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Riemann surfaces

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RIEMANN SURFACES

by

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CONTENTS

CHAPTER I

REPRESENTATIONS OF RIEMANN SURFACES

In studying algebraic functions, we are interested in a function, $w = w(z)$, where w is an <u>algebraic function</u> of z, which satisfies an equation of the form,

$$\sum_{i=0}^{n} a_i w^i = 0,$$

with $a_i \varepsilon C(z)$, the field of complex numbers with z adjoined. For the sake of convenience, this function w will often be represented as $\sum_{i=0}^{n} a_i w^i = 0$.

<u>Rational functions</u> are elements of $C(z,w)$, the field of complex numbers with z and w adjoined. The most general such function has the form,

$$R(z,w) = \frac{\sum_{i=0}^{n} b_i w^i}{\sum_{j=0}^{m} c_j w^j},$$

with b_i and $c_j \varepsilon C(z)$.

The simplest such function of degree 1 in w is of the form

$$a_1 w + a_0 = 0, \ a_1 \neq 0.$$

This function is single-valued, for to each z there corresponds one and only one w,

$$w = -\frac{a_o}{a_1}.$$

However, if we have an algebraic function of the second degree in w, of the form

$$a_2w^2 + a_1w + a_o = 0,\ a_1{}^2 - 4a_2a_o \neq 0,$$

we see

$$w = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_oa_2}}{2a_2},$$

and there are two values of w which correspond to each z. In this case, w is not a single-valued function of z.

To see this more clearly, we can simplify the expression $a_2w^2 + a_1w + a_o = 0$, by letting

$$\mu = 2a_2w + a_1.$$

The expression μ is a single-valued function of w. Then we have

$$\mu^2 = 4a_2^2w^2 + 4a_2a_1w + a_1^2 = 0.$$

Multiplying $a_2w^2 + a_1w + a_o = 0$ by $4a_2$, we have

$$4a_2^2w^2 + 4a_2a_1w + 4a_2a_o = 0,$$

$$(4a_2^2w^2 + 4a_2a_1w + a_1^2) - (a_1{}^2 - 4a_2a_o) = 0, \text{ or}$$

$$\mu^2 - (a_1^2 - 4a_2a_o) = 0.$$

Then, in general, by use of a linear transformation, we can consider every second degree equation in w as being of the form

$$\mu^2 - p(z) = 0,$$

where p(z) is the polynomial in z, $a_1^2 - 4a_2a_1$.

If $p(z) = z$, we have

$$w^2 = z = re^{i\theta}.$$

and

$$w = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}.$$

If the point z follows a path winding counterclockwise around the origin, we see that θ is constantly increasing, and if z returns to the starting point, θ has increased by 2π. Thus, on return to the starting point,

$$w = \sqrt{re^{i(\theta + 2\pi)}} = \sqrt{r}e^{\frac{i(\theta + 2\pi)}{2}} = \sqrt{r}e^{\frac{i\theta}{2} + \pi} = -\sqrt{r}e^{\frac{i\theta}{2}},$$

and w is not single-valued on the z-plane.

As Riemann realized, the simplest way to make this function single-valued is to define it on a new surface. One way is the familiar way, cutting the complex plane from 0 to infinity on the positive x-axis, placing a similarly cut plane above this "sheet", and connecting the two sheets in the following way:

Attach the "negative" side ($y < 0$) of the cut on the bottom sheet to the "positive" side ($y > 0$) of the cut on the upper sheet. Then attach the negative side of the cut on the upper sheet to the positive side of the cut on the bottom sheet.

Then when z winds once around the origin, it passes from the bottom sheet to the upper sheet, as it passes the cut from 0 to infinity. When z winds around the origin once again, it goes back to the lower sheet, across the cut from 0 to infinity. This corresponds to the fact that when z winds around the origin twice, θ increases by 4π. Thus, if $w = \sqrt{z}$, after z winds around the origin twice, we have

$$w = \sqrt{z} = \sqrt{re^{i(\theta + 4\pi)}} = \sqrt{r}e^{\frac{i(\theta + 4\pi)}{2}}$$

$$= \sqrt{r}e^{\frac{i\theta}{2} + 2\pi} = \sqrt{r}e^{\frac{i\theta}{2}} .$$

In order to make w single-valued on this surface, the lower sheet is designated as Sheet I, and each point z on this sheet is renamed

$$(z, \sqrt{z}),$$

where $z = \sqrt{r}e^{\frac{i\theta}{2}}$, and $0 \leq \theta \leq 2$. The upper sheet is Sheet II, and each point z on this sheet is renamed

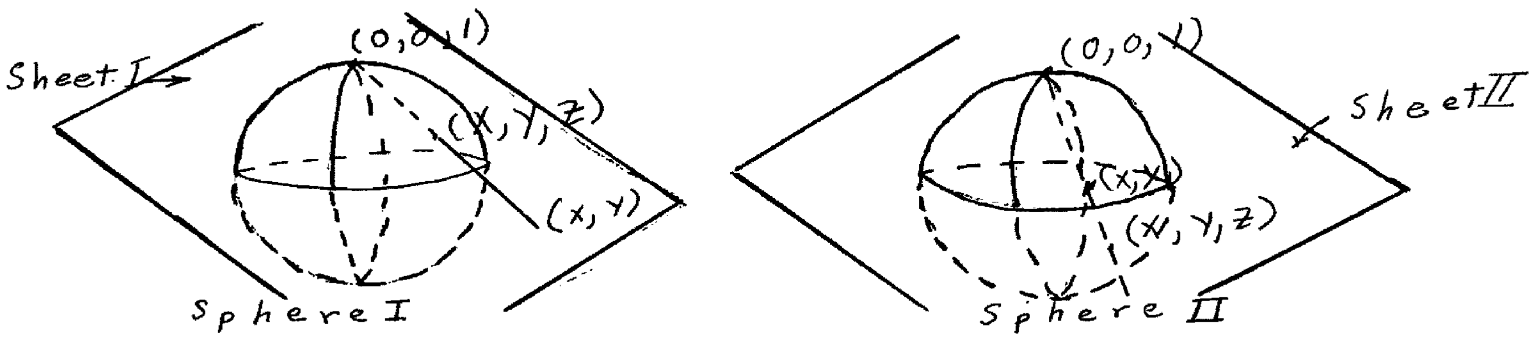
$$(z, -\sqrt{z}),$$

$-\sqrt{z}$ corresponding to $2\pi \leq \theta \leq 4\pi$. Now we have a surface corresponding to the function $w^2 = z$, with ordered pairs (z, w), and w is single valued on this surface.

The surface constructed in this way cannot be realized in three dimensional Euclidean space, E^3. It is desirable to construct a topologically equivalent surface realizable in E^3. We can do this by first mapping the two sheets, I and II, topologically onto two spheres,

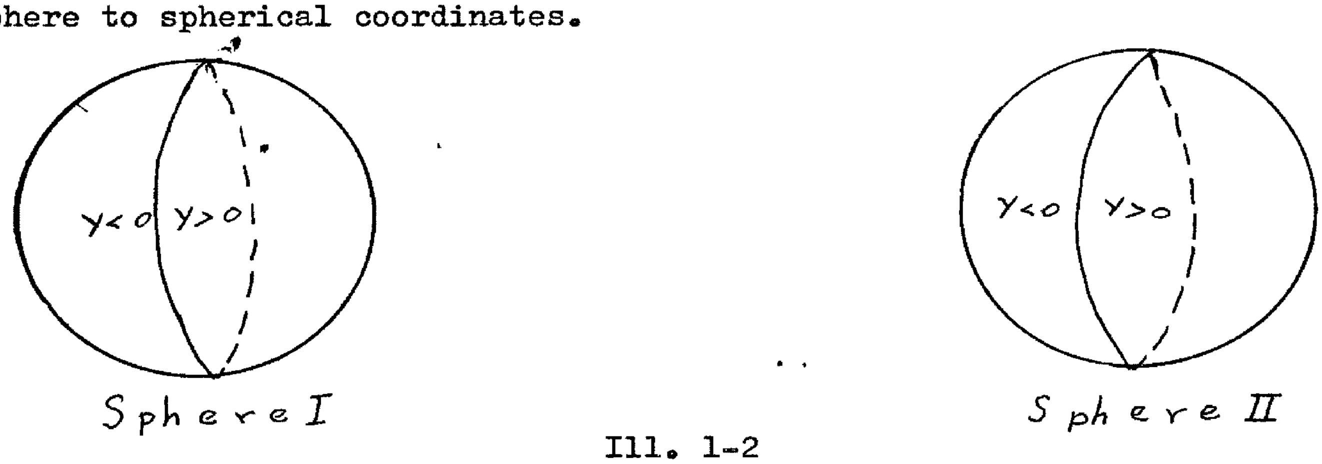
$$X^2 + Y^2 + Z^2 = 1.$$

To do this, stereographic projection is used. First, we let the z-plane coincide with the plane X = 0. Then a line is passed through the point (0, 0, 1) and the point (x, y) of the z-plane. The point (X, Y, Z), where the line cuts the sphere, is the projection of the point (x,y) on the sphere. As can be seen, the point (0, 0, 1) is the image of the point at infinity.



Ill. 1-1

Then the spheres are cut along the meridian circle from the south pole to the north pole, corresponding to the cuts along the positive x-axes. (Ill. 1-2). Next, the two spheres are mapped topologically onto the two hemispheres of a third sphere, called Sphere III. In order to do this, we first change the rectangular coordinates of the sphere to spherical coordinates.



Ill. 1-2

We know

$$X = \cos \theta \cos \varphi$$

$$Y = \sin \theta \cos \varphi$$

$$Z = \sin \varphi,$$

where φ is the angle the line on (X, Y, Z) and $(0, 0, 0)$ makes with the plane $Z = 0$, and θ is the angle made by the intersection of the line through the origin and the point (x, y) on the original (x, y)-plane, and the x-axis. Thus θ is the angle used in changing rectangular coordinates to polar coordinates, where we have

$$x = r \cos \theta,$$

$$y = r \sin \theta.$$

Now the point (X, Y, Z) on the sphere has the coordinates

$$(\theta, \varphi), \ 0 \leq \theta \leq 2\pi, \ -\pi \leq \varphi \leq \pi.$$

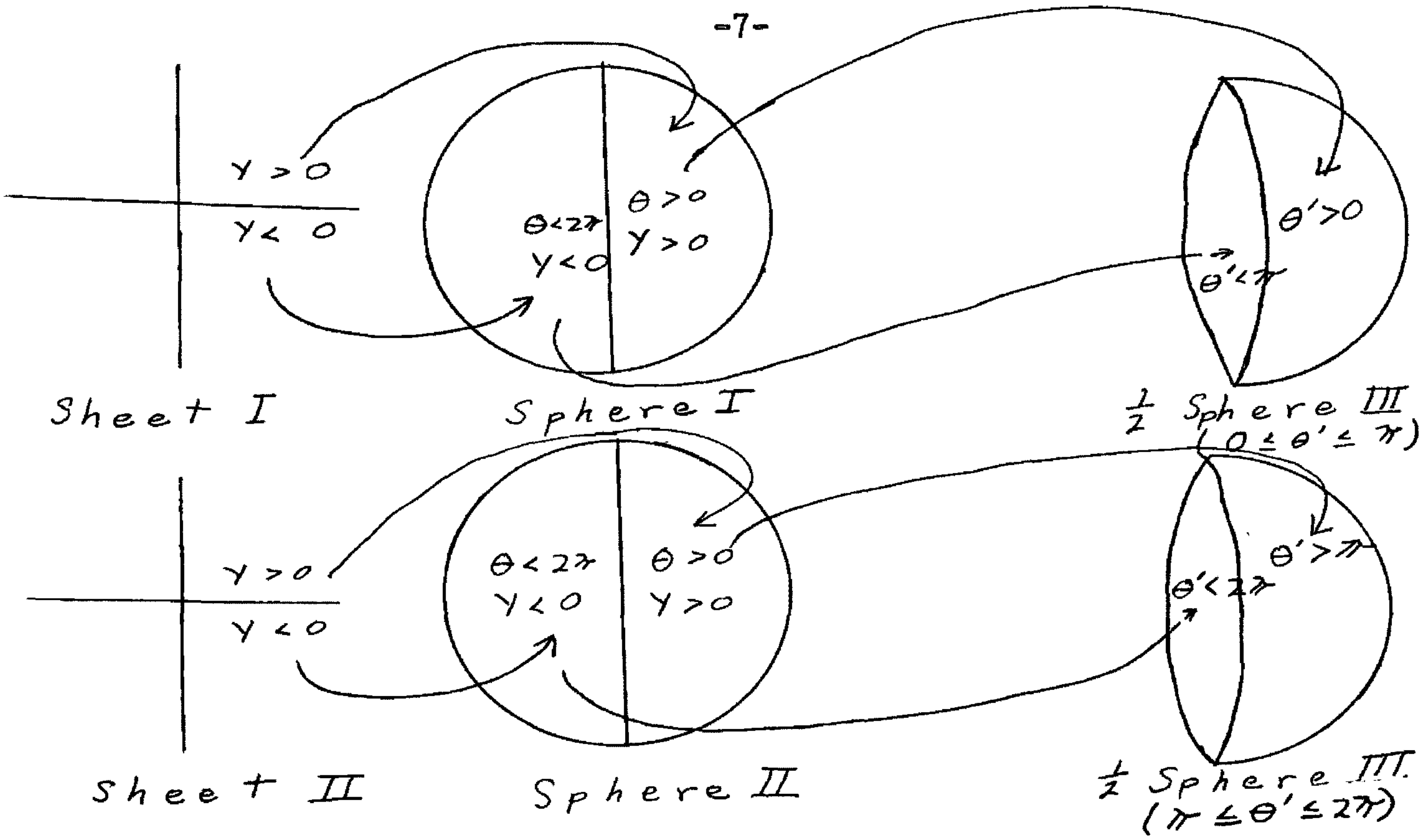
To take the points on the two spheres into points on the two hemispheres, for points on Sphere I, we make the transformation Γ_1, with

$$\Gamma_1[(\theta, \varphi)] = (\tfrac{\theta}{2}, \varphi).$$

For points on Sphere II, we make the transformation Γ_2, with

$$\Gamma_2(\theta, \varphi) = (\pi + \tfrac{\theta}{2}, \varphi).$$

Now we have a 1-1 mapping of Spheres I and II onto Sphere III, and we shall see that an image of a point z passes from the image of Sheet I to the image of Sheet II in the same way that z passes from Sheet I to Sheet II.



Ill. 1-3

The points on Sheet 1 with $y > 0$ map into points on Sphere III with $Y > 0$. The points with $y < 0$, but close to 0, map into points with $Y > 0$, but close to 0, and in spherical coordinates with θ close to, but less than, π on Sphere III. The points on Sheet II with $y > 0$, but close to 0, map into points with $Y < 0$, but close to 0, and with the spherical coordinate θ less than 2π, but near 2π.

If a point z on Sheet I has $x > 0$, and $y < 0$, but near 0, it is on the negative side of the cut along the positive x-axis, and if z continues in a counterclockwise direction, it will pass to Sheet II across the cut on the x-axis. The image of this point z on Sphere I has as its θ-coordinate, $\theta < 2\pi$, but nearly 2π. Its image in the hemisphere of Sphere III with $0 \leq \theta \leq \pi$ has θ very near π. If it continues in a counterclockwise direction

direction (θ increasing), it will soon pass to the hemisphere with $\pi \leq \theta \leq 2\pi$, which is the image of Sphere II, which is in turn, the image of Sheet II. Thus the image of z in Sphere III moves from the image of Sheet I to the image of Sheet II, and passes over the line $\theta = \pi$, which is the image of the cut over which z had to pass, traveling in a counterclockwise direction, to go from Sheet I to Sheet II. Similarly, the line $\theta = 2\pi$ or 0 is the image of the line over which z must pass to go from Sheet II to Sheet I, again traveling in a counterclockwise direction.

Thus the mapping of Sheets I and II onto Sphere III is a 1-1 mapping, under which a point z can wind around the origin in the same manner as on the original Riemann surface composed of Sheets I and II.

If the function $w^2 - p(z) = 0$ is of the form

$$w^2 = a_1 z + a_o,$$

we cannot make a cut from 0 to infinity, as before. We know

$$w = \sqrt{a_1 + a_o} = \sqrt{a_1}\sqrt{z + \frac{a_o}{a_1}}.$$

If the point z winds around the point $-\frac{a_o}{a_1}$, we can write z as

$$z = -\frac{a_o}{a_1} + re^{i\theta}.$$

As θ increases by 2π, we have

$$w = \sqrt{a_1}\sqrt{-\frac{a_o}{a_1} + re^{i\theta + 2\pi i} + \frac{a_o}{a_1}} =$$

$$\sqrt{a_1}\sqrt{re^{i(\theta + 2\pi)}} = \sqrt{a_1}\sqrt{r}e^{\frac{i\theta}{2} +} = -\sqrt{a_1}\sqrt{r}e^{\frac{i\theta}{2}}.$$

In order to make a surface on which this function is single-valued, we proceed as before, except that we make the cut from $-a_o/a_1$ to infinity, along the line through $z = 0$ and $z = -a_o/a_1$.

The function

$$w^2 = a_2z^2 + a_1z + a_o$$

can be factored, so that

$$w^2 = a_2(z-r_1)(z-r_2),$$

where r_1 and r_2 are the roots of the equation

$$a_2z^2 + a_1z + a_o = 0.$$

Then

$$w = \sqrt{a_2}\sqrt{z-r_1}\sqrt{z-r_2}.$$

If $z = r_1 + re^{i\theta}$, when z winds around r_1 but not around r_2, so that θ increases by 2π, we have

$$w = \sqrt{a_2}\sqrt{r_1 + re^{i(\theta + 2\pi)} - r_1}\sqrt{z-r_2}$$

$$= \sqrt{a_2}\sqrt{re^{i(\theta + 2\pi)}}\sqrt{z-r_2} = \sqrt{a_2}\sqrt{r}e^{\frac{i\theta}{2} + i\pi}\sqrt{z-r_2}$$

$$= - \sqrt{a_2}\,\sqrt{r}e^{\frac{i\theta}{2}}\,\sqrt{z-r_2} = - \sqrt{a_2}\sqrt{z-r_1}\,\sqrt{z-r_2}.$$

The result is the same when z winds around r_2 but not r_1. If z winds around both r_1 and r_2, then the arguments of both $z-r_1$ and $z-r_2$ are increased by 2π, or

$$w = \sqrt{a_2}\sqrt{z-r_1}e^{i\pi}\,\sqrt{z-r_2}e^{i\pi}$$

$$= \sqrt{a_2}\sqrt{z-r_1}\sqrt{z-r_2}e^{2i\pi} = \sqrt{a_2}\sqrt{z-r_1}\sqrt{z-r_2}.$$

Then the cut is made along the line from r_1 to r_2, and the branch points are r_1 and r_2. By attaching Sheets I and II along this cut, as before, and identifying points on Sheets I and II as usual, w is single-valued on this Riemann surface. To make this surface realizable in E^3, we map the two sheets onto two spheres, except that this time we map the points r_1 and r_2 into (0, 0, -1) and (0, 0, 1). This may be done by first mapping each z-plane into itself by the transformation

$$\mu = \frac{-zr_2 + r_1r_2}{zr_1 - r_1r_2},$$

so that r_1 goes into $\mu = 0$ and r_2 goes into $\mu = \infty$. These images of Sheets I and II are then mapped onto Spheres I and II, which are in turn mapped onto Sphere III, as before.

However, if

$$w^2 = \sum_{i=0}^{3} a_i z^i,$$

the situation is changed. If we factor $\sum_{i=0}^{3} a_i z^i$, we have

$$w^2 = a_3(z-r_1)(z-r_2)(z-r_3),$$

where r_1, r_2, and r_3 are the zeroes of w. Then

$$w = \sqrt{a_3}\ \sqrt{\prod_{i=1}^{3}(z-r_i)}.$$

If z winds around two of the roots, such as r_1 and r_2, then the situation is the same as in the case where p(z) was of degree 2, namely the argument of $(z-r_1)(z-r_2)$ changes by 4π and the sign of w is not changed. Thus, if we make a cut from r_1 to r_2 and attach Sheets I and II as before, when z winds counterclockwise around r_1 or r_2, but not both, and not around r_3, z passes from Sheet I to Sheet II. However, when z winds around r_1 or r_2, but not both, and also around r_3, the argument of w^2 changes by 4π, and thus the sign of w does not change. This suggests z passing from Sheet I to Sheet II along the cut from r_1 to r_2, and then in some way returning to Sheet I. Therefore, another cut is made from r_3 to infinity, along the line through z = 0 and $z = r_3$, and the two sheets are attached as usual along this cut.

