], j \in [2m+1]\} \cup g^{(2m+2)}$$

will have the following properties:

For every $h \notin \{h_1, h_2, \ldots, h_k\}$,

$$N_{D(\mathcal{G})}(g_h^{(0)}) = \{g_{h'}^{(0)} : h' \in N_{D(\mathcal{H})}(h)\}. \tag{4.1}$$

For every $i = 1, 2, \ldots, k$,

$$N_{D(\mathcal{G})}(g_{h_i}^{(0)}) = \{g_{h'}^{(0)} : h' \in N_{D(\mathcal{H})}(h_i)\} \cup g_i^{(1)}, \tag{4.2}$$

$$N_{D(\mathcal{G})}(g_i^{(1)}) = \{g_{h_i}^{(0)}, g_i^{(2)}\}, \text{ and} \tag{4.3}$$

$$N_{D(\mathcal{G})}(g_i^{(2m+1)}) = \{g_i^{(2m)}, g^{(2m+2)}\}. \tag{4.4}$$

For every $i = 1, 2, \ldots, k$ and $j = 2, 3, \ldots, 2m$,

$$N_{D(\mathcal{G})}(g_i^{(j)}) = \{g_i^{(j-1)}, g_i^{(j+1)}\}. \tag{4.5}$$

Finally,

$$N_{D(\mathcal{G})}(g^{(2m+2)}) = \{g_i^{(2m+1)} : i = 1, 2, \ldots, k\}. \tag{4.6}$$

Suppose that the collection $\mathcal{G}$ can be constructed and the properties above can be verified. Then it follows that the map

$$\psi' : V(G) \rightarrow V(D(\mathcal{G}))$$

defined by

$$\psi'(v) = \left\{ \begin{array}{ll} g^{(2m+2)} & v = v_0 \\ g^{(0)}_{\psi(v)} & v \in V(H) \end{array} \right\}$$

is a homeomorphic mapping, thus completing the proof.

We will now give the promised construction of $\mathcal{G}$ and verify (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6). First, we construct the graphs $g_h^{(0)}$, for every $h \in \mathcal{H}$.

Let $V = V(h_1)$ and define $\mathcal{M}$ to be the collection of subsets of size m from $\binom{V}{2}$; that is,

$$\mathcal{M} = \binom{\binom{V}{2}}{m}.$$

Define $\mathcal{K}$ to be the collection of subsets of size k from $\mathcal{M}$; that is,

$$\mathcal{K} = \binom{\mathcal{M}}{k}.$$

Fix a bijective mapping $\phi_h : V(h_1) \to V(h)$ for every $h \in \mathcal{H}$. The map ϕ_{h_1} may be taken to be the identity automorphism.

For each $h \in \mathcal{H}$, make the following augmentation of h for each $K \in \mathcal{K}$.

- Let $E_1, E_2, \ldots, E_k$ be the elements of K, which are viewed as edge sets of size m on $V(h)$ via the mapping ϕ_h.

- For every $i = 1, 2, \ldots, k$ and every edge $xy \in E_i$, add a vertex $v_{K,i,xy}$ to $V(h)$ and edges $v_{K,i,xy}x$ and $v_{K,i,xy}y$ to $E(h)$.

- For every $i = 1, 2, \ldots, k$, add a vertex $v_{K,i,1}$ to $V(h)$ and edges $\{v_{K,i,1}v_{K,i,xy} : xy \in E_i\}$ to $E(h)$.

- For each $i = 1, 2, \ldots, k$, add vertices $\{v_{K,i,j} : j = 2, 3, \ldots, 2k+1\} \cup v_K$ to $V(h)$ and edges $\{v_{K,i,j}v_{K,i,j+1} : j = 1, 2, \ldots, 2k\} \cup v_{K,i,2k+1}v_K$ to $E(h)$.

- Add vertices $\{v_K^j : j = 1, 2, \ldots, m\}$ to $V(h)$ and edges $\{v_K^j v_K^{j+1} : j = 1, 2, \ldots m-1\} \cup v_K v_K^1$ to $E(h)$.

The augmentation defined above is now referred to as the graph $g_h^{(0)}$. Recall that this augmentation was performed for each $K \in \mathcal{K}$, so the graphs $g_h^{(0)}$ have significantly more edges and vertices than the original graphs in $\mathcal{H}$. However, since the augmentation is performed in such a symmetric way, two graphs $h, h' \in \mathcal{H}$ will differ by an edge rotation if and only if the corresponding augmentations $g_h^{(0)}$ and $g_{h'}^{(0)}$ differ by an edge rotation. Hence, Property 4.1 holds for every graph $g_h^{(0)}$ with $h \notin \{h_1, h_2, \ldots, h_k\}$.

Our next step is to define the graphs $g_i^{(1)}$ for each $i = 1, 2, \ldots, k$. The elaborate construction that we have just undertaken was designed to ensure that some $K_0 \in \mathcal{K}$ will allow us to single out the graphs $h_1, h_2, \ldots, h_k$. Recall that $m = |E(h_1)| > 0$. For each $i = 1, 2, \ldots, k$, let $E_i = E(h_i)$. Define E_i' to be the edges in $V \times V$ corresponding to E_i via the map ϕ_{h_i}. Define

$$K_0 = \{E_1', E_2', \ldots, E_k'\}.$$

The set $K_0 \in \mathcal{K}$ has some special properties. For each $i = 1, 2, \ldots, k$, the edges E_i' correspond (via ϕ_{h_i}) precisely to the edges of h_i.

We are now ready to define the graphs $g_i^{(1)}$. Thankfully, our subsequent constructions will be defined in terms of edge rotation operations, and not in terms of further augmentations. For each $i = 1, 2, \ldots, k$, define $g_i^{(1)}$ by $V(g_i^{(1)}) = V(g_{h_i}^{(0)})$ and

$$E(g_i^{(1)}) = (E(g_{h_i}^{(0)}) \setminus v_{K_0,i,i}v_{K_0,i,i+1}) \cup v_{K_0,i,i}v_{K_0,2i+1}.$$

Now $g_i^{(1)}$ differs from $g_{h_i}^{(0)}$ by an edge rotation. If some graph $g_i^{(1)}$ differs from another $g_h^{(0)}$ by an edge rotation, then it follows that h_i is isomorphic to h. Hence, Property 4.2 holds.

