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CONCERNING PEANO SPACES

by

JOHN JOSEPH McGUIRE

B.A. Duke University, 1952

Presented in partial fulfillment of the requirements for the degree of

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J. J. M.

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INTRODUCTION

A metric space which is compact, connected and locally connected plays an important role in the field of anlysis. Such spaces are called Peano spaces. It is the intent of this thesis to investigate some of the more fundamental aspects of the structure of Peano spaces.

Giuseppe Peano (1858-1932) proved that a segment can be mapped onto a square. Peano's theorem is actually a special case of the first theorem which we shall prove. It is the famous theorem of Hahn and Mazurkiewicz that the closed unit interval $[0,1]$ can be mapped continuously onto a metric space T if and only if T is compact connected and locally connected.

Secondly, we shall introduce the concept of cut points and show that a compact connected set of which all but at most two points are cut points is a simple arc. With this and the aid of the Brouwer Reduction Theorem we shall be able to give a proof due to J. L. Kelley of the Arcwise Connectedness Theorem.

Thirdly, we shall find that Sierpinski's property S enables us to prove that every Peano space has a basis, each non-empty element of which is connected; has property S; and has a Peano space as its closure.

Finally, we will discuss cyclic element theory and leaning heavily on the results of property S, we shall prove the cyclic connectivity theorem which states that a Peano space T is cyclic if and only if for every two points a and b belonging to T there

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is a simple closed curve in T containing a and b.

In many of our theorems and definitions, the spaces concerned need not be metric spaces for the statement to hold true. However, since our primary interest is in Peano spaces which are of necessity metric spaces, all spaces shall be considered to be metric unless otherwise indicated.

Furthermore, when sets are discussed without mention of their containing space, it shall be understood that they are subsets of the general metric space S.

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CHAPTER I

INTRODUCTORY TOPOLOGY

It is the purpose of this chapter to list definitions and theorems which will be needed in later sections. Proofs will be omitted. In general, Newman's Elements of the Topology of Plane Sets of Points may be used as a reference and in such cases no reference shall be given.

Throughout this paper capital letters shall denote spaces and sets and small letters shall denote points or elements.

As a matter of notation the symbol $\in$ will mean "belongs to" or "is a member of" and $\notin$ will mean "does not belong to" or "is not a member of." For example, if a point x belongs to the set A, we will write $x \in A$ and if y does not belong to A we have $y \notin A$. The symbol $\subset$ will mean "is contained in" or "is a subset of" and $\not\subset$ will mean "is not contained in" or "is not a subset of." That is, if the set A is a subset of the set B we have $A \subset B$ and if A is not a subset of B we have $A \not\subset B$. A set A is termed a proper subset of B if $A \subset B$ but $A \neq B$.

1.1 A <u>set</u> is a collection or aggregate of objects or elements called points. Suppose E is a subset of S. The <u>complement</u> <u>$\mathcal{C}E$</u> of E is the set of all points of S not belonging to E. It is always true that $\mathcal{C}(\mathcal{C}E) = E$, and if $E \subset F$ then $\mathcal{C}F \subset \mathcal{C}E$. The <u>empty</u> or <u>null</u> set, to be denoted by $\emptyset$, is defined as being that set which has no points. A set is termed <u>non-degenerate</u> if it has at least two points.

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1.2 On any set a <u>metric</u> may be defined by associating with every pair of points x and y a non-negative number $\rho(x,y)$ called the distance between then, such that the following conditions are satisfied:

(i) $\rho(x,y) = \rho(y,x)$

(ii) $\rho(x,y) = 0$ if and only if $x = y$

(iii) $\rho(x,z) \leq \rho(x,Y) + \rho(y,z)$ (triangle inequality)

A non-empty set on which a metric has been defined is called a <u>metric space</u>.

1.3 Suppose $A \subset S$. The <u>diameter</u> $d(A)$ of A is defined as being the least upper bound of $\rho(p,q)$ where the least upper bound is taken with respect to all pairs of points p and q in A. If $d(A)$ is finite, A is bounded. If A consists of a single point, $d(A) = 0$.

1.4 Suppose A and B are sets. The <u>union</u> of A and B, written $A \cup B$ and read "A union B", is the set of all points which belong to either A or B. The <u>intersection</u> of A and B, written $A \cap B$ and read "A intersection B", is the set of all points which belong to both A and B. If the sets A and B have no points in common, they are said to be <u>disjoint</u> and we write $A \cap B = \emptyset$. As a matter of abbreviation, we shall often use AB in place of $A \cap B$ and A-B in place of $A \cap CB$.

Some identical relations involving $\cup$ and $\cap$ are:

(1) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

(2) $A \cup B = B \cup A$, $A \cap B = B \cap A$

(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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(5) $A \cup A = A$, $A \cap A = A$

(6) $A \subset A \cup B$, $A \cap B \subset A$

(7) If $A \subset C$ and $B \subset C$ then $(A \cup B) \subset C$

(8) If $C \subset A$ and $C \subset B$ then $C \subset (A \cap B)$

(9) If $\mathfrak{E}$ is any collection of sets E then $\mathcal{C}(\bigcup_{E \in \mathfrak{E}} E) = \bigcap_{E \in \mathfrak{E}} (\mathcal{C}E)$ and $\mathcal{C}(\bigcap_{E \in \mathfrak{E}} E) = \bigcup_{E \in \mathfrak{E}} (\mathcal{C}E)$.

1.5 Suppose p is a point and ϵ is a positive number. By an <u>ϵ-neighborhood</u> (or merely neighborhood) of the point p, we mean the set of all points whose distance from p is less than ϵ, and we write $N(p,\epsilon)$. Similarly, an ϵ-neighborhood $N(E,\epsilon)$ of a set E denotes the set of all points whose distance from E is less than ϵ.

1.6 A subset A of S is said to be <u>open</u> if for every point p of A there is a neighborhood of p contained in A. The following statements hold.

(1) The empty set is open and S itself is open.

(2) Any neighborhood is an open set.

(3) Any union of open sets is open.

(4) The intersection of a finite number of open sets is open.

1.7 A <u>sequence</u> $x_1, x_2, \ldots, x_n, \ldots$ or $\{x_n\}$ of points of a space is determined by assigning a point x_n to every positive integer n. Thus, it should be noted that a sequence is a set of points indexed by the positive integers and is not merely any set of points. A sequence $\{x_n\}$ is said to <u>converge</u> to a point a, and we write $x_n \to a$, if for every $\epsilon > 0$, $\rho(x_n, a) < \epsilon$ for all but a finite number of values of n. Equivalent symbolism for convergence is

$$\lim_{n \to \infty} \rho(x_n, a) = 0.$$

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A sequence can converge to only one point. As a consequence of this if $x_n \rightarrow a$, $y_n \rightarrow b$, and $x_n = y_n$ for all n then a = b.

A sequence $\{A_n\}$ of sets is said to be <u>monotone decreasing</u> if and only if $A_{n+1} \subset A_n$ for each n. The sequence $\{A_n\}$ is <u>monotone increasing</u> if and only if $A_n \subset A_{n+1}$ for each n. (The analogy with sequences of points is obvious.)

1.8 The sequence $\{y_n\}$ is a <u>subsequence</u> of $\{x_n\}$ if $y_n = x_{r_n}$ where r_n are positive integers such that $r_1 < r_2 < r_3 < \ldots$. If $x_n \rightarrow a$, then every subsequence of $\{x_n\}$ converges to a.

1.9 In terms of neighborhoods, a point q is a <u>limit point</u> of a set B if every neighborhood of q contains an infinite number of points of B. In fact, it can be shown that q is a limit point of B if and only if every neighborhood of q contains a point p of B distinct from q.

In terms of sequences, q is a <u>limit point</u> of B if and only if there is a sequence $\{x_n\}$ of distinct points of B such that $x_n \rightarrow q$.

1.10 A set B is said to be <u>closed</u> if and only if its complement S-B is open. The following statements hold.

(1) The empty set is closed and S itself is closed.

(2) A set is closed if it contains all of its limit points.

(3) Any intersection of closed sets is closed.

(4) The union of a finite number of closed sets is closed.

(5) Every finite set is closed.

1.11 The closure $\overline{E}$ of a set E is the smallest closed set containing E. More explicitly, $\overline{E}$ is the intersection of all closed sets containing E. The following statements hold.

(1) $\overline{E}$ is closed.

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(2) $E = \overline{E}$ if and only if E is closed.

(3) Always $\overline{\overline{E}} = \overline{E}$.

(4) If a single point p is considered as a set then $\overline{p} = p$.

(5) If a point $x \in \overline{E}$ then any neighborhood of x contains a point of E. Thus, for every $\alpha > 0$ there is a point y of E such that $\rho(x,y) < \alpha$.

(6) If E is bounded, $\overline{E}$ is bounded and $d(E) = d(\overline{E})$.

(7) If sets E_1 and E_2 are such that $E_1 \subset E_2$ then $\overline{E}_1 \subset \overline{E}_2$.

(8) If $E_1, \ldots, E_k$ is a finite collection of sets then $\overline{\bigcup_{i=1}^{k} E_i} = \bigcup_{i=1}^{k} \overline{E}_i$.

(9) For any collection of sets $E_1, \ldots, E_n, \ldots$, $\overline{\bigcap_{n=1}^{\infty} E_n} \subset \bigcap_{n=1}^{\infty} \overline{E}_n$ and $\bigcup_{n=1}^{\infty} \overline{E}_n \subset \overline{\bigcup_{n=1}^{\infty} E_n}$.

An important fact concerning metric spaces is that given any point p of a metric space S and any open set H of S such that $p \in H$, then there exists an open set G containing p and $p \in \overline{G} \subset H$.

1.12 The <u>interior</u> i(E) of a set E is the largest open set contained in E. More explicitly, i(E) is the union of all open sets contained in E. It follows that if E is open then i(E) = E. Any point belonging to i(E) is an <u>interior point</u> of E. The <u>frontier</u> fr(E) of E is by definition $\overline{E} - i(E)$, and if E is open $fr(E) = \overline{E} - E$. The following statements hold.

(1) $fr(\emptyset) = \emptyset$ and $fr(S) = \emptyset$.

(2) Always $E \cup fr(E) = \overline{E}$.

1.13 We also have the condition of a set A being open or closed in relation to another set M containing A. That is, suppose $A \subset M \subset S$. We say that A is <u>open in M</u> if and only if there is an open set G such that $A = M \cap G$. Similarly, A is <u>closed in M</u> if and

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only if there is a closed set F such that $A = M \cap F$. The following statements hold.

(1) If A is open (closed) then A is open (closed) in M.

(2) If M is open (closed) and A is open (closed) in M then A is open (closed).

(3) A is closed in M if $A = M \cap \overline{A}$.

1.14 Given sets S and T, a rule f is called a mapping (transformation) of S <u>into</u> T and we write $f:S \to T$, if with every point $a \in S$ we associate a point $f(a) \in T$. f(a) is called the <u>image</u> of the point a under the mapping f. A mapping f is said to map S <u>onto</u> T if $f(S) = T$. In mathematics it is grammatically correct to state that a mapping $f:S \to T$ is onto. Notice if f is an onto mapping a point $b \in T$ if and only if there is at least one point $a \in S$ such that $f(a) = b$. It is easily verified that every onto mapping is itself an into mapping though the converse does not necessarily hold.

1.15 A mapping f:S T is said to be a <u>one-to-one</u> mapping of S onto T (or a one-to-one correspondence between S and T) if and only if each point b of T is the image of precisely one point $f^{-1}(b)$ of S. The point $f^{-1}(b)$ is called the <u>inverse</u> image of b. The relation is symetrical and f^{-1} is a one-to-one mapping of T onto S.

1.16 Suppose there is a mapping $f:S \to T$. Let F be any collection of subsets E of S and let H be any collection of subsets G of T. The following statements hold.

(1) $f(\emptyset) = \emptyset$ and $f^{-1}(\emptyset) = \emptyset$.

(2) If $E_1 \subset E_2$ then $f(E_1) \subset f(E_2)$.

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(3) If $G_1 \subset G_2$ then $f^{-1}(G_1) \subset f^{-1}(G_2)$.

(4) Always $E_1 \subset f^{-1}f(E_1)$ but $E_1 = f^{-1}f(E_1)$ if and only if f is one-to-one.

(5) Always $ff^{-1}(G_1) \subset G_1$ but $ff^{-1}(G_1) = G_1$ if and only if f is onto.

(6) $\complement f(E_1) \subset f(\complement E_1)$ if f is onto but $\complement f(E_1) = f(\complement E_1)$ if and only if f is onto and one-to-one.

(7) Always $f^{-1}(\complement G_1) = \complement f^{-1}(G_1)$.

(8) Always $f(\bigcup_{E \in F} E) = \bigcup_{E \in F} f(E)$ and $f(\bigcap_{E \in F} E) \subset \bigcap_{E \in F} f(E)$

(9) Always $f^{-1}(\bigcup_{G \in H} G) = \bigcup_{G \in H} f^{-1}(G)$ and $f^{-1}(\bigcap_{G \in H} G) = \bigcap_{G \in H} f^{-1}(G)$.