In the same manner, if

$$w^2 = \sum_{i=0}^{4} a_i z^i = a_4 \prod_{i=1}^{3}(z-r_i),$$

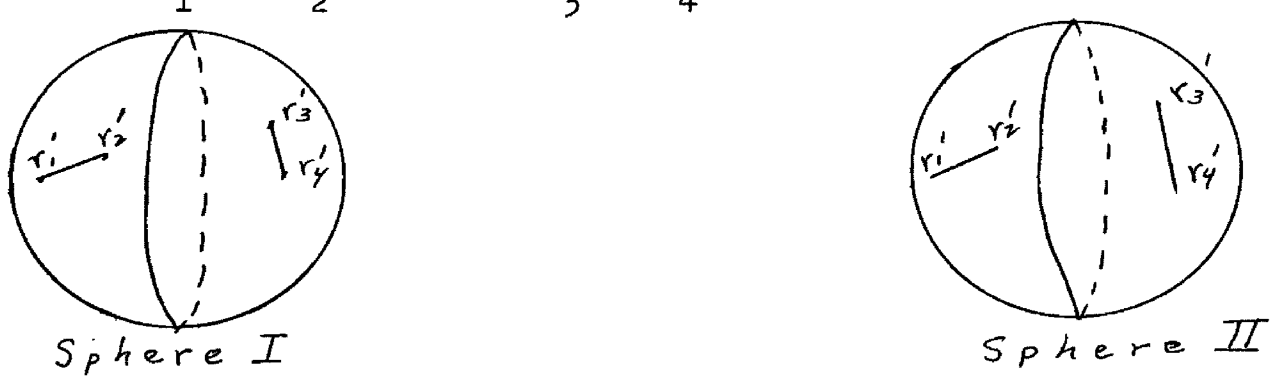
by making the cuts from r_1 to r_2 and from r_3 to r_4, and attaching Sheets I and II as usual,

$$w = \sqrt{a_4}\sqrt{\prod_{i=1}^{4}(z-r_i)}$$

is single-valued on this Riemann surface.

However, this new surface, with two cuts instead of one, is not topologically equivalent to a sphere. On a sphere, any closed curve may be deformed into a point, but a closed curve on this surface, for instance, which goes from Sheet I to Sheet II over one cut and continues back to Sheet I over the other cut, cannot be deformed into a point. However, the two sheets can be mapped onto a torus, to which the surface is topologically equivalent, as follows:

First the two planes are mapped stereographically onto Spheres I and II, as before, and then cuts are made from r_1' to r_2', the images of r_1 and r_2, and from r_3' to r_4'. (Ill. I-4). Next

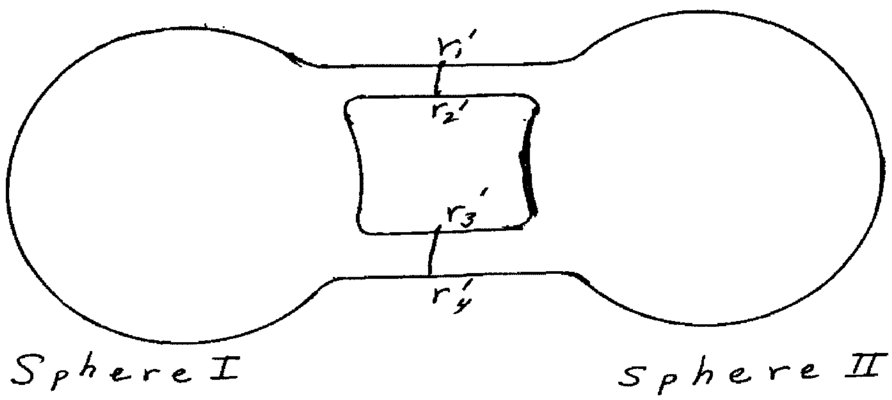


Ill. I-4

we imagine the cuts from r_1' to r_2' and from r_3' to r_4' being pulled out in tubes. (Ill. I-5). We know a surface of this type is topologically equivalent to the sphere with the cuts described. Now there are two spheres with two tubes, each, extended. Next, r_1' and r_2' on Sphere I are matched with r_1' and r_2', respectively, on Sphere II.

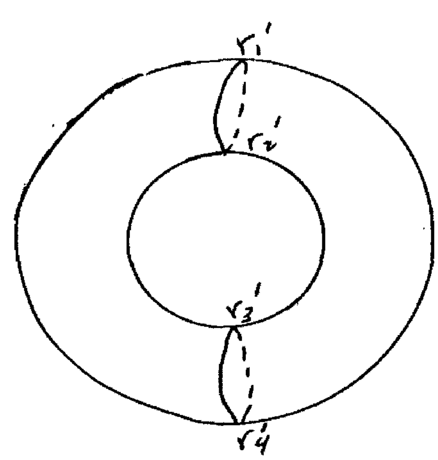
Ill. I-5

Similarly, r_3' and r_4' on Sphere I are matched with r_3' and r_4' on Sphere II respectively.



Ill. I-6

By imagining this surface to be rubber, as in a balloon, see that it can be deformed easily in the shape of a torus or doughnut. (Ill. I-7)



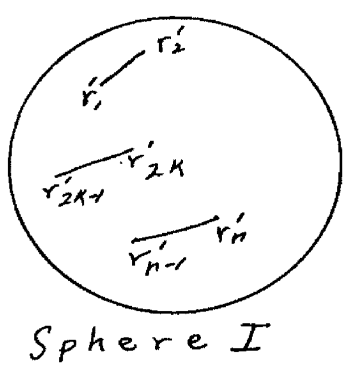
Ill. I-7

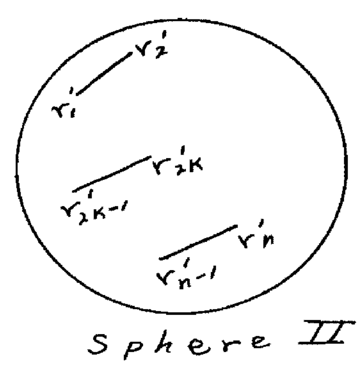
If we have

$$w^2 = \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{n} (z-r_i),\ n \geq 5,$$

to make the usual Riemann surface, we make cuts from r_1 to r_2, ..., r_{2k-1} to r_{2k}, ..., and from r_n to ∞ if n is odd, or from r_{n-1} to r_n if n is even, and attach Sheet I to Sheet II in the usual manner. Thus, if n is odd, we have the two sheets attached along $\frac{n+1}{2}$ cuts, and if n is even, along $\frac{n}{2}$ cuts.

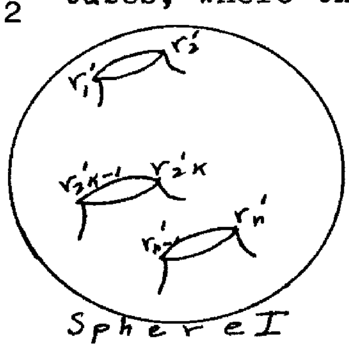
To construct a topologically equivalent surface realizable in E^3, we first map the two sheets stereographically onto two spheres, as was described for the case when $n \leq 4$. Then, as in the case when $n = 3$ or 4, the two spheres are cut along lines corresponding to the cuts on the two sheets. (Ill. I-8)

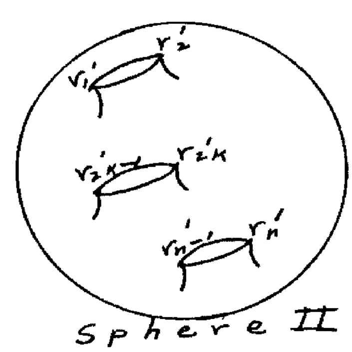




Ill. I-8

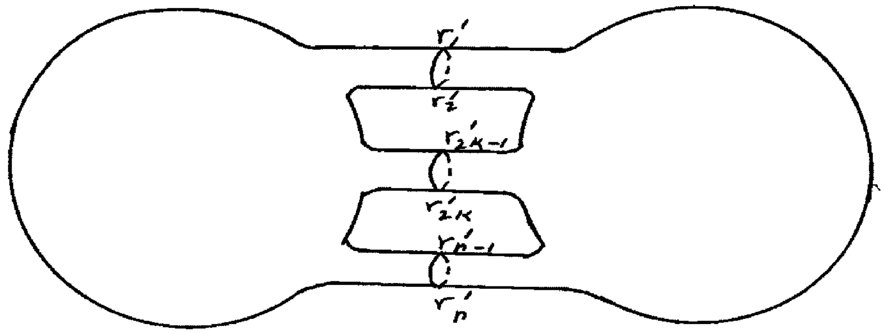
Again visualizing the two spheres as rubber balloons, we pull out $\frac{n+1}{2}$ or $\frac{n}{2}$ tubes, where the cuts were made.





Ill. I-9

The ends of the tubes are matched as before, and we now have two spheres with $\frac{n+1}{2}$ tubes, if n is odd, or $\frac{n}{2}$ tubes, if n is even, connecting the two spheres. (Ill. I-10).

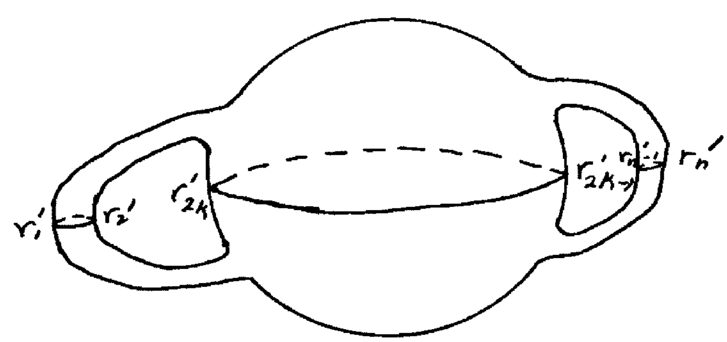


Ill. I-10

Again imagining the surfaces to be rubber, we see that the two spheres can be deformed into one along one of the cuts, and the surface is still topologically equivalent to our original Riemann surface. Thus we have as our Riemann surface for the function

$$w^2 = \sum_{i=0}^{n} a_i z^i$$

a sphere with $\frac{n+1}{2} - 1$ handles, if n is odd, and with $\frac{n}{2} - 1$ handles, if n is even. (Ill. I-11). The number of handles is designated by g,



Ill. I-11

and g is called the genus of the Riemann surface which is topologically equivalent to a sphere with g handles. The corresponding function

$$a_2 w^2 + a_1 w + a_o = 0$$

is single-valued on this surface.

It can be shown that the Riemann surface of any algebraic function is topologically a sphere with g handles, and that the algebraic function is a single-valued function of the points on this surface. [1]

1 . Ref. (7), page 11.

CHAPTER II

MANIFOLDS

In this section, we are assuming certain elementary topological ideas and concepts, such as may be found in Hall and Spencer's ELEMENTARY TOPOLOGY.

To study further the properties of Riemann surfaces, we shall define and investigate <u>2-dimensional</u> <u>manifolds</u>, and especially those manifolds which are analytic. It shall be shown that any Riemann surface of a given analytic function is an analytic manifold.

DEFINITION 2-1--A set E is said to be connected if it cannot be expressed as the union of two non-empty disjoint open sets.

DEFINITION 2-2--A <u>2-dimensional</u> <u>manifold</u> is a connected Hausdorff space M in which each point of M is contained in an open set U which is homeomorphic to an open set V in the Euclidean plane E^2.

We designate points of E^2 by ordered pairs of real numbers, (x,y). For $P \in U$, we let

$$\Phi(P) = (\varphi_1(P), \varphi_2(P)),$$

where φ_1 and φ_2 are continuous real-valued functions of P and let

$$\varphi_1(P_o) = a, \ \varphi_2(P_o) = b.$$

THEOREM 2-1--A connected Hausdorff space M is a 2-dimensional manifold if and only if every point of M is contained in an open set homeomorphic to a disk $K = \{(x,y) \mid (x-a)^2 + (y-b)^2 < r^2\}$, a, b, and r arbitrary, in E^2.

PROOF--If every point of M is contained in an open set homeomorphic to a disk in E^2, M is by definition a 2-dimensional manifold.

Conversely, if M is a 2-dimensional manifold, let $V = \Phi(U)$, as described above, with $P_o \in U$. Since V is an open set in E^2, there is a spherical neighborhood of the point (a, b),

$$K = \{(x, y) \mid (x-a)^2 + (y-b)^2 < r^2\},$$

such that $K \subset V$. Then $\Phi^{-1}(K)$ is an open set of M containing P_o and homeomorphic to an open disk of E^2

Since the mapping

$$\Phi: U \subset M \to V \subset E^2$$

is 1-1, each ordered pair $(x, y) \in V$ determines one and only one point $P \in U$, and therefore (x, y) can be used as the coordinates of P in U. The ordered pairs (x, y) are called the <u>local coordinates</u> or <u>local parameters</u> of P, under the mapping Φ. The set of points in M with local coordinates (x, y), such that

$$(x-a)^2 + (y-b)^2 < r^2$$

is called a <u>coordinate disk</u> or <u>parametric disk</u> of radius r about P_o.

Some examples of 2-dimensional manifolds are E^2 itself, the complex plane, the sphere, and the torus. The cone

$$K: \xi^2 + \eta^2 = \mu^2$$

is not a manifold, as we can see by considering any open set D in K containing the point (0, 0, 0). The set $D - \{(0, 0, 0)\}$ is obviously disconnected. However, if K were a manifold, under any homeomorphism Φ, the image of D in E^2 would contain an open disk A which would in turn contain the image of (0, 0, 0). However,

$\Phi^{-1}(A) - \{(0, 0, 0)\}$ is disconnected and hence its homeomorphic image

$$\Phi[\Phi^{-1}(A) - \{(0, 0, 0)\}] = A - \{\Phi(0, 0, 0)\}$$

is disconnected. But we know that such a punctured disk, $A - \{\Phi(0, 0, 0)\}$, is connected. Hence there is no open connected set of E^2 which is the homeomorphic image of D and therefore K is not a manifold.

In general, the set of local coordinates about the point P_0 is not unique. First, let α be a mapping such that

$$\alpha(x, y) = [\alpha_1(x, y), \alpha_2(x, y)]$$

is a homeomorphism of $V = \Phi(U)$ onto another Euclidean neighborhood W. Then $\alpha \circ O$, for $P \in U$, with

$$\alpha \circ \Phi(P) = \alpha[\varphi_1(P), \varphi_2(P)] =$$

$$\{\alpha_1[\varphi_1(P), \varphi_2(P)], \alpha_2[\varphi_1(P), \varphi_2(P)]\}$$

is another homeomorphism of U onto an open set of E^2. It can be seen that $\alpha \circ \Phi(P)$, as given above, is another set of local coordinates of the point P. In addition, if U_1 and U_2 are parametric disks containing P_0, then $U_1 \cap U_2$ is also a neighborhood of P_0. If

$$\Phi(P) = [\varphi_1(P), \varphi_2(P)]$$

is a local parameter in U_1, and

$$\Psi(P) = [\psi_1(P), \psi_2(P)]$$

is a local parameter in U_2, then both parameters are valid in $U_1 \cap U_2$, and

$$\Psi[\Phi^{-1}(x, y)]$$

defines a homeomorphism of $\Phi(U_1 \cap U_2)$ onto $\Psi(U_1 \cap U_2)$.

Let G be a region of a manifold M, i. e., an open, connected subset of M. Let U be any open set of M such that $U \cap G \neq \phi$, and U is homeomorphic to an open set V in E^2 under the mapping Φ. Then $G \cap U$ is an open set of M and $\Phi(G \cap U)$ is an open set in $V \subset E^2$ under the homeomorphic mapping Φ, for open sets map into open sets under a homeomorphism. Because G is connected, G is also a manifold. Then we see that a subregion of a 2-dimensional manifold is again a 2-dimensional manifold.

THEOREM 2-2--Every manifold is arcwise connected.

PROOF--Let A be the set of points in the manifold M that can be connected to a point P_o by a path in M. Every point P in A belongs to a parametric disk D that is the image under the homeomorphism Φ^{-1} of a disk K in E^2. Each point P_1 in D can be joined to P by a path that is the homeomorphic image of a radial line in K. Therefore every point P_1 in the parametric disk D also belongs to A. Thus A is open. But $M - A$ is open, for, if $Q \in M - A$, Q has about it a parametric disk D', and $D' \cap A = \phi$. To see this, if $D' \cap A \neq \phi$, there is a Q_1 in $D' \cap A$ such that Q_1 can be connected with P_o by a path C in M. However, since D is a parametric disk, there is also a path L in D that connects Q_1 and Q, such that L is the image of a radial line L in K', the pre-image of D', i. e., $Q \in A$. Thus $M - A$ is open. Since $M = A \cup (M - A)$ and M is connected, either A or $M - A$ must be empty. Since $P_o \in A$, $M - A$ must be empty. Therefore $M = A$, and thus M is arcwise connected.

Because each point $P \in M$ has about it a parametric disk D, the union of all these disks forms a covering of M.

THEOREM 2-3--M has a countable base if and only if M has a covering consisting of countable many parametric disks.

PROOF--Assume M has a countable base $G = \{U_n\}_{n=1}^{\infty}$. If $P \in M$, P has about it a parametric disk D, and $D = \bigcup_{n=1}^{k} U_n'$, $U_n' \in \{U_n\}_{n=1}^{\infty}$, where $k \leq \infty$, and for some n, $P \in U_n' \subseteq D$. Let U_n' be called $U_n'(P)$. Then there is a disk D_n such that $U_n'(P) \subseteq D_n$ and $\{D_n\}_{n=1}^{\infty}$ forms a countable covering of M.

Assume there is a countable covering of M by parametric disks, $\{D_n\}_{n=1}^{\infty}$. Let V_n be the image of D_n on the Euclidean plane. Let R_n be the collection of all disks in V_n with points (b_1, b_2) as centers, with b_1 and b_2 rational, and with rational radii. Then each of the disks in R_n has a pre-image in D_n. Let S_n be the collection of these preimages in D_n. Then if $G = \bigcup_{n=1}^{\infty} S_n$, G forms a countable set. To show G is a base for M, let $V \subset M$ be an open set. If $p \in V$, $p \in D_n$ for some n, and p has the local coordinates (c_1, c_2). Since V is open, there is an $\varepsilon > 0$ such that the points of the disk D, with local coordinates (x, y) satisfying

$$(x-c_1)^2 + (y-c_2)^2 < \varepsilon^2$$

is in V. Let r, b_1, and b_2 be rational numbers satisfying:

$$r < \varepsilon,$$

$$|b_1 - c_1| < \frac{r}{4},$$

$$|b_2 - c_2| < \frac{r}{4}.$$

Then the parametric disk $S_p = \left\{(x, y) \,\middle|\, (x-b_1)^2 + (y-b_2)^2 < \frac{r^2}{4}\right\}$ is an element of G, and $p \in S_p \in G$, so that

$$V = \bigcup_{p \in V} S_{\hat{p}}.$$

Thus every open set of M is the union of open sets of G and G is therefore a base.

When we are studying a function f defined on a manifold M, we may consider f as a function of the local coordinates $[\varphi_1(P), \varphi_2(P)]$, where Φ is the homeomorphism of the open set D containing P onto the open set $K = \Phi(D)$ in E^2. However, if for $(x, y) \in K$, the mapping λ, with

$$\lambda(x, y) = [\lambda_1(x, y), \lambda_2(x, y)],$$

is a homeomorphism of K onto $\lambda(K) \subset E^2$, then $\lambda \circ \Phi(P)$ represents a change of local coordinates. We must be certain that the properties we study in terms of a local coordinate system are not lost if we change to a different coordinate system, as in the homeomorphism above. For example, f is continuous in a neighborhood U of a point P_o if and only if, for the local coordinate system

$$\Phi(P) = [\varphi_1(P), \varphi_2(P)] = (x, y),$$

valid in U,

$$f[\Phi^{-1}(x, y)] = g(x, y),$$

is a continuous function of the two variables w and y in $\Phi(U)$.

If Ψ is the homeomorphism of a set $V \subset M$ into E^2, and $U \cap V \neq \phi$, then, for $P \in U \cap V$,

$$\Psi(P) = [\psi_1(P), \psi_2(P)] = (x_1, y_1)$$

is a new set of local coordinates of the point P valid in $U \cap V$, and

$$(x, y) = \Phi[\Psi^{-1}(x_1, y_1)] =$$

$$\Phi[\psi_1^{-1}(x_1, y_1), \psi_2^{-1}(x_1, y_1)] =$$

$$\{\varphi_1[\psi_1^{-1}(x_1, y_1), \psi_2^{-1}(x_1, y_1)], \varphi_2[\psi_1^{-1}(x_1, y_1), \psi_2^{-1}(x_1, y_1)]\}$$

$$= [\lambda_1(x_1, y_1), \lambda_2(x_1, y_1)]$$

is a homeomorphism of $\Psi(U \cap V)$ into $\Phi(U \cap V)$. If

$$\Phi^{-1}(x, y) = \Psi^{-1}(x_1, y_1),$$

then

$$f[\Phi^{-1}(x, y)] = f[\Psi^{-1}(x_1, y_1)]$$

$$= g[\psi_1(x_1, y_1), \psi_2(x_1, y_1)] = h(x_1, y_1)$$

is still a continuous function of (x_1, y_1).

Since we are interested in analytic functions, and we want to be able to talk about the differentiability of functions, we find it convenient to introduce the concept of a differentiable manifold.

DEFINITION 2-3--A real-valued function defined in a region $R \subset E^2$ is said to be of <u>class C^n</u> if all its partial derivatives of order $\leq n$ exist and are continuous in R.

If two real-valued functions, f_1 and f_2, are defined in a region $R \subset E^2$, and f_1 and f_2 are of class C^n, then f_1 and f_2 determine a mapping f of R into a subset of E^2,

$$f: R \rightarrow f_1(R) \times f_2(R),$$

which is of class C^n.