We will now construct the graphs $g_i^{(j)}$ for each $i = 1, 2, \ldots, k$ and $j = 2, 3, \ldots, 2m+1$:

Step (0): Fix i, let $j = 2$, and initiate $BEGIN = E_i'$ and $END = \varnothing$.

Step (1): Suppose that we have constructed $g_i^{(j-1)}$. Then there is an edge $xy \in$

$BEGIN$. Define $g_i^{(j)}$ by $V(g_i^{(j)}) = V(g_i^{(j-1)})$ and

$$E(g_i^{(j)}) = (E(g_i^{(j-1)}) \setminus xy) \cup xv_{K_0}^{j-1}.$$

Finally, put $BEGIN = BEGIN \setminus xy$ and put $END = END \cup xv_{K_0}^{j-1}$.

Step (2): If $j' = j < m+1$, then put $j = j'+1$ and repeat Step (1). Otherwise, let $END_i = END$ and stop.

After this process has been repeated for each i, we will have constructed the graphs $g_i^{(j)}$ for $j = 2, 3, \ldots, m+1$. We now use a similar procedure to construct the remaining graphs, $g_i^{(j)}$ for $j = m+2, m+3, \ldots, 2m+1$.

Step (0): Fix i, let $j = m+2$, and initiate $BEGIN = END_i$.

Step (1): Suppose that we have constructed $g_i^{(j-1)}$. Then there is an edge of the form $xv_{K_0}^{j-m-1} \in BEGIN$. Define $g_i^{(j)}$ by $V(g_i^{(j)}) = V(g_i^{(j-1)})$ and

$$E(g_i^{(j)}) = (E(g_i^{(j-1)}) \setminus xv_{K_0}^{j-m-1}) \cup v_{K_0}v_{K_0}^{j-m-1}.$$

Finally, put $BEGIN = BEGIN \setminus xv_{K_0}^{j-m-1}$.

Step (2): If $j' = j < 2m+1$, then put $j = j'+1$ and repeat Step (1). Otherwise, stop.

We have now defined all but one member of $\mathcal{G}$, namely $g^{(2m+2)}$. Before we do so, now is a good time to assess the graphs of the form $g_i^{(2m+1)}$ for each i. The edge rotation performed in the construction of $g_i^{(1)}$ from $g_{h_i}^{(0)}$ is now the only difference between the graphs $g_i^{(2m+1)}$. None of these graphs differ by an edge rotation, due to this difference. In fact, for all $j, j' \in [2m+1]$ and every $i \neq i' \in [k]$, the graphs $g_i^{(j)}$ and $g_{i'}^{(j')}$ do not differ by an edge rotation. However, for each $i = 1, 2, \ldots, k$ and each $j = 1, 2, \ldots, 2m$, the construction of $g_i^{(j)}$ from the perspective of $g_i^{(j-1)}$ ensures that these graphs differ by an edge rotation. Therefore, Properties 4.3 and 4.5 hold.

The graph $g^{(2m+2)}$ is defined by taking $V(g^{(2m+2)}) = V(g_1^{(2m+1)})$ and

$$E(g^{(2m+2)}) = (E(g_1^{(2m+1)}) \setminus v_{K_0,1,1}v_{K_0,3}) \cup v_{K_0,1,1}v_{K_0,1,2}.$$

This construction ensures that $g^{(2m+2)}$ differs from $g_1^{(2m+1)}$ by an edge rotation. In fact, $g^{(2m+2)}$ differs by an edge rotation from precisely the graphs $g_1^{(2m+1)}, g_2^{(2m+1)}, \ldots, g_k^{(2m+1)}$. Properties 4.4 and 4.6 now follow and the proof is complete.

Chapter 5

K_3-moves

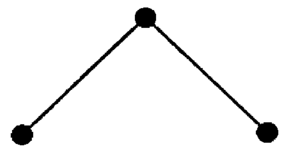
Our discussion of K_3-moves is devoted entirely to the question of whether one graph (or matroid) can be transformed into another using K_3-moves. The operation lends itself naturally to the partial ordering of graphs and of matroids.

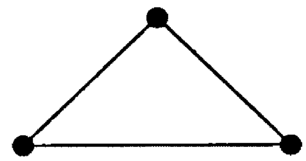
5.1 The poset $\mathcal{G}_n$

Define a relation $\preceq$ on the set of graphs by the rule:

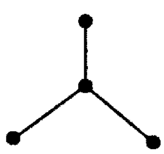
$G \preceq H$ if and only if G is obtained from H by a sequence of K_3-moves.

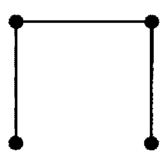
Clearly, $\preceq$ is a partial order. Consider the set of connected graphs on n vertices, which we will denote by $\mathcal{G}_n$. It is natural to think of $\mathcal{G}_n$ as the poset $(\mathcal{G}_n, \preceq)$. Hasse diagrams for $\mathcal{G}_3$, $\mathcal{G}_4$, and $\mathcal{G}_5$ are given below. The members of $\mathcal{G}_n$ are shown with their identification numbers (see [15]) to provide cross-reference with the each Hasse diagram.