1.17 A set which is either finite or can be placed in a one-to-one correspondence with the set of all positive integers is said to be <u>enumerable</u> (countable). Every subset of a countable set is countable.

1.18 Suppose $\mathcal{B}$ is a collection of open sets in the space S. $\mathcal{B}$ is said to form a <u>base</u> or <u>basis</u> for S if every open set contained in S can be expressed as the union of some of the sets in the collection .

We shall also speak of a basis at a point. Suppose p is a point of the space S and let σ be a collection of open sets in S each of which contains p. We shall say that σ is a base at the point p if and only if given any open set E of S such that $p \in E$, then there exists a set $F \in \sigma$ such that $F \subset E$. It can be easily verified that in metric spaces every point p has a countable basis at p.

1.19 A set A is <u>dense</u> <u>in</u> <u>a</u> <u>set</u> B if $A \subset B$ and every neighborhood containing a point of B also contains a point of A. If A

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is dense in S where S is the whole space, then $\bar{A} = S$. A set is separable if it has an enumerable dense subset. It can be easily shown that a metric space is separable if and only if it has an enumerable base.

1.20 By a covering of a subset E of S we mean a collection of sets $\mathcal{F}$ contained in S such that $E \subset \bigcup_{F \in \mathcal{F}} F$. Thus, if $\mathcal{F}$ covers the whole space S then $\bigcup_{F \in \mathcal{F}} F = S$. If the sets of a covering are open, closed or finite in number, the covering is termed open, closed or finite respectively. It is an ϵ-covering if each of the sets of $\mathcal{F}$ has diameter not exceeding ϵ. If $\mathcal{F}^*$ is a subfamily of $\mathcal{F}$ and also covers E then $\mathcal{F}^*$ is called a subcovering of E.

1.21 A set H is compact if and only if from every open covering of H a finite subcovering can be selected. It is easily verified that any closed interval $[a,b]$ is compact. Thus, in particular the closed unit interval $[0,1]$ is compact. The open interval $(0,1)$ is not compact, however, since no finite subcovering can be selected from the covering $\mathcal{F} = \bigcup_{n=1}^{\infty} (\frac{1}{n},1)$. The following statements hold.

(1) A set H is compact if every sequence in H has a subsequence which converges to a point of H.

(2) If H is compact then H is closed.

(3) If H is a closed subset of a compact space S then H is compact.

(4) All compact sets are separable, thus, in view of paragraph 1.19 every compact set has an enumerable dense base.

(5) A decreasing sequence of non-empty compact sets has a non-empty common part.

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1.22 A subset H of S admits of a separation if there exists sets A and B in S such that $H = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. If this is the case, we write $H = A/B$. H is termed <u>disconnected</u> if it admits of a separation. Otherwise, H is <u>connected</u>. For example, suppose H is the set of all non-zero real numbers. By letting A be the set of all positive reals and B be the set of all negative reals it is readily seen that $H = A/B$. The following statements hold.

(1) If a set E consists of a single point then E is connected.

(2) Every interval is connected and in fact the set of all real numbers R_1 is connected.

(3) Suppose E is a connected subset of R_1. If points a and b belong to E then $[a,b] \subset E$.

(4) The set H is disconnected if and only if it is the union of two non-empty disjoint sets, each of which is open in H. In particular, S is disconnected if and only if it is the union of two non-empty disjoint open sets (open may be replaced by closed throughout).

(5) If G_1 and G_2 are non-empty disjoint open sets, then $G_1 \cup G_2 = G_1/G_2$.

(6) Suppose $E \subset S$ and $E = C/D$. If E is closed, then C and D are closed.

(7) If $H = A/B$ and there is a connected set E contained in H then either $E \subset A$ or $E \subset B$. Suppose G and H are subsets of S, G and H are closed, $G \cup H$ is connected and $G \cap H$ contains at most two points. Then G is connected or H is connected.

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(8) If A is connected and $\mathcal{B}$ is any collection of connected sets such that if $B \in \mathcal{B}$ implies $A \cap B \neq \emptyset$ then $A \cup (\bigcup_{B \in \mathcal{B}} B)$ is connected.

(9) If E is connected and $E \subset H \subset \overline{E}$, then H is connected. Thus, we see that if E is connected then $\overline{E}$ is connected.

(10) If E is a non-empty subset of a space S and x is a point of S such that $E \cup (x)$ is connected, then $x \in \overline{E}$.

(11) If S is connected and has more than one point, then S is non-countable.

1.23 A compact connected set with at least two points is called a <u>continuum</u>. It is clear from part (11) of 1.22 that a continuum is non-countable.

1.24 A set G is a <u>component</u> of a set E if and only if the following conditions hold: (i) $G \neq \emptyset$, (ii) $G \subset E$, (iii) G is connected, and (iv) if $G \subset F \subset E$ and F is connected, then $G = F$. A set E is <u>totally disconnected</u> if all of its components are single points. The following statements hold.

(1) Any set is the union of its components.

(2) If G and H are components of E, then either $G = H$ or $G \cap H = \emptyset$.

(3) If G is a component of E and B is any connected subset of E then either $B \subset G$ or $B \cap G = \emptyset$.

(4) If E is connected and $E \neq \emptyset$ then there is only one component of E, namely, E itself.

(5) If a set E is disconnected then there are at least two components of E.

(6) If E is closed and A is a component of E then A is closed.

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(7) If p is a point of a set E then there is exactly one component of E containing p.

1.25 A set H is said to be <u>locally connected</u> if and only if components of open sets (open in H!) are open in H. The following statements hold.

(1) If all neighborhoods of a set H are connected then H is locally connected.

(2) A set H is locally connected if and only if for every point $a \in H$ and for every positive number ϵ, there is a positive number δ such that if a point $b \in N(a,\delta) \subset H$ then there exists a connected set C such that $a \in C$, $b \in C$ and $C \subset N(a,\epsilon) \subset H$.

(3) If for every point p of a set H and every positive number ϵ there is a positive number $\delta \leq \epsilon$ and $N(p,\delta) \subset H$ and is connected then H is locally connected.

(4) A set H is locally connected if and only if the existance of an open set G (open in H) contained in H and containing a point p implies there is a subset E of G such that E is open in G, E is connected, and $p \in E$.

(5) Any open set is a locally connected space is itself locally connected.

1.26 A <u>Peano space</u> is a compact, connected, locally connected metric space. As an example, any closed interval $[a,b]$ in R_1 is a Peano space. We have seen that $[a,b]$ is both compact and connected. Suppose G is an open set contained in $[a,b]$ and containing a point p. Thus, there is a $\delta > 0$ such that $N(p,\delta) \subset G$. $N(p,\delta)$ is open and being an interval it is connected. Certainly $p \in N(p,\delta)$ hence by part (4) of 1.25, $[a,b]$ is locally connected.

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To define a metric on $[a,b]$ we use the standard formula for distance in R_1, $\rho(x,y) = |x-y|$. Therefore, $[a,b]$ and, in particular, $[0,1]$ are Peano spaces.

Some other simple examples of Peano spaces are: a single point, the boundary of a rectangle or circle, the boundary of a rectangle or circle plus all of their interior points.

1.27 A mapping $f:S \rightarrow T$ is said to be continuous at a point a of S if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in N(a,\delta)$ in S then $f(x) \in N(f(a),\epsilon)$ in T, i.e., $f(N(a,\delta)) \subset N(f(a),\epsilon)$. The mapping f is continuous on S if it is continuous at every point of S. The following statements hold:

(1) A mapping $f:S \rightarrow T$ is continuous at a point p if and only if for every open set $G \subset T$ such that $f(p) \in G$, there is an open set $H \subset S$ such that $p \in H$ and $f(H) \subset G$.

(2) A mapping $f:S \rightarrow T$ is continuous on S if and only if for every open set $G \subset T$, $f^{-1}(G)$ is an open subset of S.

(3) A mapping $f:S \rightarrow T$ is continuous on S if and only if for every closed set $H \subset T$, $f^{-1}(H)$ is a closed subset of S.

(4) Suppose a mapping $f:S \rightarrow T$ is continuous at a point $p \in S$ and there is a sequence $\{x_n\}$ of S such that $x_n \rightarrow p$, then $f(x_n) \rightarrow f(p)$.

(5) Suppose there is a mapping $f:S \rightarrow T$ and a point $p \in S$. If every sequence $\{x_n\}$ of S is such that $x_n \rightarrow p$ implies $f(x_n) \rightarrow f(p)$ then f is continuous at p.

(6) Suppose a mapping $f:S \rightarrow T$ is continuous on S. If E is compact subset of S then $f(E)$ is compact.

(7) Suppose a mapping $f:S \rightarrow T$ is continuous on S and S is compact. If E is a closed subset of S then $f(E)$ is closed.

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(8) Suppose a mapping $f:S \to T$ is continuous on S. If E is a connected subset of S then $f(E)$ is connected.

(9) Suppose there is a continuous mapping f of S onto T. If S is compact and locally connected then T is compact and locally connected.

Thus, we see that if f is a continuous mapping of a Peano space S onto a metric space T, then T is a Peano space.

1.28 Let f be a one-to-one mapping of S onto T. Recall that for any point p of T there is exactly one point q of S such that $f(q) = p$. Let $g(p) = q$. This defines a mapping $g:T \to S$ such that $f(g(p)) = f(q) = p$. The mapping g is called the <u>inverse mapping of f</u>. It is easily verified that g is one-to-one and maps T onto S. The following statements hold:

(1) If $f:S \to T$ is a one-to-one continuous mapping of S onto T then there exists $g:T \to S$ which is a one-to-one mapping of T onto S.

(2) If $f:S \to T$ is a one-to-one continuous mapping of S onto T and S is compact then the inverse mapping g is continuous on T.

If $f:S \to T$ is a one-to-one continuous mapping of S onto T and if the inverse mapping g, in addition to being one-to-one and onto, is continuous on T then f is called a <u>topological</u> mapping of S onto T. The reader may easily verify that if f is a topological mapping of S onto T then open sets in S correspond to open sets in T and vice versa (open may be replaced by closed throughout).

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CHAPTER II

THE HAHN-MAZURKIEWICZ THEOREM

We have seen in section 1.27 that if a continuous mapping f maps a Peano space S onto a metric space T, then T is a Peano space. Indeed, this proves the necessary part of the Hahn-Mazurkiewicz Theorem that a space S is a Peano space if and only if the closed unit interval $[0,1]$ can be mapped continuously onto S. In this chapter, we shall prove the sufficiency part of the theorem after suitable introduction.

Definition 2.1 A chain of sets is a finite succession of sets $E_1, E_2, \ldots, E_k$ such that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1,2,\ldots,k-1$.

Theorem 2.2 Suppose Q is a connected set and F is the collection of sets $F_1, F_2, \ldots, F_n$ which form a finite closed covering of Q such that $Q \cap F_i \neq \emptyset$ for each i. Then any two sets F_j and F_k of F can be joined by a chain of sets in F.

Proof. Let F_k be an arbitrary set of the collection F. Let H_1 be the union of those sets which are joined to F_k by a chain of sets and let H_2 be the union of all the rest of the sets in F. Clearly $H_1 \cap H_2 = \emptyset$ and since $F_k \subset H_1$, $H_1 \neq \emptyset$. H_1 and H_2 are each a finite union of closed sets, hence are both closed. Now $Q \subset H_1 \cup H_2$, thus $Q = Q \cap (H_1 \cup H_2) = (Q \cap H_1) \cup (Q \cap H_2)$. Since Q is connected, we cannot have $Q = Q \cap H_1 / Q \cap H_2$, but notice:

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$$(\overline{Q\cap H_1})\cap(Q\cap H_2)\subset\bar{Q}\cap\bar{H}_1\cap Q\cap H_2$$
$$\subset\bar{H}_1\cap H_2$$
$$= H_1\cap H_2$$
$$= \emptyset.$$

Similarly, $(Q\cap H_1)\cap(\overline{Q\cap H_2}) = \emptyset$ and

$$(Q\cap H_1)\cap(Q\cap H_2) = Q\cap H_1\cap H_2$$
$$= \emptyset$$

and as we have seen $H_1\cap Q \neq \emptyset$. It follows that we must have $H_2\cap Q = \emptyset$ in order to preserve the connectedness of Q. Suppose $H_2 \neq \emptyset$, i.e., there is some set F_m of F such that $F_m \subset H_2$. By hypothesis $F_m\cap Q \neq \emptyset$, hence $H_2\cap Q \neq \emptyset$ which is impossible. Therefore, the assumption that $H_2 \neq \emptyset$ led to a contradiction, hence $H_2 = \emptyset$, i.e., every set F_i of F may be joined to F_k by a chain of sets in F. Q.E.D.

