2002

The effect of information on a stochastic fishery model

Greg Cripe
The University of Montana

THE EFFECT OF INFORMATION ON A STOCHASTIC FISHERY MODEL

by

Greg Cripe

presented in partial fulfillment of the requirements

for the degree of

Doctor of Mathematics

The University of Montana

May 2001

Approved by:

Chairperson

Dean, Graduate School

5-31-02

Date

UMI Number: 3051607

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

Cripe, Greg Ph.D., May 2001 Mathematics

THE EFFECT OF INFORMATION ON A STOCHASTIC FISHERY MODEL

Director: Leonid Kalachev LK

Survival rates and carrying capacities in a fishery may be strongly affected by variations in climatic factors. When the stock is under control of a single manager, information about the stochastic growth parameters leads to improved economic return. However, when the stock is transboundary, additional information concerning the stochastic parameters can lead to overharvesting and in turn to lower economic returns.

To show this, we formulate the model as an optimal control problem in a game theoretic setting. We find the optimal harvest proportions using dynamic programming, maximizing the utility at each stage of the game. We then simulate the model using the derived harvest proportions. The generated data is analyzed to determine the effect information has on the utility function.

Contents

Chapter 1

Introduction

In recent years there has been increasing recognition of the impact of environmental stochasticity on major marine fisheries worldwide. The management of a fish stock is complicated by the resulting uncertainties. The goal of this paper is to provide a better understanding of the impact of knowledge on these stochastic fisheries. The intent is to ascertain whether earlier and more accurate information regarding the stochastic environmental conditions might allow for better management of the resource. This is indeed the case when a single manager controls the resource, however we will show this is not necessarily the case when the stock is transboundary.

We examine a model with classical assumptions. In [3] Clark studied a discrete time bioeconomic model for the harvesting of a renewable animal resource. The control theory problem was solved using dynamic programming under simple assumptions concerning growth and utilization of the stock. Later in [6], Levhari and Mirman incorporated the above assumptions into a game-theoretic setting. Beverton and Holt [2] previously had noted the difficulties in using purely deterministic models. In the papers of Mann [7] , Jacquette [5] and Reed [11] , the growth parameters are considered as random variables. In some cases the time sequence of parameters were considered to be Markov chains while in other cases the parameters were taken to be independt and identically distributed (i.i.d).

There has been considerable progress in non-cooperative game theory since the above papers were written. Fudenberg and Tirole [4] state we now have a deeper understanding of the role of information and how it impacts the outcomes of games. One would expect that these developments would have been applied to fishery models. However, McKelvey in [8] states: *'It seems the full potential of the approach has yet to be achieved. In particular, the negative implications of uncertain and asymmetric information has not really been explored in harvesting models.'*

In this paper we propose a study of the effects of uncertainty in a stochastic model. We focus on the role of incomplete information and on a comparison of outcomes in game versions that incorporate alternative information structures. We start by considering the following model. Two independently operated fleets competitively harvest a fish stock. The harvesting occurs annually. Each fleet selectively chooses a harvesting policy in order to maximize their discounted long term returns. The choice is made in response to the expected competitors harvest policy. In the classical version of the game, the two fleets harvest simultaneously from a common stock. In our version, the fish stock migrates from one countries fishing grounds to another countries fishing grounds. The fish are harvested sequentially as it travels from its adult feeding grounds to its spawning grounds.

We will examine several cases of the basic model starting with a sole manager of the stock under deterministic growth conditions. We proceed to introduce stochasticity by allowing Markov stochasticity to occur in the stock-recruitment relation, loosely simulating the occurrence of El Niño marine climatic events. Later we allow multiple fleets an opportunity to harvest the stock sequentially under stochastic growth conditions. Here we allow the fleets to have ascertained different knowledge about the growth parameters. Specifically we consider three cases: a fleet will only know last years growth rate, this years growth rate or the fleet may know next years growth rate. We allow different fleets to have different knowledge in a single game.

We demonstrate that when a single manager is employed, an increase in knowledge leads to an increase in economic return. However, a wide variety of situations arise when the fishery is competitive. For example, an increase in knowledge by both players will increase the return for the first fleet while a decrease in return will be observed for the following fleet.

In the last chapter we consider the spatially separated model proposed by McKelvey [9] and McKelvey and Cripe [10]. Here the stochasticity is of two kinds. The biological growth parameter is a Markovian random variable as in the previous chapters. The second way in which stochasticity appears is through a splitting of the stock, with a fraction, θ, available for fleet 1 and the fraction, $1 - \theta$, available for fleet 2. Here the harvesting takes place simultaneously, although in different 'split streams.' We assume theta to be i.i.d. The model formulation is based loosely on the Canadian-U.S. harvest competition over Canada's Fraser river sockeye salmon stock. In that real-world fishery, the spawning run splits as it rounds Vancouver Island, with only a fraction of the fish being available for the U.S. to harvest.

This time, instead of only allowing past, previous or future knowledge concerning the stochastic parameters, we consider an imprecise measurement of the stochastic parameters. We assume a player knows the proportion of observations measured correctly. For example, if the measurement is correct with probability .5, the fleet has gained no additional knowledge, whereas if the player measures correctly with probability 1, the fleet has full knowledge of the current condition.

We include below a list of notations used throughout the paper. A generic fleet is denoted by player ν.

List of Notations

γ - discount factor

R - recruitment

R^+- recruitment in the following year

S - escapement

h - harvest proportion

$\widehat{h}$ - maximum harvest proportion

$\tilde{h}$ - calculated optimal harvest proportion in interior $(0 \leq \tilde{h} \leq \widehat{h}\,)$

τ - number of years remaining before horizon

b - growth parameter

b^{+} - growth parameter in the following year

p_{nm} - probability of state b_n moving to state b_m

ν - generic fleet

$\overline{\nu}$ - ν's opponent

$F = AS^{b}$ - growth function

$Y = h_{\nu}R_{\nu}$ - yield

$U = \sum_{t=0}^{T} \gamma^{t} \ln\left[Y(t)\right]$ - utility function

$\mu(R) = \frac{d}{dR}(U(R))$

$\mathcal{M}(R) = R \cdot \mu(R)$

$\lambda(S) = \frac{\partial}{\partial S}U(S)$

$\Lambda(S) = S\lambda$

$P = [p_{nm}]$

B - diagonal matrix with entries $b_1, \cdots b_n$

$Q = BP$

θ - fraction of fish in player α's stream

θ_{ν} - fraction of recruitment R in player ν's stream

$\tilde{\theta}_{\nu}$ - measurement of θ taken by player ν

$\tilde{b}_\nu$ - measurement of b taken by player ν

$x^\nu_{ij} = prob(b = b_j \mid \tilde{b}_\nu = b_i)$ for player ν

$\tilde{x}^\nu_{ij} = prob(\tilde{b} = b_j \mid b_\nu = b_i)$ for player ν

$q^\nu_{ij} = prob(\theta = \theta_j \mid \tilde{\theta} = \theta_i)$ for player ν

$\tilde{q}^\nu_{ij} = prob(\tilde{\theta}_\nu = \theta_j \mid \theta = \theta_i)$ for player ν

σ - total fraction of fish harvested in split stream game

$p_i = prob(b = b_i)$

$\tilde{p}_i = prob(\tilde{b} = b_i)$

$q_i = prob(\theta = \theta_i)$

$\tilde{q}^\nu_i = prob(\tilde{\theta}_\nu = \theta_i)$

$$\tilde{P} = \begin{pmatrix} \tilde{p}^{11} & \tilde{p}^{12} \\ \tilde{p}^{21} & \tilde{p}^{22} \end{pmatrix}.$$

$\bar{b}$ = average of the b_i

$j_{nm} = prob(\tilde{b}_n \cap b_m)$

$s_{nm} = prob(\tilde{\theta}_n \cap \theta_m)$

DPE - dynamic programming equation

i.i.d- independent and identically distributed

Chapter 2

Sole Management

In this chapter we determine the optimal harvest when the stock is under the control of a single manager. We first examine, as a baseline case, a model with fixed growth parameters. Next, we will generalize the model to allow for cyclic patterns in the growth function. Finally we will introduce stochasticity into our model.

Deterministic Case

We start with the purely autonomous version of our deterministic model. A sole manager controls the harvest of a single fish stock. The life-cycle of the fish is illustrated in the diagram below.

$$R \to S = (1-h)R \to R^{+} = F(S).$$

The fleet has a certain amount of fish available for harvest, called the *recruitment*, R. At the appropriate time, the fleet takes a proportion, h, of the available stock for harvesting. We assume for social, economic or physical reasons, the proportion of available stock harvested is limited, and so

$$0 \le h \le \widehat{h} < 1.$$

Associated with each harvesting season is a *yield*, $Y = hR$. After the fleet has harvested a proportion of the available stock, the escapement, $S = (1-h)R$, is some fraction of the initial recruitment R. The escapement then spawns, providing the subsequent season's recruitment R^{+}. We set $R^{+}=F(S)$, where the growth function F is typically chosen to be monotone-increasing with a fixed, unique carrying capacity K such that $F(K) = K$. In this paper we use the growth function chosen by Levhari and Mirman. Explicitly,

$$R^{+} = F(S) = AS^{b},$$

with $1 \ge A > 0$ and $0 < b < 1$. This results in a carrying capacity of

$$K = A^{\frac{1}{1-b}}.$$

We choose to normalize the number of fish by using K as the unit of measurement. In our baseline autonomous case, A and b are taken to be constant over time.

The fleet chooses a certain risk averse utility function U [1]. We choose, as Levhari

[1] A risk averse utility function f, is one where the expected utility of a fifty-fifty gamble between two alternatives has a less desirable outcome than taking the utility of the average, $f(a) + f(b) \le 2f\left(\frac{a+b}{2}\right)$.

and Mirman did,

$$U(R) = \sum_{t=0}^{T} \gamma^t \ln(Y(t))$$

where γ is a constant discount factor, $0 \leq \gamma \leq 1$ and T is the number of harvesting seasons considered.

We should note that usually economists wish to study profit and not utility. We have left out the cost of production and variable market prices from our model. This has been done to ensure the model is analytically tractable. In general, profit will not be directly related to utility. However if the number of fish being caught and sold does not vary much from season to season, the utility should be a resonable measure of profit.

The fleet then wishes to maximize its utility function by optimally choosing a harvest proportion for each year. Our goal is then to find, if it exists, an optimal harvest proportion for each year that the harvesting game is played. This can be done by choosing the harvest proportion h^t for each $t \in 0 \ldots T$. We then take the limit as $T \to \infty$ to find a time-independent solution. When this happens we say the game has an infinite time-horizon. At each stage of a finite-horizon game let τ represent the number of harvesting periods remaining prior to termination. As such the harvest policy is of the form

$$\{h^s\}_{s=0}^{\tau}.$$

At any time t, the fleet's utility function satisfies the dynamic programming equation

(DPE)

$$U(R) = \max_{0 \le h \le \widehat{h}} \{\ln(hR) + \gamma U(R^+)\}. \tag{2.1}$$

with a different harvest proportion for each cycle, possibly dependent on R. The utility function thus satisfies the DPE

$$U^\tau(R^\tau) = \max_{0 \le h^\tau \le \widehat{h}} \left\{\ln(h^\tau R^\tau) + \gamma U^{\tau^+}(R^{\tau^+})\right\}.$$

where

$$R^{\tau^+} = R^{\tau-1}$$

and

$$h^{\tau^+} = h^{\tau-1}.$$

We shall use Bellman's [1] *Principle of Optimality* which states: An optimal policy has the property that, whatever the initial state and decision (i.e., control) are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

We now proceed iteratively, working backwards in time from the horizon. In the terminal period, when $\tau = 0$, the utility for the fleet is

$$U^0(R^0) = \max_{0 \le h^0 \le \widehat{h}} \{\ln(h^0 R^0)\} = \ln(\widehat{h} R^0).$$

Therefore the optimal choice for the h^0, is

$$h^0 = \widehat{h}.$$

When $\tau = 1$ we obtain

$$\begin{aligned} U^1(R^1) &= \max_{0 \le h^1 \le \widehat{h}} \{\ln(h^1 R^1) + \gamma \cdot U^0(R^0)\} \\ &= \max_{0 \le h^1 \le \widehat{h}} \left\{\ln(h^1 R^1) + \gamma \cdot \ln\left(A\widehat{h}(1-h^1)^b R^b\right)\right\} \end{aligned} \tag{2.2}$$

Differentiating (2.2) we obtain

$$\frac{\partial U^1}{\partial \tilde{h}^1} = \frac{1}{\tilde{h}^1} - \frac{\gamma b}{1-\tilde{h}^1} = 0$$

if there is an interior maximum $\tilde{h}^1$. Hence

$$\tilde{h}^1 = \frac{1}{1+\gamma b}.$$

When $\tau = 2$ we obtain

$$\begin{aligned} U^2(R^2) &= \max_{0 \le h^2 \le \widehat{h}} \{\ln(h^2 R^2) + \gamma \cdot U^1(R^1)\} \\ &= \max_{0 \le h^2 \le \widehat{h}} \{\ln(h^2 R^2) + \gamma \ln(h^1 R^1) + \gamma^2 \ln(h^0 R^0)\} \\ &= \max_{0 \le h^2 \le \widehat{h}} \left\{\ln(h^2 R^2) + \gamma \ln(\frac{1}{1+\gamma b}) + \gamma \ln(R^1) + \gamma^2 \ln \widehat{h} + \gamma^2 \ln(R^0)\right\} \end{aligned} \tag{2.3}$$

where

$$R^1 = A(1-h^2)^b (R^2)^b$$

$$R^0 = A(1-h^1)^b (R^1)^b.$$

Differentiating (2.3) we obtain

$$\frac{\partial U^2}{\partial h^2} = \frac{1}{\tilde{h}^2} - \frac{\gamma b}{1-\tilde{h}^2} - \frac{(\gamma b)^2}{1-\tilde{h}^2} = 0$$

if there is an interior maximum. Hence

$$\tilde{h}^2 = \frac{1}{1 + \gamma b + (\gamma b)^2}.$$

Continuing in this fashion, we obtain

$$\tilde{h}^T = \frac{1}{1 + \gamma b + (\gamma b)^2 + \cdots + (\gamma b)^T} = \frac{1 - \gamma b}{1 - (\gamma b)^{T+1}}.$$

Letting $T \to \infty$ we find the steady-state harvest proportion to be

$$\tilde{h} = 1 - \gamma b \tag{2.4}$$

since $\gamma b < 1$.

We now compute the optimal harvest proportion in a more efficient manner. Define

$$\mu^\tau(R^\tau) = \frac{\partial U^\tau(R^\tau)}{\partial R^\tau}.$$

Then μ is the marginal unit asset value. We shall also find it useful to define

$$\mathcal{M}^\tau(R^\tau) = R^\tau \cdot \mu^\tau(R^\tau).$$

Using induction, we will show that h^τ and $\mathcal{M}^\tau$ are independent of R^τ. Suppose, for a given $\tau \geq 1$, $\mathcal{M}^{\tau^+}$ is independent of R^τ. We drop the reference to τ and explicitly refer to the year only when necessary. The following formulas will be used below,

$$\begin{aligned} R^+ &= A(1-h)^b R^b, \\ \frac{\partial R^+}{\partial h} &= -b \cdot A(1-h)^{b-1} R^b \\ &= -\frac{b \cdot R^+}{1-h}. \end{aligned}$$

Differentiating the right hand side of the DPE (2.1) with respect to h yields

$$\begin{aligned}\frac{\partial U}{\partial h} &= \frac{1}{h} + \gamma\mu^{+} \cdot \frac{\partial R^{+}}{\partial h} \\ &= \frac{1}{h} - \frac{\gamma \cdot b}{1-h} \cdot \mathcal{M}^{+}.\end{aligned} \tag{2.5}$$

Let

$$\tilde{h} = \frac{1}{1 + \gamma \cdot b \cdot \mathcal{M}^{+}} \tag{2.6}$$

denote the zero of expression (2.5). By the induction hypothesis, expression (2.6) is independent of R. The optimal harvest is then either $\tilde{h}$ or is the maximum allowable harvest $\widehat{h}$. That is

$$h = \min[\widehat{h}, \tilde{h}].$$

Thus $h(R)$ is a constant, independent of R. We now know that

$$\begin{aligned}\frac{\partial R^{+}}{\partial R} &= b \cdot A(1-h)^{b} R^{b-1} \\ &= \frac{b \cdot R^{+}}{R},\end{aligned}$$

and also note that

$$\begin{aligned}\frac{\partial U}{\partial R} &= \frac{1}{R} + \gamma \cdot \mu^{+} \frac{\partial R^{+}}{\partial R} \\ &= \frac{1}{R} + \frac{\gamma \cdot b \cdot \mu^{+} R^{+}}{R}.\end{aligned}$$

Therefore

$$\mathcal{M}(R) = 1 + \gamma \cdot b \cdot \mathcal{M}^{+} \tag{2.7}$$

also is independent of R. Substituting (2.7) in (2.6) gives

$$\tilde{h} = \frac{1}{\mathcal{M}}.$$

We now proceed iteratively. In the terminal period, when $\tau = 0$, the optimal choice for the h^0, is

$$h^0 = \widehat{h}.$$

and

$$\mathcal{M}^0 = R^0 \cdot \mu(R^0) = 1.$$

Iteration of the recursion equation (2.7) gives

$$\begin{aligned}
\mathcal{M}^1 &= 1 + \gamma b \cdot \mathcal{M}^0 = 1 + \gamma b, \\
\mathcal{M}^2 &= 1 + \gamma b \cdot \mathcal{M}^1 = 1 + \gamma b + (\gamma b)^2 \\
&\cdots, \\
\mathcal{M}^\tau &= 1 + \gamma b + (\gamma b)^2 + \cdots + (\gamma b)^\tau = \frac{1 - (\gamma b)^{\tau+1}}{1 - \gamma b}.
\end{aligned}$$

Letting $\tau \to \infty$, we find the limiting, time independent relations for the infinite-horizon to be

$$\begin{aligned}
\mathcal{M}^\infty &= \frac{1}{1 - \gamma b}, \\
\bar{h} &= 1 - \gamma b,
\end{aligned} \tag{2.8}$$

the same as in equation (2.4).

The steady-state recruitment, where $R = R^+$, can then be calculated. Assuming that $\bar{h} \leq \widehat{h}$ we find

$$\begin{aligned}
\bar{R} &= A[(1 - \bar{h})\bar{R}]^b \\
&= A[\gamma b \bar{R}]^b
\end{aligned}$$

or

$$\hat{R} = A^{\frac{1}{1-b}}(\gamma b)^{\frac{b}{1-b}}. \tag{2.9}$$

To summarize, we have found the optimal harvest proportion to be $h = 1 - \gamma b$ independent of R and of A.

Periodic Growth

We now generalize to a periodically cyclic growth function. In this model the biological growth function will cycle through a deterministic sequence of N distinct growth states, $n = 0, 1, \ldots, N-1$ with

$$F(S, n) = A_n \cdot S^{b_n}.$$

We proceed as before by first examining the finite-time horizon. At the beginning of period τ, the state of the system is $[R^\tau, n_\tau]$. If at a given period, $\tau > 0$, the growth state is specified to be n_τ, then at the subsequent period, $\tau^+ = \tau - 1$, the corresponding growth state is

$$n^+ = n^{\tau^+} = n^{\tau-1 \mod (N)}.$$

Let $h^\tau(n)$ denote the fleet's harvested fraction of the accessible stock R^τ. The DPE for the utility function is

$$U^\tau[(R^\tau, n)] = \max_{0 \le h \le \hat{h}} \left\{ \ln(h^\tau R^\tau) + \gamma U^{\tau^+}[(R^{\tau^+}, n^+)] \right\}. \tag{2.10}$$

As before we work backwards in time. In the terminal period, when $\tau = 0$, the maximal utility is

$$U^0[(R^0, b)] = \max_{0 \leq h^0 \leq \widehat{h}} \ln(h^0 R^0) = \ln\left\{\widehat{h} R^0\right\}.$$

Hence

$$h^0 = \widehat{h}$$

and

$$\mathcal{M}^0 = R^0 \cdot \mu^0 = 1$$

are both independent of the growth state n, and R^0.