DEFINITION 2-4--The manifold M is a differentiable of C^1 manifold

(1) if there is given a collection $\{U_i, \Phi_i\}_{i\in I}$, where for some index set I, $\{U_i\}_{i\in I}$ is an open covering of M and Φ_i is a homeomorphism if U_i onto an open set of E^2 --the mapping Φ_i defines a system of local coordinates in the set U_i --and

(2) if, when $U_i \cap U_j \neq \phi$, the $\Phi_j(\Phi_i^{-1})$ is a C^1 mapping of $\Phi_i(U_j \cap U_i)$ into $\Phi_j(U_i \cap U_j)$.

The collection $\{U_i, \Phi_i\}_{i\in I}$ is said to define a differentiable structure in the manifold M. Let $\{V_j, \Psi_j\}_{j\in J}$ be another differentiable structure defined on M. Then $\{U_i, \Phi_i\}_{i\in I}$ and $\{V_j, \Psi_j\}_{j\in J}$ are said to be the same, if the covering obtained by taking all the open sets in $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ with their respective mappings Φ_i and Ψ_j satisfies (1) and (2) above.

A <u>differentiable manifold</u> is defined as a manifold together with a set of allowed local coordinates (those defined by $[\varphi_{i1}(P), \varphi_{i2}(P)]$, with Φ_i being an allowed homeomorphism of U_i onto an open subset of E^2), which are the only local coordinates to be used.

If f is a real-valued function on a C^1 manifold M, then in each parametric disk U, f may be expressed as a function of the local coordinates in U.

DEFINITION 2-5--The function f is said to be of <u>Class C^1</u> on M when f is a C^1 function of all the allowed local coordinates of each parametric disk.

Since the changes of coordinates on a C^1 manifold are of class C^1, and a C^1 function of a C^1 function is a C^1 function, f is still of class C^1 on a set $U \subset M$ under a change of local coordinates.

We remark that even though f is differentiable on a set U with respect to a given set of local coordinates, f may not be differentiable with respect to another set of local coordinates. For example, for $P \in U$, let

$$\Phi(P) = (x, y)$$

be a set of local coordinates such that

$$f[\Phi^{-1}(x, y)] = g(x, y)$$

is a differentiable function of (x, y) in $\Phi(U)$. Then let Ψ be another homeomorphism of U into E^2, with

$$\Psi(P) = (x_1, y_1)$$

and such that

$$x = \mu_1(x_1, y_1), \; y = \mu_2(x_1, y_1),$$

with μ_1 and μ_2 continuous functions of x_1 and y_1. However

$$f[\Psi^{-1}(x_1, y_1)] = g[\mu_1(x_1, y_1), \mu_2(x_1, y_1)] = h(x_1, y_1)$$

may not have partial derivatives, for a differentiable function of a continuous function may not be differentiable. Therefore more structure on the manifold is needed.

DEFINITION 2-6-- The manifold M is called a (complex) analytic manifold or an (abstract) Riemann surface

(1) if there is given a collection $\{U_i, \Phi_i\}_{i\varepsilon I}$, where, for the index set I, $\{U_i\}_{i\varepsilon I}$ is an open covering of M and Φ_i is a homeomorphism of U_i onto an open set in the complex z-plane; and

(2) if, when $U_i \cap U_j \neq \phi$, then $\Phi_j(\Phi_i^{-1})$ is a conformal, sense-preserving mapping of $\Phi_i(U_i \cap U_j)$ onto $\Phi_j(U_i \cap U_j)$; that is, $w = \Phi_j[\Phi_i^{-1}(z)] = f(z)$ is an analytic function of z in $\Phi_i(U_i \cap U_j)$.

Since $\Phi_j(\Phi_i^{-1})$ is 1-1, $f'(z) \neq 0$. The mapping Φ_i defines local coordinates in U_i, and $\{U_i, O_i\}_{i\varepsilon I}$ defines an <u>analytic structure</u> in the manifold M. Another collection $\{V_j, \Psi_j\}_{j\varepsilon J}$ defines the same analytic structure on M if the collection of all open sets $\{U_i\}_{i\varepsilon I}$ and $\{V_j\}_{j\varepsilon J}$ together with the allowed mappings satisfy conditions (1) and (2). Analogous to the case of differentiable manifolds, a Riemann surface is a manifold together with a certain set of allowed local coordinates, and only these coordinates are to be used.

Not only may a point $P_o \varepsilon M$ have several sets of local coordinates, because P_o may belong to more than one U_i, but if $P \varepsilon U$, $\Phi(P) = z \varepsilon \Phi(U)$, and if $w = f(z)$ is a 1-1 conformal mapping of $\Phi(U)$ onto an open set of the w-plane, then $f[\Phi(P)] = w = w_1 + iw_2$ is also a set of local coordinates of P. If $\Phi(P_o) = z_o$, then the parametric disk $D = \{z \mid |z-z_o| < r\}$, for r sufficiently small, is contained in U. Setting

$$w = \frac{(z - z_o)}{r}$$

there is a new local parameter

$$w = \Psi(P)$$

with $\Psi(P_o) = 0$ and $|w| \leq 1$. Thus every point $P_o \in M$ is the center of a parametric disk $D = \{w \,|\, |w| \leq 1\}$.

DEFINITION 2-7--If f is a complex-valued function on M, then f is called <u>analytic at P_o</u> if, in terms of the local parameter, $z = \Phi(P)$, with $\Phi(P_o) = 0$, the function $f[\Phi^{-1}(z)]$ is an analytic function of z for $z < r$, $r > 0$. [Note--there is a series

$$\sum_{n=0}^{\infty} a_n z^n = f[\Phi^{-1}(z)],$$

convergent for $z < r$, as we know.]

Since a change of local coordinates involves functions of the type $\Phi_i(\Phi_j^{-1})$, which are analytic if M is an analytic manifold, f is analytic for all sets of allowed local coordinates in U if f is analytic for the set of coordinates $z = \Phi(P)$, $P \in U$, because an analytic function of an analytic function is again analytic.

While the functions considered so far have been mappings of a Riemann surface into the complex plane, we shall also consider mappings, f, which take a Riemann surface S_1 into a second Riemann surface S_2.

Let $P_o \in S_1$ and $f(P_o) = Q_o \in S_2$. Let Φ be the homeomorphism of U, containing P_o, into the z-plane. Let Ψ be the homeomorphism of V, containing Q_o, into the z-plane. Let $z = \Phi(P)$ and $w = \Psi(Q)$. Then f is said to be <u>analytic</u> on S_1 if the composite function

$$w = \Psi\{f[\Phi^{-1}(z)]\} = g(z)$$

is an analytic function of z for all $P \in S_1$.

DEFINITION 2-8--Two Riemann surfaces such as S_1 and S_2 are said to be <u>conformally</u> equivalent if there is a 1-1 analytic mapping of S_1 onto S_2.

From the definition of two conformally equivalent surfaces and the definition of an analytic complex-valued function on a manifold, it can be seen that any open set U of an abstract Riemann surface M with a given allowable mapping Φ is conformally equivalent to an open set of the z-plane, namely, $\Phi(U)$. Also, if V is another open set of M, with the allowable mapping Ψ, and with $U \cap V \neq \emptyset$, then $U \cap V$ is conformally equivalent to both $\Phi(U \cap V)$ and $\Psi(U \cap V)$. Thus, any Riemann surface consists of small neighborhoods patched together so that overlapping pieces fit together conformally.

When we study analytic functions in the z-plane, we are led to the construction of Riemann surfaces on which these functions are single-valued. Usually, these surfaces are pictured as several sheets, each a replica of the z-plane, lying over one another, and connected appropriately.

In this section, it will be shown that this Riemann surface is an abstract Riemann surface or analytic manifold. The analytic functions that have been studied so far have been of the form

$$w^2 = \sum_{n=0}^{\infty} a_n z^n,$$

or have been reducible to this form. The most general analytic function is of the form

$$P(z-a) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

These power series will form the building blocks for the Riemann surface of an analytic function.

The function $P(z-a)$ coverges either in the whole z-plane or in a disk $D = \{z \mid |z-a| < r\}$ and perhaps on part of the boundary.

DEFINITION 2-9--A <u>regular</u> <u>function</u> <u>element</u> is defined as a power series, $P(z-a)$, which converges to a regular analytic function in $D = \{z \mid |a-z| < r\}$, where r is the radius of convergence. The point $z = a$ is called the center of the function element.

Since

$$z - a = (a-b) + (b-a),$$

we have, if $|a-b| < r$,

$$P(z-a) = \sum_{n=0}^{\infty} a_n (z-a)^n =$$

$$\sum_{n=0}^{\infty} a_n (z - b + b - a)^n = \sum_{n=0}^{\infty} a_n [(z-b) + b-a)]^n.$$

Then we can use the binomial theorem to get

$$P(z-a) = Q(z-b) = \sum_{n=0}^{\infty} b_n (z-b)^n,$$

where

$$b_n = \sum_{i=n}^{\infty} a_i \binom{i}{i-n} (b-a)^{i-n}.$$

Because $|b-a| < r$ and $Q(z-b)$ is simply a rearrangement of the terms of $P(z-a)$ in the circle $\{z \mid |z-a| \leq r\}$, the radius of convergence of $Q(z-b)$ is at least as great as $r - |b-a|$, or the distance, on the line through b and a, from b to the nearest point on the circle $\{z \mid |z-a| = r\}$. If the radius of convergence of $Q(z-b)$ is greater than r, it is said the function $P(z-a)$ has been extended beyond the disk $\{z \mid |z-a| < r\}$. The function $Q(z-b)$ is called a direct analytic continuation of $P(z-a)$.

If we have been successful in continuing $P(z-a)$ beyond $\{z \mid |z-a| < r\}$, we may be successful in extending $Q(z-b)$ beyond $\{z \mid |z-b| < r_b\}$, where r_b is the radius of convergence of $Q(z-b)$. From this idea we develop the idea of a chain.

DEFINITION 2-10--A chain is a finite sequence of disks, $K_1, K_2, \ldots, K_n$, so arranged that if a_i is the center of K_i, $i = 1, 2, \ldots, n$, and r_i is the radius of K_i, then $|a_i - a_{i+1}| < r_i$, or the center a_{i+1} of the disk K_{i+1} lies within the disk K_i.

DEFINITION 2-11--<u>Analytic continuation along a chain of disks</u> -- Let $K_1, K_2, \ldots, K_n$ be a chain as defined above. Let $P_i = P_i(z-a_i)$ be a function element with K_i as its disk of convergence. If $P_{i+1}(z-a_{i+1})$ is a direct analytic continuation of P_i, $i = 1, 2, \ldots, n$, P_1 is said to have been continued analytically along the chain of disks, $K_1, K_2, \ldots, K_n$.

DEFINITION 2-12--<u>Analytic continuation along a path</u> -- Let $C = (\alpha, I)$

be a path in the z-plane, with $z = \alpha(t)$, $0 \le t \le 1$. Let $\alpha(0) = a$ and $\alpha(1) = b$ be the end-points of this path. Let

$$P_o = P_o[z-\alpha(0)] = P_o(z-a)$$

be a function element defined at $z = \alpha(0) = a$. To each $t \in I$, we can associate a function element

$$P(t) = P_t[z-\alpha(t)],$$

defined as follows:

Let $t_o \in I$ and let $r(t_o)$ be the radius of convergence of the function element P_{t_o}. If t_1 has the property that $\alpha(t) \in \{z \mid |z-(t_o)| < r(t_o)\}$, for $t_o \le t \le t_1$, we require P_{t_1} to be a direct analytic continuation of P_{t_o} along C. This, of course, excludes the necessity of P_t being a direct analytic continuation of P_{t_o} simply because C winds back into the circle of convergence after once leaving it. However, because C is continuous and because the radius of convergence of $P_{t_o}[z-(t_o)] > 0$--otherwise $P_{t_o}[z-(t_o)]$ would not be analytic in a neighborhood (analyticity is not defined for a point)--for $r(t_o) > 0$, there is a $\delta > 0$ such that $|t-t_o| < \delta$ implies $|\alpha(t) - \alpha(t_o)| < r(t_o)$.

If the conditions listed above have been satisfied, we say $P_1 = P_1[z-\alpha(1)] = P_1(z-b)$ has been obtained from $P_o = P_o[z-\alpha(0)] = P_o(z-a)$ by analytically continuing P_o along the path C. We could have instead obtained P_o from P_1 by continuing P_1 along the path C^{-1}.

THEOREM 2-4--Analytic continuation of a given function element P_o along a given curve C always leads to the same function element P_1.

PROOF--Let P_o be identically equal to Q_1, and let P_t and Q_t be the continuations of P_o and Q_o, respectively, along the path $C = (\alpha, I)$. Let E be the subset of the interval $I = \{t \mid 0 \le t \le 1\}$ consisting of those t for which $P_t \equiv Q_t$. The subset E contains the point $t = 0$, so that $E \neq \phi$. For all t_o, P_{t_o} and Q_{t_o} converge in a circle, $\{z \mid |z - (t_o)| < \mathcal{E}(t_o)\}$, where $\mathcal{E}(t_o)$ is the minimum of the radii of convergence of P_{t_o} and Q_{t_o}. Since C is continuous, there is a $\delta(t_o)$ such that if $|t-t_o| < \delta(t_o)$, then $|\alpha(t) - \alpha(t_o)| < \mathcal{E}(t_o)$. Hence, for $|t-t_o| < \delta(t_o)$, P_t and Q_t are direct analytic continuations of P_{t_o} and Q_{t_o}, respectively. If $t_o \in E$, then $P_{t_o} \equiv Q_{t_o}$, and $P_t \equiv Q_t$, for $|t-t_o| < \delta(t_o)$. Thus E is open relative to I.

If t_o is a limit point of E, then if the minimum of the radius of convergence of P_{t_o} and Q_{t_o} is $\mathcal{E}(t_o)$, there is a $\delta'(t_o)$ such that if $|t-t_o| < \delta(t_o)$, then $|\alpha(t) - \alpha(t_o)| < \frac{\mathcal{E}(t_o)}{2}$. Let t_1 be a t such that $|t_1 - t_o| < \delta(t_o)$. Then $|\alpha(t_1) - \alpha(t_o)| < \frac{\mathcal{E}(t_o)}{2}$, and $P_{t_1} = Q_{t_1}$. However, because

$\left|\alpha(t_1)-\alpha(t_o)\right| < \frac{\mathcal{E}(t_o)}{2} < \mathcal{E}(t_o)$, P_{t_1} is a direct analytic continuation of P_{t_o}, and Q_{t_1} is a direct analytic continuation of Q_{t_o}. But the radius of convergence of both P_{t_1} and Q_{t_1} are equal to or greater than $\mathcal{E}(t_o) - \left|\alpha(t_1)-\alpha(t_o)\right| \geq \mathcal{E}(t_o) - \frac{\mathcal{E}(t_o)}{2} = \frac{\mathcal{E}(t_o)}{2}$, or $\alpha(t_o)$ belongs to the circle of convergence of $P_{t_1} \equiv Q_{t_1}$, and in this circle $P_t \equiv Q_t$, so that $P_{t_o} \equiv Q_{t_o}$. Thus $t_o \in E$, and E is both open and closed, relative to I, so that E = I. Then $P_t \equiv Q_t$ for all $t \in I$, and thus $P_1 \equiv Q_1$.

THEOREM 2-5--The radius of convergence r(a) of the series P(z-a) is either identically infinite or is a continuous function of the center a.

PROOF--If $r(a) = \infty$, and Q(z-b) is a direct analytic continuation of P(z-a), then

$$r(b) \geq r(a) - |b-a| = \infty - |b-a| = \infty.$$

If $r(a) < \infty$, choose b such that $|b-a| < \frac{r(a)}{2}$, and

$$r(b) \geq r(a) - |b-a| \geq \frac{r(a)}{2}.$$

Then $|a-b| < \frac{r(a)}{2} < r(b)$, so that a lies in the circle of convergence of Q(z-b) and

$$r(a) \geq r(b) - |a-b|.$$

Then

$$r(b) - r(a) \geq -|b-a|$$

and

$$|a-b| \geq r(b)-r(a),$$

so that

$$|r(b)-r(a)| \leq |b-a|,$$

and thus for $|b-a|$ arbitrarily small, $|r(b)-r(a)|$ is arbitrarily small, and $r(a)$ is a continuous function of a.

THEOREM 2-6--If the continuation of the function element P_o along a curve $C = (\alpha, I)$ is possible, it can always be accomplished by analytic continuation along a finite chain of disks.

PROOF--Since the radius of convergence $r[\alpha(t)]$ of the function element P_t is a continuous function of t, and because $r[\alpha(t)] > 0$, it has a lower bound $\delta > 0$. Let the sequence $0 = t_o < t_1 < \ldots < t_n = 1$ be chosen such that $|\alpha(t_{i+1})-\alpha(t_i)| < \delta$, $i = 1, \ldots, n$. Then the sequence of disks $K_i = \{z \mid |z-\alpha(t_i)| < r[\alpha(t_i)]\}$, $i = 0, 1, \ldots, n$, forms a finite chain, and $P_o, P_{t_1}, \ldots P_1$ form an analytic continuation along this chain of disks.

What happens if P_o is continued along a curve $C_o(\alpha_o, I)$, from $\alpha_o(0) = a$ to $\alpha_o(1) = b$, and then Q_t, with $Q_o \equiv P_o$, is continued along a second curve, $C_1(\alpha_1, I)$ with end points $\alpha_1(0) = a$ and $\alpha_1(1) = b$? Is $Q_1 \equiv P_1$? The answer is yes, if C_1 is "sufficiently" close to C_o.

THEOREM 2-7--Let δ be the minimum of the radius of convergence $r(\alpha(t))$ of the function element P_t. Let $C_1(\alpha_1, I)$ be any other curve with $\alpha_1(0) = a$ and $\alpha_1(1) = b$, such that $\alpha_1(t) - \alpha_0(t) < \frac{\delta}{4}$ (the precise meaning of "sufficiently" close). If Q_t is the function element obtained by continuing $P_o = Q_o$ along the curve C_1, then $P_1 \equiv Q_1$.

PROOF--Let $\{t_i\}_{i=o}^{n}$, with $t_i < t_{i+1}$, $t_o = 0$, $t_n = 1$, be a sequence such that for all i, $\left|\alpha(t_i) - \alpha(t_{i+1})\right| < \frac{\delta}{4}$. If K_i is the disk $\left|z - \alpha(t_i)\right| < r(\alpha(t_i))$, $i = 0, \ldots, n$, then the chain of disks $K_o, \ldots, K_n$ gives us the continuation of P_o to P_1 by a finite succession of direct continuations. Let L_i represent the line segment joining $\alpha_o(t_i)$ to $\alpha_1(t_i)$. If we continue a function element P_t from $\alpha_o(t)$ to any point γ such that $\left|\alpha(t) - \gamma\right| < r(\alpha(t))$, along any path lying entirely within the circle $K = \{z \,|\, |z - \alpha(t)| < r(\alpha(t))\}$, the function element P will be the same, for each such direct continuation is simply a rearrangement of the terms of the original series. Therefore, if we continue P_o from a to $\alpha_o(t_1)$ and from a to $\alpha_1(t_1)$ and then to $\alpha_o(t_1)$ along L_1^{-1}, we obtain the same function element P_{t_1}. Next we continue P_{t_1} along the C_o to $\alpha_o(t_2)$. If we continue P_{t_1} along L_1, to $\alpha_1(t_1)$, then along C_1 from $\alpha_1(t_1)$ to $\alpha_1(t_2)$ and then along L_2^{-1} to $\alpha_o(t_2)$, since

$$r(\alpha_o(t_2)) \geq r\alpha_o(t_o)) - \left|\alpha_o(0) - \alpha_o(t_2)\right|$$

$$\geq r(\alpha_o(0)) - \frac{\delta}{2} \geq \frac{\delta}{2},$$

we see that all $\alpha_j(t_i)$, $j = 0, 1$, $i = 0, 1, 2$, lie within the radius of convergence of all the function elements P_{t_i}, $i = 0, 1, 2$, and Q_{t_i}, $i = 0, 1, 2$, and therefore these function elements are rearrangements of each other. Similarly, it can be shown that analytic continuation of P_{t_2} along C_0 from $\alpha_0(t_2)$ to $\alpha_0(t_3)$ and on L_2 from $\alpha_0(t_2)$ to $\alpha_1(t_2)$, on C_1 from $\alpha_1(t_2)$ to $\alpha_1(t_3)$ and from $\alpha_1(t_3)$ to $\alpha_0(t_3)$ on L_3 lead to the same function element P_{t_3}. Now we have continued P_0 along two paths to get P_{t_3}. We can continue this process for a finite number of steps, till we get to $\alpha_0(1) = b$, and $P_1 \equiv Q_1$.

If we cannot continue P_0 along a given curve $C = (\alpha, I)$ at a point $t = \tau$, (P_0 can be continued for $0 \leq t \leq t_0$ for $t < \tau$, but not for $t_0 > \tau$), the point $\alpha(\tau)$ is called a <u>singular point</u> relative to C and P_0.

DEFINITION 2-13--Analytic Function (Weierstrass)

The (complete) analytic function is the set A of <u>all</u> function elements obtainable from a given function element by analytic continuation.

It is easy to see that any function element of A can be obtained from any other element of A by analytic continuation and furthermore, from THEOREM 2-6, in a finite number of steps. Also if two such sets have one element in common, they are the same, or identical.