Corollary. Suppose Q is a connected set and F is the collection of sets $F_1, F_2, \ldots, F_n$ which form a finite closed covering of Q such that $Q\cap F_i \neq \emptyset$ for each i. Given any two sets F_j and F_k there is a chain of sets beginning with F_j and ending with F_k including all of the sets of F. (Repetitions are allowed.)

Proof. Let the sets of F be arranged so that

$$F_j = F_1, F_2, \ldots, F_n = F_k.$$

By theorem 2.2, F_r and F_{r+1} can be joined by a chain of sets for each r. Insert these chains between each of the original consecutive pairs and we have the desired chain. Q.E.D.

Theorem 2.3 If S is a compact locally connected space and $\epsilon>0$, then there is a finite ϵ-covering of S by compact connected sets.

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Proof. Consider all $\frac{1}{2}\epsilon$-neighborhoods of all points p of S. Each neighborhood is an open subset of the locally connected space S, hence their components form an open ϵ-covering of S. Due to the compactness of S we may select a finite number of these components $C_1, C_2, \ldots, C_k$ which also form a ϵ-covering of S. Their closures $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_k$ are connected, have diameters less than ϵ (see 1.11), and are closed subsets of S, hence are compact. Thus, the required covering is obtained. Q.E.D.

Theorem 2.4 (Hahn-Mazurkiewicz Theorem) The closed unit interval $[0,1]$ can be mapped continuously onto a metric space S if and only if S is compact, connected, and locally connected.

Proof. The necessary part follows immediately from section 1.27 and the fact that $[0,1]$ is a Peano space.

To prove the converse suppose S is a compact, connected, locally connected, metric space. Let a and b be any point of S. Without loss of generality assume $d(S) \leq 1$. We shall define a series of integers N_m and a series of 2^{-m}-coverings of S by chains of connected compact sets K_m^i for $i = 1,2,\ldots,N_m$. Such coverings and chains exist by virtue of theorem 2.3 and the corollary to theorem 2.2. Recall that since S is the whole space any covering $\mathcal{F}$ of S is such that $\bigcup_{F \in \mathcal{F}} F = S$.

For m = 0 we set $N_o = 1$ and $K_o^1 = S$. For m = 1 we have the $\frac{1}{2}$-covering by the chain $K_1^1, K_1^2, \ldots, K_1^{N_1}$. Let $a \in K_1^1$ and $b \in K_1^{N_1}$. Suppose we have completed the definition up to and including m with $a \in K_m^1$ and $b \in K_m^{N_m}$. Choose a point a_i in $K_m^i \cap K_m^{i+1}$ for

$$i = 1,2,\ldots,N_m-1,$$

letting $a = a_o$ and $b = a_{N_m}$.

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There is also a finite $2^{-(m+1)}$-covering of S by connected compact sets $F_1, F_2, \ldots, F_k$. Considering all i, r and m retain the sets $F_r \cap K_m^i$ which are non-empty. Clearly, these sets are finite in number, closed and also cover S, hence cover K_m^i. Therefore, for each m the sets $F_r \cap K_m^i$ can be arranged (with repetitions) as a chain of sets beginning with a set containing a_{i-1} and ending with one containing a_i. Suppose the greatest number of sets in any chain is n_m. By repeating the last set of a chain a sufficient number of times, we may assume all chains to contain exactly n_m sets. We now have chains of sets associated with $K_m^1, K_m^2, \ldots, K_m^{N_m}$, placed consecutively in this order. The union of these chains of course covers S, hence may be arranged to form a single chain of connected closed sets. We will number the sets of this single chain as K_{m+1}^i where $i = 1,2,\ldots,n_m N_m$ and $n_m N_m = N_{m+1}$. This completes our definition of K_m^i.

If integers j and i are such that $0 < j \leq N_m$ and

(1) $$n_m(j-1) < i \leq n_m j$$

then clearly $K_m^j \cap K_{m+1}^i \neq \emptyset$ and since $d(K_{m+1}^i) \leq 2^{-(m+1)}$ it follows that $K_{m+1}^i \subset N(K_m^j, 2^{-(m+1)})$. Thus, letting A_m^i denote $N(K_m^i, 2^{-m})$ we have $A_{m+1}^i \subset A_m^j$ whenever j and i satisfy (1).

For every number $0 \leq \xi < 1$ there must be an integer j such that $0 < j \leq N_m$ and

(2) $$\frac{(j-1)}{N_m} \leq \xi < \frac{j}{N_m}$$

Let $A_m(\xi)$ denote the set A_m^j for this particular value of j. $A_m(1)$ will be $A_m^{N_m}$ by definition. Suppose for this same ξ, $A_{m+1}(\xi) = A_{m+1}$, i.e.,

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$$(3) \qquad \frac{(i-1)}{N_{m+1}} \leq \xi < \frac{i}{N_{m+1}}$$

Since $N_{m+1} = n_m N_m$, we then have

$$\frac{(i-1)}{N_{m+1}} \leq \xi < \frac{j}{N_m} = \frac{n_m j}{N_{m+1}}$$

and

$$\frac{n_m(j-1)}{N_{m+1}} = \frac{(j-1)}{N_m} \leq \xi < \frac{i}{N_{m+1}}$$

hence, $i-1 < n_m j$, i.e., $i \leq n_m j$ and $n_m(j-1) < i$, i.e., $n_m(j-1) \leq i-1$. It now follows from (2) and (3) that

$$\frac{(j-1)}{N_m} \leq \frac{(i-1)}{N_{m+1}} \leq \xi < \frac{i}{N_{m+1}} \leq \frac{j}{N_m}.$$

Multiplying through by N_{m+1}, we notice that both i and j satisfy the relations (1). Thus, $A_{m+1}(\xi) \subset A_m(\xi)$ if $0 \leq \xi < 1$, and if $\xi = 1$ we also have $A_{m+1}(1) \subset A_m(1)$.

For any ξ consider the sets $\overline{A_m(\xi)}$, $m = 1,2,\ldots$. These sets are closed, hence compact and since $\overline{A_{m+1}(\xi)} \subset \overline{A_m(\xi)}$ for all m they all have at least one point in common. However, since $d(A_m(\xi)) \rightarrow 0$ as $m \rightarrow \infty$ they have exactly one point in common. We shall define this point as $f(\xi)$, i.e., $f(\xi) = \bigcap_{m=1}^{\infty} \overline{A_m(\xi)}$. Similarly, since $a \in A_m^1 = A_m(0)$ and $b \in A_m^{N_m} = A_m(1)$, we have $f(0) = \bigcap_{m=1}^{\infty} \overline{A_m(0)} = a$ and $f(1) = \bigcap_{m=1}^{\infty} \overline{A_m(1)} = b$ respectively. Thus, we have defined a mapping $f:[0,1] \rightarrow S$. To complete the proof we will show that f maps $[0,1]$ continuously onto S.

For every $\epsilon > 0$ there is an integer m such that $\epsilon > \frac{6}{2^m} > 0$. Supposing $|\xi - \eta| < \frac{1}{N_m}$ we have two cases to consider:

(i) If ξ and η belong to the same set A_m^j, then $A_m(\xi) = A_m(\eta) = A_m^j$ and $f(\xi)$ and $f(\eta)$ both belong to $\overline{A_m(\xi)} = \overline{A_m(\eta)}$. Therefore, $\rho(f(\xi), f(\eta)) \leq \frac{3}{2^m} < \epsilon$.

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(ii) Suppose ξ and η belong to consecutive sets. Without loss of generality assume $\xi<\eta$. Therefore, $f(\xi)\in\overline{A_m(\xi)} = \overline{A}_m^j$ and $f(\eta)\in\overline{A_m(\eta)} = \overline{A}_m^{j+1}$. Since $K_m^i\cap K_m^{i+1} \neq \emptyset$, $\overline{A}_m^j\cap\overline{A}_m^{j+1} \neq \emptyset$. Suppose $q\in\overline{A}_m^j\cap\overline{A}_m^{j+1}$. Then $\rho(f(\xi),q) \leq \frac{3}{2^m}$, $\rho(q,f(\eta)) \leq \frac{3}{2^m}$, and by the triangle inequality $\rho(f(\xi),f(\eta)) \leq \rho(f(\xi),q) + \rho(q,f(\eta)) \leq \frac{3}{2^m} + \frac{3}{2^m} = \frac{6}{2^m}<\epsilon$. Thus, in either case we have $\rho(f(\xi),f(\eta))<\epsilon$ if $|\xi-\eta|<\frac{1}{N_m}$, hence f is continuous.

Suppose a point $x\in A_m^{j+1}$. Since $f(\frac{j}{N_m})\in A_m(\frac{j}{N_m}) = A_m^{j+1}$, we have $\rho(x,f(\frac{j}{N_m})) \leq \frac{3}{2^m}$. That is, for any neighborhood $N(x,\epsilon)$ of x there is an integer m such that $\frac{3}{2^m}<\epsilon$ and $f(\frac{j}{N_m})\in N(x,\epsilon)$. Thus, the set of points $f(\frac{j}{N_m})$ for all m and all $j = N_m$ is dense in S and since S is the whole space, we have $\overline{f(\frac{j}{N_m})} = S$. But $\frac{j}{N_m}\subset[0,1]$ for all j and m, hence $f(\frac{j}{N_m})\subset f[0,1]$ and $f(\frac{j}{N_m})\subset\overline{f[0,1]}$. It follows that $S = \overline{f(\frac{j}{N_m})}\subset\overline{f[0,1]}$ S thus, $\overline{f[0,1]} = S$. But since $[0,1]$ is closed, $f[0,1]$ is closed. Finally, then we have $f[0,1] = \overline{f[0,1]} = S$, and f is an onto mapping. Q.E.D.

A similar discussion of the Hahn-Mazurkiewicz Theorem may be found in Newman's Elements of the Topology of Plane Sets of Points.

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CHAPTER III

SIMPLE ARCS AND ARCWISE CONNECTEDNESS

Definition 3.1 A point x of a connected set E is termed a <u>cut point</u> of E if E-x is disconnected. Otherwise x is called a <u>non-cut point</u>.

<u>Theorem 3.2 If S is connected but $S-a = H_1/H_2$ then $\overline{H}_1 = H_1 \cup a$ and $\overline{H}_2 = H_2 \cup a$.</u>

<u>Proof</u>. $S = H_1 \cup H_2 \cup a$, $H_1 \cap H_2 \cap a = \emptyset$ and since $\overline{H}_1 \cap H_2 = \emptyset$ we have $\overline{H}_1 \subset \complement H_2 = H_1 \cup a$ in S. Thus, $H_1 \subset \overline{H}_1 \subset H_1 \cup a$ and either $\overline{H}_1 = H_1$ or $\overline{H}_1 = H_1 \cup a$.

Suppose $\overline{H}_1 = H_1$. Now $H_1 \cap (H_2 \cup a) = \emptyset$, $\overline{H}_1 \cap (H_2 \cup a) = \emptyset$ and $H_1 \cap (\overline{H_2 \cup a}) = H_1 \cap (\overline{H}_2 \cup \overline{a}) = (H_1 \cap \overline{H}_2) \cup (H_1 \cap a) = \emptyset$, and clearly $H_1 \neq \emptyset$ and $(H_2 \cup a) \neq \emptyset$. Therefore, $S = H_1/H_2 \cup a$ contrary to the assumption that S is connected, hence $\overline{H}_1 \neq H_1$, i.e., $\overline{H}_1 = H_1 \cup a$. Similarly, $\overline{H}_2 = H_2 \cup a$. (It follows since $H_1 = \complement\overline{H}_2$ and $H_2 = \complement\overline{H}_1$ that H_1 and H_2 are open sets in S.) Q.E.D.

Definition 3.3. A space T is a <u>simple arc</u> if and only if there is a topological mapping f of the closed interval $[0,1]$ onto T. Thus, in view of 1.28 if T is a simple arc then open sets of T correspond to open sets of $[0,1]$ and vice versa (open may be replaced by closed throughout).

If A is a simple arc with end points a and b, for convenience we may often denote A by ab. Suppose points u and v lie on the simple arc ab. If E is the set consisting of u, v and all points

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x of ab which lie between u and v then the reader can easily verify that $E(=uv)$ is a simple arc and that $E \subset ab$. When such a situation exists we say that E is a <u>subarc</u> of ab.

In view of theorem 2.4 it is clear from definition 3.3 that a simple arc is a Peano space.

<u>Theorem 3.4 A continuum, X, of which all but at most two points are cut points is a simple arc</u>.

<u>Proof</u>. Let X be the given continuum and A the set of all non-cut points in X. Therefore, A has 0, 1 or 2 members and is not identical with X (see 1.23). Let x_o be a point of X not in A such that $X-x_o = P/Q$.

(1) We have seen in theorem 3.2 that $\underline{\overline{P} = P \cup x_o}$, $\underline{\overline{Q} = Q \cup x_o}$ and <u>P and Q are open sets</u>.