Suppose for a given $\tau \geq 1$ that $\mathcal{M}^{\tau^+}$ is independent of R^τ. Differentiating the right hand side of the DPE (2.10) with respect to h^τ yields

$$\frac{\partial U^\tau}{\partial h^\tau} = \frac{1}{h^\tau} - \frac{\gamma \cdot b}{1 - h^\tau} \cdot \mathcal{M}^{\tau^+}. \tag{2.11}$$

The optimal choice for h^τ is

$$h^\tau = \min[\widehat{h}^\tau, \bar{h}^\tau],$$

where

$$\bar{h}^\tau = \frac{1}{1 + \gamma \cdot b_n \mathcal{M}^{\tau^+}[(R^{\tau^+}, n^+)]} \tag{2.12}$$

is a zero of expression (2.11). Therefore h^τ is independent of R^τ. We then write

$$h^\tau[(R^\tau, n)] = h^\tau(n).$$

Differentiating the DPE (2.10)with respect to R^τ, we find

$$\mathcal{M}^\tau[(R^\tau, n)] = 1 + \gamma \cdot b_n \cdot \mathcal{M}^{\tau^+} \tag{2.13}$$

which also is independent of R^τ. We then write

$$\mathcal{M}^\tau[(R^\tau, n)] = \mathcal{M}^\tau(n).$$

Substituting equation (2.13) into equation (2.12) we obtain

$$\bar{h}^\tau(n) = \frac{1}{\mathcal{M}^\tau(n)}.$$

Iteration of the recursion equation (2.13)gives

$$\mathcal{M}^1(n) = 1 + \gamma b,$$

$$\mathcal{M}^2(n) = 1 + \gamma b \cdot \mathcal{M}^1(n^+) = 1 + \gamma b_n + \gamma^2 b_n b_{n^+},$$

$$\cdots.$$

In the infinite horizon limit

$$\mathcal{M}^\infty(n) = 1 + \gamma b_n \mathcal{M}^\infty(n^+). \tag{2.14}$$

Iterating expression (2.14) N times with the subscripts taken mod N, results in

$$\mathcal{M}^\infty(n) = 1 + \gamma b_n + \gamma^2 b_n b_{n-1} + \cdots + \gamma^N b_n b_{n-1} \cdots b_{n-N-1} \mathcal{M}^\infty(n-N).$$

However,

$$\mathcal{M}^\infty(n-N) = \mathcal{M}^\infty(n),$$

thus,

$$\mathcal{M}^\infty(n) = \frac{1 + \gamma b_n + \gamma^2 b_n b_{n-1} + \cdots + \gamma^{N-1} b_n b_{n-1} \cdots b_{n-N-1}}{1 - \gamma^N [b_n b_{n-1} \cdots b_{n-N}]} = \frac{1}{\bar{h}(n)}.$$

We have again found the optimal harvest proportion to be independent of R and A.

Stochastic Cases

In the next extension of our model, we introduce stochasticity in the growth function

$$F[S(t)] = A(t)S^{b(t)}.$$

The parameters $A(t)$ and $b(t)$ in the growth function are now random. The parameter $b(t)$ is declared to be a Markovian random variable, chosen from the finite set of values b_n for $n = 1, 2, \ldots N$. The sequence of random variables $b(t)$ form a Markov series with single-period transition probability distribution

$$prob(b^+ = b_m \mid b = b_n) = p_{nm}.$$

The growth parameter $A(t)$ is also random. It will be shown that the optimal control in this model is independent of $A(t)$. As such, we choose $A(t)$ to be a fixed deterministic function of $b(t)$.

We shall consider several variants of the basic model, each variant differing in the specific information that each fleet has available when it must make its harvesting decision. In describing this information, we adopt the convention that a stage of the dynamic process begins at the time of escapement. The following diagram illustrates this.

$$S \rightarrow AS^b = R \rightarrow (1-h)R \underset{\text{time step}}{=} S^+ \rightarrow A^+S^{+^{b^+}} = R^+ \rightarrow (1-h^+)R^+$$

The state variable pair (S, b) determines recruitment before harvesting is done, while the state variable pair (S^+, b^+) determine the system after the current harvest. We determine the optimal harvest proportion for the following cases.

Knowledge of b

The first case we examine is when the player knows its recruitment but does not know the value of next season's recruitment. In other words, the player knows the current value of b and all of its previous values.

As in the deterministic version of our model, we begin with the finite-horizon game. The DPE is

$$U[(R,b)] = \max_{0\le h(b)\le \widehat{h}} \left\{ \ln(hR) + \gamma \mathop{E}_{b^+|b} U^+[(R^+,b^+)] \right\}, \tag{2.15}$$

where $\mathop{E}_{b^+|b}(f(b^+))$ is the expectation of $f(b^+)$ given b.

As before we work backwards in time. In the terminal period, when $\tau = 0$, the utility is

$$U^0[(R^0,b)] = \max_{0\le h^0(b)\le \widehat{h}} \ln(h^0R^0) = \ln\left\{\widehat{h}R^0\right\}.$$

Hence

$$h^0 = \widehat{h}$$

and

$$\mathcal{M}^0 = R^0 \cdot \mu^0 = 1$$

are both independent of S^0 and R^0.

We now prove, using induction, that h^τ and $\mathcal{M}^\tau_\nu$ depend only on $b(\tau)$. Assume that $\mathcal{M}^{\tau^+}$ is independent of R^τ. The following formulas will be used below.

$$R^+ = A^+(1-h)^{b^+}R^{b^+}$$

$$\frac{\partial R^+}{\partial h} = \frac{-R^+\cdot b^+}{1-h}$$

Differentiating the utility function (2.9) with respect to h yields

$$\frac{\partial U}{\partial h} = \frac{1}{h} + \gamma \mathop{E}_{b^+|b}\left(\frac{\partial U^+}{\partial h} \cdot \frac{-R^+ \cdot b^+}{1-h}\right)$$
$$= \frac{1}{h} - \frac{\gamma}{1-h} \mathop{E}_{b^+|b}(b^+ \mathcal{M}^+). \tag{2.16}$$

Setting expression (2.16) equal to zero we obtain

$$\bar{h} = \frac{1}{1 + \gamma \mathop{E}_{b^+|b}(b^+ \mathcal{M}^+)}. \tag{2.17}$$

The right hand side of equation (2.17) does not depend on R. We now know that

$$\frac{\partial R^+}{\partial R} = b \cdot A(1-h)^{b^+} R^{b^+-1}$$
$$= \frac{b^+ \cdot R^+}{R}.$$

Differentiating the utility function with respect to R, we find that

$$\mu^\tau[(R,b)] = \frac{1}{R}\left(1 + \gamma \mathop{E}_{b^+|b}\left(b^+ \cdot R^+ \cdot \mu^+[(R^+, b^+)]\right)\right),$$

in other words

$$\mathcal{M}[(R,b)] = 1 + \gamma \mathop{E}_{b^+|b}[b^+ \mathcal{M}^+]. \tag{2.18}$$

By the induction hypothesis, the right hand side of (2.18) depends only on b. We then write

$$\mathcal{M}[(R,b)] = \mathcal{M}(b).$$

Substituting equation (2.17) in equation (2.18) we obtain

$$h(b) = \frac{1}{\mathcal{M}(b)}$$

Iterating (2.18) using $\mathcal{M}^0 = 1$ and $E[b^+ \mid b]$ as the expectation of b^+ given b we obtain

$$\mathcal{M}^1 = 1 + \gamma \underset{b^+|b}{E}(b^+\mathcal{M}^0) = 1 + \gamma E[b^+ \mid b],$$
$$\mathcal{M}^2 = 1 + \gamma \underset{b^+|b}{E}(b^+\mathcal{M}^1(b^+)) = 1 + \gamma E[b^+ \mid b] + \gamma^2 E[b^+b^{++} \mid b],$$
$$\mathcal{M}^3 = 1 + \gamma E[b^+ \mid b] + \gamma^2 E[b^+b^{++} \mid b] + \gamma^3 E[b^+b^{++}b^{+++} \mid b],$$

$$\cdots$$

If we expand these expressions, we obtain

$$\mathcal{M}^1(b_n) = 1 + \gamma \sum_{m=1}^{N} p_{nm} b_m$$
$$\mathcal{M}^2(b_n) = 1 + \gamma \sum_{m=1}^{N} p_{nm} b_m + \gamma^2 \sum_{m=1}^{N} p_{nm} b_m \sum_{m'=1}^{N} p_{mm'} b_{m'}, \text{ and so on.}$$

This series for $\mathcal{M}^\infty$ can be summed. Define the diagonal matrix

$$\mathbf{B} = [b_1, b_2, \cdots b_N]$$

and the $N \times N$ matrices

$$\mathbf{P} = [p_{nm}] \text{ and } \mathbf{Q} = \mathbf{BP}.$$

Also define the $1 \times N$ column vectors

$$\underline{\mathbf{1}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \underline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}, \text{ and } \underline{\mathcal{M}} = \begin{bmatrix} \mathcal{M}(b_1) \\ \mathcal{M}(b_2) \\ \vdots \\ \mathcal{M}(b_N) \end{bmatrix}.$$

Then

$$\underline{\mathcal{M}}^{\infty} = \underline{\mathbf{1}} + \gamma\mathbf{P}\underline{\mathbf{b}} + \gamma^2\mathbf{P}\mathbf{Q}\underline{\mathbf{b}} + \gamma^3\mathbf{P}\mathbf{Q}^2\underline{\mathbf{b}} + \cdots$$
$$= \underline{\mathbf{1}} + \gamma\mathbf{P}[\mathbf{I} - \gamma\mathbf{Q}]^{-1}\underline{\mathbf{b}}.$$

Therefore

$$h^{\tau}(b = b_i) = \frac{1}{(\underline{\mathbf{1}} + \gamma\mathbf{P}[\mathbf{I} - \gamma\mathbf{Q}]^{-1}\underline{\mathbf{b}})_i}. \tag{2.19}$$

We have found that the optimal harvest proportion is independent of R. To check the above result, we set

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $b_1 = b_2 = b$ in (2.19). We arrive at

$$h = \frac{1}{1 + \gamma\frac{1}{1-\gamma b}b}$$
$$= \frac{1}{1 + \frac{\gamma b}{1-\gamma b}}$$
$$= 1 - \gamma b,$$

which is the same as in the deterministic case (2.4).

We have found the optimal harvest proportion to be independent of R and A. We have also shown that the deterministic case is a special case of the case when b is known.

In the following sections we will calculate optimal harvest fractions for other knowledge structures. We will then be able to compare the knowledge structures by using the computed control rules in realizations of the model. The results will be analyzed and we will explore the qualitative possibilities for the outcome of the game, with the focus on the game's specific knowledge structure.

Knowledge of b^+

We now change the knowledge structure by assuming knowledge of b^+. The DPE is

$$U[(R,b,b^+)] = \max_{0\leq h\leq \hat{h}} \left\{ \ln(hR) + \gamma \mathop{E}_{b^{++}|b^+} U^+[(R^+,b^+,b^{++})] \right\}. \tag{2.20}$$

In the terminal period, the utility is

$$U_\nu^0[(R^0,b+)] = \max_{0\leq h^0\leq \hat{h}} \ln(h^0R^0) = \ln\left\{\hat{h}R^0\right\}.$$

Hence

$$h^0 = \hat{h}$$

and

$$\mathcal{M}^0 = R^0\cdot\mu^0 = 1$$

are both independent of R^0.

For the induction, we assume that $\mathcal{M}^+$ depends only on b^{++}. Differentiating the utility function (2.20) with respect to h yields

$$\frac{\partial U}{\partial h} = \frac{1}{h} - \gamma \mathop{E}_{b^{++}|b^+} \frac{\mu^+R^+b^+}{1-h}. \tag{2.21}$$

Setting expression (2.21) equal to zero we obtain

$$\bar{h} = \frac{1}{1+\gamma \mathop{E}_{b^{++}|b^+}(b^+\mathcal{M}^+)}.$$

Since the right hand side of the above equation depends only on b^+, we can conclude that $\bar{h}$ depends only on b^+. Differentiating the utility function (2.20) with respect to R, we find that

$$\mu[(R, b^+)] = \frac{1}{R}\left(1 + \gamma b^+ \underset{b^{++}|b^+}{E} \left(\mu^+[(R^+, b^{++})]\right)\right),$$

equivalently

$$\mathcal{M}[(R, b^+)] = 1 + \gamma b^+ \underset{b^{++}|b^+}{E} (\mathcal{M}^+[(R^+, b^{++})]). \tag{2.22}$$

By the induction hypothesis, the right hand side of (2.22) depends only on b^+. We can then write

$$\mathcal{M}[(R, b^+)] = \mathcal{M}(b^+).$$

Substituting equation (2.21) into equation (2.22) we obtain

$$h(b^+) = \frac{1}{\mathcal{M}(b^+)}.$$

Iterating expression (2.22), using $\mathcal{M}^0 = 1$

$\mathcal{M}^1(b^+) = 1 + \gamma b^+,$

$\mathcal{M}^2(b^+) = 1 + \gamma b^+ + \gamma^2 b^+ E[b^{++} \mid b^+],$

$\mathcal{M}^3(b^+) = 1 + \gamma b^+ + \gamma^2 b^+ E[b^{++} \mid b^+] + \gamma^3 b^+ E[b^{++} b^{+++} \mid b^+]$, and so on. The limiting value can be expressed as

$$\mathcal{M}^\infty(b^+) = 1 + \gamma b^+ \left(1 + \gamma E[b^{++} \mid b^+] + \gamma^2 E[b{+}{+}b{+}+{+} \mid b^+] + \cdots\right)$$

$$= \underline{\mathbf{1}} + \gamma \underline{\mathbf{b}}^+ \left(\underline{\mathbf{1}} + \gamma \mathbf{P}[\mathbf{I} - \gamma \mathbf{Q}]^{-1} \underline{\mathbf{b}}^+\right).$$

The optimal harvest fractions are thus given by

$$h(b^+ = b_i) = \frac{1}{\left(\underline{1} + \gamma \underline{b}^+ \left(\underline{1} + \gamma \mathbf{P}[\mathbf{I} - \gamma \mathbf{Q}]^{-1} \underline{b}^+\right)\right)_i}. \tag{2.23}$$

Again, to check, we set

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $b_1 = b_2 = b$ in (2.23). The result is

$$\begin{aligned} h &= \frac{1}{1 + \gamma b \left(1 + \frac{\gamma b}{1 - \gamma b}\right)} \\ &= \frac{1}{1 + \frac{\gamma b}{1 - \gamma b}} \\ &= 1 - \gamma b, \end{aligned}$$

which is the same as in the deterministic case eqrefE:1d.

We have found the optimal harvest proportion to be independent of R and A. We have also shown that the deterministic case is a special case of the case when b^+ is known.

Knowledge of b^-

In the next version of our model, we assume only delayed knowledge of the stock recruitment growth parameters. To ease calculations, we now use S as the state variable. The DPE is

$$U[(S, b^-)] = \max_{0 \leq h \leq \bar{h}} \left\{ \mathop{E}_{b|b^-} \left(\ln(hS^b) + \gamma U^+[(S^+, b)] \right) \right\}. \tag{2.24}$$

When $\tau = 0$ it is clear again that $h^0 = \widehat{h}$. Hence the maximum utility is

$$U^0[(S^0, b-)] = \underset{b|b^-}{E}\left(\ln(\widehat{h}S^{0^b})\right).$$

We define

$$\lambda[(S, b^-)] = \frac{d}{dS}U[(S, b^-)]$$

and

$$\Lambda[(S, b^-)] = S\lambda[(S, b^-)].$$

Then

$$\lambda^0 = \underset{b|b-}{E}\frac{b}{S^0}$$

or

$$\Lambda^0 = E[b \mid b^-].$$

Again we note that Λ^0 and h^0 depend only on b^-. For the induction proof, we assume that Λ^+ depends only on b. The following formulas will prove useful.

$$\begin{aligned} S^+ &= A(1-h)S^b \\ \frac{\partial S^+}{\partial h} &= -\frac{S^+}{1-h} \end{aligned} \tag{2.25}$$

Differentiating the utility function (2.24) with respect to h gives

$$\begin{aligned} \frac{\partial U}{\partial h} &= \underset{b|b^-}{E}\left(\frac{1}{h} - \frac{\gamma\mu^+S^+}{1-h}\right) \\ &= \frac{1}{h} - \frac{\gamma}{1-h}\underset{b|b^-}{E}\Lambda^+. \end{aligned} \tag{2.26}$$

Setting equation (2.26) equal to zero, we obtain

$$h = \frac{1}{1 + \gamma \underset{b|b^-}{E} \Lambda^+}, \tag{2.27}$$

independent of S. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{bS^+}{S}.$$

Differentiating the DPE (2.24) with respect to S, leads to

$$\lambda[(S, b^-)] = \underset{b|b^-}{E} \left(\frac{b}{S} + \frac{b\gamma\lambda^+ S^+}{S} \right).$$

or

$$\Lambda[(S, b^-)] = \underset{b|b^-}{E} \left(b + \gamma b \Lambda^+[(S^+, b)] \right). \tag{2.29}$$

Iterating (2.29), it is evident that Λ is independent of S. We obtain

$$\Lambda^1(b^-) = \underset{b|b^-}{E} [b(1 + \gamma E[b^+ \mid b])] = E[b + \gamma b b^+ \mid b^-],$$

$$\Lambda^2(b^-) = E[b + \gamma b b^+ + \gamma^2 b b^+ b^{++} \mid b^-].$$

In the limit as $\tau \to \infty$

$$\Lambda^\infty(b^-) = E[b + \gamma b b^+ + \gamma^2 b b^+ b^{++} + \cdots \mid b^-],$$

likewise

$$\Lambda^\infty(b) = E[b^+ + \gamma b^+ b^{++} + \gamma^2 b^+ b^{++} b^{+++} + \cdots \mid b].$$

Therefore

$$E[\Lambda^\infty(b) \mid b^-] = E[b^+ + \gamma b^+ b^{++} + \cdots \mid b^-] = \mathbf{P}^2[\mathbf{I} - \gamma \mathbf{Q}]^{-1} \underline{\mathbf{b}}. \tag{2.30}$$

Substituting expression (2.27) in expression (2.30), we obtain

$$h(b^- = b_i) = \frac{1}{\left(\underline{1} + \gamma \mathbf{P}^2[\mathbf{I} - \gamma \mathbf{Q}]^{-1}\underline{\mathbf{b}}^-\right)_i}. \tag{2.31}$$

As a check, we set $\mathbf{P} = \mathbf{I}$ and $b_1 = b_2 = b$ in (2.31). The result is

$$\begin{aligned} h &= \frac{1}{1 + \gamma_\nu \frac{1}{1-\gamma_\nu b} b} \\ &= 1 - \gamma_\nu b, \end{aligned}$$

which is the same as in the deterministic case eqrefE:1d.