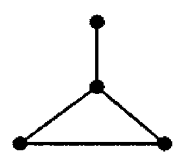

G6

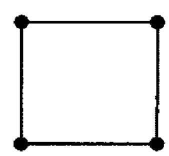

G7

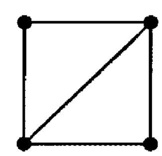
The poset $\mathcal{G}_3$

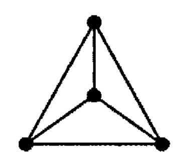

G13

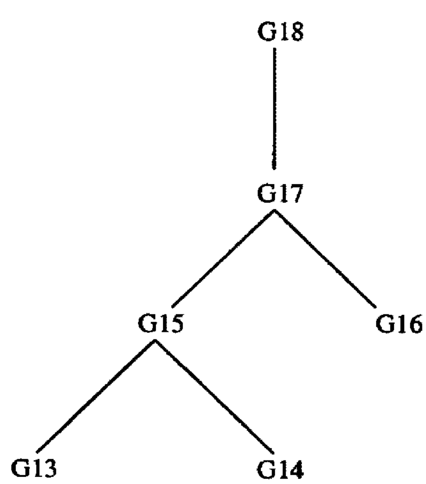

G14


G15

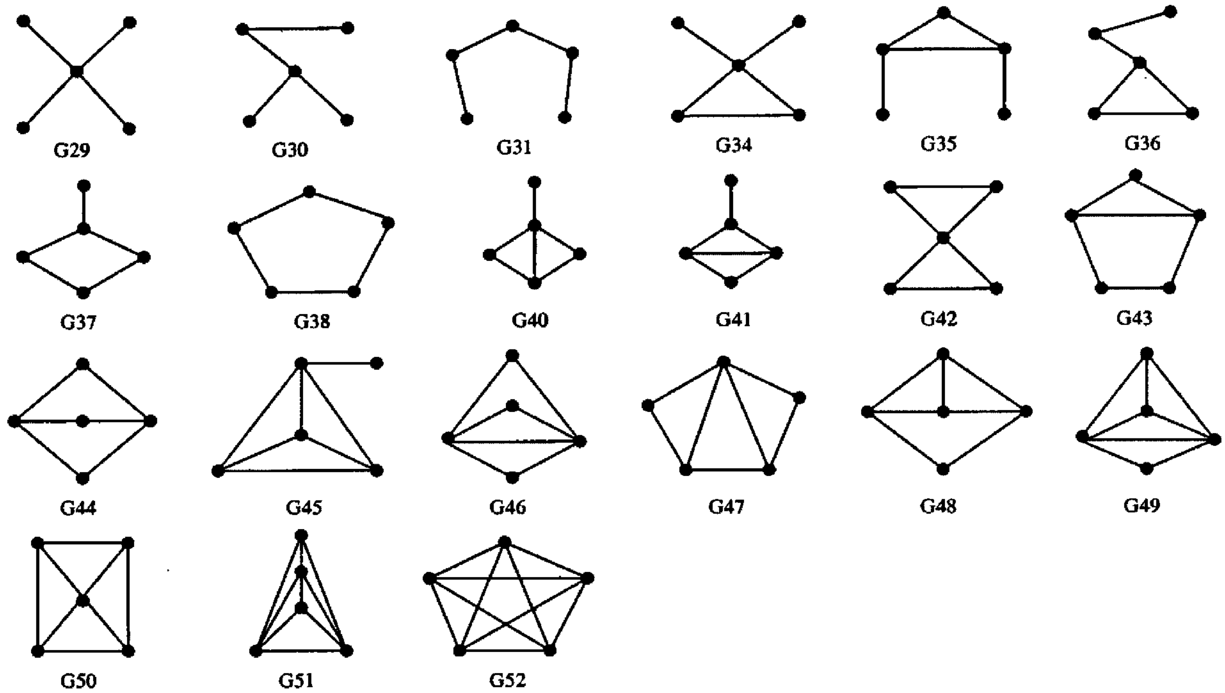

G16

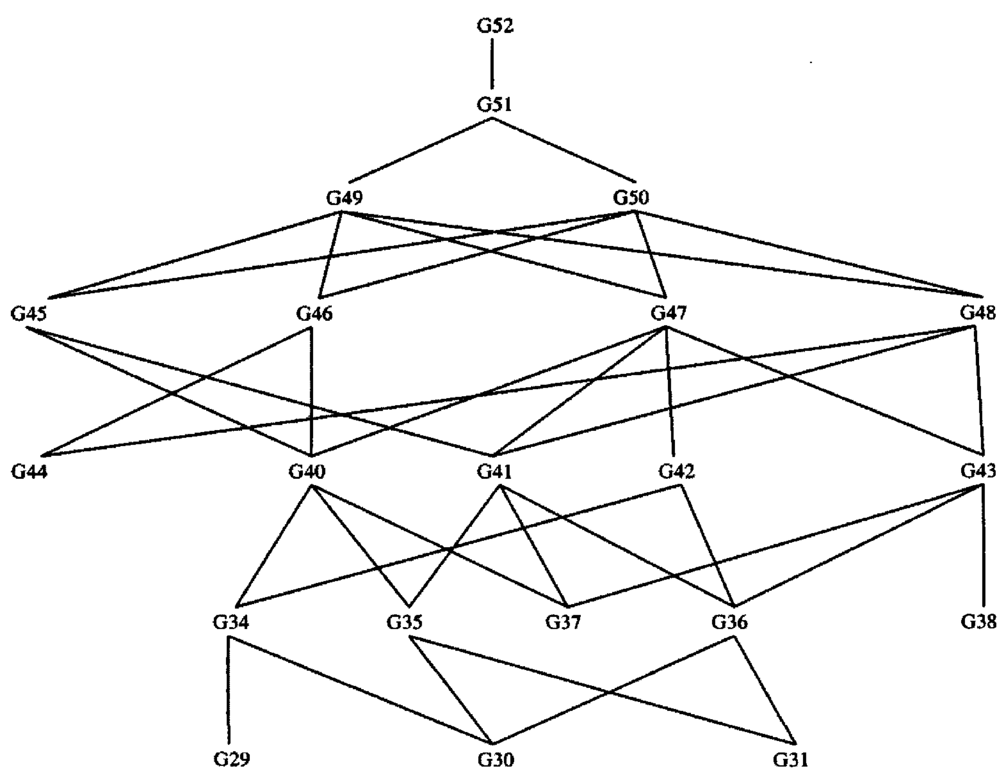

G17


G18



The poset $\mathcal{G}_4$

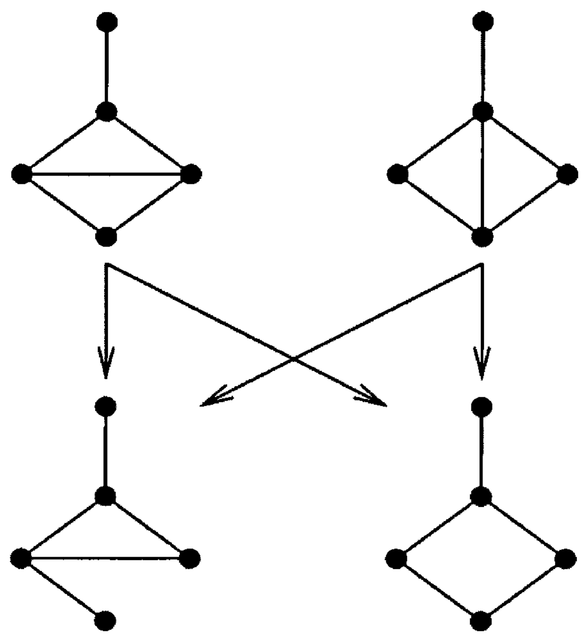




The poset $\mathcal{G}_5$

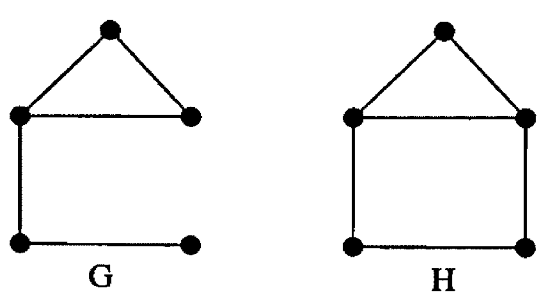
A few easy observations are in order. First, $\mathcal{G}_n$ is a connected poset, whose minimal elements are the triangle-free connected graphs on n vertices. There is exactly one maximal element, namely K_n. The sets of the form $A_{n,e}$, where $e \in \{n-1, n, \ldots, \binom{n}{2}\}$ are antichains of $\mathcal{G}_n$. The poset of connected graphs on n vertices, ordered by edge size, is a subposet of $\mathcal{G}_n$.

For every $n \geq 5$, the poset $\mathcal{G}_n$ is not a lattice. Two members do not necessarily have a unique least upper bound nor a unique greatest lower bound, as the following example shows.



For $n \geq 5$, the poset $\mathcal{G}_n$ is not a lattice

The decision problem to determine if $G \preceq H$, is at least as hard as the classic decision problem, "Determine if G is isomorphic to H". An obvious necessary condition for $G \preceq H$ is that G be a connected spanning subgraph of H. However, this property is not sufficient. Triangle-free graphs and their connected spanning proper subgraphs provide easy counterexamples to sufficiency. Another example is given below.



G spans H but $G \not\preceq H$

The next three results give sufficient conditions for $G \preceq H$. Their proofs are more or less immediate and are left to the reader.

Proposition 5.1 *Suppose that G is a connected planar graph and H is a triangulation or a near-triangulation of G. Then $G \preceq H$.* □

Proposition 5.2 *Suppose that G is a connected graph on n vertices having more than $\binom{n}{2} - \lceil \frac{n}{2} \rceil$ edges. Then $S_n \preceq G$.* □

This bound is best possible. For example, if n is even and G is obtained from K_n by the deletion of a perfect matching, then $S_n \not\preceq G$. In fact, we can say more about stars.

Proposition 5.3 *If G is a graph on n vertices, then the following are equivalent:*

- $S_n \preceq G$
- *G has a vertex of degree $n-1$*
- *S_n is isomorphic to a subgraph of G* □

There are a number of other immediate sufficient conditions, but we will save the discussion for Section 5.4. For now, we will delve a little deeper into the structure of $\mathcal{G}_n$.

5.2 Embedding posets in $\mathcal{G}_n$