(2) <u>$\overline{P}$ and $\overline{Q}$ are connected</u>. For if not let $P = H_1/H_2$ and let $x_o \in H_1$. Now $X = H_1 \cup \overline{Q} \cup H_2$, $H_1 \cup \overline{Q} \neq \emptyset$, $H_2 \neq \emptyset$,

$$H_2 \cap (H_1 \cup \overline{Q}) = (H_2 \cap H_1) \cup (H_2 \cap \overline{Q}) = \emptyset,$$
$$\overline{H}_2 \cap (H_1 \cup \overline{Q}) = (\overline{H}_2 \cap H_1) \cup (\overline{H}_2 \cap \overline{Q}) = \emptyset,$$

and

$$H_2 \cap (\overline{H_1 \cup Q}) = H_2 \cap (\overline{H}_1 \cup \overline{Q}) = (H_2 \cap \overline{H}_1) \cup (H_2 \cap \overline{Q}) = \emptyset.$$

Thus, $X = H_1/H_2 \cup \overline{Q}$ contrary to the assumption that X is connected. Hence, $\overline{P}$ is connected and, similarly, $\overline{Q}$ is connected.

(3) <u>If $y \in P$ and P_1/Q_1 is an arbitrary partition of X-y, either P_1 or Q_1 but not both is contained in P</u>. Since $y \in P$, $\overline{Q} \subset X-y = P_1/Q_1$. Now $\overline{Q}$ is connected, hence by paragraph 1.22, either $\overline{Q} \subset P_1$ or $\overline{Q} \subset Q_1$. If $\overline{Q} \subset P_1$, we have $\overline{Q} \cap Q_1 = \emptyset$ and since $x_o \in \overline{Q}$, $x_o \notin Q_1$. Therefore, $Q_1 \subset X-x_o = Q \cup P$ and we must have $Q_1 \subset P$. Suppose also that $P_1 \subset P$. Thus, $P_1 \cup Q_1 \subset P$ and $X = P_1 \cup Q_1 \cup y \subset P$

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which is contrary to the assumption that $X-x_o = P/Q$, hence $P_1 \not\subset P$. Similarly, the assumption that $\overline{Q} \subset Q_1$ gives $P_1 \subset P$ and $Q_1 \not\subset P$.

(4) <u>P contains at least one non-cut point</u>. For if not then $\overline{P}(=P \cup x_o)$ also contains no non-cut points. Since $\overline{P}$ is closed, it is compact and hence contains an enumerable dense set of points $x_1, x_2, \ldots, x_n, \ldots$ (see 1.21) each different from x_o. Therefore, $x_1, x_2, \ldots, x_n, \ldots$ is an enumerable dense subset of P. Let $X-x_1 = P_1/Q_1$ and assume that P_1 is the part contained in P. We now inductively define a series of integers n_r and of sets P_r such that for all r, $X-x_{n_r} = P_r/Q_r$ and $P_r \subset P_{r-1} \subset \ldots \subset P$. Each P_r is a non-empty open (see part (1) above) subset of P and, therefore, contains a neighborhood which itself contains at least one of the points x_n. Let n_{r+1} be the least integer such that $x_{n_{r+1}} \in P_r$. Since $P_r \subset P$, $x_{n_{r+1}}$ is a cut point and thus $X-x_{n_{r+1}} = P_{r+1}/Q_{r+1}$, where again $P_{r+1} \subset P_r \subset \ldots \subset P$. It follows that $X_{n_{r+1}} \in P_r$ but $x_{n_{r+1}} \notin P_{r+1}$ and thus all of the integers n_r are distinct. This completes the definition.

We now have a decreasing sequence of compact sets $\overline{P}, \overline{P}_1, \overline{P}_2, \ldots, \overline{P}_r, \ldots$ such that $\overline{P}_{r+1} \subset \overline{P}_r$ for all r and the common part of all of them, P_∞, is non-empty (see 1.21). Now

$$\overline{P}_{s+1} = P_{s+1} \cup x_{n_{s+1}} \subset \overline{P}_s$$

and P_∞ is also the common part of all sets P_s. Let z be a point of P_∞. Since $z \in P$, it is a cut point. Therefore, $X-z = H_1/H_2$ and every P_m contains either H_1 or H_2 (see part (3) above). Let H_1 be the part contained in P_m for all m. Hence, $H_1 \subset P_\infty$. By part (1) H_1 is open and again contains a neighborhood which itself contains some point x_k of the set $x_1, x_2, \ldots, x_n, \ldots$. Letting n_{s+1}

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be the first of the suffixes $n_2, n_3, \ldots$ that exceeds k we have $x_k \in P_s$. But this implies that n_{s+1} is not the least integer such that $x_{n_{s+1}} \in P_s$ which is contrary to our earlier definition in the previous paragraph. Thus, the assumption that P contained no non-cut points led to a contradiction and we have at least one non-cut point in P. Similarly, Q contains at least one non-cut point. We have, therefore, proved that X has exactly two non-cut points. Denote them by a and b.

(5) <u>Suppose x is a cut point</u>. <u>Then X-x has two components, each containing one of the non-cut points</u>. Let $X-x = P/Q$. We have seen above that a and b belong to P and Q respectively. We shall prove that P and Q are components of X-x. Clearly, $P \neq \emptyset$, $Q \neq \emptyset$ and P and Q are both contained in X-x. If there is a connected set F such that $P \subset F \subset X-x$ then either $F \subset Q$ or $F \subset P$. Suppose $F \subset Q$. Then $P \subset F \subset Q$ which is impossible since $P \cap Q = \emptyset$. Thus, $F \subset P$ and we have $P = F$. Similarly, the assumption that a connected set G contains Q and is contained in X-x implies $Q = G$. We must yet show that P and Q are connected sets. Suppose $P = H_1/H_2$ and without loss of generality let $a \in H_1$. Then $a \notin H_2$ and since $b \in Q$, $b \notin H_2$. We will show that X-x has the separation $H_2/(X-x)-H_2$. Clearly, $H_2 \neq \emptyset$ and $(X-x) \cap \complement H_2 \neq \emptyset$. Also, the reader can easily verify that $H_2 \cup [(X-x) \cap \complement H_2] = X-x$ and $H_2 \cap [(X-x) \cap \complement H_2] = \emptyset$. Furthermore,

$$\begin{aligned}
\overline{H}_2 \cap [(X-x) \cap \complement H_2] &= \overline{H}_2 \cap [(H_1 \cup Q \cup H_2) \cap \complement H_2] \\
&= \overline{H}_2 \cap [(H_1 \cap \complement H_2) \cup (Q \cap \complement H_2)] \\
&= [\overline{H}_2 \cap (H_1 \cap \complement H_2)] \cup [\overline{H}_2 \cap Q \cap \complement H_2] \\
&\subset \emptyset \cup [\overline{P} \cap Q] \qquad (\text{since } H_2 \subset P) \\
&= \emptyset,
\end{aligned}$$

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and by similar reasoning $H_2 \cap \left[\overline{(X-x) \cap \mathcal{C}H_2}\right] = \emptyset$. Thus

$$X-x = H_2/(X-x)-H_2,$$

but this is impossible since H_2 contains neither a nor b, contrary to part (4). Therefore, the assumption that P is disconnected is false. Similarly, Q is connected.

Before continuing the proof of theorem 3.4, we shall set up an <u>order</u> in X. A relation $\prec$ will be termed a total ordering if it holds between certain elements of a set E such that the following conditions are satisfied.

(i) for no x is $x \prec x$;

(ii) if $x \neq y$ then either $x \prec y$ or $y \prec x$;

(iii) if $x \prec y$ and $y \prec z$ then $x \prec z$.

It is clear from (i) and (iii) that we cannot have both $x \prec y$ and $y \prec x$.

Consider any point x of X. Define L_x to be the component of $X-x$ containing a if $x \neq a$. If $x = a$ let $L_x = \emptyset$. Define R_x similarly with b replacing a throughout. No L_x shall contain b and no R_x contains a. Therefore, for any point x, $X = L_x \cup x \cup R_x$.

(6) <u>$x \in L_y$ if and only if $L_x \subset L_y$ but $L_x \neq L_y$</u>. First suppose $x \in L_y$, i.e., x is in the component of $X-y$ containing a. Since $x \notin L_x$ we have $L_x \neq L_y$ but since L_x is a component of $X-x$ also containing a we have $L_x \cap L_y \neq \emptyset$. The point y cannot be a since $L_y \neq \emptyset$ and $L_a = \emptyset$. It is always true that $b \notin L_x$ and $R_b = \emptyset$, hence if $y = b$ we have $L_x \subset X-b = L_b = L_y$, i.e., $L_x \subset L_y$. Suppose $y \neq b$. Thus, since $X-x = L_x/R_x$ and $x \in L_y$, we see by part (3) that either $L_x \subset L_y$ or $R_x \subset L_y$, but b belongs to R_x and not to L_y, hence again we must have $L_x \subset L_y$.

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To prove the converse suppose $L_x \subset L_y$ but $L_x \neq L_y$. Again the point y cannot be a for if it were then L_y would be empty and since $L_x \subset L_y$, we would also have $L_x = \emptyset$. Thus, $L_x = L_y$ contrary to our hypothesis. If $x = a$, $x \in L_y$ by definition. Suppose $x \neq a$. Since $X-x = L_x/R_x$, it follows from part (1) that

$$L_x \cup x = \overline{L}_x \subset \overline{L}_y = L_y \cup y,$$

i.e., $L_x \cup x \subset L_y \cup y$. But $x \neq y$ (since $L_x \neq L_y$), hence $x \in L_y$. Similarly, $x \in R_y$ if and only if $R_x \subset R_y$ but $R_x \neq R_y$.

For all points of X we shall define the relation $x \prec y$ to mean "$x \in L_y$." Thus, if $x \neq a$, $a \prec x$ and if $x \neq b$, $x \prec b$ and for no x does $x \prec a$ or $b \prec x$. The symbol $x \prec y$ may be read "x precedes y", and we shall use such expressions as "first point", "between" and "successor" accordingly.

(7) <u>The relation $\prec$ is a total ordering</u>. Condition (i) is obviously satisfied since $x \notin L_x$ by definition. To verify that condition (ii) is satisfied suppose $x \neq y$. We must show that either $x \prec y$ or $y \prec x$. Suppose $x \not\prec y$, i.e., $x \notin L_y$, hence

$$x \in X - L_y = R_y \cup y.$$

Since $x \neq y$ we have $x \in R_y$ and by virtue of part (6), this is equivalent to $R_x \subset R_y$. Thus,

$$L_y \cup y = \complement R_y \subset \complement R_x = L_x \cup x$$

i.e., $L_y \cup y \subset L_x \cup x$, and therefore, $L_y \subset L_x \cup x$. But $x \notin L_y$, hence $L_y \subset L_x$. Again by part (6) this means $y \in L_x$, i.e., $y \prec x$. Similarly, the assumption that $y \not\prec x$ would lead us to conclude that $x \prec y$. Verification of condition (iii) follows immediately from part (6) since if $x \prec y$ and $y \prec z$ we have $L_x \subset L_y \subset L_z$, i.e., $L_x \subset L_z$ which implies that $x \prec z$.

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The set $[x \prec p]$ shall denote the set of all points x which precede a fixed point p. Clearly, then $[x \prec p] = L_p$ and by part (1) $[x \prec p]$ is an open set. The set $[p \prec x]$ is the set of all points x which p precedes. $[p \prec x]$ is the complement of $[x \prec p]$ in X-p, i.e., $[p \prec x] = R_p$ and, therefore, it is also open. Thus, the intersection $[p \prec x \prec q]$ of two such sets for points p and q is an open set. We will denote it by $\prec p,q \succ$.

(8) <u>If $p \prec q$ the set $\prec p,q \succ$ is not empty</u>. Suppose $\prec p,q \succ$ is empty. We will show that this assumption leads to the contradiction that X has the partition $L_p \cup p / R_q \cup q$. Clearly, $X = L_p \cup p \cup R_q \cup q$, $L_p \cup p \neq \emptyset$ and $R_q \cup q \neq \emptyset$. $L_p \cup p$ and $R_q \cup q$ have no point in common since $p \prec q$ implies $p \neq q$, $p \notin R_q$ and $q \notin L_p$. Thus, if the sets had a common point it would have to belong to both L_p and R_q which contradicts condition (iii). Now $[p \prec x] = R_p$ which is open. But $\complement R_p = L_p \cup p$, hence $L_p \cup p$ is closed. Similarly, $[x \prec q]$ is the open set L_q and since $\complement L_q = R_q \cup q$, we have $R_q \cup q$ closed. It readily follows that

$$(\overline{L_p \cup p}) \cap (R_q \cup q) = \emptyset$$

and

$$(L_p \cup p) \cap (\overline{R_q \cup q}) = \emptyset.$$

We now have our partition, hence $\prec p,q \succ \neq \emptyset$.