We have found the optimal harvest proportion to be independent of R and A. We have also shown that the deterministic case is a special case of the case when b^- is known.

Numerical Simulation Results

In this section we compare the different knowledge structures using numerical simulations of the models. We obtain explicit quantitative results regarding the harvest proportions and the expected payoffs. We use the following notations for the different knowledge structures.

$b-$	knowledge of b^-
b	knowledge of b
$b+$	knowledge of b^+

In the figures we display W, the historically preferred utility function. Specifically, we set

$$W = \exp[(1-\gamma)U] \tag{2.32}$$
$$= \exp\left\{E \ln\left[\prod_{t=0}^{\infty} Y^{\kappa_t}(t)\right]\right\}$$

where

$$\kappa_t(\gamma) = (1-\gamma)\gamma^t.$$

That is, W, is a weighted geometric mean of the harvests.

The following is a realization of 10000 simulations each fifty years long. The average of W is given for each knowledge structure. The independent variable is $b1$, one of the two possible growth rates.

In **Figure 2.1** we see that the economic return is higher, in the sole manger model, when the fleet has more knowledge concerning the stochastic parameters. We also see the return is lowered when the biological growth is poorer.

In **Figure 2.2** we see the management of the stock can be more aggressive when the manager has additional information.

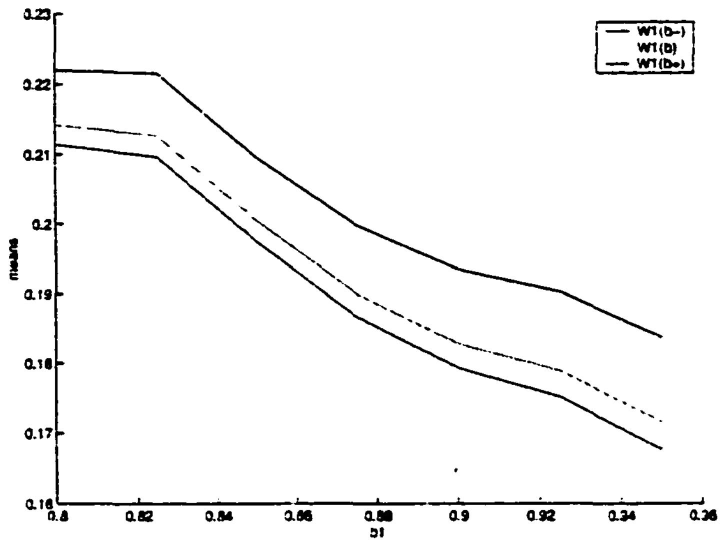

Figure 2.1: Comparison of W

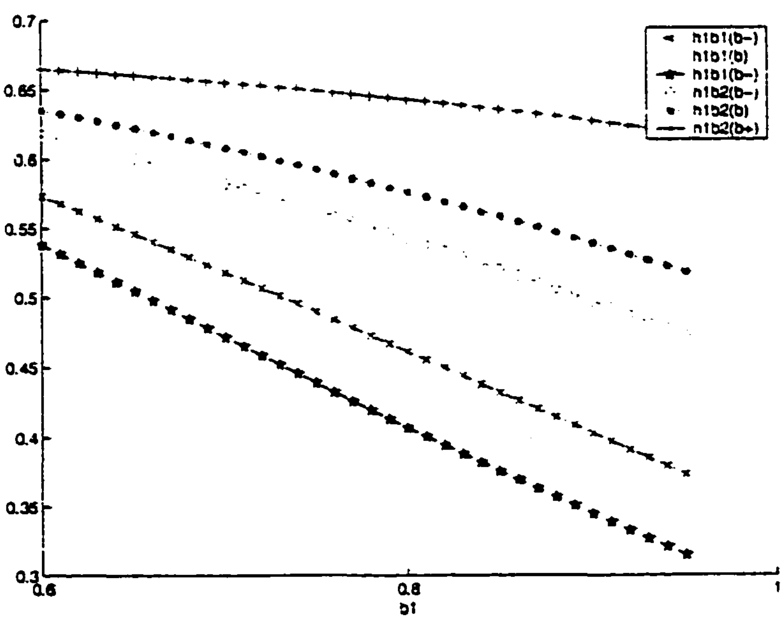

Figure 2.2: Comparison of harvest proportions

Chapter 3

Competitive Fishery

We now wish to examine the impact of information on the competitive fishery. Here the interaction between the competing fleets leads to more interesting outcomes of the game. As before, we first start with a deterministic model before proceeding to the stochastic cases.

Deterministic Case

We start with the purely autonomous version of our deterministic model. Two fishing fleets, the α-fleet and the β-fleet, compete over the harvest of a single fish stock. The life-cycle of the fish is illustrated in the diagram below.

$$R = R_\alpha \to S_\alpha = (1 - h_\alpha)R_\alpha = R_\beta \to S_\beta = (1 - h_\beta)R_\beta = S \to R_\alpha^+ = F(S).$$

At the beginning of the harvesting season, the total fish-stock biomass is available only to the α-fleet. The α-fleet harvests the proportion h_α of this stock. The remaining unharvested stock, S_α becomes available for the β-fleet to harvest. The β-fleet, in turn, takes the proportion h_β of its accessible stock R_β, leaving the local escapement S_β.

After each fleet has harvested their proportion of available stock, the total escapement, equivalent to the β-fleet's local escapement, is some fraction of the initial recruitment R. Thus,

$$S = \sigma R \text{ where } \sigma = (1 - h_\alpha)(1 - h_\beta).$$

The total escapement then spawns, providing the subsequent season's recruitment R^+. Our competitive model leads to the following DPE

$$U_\nu[R_\nu \mid h_{\bar{\nu}}] = \sum_{t=0}^{T} \gamma_\nu^t \ln[h_\nu R_\nu] \text{ for } \nu = \alpha \text{ or } \beta.$$

Note that this scenario is not neutral with respect to the two fleets since the fish stock available for harvesting is always larger for the α fleet.

Each fleet is assumed to have complete knowledge of the structure of the game, including the growth function and the initial recruitment. Both fleets know their competitor's objective function as well as their own. The individual fleet then wishes to maximize its objective function by optimally choosing a harvest proportion for each year. Each fleet chooses a policy that is the optimal response by that fleet to the policy it expects will be chosen by its opponent.

A *Nash-equilibrium policy pair* is a pair of policies, one chosen by each fleet, such that each is the optimal response by that fleet to the policy it expects to be chosen by its opponent. In other words, each fleet can not improve their yield once the other fleet commits to its policy. If such a pair of policies exist, the model is said to have a *solution*.

Our goal is then to find, if it exists, a Nash-equilibrium point for each year that the harvesting game is played. This can be done by choosing the harvest value h_ν^t for each $t \in 0 \ldots T$. We then take the limit as $T \to \infty$ to find a time-independent solution. When this happens we say the game has an infinite time-horizon.

We use standard reaction analysis to calculate the Nash-equilibrium policy pair. Let us denote the competitor to the ν-fleet by $\bar{\nu}$, that is $\bar{\nu} = \beta$ or α when $\nu = \alpha$ or β respectively. At any time t, and conditional on $h_{\bar{\nu}}$, the $\bar{\nu}$-fleet's value function for the infinite time-horizon satisfies the dynamic programming equation

$$U_\nu[R_\nu \mid h_{\bar{\nu}}] = \max_{0 \le h_\nu \le \widehat{h_\nu}} \{\ln(h_\nu R_\nu) + \gamma_\nu U_\nu[R_\nu^+ \mid h_{\bar{\nu}}^+]\}. \tag{3.1}$$

We now proceed iteratively, working backwards in time from the horizon. In the terminal period, when $\tau = 0$, the utility for the ν-fleet is

$$U_\nu^0[R_\nu^0 \mid h_{\bar{\nu}}^0] = \max_{0 \le h_\nu^0 \le \widehat{h}_\nu} \{\ln(h_\nu^0 R_\nu^0)\} = \ln(\widehat{h}_\nu R_\nu^0),$$

independent of any action taken by the competing $\bar{\nu}$-fleet. Therefore the optimal choice for the h_ν^0, is

$$h_\nu^0 = \widehat{h}_\nu,$$

and

$$\mathcal{M}_\nu^0 = R_\nu^0 \cdot \mu_\nu[R_\nu^0 \mid h_{\bar\nu}^0] = 1.$$

Note for $\nu = \alpha$ or β, h_ν^0 and $\mathcal{M}_\nu^0$ are independent of the recruitment R_ν^0 and of the competitor's policy $h_{\bar\nu}^0$. Using induction, we will show that h_ν^τ and $\mathcal{M}_\nu^\tau$ are independent of R_ν^τ and of the competitor's policy $h_{\bar\nu}^\tau$.

Suppose, for a given $\tau \geq 1$, $\mathcal{M}_\nu^{\tau^+}$ is independent of R_ν^τ. The following formulas will be used below.

$$\begin{aligned} R_\alpha^+ &= A(1-h_\alpha)^b(1-h_\beta)^b R_\alpha^b, \\ \frac{\partial R_\alpha^+}{\partial h_\alpha} &= -b \cdot A(1-h_\alpha)^{b-1}(1-h_\beta)^b R_\alpha^b \\ &= -\frac{b \cdot R_\alpha^+}{1-h_\alpha}. \end{aligned}$$

Differentiating the right hand side of the DPE (3.1) with respect to h_α yields

$$\begin{aligned} \frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha} + \gamma_\alpha \mu_\alpha^+ \cdot \frac{\partial R_\alpha^+}{\partial h_\alpha} \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha \cdot b}{1-h_\alpha} \cdot \mathcal{M}_\alpha^+. \end{aligned} \tag{3.2}$$

Let

$$\bar{h}_\alpha = \frac{1}{1 + \gamma_\alpha \cdot b \cdot \mathcal{M}_\alpha^+} \tag{3.3}$$

denote the zero of expression (3.2). By the induction hypothesis, expression (3.3) is independent of R_α and h_β. The optimal harvest is then either $\bar{h}_\alpha$ or is the maximum allowable harvest $\widehat{h}_\alpha$. That is

$$h_\alpha = \min[\widehat{h}_\alpha, \bar{h}_\alpha].$$

Thus $h_\alpha(R_\alpha \mid h_\beta)$ is a constant, independent of R_α and h_β. We now know that

$$\begin{aligned}\frac{\partial R_\alpha^+}{\partial R_\alpha} &= b \cdot A(1-h_\alpha)^b(1-h_\beta)^b R_\alpha^{b-1} \\ &= \frac{b \cdot R_\alpha^+}{R_\alpha},\end{aligned}$$

and also note that

$$\begin{aligned}\frac{\partial U_\alpha}{\partial R_\alpha} &= \frac{1}{R_\alpha} + \gamma_\alpha \cdot \mu_\alpha^+ \frac{\partial R_\alpha^+}{\partial R_\alpha} \\ &= \frac{1}{R_\alpha} + \frac{\gamma_\alpha \cdot b \cdot \mu_\alpha^+ R_\alpha^+}{R_\alpha}.\end{aligned}$$

Therefore

$$\mathcal{M}_\alpha(R_\alpha \mid h_\beta) = 1 + \gamma \cdot b \cdot \mathcal{M}_\alpha^+ \tag{3.4}$$

also is independent of R_α and h_β^τ. Substituting (3.4) in (3.3) gives

$$\bar{h}_\alpha = \frac{1}{\mathcal{M}_\alpha}.$$

Iteration of the recursion equation (3.4) gives

$$\begin{aligned}&\mathcal{M}_\alpha^1 = 1 + \gamma_\alpha b \cdot \mathcal{M}_\alpha^0 = 1 + \gamma_\alpha b, \\ &\mathcal{M}_\alpha^2 = 1 + \gamma_\alpha b \cdot \mathcal{M}_\alpha^1 = 1 + \gamma_\alpha b + (\gamma_\alpha b)^2 \\ &\qquad \cdots, \\ &\mathcal{M}_\alpha^\tau = 1 + \gamma_\alpha b + (\gamma_\alpha b)^2 + \cdots + (\gamma_\alpha b)^\tau = \frac{1 - (\gamma_\alpha b)^{\tau+1}}{1 - \gamma_\alpha b}.\end{aligned}$$

Letting $\tau \to \infty$, we find the limiting, time independent relations for the infinite-horizon to be

$$\mathcal{M}_\alpha^\infty = \frac{1}{1-\gamma_\alpha b},$$

$$\bar{h}_\alpha = 1 - \gamma_\alpha b. \tag{3.5}$$

For the β fleet we obtain the following formulas

$$\begin{aligned} R_\beta^+ &= A(1-h_\beta)^b R_\beta^b (1-h_\alpha^+), \\ \frac{\partial R_\beta^+}{\partial R_\beta} &= b \cdot A(1-h_\beta)^b (1-h_\alpha^+) R_\beta^{b-1} \\ &= \frac{b \cdot R_\beta^+}{R_\beta}. \end{aligned}$$

Using the same induction hypothesis and differentiating the right hand side of the DPE (3.1) with respect to h_β yields

$$\begin{aligned} \frac{\partial U_\beta}{\partial h_\beta} &= \frac{1}{h_\beta} + \gamma_\beta \mu_\beta^+ \cdot \frac{\partial R_\beta^+}{\partial h_\beta} \\ &= \frac{1}{h_\beta} - \frac{\gamma_\beta \cdot b}{1-h_\beta} \cdot \mathcal{M}_\beta^+. \end{aligned} \tag{3.6}$$

Let

$$\bar{h}_\beta = \frac{1}{1+\gamma_\beta \cdot b \cdot \mathcal{M}_\beta^+} \tag{3.7}$$

denote the zero of expression (3.6). By the induction hypothesis, expression (3.7) is independent of R_β and h_α. Thus $h_\beta(R_\beta \mid h_\alpha)$ is a constant, independent of R_β and h_α. We now know that

$$\begin{aligned} \frac{\partial R_\beta^+}{\partial R_\beta} &= b \cdot A(1-h_\alpha^+)(1-h_\beta)^b R_\beta^{b-1} \\ &= \frac{b \cdot R_\beta^+}{R_\beta}, \end{aligned}$$

and also note that

$$\frac{\partial U_\beta}{\partial R_\beta} = \frac{1}{R_\beta} + \gamma_\beta \cdot \mu_\beta^+ \frac{\partial R_\beta^+}{\partial R_\beta}$$
$$= \frac{1}{R_\beta} + \frac{\gamma_\beta \cdot b \cdot \mu_\beta^+ R_\beta^+}{R_\beta}.$$

Therefore

$$\mathcal{M}_\beta(R_\beta \mid h_\alpha) = 1 + \gamma \cdot b \cdot \mathcal{M}_\beta^+ \tag{3.8}$$

also is independent of R_β and h_α. Substituting (3.8) in (3.7) gives

$$\tilde{h}_\beta = \frac{1}{\mathcal{M}_\beta}$$

so

$$\tilde{h}_\beta = 1 - \gamma_\beta b. \tag{3.9}$$

The steady-state recruitment, where $R = R^+$, can then be calculated. Assuming that $\tilde{h}_\nu \leq \widehat{h}_\nu$

$$\bar{R} = A[(1 - \tilde{h}_\alpha)(1 - \tilde{h}_\beta)\bar{R}]^b$$
$$= A[\gamma_\alpha b \gamma_\beta b \bar{R}]^b$$

or

$$\bar{R} = A^{\frac{1}{1-b}}(\gamma_\alpha\gamma_\beta b^2)^{\frac{b}{1-b}}. \tag{3.10}$$

The value of a single year's harvest, $CompVal_\nu$, is then

$$CompVal_\alpha = \ln\left((1 - \gamma_\alpha b)A^{\frac{1}{1-b}}(\gamma_\alpha\gamma_\beta b^2)^{\frac{b}{1-b}}\right)$$
$$CompVal_\beta = \ln\left((1 - \gamma_\beta b)(1 - \gamma_\alpha b)A^{\frac{1}{1-b}}(\gamma_\alpha\gamma_\beta b^2)^{\frac{b}{1-b}}\right).$$

The total value of a single year's harvest, $CompVal$, is therefore

$$CompVal = \ln\left((1-\gamma_\alpha b)^2(1-\gamma_\beta b)A^{\frac{2}{1-b}}(\gamma_\alpha\gamma_\beta b^2)^{\frac{2b}{1-b}}\right).$$

Summarizing our work for the deterministic model, we have found that the optimal steady-state harvest fraction is independent of both the recruitment R_ν and the opponents policy $h_{\bar{\nu}}$. The optimal harvest fraction was found to be

$$\bar{h}_\nu = 1 - \gamma_\nu b.$$

We note that this is the same expression as we found before in the sole manager model (2.4).

Periodic Growth

We now generalize to a periodically cyclic growth function. In this model the biological growth function will cycle through a deterministic sequence of N distinct growth states, $n = 0, 1, \ldots, N-1$ with

$$F(S,n) = A_n \cdot S^{b_n}.$$

We proceed as before by first examining the finite-time horizon. At the beginning of period τ, the state of the system is $[R^\tau, n_\tau]$.

Let $h_\nu^\tau(n)$ denote the ν-fleet's harvested fraction of the accessible stock R_ν^τ. The DPE for the utility function is

$$U_\nu^\tau[(R_\nu^\tau, n) \mid h_{\bar{\nu}}^\tau] = \max_{0 \le h_\nu \le \bar{h}_\nu} \left\{ \ln(h_\nu^\tau R_\nu^\tau) + \gamma_\nu U_\nu^{\tau^+}[(R_\nu^{\tau^+}, n^+) \mid h_{\bar{\nu}}^{\tau^+}] \right\}. \qquad (3.11)$$

As before we work backwards in time. In the terminal period, when $\tau = 0$, the maximal utility for the ν-fleet is

$$U_\nu^0[(R_\nu^0, b) \mid h_{\bar{\nu}}^0] = \max_{0 \le h_\nu^0 \le \widehat{h}_\nu} \ln(h_\nu^0 R_\nu^0) = \ln\left\{\widehat{h}_\nu R_\nu^0\right\}.$$

Hence

$$h_\nu^0 = \widehat{h}_\nu$$

and

$$\mathcal{M}_\nu^0 = R_\nu^0 \cdot \mu_\nu^0 = 1$$

are both independent of the growth state n, $h_{\bar{\nu}}^0$ and R_ν^0.