If $P(z-a)$, with

$$P(a-z) = a_o + \sum_{i=1}^{\infty} a_i (z-a)^i,$$

belongs to an analytic function A, then a_o is called a <u>value</u> of A at the point z = a. Let $P_{1,0}(z-b)$ be a function element continued analytically along a given path, $C_o = (\alpha_o, I)$, with endpoints $\alpha_o(0) = b$ and $\alpha_o(1) = a$. Obviously, $P_{1,0}$ can be continued analytically along any path, $C_i = (\alpha_i, I)$, with $\alpha_i(0) = b$ and $\alpha_i(1) = a$, so that we may have several different function elements of A with different values at z = a. Let $P_{i,j}(z-\alpha_i(t_j))$ be the function element defined at $\alpha_i(1) = a$. Thus

$$P_{i1}(z-a) = a_{i,o} + \sum_{i=1}^{\infty} a_{i,n}(z-a)^i, \quad i = 1, 2, \ldots,$$

and the value of p_{i1} at z = a is $a_{i,o}$. Therefore the analytic function A is multiple-valued at z = a just as was the analytic function w for which $w^2 = z$, the first such function we studied.

Because we want to study A as a single-valued function, we shall associate with A a manifold, M_A, on which A <u>will</u> be a single-valued function.

Since A is the totality of function elements P(z-a) derived from a given function element P, we see that the multiple-valuedness of A arises from different continuations of P along different paths, giving rise to different function elements, P(z-a), Q(z-a), etc., at z = a. Thus we see we can consider the <u>set of ordered pairs</u> <u>(a, P(z-a))</u>. Denote this set M_A.

DEFINITION 2-14--We shall call two pairs, (a, P(z-a)) and (b, Q(z-a)) equivalent if

(1) $a = b$

(2) $P(z-a) = Q(z-b)$ in their common circle of convergence. To see that this is an equivalence relation,

(1) Clearly $(a, P(z-a)) = (a, P(z-a))$ for $a = a$ and $P(z-a) \equiv P(z-a)$ in $\{z \mid |z-a| < r(a)\}$.

(2) If $(a, P(z-a)) = (b, Q(z-b))$, then $a = b$ means $b = a$, and if $P(z-a) \equiv Q(z-b)$ in their common circle of convergence, then $Q(z-b) \equiv P(z-a)$ in their common circle of convergence, so that $(b, Q(z-b)) = (a, P(z-a))$.

(3) If $(a, P(z-a)) = (b, Q(z-b))$ and $(b, Q(z-b)) = (c, R(z-c))$, then $a = b$ and $b = c$ implies $a = c$, while if $r(a) \leq r(b)$, the common circle of convergence of $P(z-a)$ and $Q(z-b)$ is $|z-a| < r(a)$, while if $r(a) \geq r(b)$, the common circle of convergence is $|z-b| < r(b)$. Similarly, if $r(b) \leq r(c)$, then the common circle of convergence of $Q(z-b)$ and $R(z-c)$ is $|z-b| < r(b)$, while if $r(b) \geq r(c)$, the common circle of convergence is $|z-c| < r(c)$. If $r(a) \leq r(c) \leq r(b)$, certainly $P(z-a) \equiv R(z-c)$ inside $|a-z| < r(a) \subseteq |z-c| < r(c)$, the common circle of convergence of $Q(z-b)$ and $R(z-c)$. Similarly, if $r(a) \leq r(b) \leq r(c)$, $P(z-a) \equiv R(z-c)$ inside $|z-a| < r(a)$. In fact, inside $|z-a| < r(\delta) = \min\,(r(a), r(b), r(c))$, we know $P(z-a) \equiv Q(z-b)$, and $Q(z-b) \equiv R(z-c)$, so that in this circle, $P(z-a) \equiv R(z-b)$, so that $(a, P(z-a)) = (c, R(z-c))$.

To make M_A an analytic manifold, not just a manifold, we define a topology in M_A.

Let $(a, P(z-a)) \in M_A$, and let $K_\rho(a)$ in the z-plane be any disk $|z-a| < \rho$, $\rho < r(a)$, the radius of convergence of the function element $P(z-a)$. A <u>disk</u> D about $(a, P(z-a))$ is the set of points $\{(b, Q(z-b) \mid b \in K_\rho(a)$ and $Q(z-b)$ is a direct continuation of $P(z-a)\}$. Then we say $V \subset M_A$ is <u>open</u> if for each point $(a, P(z-a)) \in V$, there is a disk D about $(a, P(z-a))$ such that $D \subsetneq V$. To show the disk D described above is open, let $(b, Q(z-b)) \in D$ be any point of D. Then $\{z' \mid |z'-b| < \rho - |a-b|\} \neq \emptyset$, for $(b, Q(z-b)) \in D$ means $|b-a| > 0$. Then, because any such z' lies within $K_\rho(a)$, there is a function element $R(z-z')$ that is a rearrangement of the terms of $P(z-a)$. Because z' lies within the circle of convergence of $Q(z-b)$ and since $Q(z-b)$ is a rearrangement of the terms of $P(z-a)$, $R(z-z')$ is a rearrangement of the terms of $Q(z-b)$ and thus a direct analytic continuation of $Q(z-b)$. Then any point in D has about it a parametric disk of the same kind as D, and we see D is open.

We now show that this definition makes M_A a topological space.

(a) The empty set $\emptyset$ is an open set, since no element of $\emptyset$ fails to satisfy the condition.

(b) The whole set M_A is an open set, for if a point $(a, P(z-a)) \in M_A$, then a disk D about $(a, P(z-a))$ contains only elements of the form $(b, Q(z-b))$ satisfying the above condition. However every element of the form $(b, Q(z-b))$ is an element of M_A, so that, since $(b, Q(z-b)) \in D$, $D \subset M_A$. Then by the definition given above of an open set, M_A is open.

(c) If $A = \bigcup_{\alpha \in I} A_\alpha$, A_α open, then A is open. For if $(a, P(z-a)) \in A_\alpha$, $\alpha \in I$, there is a $D \subset A$ such that $(a, P(z-a)) \in D \subsetneq A_\alpha \subsetneq A$, and thus A is open.

(d) If $A = \bigcap_{i=1}^{n} A_i$, with A_i open for all i, then A is open. To show this, let $(a, P(z-a)) \in A$. Then $(a, P(z-a)) \in A_i$, for $i = 1, 2, \ldots, n$. Each such A_i also contains points (b, Q), with $|b-a| < \rho_i < r(a)$ and Q a direct analytic continuation of P. Then A contains all point (b, Q) with $|b-a| < \min\{\rho_i\}_{i=1}^{n}$ and with Q a direct analytic continuation of P, and the set of these points forms a disk about (a, P) in A.

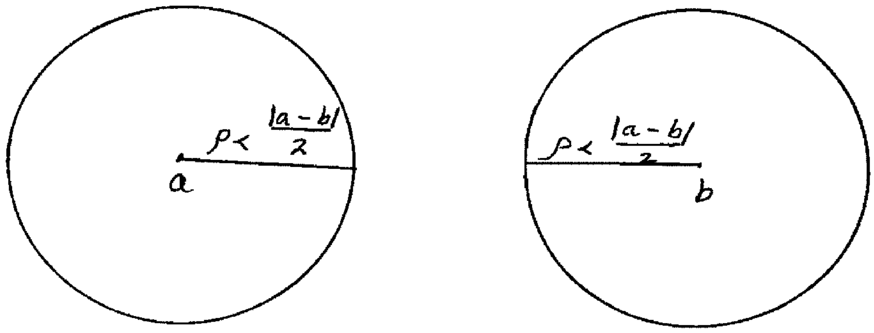
To show M_A is not only a topological, but a Hausdorff space, we must show if $(a, P) = (a, P(z-a))$ and $(b, Q) = (b, Q(z-b))$ are two points of M_A, then there are two disjoint open sets, V_P and V_Q, containing (a, P) and (b, Q), respectively.

We must consider two cases, namely:

(1) $a \neq b$ and

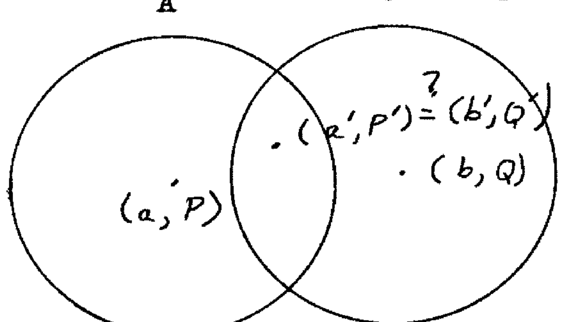
(2) $a = b$, but $P \not\equiv Q$.

(1) if $a \neq b$, we can find two disjoint disks in E^2, $D_1 = \{z \mid |z-a| < \frac{a-b}{2}\}$, $D_2 = \{z \mid |z-b| < \frac{a-b}{2}\}$, and in D_1, we can find contained in D_1 a disk $K(a)$, with P converging in $K(a)$ and in D_2, a disk $K(b)$, with Q converging in $K(b)$. Then, in M_A, let U be the set of points (a_1, P_1), where $a_1 \in K(a)$ and P_1 is a direct continuation of P, and let V be the set of points (b_1, Q_1) where $b_1 \in K(b)$ and Q_1 is a direct continuation of Q. Obviously $(a, P) \in U$, and $(b, Q) \in V$. Then $U \cap V = \emptyset$, because they are the homeomorphic image of $K(a) \cap K(b) = \emptyset$.



Ill. 2-1

In the second case, (2), if $a = b$, $P \not\equiv Q$, let $K\rho(a)$ be the disk containing $a = b$, on which both P and Q converge. Let U be the open set consisting of points (a_1, P_1) where $a_1 \in K\rho(a)$ and P_1 is a direct continuation of P. Let V be the open set consisting of points (b_1, Q_1) where $b_1 \in K\rho(a)$, and Q_1 is a direct continuation of Q. Then $U \cap V = \emptyset$, for if $U \cap V \neq \emptyset$, there is a point $(a', P') = (b', Q') \in U \cap V$, with $a' = b'$, $P' \equiv Q'$. But this means that, in the z-plane, we have continued P from a to a' and then back to $a = b$, never leaving the circle of convergence $K\rho(a)$, and yet have arrived at a different function element Q. But this is impossible, since every function element arrived at without leaving $K\rho(a)$ is only a rearrangement of P. Then $(a, P) \in U$, $(b, Q) \in V$, and $U \cap V = \emptyset$. Thus we have shown M_A is not only a topological, but a Hausdorff space.

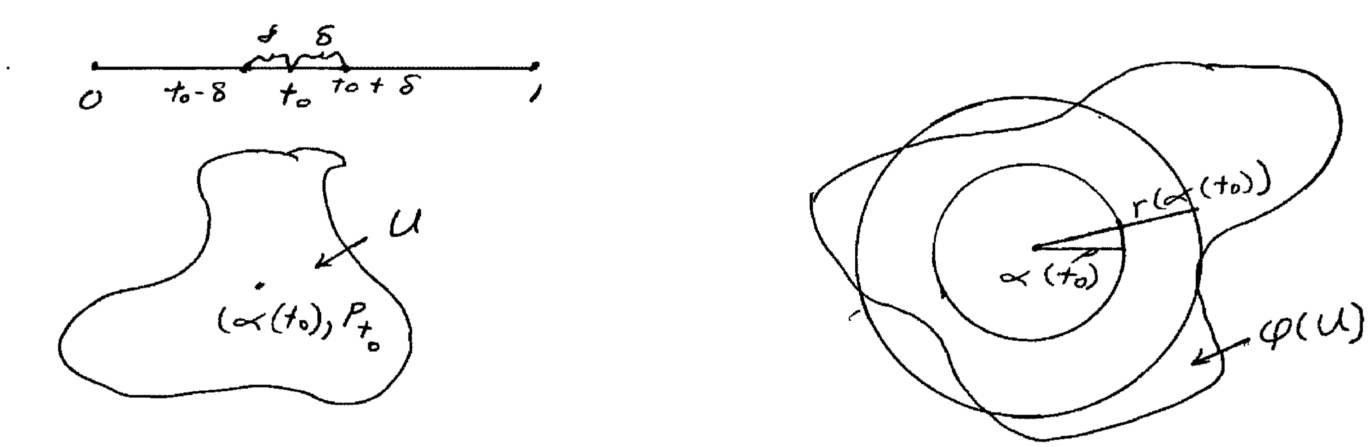


Ill. 2-2

DEFINITION 2-15--If the point $(a, P) \in M_A$, then the point a in the z-plane is called the projection of the point (a,P) on the z-plane. If V is a set of points in M_A, then the projection of V on the z-plane is the set of points $\{z = a \mid (a, P) \in V\}$.

THEOREM 2-8--M_A is an analytic manifold. First we show that M_A is a manifold. The projection mapping φ described above takes each parametric disk D into the corresponding disk $K_\rho(a) = \varphi(D)$, where (a, P) is the center of the disk D and $|b-a| < \rho$, for any $(b, Q) \in D$, and with $\varphi_D(a_1, P_1) = a_1$, for all points $(a_1, P_1) \in D$. We want to show M_A is connected. To do this we show M_A is arcwise connected.

Let (a, P) and (b, Q) be two points of M_A. Then there is a path $C = (\alpha, I)$ in the z-plane such that $\alpha(0) = a$, $\alpha(1) = b$, and $P \equiv P_0$, continued along C by analytic continuation, gives $P_1 \equiv Q$ at $\alpha(1) = b$. We want to consider the points $(\alpha(t), P_t)$ M_A and show that the set of these points is indeed a path joining (a,P) and (b, Q). To show $\gamma = \{(\alpha(t), P_t) \mid 0 \le t \le 1\}$, is indeed a path, we show $(\alpha(t), P_t)$ is a continuous mapping of



Ill. 2-3

$I = 0,1$ into M_A. If U is a neighborhood of a point $((t_o), P_{t_o} \quad M_A$, there is a disk, $K \quad (\quad (t_o))$, lying within the projection $\varphi(U)$ of U on the z-plane. Let ρ be small enough, also, so that $K_\rho(\alpha(t_o))$ lies within the disk of convergence of the function element P_{t_o}. That is,

$K_\rho(\alpha(t_o)) \subseteq \varphi(U) \cap \{z \,|\, |z- \ (t_o)| < \ r(\alpha(t_o))$, where $r(\alpha(t_o))$ is the radius of convergence of the function element $P_{t_o}(z-\alpha(t_o))$.

Then there is a $\delta > 0$ such that $\alpha(t) \in K_\rho(\alpha(t_o))$, when $|t-t_o| < \delta$. If $|t-t_o| < \delta$, we know, then, that P_t is a direct continuation of P_{t_o}, and because $\alpha(t) \in K_\rho(\alpha(t_o)) \subseteq \varphi(U)$, we know $(\alpha(t), P_t) \in U$. Thus $(\alpha(t), P_t)$, for $0 \leq t \leq 1$, is a continuous mapping of $I = [0, 1]$ into M_A, and M_A is arcwise connected and therefore connected.

Therefore M_A is a manifold because it is a connected Hausdorff space, each of whose points is contained in an open set homeomorphic to an open set in the z-plane which is homeomorphic to E^2.

To show M_A is an analytic manifold, we must show M_A satisfies:

(1) there is a collection (U_i, φ_i) such that the U_i form an open covering of M_A and φ_i is a homeomorphism of U_i onto an open set of the z-plane, and

(2) if $U_i \cap U_j \neq \emptyset$, then $\varphi_j(\varphi_i^{-1})$ is a conformal mapping of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$.

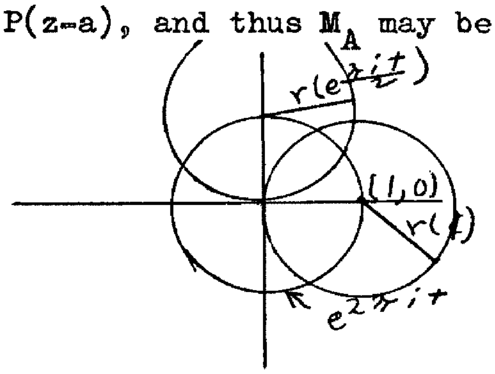
To show (1) we take for the set of $U_i \subset M_A$ the parametric disks about the points (a_i, P_i) of M_A, and for the mappings φ_i the projection mappings such that $\varphi_i(a_i, p_i) = a_i = z$. To verify (2), if $U_i \cap U_j \neq \phi$, there is a point $(a_i, P_i) = (a_j, P_j) \in U_i \cap U_j$ and $\varphi_i(a_i, P_i) = a_i$, $\varphi_j(a_j, P_j) = a_j = a_i$. Then

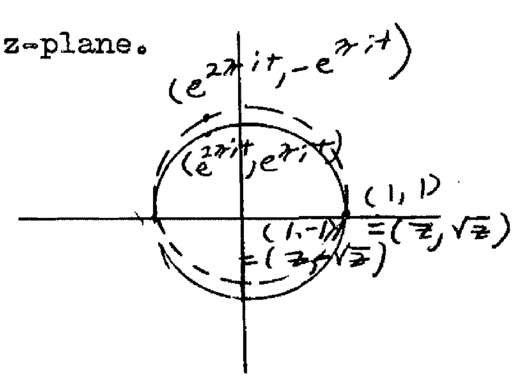
$\varphi_j(\varphi_i^{-1}(a_i)) = a_j = \varphi_j(\varphi_i^{-1}(a_j)) = \varphi_j(a_i, P_i) = \varphi_j(a_j, P_j) = a_j$.

Thus if $U_i \cap U_j \neq \phi$, $\varphi_j(\varphi_i^{-1})$ is simply the identity mapping, which is certainly a conformal mapping.

DEFINITION 2-16--The analytic manifold M_A is called the <u>analytic manifold of the regular function elements of A</u>.

If $P(z-a)$ is an entire function, that is, if $P(z-a)$ converges for $|z-a| < \infty$, then the analytic manifold M_A associated with $P(z-a)$ is the z-plane, for the pro ection of the disk of convergence of any function element (a, P) is the disk $K_{r(a)}(a)$, where r(a) is the radius of convergence of $P(z-a)$, but in the case of an entire function, $r(a) = \infty$, so $K_{r(a)}(a)$ is simply the z-plane, and thus the z-plane is conformal to M_A, the analytic manifold of the function $P(z-a)$, and thus M_A may be considered as the z-plane.





Ill. 2-4

However, the function $\sqrt{z}$, which we considered before, is not single-valued on the finite plane. Expanding $P(z-1) = \sqrt{z}$, we have

$$P(z-1) = \sqrt{z} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 + \ldots$$

and the radius of convergence of this function is 1.

If $z = 1 + re^{i\theta}$, $P(z-1) = P(re^{i\theta}) = \sqrt{r}e^{\frac{i\theta}{2}}$. If we continue $P(z-1)$ along the path, with $r(t) = e^{2\pi it}$, $0 \leq \quad \leq 1$, we find P_t converges in $\{z \mid |z-e^{2\pi it}| < 1\}$ and

$$P_t = e^{\pi it} + \frac{1}{2}e^{-\pi it}(z-e^{2\pi it}) - \frac{1}{8}e^{-3\pi it}(z-e^{2\pi it})^2 + \ldots$$

If $t = 1$,

$$z(1) = P_1 = e^{2\pi i(1)} = e^{i\pi} = -1.$$

Therefore $P_1 \equiv -P_o$. Then the two points $(1, P_o)$ and $(1, -P_o)$ of the analytic manifold of the function z have the same projection on the z-plane. If P_1 is continued along γ' with $\gamma'(t) = e^{2\pi it+2\pi}$, we find

$$P_{1'} \equiv -P_1 \equiv P_o.$$

Then the point $(1, P_{1'}) = (1, P_o)$. Thus we have continued P_o along the path γ'' with $\gamma''(t) = e^{4\pi it}$, and have arrived at the original function P_o. If we take any point $z_1 \neq 0$, we have the same situation, with the two points $(z, \sqrt{z})$ and $(z, -\sqrt{z}) \in M_{\sqrt{z}}$ projecting into the point z of the z-plane, and as z winds around the origin once, the corresponding point on $M_{\sqrt{z}}$ goes from $(z, \sqrt{z})$ to $(z, -\sqrt{z})$, and when z winds around the origin a second time, goes from $(z, -\sqrt{z})$ to $(z, \sqrt{z})$. Thus $M_{\sqrt{z}}$ has as one familiar model the two planes, connected as usual along the branch line $0 \leq x \leq \infty$.

Up to now, we have considered functions that can be represented in the form

$$f(z) = P(z-a) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

which are regular for

$$|z-a| < r(a).$$

Now we are going to consider functions or elements in which a is a singular point. In such a function we have one or more terms of the function of the form

$$a_{-n}(z-a)^{-n}, \text{ n an integer} > 0,$$

so that the function approaches ∞ as z approaches a. Such a function is called a singular function element and the point a is called a singular point of this function element. We may represent a singular function element as

$$S(\mu) = \sum_{n=\nu}^{\infty} a_n \mu^n, \ \nu \text{ an integer},$$

and consider the set of ordered pairs

$$(a, S(\sqrt[k]{z-a})) = (a, S)_k,$$

where k is a positive integer, and a is a singular point of the function $S(\mu) = S(\sqrt[k]{z-a})$. If the point at infinity is a singular point, we consider the ordered pairs,

$$(\infty, S\sqrt{\tfrac{1}{z}})) = (\infty, S)_k.$$

Obviously, if k = 1 and $\nu \geq 0$, $S(\sqrt[k]{z-a})$ is the regular function element P(z-a) that we considered before.