Since compact sets are separable (1.21) and X is compact, there is an enumerable dense set $C = (y_1, y_2, \ldots)$ in X. We want C to contain neither a nor b. To show this is possible, let p be an arbitrary point of X different from both a and b. Since X is open there is an $\epsilon > 0$ such that $N(p,\epsilon) \subset X$. If $N(p,\epsilon)$ contains neither a nor b then since C is dense in X there is a point y_r of C, $y_r \neq a$,

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$y_r \neq b$ and $y_r \in N(p,\epsilon)$. Suppose $N(p,\epsilon)$ contains a and not b (if $N(p,\epsilon)$ contained b and not a an entirely similar discussion would follow). Thus, $\rho(p,a) = \lambda$ where $\lambda > 0$ and $a \notin N(p,\lambda)$ and $b \notin N(p,\lambda)$. There is, therefore, a point y_j of C such that $y_j \in N(p,\lambda)$, $y_j \neq a$ and $y_j \neq b$.

Suppose $N(p,\epsilon)$ contains both a and b, thus $\rho(p,a) = \lambda$ and $\rho(p,b) = \mu$ where $\lambda > 0$ and $\mu > 0$. Let $\delta = \lfloor \lambda,\mu \rfloor$. Therefore, $N(p,\delta)$ contains a point y_m of C where $y_m \neq a$ and $y_m \neq b$.

Suppose $p = a$. (A similar discussion would follow if $p = b$). Since X is open, there is an $\epsilon > 0$ such that $N(a,\epsilon) \subset X$. Suppose $b \notin N(a,\epsilon)$. $N(a,\epsilon) \neq a$ since $N(a,\epsilon)$ is open and the single point a is closed. That is, if $N(a,\epsilon) = a$, $N(a,\epsilon)$ would be both open and closed, hence $X-N(a,\epsilon)$ would be both open and closed. Thus, X would be the union of two non-empty disjoint open sets $N(a,\epsilon)$ and $X-N(a,\epsilon)$. Therefore, $X = N(a,\epsilon)/X-N(a,\epsilon)$ by 1.22 which is impossible since X is connected. Hence, there is a point d of $N(a,\epsilon)$ and $d \neq a$, i.e., $\rho(d,a) = \alpha$ where $\alpha > 0$. Let $\gamma = \lfloor \alpha, \epsilon-\alpha \rfloor$. Now $N(d,\gamma) \subset N(a,\epsilon)$, $a \notin N(d,\gamma)$ and there is a point y_n of C such that $y_n \in N(d,\gamma)$, i.e., $y_n \in N(a,\epsilon)$ and $y_n \neq a$ and $y_n \neq b$. If $b \in N(a,\epsilon)$ then $N(a,\epsilon)$ contains a and b but $N(a,\epsilon) \neq a \cup b$, for if $N(a,\epsilon) = a \cup b$ we would, by the same reasoning as above, again have the connected set $X = N(a,\epsilon)/X-N(a,\epsilon)$ which is impossible. Therefore, there is a point $c \in N(a,\epsilon)$, $c \neq a$, $c \neq b$, $\rho(a,c) = \theta$ where $\theta > 0$, and $\rho(b,c) = \beta$ where $\beta > 0$. Let $\sigma = \lfloor \theta,\beta,\epsilon-\theta \rfloor$. Now $N(c,\sigma) \subset N(a,\epsilon)$, $a \notin N(c,\sigma)$ and $b \notin N(c,\sigma)$. Thus, there is a point y_i of C such that $y_i \in N(c,\sigma)$, $y_i \neq a$ and $y_i \neq b$. Thus, every neighborhood of an arbitrary point p contains a point of C different from a and b, i.e., $C-(a \cup b)$ is

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dense in X. But $C-(a \cup b)$ is a subset of the enumerable set C, hence $C-(a \cup b)$ is also enumerable (see 1.18). Let $E_o = C-(a \cup b)$ and let the points of E_o be $x_1, x_2, \ldots$.

If $p \prec q$, the non-empty open set $\prec p,q \succ$ contains at least one point of E_o, thus, in particular, there is a point of E_o between any two points of E_o itself.

Let $\alpha_1, \alpha_2, \ldots,$ be any enumeration of the rational points of the open interval (0,1). We now construct two sequences

$$y_1, y_2, \ldots \text{ of points of } E_o$$
$$\beta_1, \beta_2, \ldots \text{ of rational points of } (0,1),$$

as follows. Let $y_1 = x_1$, and $\beta_1 = \alpha_1$. Suppose that $y_1, y_2, \ldots, y_{n-1}$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ have been defined. If n is even let y_n be the point x_k of lowest k not already in the set $y_1, y_2, \ldots, y_{n-1}$, and let β_n be the α_k of lowest k having the same relations to $\beta_1, \beta_2, \ldots, \beta_{n-1}$, relative to $<$, as y_n has to $y_1, y_2, \ldots, y_{n-1}$ relative to $\prec$. That is, if y_n is such that

$$\cdots \prec y_g \prec y_n \prec y_h \prec \cdots$$

the β_n is such that

$$\cdots < \beta_g < \beta_n < \beta_h < \cdots.$$

If n is odd we reverse the roles of x's and α's, y's and β's: β_n will be the rational number α_k of lowest k not among the set $\beta_1, \beta_2, \ldots, \beta_{n-1}$, and y_n the point x_k of lowest k with the appropriate $\prec$-relations. That is, if β_n is such that

$$\cdots < \beta_e < \beta_n < \beta_f < \cdots$$

then y_n is such that

$$\cdots \prec y_e \prec y_n \prec y_f \prec \cdots.$$

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We are assured of such a point y_n existing since $y_e \prec y_f$ implies $\prec y_e, y_f \succ \neq \emptyset$.

We shall now define a mapping f. It is clear that every point x_i appears once and only once as a y_j and every rational number α_i once and only once as a β_j. Define $f(y_i)$ to be β_i. Since E_o is an enumerable set and the rational numbers of (0,1) are also enumerable we see that f is a one-to-one mapping of E_o onto the rationals of (0,1). In addition to this, we have so defined y_n and β_n so that f is an order preserving mapping, i.e., if $y_r \prec y_s$ then $f(y_r) \prec f(y_s)$.

Let us now define a subset Λ of E_o to be a <u>section</u> if

(o_1) Λ has no last point,

(o_2) and if $x \in \Lambda$ all predecessors of x in E_o also belong to Λ.

(Clearly, Λ may be either the empty set or the whole of E_o). The analyogy with sections of rational numbers in (0,1) is obvious.

(9) <u>Let Λ be a section of E_o and K the set of all points of X not followed by any point of Λ</u>. <u>Then K has a first point</u>. K cannot be empty since it contains b. If K = X our required first point is a. Therefore, suppose K $\neq$ X. Consider the set X-K and let x be one of its points. If x $\neq$ a there is a point y of Λ following x, and $\prec a,y \succ$ is an open set containing x and contained in X-K. Thus, there is a $\epsilon > 0$ such that $N(x,\epsilon) \subset \prec a,y \succ \subset X-K$, i.e., X-K is open. Similarly, if x = a we may again use the point y of Λ and we have $x \in L_y \subset X-K$. L_y is open, hence there is a $\delta > 0$ such that $N(x,\delta) \subset L_y \subset X-K$. Thus, in either case X-K is an open set. Suppose K has no first point. We shall show that this

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assumption leads to the contradiction that the connected set X has the partition K/X-K. Clearly, $K \neq \emptyset$, $X-K \neq \emptyset$ and $K \cap (X-K) = \emptyset$. Since K has no first point, K is open for if $x \in K$ there is a point y of K such that $y \prec x$ and y follows all points of Λ. Now $\prec y,b \succ$ is an open set contained in K and containing x, hence there is an $\epsilon_1 > 0$ such that

$$N(x,\epsilon_1) \subset \prec y,b \succ \subset K.$$

Therefore, K is open and X is the union of two non-empty disjoint open sets K and X-K, hence X = K/X-K (see 1.22).

We have now shown that for every section of E_0 there corresponds a unique point of X, namely the first point of K, which we will call the point determined by the section. It follows immediately that points determined by different sections are different, for if $x \in \Lambda_1$ but $x \notin \Lambda_2$ then x precedes the point determined by Λ_1 but does not precede the point determined by Λ_2. It is also true that every point x of X is determined by a section and different points are determined by different sections. To verify this let Λ be the subset of E_0 consisting of all points of E_0 which precede x. Again let K be the set of points of X not followed by any point of Λ. Clearly, $x \in K$ but we must show that x is the first point of K. Suppose $y \prec x$ and $y \in K$. There is a point z of E_0 such that $z \in \prec y,x \succ$, i.e., $y \prec z \prec x$. Hence, $z \in \Lambda$ and $y \prec z$ which contradicts the assumption that $y \in K$. Thus, x is the first point of K and is determined by Λ. Suppose x and y are different points of X and are determined by sections Λ_x and Λ_y respectively. Without loss of generality assume $y \prec x$. Again there is a point z of E_0 such that $z \in \prec y,x \succ$, i.e., $y \prec z \prec x$.

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Now $y \prec z$, hence $z \notin \Lambda_y$ but $z \prec x$ implies $z \in \Lambda_x$ and we have $\Lambda_x \neq \Lambda_y$. It is now clear that we have set up a one-to-one correspondence between the sections of E_0 and the points of X.

This result enables us to extend the mapping f of E_0 onto the rationals of (0,1), to be a mapping of X onto all points of (0,1). Suppose x is any point of X and Λ_x is the section determining it. Since f is order preserving it clearly maps Λ_x onto a section of the rationals of (0,1). Define f(x) to be the real number determined by this section. Letting f(a) = 0 and f(b) = 1 it follows that f is a one-to-one mapping of X onto [0,1] and if $y \in E_0$ the new definition of f(y) agrees with the old.

(10) <u>The mapping f, so extended, is order preserving</u>, i.e., if $x \prec y$ then $f(x) < f(y)$. Suppose x and y are points of X and $x \prec y$. If both x and y belong to E_0 it follows from the original definition of f that $f(x) < f(y)$. Suppose $x \in E_0$ but $y \notin E_0$. Since $x \prec y$, $x \in \Lambda_y$ and it is always true that $y \notin \Lambda_y$. Now $f(x) \in f(\Lambda_y)$ and there is a point z of E_0 such that $z \in \prec x,y \succ$, i.e., $x \prec z \prec y$ and $z \in \Lambda_y$, hence $f(x) < f(z)$ and $f(z) \in f(\Lambda_y)$. By definition f(y) is the first real number of (0,1) not followed by any number in $f(\Lambda_y)$. Clearly then $f(y) \not< f(x)$ and we must have either $f(x) = f(y)$ or $f(x) < f(y)$. If $f(x) = f(y)$ it follows from $f(x) \in f(\Lambda_y)$ that $f(y) \in f(\Lambda_y)$ which is impossible since $y \notin \Lambda_y$. Therefore, $f(x) < f(y)$. Suppose $y \in E_0$ but $x \notin E_0$. Since $x \prec y$, $y \notin \Lambda_x$. Let z be any point of Λ_x, i.e., $z \in E_0$ and $z \prec x \prec y$, hence $f(z) \in f(\Lambda_x)$ and $f(z) < f(y)$. Thus, f(y) is greater than every number in $f(\Lambda_x)$. If $f(y) < f(x)$, $y \in E_0$ implies that $f(y) \in f(\Lambda_x)$, but this is impossible since $y \notin \Lambda_x$.

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Therefore, $f(x) \leq f(y)$. There is a point w of E_o such that $w \in \prec x,y \succ$, i.e., $x \prec w \prec y$. Now the only restrictions we have on y is that $y \in X$, $x \prec y$ and $y \in E_o$. In view of the fact that w also satisfies these restrictions, we have $f(x) \leq f(w)$. However, $f(w) < f(y)$ since both w and y belong to E_o. Hence $f(x) < f(y)$. Suppose neither x nor y belongs to E_o. Again there is a point z of E_o and $x \prec z \prec y$, i.e., $z \in \Lambda_y$ but $z \notin \Lambda_x$. In a manner entirely similar to the previous case we have $f(x) < f(z)$. Also, $f(y) \not< f(z)$ for if it were, $f(z)$ would not belong to $f(\Lambda_y)$ which contradicts the fact that $z \in E_o$ and $z \in \Lambda_y$. Therefore, $f(z) \leq f(y)$, i.e., $f(x) < f(y)$.