Suppose for a given $\tau \ge 1$ that $\mathcal{M}_\nu^{\tau^+}$ is independent of R_ν^τ. Differentiating the right hand side of the DPE (3.11) with respect to h_α^τ yields

$$\frac{\partial U_\alpha^\tau}{\partial h_\alpha^\tau} = \frac{1}{h_\alpha^\tau} - \frac{\gamma_\alpha \cdot b}{1 - h_\alpha^\tau} \cdot \mathcal{M}_\alpha^{\tau^+}. \tag{3.12}$$

The optimal choice for h_α^τ is

$$h_\alpha^\tau = \min[\widehat{h}_\alpha^\tau, \tilde{h}_\alpha^\tau],$$

where

$$\tilde{h}_\alpha^\tau = \frac{1}{1 + \gamma_\alpha \cdot b_n \mathcal{M}_\alpha^{\tau^+}[(R_\alpha^{\tau^+}, h_{\bar{\alpha}}^{\tau^+}, n^+) \mid h_{\bar{\alpha}}^\tau]} \tag{3.13}$$

is a zero of expression (3.12). Therefore h_α^τ is independent of R_α^τ and $h_{\bar{\alpha}}^\tau$. We then write

$$h_\alpha^\tau[(R_\alpha^\tau, n) \mid h_{\bar{\alpha}}] = h_\alpha^\tau(n).$$

Differentiating the DPE (3.11) with respect to R_α^τ, we find

$$\mathcal{M}_\alpha^\tau[(R_\alpha^\tau, n) \mid h_{\bar{\alpha}}^\tau] = 1 + \gamma \cdot b_n \cdot \mathcal{M}_\alpha^{\tau^+} \tag{3.14}$$

which also is independent of R_α^τ and $h_{\bar{\alpha}}^\tau$. We then write

$$\mathcal{M}_\alpha^\tau[(R_\alpha^\tau, n) \mid h_{\bar{\alpha}}^\tau] = \mathcal{M}_\alpha^\tau(n).$$

Substituting equation (3.14) into equation (3.13) we obtain

$$\bar{h}_\alpha^\tau(n) = \frac{1}{\mathcal{M}_\alpha^\tau(n)}.$$

Iteration of the recursion equation (3.14)gives

$$\mathcal{M}_\alpha^1(n) = 1 + \gamma_\alpha b,$$

$$\mathcal{M}_\alpha^2(n) = 1 + \gamma_\alpha b \cdot \mathcal{M}_\alpha^1(n+) = 1 + \gamma_\alpha b_n + \gamma_\alpha^2 b_n b_{n+},$$

$$\cdots .$$

In the infinite horizon limit

$$\mathcal{M}_\alpha^\infty(n) = 1 + \gamma_\alpha b_n \mathcal{M}_\alpha^\infty(n+). \tag{3.15}$$

Iteration of the expression (3.15) N times with the subscripts taken mod N, results in

$$\mathcal{M}_\alpha^\infty(n) = 1 + \gamma_\alpha b_n + \gamma_\alpha^2 b_n b_{n-1} + \cdots + \gamma_\alpha^N b_n b_{n-1} \cdots b_{n-N-1} \mathcal{M}_\alpha^\infty(n-N).$$

However,

$$\mathcal{M}_\alpha^\infty(n-N) = \mathcal{M}_\alpha^\infty(n),$$

thus,

$$\mathcal{M}_\alpha^\infty(n) = \frac{1 + \gamma_\alpha b_n + \gamma_\alpha^2 b_n b_{n-1} + \cdots + \gamma_\alpha^{N-1} b_n b_{n-1} \cdots b_{n-N-1}}{1 - \gamma^N [b_n b_{n-1} \cdots b_{n-N}]} = \frac{1}{h_\alpha(n)}.$$

The calculations for the β fleet are similar and lead to the same result. We have shown the optimal harvest proportion to be independent of R and A.

Stochastic Cases

In the next extension of our model, we introduce stochasticity in the growth function

$$F[S(t)] = A(t)S^{b(t)}.$$

We adopt the convention that a stage of the dynamic process begins at the time of total seasonal escapement and ends at the specification of player $\beta's$ escapement. Note that even though $S_\beta = S^+$, we consider S_β to be in the season prior to the total escapement. The following diagram illustrates this.

$$S \to AS^b = R_\alpha \to S_\alpha = R_\beta \to S_\beta \underset{\text{time step}}{=} S^+ \to A^+S^{+b^+} = R_\alpha^+ \to S_\alpha^+ = R_\beta^+ \to S_\beta^+$$

The state variable pair (S, b) determines recruitment before harvesting is done, while the state variable pair (S^+, b^+) determine the system after the current harvest.

Knowledge of b

The first case we examine is when both players know their recruitment but do not know the value of next season's recruitment. In other words, both players know the current value of b and all of its previous values. In addition both players, after having calculated their optimal harvest fractions, will be able to deduce the value S_β.

As in the deterministic version of our model, we begin with the finite-horizon game. The DPE for the α-fleet is

$$U_\alpha[(R_\alpha, b) \mid h_\beta] = \max_{0 \le h_\alpha \le \hat{h}_\alpha} \left\{ \ln(h_\alpha R_\alpha) + \gamma_\alpha \underset{b^+|b}{E} U_\alpha^+[(R_\alpha^+, b^+) \mid h_\beta^+] \right\}, \tag{3.16}$$

where $\underset{b^+|b}{E}(f(b^+) \mid b)$ is the expectation of $f(b^+)$ given b.

As before we work backwards in time. In the terminal period, when $\tau = 0$, the utility for the ν-fleet is

$$U_\nu^0[(R_\nu^0, b) \mid h_{\bar{\nu}}^0] = \max_{0 \le h_\nu^0 \le \hat{h}_\nu} \ln(h_\nu^0 R_\nu^0) = \ln\left\{\hat{h}_\nu R_\nu^0\right\}.$$

Hence

$$h_\nu^0 = \hat{h}_\nu$$

and

$$\mathcal{M}_\nu^0 = R_\nu^0 \cdot \mu_\nu^0 = 1$$

are both independent of $h_{\bar{\nu}}^0$, S_ν^0 and R_ν^0.

We now prove, using induction, that h_ν^τ and $\mathcal{M}_\nu^\tau$ depend only on $b(\tau)$. Assume that $\mathcal{M}_\nu^+$ is independent of R_ν. The following formulas will be useful.

$$R_\alpha^+ = A^+(1-h_\beta)^{b^+} R_\beta^{b^+} = A^+(1-h_\beta)^{b^+}(1-h_\alpha)^{b^+} R_\alpha^{b^+}$$
$$\frac{\partial R_\alpha^+}{\partial h_\alpha} = \frac{-R_\alpha^+ \cdot b^+}{1-h_\alpha}$$

Differentiating the utility function (3.16) with respect to h_α yields

$$\begin{aligned}\frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha} + \gamma_\alpha \underset{b^+|b}{E}\left(\frac{\partial U_\alpha^+}{\partial h_\alpha} \cdot \frac{-R_\alpha^+ \cdot b^+}{1-h_\alpha}\right) \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha}{1-h_\alpha} \underset{b^+|b}{E}(b^+ \mathcal{M}_\alpha^+).\end{aligned} \tag{3.17}$$

Setting expression (3.17) equal to zero we obtain

$$\bar{h}_\alpha = \frac{1}{1 + \gamma_\alpha \underset{b^+|b}{E}(b^+ \mathcal{M}_\alpha^+)}. \tag{3.18}$$

The right hand side of equation (3.18) does not depend on R_α or h_β. We now know that

$$\begin{aligned}\frac{\partial R_\alpha^+}{\partial R_\alpha} &= b \cdot A(1-h_\alpha)^{b^+}(1-h_\beta)^{b^+} R_\alpha^{b^+-1} \\ &= \frac{b^+ \cdot R_\alpha^+}{R_\alpha}.\end{aligned}$$

Differentiating the utility function with respect to R_α, we find that

$$\mu_\alpha[(R_\alpha, b) \mid h_\beta] = \frac{1}{R_\alpha}\left(1 + \gamma_\alpha \underset{b^+|b}{E}\left(b^+ \cdot R_\alpha^+ \cdot \mu_\alpha^+[(R_\alpha^+, b^+) \mid h_\beta^+]\right)\right),$$

in other words

$$\mathcal{M}_\alpha[(R_\alpha, b) \mid h_\beta] = 1 + \gamma_\alpha \underset{b^+|b}{E}[b^+ \mathcal{M}_\alpha^+]. \tag{3.19}$$

By the induction hypothesis, the right hand side of (3.19) depends only on b. We then write

$$\mathcal{M}_\alpha[(R_\alpha, b) \mid h_\beta] = \mathcal{M}_\alpha.$$

Substituting equation (3.18) in equation (3.19) we obtain

$$h_\alpha = \frac{1}{\mathcal{M}_\alpha}$$

Iterating (3.19) using $\mathcal{M}_\alpha^0 = 1$ and $E[b^+ \mid b]$ as the expectation of b^+ given b we obtain

$$\begin{aligned}
\mathcal{M}_\alpha^1 &= 1 + \gamma_\alpha \underset{b^+|b}{E}(b^+ \mathcal{M}_\alpha^0) = 1 + \gamma_\alpha E[b^+ \mid b], \\
\mathcal{M}_\alpha^2 &= 1 + \gamma_\alpha \underset{b^+|b}{E}(b^+ \mathcal{M}_\alpha^1(b^+)) = 1 + \gamma_\alpha E[b^+ \mid b] + \gamma_\alpha^2 E[b^+ b^{++} \mid b], \\
\mathcal{M}_\alpha^3 &= 1 + \gamma_\alpha E[b^+ \mid b] + \gamma_\alpha^2 E[b^+ b^{++} \mid b] + \gamma_\alpha^3 E[b^+ b^{++} b^{+++} \mid b],
\end{aligned}$$

Then

$$\underline{\mathcal{M}_\alpha^\infty} = \underline{1} + \gamma_\alpha \mathbf{P}\underline{\mathbf{b}} + \gamma_\alpha^2 \mathbf{PQ}\underline{\mathbf{b}} + \gamma_\alpha^3 \mathbf{PQ}^2\underline{\mathbf{b}} + \cdots$$
$$= \underline{1} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}.$$

Therefore

$$h_\alpha^\tau(b = b_i) = \frac{1}{(\underline{1} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}})_i}. \tag{3.20}$$

For the β-fleet, we can calculate its harvest fraction similarly. The induction proof that h_β^τ and $\mathcal{M}_\beta^\tau$ depend only on $b(\tau)$ is identical to the previous proof.

$$U_\beta^\tau[(R_\beta^\tau, b) \mid h_\alpha^\tau] = \max_{0 \le h_\beta^\tau \le \hat{h}_\beta} \left\{ \ln(h_\beta^\tau R_\beta^\tau) + \gamma_\beta \mathop{E}_{b^+|b} U_\beta^{\tau^+}[(R_\beta^{\tau^+}, b^+) \mid h_\alpha^{\tau^+}] \right\}.$$

The following formulas apply.

$$R_\beta^+ = A(1 - h_\alpha)R_\alpha^+ = A(1 - h_\alpha)(1 - h_\beta)^{b^+} R_\beta^{b^+},$$
$$\frac{\partial R_\beta^+}{\partial h_\beta} = \frac{-R_\beta^+ \cdot b^+}{1 - h_\beta}.$$

Differentiating the utility function with respect to h_β, yields

$$\frac{\partial U_\beta^+}{\partial h_\beta} = \frac{1}{h_\beta} + \gamma_\beta \mathop{E}_{b^+|b}\left[\frac{\partial U_\beta^+}{\partial h_\beta} \cdot \frac{-R_\beta^+ \cdot b^+}{1 - h_\beta}\right]$$
$$= \frac{1}{h_\beta} - \frac{\gamma_\beta}{1 - h_\beta} \mathop{E}_{b^+|b}[b^+ \mathcal{M}_\beta^+] \tag{3.21}$$

Setting expression (3.21) equal to zero we obtain

$$\bar{h}_\beta = \frac{1}{1 + \gamma_\beta \mathop{E}_{b^+|b}(b^+ \mathcal{M}_\beta^+)}. \tag{3.22}$$

The right hand side of equation (3.22) does not depend on R_β or h_α, therefore h_β depends only on b. We now know that

$$\frac{\partial R_\beta^+}{\partial R_\beta} = \frac{R_\beta^+ \cdot b^+}{R_\beta}.$$

Differentiating the utility function with respect to R_β, we find that

$$\mu_\beta(R_\beta, b \mid h_\alpha) = \frac{1}{R_\beta}\left(1 + \gamma_\beta \underset{b^+|b}{E}\left(b^+ \cdot R_\beta^+ \cdot \mu_\beta^+[(R_\beta^+, b^+) \mid h_\alpha^+]\right)\right),$$

in other words

$$\mathcal{M}_\beta[(R_\beta, b) \mid h_\alpha] = 1 + \gamma_\beta \underset{b^+|b}{E}\left(b^+ \mathcal{M}_\beta^+[(R_\beta^+, b^+) \mid h_\alpha^+]\right). \tag{3.23}$$

The induction hypothesis shows that the right hand side of the expression (3.23) depends only on b. Substituting equation (3.23) into equation (3.22), we obtain

$$h_\beta(b) = \frac{1}{\mathcal{M}_\beta(b)}$$

The iteration process leads to the result

$$\underline{\mathcal{M}}_\beta^\infty = \mathbf{1} + \gamma_\beta \mathbf{P}[\mathbf{I} - \gamma_\beta \mathbf{Q}]^{-1}\underline{\mathbf{b}}.$$

We have found that the optimal harvest proportion is independent of R_ν and of $h_{\bar{\nu}}$. The optimal harvest fraction is

$$h_\nu(b = b_i) = \frac{1}{(\underline{\mathbf{1}} + \gamma_\nu \mathbf{P}[\mathbf{I} - \gamma_\nu \mathbf{Q}]^{-1}\underline{\mathbf{b}})_i}. \tag{3.24}$$

We note that this expression is the same as in the sole manager case (2.19).

Knowledge of b^+

We now change the knowledge structure by assuming both fleets have knowledge of b^+. The DPE for the α -fleet is

$$U_\alpha[(R_\alpha, b, b^+) \mid h_\beta] = \max_{0 \le h_\alpha \le \widehat{h}_\alpha} \left\{ \ln(h_\alpha R_\alpha) + \gamma_\alpha \mathop{E}_{b^{++}|b^+} U_\alpha^+[(R_\alpha^+, b^+, b^{++}) \mid h_\beta^+] \right\}. \quad (3.25)$$

In the terminal period, the utility for the ν-fleet is

$$U_\nu^0[(R_\nu^0, b+) \mid h_{\bar{\nu}}^0] = \max_{0 \le h_\alpha^0 \le \widehat{h}_\alpha} \ln(h_\nu^0 R_\nu^0) = \ln\left\{ \widehat{h}_\nu R_\nu^0 \right\}.$$

Hence

$$h_\nu^0 = \widehat{h}_\nu$$

and

$$\mathcal{M}_\nu^0 = R_\nu^0 \cdot \mu_\nu^0 = 1$$

are both independent of $h_{\bar{\nu}}^0$ and R_ν^0.

For the induction, we assume that $\mathcal{M}_\nu^+$ depends only on b^{++}. Differentiating the utility function (3.25) with respect to h_α yields

$$\frac{\partial U_\alpha}{\partial h_\alpha} = \frac{1}{h_\alpha} - \gamma_\alpha \mathop{E}_{b^{++}|b^+} \frac{\mu_\alpha^+ R_\alpha^+ b^+}{1 - h_\alpha}. \quad (3.26)$$

Setting expression (3.26) equal to zero we obtain

$$\bar{h}_\alpha = \frac{1}{1 + \gamma_\alpha \mathop{E}_{b^{++}|b^+} (b^+ \mathcal{M}_\alpha^+)}.$$

Since the right hand side of the above equation depends only on b^+, we can conclude that $\bar{h}_\alpha$ depends only on b^+. Differentiating the utility function (3.25) with respect to R_α, we find that

$$\mu_\alpha[(R_\alpha, b^+) \mid h_\beta] = \frac{1}{R_\alpha}\left(1 + \gamma_\alpha b^+ \underset{b^{++}|b^+}{E} \left(\mu_\alpha^+[(R_\alpha^+, b^{++}) \mid h_\beta^+]\right)\right),$$

equivalently

$$\mathcal{M}_\alpha[(R_\alpha, b^+) \mid h_\beta] = 1 + \gamma_\alpha b^+ \underset{b^{++}|b^+}{E} (\mathcal{M}_\alpha^+[(R_\alpha^+, b^{++}) \mid h_\beta^+]). \tag{3.27}$$

By the induction hypothesis, the right hand side of (3.27) depends only on b^+. We can then write

$$\mathcal{M}_\alpha[(R_\alpha, b^+) \mid h_\beta] = \mathcal{M}_\alpha(b^+).$$

Substituting equation (3.26) into equation (3.27) we obtain

$$h_\alpha(b^+) = \frac{1}{\mathcal{M}_\alpha(b^+)}.$$

Iterating expression (3.27), using $\mathcal{M}_\alpha^0 = 1$

$\mathcal{M}_\alpha^1(b^+) = 1 + \gamma_\alpha b^+,$

$\mathcal{M}_\alpha^2(b^+) = 1 + \gamma_\alpha b^+ + \gamma_\alpha^2 b^+ E[b^{++} \mid b^+],$

$\mathcal{M}_\alpha^3(b^+) = 1 + \gamma_\alpha b^+ + \gamma_\alpha^2 b^+ E[b^{++} \mid b^+] + \gamma_\alpha^3 b^+ E[b^{++} b^{+++} \mid b^+]$, and so on. The limiting value can be expressed as

$$\begin{aligned}\mathcal{M}_\alpha^\infty(b^+) &= 1 + \gamma_\alpha b^+ \left(1 + \gamma_\alpha E[b^{++} \mid b^+] + \gamma_\alpha^2 E[b{+}{+}b{+}++ \mid b^+] + \cdots\right) \\ &= \underline{\mathbf{1}} + \gamma_\alpha \underline{\mathbf{b}}^+ \left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1} \underline{\mathbf{b}}^+\right).\end{aligned}$$

The optimal harvest fraction is given by

$$h_\alpha(b^+ = b_i) = \frac{1}{\left(\underline{\mathbf{1}} + \gamma_\alpha \underline{\mathbf{b}}^+ \left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1} \underline{\mathbf{b}}^+\right)\right)_i}. \tag{3.28}$$

The calculations for the β-fleet lead to the same result for $\bar{h}_{\beta}^{\tau}$. We note that the optimal harvest proportion is independent of R and A and that the above expression (3.28) is the same as in the sole manager case (2.23).