If $k \neq 1, \mu = \sqrt{z-a}$ is not single valued, for if

$$z = a+re^{i\theta} = a+re^{i(\theta+2n\pi)},$$

$$\mu = \sqrt{r}e^{\frac{i(\theta+2n\pi)}{k}}, \; n = 0, 1, \ldots, k-1.$$

Thus μ takes on k different values in any sufficiently small neighborhood of $z = a$. However, we cannot say this, with certainty, about $S(\mu)$. As a simple example, let us consider $S(\mu) = \mu^2$, with $k = 4$. Then

$$S(\sqrt[4]{z-a}) = (\sqrt[4]{z-a})^2 = \sqrt{z-a}.$$

Thus in a sufficiently small neighborhood of $z = a$, $S(\mu) = (z-a)^2$ takes on only two values, instead of the four values $\sqrt[4]{z-a} = \mu$ takes on in the corresponding neighborhood.

In order to take care of the general situation of this type, we consider $S(\mu) = \sum_{n=\nu}^{\infty} a_n \mu^{n\ell}$, ℓ a positive integer, and for the ordered pair $(a, S)_k$, let $(\ell, k) = m \neq 1$. Then $\ell = \lambda m$ and $k = \kappa m$. If we let ε be a primitive k-th root of 1 ($\varepsilon^k = 1, \varepsilon^t \neq 1, t < k$), we have

$$S(\varepsilon^{\kappa}\mu) \equiv \sum_{n=\nu}^{\infty} a_n (\varepsilon^{\kappa}\mu)^{n\ell} \equiv \sum_{n=\nu}^{\infty} a_n (\varepsilon^{\kappa}\mu)^{nm\lambda}$$

$$\equiv \sum_{n=\nu}^{\infty} a_n (\varepsilon^{n\kappa m\lambda})(\mu^{nm\lambda}) \equiv \sum_{n=\nu}^{\infty} a_n \mu^{n\ell} \equiv S(\mu)$$

from above, since $\varepsilon^k = 1$. If

$$T(\mu) = \sum_{n=\nu}^{\infty} a_n \mu^{n\lambda},$$

and if

$$S(\sqrt[k]{z-a}) = \sum_{n=\gamma}^{\infty} a_n((z-a)^{\frac{1}{k}})^{n\ell},$$

from above, then, with $k = \kappa m$,

$$T(\sqrt[\kappa]{z-a}) \equiv \sum_{n=\gamma}^{\infty} a_n[(z-a)^{\frac{1}{\kappa}}]^{n\lambda} \equiv \sum_{n=\gamma}^{\infty} a_n(z-a)^{\frac{n\lambda}{\kappa}} \equiv \sum_{n=\gamma}^{\infty} a_n(z-a)^{\frac{n\lambda m}{\kappa m}}$$

$$\equiv \sum_{n=\gamma}^{\infty} a_n(z-a)^{\frac{n\ell}{k}} \equiv \sum_{n=\gamma}^{\infty} a_n[(z-a)^{\frac{1}{k}}]^{n\ell} \equiv S(\sqrt[k]{z-a})$$

so that $T(\sqrt[\kappa]{z-a})$ and $S(\sqrt[k]{z-a})$ represent the same function. In order to eliminate this occurence, we shall assume $S(\mu) \equiv S(\varepsilon\mu)$ if $\varepsilon^k = 1$, and $\varepsilon \neq 1$.

Similarly for the definition of equivalence of ordered pairs in M_A, we shall define two ordered pairs $(a, S)_k$ and $(b, T)_\ell$, S and T singular, to be equivalent (written $(a, S)_k \cong (b, T)_\ell$) if and only if

(1) $a = b$, $k = \ell$, and

(2) there is an ε with $\varepsilon^k = 1$ such that $S(\mu) = T(\varepsilon\mu)$. This is an equivalence relation, with each of the ordered pairs $(a, S)_k$ defining an equivalence class. We denote by R the set of equivalence classes of these ordered pairs. Obviously, as noted before, if $\gamma \geq 0$ and $k = 1$, $(a, S)_k = \sum_{n=\gamma}^{\infty} a_n(z-a)^n$ is a regular function element, and we see that some points of R are regular function elements. If $(a, S)_k$ is not such a regular function element, then it is called a <u>singular function element</u>.

If $S(\mu) = \sum_{n=\gamma}^{\infty} a_n\mu^n$, then terms of the form $a_n\mu^n$, with $n \geq 0$, are certainly regular function elements. Let $r(S)$ be the radius of convergence of the regular terms $\sum_{n=0}^{\infty} a_n\mu^n$. We define a <u>disk</u> D about the point $(a, S)_k \in R$, $a \neq \infty$, to be $(a, S)_k$ and the set of regular function elements $(b, P) \in R$, with $|b-a| < \rho^{\kappa}$, $\rho < r(S)$, and $P = P(z-a)$ converging to a function identically equal to one of the k determination of $S(\sqrt[\kappa]{z-a})$ in their common region of definition. Thus $P(z-b) \equiv S(\varepsilon \sqrt[\kappa]{z-a})$, where $\varepsilon^k = 1$. The length ρ is called the radius of the disk D. If $(a, S)_k = (\infty, S)_k$, we take as the disk D the center $(\infty, S)_k$ and all regular function elements (b, P), with $|b|^{-1} < \rho^{\kappa}$, $\rho < r(S)$, and $P = P(z-b)$ converging to a function identically equal to one of the k determinations of $S(\sqrt[\kappa]{\frac{1}{z}})$ in their common region of definition.

An open set $V \subset R$ is one in which each point $(p, S)_k \in V$ has about it a disk, D_p, as described above, with $D \subset V$. It can be shown, as was done for $D \subset M_A$, that D_p itself is open.

Again, as for the set M_A, the above definition makes R a topological space. The first four axioms are easily checked. To show R is a Hausdorff space, we take two point, $(a, S)_k$ and $(b, T)_\ell$, such that $(a, S)_k \not\equiv (b, T)_\ell$. If $a \neq b$, we can find two disks, D_1 and D_2, $D_1 = \left\{z \middle| |z-a| < \min\left[\frac{|a-b|}{2}, r(S)\right]\right\}$, and $D_2 = \left\{z \middle| |z-b| < \min\left[\frac{|b-a|}{2}, r(T)\right]\right\}$, and the proof follows as the proof, with $a \neq b$, for $(a, P) \neq (b, Q) \in M_A$.

If $a = b$, let $(a, S)_k \in C = \left\{z \middle| |z-a| < \rho < r(S)\right\}$, $(b, T) \in V$, $V = \left\{z \middle| |z-b| < \rho < r(T)\right\}$. If $U \cap V \neq \emptyset$, there is a point

$(c, P) \in U \cap V$, and because $(c, P) \in U$, $P(z-c) \equiv S(\varepsilon_1 \sqrt[k]{z-c})$ and because $(c, P) \in V$, $P(z-c) \equiv T(\varepsilon_2 \sqrt[k]{z-c})$, both in some neighborhood of $z = c$.

To show, if $U \cap V \neq \phi$, $(a, S)_k \cong (b, T)$, we let t be an ℓth-root of $\sqrt[k]{z-a}$ chosen so that near $[(c-a)^{\frac{1}{k}}]^{\frac{1}{\ell}}$, $t^{\ell} = \sqrt[k]{z-a}$ and $\sqrt[\ell]{z-a} = t^k \varepsilon_\ell^{\alpha}$, where ε_ℓ is a primitive ℓ-th root of 1 and $0 < \alpha < \ell$. That is, t^k is one of the ℓ-th roots of $z-a$. Then since $S(\sqrt[k]{z-a}) \equiv T(\sqrt[\ell]{z-a})$ in some neighborhood of $z = c$, and hence in some neighborhood of $[(c-a)^{\frac{1}{k}}]^{\frac{1}{\ell}}$,

$$S(\sqrt[k]{z-a}) \equiv S(t^{\ell}) \equiv T(\sqrt[\ell]{z-a}) \equiv T(t^k \varepsilon_\ell^{\alpha}).$$

But then $S(\sqrt[k]{z-a}) \equiv T(\sqrt[\ell]{z-a})$ in the larger disk about $z = a$ containing $z = a$, for $r(S)^k \geq |c-a|$, or $r(S) \geq |c-a|^{\frac{1}{k}}$ Replacing t by $\varepsilon_\ell t$, we have

$$T(t^k \varepsilon_\ell^k \varepsilon_\ell^{\alpha}) \equiv S((\varepsilon_\ell t)^{\ell}) \equiv S(\varepsilon_\ell^{\ell} t^{\ell}) \equiv S(t^{\ell}) \quad T(t^k \varepsilon_\ell^{\alpha}).$$

Let $\mu = \varepsilon_\ell^{\alpha} t^k$. Then

$$T(\mu) \equiv T(\varepsilon_\ell^{k} \mu)$$

We have $(\varepsilon_\ell^k)^{\ell} = (\varepsilon_\ell^{\ell})^k = 1^k = 1$, so that for some $\varepsilon = \varepsilon_\ell^k$,

$$T(\mu) \equiv T(\varepsilon\mu).$$

But previously, we had stated that $S(\mu) \not\equiv S(\varepsilon\mu)$ if $\varepsilon^k = 1$ and $\varepsilon \neq 1$. Therefore $\varepsilon = 1$. Then $\varepsilon_\ell^k = 1$ implies there is an integer m such that $m\ell = k$, i. e., k is a multiple of ℓ.

If we chose t to be a k-th root of $\sqrt[\ell]{z-a}$ such that $t^k = \sqrt[\ell]{z-a}$ and $t^{\ell} = \varepsilon_k^{\beta} \sqrt[k]{z-a}$, where ε_k is a primitive k-th root of 1 and $0 < \beta < k$, we can show, by the same procedure,

$nk = \ell$, n an integer.

Then $\ell = nk = nm\ell$ and m and n are two integers such that $mn = 1$. But this means $m = n = 1$, and $k = \ell$. Therefore, in a neighborhood of $z = a$, we have

$$S(\sqrt[k]{z-a}) \equiv T(\eta\sqrt[k]{z-a})$$

where $\eta^k = 1$. Then from the definition of equivalent points on R, $(a, S)_k \cong (b, T)_\ell$, and $(a, S)_k$ and $(b, T)_\ell$ represent the same point on R, contrary to the statement that $(a, S)_k \not\cong (b, T)_\ell$, and thus we have shown that our assumption $U \cap V \neq \emptyset$ is false.

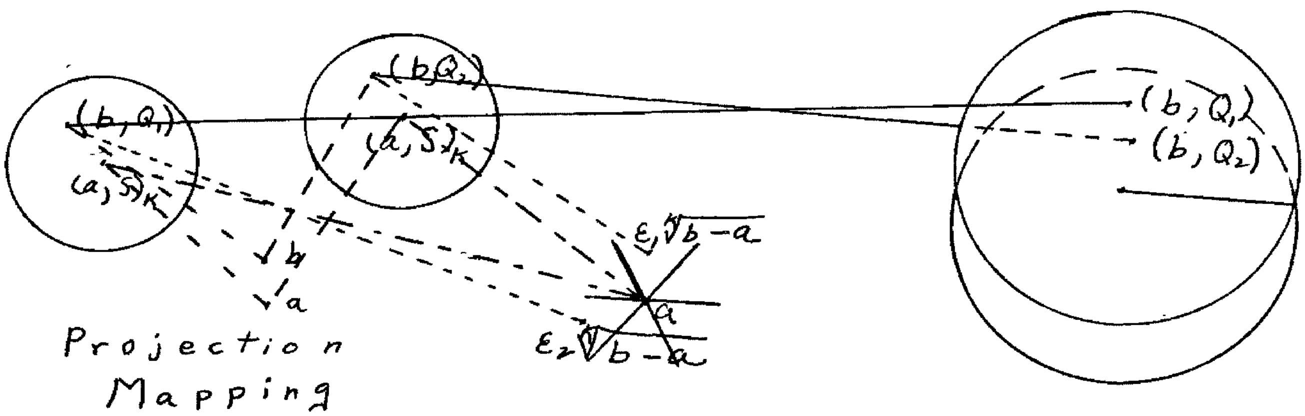
Let $P(z-a)$ be a regular function element. By definition, the analytic function A containing $P(z-a)$ is the set of regular function elements $Q(z-b)$ which can be obtained from $P(z-a)$ by analytic continuation. Since $Q(z-b) = \sum_{n=0}^{\infty} a_n(z-b)^n = (b, Q)$, $Q(z-b)$ is a regular function element of R. We know that the set of pairs (b, Q), where Q is an analytic continuation of $P = P(z-a)$, is the analytic manifold of the regular function elements of A. Since the definition of $(a, P) \equiv (b, Q)$ in M_A is the same as the definition of $(a, S)_k \cong (b, T)_\ell$ in R, with $k = \ell = 1$, we see the set of ordered pairs (b, Q) in M_A is the set of regular function elements in R. Therefore M_A is a subset of R, which we will call R_A. If (b, Q) is a point in R_A, a disk D about (b, Q) consists of the same elements as a disk about (b, Q) in M_A. Then, a set in R_A is open if and only if it is open in R. Since $M_A = R_A$ is arcwise connected, R_A lies in one component of R.

DEFINITION 2-17--The totality of function elements (regular or singular) in the component R_A of R which contains the function elements in the complete analytic function A is called the analytic configuration of the analytic function A.

THEOREM 2-9--The analytic configuration of an analytic function A together with the structure given in terms of the disks in R defined above is an analytic manifold.

PROOF--Because R_A is a component of R, R_A is connected. We know R, and therefore R_A as a subset of R, is a Hausdorff space. We must show there is a homeomorphism of the disks of R_A to disks in E^2, and we must show we can define an analytic structure on R_A.

If (a, P) is a regular function element of R_A, the projection $(a, P) \rightarrow a$ is a homeomorphism of a disk D containing (a, P) onto an Euclidean disk. However, if $(a, S)_k$ is a singular function element with $k \neq 1$, the projection $(a, S)_k \rightarrow a$, since there are k k-th roots of (z-a), would have k points projecting to $b \neq a$, corresponding to each $P(z-b) \equiv S(\varepsilon \sqrt[k]{z-b})$, where $\varepsilon^k = 1$, in the common region of definition of P(z-b) and $S(\varepsilon \sqrt[k]{z-a})$. To make a 1-1 mapping, let (b, P) be a point in the disk of radius ρ about $(a, S)_k$. Then we know $P(z-b) \equiv S(\varepsilon \sqrt[k]{z-a})$, in their common region of definition, where ε, with $\varepsilon^k = 1$, fixes one of the k roots of z-a in a neighborhood of z = b. Then we map the point (b, P) into $\varepsilon \sqrt{b-a}$ of the disk $|z| < \rho$, and the point $(a, S)_k$ goes to 0. If $a = \infty$, (b, P) goes into $\varepsilon \sqrt[k]{\frac{1}{b}}$, with $(\infty, S)_k$ going into 0. Then instead of k points of R mapping into b, we have each of these k points mapping into a different k-th root of (b-a). To make this mapping a little more graphic, a disk about $(a, S)_k$ can be thought of as a ramp circling about $(a, S)_k$ k times, then penetrating to the bottom surface, just as an example, we think of the Riemann surface of $w = \sqrt{z}$ about $z = 0$ as a spiral ramp circling around twice and penetrating again to the bottom sheet. Then we have the homeomorphism



Ill. 2-5

$(b, P) \to \varepsilon\sqrt[k]{b-a}$, where (b, P) is a regular function element of the disk D about $(a, S)_k$, and ε would correspond to the sheet on which (b, P) was situated. Then, if we think of the $(\ell+1)$-th sheet in the disk D, because

$$b = a+re^{i(\theta+2\ell\pi)} = a+re^{i\theta},$$

we have

$$(b, P) \to \sqrt[k]{a+re^{i(\theta+2\ell\pi)}-a} = e^{\frac{2\ell\pi i}{k}} \sqrt[k]{re^{i\theta}}$$

$$= e^{\frac{2\ell\pi i}{k}} \sqrt[k]{a+re^{i\theta}-a} = e^{\frac{2\ell\pi i}{k}} \sqrt[k]{b-a},$$

and we see $D \cap$ Sheet $(\ell+1)$ maps into the sector

$$\frac{2\pi\ell}{k} \leq \theta \leq \frac{2\pi(\ell+1)}{k},$$

with $|b - a| < \rho^k < [r(S)]^k$.

Next, we want to show that these local coordinates make R_A into an analytic manifold. $\varphi(a, P)_k = a$ if $k = 1$ and (a, P) is the center of a disk U, and $\varphi(b, P) = \varepsilon\sqrt[k]{b-a}$ if (b, P) belongs to a disk D with center $(a, S)_k$. If two disks, D_ℓ and D_k, about two singular elements, $(b, T)_\ell$ and $(a, S)_k$, respectively, have elements in common, from the definition of the disks D_ℓ and D_k, all of the elements in $D_\ell \cap D_k$ are regular function elements. If $(b, P) \varepsilon D_\ell \cap D_k$, the parameter in D at (c, P) is $z = \varepsilon_\ell \sqrt[\ell]{c-a}$, with $\varepsilon_\ell^\ell = 1$, and the parameter in D_k is $w = \varepsilon_k \sqrt[k]{c-b}$, with $\varepsilon_k^k = 1$. Since $c \neq a$, and $c \neq b$, we know $z^\ell = c-a$ or $c = a+z^\ell$, and $w = \varepsilon_k \sqrt[k]{z^\ell + a - b}$. Then w is an analytic function of z in $D_\ell \cap D_k$, if $(a, S)_k$ is at the center of D_k and $(b, T)_\ell$ is at the center of D_ℓ. If (a, P) and (b, Q) are regular function elements at the center of U and V, respectively, with $U \cap V \neq \phi$, then the projection mapping, $(a, P) \to a$, as described for M_A, gives a mapping such that $\varphi_i(\varphi_j^{-1})$, being the identity mapping, is analytic, as previously described.

DEFINITION 2-18--<u>The Riemann Surface of the analytic function A</u> is the analytic manifold R_A obtained by putting the above analytic structure on the analytic configuration of A.

THEOREM 2-10--The regular function elements of the analytic configuration of an analytic function A are function elements of A.

PROOF--If (b, Q) is a regular element of R_A, then (b, Q) can be joined to any fixed element (a, P) of A by a curve on R_A. We want to show (b, Q) itself belongs to A. In order to show this, we show the path from (b, Q) to (a, P) lies in $M_A \subseteq R_A$. If $(c, R)_k$ is a function element (regular or singular) of R_A, then about $(c, R)_k$ is a disk D, of ordered pairs (d, S), all of which are regular function elements. Then $D - \{(c, R)_k\}$ is open and $F = \{(c, R)_k \mid (c, R)_k$ is a singular function element$\}$ is a set of isolated points on R_A, and has no cluster point on R_A. Thus $G = R_A - F$ is open, since it is the union of all the disks $D - \{(c, R)_k\}$ and the union of all disks D' with regular function elements as centers. We want to show G is connected.

Assume G is not connected. Then $G = G_1 \cup G_2$, such that $G_1 \cap G_2 = \emptyset$ and G_1 and G_2 are open. But $\overline{G_1} \cap \overline{G_2} \neq \emptyset$, because, since each element $(c, R)_k$ of F has a deleted neighborhood, $D - \{(c, R)_k\}$ in G, $R_A = \overline{G} = \overline{G_1 \cup G_2} = \overline{G_1} \cup \overline{G_2}$. If $\overline{G_1} \cap \overline{G_2} = \emptyset$, then R_A is the union of two closed disconnected sets, contrary to the statement R_A is connected. If $p \in \overline{G_1} \cap \overline{G_2}$, and if $p \in G_1$, for example, then G_1 is a neighborhood of p which contains no points of G_2, because G_1 is open and $G_1 \cap G_2 = \emptyset$. Then $p \notin \overline{G_2}$, because every neighborhood of every point of $\overline{G_2}$ has non-empty intersection with G_2. Similarly, if $p \in G_2$, $p \notin \overline{G_1}$. Thus, if $p \in \overline{G_1} \cap \overline{G_2}$, $p \notin G_1 \cup G_2$, and thus $p \in R_A-(G_1 \cap G_2) = F$, or $\overline{G_1} \cap \overline{G_2} \subseteq F$. Let $p \in \overline{G_1} \cap \overline{G_2} \subseteq F$. Let D be a disk containing p such that $D-p \subset G$. That is, $D \subset R_A$, so that $D-p \subset R_A-F = G$. Then

$$D-p = D \cap (G_1 \cup G_2) = (D \cap G_1) \cup (D \cap G_2),$$

which is the union of two open sets with empty intersection, and is thus disconnected. However, D-p is the homeomorphic image of a punctured disk in the Euclidean plane and therefore is connected. Therefore the assumption R_A-F is not connected leads to the false conclusion that the homeomorphic image of a punctured disk in the Euclidean plane is not connected. Thus we must conclude that R_A-F is connected. Then R_A-F is a manifold and therefore, from a previous theorem, is arcwise connected. Hence, any two points in R_A-F (the set of regular function elements) may be jointed by a path in R_A-F, or by a path composed only of regular function elements, so that $R_A-F = M_A$.

If D is a disk about $(a, S)_k$, where $(a, S)_k$ is a singular function element of R_A, then $D-(a, S)_k$ contains only regular function elements, from the definition of the disk D. Then, if $(b, P) \in D$, (b, P) has the local coordinates $\varepsilon\sqrt[k]{b-a}$, with $\varepsilon^k = 1$, and $P(z-b) \equiv S(\varepsilon\sqrt{b-a})$ in their common region of definition.