(11) <u>f is a topological mapping of X onto [0,1]</u>. We have seen that f is a one-to-one mapping of X onto $[0,1]$. To prove f is continuous on X we will show that if G is any open set in $[0,1]$ then $f^{-1}(G)$ is an open set in X (paragraph 1.27). Since $[0,1]$ is locally connected, components of G are open sets. Now G is equal to the union of its components, hence $f^{-1}(G)$ is equal to the union of the inverse images of the components of G. Thus, if the components of G map onto open sets in X it will follow that $f^{-1}(G)$ is the union of open sets hence is itself an open set (paragraph 1.6). To begin, it is clear that since components are connected sets, the components of G must be open intervals (open in $[0,1]$). The typical open intervals in $[0,1]$ are

$$(\gamma,\delta),\ (\gamma,1],\ \text{and}\ [0,\gamma).$$

Let $D = (\gamma,\delta)$. Since f is one-to-one, there are points t and u of X such that $f(t) = \gamma$ and $f(u) = \delta$. Clearly, $f(t) < f(u)$ and since f is order preserving $t \prec u$. Recall that $\prec t,u \succ$ is by

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definition the open set $L_u \cap R_t$ in X (part 7). We will show that

$$f^{-1}(D) = L_u \cap R_t$$

thus proving $f^{-1}(D)$ is open in X. Suppose p is any point of $f^{-1}(D)$. There is a point q of D such that $f(p) = q$. Thus

$$f(t) = \gamma < q < \delta = f(u),$$

i.e.,

$$f(t) < f(p) < f(u)$$

which implies that $t \prec p \prec u$, hence $p \in L_u \cap R_t$. Suppose a point r belongs to $L_u \cap R_t$, i.e., $t \prec r \prec u$, and hence

$$\gamma = f(t) < f(r) < f(u) = \delta$$

Letting $f(r) = s$ we have $\gamma < s < \delta$, i.e., $s \in D$ which implies $f^{-1}(s) \in f^{-1}(D)$. But $f^{-1}(s) = f^{-1}f(r) = r$, hence $r \in f^{-1}(D)$ and we have $f^{-1}(D) = L_u \cap R_t$.

Let $E = (\gamma, 1]$. Again $f(t) = \gamma$ where $t \in X$. We will show $f^{-1}(E)$ equals the open set R_t in X. Suppose m is any point of $f^{-1}(E)$. There is a point c of E such that $f(m) = c$. Therefore,

$$f(t) = \gamma < c = f(m),$$

i.e.,

$$f(t) < f(m)$$

which implies $t \prec m$ hence $m \in R_t$. Let n be any point of R_t, i.e., $t \prec n$. Therefore,

$$\gamma = f(t) < f(n)$$

hence $f(n) \in E$ and it follows that $n \in f^{-1}(E)$.

Letting $F = [0, \gamma)$ it can be shown in a manner entirely similar to the previous case that $f^{-1}(F)$ is equal to the open set L_t in X. Thus, in very instance the inverse image of a component of G is an open set hence G is open and f is continuous on X.

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Letting g be the mapping f^{-1} and since X is compact, we have in g a continuous one-to-one mapping of $[0,1]$ onto X (paragraph 1.28). Therefore, f is a topological mapping of X onto $[0,1]$. This result enables us to state that X is a simple arc (paragraph 3.3) and the theorem is proved. Q.E.D.

Corollary. a and b are the end points.

Theorem 3.5 Every continuum has at least two points that are non-cut points. (Proved in part 4 of 3.4)

For a similar discussion of theorems 3.2 and 3.4, the reader may refer to Newman's Elements of the Topology of Plane Sets of Points.

Definition 3.6 A set of points E is said to be irreducible with respect to a given property P provided the set E has property P but no non-empty closed proper subset of E has property P.

Definition 3.7 A property P is said to be inducible provided that when each set of a monotone decreasing sequence $A_1, A_2, \ldots$ of compact sets has property P then so also does their intersection $A = \bigcap_{i=1}^{\infty} A_i$. As an example, the property of being non-empty is inducible for the intersection of a monotone decreasing sequence of non-empty compact sets is itself non-empty (1.21).

Theorem 3.8 (Brouwer Reduction Theorem) If property P is inducible and K is a non-empty compact space having property P, then there is a non-empty compact subset Q of K such that Q is irreducible with respect to property P.

Proof. (We shall assume throughout that K itself is not irreducible with respect to property P.) Let $R_1, R_2, \ldots$ be an enumerable base for K (1.21). Let n_1 be the least integer such

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that K contains a non-empty compact set K_1, $K_1 \neq K$, K_1 has property P and $K_1 \cap R_{n_1} = \emptyset$. To assure us that such a set exists, recall that K is not irreducible with respect to property P. This implies there exists a set K_1 such that $K_1 \neq \emptyset$, $K_1 \subset K$, $K_1 \neq K$ and K_1 is compact and has property P. Thus, there is a point a_1 of $K-K_1$, i.e., $a_1 \in \complement K_1$. Since K_1 is closed, $\complement K_1$ is open. Therefore, $\complement K_1$ is the union of some of the sets R_m, $m = 1,2,\ldots$, and a_1 belongs to at least one of these, say R_s. Now $a_1 \in R_s \subset \complement K_1$, i.e., $K_1 \cap R_s = \emptyset$. We therefore have at least one set of our base which is disjoint from K_1 and we let n_1 be the least integer such that $K_1 \cap R_{n_1} = \emptyset$. Let n_2 be the least integer greater than n_1 such that K_1 contains a non-empty compact set K_2, $K_2 \neq K_1$, K_2 has property P and $K_2 \cap R_{n_2} = \emptyset$. We assume there does exist a set K_2 such that $K_2 \neq \emptyset$, $K_2 \subset K_1$, $K_2 \neq K_1$ and K_2 is compact and has property P, for if not, then K_1 has property P irreducibly and the theorem is proved. In addition, there is a point a_2 of K_1-K_2, i.e., a_2 belongs to the open set $\complement K_2$. And, since $\complement K_2$ is the union of some of the sets R_m, $m = 1,2,\ldots$, there is at least one set R_t contained in $\complement K_2$ such that $a_2 \in R_t$, i.e., $a_2 \in R_t \subset \complement K_2$ hence $K_2 \cap R_t = \emptyset$. Also, $t > n_1$. To show this, notice that since $a_2 \in K_1$ and $a_2 \in R_t$ we have $K_1 \cap R_t \neq \emptyset$. If $t = n_1$ this implies $K_1 \cap R_{n_1} \neq \emptyset$ which is impossible since $K_1 \cap R_{n_1} = \emptyset$. Thus, $t \neq n_1$. Nor can t be less than n_1 for recall that $K_2 \neq \emptyset$, K_2 is compact, K_2 has property P and $K_2 \cap R_t = \emptyset$, and since $K_2 \subset K_1$ we have $K_2 \subset K$ and $K_2 \neq K$. That is, K_2 is a set satisfying all of the appropriate properties that K_1 satisfied, and if $t < n_1$ we see that n_1 was not the least integer such that some set satisfied these properties. Thus again we have a contradiction,

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hence $s > n_1$. We, therefore, conclude that there is at least one integer (namely t) greater than n_1 such that $K_2 \cap R_t = \emptyset$. Hence there is certainly a least integer n_2 greater than n_1 such that $K_2 \cap R_{n_2} = \emptyset$.

In general let n_i be the least integer greater than n_{i-1} such that K_{i-1} contains a non-empty compact set K_i, $K_i \neq K_{i-1}$, K_i has property P and $K_i \cap R_{n_i} = \emptyset$. And, we assume there does exist a set K_i such that $K_i \neq \emptyset$, $K_i \subset K_{i-1}$, $K_i \neq K_{i-1}$ and K_i is compact and has property P, for if not, then K_{i-1} has property P irreducibly and the theorem is proved. Now by the same reasoning as before there is a point a_i of $K_{i-1} - K_i$, i.e., a_i belongs to the open set $\complement K_i$ which in turn is the union of some of the sets R_m, $m = 1,2,\ldots.$ Thus, there is at least one set R_u such that $a_i \in R_u \subset \complement K_i$, hence $K_i \cap R_u = \emptyset$. Also, $u > n_{i-1}$, for if not then there are three possibilities to consider.

(i) Suppose $u = n_j$ where $j \leq i-1$. Following our general procedure $K_{i-1} \subset K_j$ and $a_i \in K_{i-1} \cap R_u$, hence $K_{i-1} \cap R_u \neq \emptyset$. But with $u = n_j$ we have

$$K_{i-1} \cap R_u \subset K_j \cap R_u = K_j \cap R_{n_j},$$

i.e., $K_j \cap R_{n_j} \neq \emptyset$ which is impossible since we know that $K_j \cap R_{n_j} = \emptyset$.

(ii) Suppose $n_{j-1} < u < n_j$ where $j \leq i-1$. Recall that $K_i \neq \emptyset$, K_i is compact, K_i has property P and $K_i \cap R_u = \emptyset$. Since $K_i \subset K_{i-1} \subset K_j \subset K_{j-1}$ we also have $K_i \subset K_{j-1}$ and $K_i \neq K_{j-1}$. That is, K_i, in conjunction with u satisfies all of the appropriate properties satisfied by K_j, in conjunction with n_j, and since $n_{j-1} < u < n_j$ we see that n_j is not the least integer greater than n_{j-1} such that some set has these properties. Thus again we have a contradiction.

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(iii) Suppose $u < n_1$. By the same reasoning as in (ii), K_i, in conjunction with u satisfies all of the appropriate properties satisfied by K_1, in conjunction with n_1, and since $u < n_1$, n_1 is not the least integer such that some set satisfies these properties. Therefore, $u \not< n_1$.

Thus in each case we have a contradiction, hence $u > n_{i-1}$ and we conclude that there is at least one integer (namely u) greater than n_{i-1} such that $K_i \cap R_u = \emptyset$. Hence there certainly is a least integer n_i such that $K_i \cap R_{n_i} = \emptyset$.

We now have a sequence $\{K_i\}$ of sets such that for all i; $K_i \neq \emptyset$, $K_i \subset K_{i-1} \subset \ldots \subset K$, $K_i \neq K_{i-1}$, K_i is compact, has property P and $K_i \cap R_{n_i} = \emptyset$. Let $Q = \bigcap_{i=1}^{\infty} K_i$. Each K_i is compact, hence closed. Thus Q is a closed subset of the compact space K, hence Q is compact. The sequence $\{K_i\}$ is a decreasing sequence of compact sets. Therefore, their common part Q is non-empty (1.21). Since P is inducible and each set K_i has property P so also does their intersection Q. Now if Q does not contain a non-empty closed proper subset which also has property P, then Q is irreducible relative to property P and hence is our required set. To show this is the case suppose there is a subset S of Q such that $S \neq \emptyset$, $S \neq Q$, S is closed and has property P. There is a point p of Q-S, i.e., $p \in \complement S$ which is open, hence is the union of some of the sets R_m, m = 1,2,.... In particular, there is a set R_v of the basis such that $p \in R_v \subset \complement S$, thus $Q \cap R_v \neq \emptyset$ and $S \cap R_v = \emptyset$. There are four possibilities for v.

(o_1) If $v > n_i$ for all i then there clearly is a least integer $v_1 > n_i$ for all i. Thus S satisfies all of the appropriate properties required in order to belong to the sequence $\{K_i\}$.

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Therefore, $Q \subset S$ and since $S \subset Q$ we have $S = Q$ which is contrary to our assumption that $S \neq Q$.

(o_2) If $v = n_j$ where $j \leq i$ then $S \cap R_{n_j} = \emptyset$ and $Q \cap R_{n_j} \neq \emptyset$. But $Q \subset K_j$ and $K_j \cap R_{n_j} = \emptyset$, hence $Q \cap R_{n_j} = \emptyset$. This is impossible, hence $v \neq n_j$ where $j \leq i$.

(o_3) Suppose $n_{j-1} < v < n_j$ where $j \leq i$. Recall that $S \neq \emptyset$, S is compact, S has property P and $S \cap R_v = \emptyset$. Since $S \subset Q$, we have $S \subset K_{j-1}$ and $S \neq K_{j-1}$. Thus, S, in conjunction with v satisfies all of the properties that K_j satisfies, in conjunction with n_j, and since $n_{j-1} < v < n_j$ we see that n_j is not the least integer greater than n_{j-1} such that some set has these properties. This, of course, contradicts the way n_j was set up.

(o_4) Suppose $v < n_1$. By the same reasoning as in (o_3), S, in conjunction with v satisfies all of the appropriate properties satisfied by K_1, in conjunction with n_1. But $v < n_1$, hence n_1 is not the least integer such that some set has these properties.

In each case we have arrived at a contradiction, hence no such set S exists and Q is irreducible with respect to property P. Therefore, Q is our required set. Q.E.D.

The name <u>arcwise connected</u> is given to a set H provided every two points of H can be joined by a simple arc lying entirely in H. Leaning heavily on the Hahn-Mazurkiewicz Theorem (2.4) we are now able to prove the Arcwise Connectedness Theorem.

<u>Theorem 3.9 (Arcwise Connectedness) Every two points a and b of a compact, connected, locally connected space S can be joined in S by a simple arc.</u>

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Proof. Let $I = [0,1]$. In view of theorem 2.4 there is a continuous mapping f of I onto S such that $f(0) = a$ and $f(1) = b$. We shall say that a closed subset F of I has property P provided $a \cup b \subset f(F)$ and if (xy) is any maximal segment of I-F, then $f(x) = f(y)$. Clearly, I has property P.