Knowledge of b^-

In the next version of our model, we assume that both fleets have only delayed knowledge of the stock recruitment growth parameters. To ease calculations, we now use S as the state variable. The DPE for the α-fleet is

$$U_{\alpha}[(S,b^-) \mid h_{\beta}] = \max_{0 \le h_{\alpha} \le \widehat{h}_{\alpha}} \left\{ \underset{b|b^-}{E} \left(\ln(h_{\alpha}S^b) + \gamma_{\alpha}U_{\alpha}^{+}[(S^+,b) \mid h_{\beta}^{+}]\right) \right\}. \tag{3.29}$$

When $\tau = 0$ it is clear again that $h_{\alpha}^{0} = \widehat{h}_{\alpha}$. Hence the utility for the α-fleet is

$$U_{\alpha}^{0}[(S^0, b-) \mid h_{\beta}^{\tau}] = \underset{b|b^-}{E} \left(\ln(\widehat{h}_{\alpha}(S^0)^b)\right).$$

We define

$$\lambda_{\alpha}^{\tau}[(S{\cdot}b^-) \mid h_{\beta}^{\tau}] = \frac{\partial}{\partial S}U_{\alpha}[(S,b^-) \mid h_{\beta}]$$

and

$$\Lambda_{\alpha}[(S,b^-) \mid h_{\beta}] = S\lambda[(S,b-) \mid h_{\beta}].$$

Then

$$\lambda_{\alpha}^{0} = \underset{b|b-}{E} \frac{b}{S^0}$$

or

$$\Lambda_{\alpha}^{0} = E[b \mid b^-].$$

Again we note that Λ_α^0 and h_α^0 depend only on b^-. For the induction proof, we assume that Λ_α^+ depends only on b. The following formulas will prove useful.

$$S^+ = A(1-h_\alpha)(1-h_\beta)S^b$$
$$\frac{\partial S^+}{\partial h_\alpha} = -\frac{S^+}{1-h_\alpha} \tag{3.30}$$

Differentiating the utility function (3.29) with respect to h_α gives

$$\frac{\partial U_\alpha}{\partial h_\alpha} = \underset{b|b^-}{E}\left(\frac{1}{h_\alpha} - \frac{\gamma_\alpha \mu_\alpha^+ S^+}{1-h_\alpha}\right)$$
$$= \frac{1}{h_\alpha} - \frac{\gamma_\alpha}{1-h_\alpha}\underset{b|b^-}{E}\Lambda_\alpha^+. \tag{3.31}$$

Setting equation (3.31) equal to zero, we obtain

$$h_\alpha = \frac{1}{1+\gamma_\alpha \underset{b|b^-}{E}\Lambda_\alpha^+}, \tag{3.32}$$

independent of S and h_β. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{bS^+}{S}.$$

Differentiating the DPE (3.29) with respect to S, leads to

$$\lambda_\alpha[(S,b^-) \mid h_\beta] = \underset{b|b^-}{E}\left(\frac{b}{S} + \frac{b\gamma_\alpha \lambda_\alpha^+ S^+}{S}\right).$$

or

$$\Lambda_\alpha[(S,b^-) \mid h_\beta] = \underset{b|b^-}{E}\left(b + \gamma_\alpha b \Lambda_\alpha^+[(S^+,b) \mid h_\beta^+]\right). \tag{3.34}$$

Iterating (3.34), it is evident that Λ_α is independent of S. We obtain

$$\Lambda_\alpha^1(b^-) = \underset{b|b^-}{E}[b(1+\gamma_\alpha E[b^+ \mid b])] = E[b + \gamma_\alpha bb^+ \mid b^-],$$

$$\Lambda_\alpha^2(b^-) = E[b + \gamma_\alpha bb^+ + \gamma_\alpha^2 bb^+b^{++} \mid b^-].$$

In the limit as $\tau \to \infty$

$$\Lambda_\alpha^\infty(b^-) = E[b + \gamma_\alpha bb^+ + \gamma_\alpha^2 bb^+b^{++} + \cdots \mid b^-],$$

likewise

$$\Lambda_\alpha^\infty(b) = E[b^+ + \gamma_\alpha b^+b^{++} + \gamma_\alpha^2 b^+b^{++}b^{+++} + \cdots \mid b].$$

Therefore

$$E[\Lambda_\alpha^\infty(b) \mid b^-] = E[b^+ + \gamma_\alpha b^+b^{++} + \cdots \mid b^-] = \mathbf{P}^2[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}. \tag{3.35}$$

Substituting expression (3.32) in expression (3.35), we obtain

$$h_\alpha(b^- = b_i) = \frac{1}{\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}^2[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}^-\right)_i}.$$

The DPE for the β-fleet is

$$U_\beta[(S, b^-) \mid h_\alpha] =$$

$$\max_{0 \le h_\beta \le \widehat{h}_\beta} \left\{ \mathop{E}_{b|b^-} \left(\ln\left((1-h_\alpha)(1-h_\beta)S^b\right) + \gamma_\beta U_\beta^+[(S^+, b) \mid h_\alpha^+]\right) \right\} \tag{3.36}$$

When $\tau = 0$ it is clear again that $h_\beta^0 = \widehat{h}_\beta$. Hence the utility for the β-fleet is

$$U_\beta^0[(S^0, b-) \mid h_\alpha^\tau] = \mathop{E}_{b|b^-} \left(\ln(\widehat{h}_\alpha \widehat{h}_\beta (S^0)^b)\right).$$

Then

$$\lambda_\beta^0 = \mathop{E}_{b|b-} \frac{b}{S^0}$$

or

$$\Lambda_\beta^0 = E[b \mid b^-].$$

Again we note that Λ_β^0 and h_β^0 depend only on b^-. For the induction proof, we assume that Λ_β^+ depends only on b. The following formulas will be used.

$$\begin{aligned} S^+ &= A(1-h_\alpha)(1-h_\beta)S^b \\ \frac{\partial S^+}{\partial h_\beta} &= -\frac{S^+}{1-h_\beta} \end{aligned} \tag{3.37}$$

Differentiating the utility function (3.36) with respect to h_β gives

$$\begin{aligned} \frac{\partial U_\beta}{\partial h_\beta} &= \underset{b|b^-}{E}\left(\frac{1}{h_\beta} - \frac{\gamma_\beta \mu_\beta^+ S^+}{1-h_\beta}\right) \\ &= \frac{1}{h_\beta} - \frac{\gamma_\beta}{1-h_\beta}\underset{b|b^-}{E}\Lambda_\beta^+. \end{aligned} \tag{3.38}$$

Setting equation (3.38) equal to zero, we obtain

$$h_\beta^T = \frac{1}{1+\gamma_\beta \underset{b|b^-}{E}\Lambda_\beta^+}, \tag{3.39}$$

independent of S and h_α. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{bS^+}{S}.$$

Differentiating the DPE (3.36) with respect to S, leads to

$$\lambda_\beta[(S,b^-) \mid h_\alpha] = \underset{b|b^-}{E}\left(\frac{b}{S} + \frac{b\gamma_\beta \lambda_\beta^+ S^+}{S}\right).$$

or

$$\Lambda_\beta[(S,b^-) \mid h_\alpha] = \underset{b|b^-}{E}\left(b + \gamma_\beta b \Lambda_\beta^+[(S^+,b) \mid h_\alpha^+]\right). \tag{3.41}$$

We note that the expressions are the same for the β-fleet as the α-fleet. We can conclude the expression for h_β is similar to the expression for h_α. The optimal harvest fractions are given by

$$h_\nu(b^- = b_i) = \frac{1}{\left(\underline{1} + \gamma_\nu \mathbf{P}^2[\mathbf{I} - \gamma_\nu \mathbf{Q}]^{-1}\underline{\mathbf{b}}^-\right)_i}. \tag{3.42}$$

We note that the optimal harvest proportion (3.42) is independent of R and A and is the same as in the sole manager case (2.31).

Asymmetric Knowledge

We now consider an asymmetric version of the model. Here, the two fleets will have different knowledge of the stochastic growth parameters. In our previous work, we have demonstrated that the ν-fleet's harvest policy was completely independent of the $\bar{\nu}$-fleet's policy. All of the calculations are identical. We can conclude that the harvest proportions for the ν-fleet in the asymmetric game is equal to the harvest proportion for the ν-fleet in the symmetric game.

Numerical Simulation Results

In this section we compare the different knowledge structures by running simulations. We use the following notations for the different knowledge structures.

$b-$	symmetric knowledge of b^-
b	symmetric knowledge of b
$b+$	symmetric knowledge of b^+
$b-vb$	asymmetric knowledge, α knows b^-, β knows b
$b-vb+$	asymmetric knowledge, α knows b^-, β knows b^+
$bvb-$	asymmetric knowledge, α knows b, β knows b^-
$bvb+$	asymmetric knowledge, α knows b, β knows b^+
$b+vb-$	asymmetric knowledge, α knows b^+, β knows b^-
$b+vb$	asymmetric knowledge, α knows b^+, β knows b

The following is a realization of 10000 simulations each fifty years long. The average of W, the historically preferred utility function, (see (2.32)) is given for each knowledge structure. The independent variable is $b1$.

Figure 3.1 Here we look at the symmetric knowledge cases. Player one, P_α receives a higher utility with additional knowledge. However, P_β performs worse with additional knowledge.

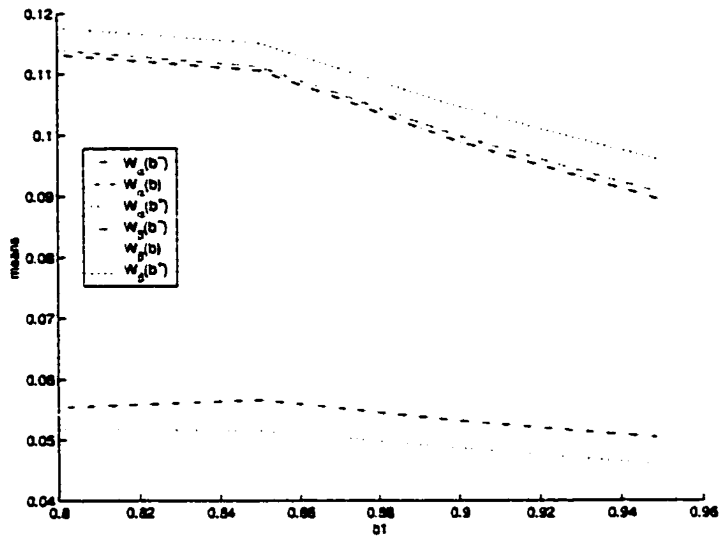

Figure 3.1: Comparison of W_α with W_β

Figures 3.2, 3.3, 3.4 Here we examine asymmetric knowledge. P_α has the same knowledge within each figure, whereas P_β's knowledge is allowed to vary. In these figures, P_β performs better with additional knowledge. However, P_α's return decreases when P_β gains this additional knowledge.

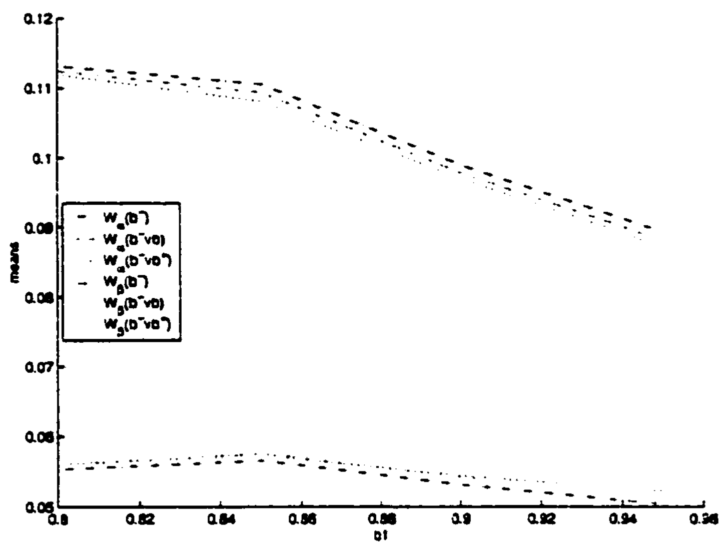

Figure 3.2: Comparison of W_α with W_β when β-fleet knowledge is increasing

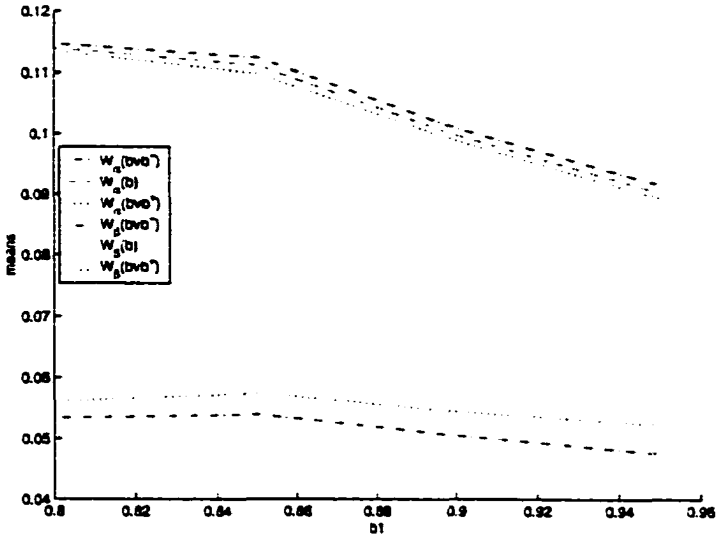

Figure 3.3: Comparison of W_α with W_β when β-fleet knowledge is increasing

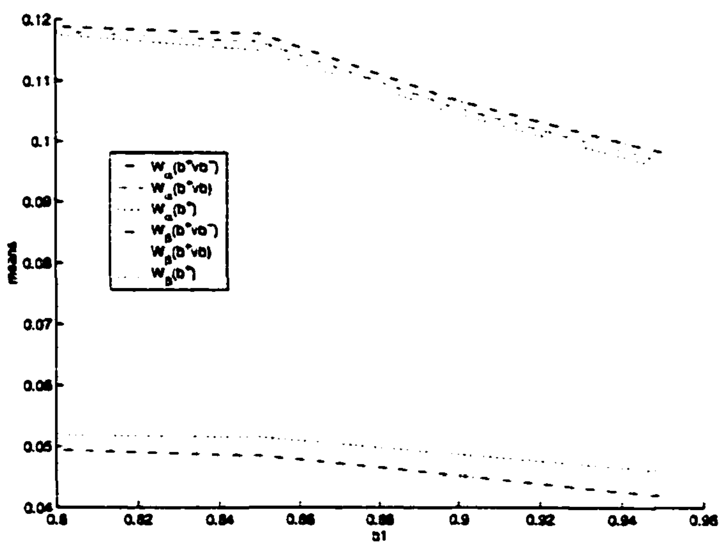

Figure 3.4: Comparison of W_α with W_β when β-fleet knowledge is increasing

Figures 3.5, 3.6, 3.7 Here the situation is reversed. P_β has the same knowledge within each figure, whereas P_α's knowledge is allowed to vary. In these figures, P_α performs better with additional knowledge. However, P_β's return decreases when P_α gains this additional knowledge.

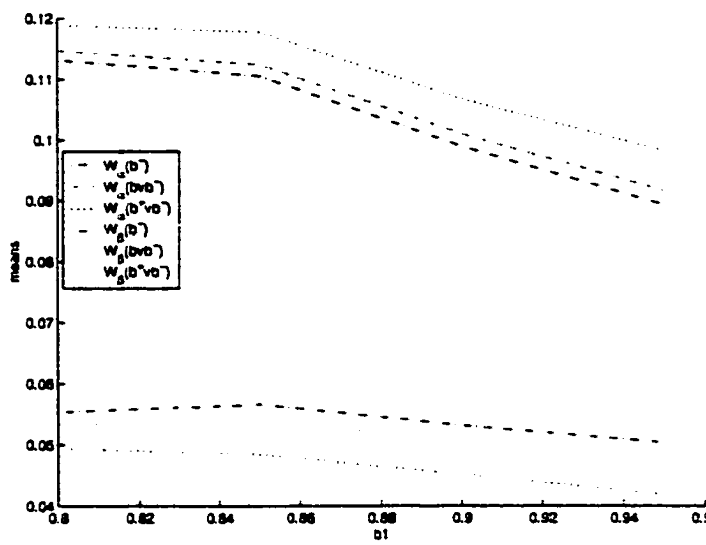

Figure 3.5: Comparison of W_α with W_β when α-fleet knowledge is increasing

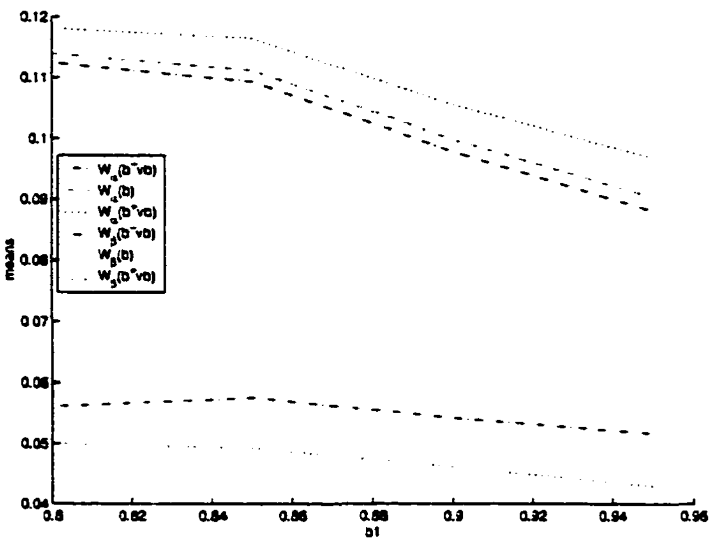

Figure 3.6: Comparison of W_α with W_β when α-fleet knowledge is increasing

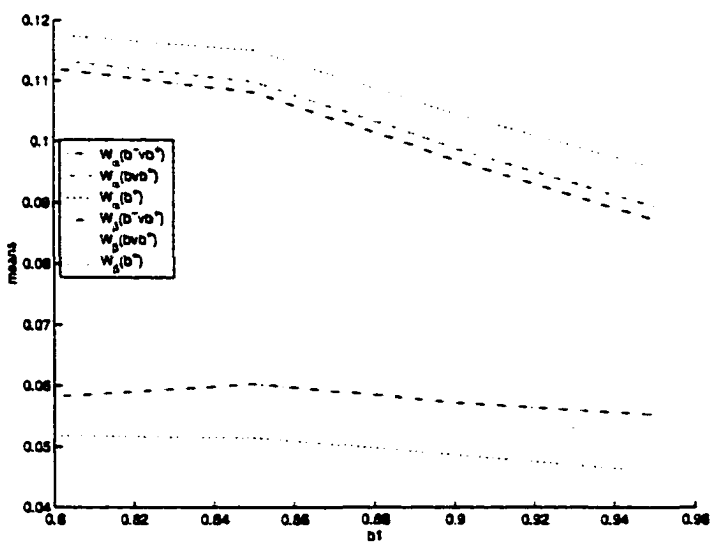

Figure 3.7: Comparison of W_α with W_β when α-fleet knowledge is increasing

Figure 3.8 This figure shows the various harvest rates for the different knowledge structures and different values of b. When the b value is b_1 (indicating poor growth) the harvest fractions are lower than when b is b_2. However the ordering based on knowledge is inverted when b is changed from b_1 to b_2. In other words, since the utility function is risk-averse, with poor knowledge the players tend to be conservative. However, if the player has a high degree of knowledge the two harvest rates will be further apart than if the player possesses a lesser degree of knowledge, indicating the player takes a more aggressive approach to harvesting. This in turn benefits the first player since he is in the position to harvest first.

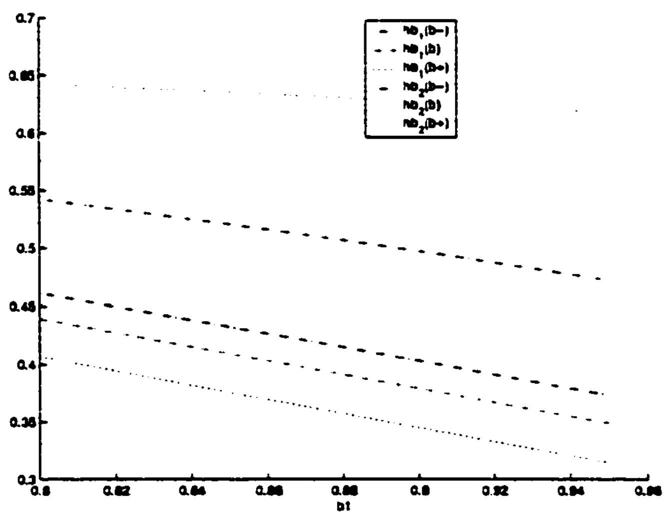

Figure 3.8: Comparison of harvest proportions for the two possible states of b

Chapter 4

Fairness

Until now, the α-fleet has had an inherent advantage in the fishery. We may wonder if there is a way to address this issue by limiting the α-fleet's harvest. Suppose that the α fleet has only a fraction, δ, of R available for harvesting. We show what effects this added variant will has on the outcome of the game and determine what value of δ equalizes the game.