If X is a collection of graphs, then the poset $(X, \preceq)$ is identified as a *K_3-move poset.* Is every poset isomorphic to a K_3-move poset? The answer is "yes". Every finite poset is embeddable in $\mathcal{G}_n$, for n sufficiently large. In fact, we can say a little more.

Theorem 5.4 *If $(X, \leq)$ is a finite nonempty poset and $n = 2(|X|+1)$, then $(X, \leq)$ is isomorphic to an induced subposet of $\mathcal{G}_n$. If $(X, \leq)$ has a $\hat{1}$, then X is isomorphic to an induced subposet of $\mathcal{G}_{n-2}$.*

A poset $(A, \preceq)$ is an *induced subposet* of a poset $(B, \preceq')$ if $A \subseteq B$ and $a_1, a_2 \in A$ satisfy $a_1 \preceq a_2$ if and only if $a_1 \preceq' a_2$. The element $\hat{1}$ of a poset $(A, \preceq)$, if it exists, satisfies $x \preceq \hat{1}$ for every $x \in A$.

In completing the proof, it becomes necessary to introduce a family of labelled graphs. For $X = \{x_1, x_2, \ldots, x_k\}$, define a graph P_X on vertex set

$$V(P_X) = \{x_1, x_2, \ldots, x_k\} \cup \{y_0, y_1, \ldots, y_{k+1}\}$$

and with edge set

$$E(P_X) = \{x_i y_i, x_i y_{i+1}, y_i y_{i+1} : i = 1, 2, \ldots, k\} \cup y_0 y_1.$$

Proof of Theorem 5.4. Let $X = \{x_1, x_2, \ldots, x_k\}$. For each $x \in X$, let U_x denote the set of edges $\{x_i y_i : x \leq x_i\}$, and consider the graph $P_X \setminus U_x$. We will show that $(\{P_X \setminus U_x : x \in X\} \cup P_X, \preceq)$ and $(X, \leq) \cup \hat{1}$ are isomorphic as posets. If $P_X \setminus U_x \preceq P_X \setminus U_w$ for some $x, w \in X$, then $U_w \subseteq U_x$ and $\{z \in X : z \leq w\} \subseteq \{z \in X : z \leq x\}$. Therefore, $x \leq w$. Suppose conversely, that $x \leq w$. Then $\{z \in X : z \leq w\} \subseteq \{z \in X : z \leq x\}$ and $P_X \setminus U_x \preceq P_X \setminus U_w$. Hence, the two posets are isomorphic. All of the graphs $P_X \setminus U_x$ have n vertices, and so live in $\mathcal{G}_n$. Therefore $(X, \leq)$ is isomorphic to an induced subposet of $\mathcal{G}_n$. The graph P_X acts as the $\hat{1}$ in $(X, \leq) \cup \hat{1}$. Moreover, if $(X, \leq)$ has a $\hat{1}$, then we could have taken the poset $X' = X \setminus \hat{1}$ in the hypothesis of the theorem. In this case, $X' \cup \hat{1} = X$ is isomorphic to an induced subposet of $\mathcal{G}_{n-2}$. $\square$

5.3 Upper bounds for the trees

Trees are as fundamental to connected graphs as prime numbers are to integers. The stars and paths are exceptional trees, and the wheel, W_{n-1}, has the property that $S_n \preceq W_{n-1}$ and $P_n \preceq W_{n-1}$. However, for n sufficiently large, there can be arbitrarily many trees T on n vertices such that $T \not\preceq W_{n-1}$. For example, if T is a tree on n

vertices having two vertices of degree at least 4, then $T \not\preceq W_{n-1}$, since the wheel has at most one vertex of degree greater than three, namely the center vertex.