DEFINITION 2-17--A path $(\alpha(t), P_t)_{t\in I}$, in D, is the line segment $\varepsilon\sqrt[k]{\alpha(t)-a}$, $t \in I$, from $\varepsilon\sqrt{b-a}$ to 0, with $P_t(z-\alpha(t)) \equiv S(\varepsilon\sqrt[k]{z-a})$, $0 \leq t < 1$, in their common region of definition, while $P_1(z-a) \equiv S(\sqrt[k]{z-a})$.

We have shown that, if $(a, S)_k$ is any singular function element of R_A, $(a, S)_k$ can be joined by a path in D to any regular function element in the disk D about $(a, S)_k$. We have previously shown any regular function element in R_A can be joined by a path in R_A to any other regular function element in R_A. Thus we see that any function element, regular or singular, can be joined to any other function element by a path in R_A.

DEFINITION 2-18--A function element, $(a, S)_k$ (regular or singular), is said to be joined analytically to a regular function element, (b, P), if there is a path (α, I), $\alpha(0) = b$, $\alpha(1) = a$, in the complex plane (or sphere) such that $(\alpha(t), P_t)$, $0 \leq t < 1$, are regular function elements forming an analytic continuation of P and if for all t sufficiently near 1, P_t is identically equal to a fixed determination of S, while $(a, P_1) \equiv (a, S)_k$.

Then M_A is a submanifold of regular function elements of the Riemann surface R_A of the analytic function A. The singular function elements $(a, S)_k$, $k > 1$, are called the algebraic branch points of R_A. At such a point $(a, A)_k$, we say the analytic function

$$(a, S)_k = \sum_{n=\gamma}^{\infty} a_n \mu^n$$

has the value a_o if $\gamma \geq 0$, and has the value ∞, if $\gamma < 0$.

CHAPTER III

DIRICHLET'S PROBLEM

From the study of analytic functions on a Riemann surface, we are naturally led to the study of harmonic functions. If $f(z) = f(x,y) = u(x,y) + iv(x,y)$, and if $f(z)$ is analytic, we know $u(x,y)$ and $v(x,y)$ are harmonic, or $\Delta u = 0$ and $\Delta v = 0$. However, if we have a real-valued function $u(x,y)$, such that $\Delta u = 0$, we may not be able to find a function v such that $\Delta v = 0$ in the entire region under consideration. Therefore our study of analytic functions leads us to the more general area of harmonic functions.

In this section, we shall study the Dirichlet Problem, the existence of a solution, and the solution, when it exists.

THE DIRICHLET PROBLEM--Given a region W, and a real-valued function, f, continuous on W', the boundary of W, is it possible to find a harmonic function u, such that $u \equiv f$ on W' and u is harmonic in W?

In order to show the solution, when it exists, we shall study Poisson's Integral, a generalization of Harnack's principle, and subharmonic functions, leading to the solution of Dirichlet's problem by Perron's method.

Recalling <u>Cauchy's Integral formula</u>, if $f(z)$ is analytic,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z')\, dz'}{z'-z}$$

where γ is the circle $\gamma = \{z \mid |z-z_0| = r\}$. Since

$$z' = z_o + re^{i\theta}, \; dz' = ire^{i\theta}d\theta,$$

and we have

$$f(z_o) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_o+re^{i\theta})ire^{i\theta}d\theta}{z_o+re^{i\theta}-z_o}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_o+re^{i\theta})d\theta.$$

If $f(re^{i\theta}) = U(re^{i\theta}) + iV(re^{i\theta})$, where U and V are real-valued functions of $re^{i\theta}$,

$$f(z_o) = \frac{1}{2\pi}\int_0^{2\pi} U(z_o + re^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} iV(z_o + re^{i\theta})d\theta.$$

Equating real and imaginary parts, we have then

$$U(z_o) = \frac{1}{2\pi}\int_0^{2\pi} U(z_o + re^{i\theta})d\theta,$$

when $U(z)$ is the real part of an analytic function $f(z)$. Because $f(z)$ is analytic, $U(z)$ is harmonic, and we are led to the <u>maximum principle</u> for harmonic functions.

THEOREM 3-1--A non-constant harmonic function has neither a maximum nor minimum in its region of definition. Therefore, the maximum and minimum on a closed, bounded set E are taken on its boundary.

NOTE--As we observed before, if we have a harmonic function $U(z)$ given, and the region under consideration is not simply connected, we may not be able to find a function $V(z)$ such that $U(z) + iV(z)$ is analytic in the whole region under consideration. However, in this proof, we use only simply connected subsets, namely disks, of the region under consideration, and in such subsets, $U(z)$ is the real part of a function analytic in the entire disk under consideration.

PROOF--It is sufficient to show that if $U(z)$ is not a constant, its maximum is taken on the boundary of any closed, bounded set.

Given a closed, bounded set E, assume there is a z_0 such that $U(z_0) \geq U(z)$, $z \in E$, $z_0 \notin E'$, the boundary of E. Then there is a disk $\gamma = \{z \mid |z-z_0| \leq r\} \subset E - E'$, for r sufficiently small, such that

$$U(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\theta})d\theta.$$

Also because $U(z_0) \geq U(z_0 + re^{i\theta})$, we have

$$U(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\theta})d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} U(z_0)d\theta = U(z_0).$$

Then $\frac{1}{2\pi}\int_0^{2\pi}\{U(z_0) - U(z_0 + re^{i\theta})\}d\theta = 0$, and since $\{U(z_0) - U(z_0 + re^{i\theta})\} \geq 0$, the integral can be zero only if the integrand is equal to zero. This means $U(z_0) = U(z_0 + re^{i\theta})$ for all θ, or $U(z)$ is a constant.

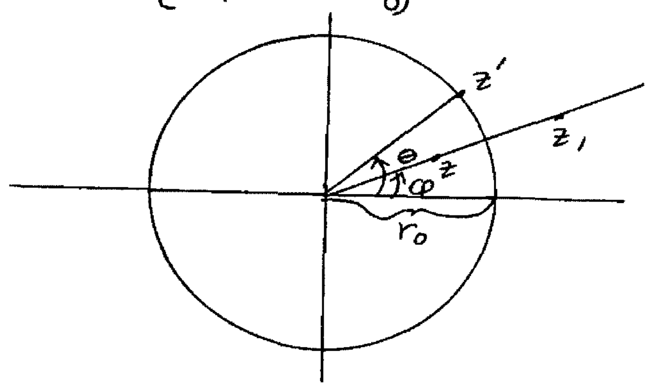
From this maximum and minimum principle, if two given harmonic functions, U_1 and U_2, are equal on the boundary of their region of definition, E, they are identical. To see this, if $U_1 - U_2$ and $U_2 - U_1 = 0$ on the boundary of E, and both are harmonic, then 0 is both the maximum and minimum of $U_1 - U_2$, and thus $U_1 \equiv U_2$ in E.

Therefore, if Dirichlet's problem, described above, has a solution, it is of necessity unique.

Again going back to Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z')}{z'-z}\, dz',$$

we can let γ be the circle $\{z \mid |z'| = r_0\}$ and let $z = re^{i\varphi}, r < r_0$.



Ill. 3-1

The inverse z_1 of the point z with respect to the circle can be written

$$z_1 = \frac{r_0^2}{r} e^{i\varphi} = \frac{r_0^2}{re^{-i\varphi}} = \frac{z'\overline{z'}}{\overline{z}} .$$

If f is analytic everywhere in and on the circle, which implies the real part of f is harmonic,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z')}{z'-z}\, dz',$$

but if we replace z by z_1,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z')}{z'-z_1}\, dz' = 0,$$

for $\frac{f(z')}{z'-z_1}$ is holomorphic within and on the circle

$$\gamma = \left\{ z \,\middle|\, |z| = r_o \right\}$$

and, because this circle is a closed contour, the integral of a function holomorphic within and on this circle is 0. Since $z' = r_o e^{i\varphi}$ and $dz' = ir_o e^{i\varphi} d\varphi = iz'd\varphi$, we have

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \left\{ \frac{f(z')}{z'-z} - \frac{f(z')}{z'-z_1} \right\} iz'd\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{ \frac{z}{z'-z} - \frac{z}{z'-z_1} \right\} f(z')d\varphi.$$

Looking at $\frac{z'}{z'-z} - \frac{z'}{z'-z_1}$, and remembering the value of z_1, we have

$$\frac{z'}{z'-z} - \frac{z'}{z'-z_1} = \frac{z'}{z'-z} - \frac{z'}{z'-\frac{z'\bar{z}'}{\bar{z}}} = \frac{z'}{z'-z} - \frac{1}{1-\frac{\bar{z}'}{\bar{z}}} = \frac{z'}{z'-z} - \frac{\bar{z}}{\bar{z}-\bar{z}'} = \frac{z'}{z'-z} + \frac{\bar{z}}{\bar{z}'-\bar{z}}$$

then $\frac{z'}{z'-z} + \frac{\bar{z}}{\bar{z}'-\bar{z}} = \frac{z'\bar{z}'-z'\bar{z} + z'\bar{z} - z\bar{z}}{|z'-z|^2} = \frac{r_o^2 - r^2}{|z'-z|^2}$. Therefore we can write

$$f(z) = f(re^{i\theta}) = \frac{r_o^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(re^{i\varphi})}{|z'-z|^2}\, d\varphi$$

$$= \frac{r_o^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(re^{i\varphi})}{r_o^2 - 2r_o r\cos(\varphi-\theta) + r^2}\, d\varphi.$$

If $f(z) = f(re^{i\theta}) = U(re^{i\theta}) + iV(re^{i\theta})$, we have

$$U(re^{i\theta}) = \frac{r_o^2 - r^2}{2\pi}\int_0^{2\pi} \frac{U(r_o e^{i\varphi})}{r_o^2 - 2r_o r\cos(\varphi-\theta) + r^2}\, d\varphi.$$

This is Poisson's integral formula in polar coordinates, or in terms of z in the denominator, we have

$$U(r, \ ) = \frac{r_o^2 - r^2}{2\pi}\int_0^{2\pi} \frac{U(r_o, \varphi)}{|z'-z|^2}\, d\varphi.$$

Since $\mathrm{Re}\left[\frac{z' + z}{z' - z}\right] = \frac{r_o^2 - r^2}{|z'-z|^2}$, we make the following definition:

DEFINITION 3-1--For any piecewise continuous function $U(\theta)$ in $0 \leq \theta \leq 2$, with $|z| < 1$,

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta} + z}{e^{i\theta} - z}\right] U(\theta)d\theta.$$

We readily see

$$P_{U+V} = P_U + P_V$$

and

$$P_{cV} = cP_V, \text{ c a constant.}$$

Moreover, for $U(\theta) = 1$, we have

$$P_U(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{i\theta}-z}\, d\theta = 1,$$

because

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] U(\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+|z|^2}{|e^{i\theta}-z|^2}\, U(\theta)d\theta .$$

Then remembering how the Poisson integral was derived, we know that for $|z'| = 1$, and $U(\theta) = f(z') \quad 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, U(\theta)d\theta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z')}{z'-z}\, dz'.$$

If $f(z') \equiv 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, 1\, d\theta = \frac{1}{2\pi i}\int_\gamma \frac{1}{z'-z}\, dz' = 1.$$

so that if c is a constant,

$$P_c = c.$$

Therefore, if

$$m \leq U \leq M,$$

Because

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} \geq 0,$$

we have

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2}\, m \leq \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, U \leq \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, M.$$

Then

$$P_U - m = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, U(\theta) - \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, m\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, (U(\theta) - m)\, d\theta \geq 0,$$

for the integral of the product of non-negative quantities is non-negative. Similarly,

$$P_U \leq M$$

so that we have

$$m \leq P_U \leq M.$$

THEOREM 3-2--The function $P_U(z)$ is harmonic for $|z| < 1$, and

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$$

if $U(\theta)$ is continuous at θ_0.

PROOF--If we differentiate

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, U(\theta)d\theta$$

under the integral with respect to z, we see that, because $U(\theta)$ and $d\theta$ are not changed by a change in z, the only quantity which is affected by differentiation is

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2}.$$

However, we know

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} = \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$$

is the real part of a function analytic for z 1. Then as the real part of an analytic function, $\frac{1-z^2}{e^{i\theta}-z^2}$ is certainly harmonic for $|z|<1$, i.e., $\Delta\left[\frac{1-|z|^2}{e^{i\theta}-z^2}\right] = 0$. Therefore,

$$\Delta U_P(z) = \frac{1}{2\pi}\int_0^{2\pi} \Delta\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right] U(\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 0 \quad U(\theta)d\theta = 0.$$

Thus the integral $P_U(z)$ is also harmonic for $|z| < 1$.

To show

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0),$$

since $U(\theta_o)$ is a constant, we have $P_U(z) - U(\theta_o)$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,(U(\theta) - U(\theta_o'))d\theta.$$ Because

$U(\theta)$ is continuous, for a given $\mathcal{E}$, if $|\theta-\theta_o| < \delta(\mathcal{E})$, $|U(\theta) - U(\theta_o)| < \mathcal{E}$.

Since $\dfrac{1 - r^2}{1-2r\cos(\theta-\theta\ ) + r^2} = \dfrac{1-|z|^2}{|e^{i\theta}-z|^2}$ has a period of 2 ,

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta)-U(\theta_o)\right]d\theta =$$

$$\frac{1}{2\pi}\int_{\theta_o-\delta}^{2\pi+\theta_o-\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta)-U(\theta_o)\right]d\theta \le \frac{1}{2\pi}\int_{\theta_o-\delta}^{\theta_o+\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\mathcal{E}\,d\theta$$

$$+ \frac{1}{2\pi}\int_{\theta_o+\delta}^{2\pi+\theta_o-\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta)-U(\theta_o)\right]d\theta = \mathcal{E} + \frac{1}{2\pi}\int_{\theta_o+\delta}^{2\pi+\theta_o-\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta) - U(\theta_o)\right]d\theta.$$

If $|\theta-\theta_o| \ge \delta$, there is an $m(\delta)$ such that

$$|e^{i\theta}-z|^2 = |e^{i\theta}-re^{i\theta_o}|^2 \ge m(\delta).$$

Also, since $U(\theta)$ is sectionally continuous, there is an M such that $|U(\theta) - U(\theta_o)| \le M$. Then for $1-r < \frac{m(\delta)}{2M}\mathcal{E}$, we have

$$\frac{1}{2\pi}\int_{\theta_o+\delta}^{2\pi+\theta_o-\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta) - U(\theta_o)\right]d\theta \le \frac{1}{2\pi}\,\frac{1-r^2}{m(\delta)}\,M(2\pi)$$

$$\le \frac{2(1-r)2M}{2\,m(\delta)} < \frac{2M}{m(\delta)}\,\frac{m(\delta)}{2M}\,\mathcal{E} = \mathcal{E}.$$ Then

$$\frac{1}{2\pi}\int_{\theta_o-\delta}^{2\pi+\theta_o-\delta}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]\left[U(\theta) - U(\theta_o)\right]d\theta < 2\mathcal{E},$$

for an arbitrary ε, when $0 < 1 - r < \frac{m(\varepsilon)}{2M}\varepsilon$.

If $r = 0$, we have $P_U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\, d\theta$, so that <u>the value of a harmonic function at the center of the circle is the average of its boundary values on the circle</u>.

While we now have a device for solving the Dirichlet problem in the unit circle for given values on the boundary, we would like to be able to solve it for the boundary of certain given regions. To do this, we first solve it for a given circle with center z_o and radius ρ. Let $U(\theta)$, with $U(0) = U(2\pi)$, be a continuous function for the boundary $|z-z_o| = |z_o + e^{i\theta} - z_o| = \rho$. We wish to find a function $U'(z)$, harmonic in $|z-z_o| < \rho$, continuous on $|z-z_o| = \rho$ and such that

$$U'(z_o + \rho e^{i\theta}) = U(\theta).$$

From Theorem 3-2, $U'(z)$ is given by

$$u'(z) = P_u\left(\frac{z-z_o}{\rho}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - \left|\frac{z-z_o}{\rho}\right|^2}{\left|e^{i\theta} - \frac{z-z_o}{\rho}\right|^2}\, U(\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\, U(\theta)\, d\theta$$

and we know, from Theorem 3-1, that this function is unique.

THEOREM 3-3--A continuous function $u(z)$ in a region Ω which at all points $z_o \in \Omega$ satisfies

$$u(z_o) = \frac{1}{2\pi}\int_0^{2\pi} u(z_o + re^{i\theta})\, d\theta$$

for all sufficiently small r is necessarily harmonic.

PROOF--If $z_o \in \Omega$, there is a ρ sufficiently small that

$$u(z_o) = \frac{1}{2\pi}\int_0^{2\pi} u(z_o + \rho e^{i\theta}) d\theta.$$

If in the disk $D = \{z \mid |z-z_o| < \rho\}$ there were a z_1 such that $u(z_1) \geq u(z)$ for $z \in \overline{D} = \{z \mid |z-z_o| \leq \rho\}$, we could show, as in the proof of the maximum principle for harmonic functions, using a sufficiently small ρ_1, that u(z) is a constant in $\{z \mid |z-z_1| \leq \rho_1\}$. Thus the maximum principle applies to u(z) in D. Because the maximum principle applies to any harmonic function defined in D, it applies to the difference between u(z) and any such harmonic function.

Therefore let

$$v(z) = P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\, u(z_o + \rho e^{i\theta}) d\theta.$$

Then $u(z) - v(z) = 0$ on $\{z \mid |z-z_o| = \rho\}$ and $u(z) - v(z)$ has no maximum or minimum in the interior of D. Thus $u(z) - v(z) \equiv 0$ or $u(z) \equiv v(z)$ for $z \in D$. However, since z_o was arbitrary, we see u(z) is identically equal to a harmonic function in all of Ω, or u(z) is harmonic in Ω.

HARNACK'S PRINCIPLE

Before proving Harnack's principle as a theorem, we want to prove the following lemma.

LEMMA 3-1--If $u(z)$ is harmonic for $|z|<\rho$, then

$$|u(z)| \leq \frac{\rho+r}{\rho-r}|u(0)|.$$

Further, if $u(z) \geq 0$ for $|z| \leq \rho$, then

$$\frac{\rho-r}{\rho+r}u(0) \leq u(z) \leq \frac{\rho+r}{\rho-r}u(0).$$

First,

$$\rho-r \leq |\rho e^{i\theta}-z| \leq \rho+r$$

implies

$$(\rho-r)^2 \leq |e^{i\theta}-z|^2 \leq (\rho+r)^2,$$

and

$$\frac{1}{(\rho+r)^2} \leq \frac{1}{|\rho e^{i\theta}-z|^2} \leq \frac{1}{(\rho-r)^2},$$

and thus

$$\frac{\rho^2-r^2}{(\rho+r)^2} \leq \frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2} \leq \frac{\rho^2-r^2}{(\rho-r)^2},$$

or

$$\frac{\rho-r}{\rho+r} \leq \frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2} \leq \frac{\rho+r}{\rho-r}.$$

Then if $u(z)$ is harmonic,

$$|u(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2}u(\rho e^{i\theta})d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2}|u(\rho e^{i\theta})|d\theta$$

$$\leq \frac{1}{2\pi}\frac{\rho+r}{\rho-r}\int_0^{2\pi} u(\rho e^{i\theta})\, d\theta = \frac{\rho+r}{\rho-r}\; u(0) \; .$$

If $u(z) \geq 0$, we have

$$|u(z)| = u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2}\; u(\rho e^{i\theta})d\theta.$$

$$\geq \frac{1}{2\pi}\frac{\rho-r}{\rho+r}\int_0^{2\pi} u(\rho e^{i\theta})\, d\theta,$$

or, since

$$\int_0^{2\pi} u(\rho e^{i\theta})d\theta = \int_0^{2\pi} |u(\rho e^{i\theta})|\, d\theta,$$

when $u(z) \geq 0$ for $z \in |z| \leq \rho$,

$$\frac{1}{2\pi}\frac{\rho-r}{\rho+r}\int_0^{2\pi} u(\rho e^{i\theta})d\theta \leq u(z) \leq \frac{\rho+r}{\rho-r}\int_0^{2\pi} u(\rho e^{i\theta})d\theta.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} u(\rho\; e^{i\theta})d\theta = u(0),$$

we have

$$\frac{\rho-r}{\rho+r}\; u(0) \leq u(z) \leq \frac{\rho+r}{\rho-r}\; u(0).$$

We can apply this inequality to a series of positive terms, or to the differences between successive terms of an increasing sequence, of harmonic functions.

THEOREM 3-4 (Harnack's Principle)--Consider a sequence of functions $u_n(z)$, each defined and harmonic in a certain region Ω_n. Let Ω be a region such that every point in Ω has a neighborhood contained in all but a finite number of the Ω_n, and assume moreover that in this neighborhood $u_n(z) \leq u_{n+1}(z)$ as soon as n is sufficiently large. Then there are only two possibilities: Either $u_n(z)$ tends uniformly to ∞ on every compact (i. e., closed and bounded) subset of Ω, or $u_n(z)$ tends to a harmonic limit function $u(z)$ in Ω, uniformly on compact sets.