Deterministic Case

We start with the purely autonomous version of our deterministic model. Two fishing fleets, the α-fleet and the β-fleet, compete over the harvest of a single fish stock. The life-cycle of the fish is illustrated in the diagram below.

$$\begin{array}{llll}
S \to AS^b & \to & R_\alpha = \delta AS^b \to S_\alpha & \\
 & \searrow & & \searrow \\
 & & \longrightarrow & R_\beta = (1-\delta)AS^b + S_\alpha \to S_\beta = S^+
\end{array}$$

The α-fleet's utility function thus satisfies the DPE

$$U_\alpha[S \mid h_\beta] = \max_{0 \le h_\alpha \le \widehat{h}_\alpha} \left\{ \ln(\delta h_\alpha AS^b) + \gamma_\alpha U_\alpha^+[S^+ \mid h_\beta^+] \right\}. \tag{4.1}$$

We now proceed iteratively, working backwards in time from the horizon. In the terminal period, when $\tau = 0$, the utility for the α-fleet is

$$U_\alpha^0[S^0 \mid h_\beta^0] = \max_{0 \le h_\alpha^0 \le \widehat{h}_\alpha} \left\{ \ln(\delta h_\alpha^0 AS^{0^b}) \right\} = \ln(\delta \widehat{h}_\alpha AS^{0^b}).$$

Therefore the optimal choice for the h_α^0, is

$$h_\alpha^0 = \widehat{h}_\alpha,$$

and

$$\Lambda_\alpha^0 = b,$$

independent of any action taken by the β-fleet. Using induction, we will again show that h_α^τ and Λ_α^τ are independent of S^τ and of the competitor's policy h_β^τ.

Suppose, for a given $\tau \ge 1$, Λ_α^+ is independent of S. The following formulas will be used below

$$S_\alpha = (1 - h_\alpha)\delta A S^b$$

$$R_\beta = (1-\delta)AS^b + S_\alpha$$

$$= (1 - \delta h_\alpha)AS^b$$

$$S_\beta = (1-\delta h_\alpha)(1-h_\beta)AS^b = S^+$$

$$\frac{\partial S^+}{\partial h_\alpha} = \frac{-\delta S^+}{(1-\delta h_\alpha)}.$$

Differentiating the right hand side of the DPE (4.1) with respect to h_α yields

$$\begin{aligned}\frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha} + \gamma_\alpha \lambda_\alpha^+ \frac{\partial S^+}{\partial h_\alpha} \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha \delta \Lambda_\alpha^+}{1-\delta h_\alpha}.\end{aligned} \tag{4.2}$$

Let

$$\bar{h}_\alpha = \frac{1}{\delta(1+\gamma_\alpha \Lambda_\alpha^+)} \tag{4.3}$$

denote the zero of expression (4.2). By the induction hypothesis, expression (4.3) is independent of S and h_β. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{bS^+}{S}$$

and also note that

$$\frac{\partial U_\alpha}{\partial S} = \frac{b}{S} + \gamma_\alpha b \frac{\lambda_\alpha^+ S^+}{S}.$$

Therefore

$$\Lambda_\alpha^\tau(S \mid h_\beta) = b(1+\gamma_\alpha \Lambda_\alpha^+) \tag{4.4}$$

also is independent of S and h_β. Substituting (4.4) in (4.3) gives

$$\tilde{h}_\alpha = \frac{b}{\delta \Lambda_\alpha}$$

Iteration of the recursion equation (4.4) gives

$$\begin{aligned}
\Lambda_\alpha^1 &= b(1+\gamma_\alpha b \Lambda_\alpha^0) = b(1+\gamma_\alpha b), \\
\Lambda_\alpha^2 &= b(1+\gamma_\alpha b \Lambda_\alpha^1) = b(1+\gamma_\alpha b(1+\gamma_\alpha b)) \\
&= b(1+\gamma_\alpha b + \gamma_\alpha^2 b^2)
\end{aligned}$$

Letting $\tau \to \infty$, we find the limiting, time independent relations for the infinite-horizon to be

$$\begin{aligned}
\Lambda_\alpha^\infty &= \frac{b}{1-\gamma_\alpha b}, \\
\tilde{h}_\alpha &= \frac{1-\gamma_\alpha b}{\delta}.
\end{aligned} \tag{4.5}$$

As a check, if $\delta = 1$ in the above equation then

$$h_\alpha = 1 - \gamma_\alpha b,$$

the same result as previously obtained in the sole manager case (2.4).

The β-fleet's utility function satisfies the DPE

$$U_\beta[S^\tau \mid h_\alpha] = \max_{0 \le h_\beta \le \widehat{h}_\beta} \left\{ \ln(h_\beta A S^b) + \gamma_\beta U_\beta^+[S^+ \mid h_\alpha^+] \right\}. \tag{4.6}$$

The optimal choice for the h_β^0, is

$$h_\beta^0 = \widehat{h}_\beta,$$

and

$$\Lambda_\beta^0 = b,$$

independent of any action taken by the α-fleet. Using induction, we will again show that h_β and Λ_β are independent of S and of the competitor's policy h_α.

Suppose, for a given $\tau \geq 1$, Λ_β^+ is independent of S. Using

$$\frac{\partial S^+}{\partial h_\beta} = \frac{-S^+}{(1-h_\beta)},$$

and differentiating the right hand side of the DPE (4.6) with respect to h_β yields

$$\begin{aligned} \frac{\partial U_\beta}{\partial h_\beta} &= \frac{1}{h_\beta} + \gamma_\beta \lambda_\beta^+ \frac{\partial S^+}{\partial h_\beta} \\ &= \frac{1}{h_\beta} - \frac{\gamma_\beta \Lambda_\beta^+}{1-h_\beta}. \end{aligned} \tag{4.7}$$

Let

$$\bar{h}_\beta = \frac{1}{1+\gamma_\beta \cdot \Lambda_\beta^+} \tag{4.8}$$

denote the zero of expression (4.7). By the induction hypothesis, expression (4.8) is independent of S and h_α. We now know that

$$\frac{\partial S^+}{\partial S} = -\frac{bS^+}{S}$$

and also note that

$$\frac{\partial U_\beta}{\partial S} = \frac{b}{S} + \gamma_\beta b \frac{\lambda_\beta^+ S^+}{S}.$$

Therefore

$$\Lambda_\beta(S \mid h_\alpha) = b(1+\gamma_\beta \Lambda_\beta^+) \tag{4.9}$$

also is independent of S and h_α. Iterating the recursion equation (4.9) we find

$$\Lambda_\beta^\infty = \frac{b}{1-\gamma_\beta b},$$

$$\bar{h}_\beta = 1 - \gamma_\beta b.$$

The steady-state escapement, where $R = R^+$, can then be calculated. Assuming that $\bar{h}_\nu \leq \widehat{h}_\nu$

$$\bar{R} = A[(1-\delta\bar{h}_\alpha)(1-\bar{h}_\beta)\bar{R}]^b.$$

Therefore

$$\begin{aligned}\bar{R} &= A^{\frac{1}{1-b}}(1-\delta h_\alpha)^{\frac{b}{1-b}}(1-h_\beta)^{\frac{b}{1-b}} \\ &= A^{\frac{1}{1-b}}\left(\gamma_\alpha\gamma_\beta b^2\right)^{\frac{b}{1-b}}\end{aligned}$$

The harvest levels are

$$\begin{aligned}\bar{Y}_\alpha &= h_\alpha\delta\bar{R} \\ &= (1-\gamma_\alpha b)\bar{R} \\ \bar{Y}_\beta &= h_\beta(1-\delta h_\alpha)R \\ &= (1-\gamma_\beta b)\gamma_\alpha b\bar{R}.\end{aligned}$$

If we try to solve $\bar{Y}_\alpha = \bar{Y}_\beta$ we find it impossible to equalize this game by adjusting δ. However if $\bar{h}_\nu \geq \widehat{h}_\nu$

$$\bar{R} = A[(1-\delta\widehat{h}_\alpha)(1-\bar{h}_\beta)\bar{R}]^b.$$

Therefore

$$\bar{R} = A^{\frac{1}{1-b}}(1 - \delta\widehat{h}_\alpha)^{\frac{b}{1-b}}(1 - h_\beta)^{\frac{b}{1-b}}$$
$$= A^{\frac{1}{1-b}}\left(\gamma_\beta b(1 - \delta\widehat{h}_\alpha)\right)^{\frac{b}{1-b}}$$

The harvest levels are then

$$\bar{Y}_\alpha = \delta\widehat{h}_\alpha\bar{R}$$
$$\bar{Y}_\beta = h_\beta(1 - \delta h_\alpha)R$$
$$= (1 - \gamma_\beta b)(1 - \delta\widehat{h}_\alpha)\bar{R}.$$

In this case

$$\delta = \frac{1 - \gamma_\beta b}{(2 - \gamma_\beta b)\widehat{h}_\alpha}$$

equalizes the game.

We have shown that the optimal harvest proportion is independent of R and A. The harvest proportion is the same as calculated in the sole manager case (2.4). Therefore the δ constraint can not equalize the game unless the first player's optimal harvest proportion is is constrained by $\hat{h}$.

Stochastic Cases

Knowledge of b

The first case we examine is when both players know their recruitment but do not know the value of next season's recruitment. In other words, both players know the current value of b and all of its previous values. In addition both players, after having calculated their optimal harvest fractions, will be able to deduce the value S_β.

As in the deterministic version of our model, we begin with the finite-horizon game. The DPE for the α -fleet is

$$U_\alpha[(S,b) \mid h_\beta] = \max_{0 \le h_\alpha(b) \le \widehat{h}_\alpha} \left\{ \ln(\delta h_\alpha A S^{\tau^b}) + \gamma_\alpha \mathop{E}_{b^+|b} U_\alpha^+[(S^+, b^+) \mid h_\beta^+] \right\}. \tag{4.10}$$

As before we work backwards in time. Again

$$h_\nu^0 = \widehat{h}_\nu$$

and

$$\Lambda_\nu^0 = b$$

are both independent of h_β^0 and S^0.

We now prove, using induction, that h_ν^τ and Λ_ν^τ depend only on b. Differentiating the utility function (4.10) with respect to h_α yields

$$\begin{aligned} \frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha} + \gamma_\alpha \mathop{E}_{b^+|b} \left(\lambda_\alpha^+ \frac{\partial S^+}{\partial h_\alpha} \right) \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha \delta}{1 - \delta h_\alpha} \mathop{E}_{b^+|b} (\Lambda_\alpha^+). \end{aligned} \tag{4.11}$$

Setting expression (4.11) equal to zero we obtain

$$\bar{h}_\alpha = \frac{1}{\delta \left(1 + \gamma_\alpha \mathop{E}_{b^+|b} (\Lambda_\alpha^+) \right)}. \tag{4.12}$$

The right hand side of equation (4.12) does not depend on S or h_β. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{b S^+}{S}.$$

Differentiating the utility function with respect to S, we find that

$$\Lambda_\alpha[(S,b) \mid h_\beta] = b(1 + \gamma_\alpha b \mathop{E}_{b^+|b}[\Lambda_\alpha^+]). \tag{4.13}$$

By the induction hypothesis, the right hand side of (4.13) depends only on b and b^+. Substituting (4.13) into (4.12) we obtain

$$h_\alpha = \frac{b}{\delta \Lambda_\alpha^\tau}$$

is a function of b only. Iterating (4.13) using $\Lambda_\alpha^0 = b$ we obtain

$$\Lambda^1 = b(1 + \gamma_\alpha \mathop{E}_{b^+|b}(\Lambda_\alpha^0)) = b(1 + \gamma_\alpha E[b^+ \mid b]),$$

$$\Lambda_\alpha^2 = b(1 + \gamma_\alpha E[b^+ \mid b] + \gamma_\alpha^2 E[b^+ b^{++} \mid b]).$$

Then

$$\underline{\Lambda}_\alpha^\infty = \underline{\mathbf{b}}\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1} \underline{\mathbf{b}}\right).$$

Therefore

$$h_\alpha(b = b_i) = \frac{1}{(\delta\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1} \underline{\mathbf{b}}\right))_i}. \tag{4.14}$$

Note that if $\delta = 1$ then (4.14) becomes

$$h_\alpha^\tau(b = b_i) = \frac{1}{(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1} \underline{\mathbf{b}})_i},$$

the same result as previously obtained without the δ constraint in (2.19).

The DPE for the β-fleet is

$$U_\beta[(S,b) \mid h_\alpha] = \max_{0 \le h_\beta(b) \le \hat{h}_\beta} \left\{ \ln(h_\beta A S^b) + \gamma_\beta \mathop{E}_{b^+|b} U_\alpha^+[(S^+, b^+) \mid h_\alpha^+] \right\}. \tag{4.15}$$

Differentiating the utility function (4.15) with respect to h_β yields

$$\frac{\partial U_\beta}{\partial h_\beta} = \frac{1}{h_\beta} - \frac{\gamma_\beta}{1 - h_\beta} \mathop{E}_{b^+|b}(\Lambda_\beta^+). \tag{4.16}$$

Setting expression (4.16) equal to zero we obtain

$$\bar{h}_\beta^r = \frac{1}{1 + \gamma_\beta \mathop{E}_{b^+|b}(\Lambda_\beta^+)}. \tag{4.17}$$

We now know that

$$\frac{\partial S^+}{\partial S} = \frac{bS^+}{S}.$$

Differentiating the utility function with respect to S, we find that

$$\Lambda_\beta[(S, b) \mid h_\alpha] = b(1 + \gamma_\beta b \mathop{E}_{b^+|b}[\Lambda_\beta^+]). \tag{4.18}$$

Substituting (4.18) into (4.17) we obtain

$$h_\beta(b) = \frac{b}{\Lambda_\beta(b).}$$

Iterating (4.13) using $\Lambda_\beta^0 = b$ we obtain

$$\underline{\Lambda}_\beta^\infty = \underline{\mathbf{b}} \left(\underline{\mathbf{1}} + \gamma_\beta \mathbf{P}[\mathbf{I} - \gamma_\beta \mathbf{Q}]^{-1} \underline{\mathbf{b}} \right).$$

Therefore

$$h_\beta(b = b_i) = \frac{1}{(\underline{\mathbf{1}} + \gamma_\beta \mathbf{P}[\mathbf{I} - \gamma_\beta \mathbf{Q}]^{-1} \underline{\mathbf{b}})_i}, \tag{4.19}$$

the same result as previously obtained without a δ constraint in (2.19).

Knowledge of $b+$

The DPE for the α-fleet is

$$U_\alpha[(S,b^+) \mid h_\beta] = \max_{0\le h_\alpha(b)\le \widehat{h}_\alpha} \left\{ \ln(\delta h_\alpha A S^b) + \gamma_\alpha \mathop{E}_{b^{++}|b^+} U_\alpha^+[(S^+, b^+, b^{++}) \mid h_\beta^+] \right\}. \quad (4.20)$$

As before we work backwards in time.

$$h_\nu^0 = \widehat{h}_\nu$$

and

$$\Lambda_\nu^0 = b$$

are both independent of $h_{\bar{\nu}}^0$ and S^0. Differentiating the utility function (4.20) with respect to h_α yields

$$\begin{aligned} \frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha^\tau} + \gamma_\alpha \mathop{E}_{b^{++}|b^+} \left(\lambda_\alpha^+ \frac{\partial S^+}{\partial h_\alpha} \right) \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha \delta}{1-\delta h_\alpha} \mathop{E}_{b^{++}|b^+} (\Lambda_\alpha^+). \end{aligned} \quad (4.21)$$

Setting expression (4.21) equal to zero we obtain

$$\tilde{h}_\alpha = \frac{1}{\delta\left(1 + \gamma_\alpha \mathop{E}_{b^{++}|b^+} (\Lambda_\alpha^+(b^+))\right)}. \quad (4.22)$$

The right hand side of equation (4.22) does not depend on S or h_β. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{b S^+}{S}.$$

Differentiating the utility function with respect to S, we find that

$$\Lambda_\alpha[(S, b^+) \mid h_\beta] = b(1 + \gamma_\alpha b \underset{b^{++}|b^+}{E}[\Lambda_\alpha^+]). \tag{4.23}$$

By the induction hypothesis, the right hand side of (4.13) depends only on b and b^+. Substituting (4.23) into (4.22) we obtain

$$h_\alpha(b^+) = \frac{b}{\delta\Lambda_\alpha(b^+).}$$

Iterating (4.23) using $\Lambda_\alpha^0 = b$ we obtain

$$\Lambda^1 = b(1 + \gamma_\alpha \underset{b^+|b}{E}(\Lambda_\alpha^0)) = b(1 + \gamma_\alpha b^+),$$

$$\Lambda_\alpha^2 = b(1 + \gamma_\alpha b^+ + \gamma_\alpha^2 b^+ E[b^{++} \mid b^+]).$$

Then

$$\underline{\Lambda}_\alpha^\infty = \mathbf{\underline{b}}\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\mathbf{\underline{b}}\right).$$

Therefore

$$h_\alpha(b^+ = b_i) = \frac{1}{(\delta\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\mathbf{\underline{b}}\right))_i}. \tag{4.24}$$

Note that if $\delta = 1$ then (4.24) becomes

$$h_\alpha(b^+ = b_i) = \frac{1}{\left(\underline{(\mathbf{1}} + \gamma_\alpha \mathbf{P}[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\mathbf{\underline{b}}\right)_i},$$

the same result as previously obtained without a δ constraint in (2.23). We have shown the optimal harvest proportion to be independent of R and A.

Knowledge of b^-

The DPE for the α-fleet is

$$U_\alpha[(S,b^-) \mid h_\beta] = \max_{0\le h_\alpha \le \hat{h}_\alpha} \left\{ \underset{b|b^-}{E} \left(\ln(\delta h_\alpha A S^b) + \gamma_\alpha U_\alpha^+[(S^+,b) \mid h_\beta^+]\right) \right\}. \tag{4.25}$$

Differentiating the utility function (4.25) with respect to h_α yields

$$\begin{aligned} \frac{\partial U_\alpha}{\partial h_\alpha} &= \frac{1}{h_\alpha} + \gamma_\alpha \underset{b|b^-}{E} \left(\lambda_\alpha^+ \frac{\partial S^+}{\partial h_\alpha}\right) \\ &= \frac{1}{h_\alpha} - \frac{\gamma_\alpha}{1-\delta h_\alpha} \underset{b|b^-}{E} \left(\delta(\Lambda_\alpha^+)\right). \end{aligned} \tag{4.26}$$

Setting expression (4.26) equal to zero we obtain

$$\tilde{h}_\alpha = \frac{1}{\delta\left(1 + \gamma_\alpha \underset{b|b^-}{E} (\Lambda_\alpha^+)\right)}. \tag{4.27}$$

Differentiating the utility function with respect to S, we find that

$$\Lambda_\alpha[(S,b^-) \mid h_\beta] = \underset{b|b^-}{E} \left(b(1+\gamma_\alpha b \Lambda_\alpha^+)\right). \tag{4.28}$$

Iterating (4.28) using $\Lambda_\alpha^0 = \underset{b|b^-}{E}(b)$ we obtain as before

$$E[\Lambda_\alpha^\infty] = \mathbf{P}^2[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}.$$

Therefore

$$h_\alpha(b^- = b_i) = \frac{1}{\left(\delta\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}^2[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}\right)\right)_i}. \tag{4.29}$$

Note that if $\delta = 1$ then (4.29) becomes

$$h_\alpha(b^- = b_i) = \frac{1}{\left(\underline{\mathbf{1}} + \gamma_\alpha \mathbf{P}^2[\mathbf{I} - \gamma_\alpha \mathbf{Q}]^{-1}\underline{\mathbf{b}}\right)_i},$$

the same result as previously obtained without a δ constraint in (2.31).

In this chapter we have tried to neutralize the advantage of the first harvester by restricting the allowing that harvester access to only a fraction of the fish. We have shown that this player compensates by increasing its harvest proportion. Therefore in order to make the game fair, the δ constraint must be small enought to ensure that the first harvester is constrained by $\hat{h}$. The optimal proportion is independent of R and A.