The wheel is an upper bound on the star and path. We want more than that. If $\{H, G_1, G_2, \ldots, G_k\}$ are members of $\mathcal{G}_n$, then we say that H is an *upper bound of minimal size on* $\{G_1, G_2, \ldots, G_k\}$ if $G_i \preceq H$ for all $i = 1, 2, \ldots, k$ and for every graph H' with $|E(H')| < |E(H)|$ there is some i such that $G_i \not\preceq H'$. Notice that this definition contrasts the common notion of a least upper bounds in general posets.

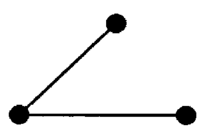
Certainly, such graphs exist, for K_n is an upper bound on every tree. We are interested in graphs with this property but which are as small (in edge size) as possible. We are led to the following tantalizing problem.

Problem 5.5 *For each $n \geq 3$, find an upper bound of minimal size on the collection of trees, $A_{n,n-1}$.*

We have solved the problem for $n = 3, 4, \ldots, 7$. The following theorem summarizes the findings.

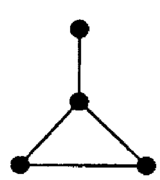
Theorem 5.6

- *The upper bound of minimal size on $A_{3,2}$ has size 2 and is the graph, $G6$.*



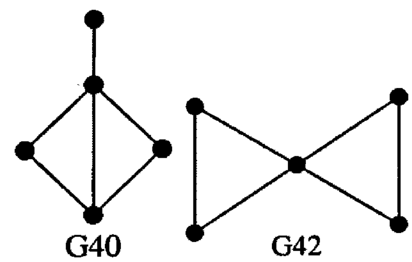
G6

- *The upper bound of minimal size on $A_{4,3}$ has size 4 and is the graph, $G15$.*

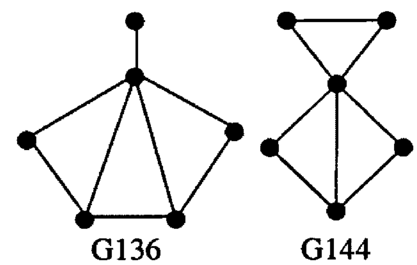


G15

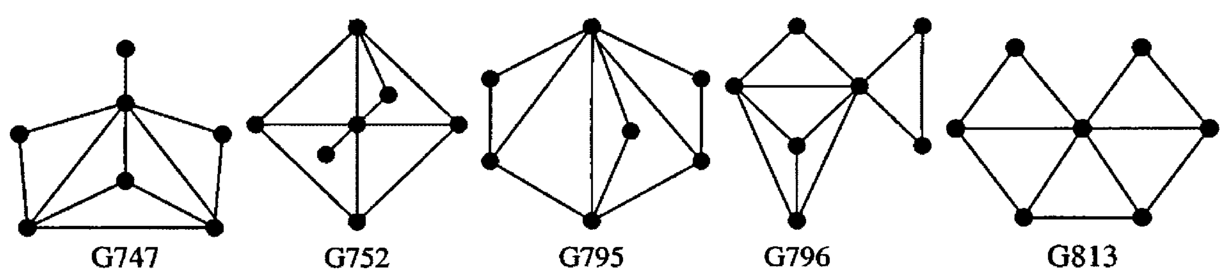
- *The upper bounds of minimal size on $A_{5,4}$ have size 6 and are the graphs, $G40$ and $G42$.*



- *The upper bounds of minimal size on $A_{6,5}$ have size* 8 *and are the graphs,* $G136$ *and* $G144$.



- *The upper bounds of minimal size on $A_{7,6}$ have size* 11 *and are the graphs,* $G747$, $G752$, $G795$, $G796$, *and* $G813$.



The proof of Theorem 5.6 requires the manual checking of a number of candidates; however, most cases are dealt with handily using the following necessary condition.

Lemma 5.7 *If G is an upper bound on the collection of trees on n vertices, $A_{n,n-1}$, then*

- *G has a Hamilton path and*
- *G has a vertex of degree $n-1$.*

Proof. In general, G must have every tree on n vertices as a spanning subgraph. In particular, G contains a copy of P_n and S_n. □

It is left to the reader to verify that the graphs given in Theorem 5.6 are upper bounds on the specified class of trees. We will now discuss why these are in fact the only graphs of minimal size having this property.

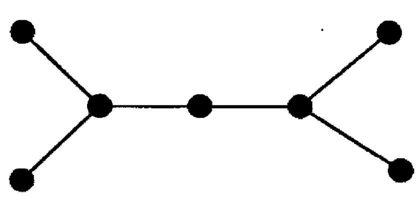
$n = 3$ *case:* It is clear that $G6$ is the upper bound of minimal size on $A_{3,2}$. It is the only tree on 3 vertices and is an upper bound on itself!

$n = 4$ *case:* Of all graphs on 4 vertices and 4 edges, the graph $G15$ is the only graph which does not violate the conclusion of Lemma 5.7 and no graph with strictly fewer edges can be an upper cound on $A_{4,3}$.

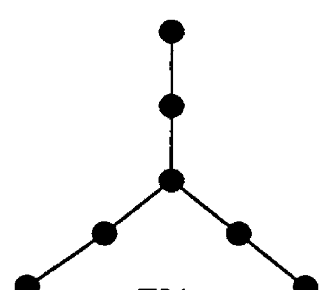
$n = 5$ *case:* Lemma 5.7 rules out all graphs with 5 vertices and 5 edges. The only graphs on 5 vertices and 6 edges not ruled out by Lemma 5.7 are $G40$ and $G42$.

$n = 6$ *case:* The method of argument is identical to the previous two cases.

$n = 7$ *case:* The situation gets a little more interesting in this final case because Lemma 5.7 fails to rule out some of the graphs which are not upper bounds on $A_{7,6}$. Three graphs on 10 edges and five graphs on 11 edges must be ruled out by some other means. For each such graph G we give a counterexample by exhibiting a tree T for which $T \not\preceq G$. The only two counterexamples needed, the trees $T20$ and $T24$, are shown below.