We will show that property P is inducible. Let $F_1, F_2, \ldots$ be a monotone decreasing sequence of sets in I, i.e., $F_{n+1} \subset F_n$ for all n, such that each F_n is compact and has property P. Let $F = \bigcap_{n=1}^{\infty} F_n$ and let (xy) be a maximal open segment in I-F. Now $a \cup b \subset f(F_n)$ for all n, hence $a \cup b \subset \bigcap_{n=1}^{\infty} f(F_n)$. But

$$\bigcap_{n=1}^{\infty} f(F_n) \subset f(\bigcap_{n=1}^{\infty} F_n) = f(F),$$

for since $a \in \bigcap_{n=1}^{\infty} f(F_n)$ there is a point x_n of F_n such that $f(x_n) = a$ for all n. Thus we have a sequence of points $\{x_n\}$ in the compact set I. Hence there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \rightarrow z_o$ where $z_o \in I$. Since $\{F_n\}$ is a monotone decreasing sequence of sets, if $m \geq n$ we have $x_{k_m} \in F_{k_m} \subset F_m \subset F_n$ and since $x_{k_n} \rightarrow z_o$, $z_o \in \overline{F}_n$ (1.9). But each F_n is closed, hence $z_o \in F_n$. By letting m be $\geq$ n for each n we have $z_o \in F_n$ for all n, i.e., $z_o \in \bigcap_{n=1}^{\infty} F_n = F$. Now since f is continuous and $x_{k_n} \rightarrow z_o$, $f(x_{k_n}) \rightarrow f(z_o)$. But $f(x_n) = a$ for all n, thus the sequence $\{f(x_n)\}$ converges to a. And since $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$, $\{f(x_{k_n})\}$ is a subsequence of $\{f(x_n)\}$ hence $f(x_{k_n}) \rightarrow a$ (1.8). Therefore, $f(z_o) = a$ (1.7) and since $z_o \in F$, $f(z_o) \in f(F)$, i.e., $a \in f(F)$. Similarly, since $b \in \bigcap_{n=1}^{\infty} f(F_n)$, $b \in f(F)$ and thus $a \cup b \subset f(F)$. Now $\mathcal{C}F = \mathcal{C}\bigcap_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \mathcal{C}F_n$, i.e., $I-F = \bigcup_{n=1}^{\infty}(I-F_n)$. Therefore, for any point d of (xy) there is a k such that $d \in I-F_k$. F_k is closed, hence $I-F_k$ is open and there is an $\epsilon > 0$ such that

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$N(d,\epsilon) \subset I-F_k$. Let x_k be the first point of F_k going from d to x and let y_k be the first point of F_k going from d to y. Thus $(x_k y_k)$ is a maximal segment of $I-F_k$, $x_k \leq d-\epsilon$, $y_k \geq d+\epsilon$ and since F_k has property P, $f(x_k) = f(y_k)$. Similarly, if x_{k+1} and y_{k+1} are the first points of F_{k+1} going from d to x and y respectively, $(x_{k+1} y_{k+1})$ is a maximal segment of $I-F_{k+1}$, $f(x_{k+1}) = f(y_{k+1})$ and

$$x \leq x_{k+1} \leq x_k < y_k \leq y_{k+1} \leq y.$$

Continuing this process we obtain $(x_{k+j} y_{k+j})$ as a maximal segment of $I-F_{k+j}$, $f(x_{k+j}) = f(y_{k+j})$, and

$$x \leq x_{k+j} \leq \ldots \leq x_k < y_k \leq \ldots \leq y_{k+j} \leq y.$$

Clearly, x is a lower bound for the sequence $\{x_{k+j}\}$ and y is an upper bound for the sequence $\{y_{k+j}\}$. We shall show that these sequences converge to x and y respectively. Suppose not, i.e., suppose $\{x_{k+j}\}$ converges to a point q of $q > x$. Since $q \in (xy)$, $q \in I-F = \bigcup_{n=1}^{\infty}(I-F_n)$, thus there is an h such that $q \in I-F_h$. $I-F_h$ is open, hence there is a $\lambda_1 > 0$ such that $N(q,\lambda_1) \subset I-F_h$. Let λ be less than the minimum of λ_1 and $y_h - q$. Thus $N(q,\lambda) \subset I-F_h$. Since the sequence $x_k, x_{k+1}, \ldots$ converges to q, there is an m such that if $s \geq m$ then $q \leq x_s < q+\lambda$. Let r equal the maximum of h and m, i.e., $r \geq h$ and $r \geq m$. Therefore, $q \leq x_r < q+\lambda$ and (x_r, y_r) is a maximal segment of $I-F_r$. In addition to this since $r \geq h$ we have $\lambda < y_r - q$, thus $N(q,\lambda) \subset I-F_r$. Now $(x_r y_r)$ is a non-empty interval, hence is a non-empty connected set. Therefore, there is only one component of $(x_r y_r)$, namely $(x_r y_r)$ itself (1.24). It is easily verified that

$$(q-\lambda, y_r) = (q-\lambda, q+\lambda) \cup (x_r, y_r),$$

and since $q \leq x_r$, this implies that (x_r, y_r) is a proper subset of $(q-\lambda, y_r)$. But recall $N(q,\lambda) \subset I-F_r$ hence $(q-\lambda, y_r) \subset I-F_r$ and since

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$(q-\lambda, y_r)$ is an interval, it is a connected set. This is impossible, for $(x_r y_r)$ is a component of $I-F_r$ hence cannot be the proper subset of a connected set of $I-F_r$. Thus we have a contradiction, i.e., $\{x_{k+j}\}$ does not converge to a point $q > x$. Therefore, $x_{k+j} \rightarrow x$. Similarly $y_{k+j} \rightarrow y$ and since f is continuous we have $f(x_{k+j}) \rightarrow f(x)$ and $f(y_{k+j}) \rightarrow f(y)$. But $f(x_{k+j}) = f(y_{k+j})$ for all k + j, hence $f(x) = f(y)$ (1.7) and property P is inducible.

Now I is a non-empty compact set having property P, thus by the Brouwer Reduction Theorem (3.8) there is a non-empty compact subset A of I which is irreducible with respect to property P. Let $f(A) = T$. We shall show that T is our required simple arc in S joining a and b.

T is compact (1.27). T is also connected, for if not then T has a separation E_1/E_2. Since T is closed $\overline{E}_1 \subset \overline{T} = T$ and

$$\begin{aligned}\overline{E}_1 = \overline{E}_1 \cap T &= \overline{E}_1 \cap (E_1 \cup E_2)\\ &= (\overline{E}_1 \cap E_1) \cup (\overline{E}_1 \cap E_2)\\ &= E_1,\end{aligned}$$

i.e., E_1 is closed. Similarly, E_2 is closed and by the continuity of f, both $A \cap f^{-1}(E_1)$ and $A \cap f^{-1}(E_2)$ are closed sets. They are also disjoint since

$$f^{-1}(E_1) \cap f^{-1}(E_2) = f^{-1}(E_1 \cap E_2) = f^{-1}(\emptyset) = \emptyset.$$

Furthermore, $A = (A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2))$ for if c is a point of A then $f(c) \in T = E_1 \cup E_2$. Without loss of generality let $f(c) \in E_1$. Therefore, $c \in f^{-1}(E_1)$, i.e., $c \in (A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2))$, and clearly, if d is a point of $(A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2))$, then $d \in A$, hence we have the equality. Thus, there is a maximal segment (xy) in I-A such that $x \in f^{-1}(E_1)$ and $y \in f^{-1}(E_2)$. But this implies that

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$f(x) \in E_1$ and $f(y) \in E_2$ which is impossible since $E_1 \cap E_2 = \emptyset$ but A has property P, i.e., in particular, $f(x) = f(y)$.

Let $p \in T-(a \cup b)$ and $P' = A \cap f^{-1}(p)$. $P' \neq \emptyset$ since $p \in f(A)$ implies there is a point t of A such that $f(t) = p$, i.e., $t \in f^{-1}(p)$, thus $t \in A \cap f^{-1}(p)$. Similarly, $A \cap f^{-1}(a)$ and $A \cap f^{-1}(b)$ are non-empty sets. Let a_o be the first point of $[0,1]$ in A. If $a_o = 0$ then $f(a_o) = f(0) = a$. If $a_o > 0$ then $[0,a_o)$ is a maximal segment of I-A and again $f(a_o) = f(0) = a$. Thus in either case $f(a_o) = a$, i.e., $a_o \in f^{-1}(a)$ hence $a_o \in A \cap f^{-1}(a)$. Similarly, if b_o is the last point of $[0,1]$ in A we have $f(b_o) = b$ and $b_o \in f^{-1}(b)$ hence $b_o \in A \cap f^{-1}(b)$. It is readily seen that $A \subset [a_o,b_o]$ and $a_o \notin P'$ and $b_o \notin P'$.

Let p_1 be the first point of P' in $[a_o,b_o]$ and let p_2 be the last point of P' in $[a_o,b_o]$. If $p_1 = p_2$ then $P' = p_1 = p_2$. If $p_1 \neq p_2$ then $p_1 < p_2$. Suppose there is a point p_3 of A such that $p_1 < p_3 < p_2$. Let

$$D = A \cap ([a_o,p_1] \cup [p_2,b_o]).$$

D is a proper subset of A since $p_3 \in A$ but $p_3 \notin D$, and $(a_o \cup b_o) \subset D$ thus $(a \cup b) \subset f(D)$. Let (xy) be a maximal segment of I-D. If (xy) is $[0,a_o)$ then $f(0) = a = f(a_o)$, i.e., $f(x) = f(y)$. Similarly, if (xy) is $(b_o,1]$, $f(b_o) = b = f(1)$ and $f(x) = f(y)$. If (xy) is (p_1,p_2) then since p_1 and p_2 belong to P', $p_1 = f^{-1}(p)$ and $p_2 = f^{-1}(p)$, i.e., $f(p_1) = p = f(p_2)$ and again $f(x) = f(y)$. Suppose $a_o \leq x < y \leq p_1$ and let w be an arbitrary point of (x,y). Then $w \in I-D$ and $w \in [a_o,p_1] \cup [p_2,b_o]$. But $w \notin D$, hence $w \notin A$, i.e., $w \in I-A$. Thus $(x,y) \subset I-A$ and since (xy) is a maximal segment of I-D, x and y belong to D which implies that x and y belong to A. Therefore,

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(xy) is a maximal segment of I-A and $f(x) = f(y)$. Similarly, if $p_2 \leq x < y \leq b_o$, $f(x) = f(y)$. Now in every case $f(x) = f(y)$, hence D has property P. This is impossible since D is a proper subset of A and A has property P irreducibly. Thus, the assumption that there is a point p_3 of A such that $p_1 < p_3 < p_2$ is false and we see that $P' = p_1 \cup p_2$ where p_1 may equal p_2.

Let $A_1 = A \cap [a_o, p_1)$ and $A_2 = A \cap (p_2, b_o]$. Now

$$A_1 \cup P' = (A \cap [a_o, p_1]) \cup p_2$$

and

$$A_2 \cup P' = (A \cap [p_2, b_o]) \cup p_1.$$

Let $T_1 = f(A_1)$ and $T_2 = f(A_2)$. We shall show that $T - p = T_1/T_2$ thus showing that any arbitrary point p of $T-(a \cup b)$ is a cut point of T. First it is easily verified that

$$[(A \cap [a_o, p_1]) \cup p_2] \cup [(A \cap [p_2, b_o]) \cup p_1] = A,$$

i.e., $A = A_1 \cup A_2 \cup P'$. Therefore, $T = f(A) = f(A_1) \cup f(A_2) \cup f(P') = T_1 \cup T_2 \cup p$. Now, $p \notin T_1$ for if it did then $p \in f(A_1)$ hence there is a point s of A_1 such that $f(s) = p$. Now p_1 and p_2 do not belong to A_1 hence $s \neq p_1$ and $s \neq p_2$. Therefore, $s \notin P'$, i.e.,

$$s \in \mathcal{C}P' = \mathcal{C}(A \cap f^{-1}(p)) = \mathcal{C}A \cap \mathcal{C}f^{-1}(p).$$

Hence $s \in \mathcal{C}A$. But since $s \in A_1$, $s \in A$ and we have a contradiction. Thus, $p \notin T_1$. Similarly, $p \notin T_2$ and since $T' = T_1 \cup T_2 \cup p$ we have $T - p = T_1 \cup T_2$.

Clearly $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$. To show that $T_1 \cap T_2 = \emptyset$ suppose $q \in T_1 \cap T_2$. Thus, $q \in f(A_1)$ and $q \in f(A_2)$, and there are points q_1 of A_1 and q_2 of A_2 such that $f(q_1) = q = f(q_2)$ where $q_1 \neq p_1$ and $q_2 \neq p_2$ since $p_1 \notin A_1$ and $p_2 \notin A_2$. Let $E = A \cap ([a_o, q_1] \cup [q_2, b_o])$. Now the identical reasoning by which we showed that D was a proper

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subset of A and had property P, may be used to conclude that E also is a proper subset of A and has property P. Again this is impossible, hence the assumption that $T_1 \cap T_2 \neq \emptyset$ is false, i.e., $T_1 \cap T_2 = \emptyset$. Clearly, $A_1 \cup P'$ is closed, hence $f(A_1 \cup P')$ is closed (1.27). But

$$f(A_1 \cup P') = f(A_1) \cup f(P') = T_1 \cup p,$$

thus $T_1 \cup p$ is closed. Since $T_1 \subset T_1 \cup p$ we have

$$\overline{T}_1 \subset \overline{T_1 \cup p} = T_1 \cup p,$$

but $p \notin T_2$ hence $\overline{T}_1 \cap T_2 = \emptyset$. Similarly, $T_1 \cap \overline{T}_2 = \emptyset$ and we have shown that any point p of $T-(a \cup b)$ is a cut point of T. In summary, $a \cup b \subset T$, $T \subset S$ and we have just shown that T is a continuum of which all but at most two points (namely a and b) are cut points. Thus by theorem 3.4 T is a simple arc. Q.E.D.