Chapter 5

Split Streams

In this chapter we consider a new model and a new type of information. The two fleets compete over the harvest of a single stock. The life cycle is illustrated in the diagram below.

$$
\begin{tikzcd}
 & R_\alpha = \theta_\alpha R \rightarrow S_\alpha = (1 - h_\alpha) R_\alpha \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & S^+ = S_\alpha + S_\beta \rightarrow R^+ \\
 & R_\beta = \theta_\beta R \rightarrow S_\beta = (1 - h_\beta) R_\beta \arrow[ur] &
\end{tikzcd}
$$

At the beginning of the season, the total stock splits into two streams, with stock $R_\alpha = \theta_\alpha R$ available to player α and $R_\beta = \theta_\beta R$ available to player β where $\theta_\alpha + \theta_\beta = 1$. At the end of the season, the unharvested stocks S_α and S_β reunite to form the total escapement $S = S_\alpha + S_\beta = \sigma R$, where $\sigma = 1 - \theta_\alpha h_\alpha - \theta_\beta h_\beta$.

In the following cases we change the knowledge structure by assuming the players obtain partial knowledge of the stochastic parameters obtained by making imperfect observations of the parameter in question. We find the optimal harvest proportions under various scenarios and use numerical simulations to compare the effect of the imperfect observations.

Imperfect Knowledge of θ

We first consider the case where the split, θ, is imperfectly measured. We assume that each player knows the distributions, $\tilde{q}_{ij}^{\nu} = prob\{\theta = \theta_j | \tilde{\theta}_\nu = \theta_i\}$, where $\tilde{\theta}_\nu$ is the measurement made by player ν, and $q_j = prob\{\theta = \theta_j\}$. We also assume that each player knows the distribution of b, where b is now assumed to be iid. The ν-fleet's utility function thus satisfies the DPE

$$U_\nu^\tau[(R, b, \tilde{\theta}_\nu) \mid h_{\bar{\nu}}] = \max_{0 \le h_\nu \le \widehat{h}_\nu} \underset{\theta_\nu | \tilde{\theta}_\nu}{E} \left\{ \ln(\theta_\nu h_\nu R) + \gamma_\nu \underset{b^+}{E} \underset{\tilde{\theta}_\nu^+}{E} (U_\nu^+[(R^+, b^+, \tilde{\theta}_\nu^+) \mid h_{\bar{\nu}}^+]) \right\}. \tag{5.1}$$

We now proceed iteratively, working backwards in time from the horizon. In the terminal period, when $\tau = 0$, the utility for the ν-fleet is

$$U_\nu^0 = \max_{0 \le h_\nu \le \widehat{h}_\nu} \underset{\theta_\nu | \tilde{\theta}_\nu}{E} \ln(\theta_\nu h_\nu R)$$

Therefore the optimal choice for the h_ν^0, is

$$h_\nu^0 = \widehat{h}_\nu,$$

and

$$\mathcal{M}_\nu^0 = 1,$$

independent of any action taken by the $\bar{\nu}$-fleet. Using induction, we will again show that h_ν^τ and $\mathcal{M}_\nu^\tau$ are independent of R^τ and of the competitor's policy $h_{\bar{\nu}}^\tau$.

Suppose, for a given $\tau \geq 1, \mathcal{M}_\nu^+$ is dependent only on b^+. The following formulas will be useful

$$\begin{aligned} R^+ &= A[\sigma R]^{b^+} \\ \frac{\partial R^+}{\partial h_\nu} &= -b^+ A[\sigma R]^{b^+-1}(\theta_\nu R) \\ &= \frac{-\theta_\nu b^+ R^+}{\sigma}. \end{aligned}$$

Differentiating the right hand side of the DPE (5.1) with respect to h_ν^τ yields

$$\frac{\partial U_\nu}{\partial h_\nu} = \underset{\theta_\nu|\bar{\theta_\nu}}{E}\left(\frac{1}{h_\nu} - \gamma_\nu \underset{\bar{\theta}_\nu^+}{E}\left(\frac{\theta_\nu}{\sigma}\underset{b^+}{E}(b^+\mathcal{M}_\nu{}^+)\right)\right). \tag{5.2}$$

Setting (5.2) equal to zero, we obtain the interior solution

$$1 = \underset{\theta_\nu|\bar{\theta}_\nu}{E}\left(\frac{\theta_\nu h_\nu}{\sigma}\gamma \underset{b^+}{E}(b^+\mathcal{M}_\nu{}^+)\right). \tag{5.3}$$

Since (5.3) holds for $\nu = \alpha$ or β, by the induction hypothesis we find that h_ν is indeed independent of R and $h_{\bar{\nu}}$. We now know that

$$\frac{\partial R^+}{\partial R} = b^+ A[\sigma R]^{b^+-1}.$$

We also note that

$$\frac{\partial U_\nu}{\partial R} = \frac{1}{R}\underset{\bar{\theta}_\nu}{E}\left(1 + \gamma_\nu \underset{\theta_\nu|\bar{\theta}_\nu^+}{E}\underset{b^+}{E}(b^+\mu_\nu^+ R^+)\right).$$

Therefore we can write

$$\mathcal{M}_\nu = 1 + \gamma_\nu \underset{b^+}{E}(b^+\mathcal{M}_\nu^+) \tag{5.4}$$

independent of R, b and $h_{\bar{\nu}}$. Substituting (5.4) in (5.3), we obtain

$$1 = \underset{\theta_\nu|\bar{\theta}_\nu}{E}\left(\frac{\theta_\nu h_\nu}{\sigma}\right)(\mathcal{M}_\nu - 1). \tag{5.5}$$

Iterating (5.4) with $\mathcal{M}_\nu^0 = 1$ and $E(b) = \bar{b}$ gives

$$\mathcal{M}_\nu^1 = 1 + \gamma_\nu E(b^+),$$

$$\mathcal{M}_\nu^2 = 1 + \gamma_\nu E(b + (1 + \gamma_\nu E(b^{++}))) = 1 + \gamma_\nu \bar{b} + \gamma_\nu^2 \bar{b}^2,$$

$$\ldots$$

Letting $\tau \to \infty$, we find the limiting, time independent relations for the infinite-horizon:

$$\mathcal{M}_\nu^\infty = 1 + \frac{\gamma\bar{b}}{1 - \gamma\bar{b}} \tag{5.6}$$

and

$$1 = \underset{\theta_\nu|\bar{\theta}_\nu}{E}\left(\frac{\theta_\nu h_\nu(\bar{\theta}_\nu)}{\sigma}\right)\frac{\gamma_\nu\bar{b}}{1 - \gamma_\nu\bar{b}}. \tag{5.7}$$

Imperfect Knowledge of b

We now consider the case where the growth parameter, b, is imperfectly measured. We assume that each player knows the distributions, $\bar{p}_{b_{ij}}^{\nu} = prob\{b = b_j \mid \bar{b}_\nu = b_i\}$, where $\bar{b}_\nu$ is the measurement of b made by player ν and also $p_j = prob\{b = b_j\}$. We also assume that each player knows the distribution of θ, where θ is assumed to be iid. The DPE for the ν -fleet is

$$U_\nu[(S, \bar{b}_\nu, \theta_\nu) \mid h_{\bar{\nu}}] = \max_{0 \le h_\nu \le \bar{h}_\nu}\left\{\underset{b|\bar{b}_\nu}{E}\left(\ln(\theta_\nu h_\nu A S^b) + \gamma_\nu \underset{\bar{b}_\nu^+ \theta_\nu^+}{E\,E}\left(U_\nu^+[(S^+, \bar{b}_\nu^+, \theta_\nu^+) \mid h_{\bar{\nu}}^+]\right)\right)\right\}. \tag{5.8}$$

In the terminal period, when $\tau = 0$, the utility for the ν-fleet is

$$U_\nu^0 = \max_{0 \le h_\nu \le \widehat{h}_\nu} \mathop{E}_{b|\bar{b}_\nu} \ln(\theta_\nu h_\nu A S^b)$$

Therefore the optimal choice for the h_ν^0, is

$$h_\nu^0 = \widehat{h}_\nu,$$

and

$$\Lambda_\nu^0(\bar{b}_\nu) = E(b \mid \bar{b}_\nu),$$

independent of any action taken by the $\bar{\nu}$-fleet and of S. Using induction, we will again show that h_ν^τ and Λ_ν^τ are independent of S^τ and of the competitor's policy.

Suppose, for a given $\tau \ge 1$, Λ_ν^+ is dependent only on $\bar{b}_\nu^+$. Differentiating the right hand side of the DPE (5.8) with respect to h_ν^τ yields

$$\frac{\partial U_\nu}{\partial h_\nu} = \mathop{E}_{b|\bar{b}_\nu} \left(\frac{1}{h_\nu} - \frac{\gamma_\nu \theta_\nu}{\sigma} \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\theta_\nu^+} (\Lambda_\nu^+) \right). \tag{5.9}$$

Setting (5.9) equal to zero, we obtain

$$1 = \mathop{E}_{b|\bar{b}_\nu} \left(\frac{\gamma_\nu \theta_\nu h_\nu}{\sigma} \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\theta^+} (\Lambda_\nu^+) \right). \tag{5.10}$$

Since (5.10) holds for $\nu = \alpha$ or β, using the induction hypothesis, we find that h_ν is independent of S and $h_{\bar{\nu}}^\tau$. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{b S^+}{S}$$

and

$$\frac{\partial U}{\partial S} = \mathop{E}_{b|\bar{b}_\nu} \frac{b}{S} (1 + \gamma_\nu \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\theta_\nu^+} (\lambda_\nu^+ S^+)). \tag{5.11}$$

Therefore we can write

$$\Lambda_\nu(\tilde{b}_\nu) = \underset{b|\tilde{b}_\nu}{E}\left(b(1+\gamma_\nu \underset{\tilde{b}_\nu^+}{E}(\Lambda_\nu^+)\right), \tag{5.12}$$

independent of S and $h_{\tilde{\nu}}$.

Let

$$\tilde{\mathbf{P}}_\nu = \begin{pmatrix} \tilde{p}_{11}^\nu & \tilde{p}_{12}^\nu \\ \tilde{p}_{21}^\nu & \tilde{p}_{22}^\nu \end{pmatrix}$$

be the matrix containing the probabilities of b based on the measurement of b.

Let $\mathbf{J}_{nm}$ be the matrix that contains the joint distributions of $\tilde{b}$ and b,

$$\mathbf{J}_{nm} = prob(\tilde{b}_n \cap b_m).$$

It should be noted that $\mathbf{J}$ can be calculated from the $\tilde{\mathbf{P}}$ and the p_j.

Let

$$\begin{aligned} \hat{b}_\nu &= \underset{\tilde{b}}{E} b \\ &= \sum_j \underset{\tilde{b}_\nu = b_j}{E} b \cdot prob(\tilde{b}_\nu = b_j) \\ &= \sum_k \sum_j \mathbf{J}_{jk}^\nu b_k \\ &= \bar{b}, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \hat{b}_i^\nu &= \underset{\tilde{b}_\nu = b_i}{E} b \\ &= \sum_k \tilde{P}_{ik}^\nu b_k. \end{aligned}$$

Iterating (5.9), using $\Lambda_\nu^0 = E(b \mid \bar{b}_\nu)$, gives

$$\Lambda_\nu^1(\bar{b}_\nu) = \mathop{E}_{b|\bar{b}_\nu} b(1 + \gamma_\nu \mathop{E}_{\bar{b}_\nu^+}(E[b^+ \mid \bar{b}_\nu^+])),$$

or equivalently,

$$\Lambda_\nu^1(\bar{b}_\nu = b_i) = \hat{b}_i^\nu(1 + \gamma_\nu \bar{b}_\nu)$$

Letting $\tau \to \infty$, we find the limiting, time independent relations for the infinite-horizon:

$$\Lambda_\nu(\bar{b}_\nu = b_i) = \hat{b}_i^\nu(1 + \gamma_\nu \bar{b}_\nu + \gamma_\nu^2 \bar{b}_\nu^2 + \cdots) = \frac{\hat{b}_i^\nu}{1 - \gamma_\nu \bar{b}_\nu}$$

so

$$\mathop{E}_{\bar{b}^+} \Lambda_\nu^{+\infty} = \frac{\bar{b}_\nu}{1 - \gamma_\nu \bar{b}_\nu}. \tag{5.14}$$

Substituting (5.14) into (5.10) we obtain

$$1 = \frac{\gamma_\nu \bar{b}_\nu \theta_\nu h_\nu}{\sigma(1 - \gamma_\nu \bar{b}_\nu)}. \tag{5.15}$$

We note that h is independent of $\bar{b}$.

Imperfect Knowledge of b and θ

We now consider the case where both b and θ, are imperfectly measured and iid. We assume that each player knows the distributions, $\bar{p}_{ij}^\nu, \bar{q}_{ij}^\nu$, and the true distributions of θ and b. The DPE for the ν -fleet is

$$U_\nu[(S, \bar{b}_\nu, \bar{\theta}_\nu) \mid h_{\bar{\nu}}] =$$

$$\max_{0 \le h_\nu \le \hat{h}_\nu} \left\{ \mathop{E}_{b|\bar{b}_\nu} \mathop{E}_{\theta_\nu|\bar{\theta}_\nu} \left(\ln(\theta_\nu h_\nu A S^b) + \gamma_\nu \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\bar{\theta}_\nu^+} U_\nu^+[(S^+, \bar{b}_\nu^+, \theta_\nu^+) \mid h_{\bar{\nu}}^+] \right) \right\}. \tag{5.16}$$

In the terminal period, when $\tau = 0$, the utility for the ν-fleet is

$$U_\nu^0 = \max_{0 \le h_\nu \le \widehat{h}_\nu} \mathop{E}_{b|\bar{b}_\nu} \mathop{E}_{\theta_\nu|\bar{\theta}_\nu} \ln(\theta_\nu h_\nu A S^b).$$

Therefore the optimal choice for the h_ν^0, is

$$h_\nu^0 = \widehat{h}_\nu,$$

and

$$\Lambda_\nu^0 = E(b \mid \bar{b}_\nu),$$

independent of any action taken by the $\bar{\nu}$-fleet and of S. Using induction, we again show that h_ν^τ and Λ_ν^τ are independent of S^τ and of the competitor's policy.

Suppose, for a given $\tau \ge 1$, Λ_ν^+ is dependent only on $\bar{b}_\nu^+$. Differentiating the right hand side of the DPE (5.16) with respect to h_ν^τ yields

$$\frac{\partial U_\nu}{\partial h_\nu} = \mathop{E}_{b|\bar{b}_\nu} \mathop{E}_{\theta_\nu|\bar{\theta}_\nu} \left(\frac{1}{h_\nu} - \frac{\gamma_\nu \theta_\nu}{\sigma} \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\bar{\theta}_\nu^+} (\Lambda_\nu^+) \right). \tag{5.17}$$

Setting (5.17) equal to zero, we obtain

$$1 = \mathop{E}_{\theta_\nu|\bar{\theta}_\nu} \mathop{E}_{b|\bar{b}_\nu} \left(\frac{\gamma_\nu \theta_\nu h_\nu}{\sigma} \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\bar{\theta}_\nu^+} (\Lambda_\nu^+) \right). \tag{5.18}$$

Since (5.18) holds for $\nu = \alpha$ or β, using the induction hypothesis we find that h_ν is independent of S and $h_{\bar{\nu}}$. We now know that

$$\frac{\partial S^+}{\partial S} = \frac{b S^+}{S},$$

and

$$\frac{\partial U}{\partial S} = \mathop{E}_{b|\bar{b}_\nu} \frac{b}{S} (1 + \gamma_\nu \mathop{E}_{\bar{b}_\nu^+} \mathop{E}_{\bar{\theta}_\nu^+} (\lambda_\nu^+ S^+)). \tag{5.19}$$

Therefore we can write

$$\Lambda_\nu(\bar{b}_\nu) = \underset{b|\bar{b}_\nu}{E}\left(b(1 + \gamma_\nu \underset{\bar{b}^+}{E}(\Lambda_\nu^+))\right), \tag{5.20}$$

independent of S and $h_{\bar{\nu}}$.

Iterating (5.20), we again obtain

$$\underset{\bar{b}_\nu^+|\bar{b}_\nu}{E} \Lambda_\nu^+(\infty) = \frac{\bar{b}}{1 - \gamma_\nu \bar{b}}. \tag{5.21}$$

Substituting (5.21) into (5.18) we obtain

$$\frac{1 - \gamma_\nu \bar{b}}{\gamma_\nu \bar{b}} = \underset{\theta_\nu|\bar{\theta}_\nu}{E} \frac{\theta_\nu h_\nu(\bar{\theta}_\nu)}{\sigma}. \tag{5.22}$$

Imperfect Knowledge of θ with b Markovian

For our last case we assume θ is imperfectly measured, the growth parameter b has a known Markov distribution, and that the value of b^- is known while the value of b is unknown. The ν-fleet's utility function satisfies the DPE

$$U_\nu^\tau[(S, b-, \bar{\theta}_\nu) \mid h_{\bar{\nu}}] = \max_{0 \le h_\nu \le \hat{h}_\nu} \underset{\theta_\nu|\bar{\theta}_\nu}{E} \underset{b|b^-}{E} \left\{ \ln(\theta_\nu h_\nu A S^b) + \gamma_\nu \underset{b^+|b}{E} \underset{\bar{\theta}_\nu^+}{E} (U_\nu^+[(S^+, b, \bar{\theta}_\nu^+) \mid h_\nu^+]) \right\}. \tag{5.23}$$

In the terminal period when $\tau = 0$ we find the utility for the ν-fleet to be

$$U_\nu^0 = \underset{\theta_\nu|\bar{\theta}_\nu}{E} \underset{b|b^-}{E} (\ln(\theta_\nu h_\nu A S^b)).$$

Therefore the optimal choice for the h_ν^0 is

$$h_\nu^0 = \hat{h}_\nu,$$

and

$$\Lambda_\nu([S, b^-, \bar{(\theta)}_\nu] \mid h_{\bar{\nu}}) = E(b \mid b^-),$$

independent of any action taken by the opponent's fleet.

Suppose for a given $\tau > 1$, Λ_ν^+ is dependent only on b^+. Differentiating the right hand side of the DPE (5.23) with respect to h_ν, we obtain

$$\frac{\partial U_\nu}{\partial h_\nu} = \underset{b|b^-}{E} \underset{\theta_\nu|\bar{\theta}_\nu}{E} \frac{1}{h_\nu} - \gamma_\nu \underset{b^=}{E} \underset{|b\bar{\theta}_\nu^+}{E} \left(\frac{\theta_\nu}{\sigma}\right). \tag{5.24}$$

Setting (5.24) equal to zero, we obtain

$$1 = \gamma_\nu \underset{b|b^-}{E} \underset{\theta_\nu|\bar{\theta}_\nu}{E} \frac{\theta_\nu h_\nu}{\sigma} \Lambda_\nu^+(b). \tag{5.25}$$

Since (5.25) holds for $\nu = \alpha$ or β we find that h_ν is independent of S and $h_{\bar{\nu}}$. We now know that

$$\frac{\partial U_\nu}{\partial S} = \underset{b|b^-}{E} \underset{\theta_\nu|\bar{\theta}_\nu}{E} \left(\frac{b}{S}(1 + \gamma_\nu \underset{b+}{E} \underset{|b\bar{\theta}_\nu^+}{E} S^+ \lambda_\nu^+)\right).$$

Therefore we can write

$$\Lambda_\nu(b^-) = \underset{b|b^-}{E} b(1 + \gamma_\nu \underset{b^+|b}{E} \Lambda_\nu^+(b)). \tag{5.26}$$

independent of S and h_{bar_ν}. Substituting (5.26) in (5.25) gives

$$1 = \underset{b|b^-}{E} \underset{\theta_\nu|\bar{\theta}_\nu}{E} \frac{\theta_\nu h_\nu}{\sigma} \Lambda_\nu^+. \tag{5.27}$$

Iterating (5.26) gives

$$E[\Lambda_\nu^{\infty +} \mid (b^- = b_i)] = (\mathbf{P}^2[\mathbf{I} - \gamma_\nu \mathbf{Q}]^{-1}\underline{\mathbf{b}})_i.$$

Numerical Simulation Results

In the following figures we show the average of W, the historically preferred utility function (see (2.32)). **Figure 5.1** In this figure player one has no knowledge of θ, that is, player one's measurement is no better than tossing a coin to determine the value of θ. The independent variable is the knowledge of θ by player two. A scale of 0 to 1 is used, with 0 representing no knowledge and 1 representing full knowledge. The parameters were set to heavily favor player one. When player two takes more accurate measurements, player two's utility is greatly increased. At the same time, player one's utility decreases although not as sharply as the increase for player two.