PROOF--First, assume there is at least one point, z_o, where $\lim u_n(z_o) = \infty$. From the assumptions made above, there is an $r > 0$ and an m such that for $|z-z_o| < r$ and $n \geq m$, $u_n(z)$ is harmonic and $u_n(z) \leq u_{n+1}(z)$. Then applying the left hand estimate above to $u_n(z) - u_m(z) \geq 0$, we have, inside the disk, $\{z \mid |z-z_o| \leq \frac{r}{2}\}$,

$$\frac{r-\frac{r}{2}}{r+\frac{r}{2}}(u_n(z_o) - u_m(z_o)) \leq u_n(z) - u_m(z),$$

or

$$\frac{1}{3}(u_n(z_o) - u_m(z_o)) \leq u_n(z) - u_m(z).$$

Thus $u_n(z)$ goes to infinity uniformly in the disk $\{z \mid |z-z_o| \leq \frac{r}{2}\}$. If we have a point z_1, such that the $\lim u_n(z_1) < \infty$, then there are an r' and an m' such that inside the disk $\{z \mid |z-z_1| < r'\}$, and for $n \geq m'$, $u_n(z)$ is harmonic and $u_n(z) \leq u_{n+1}(z)$. Then applying the right hand inequality to $u_n(z) - u_{m'}(z) \geq 0$, we have

$$u_n(z) - u_{m'}(z) \leq \frac{r+\frac{r}{2}}{r-\frac{r}{2}}(u_n(z_1) - u_{m'}(z_1))$$

or

$$u_n(z) - u_{m'}(z) \leq 3(u_n(z_1) - u_{m'}(z_1)).$$

Then $u_n(z) \leq 3u_n(z_1) + u_{m'}(z) - 3u_{m'}(z_1)$.

Thus for $|z-z_1| \leq \frac{r}{2}$, $u_n(z)$ is bounded, for $u_n(z_1)$ and $u_{m'}(z_1)$ are bounded by our assumption, and $u_m(z)$ is bounded because it is harmonic for $|z-z_1| < r$, and thus continuous in this disk. Therefore it is continuous and hence bounded in the compact disk, $\{z \mid |z-z_1| \leq r\}$. This shows that the sets in which $\lim_{n\to\infty} u_n(z) < \infty$ are open and the sets in which $\lim_{n\to\infty} u_n(z) = \infty$ are also open.

Since Ω is a connected region and is the union of the two sets, one of them must be empty. Then, if $\lim_{n\to\infty} u_n(z) = \infty$ for any $z \in \Omega$, $\lim_{n\to\infty} u_n(z) = \infty$ for all $z \in \Omega$. The uniform convergence to ∞ on compact sets follows by use of the Heine-Borel theorem.

If $\lim_{n\to\infty} u_n(z) < \infty$ at any $z \in \Omega$, we know $\lim_{n\to\infty} u_n(z) < \infty$ for all $z \in \Omega$, and we want to show the convergence is uniform. Because

$$u_{n+p}(z) - u_n(z) \leq 3\, u_{n+p}(z_1) + u_n(z_1)$$

for $|z-z_1| \leq \frac{r'}{2}$ and $n > m'$, we have uniform convergence in some neighborhood of every point. By use of the Heine-Borel theorem, we therefore have uniform convergence on any compact subset of Ω.

To show $\lim u_n(z) < \infty$ is harmonic, we consider, since $u_n(z)$ is harmonic for every n,

$$u_n(z) = P_{u_n}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\, u_n(\rho e^{i\theta} + z_o)d\theta.$$

Also, for $z \in \Omega$ and suitable ρ, we can construct $P_u(z)$, defined for $z \in \{z \mid |z-z_o| < \rho\} \subset \Omega$. If we can show $P_u(z) \equiv u(z)$ for all $z \in \Omega$, then we know u(z) is harmonic. For u and u_n defined on the circumference of the disk $D = \{z \mid |z-z_o| < \rho\} \subset \Omega$, we have

$$P_u(z) - \lim_{n\to\infty} P_{u_n}(z) = \lim_{n\to\infty}\left[P_u(z) - P_{u_n}(z)\right],$$

since $P_u(z)$ can be treated as a constant in this case. Then

$$P_u(z) - P_{u_n}(z)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\, u(z_o + \rho e^{i\theta})d\theta -$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\, u_n(z_o + \rho e^{i\theta})d\theta.$$

Therefore $\lim_{n\to\infty}\left[P_u(z) - P_{u_n}(z)\right] =$

$$\lim \left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2}\left[u(\rho e^{i\theta} + z_o) - u_n(\rho e^{i\theta} + z_o)\right]d\theta\right\}.$$

Then, because the closure $\bar{D}$ is a compact set, for any $\varepsilon > 0$ there is an N such that if $n \geq N$,

$$0 \leq \left[u(\rho e^{i\theta} + z_o) - u_n(\rho e^{i\theta} + z_o)\right] < \mathcal{E},$$

for all θ.

Then

$$\lim_{n \to \infty} \left[P_u(z) - P_{u_n}(z)\right] <$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2} \mathcal{E} \quad d\theta = \mathcal{E}.$$

Since $\left[u(\rho e^{i\theta} + z_o) - u_n(\rho e^{i\theta} + a_o)\right] \geq 0$ for $n \geq N$,

$$0 \leq \lim_{n \to \infty} \left[P_u(z) - P_{u_n}(z)\right] < \mathcal{E},$$

for arbitrary $\mathcal{E}$, so that

$$0 = P_u(z) - \lim_{n \to \infty} P_{u_n}(z) = P_u(z) - \lim_{n \to \infty} u_n(z),$$

since, because $u_n(z)$ is harmonic, $P_{u_n}(z) = u_n(z)$, for all n.

Therefore

$$0 = P_u(z) - \lim_{n \to \infty} u_n(z) = P_u(z) - u(z),$$

or

$$P_u(z) = u(z),$$

for all $z \in \{z \mid |z-z_o| < \rho$, so that $u(z)$ is harmonic in $\{z \mid |z-z_o| < \rho\}$. However, z_o and ρ are arbitrary, so that $u(z)$ is harmonic in Ω.

Next, we use Harnack's Principle to prove the following more general theorem:

THEOREM 3-5--Suppose that a family $\mathcal{U}$ of harmonic functions on a Riemann surface W satisfies the following condition:

(A) For any u_1 and u_2 belonging to $\mathcal{U}$ there is a u belonging to $\mathcal{U}$ with $u \geq \max(u_1, u_2)$ on W. Then the function

$$U(z) = \sup_{u \in \mathcal{U}} u(z)$$

is either harmonic or constantly equal to $+\infty$.

PROOF--Let z_o be an arbitrary point of W. Then there is a sequence, $\{u_n\}_{n=1}^{\infty}$ of functions, $u_n \in \mathcal{U}$, with

$$\lim_{n\to\infty} u_n(z_o) = U(z_o) .$$

Let $\overline{u_1} = u_1$, and for each n, choose $\overline{u_{n+1}}$ such that

$\overline{u_{n+1}} \geq \max(u_{n+1}, \overline{u_n})$. Then, since each $\overline{u_n} \in \mathcal{U}$,

$$\lim_{n\to\infty} \overline{u_n}(z_o) = U(z_o).$$

Also, the sequence $\{\overline{u_n}\}_{n=1}^{\infty}$ is non-decreasing. Then, applying Harnack's Principle to the sequence $\{\overline{u_n}\}_{n=1}^{\infty}$, we have

$$U_o(z) = \lim_{n\to\infty} \overline{u_n}(z)$$

is either harmonic or identically $+\infty$. Obviously $U_o(z_o) = U(z_o)$.

Let z_o' be another point of W. Then there is a sequence of functions $u_n' \in \mathcal{U}$ such that

$$\lim_{n\to\infty} u_n'(z_o') = U(z_o').$$

As before, we let $\overline{u_n'} \in \mathcal{U}$ be such that $\overline{u_n'} \geq \max(\overline{u_n'}, \overline{u_n}, \overline{u_{n-1}})$.

Then the limit function

$$U_o'(z) = \lim_{n \to \infty} \overline{u_n'}(z)$$

will satisfy $U_o' \geq U_o$ and $U_o'(z_o) = U_o(z_o)$ and $U_o'(z_o') = U(z_o')$, also.

If U_o and U_o' are finite, then since $U_o - U_o' \leq 0$ and $(U_o(z_o) - U_o'(z_o)) = 0$, $U_o - U_o'$ has a maximum at z_o, and since W is a region, this means by use of the maximum principle, $U_o \equiv U_o'$ in W. Because $U(z_o') = U_o(z_o')$, and because z_o' is an arbitrary point of W, $U \equiv U_o$. Further, U is harmonic, since U_o is harmonic according to Harnack's Principle.

If $U_o = +\infty$, then $U_o'(z_o) = +\infty$, and since $U_o'(z_o') = U(z_o')$, $U \equiv +\infty$, since z_o' is an arbitrary point of W.

SUBHARMONIC FUNCTIONS

DEFINITION 3-2--If v is a real-valued function, v is said to be subharmonic in a plane region W if

(A1) v is upper semicontinuous (u.s.c.) in W, i. e., ($v(z') \geq \overline{\lim}\ v(z)$ for all $z' \in W$).

(A2) If u is a function harmonic in $W' \subset W$, then v-u is either constant or fails to have a maximum in W'.

Because v is real-valued and hence finite for any given z', v is bounded on any compact subset of W. Conventionally, a function v' which takes on the value $-\infty$ at any or all points in the region under discussion is also admitted as an u. s. c. function. Thus an u. s. c. function may take on all finite values and $-\infty$, as well, but not $+\infty$. If V is any compact subset of W, and v is an u. s. c. function not identically $-\infty$ in V, v has a finite maximum in V. Also, if v is u. s. c.,

$$v = \lim_{n \to \infty} f_n,$$

where $\{f_n\}_{n=1}^{\infty}$ is a non-increasing sequence of continuous functions.[1] (The proof is shown for lower semicontinuous functions, but by suitably changing terminology, the theorem applies to u. s. c. functions).

DEFINITION 3-3--The function v is <u>superharmonic</u> if $-v$ is subharmonic.

If u is harmonic, u is both subharmonic and superharmonic. The converse is also true, but needs further proof.

Since the definition of subharmonicity is of local character, a function which is subharmonic in a neighborhood of every point of W is subharmonic on W. Also, subharmonicity is invariant under conformal mappings. Thus, if v is a subharmonic function defined in a region W, and if φ is a conformal mapping of W onto a region W_1 such that for

$$\varphi(x, y) = (x_1, y_1),$$

$$v(x, y) = v_1(x_1, y_1) = v_1\varphi(x, y),$$

for all $(x, y) \in W$, then V_1 is a subharmonic function on W_1.

If we wish to consider an arbitrary Riemann surface, W, we can apply the above definition of subharmonicity without change. Thus a real-valued function v is subharmonic on an arbitrary surface W, if and only if it is subharmonic when expressed in terms of a local variable, this local variable z being the value assigned

[1] Ref. (4), pg. 103.

a point $w \in W$ under one of the allowed conformal mappings, φ, of a neighborhood of w onto an open set of the z-plane.

LEMMA 3-2--If v_1 and v_2 are subharmonic, then $v = \max (v_1, v_2)$ is also subharmonic.

PROOF--(A1) If $v_1(z) \geq \overline{\lim}_{z' \to z} v_1(z')$ and $v_2(z) \geq \overline{\lim}_{z' \to z} v_2(z')$, then $v(z) \geq v_1(z)$ and $v(z) \geq v_2(z)$ implies $\overline{\lim}_{z' \to z} v(z')$ $= \max(\overline{\lim}\ v_1(z'), \overline{\lim}\ v_2(z')) \leq v(z)$.

(A2) Let u be harmonic in W', the region of definition of both v_1 and v_2, and assume further v-u has a maximum in W', say at z_o. Suppose, for example, that $v = v_1$ at $z = z_o$. Then for all $z \in W'$,

$$v_1(z) - u(z) \leq v(z) - u(z) \leq v(z_o) - u(z_o)$$
$$= v_1(z_o) - u(z_o),$$

so that v_1-u has a maximum in W', and thus is a constant c, so that we have also

$$c \leq v(z) - u(z) \leq c,$$

or

$$v(z) - u(z) = c.$$

We can extend this result to any finite number of subharmonic functions, but if we attempt to extend it to an infinite family of subharmonic functions, we fail; we cannot show $v = \max \{v_n\}_{n=1}^{\infty}$ is u. s. c.

We can learn more about subharmonic functions by the use of the

Poisson integral. If v is subharmonic and continuous on the circumference of a disk Δ with center z_o and radius ρ, we know the Poisson integral of v with respect to Δ is

$$P_v(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|z'-z|^2}\, v(z')d\theta,$$

where $z' = z_o + \rho e^{i\theta}$. We also remember $P_v(z)$ is harmonic inside Δ and

$$\lim_{z \to z'} P_v(z) = v(z').$$

If v is u. s. c., $P_v(z)$ can be interpreted as a Lesbesgue integral. However, we can also use the fact that v is the limit of a non-increasing sequence of continuous functions. Then we set

$$P_v(z) = \inf P_W(z) = \inf \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|z'-z|^2}\, W(z')d\theta,$$

where W ranges over all continuous functions such that $W(z) \geq v(z)$ for all z for which v is defined. If v is u. s. c., instead of

$$\lim_{z \to z'} P_v(z) = v(z'),$$

we have

$$\overline{\lim_{z \to z'}}\, P_v(z) \leq v(z').$$

Then v', defined as $v' = v$ on $|z-z_o| = \rho$, and $v' = P_v$ in $|z-z_o| < \rho$, is u. s. c. for $|z-z_o| < \rho$. By applying Harnack's Principle, we see that P_v is either harmonic or identically $-\infty$ in $|z-z_o| \leq \rho$.

To show the elementary character of the Poisson integral, we show

$$P_{v_1 + v_2} = P_{v_1} + P_{v_2}$$

without use of the Lebesgue integral.

If v_1 and v_2 are continuous, the relation is obvious. If v_1 and/or v_2 are u. s. c., let w_1 and w_2 be continuous majorants of v_1 and v_2. Then

$$P_{v_1 + v_2} \leq P_{w_1 + w_2} = P_{w_1} + P_{w_2}.$$

We have $w_i \geq v_i$, $i = 1, 2$, and for $z \in \left\{z \,\middle|\, |z-z_o| < \rho\right\}$,

$$\left[P_{w_i} - P_{v_i}\right](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2} \left\{\left[w_i - v_i\right](e^{i\theta} + z_o)\right\} d\theta.$$

However, we can find for our w_i a continuous function such that $w_i(z) \geq v_i(z)$ for all $z \in \left\{z \,\middle|\, |z-z_o| = \rho\right\}$, and such that for an arbitrary $\varepsilon > 0$, $w_i(z) - v_i(z) < \varepsilon$, except on a subset of measure 0, so that for this w_i,

$$\left[P_{w_i} - P_{v_i}\right](z) < \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-z_o|^2}{|\rho e^{i\theta} - (z-z_o)|^2} \varepsilon \; d\theta = \varepsilon,$$

or $P_{w_i} = P_{v_i}$, for all $z \in \left\{z \,\middle|\, |z-z_o| < \rho\right\}$. Then we have

$P_{v_1+v_2} \leq P_{w_1} + P_{w_2} = P_{v_1} + P_{v_2}$, or $P_{v_1+v_2} \leq P_{v_1} + P_{v_2}$.

To prove the inequality in the other direction, let w be a continuous function such that

$$w \geq v_1 + v_2.$$

Then

$$\overline{\lim_{z \to z'}} (P_{v_1} + P_{v_2})(z) \leq v_1(z') + v_2(z')$$

$$w(z') = \lim_{z \to z'} P_w(z)$$

Then by use of the maximum-minimum principle we see that for $|z-z_o| < \rho$,

$$P_w - (P_{v_1} + P_{v_2}) \geq 0,$$

because $P_w - (P_{v_1} + P_{v_2})$, as harmonic function takes its minimum on the boundary, $\{z \mid |z-z_o| = \rho\}$, or

$$P_w \geq P_{v_1} + P_{v_2},$$

and thus

$$P_{v_1} + P_{v_2} \leq P_{v_1+v_2}.$$

Therefore

$$P_{v_1+v_2} = P_{v_1} + P_{v_2}.$$

Obviously, if $v \leq 0$, $P_v \leq 0$, and if also $v \not\equiv 0$ on $|z-z_o| = \rho$, $P_v < 0$ in $|z-z_o| < \rho$, by use of the maximum principle.

THEOREM 3-6--An u. s. c. function v is subharmonic in a plane region W if and only if

$$v(z) \leq P_v(z)$$

in all disks Δ with $\overline{\Delta} \subset W$.

PROOF--Suppose v is subharmonic. Let w be a continuous majorant of v on the boundary of Δ. Because v is u. s. c., we have

$$\overline{\lim_{z \to z'}} v(z) \leq v(z') \leq w(z') = \lim_{z \to z'} P_w(z).$$

If $v - P_w \not\equiv c$, a constant, $v - P_w$ does not have a maximum in Δ, and therefore

$$v \leq P_w$$

in Δ, and thus

$$v \leq P_v,$$

because as was shown previously, if v is u. s. c., there can be found a continuous majorant of v, namely w, such that $P_w(z) = P_v(z)$, for all z in W. To show the proof in the opposite direction, assume

$$v(z) \leq P_v(z)$$

in all disk Δ with $\overline{\Delta} \subset W$. Then let u be harmonic in $W' \subset W$, and assume v-u has a maximum at $z_o \in W'$. Since $v(z) \leq P_v(z)$, we have

$$v-u \leq P_{v-u},$$

so we can let u be identically equal to zero, to simplify calculations. Further, since

$$v(z) - u(z) = v(z) - 0 \leq v(z_o) - u(z_o) = v(z_o),$$

if $v(z) \nleq 0$, for all z, we can let

$$v'(z) = v(z) - v(z_o) \leq 0,$$

so that, without loss of generality, we can specify $v(z) \leq 0$, and $v(z_o) = 0$. Then, for ρ sufficiently small, we consider the disk $\{z \mid |z-z_o| < \rho\}$, and we have, for $|z'-z_o| = \rho$,

$$v(z') \leq 0,$$

and

$$0 = v(z_o) \leq P_v(z_o) \leq 0.$$

But $P_v(z_o) = 0$ implies that $v(z') \equiv 0$ on the boundary of Δ. Then $v \equiv 0$ in a neighborhood of z_o, by use of the maximum principle, and because W' is connected, we can again construct a disk Δ' with another point z_o' belonging to the boundary of as center, and show in Δ' $v = 0$. Then if v-u has a maximum in W, at $z = z_o$, v-u is constant and equal to $v(z_o) - u(z_o)$ on an open subset of W, for we can show that any point at which v-u is equal to 0 is the center of an open disk in which v-u is identically equal to 0. However, it can be shown that the set in which v-u is a maximum is closed. We know u is continuous and v is u. s. c., so v-u is u. s. c. Let z' be a limit point of the set on which v-u is a maximum. We know

$$v(z') - u(z') \geq \overline{\lim_{z \to z'}} \left[v(z) - u(z) \right].$$

Then every neighborhood of z' has a point z_o such that v-u is at a maximum at z_o, since z' is a limit point of such a set. Then

$$v(z') - u(z') \geq v(z_o) - u(z_o) \geq v(z) - u(z)$$

for $z \in W$. Then $z' \in \{z \mid v(z) - u(z) = \max (v-u) \text{ in } W\}$. Thus the set in W in which v-u is equal to the maximum is both open and closed, and thus is either W or $\emptyset$. Then v is subharmonic, since it has already been stated v is u. s. c.

If v is continuous, Theorem 3-6 leads to the mean-value property, for we have

$$v(z) \leq P_v(z),$$

so that

$$v(z_o) \leq P_v(z_o) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z_o - z_o|^2}{|z_o + \rho e^{i\theta} - z_o|^2} v(z_o + \rho e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} v(z_o + \rho e^{i\theta})d\theta.$$

f v is only u. s. c. and not continuous, we can replace the Riemann integral by the Lebesgue integral and this inequality holds for any subharmonic v.

THEOREM 3-7--If v is both subharmonic and superharmonic, then v is harmonic.

PROOF--If v is subharmonic, then in any disk $D = \{z \,\|\, |z - z_o| < \rho\}$,

$$v(z_o) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_o + \rho e^{i\theta})\, d\theta$$

If v is superharmonic, -v is subharmonic, so that

$$-v(z_o) \leq \frac{1}{2\pi}\int_0^{2\pi} [-v(z_o + \rho e^{i\theta})]d\theta,$$

or $v(z_o) \geq \frac{1}{2\pi}\int_0^{2\pi} v(z_o + \rho e^{i\theta})d\theta$, so that we have

$$v(z_o) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_o + \rho e^{i\theta})d\theta \leq v(z_o),$$

or

$$v(z_o) = \frac{1}{2\pi}\int_0^{2\pi} v(z_o + \rho e^{i\theta})d\theta,$$

and since, for ρ sufficiently small, any z in W may be the center of such a disk D, z_o is arbitrary, so we know from Theorem 3-3 that v is harmonic in its region of definition.

If v_1 and v_2 and sub harmonic functions, then $v_1(z) \leq P_{v_1}(z)$ and $v_2(z) \leq P_{v_2}(z)$ and $P_{v_1+v_2} = P_{v_1} + P_{v_2}$, so that we have

$$v_1(z) + v_2(z) \leq P_{v_1}(z) + P_{v_2}(z) = P_{v_1+v_2}(z);$$

thus $(v_1 + v_2)$ is subharmonic if v_1 and v_2 are subharmonic.