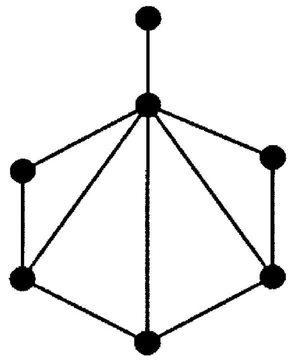


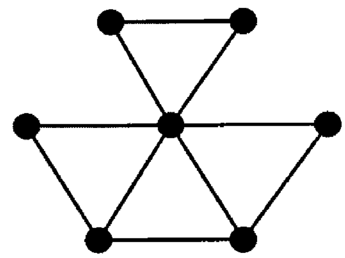
T20



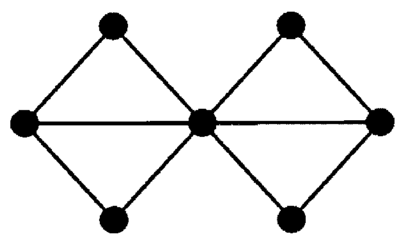
T24

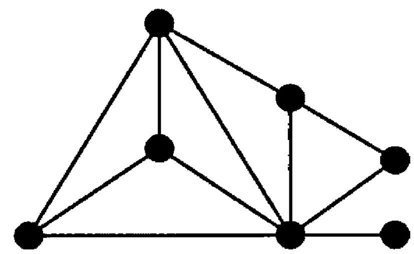
The final eight graphs may now be ruled out.



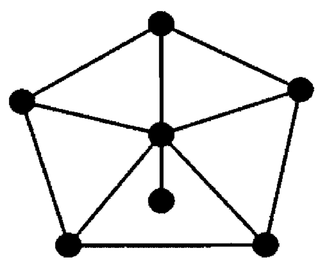


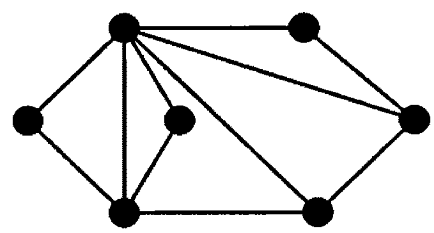
$T20 \not\preceq G618$ G618 G671 $T20 \not\preceq G671$



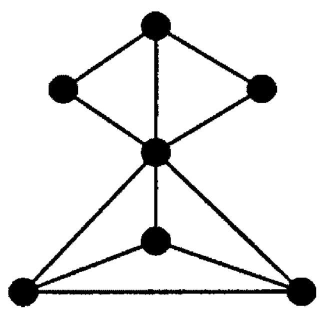


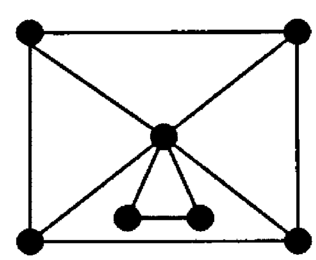
$T24 \not\preceq G672$ G672 G753 $T24 \not\preceq G753$





$T20 \not\preceq G775$ G775 G794 $T24 \not\preceq G794$





$T24 \not\preceq G814$ G814 G815 $T20 \not\preceq G815$

We are not hopeful that a complete answer to Problem 5.5 will be found. However, an investigation into the asymptotic size of these special graphs may bear fruit.

Several of the graphs in Theorem 5.6 have interesting chromatic polynomials, which are given below in "tree form". The *tree form* of a polynomial in λ is found by factoring the polynomial into terms of the type $\lambda(\lambda - 1)^n$, for $n = 0, 1, \ldots$.

$G6$:

$$\lambda(\lambda-1)^2$$

$G15$:

$$\lambda(\lambda-1)^3-\lambda(\lambda-1)^2$$

$G40$ and $G42$:

$$\lambda(\lambda-1)^4-2\lambda(\lambda-1)^3+\lambda(\lambda-1)^2$$

$G136$ and $G144$:

$$\lambda(\lambda-1)^5-3\lambda(\lambda-1)^4+3\lambda(\lambda-1)^3-\lambda(\lambda-1)^2$$

$G795$ and $G813$:

$$\lambda(\lambda-1)^6-5\lambda(\lambda-1)^5+10\lambda(\lambda-1)^4-10\lambda(\lambda-1)^3+5\lambda(\lambda-1)^2-\lambda(\lambda-1)$$

Observe that the absolute values of the coefficients are binomial coefficients. In particular, they are of the form

$$\left\{\binom{n}{k}\right\}_{k=0}^{n}$$

for some n. In other words, they appear as a line in Pascal's triangle. If a chromatic polynomial (expressed in tree form) has this property, we refer to it as *Pascal.*

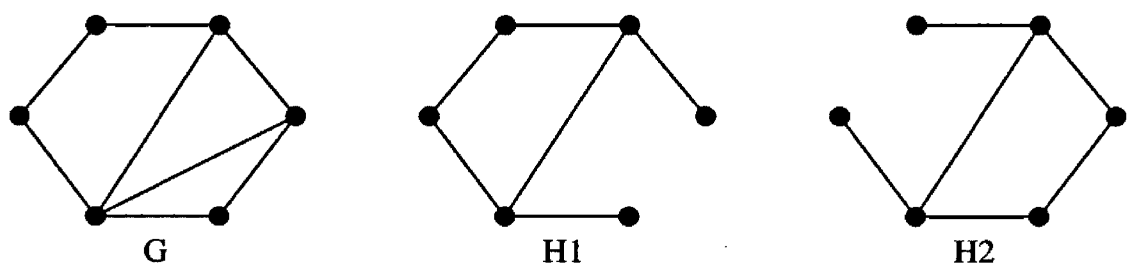
There is an easy method for constructing graphs with Pascal chromatic polynomials of every order. All trees have Pascal chromatic polynomials. Indeed, the chromatic polynomial of any tree on n vertices is $\lambda(\lambda-1)^{n-1}$. If G has a Pascal chromatic polynomial and xy is an edge of G, then by adding a vertex z to $V(G)$ and edges xz and yz to $E(G)$, we obtain another graph having a Pascal chromatic polynomial.

Read recognized this construction and conjectured that every graph with a Pascal chromatic polynomial can be constructed in this way [16]. It is not surprising that they should appear in the discussion of upper bounds on trees. Indeed, they have everything to do with triangles and trees.

5.4 Δ-good sets

Suppose that G is a connected graph and H is a spanning subgraph of G. Is $H \preceq G$? We desire to characterize the subgraphs for which the relation is true.