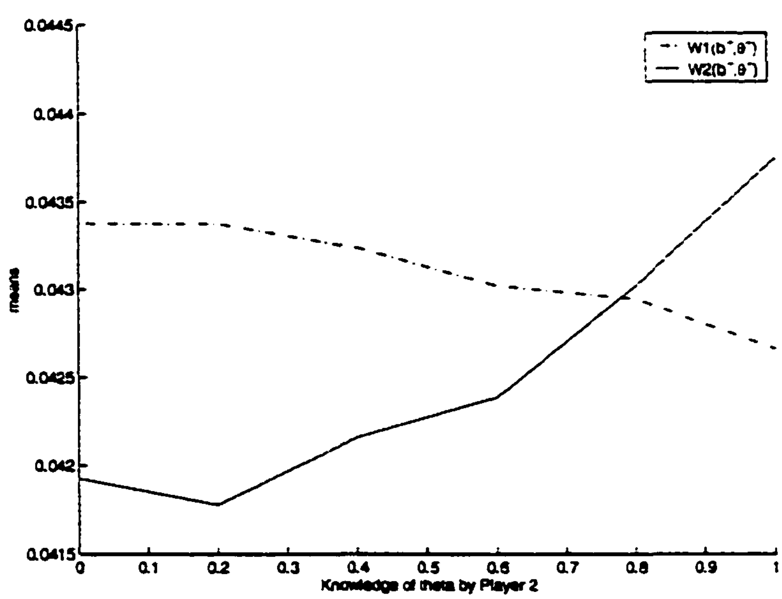

Figure 5.1: Comparison of $W1$ with $W2$

Figure 5.2 In this figure player one has full knowledge of θ. The independent variable is the knowledge of θ by player two. The parameters were chosen to be symmetric among the two players. As expected, player two's utility increases as player two gains knowledge.

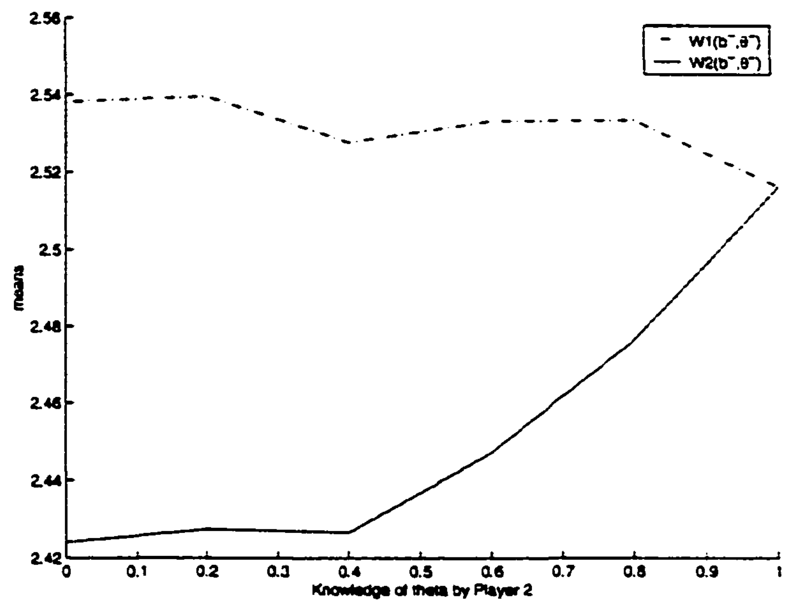

Figure 5.2: Comparison of $W1$ with $W2$

Figure 5.3 In this figure player one has some knowledge of θ. The independent variable is the knowledge of θ by player two. The parameters were chosen to be symmetric among the two players. The two average values for the utilities are equal at the point where both players measure θ with the same accuracy.

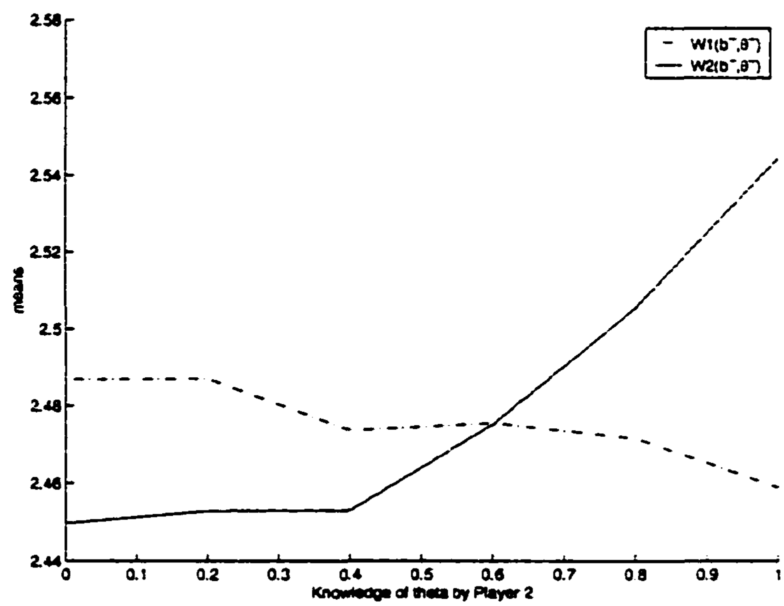

Figure 5.3: Comparison of $W1$ with $W2$

Figure 5.4 In this figure the independent variable is the knowledge of θ by player one and two. The parameters were chosen to be symmetric among the two players. When θ is measured with better accuracy, both player's utilities increases.

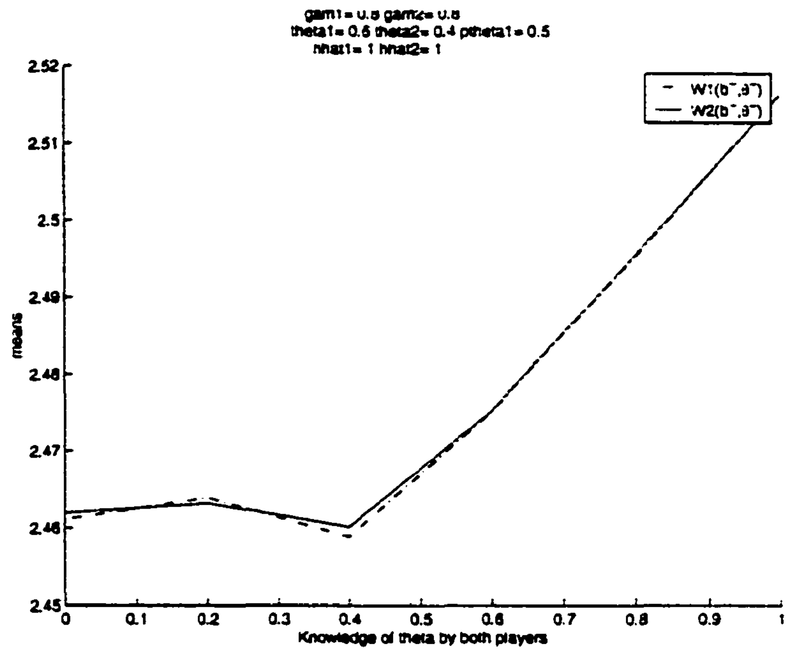

Figure 5.4: Comparison of $W1$ with $W2$

Chapter 6

Conclusion

In this paper we analyzed extensions of a classical fishery model. We derived the Nash equilibrium harvest strategies in closed form analytical expressions as explicit functions of fundamental biological and economic parameters. We also incorporated a wide range of possible information structures.

In all cases we found the optimal harvest proportions to be multiplicative, independent of the recruitment R. This relates to the risk averse utility function and is in contrast to models that are risk neutral and lead to an optimal constant escapement. We also found that optimal harvest proportions exhibit a certainty equivalence property with respect to the multiplicative factor A in the growth function. This implies that the knowledge of A has no significance on the outcome of the game.

Numerical explorations of the models show the amount of information does have an effect on the economic returns. In the sole manager game, additional knowledge always leads to an increase in return. However, in the competitive fishery, additional knowledge may lead the competitors to over-harvest, in turn lowering the returns.

The optimal harvest fractions and a summary of the numerical studies are given below.

Sole Harvester

For the deterministic version we found the optimal harvest rate to be

$$h = 1 - \gamma b.$$

For the stochastic version we found the optimal harvest rates to be

$$h(b = b_i) = \frac{1}{(\underline{1} + \gamma \mathbf{P}[\mathbf{I} - \gamma \mathbf{Q}]^{-1}\underline{\mathbf{b}})_i},$$

when b was known,

$$h(b^+ = b_i) = \frac{1}{\left(\underline{1} + \gamma \underline{\mathbf{b}}^+ \left(\underline{1} + \gamma \mathbf{P}[\mathbf{I} - \gamma \mathbf{Q}]^{-1}\underline{\mathbf{b}}^+\right)\right)_i},$$

when b^+ was known, and

$$h(b^- = b_i) = \frac{1}{\left(\underline{1} + \gamma \mathbf{P}^2[\mathbf{I} - \gamma \mathbf{Q}]^{-1}\underline{\mathbf{b}}^-\right)_i},$$

when b^- was known.

Numerical explorations led to the conclusion that for the single player game, additional knowledge always results in higher economic returns.

Competitive Fishery

In the competitive fishery we found the optimal harvest rate for each player is the same as calculated in the single player game. The simulations led us to conclude that a symmetric increase in knowledge benefits the first harvester at the expense of the second harvester. If the increase in information is asymmetric, the player who has the additional knowledge will receive a higher return than with lesser knowledge, the other player will have a decrease in economic return when its opponent gains knowledge. Therefore there is *no incentive inherent in the game for a player to share knowledge with its opponent.*

We also found that we can not easily split the resource fairly by allowing a player access to only a fraction of the fish. The player will adjust his optimal harvest fraction to take the same amount as before he was constrained *if* enough fish are present. To ensure fairness we must be certain that the first player is constrained by his maximum harvest fraction. In this scenario the player will choose to take as much of the fish as he can but will leave enough stock for the second player to harvest his fair share while leaving enough stock to spawn for the next season.

Our last model incorporated a spatial instead of a temporal split and introduced imperfect measuring of the stochastic parameters. This allows us to vary the amount of knowledge by small increments instead of b, b^+, b^- as in the previous work. Numerical simulations of this model provided a variety of possible outcomes.

Calculation of Joint Distribution

Calculation of J

Given $p_i = prob(b = b_i)$ and $x_{ij} = prob(b = b_j \mid \tilde{b} = b_i)$, we wish to find the $j_{nm} = prob(\tilde{b}_n \cap b_m)$. Let $\tilde{p}_i = prob(\tilde{b}_i)$ and $p_i = prob(b_i)$ then

$$\sum_{k=1}^{n} j_{ik} = \tilde{p}_i, \quad \sum_{i=1}^{n} j_{ik} = p_i, \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} \tilde{p}_i = 1.$$

We can write $x_{ik} = \frac{j_{ik}}{\tilde{p}_i}$ and use these relations to solve for j_{ik}. For example when $n = 2$, we have $j_{11} = \tilde{p}_1 x_{11}$ and $j_{21} = (1 - \tilde{p}_1) x_{21}$. Adding these expressions we find

$$p_1 = j_{11} + j_{21} = \tilde{p}_1 (x_{11} - x_{21}) + x_{21} \text{ or } \tilde{p}_1 = \frac{p_1 - x_{21}}{x_{11} - x_{21}}.$$

Therefore we can find the joint distribution matrix whose entries are

$$j_{11} = \tilde{p}_1 x_{11}, \quad j_{12} = \tilde{p}_1 \tilde{p}_{12}, \quad j_{21} = (1 - \tilde{p}_1) x_{21}, \quad j_{22} = (1 - \tilde{p}_1) x_{22}.$$

Calculation of x_{jk}

Given $p_i = prob(b = b_i)$ and $\tilde{x}_{ij} = prob(\tilde{b} = b_j \mid \tilde{b} = b_i)$, we wish to find the $x_{jk} = prob(b = b_k \mid \tilde{b} = b_j)$. Then $j_{ij} = p_i\tilde{x}_{ij}$, $\quad \tilde{p}_j = \sum_i p_i\tilde{x}_{ij}$ and

$$x_{jk} = \frac{j_{kj}}{\tilde{p}_j} = \frac{p_k\tilde{x}_{kj}}{\sum_i p_i\tilde{x}_{ij}}.$$

Likewise given $q_i = prob(\theta = \theta_i)$ and $\tilde{q}_{ij} = prob(\tilde{\theta} = \theta_j \mid \theta = \theta_i)$, we wish to find the $q_{jk} = prob(\theta = \theta_k \mid \tilde{\theta} = \theta_j)$. We find $s_{ij} = q_i\tilde{q}_{ij}$, $\quad \tilde{q}_j = \sum_i q_i\tilde{q}_{ij}$ and

$$q_{jk} = \frac{s_{kj}}{\tilde{q}_j} = \frac{q_k\tilde{q}_{kj}}{\sum_i q_i\tilde{q}_{ij}}.$$

Two Independent observations

Now consider two independent observations of b, namely $\tilde{b}_\nu$ and $\tilde{b}_{\bar{\nu}}$. Define

$$\begin{aligned}
\tilde{x}_{ijk} &= prob(\tilde{b}_\nu = b_j, \tilde{b}_{\bar{\nu}} = b_k \mid b = b_i) \\
&= prob[\tilde{b}_\nu = b_j \mid b = b_i] \cdot prob[\tilde{b}_{\bar{\nu}} = b_k \mid b = b_i] \\
&= \tilde{x}^{\nu}_{ij}\tilde{x}^{\bar{\nu}}_{ik}, \\
j_{ijk} &= prob[b = b_i, \tilde{b}_\nu = b_j, \tilde{b}_{\bar{\nu}} = b_k] \\
&= p_i\tilde{x}_{ijk} \\
&= p_i\tilde{x}^{\nu}_{ij}\tilde{x}^{\bar{\nu}}_{ik} \quad \text{and} \\
x^{\nu}_{ijk} &= prob[b = b_j, \tilde{b}_{\bar{\nu}} = b_k \mid \tilde{b}_\nu = b_j] \\
&= \frac{prob[\tilde{b}_\nu = b_i, b = b_j, \tilde{b}_{\bar{\nu}} = b_k]}{prob[\tilde{b}_\nu = b_i]}
\end{aligned}$$

where $prob[\tilde{b}_\nu = b_i] = \sum_j \sum_k x^\nu_{jik}$. So

$$x^\nu_{ijk} = \frac{j_{ijk}}{\sum_j \sum_k x^\nu_{jik}}.$$

We can also consider two independent observations of θ, namely $\tilde{\theta}_\alpha$ and $\tilde{\theta}_beta$. Define

$$\begin{aligned}
\bar{s}_{ijk} &= prob(\tilde{\theta}_\alpha = \theta_j, \tilde{\theta}_\beta = \theta_k \mid \theta = \theta_i) \\
&= prob[\tilde{\theta}_\alpha = \theta_j \mid \theta = \theta_i] \cdot prob[\tilde{\theta}_\beta = \theta_k \mid \theta = \theta_i] \\
&= \bar{s}^\alpha_{ij} \bar{q}^\beta_{ik}, \\
s_{ijk} &= prob[\theta = \theta_i, \tilde{\theta}_\alpha = \theta_j, \tilde{\theta}_\beta = \theta_k] \\
&= q_i \bar{s}_{ijk} \\
&= q_i \bar{s}^\alpha_{ij} \bar{q}^\beta_{ik} \quad \text{and} \\
q_{ijk} &= prob[\theta = \theta_j, \tilde{\theta}_\beta = \theta_k \mid \tilde{\theta}_\alpha = \theta_j] \\
&= \frac{prob[\tilde{\theta}_\alpha = \theta_i, \theta = \theta_j, \tilde{\theta}_\beta = \theta_k]}{prob[\tilde{\theta}_\alpha = \theta_i]}
\end{aligned}$$

where $prob[\tilde{\theta}_\alpha = \theta_i] = \sum_j \sum_k q^\alpha_{jik}$.

b Markovian

Suppose b is Markovian and $b^- = b_m$ is known but b is observed through $\tilde{b}$. Then in part A.2, replace $p_i = prob[b = b_i]$ by $p_{mi} = prob[b = b_i \mid b^- = b_m]$. Now suppose

$prob[\bar{b} \mid b]$ is also known. We can calculate

$$
\begin{aligned}
j_{jk|i} &= prob[b = b_j, \bar{b} = b_k \mid b^- = b_i] \\
&= prob[b = b_j \mid b^- = b_i] \cdot prob([\bar{b} = b_k \mid b = b_j] \mid b^-) \\
&= p_{ij}\bar{x}_{jk}
\end{aligned}
$$

and

$$
x_{jk|i} = prob[(b = b_i \mid \bar{b} = b_k) \mid b^- = b_i) = \frac{j_{jk|i}}{\sum_k j_{jk|i}}.
$$

Bibliography

[1] Bellman, R., 1957, *Dynamic Programming* Princeton University Press, Princeton.

[2] Beverton, R.J.H. and S.J. Holt, 1957, On the dynamics of exploited fish populations, *Fish. Invest. Lond.* **2**,19.

[3] Clark, C.W., 1971, Economically optimal policies for the utilization of biologically renewable resources, *Math. Biosci.*, **13**, 245-260.

[4] Fudenberg, D and J. Tirole, 1991, *Game Theory*, MIT Press, Cambridge.

[5] Jaquette, D., 1972, A discrete time population control model, *Math. Biosci.* **15**,231-252.

[6] Levhari, D. and L.J. Mirman, 1980, The great fish war: an example using a dynamic Cournot-Nash solution, *Bell Journal of Economics* **11**,322-344.

[7] Mann, S., 1970, A mathematical theory for the harvest of natural animal populations when birth rates are dependent on total population size, *Math. Biosci.* **7**,97-110.

[8] McKelvey, R., 1997, Game-theoretic insights into the international management of fisheries, *Natural Resource Modeling* **10**,7,129-171.

[9] McKelvey, R., 2001, The split-stream haresting game, NCAR Technical Report, NCAR/TN-449+STR, Boulder, Colorado.

[10] McKelvey, R. and Greg Cripe, 2001, The split-stream haresting game:Numerical and Simulation Studies, NCAR Technical Report, NCAR/TN-449+STR, Boulder, Colorado.

[11] Reed, W.J., 1974, A stochastic model for the economic management of a renewable animal resource, *Math. Biosci.* **22**,313-337.