Since we are interested in harmonic functions defined not only in a region of the Euclidean plane, but in harmonic functions defined on arbitrary Riemann surfaces, we make the following definition:

DEFINITION 3-4--Let v be subharmonic on a Riemann surface W, and let Δ be a parametric disk in W. Then P_v is the Poisson integral of v in Δ which is formed by means of a specific conformal mapping of Δ onto a circular disk.

THEOREM 3-8--The function v_o which is equal to P_v in Δ and equal to v on $W-\Delta$ is subharmonic on W.

PROOF--On $W-\Delta$, $v_o = v$ is u. s. c. Since

$$\overline{\lim_{z \to z'}} P_v(z) \leq v(z'),$$

where z' belongs to the boundary of Δ, v_o is also u. s. c. in $\overline{\Delta}$. To prove (A2), let u be harmonic in $W' \subset W$. Suppose v_o-u has a maximum at $z_o \in W'$. If $z_o \in \Delta$, then we see v_o-u is constant in a component of $W' \cap \Delta$, for in Δ, v_o is harmonic and thus v_o-u is harmonic, and does not have a maximum on any open set, unless it is a constant. If $W' = W' \cap \Delta$, we are through, for since W' is connected, $W' \cap \Delta$ is connected and hence there is only one component of W' in Δ. If $W' \neq W' \cap \Delta$, then Δ has a boundary point in W', and since v_o-u is u. s. c., the maximum is attained on the boundary. Thus, unless $v_o - u$ is a constant, it cannot have a maximum in Δ. If $z_o \in W' \cap (W - \bar{\Delta})$, then by the same reasoning, we have $v_o - u$ constant in a component of $W' \cap (W - \bar{\Delta})$.

If $W' = W' \cap (W - \bar{\Delta})$, we are through, but if $W' \neq W' \cap (W - \bar{\Delta})$, we know that, because W' is connected, W' has a point in common with Δ, and hence with the boundary of Δ, and that the maximum of $v_o - u$ is taken at that boundary point. Thus we have shown we need only consider the case where the maximum of $v_o - u$ is taken on the boundary of Δ. There we have

$$v(z) - u(z) \leq v_o(z) - u(z) \leq$$
$$v_o(z_o) - u(z_o) \leq v(z_o) - u(z_c).$$

Because $v(z) - u(z)$ then has a maximum, at z_o in $W' \cap W$, it is constant there, so that as a result of the double inequality, $v_o(z) - u(z)$ is also a constant, and we have proved condition (A2).

THE SOLUTION OF THE DIRICHLET PROBLEM

DIRICHLET'S PROBLEM--Given a continuous real-valued function f on Γ, the boundary of a subregion G of a Riemann surface W, we are required to construct a continuous function u on $\bar{G} = G \cup \Gamma$ with $u \equiv f$ on Γ and u harmonic in G.

Obviously, if u_1 and u_2 are two such functions, $u_1 \equiv u_2$, as was shown as a result of Theorem 3-1.

The following solution of Dirichlet's problem is by use of Perron's method. Perron's method was published in a paper, "Uber die Behandlung der ersten Randveraufgabe fur $\Delta^2 u = 0$," published in MATHEMATISCHE ZEITUNG, 18, pages 42-54, in 1923.[2] We find, in attempting to solve Dirichlet's problem by this method that, whether or not there exists a solution to this problem, there is associated with every function f, defined on the boundary Γ, and whether continuous or not, a function u which is either harmonic or completely degenerate ($u \equiv \pm\infty$).

To find a candidate for the desired function u, let $\mathcal{V}(f)$ be the class of all subharmonic functions v in G such that

$$\overline{\lim_{z \to z'}}\ v(z) \quad f(z')$$

for all $z' \varepsilon \Gamma$. The function f is real-valued, but otherwise may take on any desired values, even $+\infty$ or $-\infty$.

THEOREM 3-9--The function u, defined by

$$u(z) = \sup_{v \in \mathcal{V}(z)} v(z),$$

is either harmonic, identically $+\infty$ or identically $-\infty$ in G.

PROOF--Because the function which is identically $-\infty$ is in $\mathcal{V}(f)$, $\mathcal{V}(f) \neq \phi$. If this function is the only element of $\mathcal{V}(f)$, then we can see $u \equiv -\infty$, also.

If $\mathcal{V}(f)$ has other members besides the function $v \equiv -\infty$ mentioned above, we proceed as follows. Let Δ be a parametric disk such that $\Delta \subset G$. If $v \in \mathcal{V}(f)$, we can form the associated function v_o, with $v_o = P_v$ in Δ, and $v_o = v$ on $G - \Delta$. Then from Theorem 3-7, we know v_o is subharmonic in G and thus $v_o \in \mathcal{V}(f)$, and from Theorem 3-6, $v \leq v_o$. Then

[2] See Ref. (6).

$$u(z) = \sup_{v_o = P_v^o, v \in \mathcal{V}(f)} v_o(z)$$

in Δ.

All the v_o are subharmonic in G, and hence in Δ. Then we have two possible cases, $v_o \equiv -\infty$ for all v_o in Δ, and hence $u \equiv -\infty$, or there exists at least one v_o that is finite in Δ. Then we know by its construction v_o is harmonic in Δ. Then let us consider the class of all v_o finite and hence harmonic in Δ. These satisfy Theorem 3-5 and thus

$$u = \sup_{v_o = P_v^o, v \in \mathcal{V}(f)} v_o$$

is either harmonic or identically equal to $+\infty$ in Δ.

Then u is harmonic or identically equal to $-\infty$ or $+\infty$ in each parametric disk. Because G is connected, only one of these conditions can occur, so that the theorem is proved.

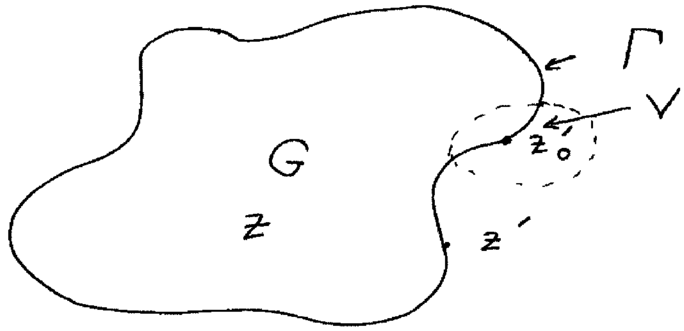
In order to determine the conditions under which a solution to Dirichlet's problem exists, we shall study the boundary behavior of the function u. In this study, we shall be interested only in the case when f is bounded, so that $|f| \leq M$, for some M.

DEFINITION 3-5--A function β in G is called a <u>barrier</u> at $z_o' \in \Gamma$, the boundary of G, if it satisfies:

(B1) β is subharmonic in G,

(B2) $\lim_{z \to z_o} \beta(z) = 0$,

(B3) $\overline{\lim_{z \to z_o}} \beta(z) < 0$ for all $z' \neq z_o'$, $z' \in \Gamma$.



Ill. 3-2

A boundary point z_o' is called <u>regular</u> if and only if there is a barrier at z_o'. Let V be a neighborhood of z_o' and β a barrier function as described. Then because V is open, β is strictly less than 0 everywhere in V, and outside V, there is a $-m$, $m > 0$, such that $\beta(z') \leq -m$ for all $z' \in \Gamma \cap (G-V)$. Let β_V be the function such that $\beta_V(z') = \max\left(\frac{\beta(z')}{m}, -1\right)$. Then $\beta_V(z') < 0$ for all $z' \in \Gamma$ and $\beta_V(z') \equiv -1$ for $z' \notin V$. β_V is called a normalized barrier with respect to V. If G' is a region such that $G' \cap V = G \cap V$, β_V can be used as a barrier for G', if we define $\beta_V \equiv -1$ in $G' - (G' \cap V)$. Thus the existence of a barrier at a point z_o' is a local property, and depends only upon the geometric properties of G in a sufficiently small neighborhood of z_o'.

THEOREM 3-10--At a regular point z_o' the function u, introduced in Theorem 3-8, satisfies

$$\varliminf_{z' \to z_o'} f(z') \leq \varliminf_{z \to z_o'} u(z) \leq \varlimsup_{z \to z_o'} u(z) \leq \varlimsup_{z' \to z_o'} f(z')$$

provided that f is bounded.

PROOF--Let $A = \varlimsup_{z' \to z_o'} f(z')$ and let V be a closed neighborhood of z_o' such that $f(z') < A + \varepsilon$, for a given $\varepsilon > 0$. If $v \in \mathcal{V}(f)$, the function φ, with

$$\varphi = v - A + (M - A)\beta_V$$

is subharmonic and

$$\overline{\lim_{z \to z'}}\, \varphi(z) < \varepsilon ,$$

for all $z' \in \Gamma$, whether inside or outside of V. If $z \in V$, we have $\overline{\lim_{z \to z'}}\, v(z) < A + \varepsilon \leq M$, because $\overline{\lim_{z \to z'}}\, v(z) \leq f(z')$ and $f(z') \leq M$ for $z' \in \Gamma$, and $-1 \leq \beta_V(z) \leq 0$ for $z \in G$. Then if $z \in V$,

$$\varphi(z) \leq A + \varepsilon - A = \varepsilon ,$$

because $(M - A)\beta_V \leq 0$. If $z \in G - V$, $z' \in \Gamma - (\Gamma \cap V)$ and $\overline{\lim_{z \to z'}}\, v(z) \leq M$, while $\beta_V = -1$. Then

$$\varphi(z) \leq M - A - (M - A) = 0.$$

Then $\varphi(z) < \varepsilon$ in G, and because v is arbitrary, it is true for all $v \in \mathcal{V}(f)$. Since

$$u(z) = \sup_{v \in \mathcal{V}(f)} v(z),$$

we have

$$u(z) - A + (M - A)\beta_V \leq \varepsilon$$

or

$$u(z) \leq A - (M - A)\beta_V + \varepsilon.$$

As z tends to z_o', we have

$$\overline{\lim_{z \to z_o'}}\, u(z) < A + \varepsilon.$$

Then

$$\overline{\lim_{z \to z_o'}}\, u(z) \leq \overline{\lim_{z' \to z_o'}}\, f(z').$$

Now to show

$$\underline{\lim}_{z'\to z_0'} f(z') \leq \underline{\lim}_{z\to z_0'} u(z),$$

we let

$$\Psi = (B + M)\beta_V + B - \mathcal{E},$$

where $B = \underline{\lim}_{z'\to z_0'} f(z')$ and $f(z') > B - \mathcal{E}$ in V, a closed neighborhood of z_0'. Again Ψ is subharmonic because β_V is subharmonic and we have

$$\overline{\lim}_{z\to z_0'} \Psi(z) \leq B - \mathcal{E} < f(z'),$$

for $z' \in V$, since $f(z') < M$ and thus

$$-M \leq B \leq M \text{ or } M + B \geq 0,$$

while $(M + B)\beta_V < 0$. If $z' \in \Gamma - (\Gamma \cap V)$, $\beta_V = -1$, so that

$$\overline{\lim}_{z\to z'} \Psi(z) = -M - B + B - \mathcal{E} = -M - \mathcal{E} < f(z').$$

Then since Ψ is subharmonic in G, and

$$\overline{\lim}_{z\to z'} \Psi(z) \leq f(z'), \quad z' \in \Gamma,$$

we know

$$u(z) \geq \Psi(z).$$

Therefore when z tends to z_0', we have

$$\underline{\lim}_{z\to z_0'} u(z) > B - \mathcal{E},$$

and thus

$$\underline{\lim}_{z'\to z_0'} f(z') \leq \underline{\lim}_{z\to z_0'} u(z),$$

so that since

$$\underline{\lim}_{z\to z_0'} u(z) \leq \overline{\lim}_{z\to z_0'} u(z),$$

the theorem is proved.

COROLLARY--If f is continuous and G is a region with only regular points, then the Dirichlet problem has a solution. Conversely, if the Dirichlet problem has a solution for arbitrary continuous f, then every boundary point is regular.

We cannot state here necessary and sufficient conditions for regularity of a boundary point. While they are known, they cannot be given in a useful form. However, we can give the following theorem which is general enough for many cases.

THEOREM 3-11--The point z_o is a regular boundary point of G whenever the component of the boundary Γ which contains z_o does not reduce to a point.

PROOF--Because regularity is a local property, we can consider the case of a subset G of the Riemann sphere. From the assumptions stated in the theorem, the component on the boundary Γ containing z_o contains another point $z_1 \neq z_o$. We can select a simply connected subset E of the complement of G containing both z_o and a suitable z_1. By making an auxiliary linear transformation, we can choose $z_o = \infty$, and $z_1 = 0$. Because E is simply connected, we know the complement of E is also simply connected. Therefore we can define a single-valued branch of the function

$$s = \sigma + i\tau = \log z$$

in G. We know $2n\pi \leq \tau \leq 2(n+1)\pi$. For the sake of simplicity, let n = 0. Then this function maps G onto G', with $0 \leq \tau < 2\pi$. Then the intersection of G', with any line $\sigma = \sigma_o$ is a union of segments of total length equal to or less than 2π. To see that the number of such segments in any one line $\sigma = \sigma_o$ is at most countable, we note there are at most finitely many segments of length $\geq \frac{1}{n}$ for each positive integer n. Letting σ_o be fixed, we know

$$\sigma_o \cap G = \bigcup_{i=1}^{\infty} S_i,$$

where S_i is a segment of $\sigma_o \cap G$. Let s_i' and s_i'' be the endpoints of the line segment S_i, with

$$\operatorname{Im} s_i' < \operatorname{Im} s_i''.$$

Then if $\sigma \geq \sigma_o$, let us define

$$\mu_i = \arg \frac{s_i' - s}{s_i'' - s} = \arg (s_i' - s) - \arg (s_i'' - s),$$

$$0 \leq \mu_i \leq \pi.$$

Then we define the function

$$\alpha(s) = -\frac{1}{\pi} \sum_{i=1}^{\infty} u_i(s).$$

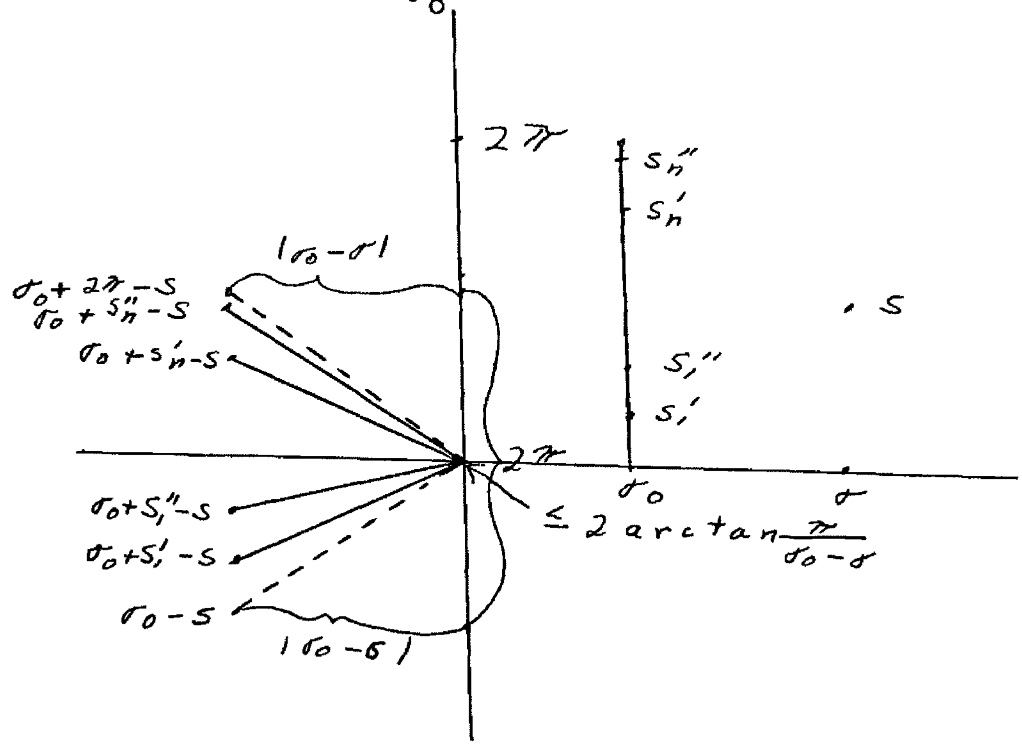
This function is harmonic because the sequence $\left\{\sum_{i=1}^{n} u_i(s)\right\}_{n=1}^{\infty}$ of sums is non-decreasing, and yet bounded, so that the limit is harmonic by Harnack's principle. The function $\mu_i(s)$ is harmonic because it is the imaginary part of the analytic function

$$\varphi = \log \frac{s_i' - s}{s_i'' - s} = \log \frac{\sigma_o + i\tau_o' - \sigma - i\tau}{\sigma_o + i\tau_o'' - \sigma - i\tau} =$$

$$\log \frac{\sigma_o + i\tau_o' - \log z}{\sigma_o + i\tau_o'' - \log z}.$$

It is easy to see (consult the following illustration) that

$$-\frac{2}{\pi}\arctan\frac{\pi}{\sigma-\sigma_0} \leq \alpha(s) \leq 0.$$



Ill. 3-3

If $\sigma=\sigma_0$, we have

$$\mathcal{M}_i(s) = \arg\frac{s_i'-s}{s_i''-s} = \arg\frac{\sigma_0+i\tau_i'-\sigma-i\tau}{\sigma_0+i\tau_i''-\sigma-i\tau}$$

$$= \arg\frac{i(\tau_i'-\tau)}{i(\tau_i''-\tau)} = \arg\frac{\tau_i'-\tau}{\tau_i''-\tau}.$$

Then because $\frac{\tau_i'-\tau}{\tau_i''-\tau}$ is a real number, its argument is a multiple of π. If $\tau>\tau_i''$, $\frac{\tau_i'-\tau}{\tau_i''-\tau}$ is the quotient of two negative numbers, and hence positive. If $\tau<\tau_i'$, $\frac{\tau_i'-\tau}{\tau_i''-\tau}$ is the quotient of two non-negative numbers and hence is non-negative. In these two cases,

$$\arg \frac{\tau_i' - \tau}{\tau_i'' - \tau} = 0.$$

If $\tau_i' < \tau < \tau_i''$,

$$\frac{\tau_i' - \tau}{\tau_i'' - \tau} < 0 \text{ and } \arg \frac{\tau_i' - \tau}{\tau_i'' - \tau} = \pi ,$$

so that

$$-\frac{1}{\pi} \sum_{n=1}^{\infty} \mu_i(s) = -\frac{1}{\pi}\left[\sum_{n=1}^{i-1} 0 + \pi + \sum_{n=i+1}^{\infty} 0\right] = -1.$$

If $\sigma = \sigma_0$ and $\tau = \tau_i'$ or τ_i'', $\mu_i(s)$ is defined to be 0. Then if we define α as identically equal to -1 for $\sigma < \sigma_0$, α is subharmonic in G.

However, we cannot yet say

$$\alpha(s) = \alpha(\log z)$$

is a barrier at z_0, for, even though it is subharmonic, negative in the interior of G, and has the limit 0, as $z \to \infty$, for

$$\lim_{\sigma \to \infty} \frac{2}{\pi} \arctan \frac{\pi}{\sigma - \sigma_0} = 0,$$

it may be that α goes to 0 at a finite boundary point. To construct a function not equal to zero at any finite boundary point, we let $\{\sigma_n\}_{n=0}^{\infty}$ be a sequence of real numbers tending to $+\infty$. Let σ_0 be replaces by σ_n in the definition of α and let this new function be defined as α_n. Then let β be the function defined as

$$\beta(z) = \sum_{n=0}^{\infty} 2^{-n} \alpha_n(\log z) = \sum_{n=0}^{\infty} 2^{-n}\left(-\frac{1}{\pi}\sum_{i=1}^{\infty} \mu_i(s)\right)$$

$$= -\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{i=1}^{\infty} 2^{-n} \arg \frac{s_{in} - s}{s''_{in} - s}, \text{ where } s_{in} = \sigma_n + i\tau_{in}.$$

Then

$$-\frac{2}{\pi}\sum_{n=0}^{\infty} 2^{-n} \arctan \frac{\pi}{\sigma - \sigma_n} \leq \beta(z) \leq 0$$

or

$$-\sum_{n=0}^{\infty} 2^{-n} \leq \beta(z) \leq 0.$$

Then $\beta(z)$ converges uniformly in G', and for z in the neighborhood of a finite boundary point, we have, for some N,

$$z = \sigma < \sigma_n, \; n \geq N,$$

so that $\sigma_n(z) = -1$ for $n \geq N$. Then $\overline{\lim_{z \to z_0}} \beta(z) < 0$, for z_0 a finite boundary point, and β is a barrier.

In conclusion, in the first section, we have shown the construction of Riemann surfaces for certain given functions. In the second section, manifolds were defined, and an abstract Riemann surface was defined in terms of these manifolds. In addition, we showed that the Riemann surfaces we constructed were just such abstract Riemann surfaces. In the third section, we studied the Poisson Integral, Harnack's Principle, and subharmonic functions, in order to determine the solution of Dirichlet's Problem, and gave a sufficient condition for the existence of a solution of this problem.

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