We are entering an investigation of subgraphs, and the graph G serves as an important point of reference. It is understood, therefore, that a subgraph H satisfies $H \preceq G$ only if there is a sequence of K_3-moves starting with G and ending with H. It is not enough for H to be *isomorphic* to such a subgraph of G. Consider, for example, the spanning subgraphs H_1 and H_2 of the graph G below.



Observe that $H_1 \preceq G$ and H_1 is isomorphic to H_2, yet $H_2 \not\preceq G$. There is no sequence of edges of G that can be successively deleted by K_3-moves to get H_2. Clearly, we can not look to $(\mathcal{G}_n, \preceq)$, a poset of *isomorphism* classes, for answers to these questions; rather, we need to distinguish the subgraphs of G more carefully.

For each edge $e \in E(G)$, define $\Delta(e)$ to be the collection of triangles of G that contain e. Suppose that the sequence $(e_1, e_2, \ldots, e_k)$ consists of some edges of G. We say that $(e_1, e_2, \ldots, e_k)$ is Δ-*good* in G if

$$\text{for all } i = 1, 2, \ldots, k,\ \Delta(e_i) \not\subseteq \bigcup_{j=1}^{i-1} \Delta(e_j). \tag{5.1}$$

We say that a subset $I \subseteq E(G)$ is Δ-*good* in G if there is a Δ-good sequence consisting of the elements of I. The empty set is proclaimed to be Δ-good.

Theorem 5.8 *For every $I \subseteq E(G)$, the relation $G \setminus I \preceq G$ holds if and only if I is Δ-good in G.*

Proof. Let $I = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$ be Δ-good. Then the elements of I can be arranged in a sequence, say $(e_1, e_2, \ldots, e_k)$ such that (5.1) holds. For each $i = 1, 2, \ldots, k$, the edge e_i is contained in a triangle of $G \setminus \{e_j : 1 \leq j < i\}$, whence $G \setminus \{e_j : 1 \leq j \leq i\} \preceq G \setminus \{e_j : 1 \leq j < i\}$. By the transitivity of $\preceq$, we have $G \setminus I \preceq G$.

Suppose that $G \setminus I \preceq G$. Then there is a sequence of K_3-moves starting with G and ending with $G \setminus I$. Suppose the deleted edges, in the order of their deletion, are $e_1, e_2, \ldots, e_k$. For each $i = 1, 2, \ldots, k$, $\Delta(e_i)$ is nonempty in $G \setminus \{e_j : 1 \leq j < i\}$ because the edge e_i can be deleted from $G \setminus \{e_j : 1 \leq j < i\}$. Therefore, in G, we have $\Delta(e_i) \not\subseteq \bigcup_{j=1}^{i-1} \Delta(e_j)$. Now I is Δ-good in G. □

In the next section, we will investigate Δ-good sets in more detail. Our approach will be from the perspective of matroids; however, the discussion simplifies unambiguously to graphs, as is always the case with matroids!

5.5 New matroids from old

The fundamental question of this section is the combinatorics of Δ-good subsets of matroids. Before going on, we are reminded that all matroids in this thesis are finite and simple, unless otherwise stated.

The reader may have realized by now that the K_3-move, defined as an operation for graphs, is easily generalized to an operation for matroids. If an element of a matroid is contained in a circuit of size three, then the deletion of the element is referred to as a K_3-move. The notions of Δ-good sequences and sets are defined for matroids in the natural way.

The partial order, $\preceq$, extends to matroids as well. For a submatroid N of a matroid M, we write $N \preceq M$ if $E(M) \setminus E(N)$ is Δ-good in M. We do not merely mean that N is isomorphic to a submatroid N' of M such that $E(M) \setminus E(N')$ is Δ-good in M.

In Lemma 2.2, we noted the fundamental property of K_3-moves (for graphs); that is, 1-connectivity is preserved under K_3-moves. From the perspective of matroids, we can distill the essentials of this result. The fundamental property of K_3-moves (for matroids) is that rank is preserved by K_3-moves.

Lemma 5.9 *If M is a matroid, $e \in E(M)$, and $M \setminus e \preceq M$, then the rank of $M \setminus e$ is equal to the rank of M.*

Proof. $M \setminus e \preceq M$ implies that e is in a circuit of M. □

Corollary 5.10 *If M is a matroid and N is a submatroid of M, then $N \preceq M$ only if N spans M; that is, $cl_M(E(N)) = M$.* □

Let $\mathcal{I}_\Delta(M)$ denote the Δ-good subsets of $E(M)$ and define

$$M_\Delta(M) = (E(M), \mathcal{I}_\Delta(M)).$$

We are led to ask the following question for each matroid, M:

Question 5.11 *Is $M_\Delta(M)$ a matroid?*

Of course, if M has no circuits of size three, then $\mathcal{I}_\Delta(M)$ is empty and $M_\Delta(M)$ consists entirely of loops. We will now answer the question in the affirmative for a few classes of matroids. The following result will be helpful in doing so.

Lemma 5.12 *If M is a matroid and $K \preceq M$ for every spanning submatroid K of M, then $M_\Delta(M) = M^*$.*

Proof. Under the hypothesis of the lemma, it follows by the necessary condition supplied in Corollary 5.10 that the Δ-good subsets of M are precisely the complements of the spanning subsets of M. Therefore, the maximal Δ-good sets are precisely the complements of bases of M, whence $M_\Delta(M) = M^*$. □

Theorem 5.13 *$M_\Delta(M(K_n))$ is a matroid and is equal to $M^*(K_n)$.*

Proof. Let H be a connected spanning subgraph of K_n. The poset $(\mathcal{G}_n, \preceq)$ has a unique maximal element, namely K_n. Therefore, $H \preceq K_n$. The result follows from Lemma 5.12. $\square$

In some sense, the projective geometries $PG(r-1,q)$ are the matroid analogues of complete graphs. It is not surprising, therefore, that a similar result holds for this class of matroids.

Theorem 5.14 *$M_\Delta(PG(r-1,q))$ is a matroid and is equal to $PG(r-1,q)^*$.*

Proof. Let H be a spanning submatroid of $PG(r-1,q)$. We will show that $H \preceq PG(r-1,q)$. If $H = PG(r-1,q)$, then we are done. Assume inductively that $H' \preceq PG(r-1,q)$ for all spanning submatroids H' of $PG(r-1,q)$ with $|H| = |H'|-1$. There is an element $v \in PG(r-1,q) \setminus H$. Since H is spanning, there exist elements $h_1, h_2, \ldots, h_k$ in $E(H)$ such that $C = \{v, h_1, h_2, \ldots, h_k\}$ is a circuit of $PG(r-1,q)$. Since $PG(r-1,q)$ is a simple matroid, k is greater than one. We assume that v and $h_1, h_2, \ldots, h_k$ are chosen so that k is minimal. Two cases may be distinguished.

Case $k = 2$: In this case, $\{v, h_1, h_2\}$ is a circuit of $H \cup v$, whence $H \preceq H \cup v$. It follows by induction and the transitivity of $\preceq$ that $H \preceq PG(r-1,q)$.

Case $k \geq 3$: We show by contradiction that this case cannot occur. It is a property of projective geometries that for every pair of elements a and b, there is a third element c in the span of $\{a,b\}$; that is, $\{a,b,c\}$ is a circuit of size three. Thus there exists an element h' of $PG(r-1,q)$ such that $\{h', h_1, h_2\}$ is a circuit of size three. If $h' \in PG(r-1,q) \setminus H$, then we arrive at a contradiction with the minimality of k. The alternative is that $h' \in H$. Now $C \setminus v$ is independent; therefore, $h' \in H \setminus \{h_1, h_2, \ldots, h_k\}$. Using strong circuit elimination, we find a circuit $C' \subseteq C \cup \{h_1, h_2, h'\}$ such that $v \in C'$ and $h_1 \notin C'$. If $|C'| < |C|$, then the minimality of k is contradicted. The only alternative is that $C' = \{v, h', h_2, h_3, \ldots, h_k\}$. If this is the case, then the symmetric difference of C' and C is the pair $\{h', h_1\}$, a contradiction, because $PG(r-1,q)$ is simple. We conclude, therefore, that $h' \in PG(r-1,q) \setminus H$, in which case it follows that $H \preceq PG(r-1,q)$.

H was chosen arbitrarily; thus, $H \preceq PG(r-1,q)$ for every spanning matroid H. The conclusion of the theorem now follows by Lemma 5.12. $\square$

The only property of the projective geometries used in the proof is that every pair of elements is contained in a circuit of size three. In light of this, the following result is self-evident.

Theorem 5.15 *$M_\Delta(U_{2,n})$ is a matroid and is equal to $U^*_{2,n}$.*

We conclude the list of examples with the wheels.

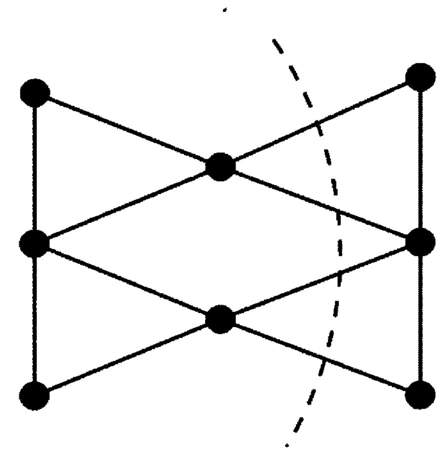
Theorem 5.16 *$M_\Delta(M(W_n))$ is a matroid and is equal to $M^*(W_n)$.*

Proof. Let H be a connected spanning subgraph of W_n. Then either $H = W_n$ or there is an edge $e \in W_n \setminus H$ such that e is contained in a triangle of H. Therefore, $H \preceq H \cup e$. By induction and transitivity, we have $H \preceq W_n$. Therefore, $H \preceq M(W_n)$ for every spanning submatroid H of $M(W_n)$. The result now follows from Lemma 5.12. $\square$

Admittedly, we have answered Question 5.11 in the affirmative only for some well-behaved classes of matroids. A systematic assessment of all matroids has proven to be difficult; yet, we believe that the following will be shown in further research.

Conjecture 5.17 *For every matroid M, $M_\Delta(M)$ is a matroid.*

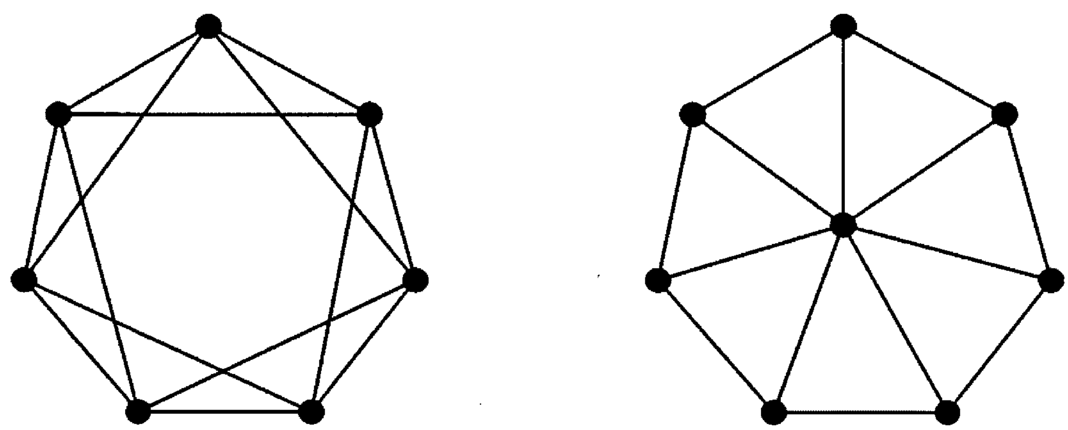
It will not be the case, however, that $M_\Delta(M)$ is always the dual of M. We illustrate this fact with the following graph G.



$$M_\Delta(M(G)) \neq M(G)^*$$

The cocircuit of $M(G)$ indicated in the diagram does not correspond to a minimal non-Δ-good subset. Thus, $M_\Delta(G) \neq M^*(G)$. However, $M_\Delta(G)$ is isomorphic to the direct sum, $M^*(K_3) \oplus M^*(K_3) \oplus M^*(K_3) \oplus M^*(K_3)$, a fact which is left to the reader to verify.

There are non-isomorphic matroids M_1 and M_2 for which the constructions $M_\Delta(M_1)$ and $M_\Delta(M_2)$ are isomorphic matroids. Consider the following example.



The matroids $M(G)$ and $M(W_7)$ are non-isomorphic; however, $M_\Delta(M(G))$ is isomorphic to $M_\Delta(M(W_7))$.

This line of investigation appears to have a ways to go. The thesis, however, ends here.

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