A similar presentation of the Brouwer Reduction and Arcwise Connectedness Theorems may be found in Whyburn's Analytic Topology.

Since every Peano space satisfies the hypotheses of theorem 3.9 we see that every Peano space is arcwise connected, and with the aid of 1.22, part (6), the reader may easily verify the following theorem.

<u>Theorem 3.10 Every arcwise connected space is connected</u>.

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CHAPTER IV

SIMPLE CLOSED CURVES AND PROPERTY S

The orientation and presentation of the definitions and theorems of this chapter are similar to those found in Hall and Spencer's Elementary Topology.

Definition 4.1 The <u>unit circle</u> is the subspace of R_2 consisting of all points (x,y) that satisfy the equation $x^2 + y^2 = 1$. (Throughout this chapter we shall let J denote the unit circle.)

We have seen that J is a Peano space (1.26). In addition, it is easily verified that every point of J is a non-cut point and if a and b are any two points of J then $J-(a \cup b)$ is not connected.

Definition 4.2 A space C is a <u>simple closed curve</u> if and only if there is a topological mapping f of the unit circle J onto C. Thus in view of paragraph 1.28 if C is a simple closed curve then open sets of C correspond to open sets of J and vice versa (open may be replaced by closed throughout).

We now see that just as the simple arc is related to the unit interval $[0,1]$ in R_1, so is the simple closed curve related to the unit circle in R_2.

Definition 4.3 Suppose a and b are distinct points of S and F_1 and F_2 are simple arcs in S, each having a and b as its end points. Then we say that F_1 and F_2 are <u>independent arcs from a to b</u> if and only if $F_1 \cap F_2 = a \cup b$.

<u>Theorem 4.4 Suppose C is a non-degenerate space. Then C is</u>

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a simple closed curve if and only if for every two distinct points a and b of C, the space C may be expressed as the union of two independent arcs from a to b.

Proof. First suppose C is a simple closed curve and let a and b be distinct points of C. There is a topological mapping f of the unit circle J onto C. Let $a' = f^{-1}(a)$ and $b' = f^{-1}(b)$. Therefore, a' and b' belong to J and since f is one-to-one, $a' \neq b'$. Clearly, there exists two independent arcs E_1 and E_2 in J joining a' and b', i.e., $J = E_1 \cup E_2$ and $E_1 \cap E_2 = a' \cup b'$. Let $f(E_1) = F_1$ and $f(E_2) = F_2$. Now

$$\begin{aligned} f(E_1) \cap f(E_2) &= f(E_1 \cap E_2) \\ &= f(a' \cup b') \\ &= f(a') \cup f(b') \\ &= a \cup b, \end{aligned}$$

i.e., $F_1 \cap F_2 = a \cup b$ and obviously $F_1 \cup F_2 = C$.

To show the converse let a and b be distinct points of C and suppose C is the union of two independent arcs F_1 and F_2 in C joining a and b, i.e., $C = F_1 \cup F_2$ and $F_1 \cap F_2 = a \cup b$. We must show that C is a simple closed curve. Let $I = [0,1]$. Since F_1 and F_2 are simple arcs there exist topological mappings f_1 of F_1 onto I and f_2 of F_2 onto I such that $f_1(a) = 0$, $f_2(a) = 0$, $f_1(b) = 1$ and $f_2(b) = 1$. Clearly, J can be expressed as the union of two independent arcs J_1 and J_2 joining the points $(-1,0)$ and $(1,0)$. Therefore, there exist topological mappings g_1 of I onto J_1 and g_2 of I onto J_2 such that $g_1[(1,0)] = (1,0)$, $g_2[(1,0)] = (1,0)$, $g_1[(0,0)] = (-1,0)$ and $g_2[(0,0)] = (-1,0)$. We shall now define a

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mapping f of C onto J. If $x \in F_1$ let $f(x) = g_1[f_1(x)]$ and if $x \in F_2$ let $f(x) = g_2[f_2(x)]$. It readily follows that f is a topological mapping of C onto J, hence C is a simple closed curve. Q.E.D.

Corollary. A space C is a simple closed curve if C contains two distinct points a and b and two independent arcs F_1 and F_2 from a to b such that $C = F_1 \cup F_2$.

Theorem 4.5 A space C is a simple closed curve if and only if C is a compact connected space such that $C-(a \cup b)$ is not connected.

Proof. Suppose C is a simple closed curve. Then there exists a topological mapping f of the unit circle J onto C, thus $f(J) = C$ is compact and connected (1.27). Suppose a and b are points of C. We must show that $C-(a \cup b)$ is not connected. Since f is a topological mapping there is a continuous, one-to-one mapping g of C onto J, hence there are points u and v of J such that $g(a) = u$ and $g(b) = v$. Suppose $C-(a \cup b)$ is connected. Therefore, $g[C-(a \cup b)]$ is connected, but notice

$$\begin{aligned} g[C-(a\cup b)] &= g[C \cap \mathcal{C}(a \cup b)] \\ &= g(C) \cap g\mathcal{C}(a \cup b) \\ &= J \cap \mathcal{C}g(a \cup b) \\ &= J \cap \mathcal{C}(g(a) \cup g(b)) \\ &= J \cap \mathcal{C}(u \cup v) \\ &= J-(u \cup v), \end{aligned}$$

i.e., $g[C-(a \cup b)] = J-(u \cup v)$ but $J-(u \cup v)$ is not connected (4.1). Thus, the assumption that $C-(a \cup b)$ is connected is false, i.e., $C-(a \cup b)$ is not connected.

To show the converse suppose C is a compact connected space such that for every pair of points a and b of C the set $C-(a \cup b)$ is

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disconnected. First, we shall show that every point of C is a non-cut point of C. Suppose otherwise, i.e., suppose there is a point z of C such that C-z is disconnected. Thus C-z has some separation P/Q. Now $\overline{P} = P \cup z$, $\overline{Q} = Q \cup z$ and $\overline{P}$ and $\overline{Q}$ are connected sets (3.4). Since they are closed subsets of the compact space C, $\overline{P}$ and $\overline{Q}$ are also compact. Thus each set $\overline{P}$ and $\overline{Q}$ has at least two non-cut points (3.5). Therefore, there exists points e and d such that $e \neq z$, $d \neq z$, $e \in \overline{P}-z$, $d \in \overline{Q}-z$ and both of the sets $\overline{P}-e$ and $\overline{Q}-d$ are connected. It follows that

$$\begin{aligned} C-(e \cup d) &= (\overline{P} \cup \overline{Q}) \cap \mathcal{C}(e \cup d) \\ &= [\overline{P} \cap \mathcal{C}(e \cup d)] \cup [\overline{Q} \cap \mathcal{C}(e \cup d)] \\ &= [\overline{P} \cap \mathcal{C}e \cap \mathcal{C}d] \cup [\overline{Q} \cap \mathcal{C}e \cap \mathcal{C}d] \\ &= [\overline{P} \cap \mathcal{C}e] \cup [\overline{Q} \cap \mathcal{C}d] \\ &= (\overline{P}-e) \cup (\overline{Q}-d). \end{aligned}$$

But $\overline{P}-e$ and $\overline{Q}-d$ clearly have the point z in common, hence

$$(\overline{P}-e) \cup (\overline{Q}-d) = C-(e \cup d)$$

is a connected set which is a contradiction since $C-(e \cup d)$ is disconnected. Thus every point of C is a non-cut point.

Recall that a and b are arbitrary points of C and that there exists a separation $C-(a \cup b) = A/B$. By a proof similar to that of theorem 3.4 it is easily seen that $\overline{A} = A \cup a \cup b$ and $\overline{B} = B \cup a \cup b$. Now $\overline{A}$ and $\overline{B}$ are closed, $\overline{A} \cup \overline{B} = C$, a connected set, and $\overline{A} \cap \overline{B} = a \cup b$ hence by paragraph 1.22 $\overline{A}$ or $\overline{B}$ is connected. Without loss of generality let $\overline{A}$ be the connected set. We shall show that $\overline{B}$ is also connected, for if not then there exists a separation $\overline{B} = B_1/B_2$. There are two distinct cases to consider.

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First, if $a \in B_1$ and $b \in B_1$ the reader can easily verify that C has the separation $(\bar{A} \cup B_1)/B_2$. But this is impossible since C is connected. Similarly, if $a \in B_2$ and $b \in B_2$ we arrive at the contradiction that C is disconnected.

Therefore, suppose $a \in B_1$ and $b \in B_2$. Since we have $B \neq \emptyset$, $a, b \notin B$, $\bar{B}$ must contain at least three points. Let B_2 be the set with at least two points (a similar discussion would follow if B_1 were the set with at least two points). It can now be easily verified that

$$C-b = (\bar{A}-b) \cup B_1/B_2-b.$$

This is, of course, impossible since no point of C is a cut point. Similarly, if $a \in B_2$ and $b \in B_1$ we arrive at a contradiction.

Therefore, both sets $\bar{A}$ and $\bar{B}$ are connected. Furthermore, since they are closed subsets of the compact space C, they are compact. We shall show that each of these sets is an independent arc from a to b. First, suppose that neither set is an arc from a to b. Since each set $\bar{A}$ and $\bar{B}$ is a continuum, they both have at least two non-cut points (3.5). Thus, if neither a nor b, or if just one of the points a or b is a non-cut point of $\bar{A}$ then there is a point $r \in \bar{A}-(a \cup b)$ such that $\bar{A}-r$ is connected. Now if both a and b are non-cut points of $\bar{A}$ the non-cut point r of $\bar{A}-(a \cup b)$ still exists. For if not then by theorem 3.4 $\bar{A}$ is a simple arc, and by the corollary to theorem 3.4, a and b are its end points which contradicts our assumption that $\bar{A}$ was not an arc from a to b. Thus we exclude this possibility and let r of $\bar{A}-(a \cup b)$ be such that $\bar{A}-r$ is connected. Similarly, there is a point $s \in \bar{B}-(a \cup b)$ such that $\bar{B}-s$ is connected. Since $\bar{A}-r$ and $\bar{B}-s$ have the common points a and b,

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$(\overline{A}-r) \cup (\overline{B}-s)$ is a connected set. But notice

$$(\overline{A}-r) \cup (\overline{B}-s) = (\overline{A} \cup \overline{B}) \cap (\mathcal{C}r \cup \overline{B}) \cap (\overline{A} \cup \mathcal{C}s) \cap (\mathcal{C}r \cup \mathcal{C}s)$$
$$= C \cap \mathcal{C}r \cap \mathcal{C}s \cap C$$
$$= C-(r \cup s),$$

i.e., $C-(r \cup s)$ is connected which is impossible. Thus, one of the sets $\overline{A}$ or $\overline{B}$ is an arc from a to b. Without loss of generality let $\overline{A}$ be this set. Therefore, a and b are the only non-cut points of $\overline{A}$. Let w be any point of $\overline{A}-(a \cup b)$. Now w is a cut point, hence $\overline{A}-w$ consists of exactly two components A_1 and A_2 containing a and b respectively (3.4, part (5)). Suppose $\overline{B}$ is not an arc from a to b. Then as before, there is a point s of $\overline{B}-(a \cup b)$ such that $\overline{B}-s$ is connected and $(a \cup b) \subset \overline{B}-s$. Since $b \in A_2$ and $b \in (\overline{B}-s)$, $A_2 \cup (\overline{B}-s)$ is the union of two connected sets which meet, thus is itself a connected set. Also $a \in A_1$ and $a \in [A_2 \cup (\overline{B}-s)]$ hence $A_1 \cup A_2 \cup (\overline{B}-s)$ is the union of two connected sets, thus is itself connected. But $A_1 \cup A_2 \cup (\overline{B}-s) = (\overline{A}-w) \cup (\overline{B}-s)$ and as we have seen above

$$(\overline{A}-w) \cup (\overline{B}-s) = C-(w \cup s).$$

This is impossible since $(\overline{A}-w) \cup (\overline{B}-s)$ is connected and $C-(w \cup s)$ is disconnected. Finally, then both $\overline{A}$ and $\overline{B}$ are arcs from a to b and since $\overline{A} \cap \overline{B} = a \cup b$, they are independent arcs. Also $C = \overline{A} \cup \overline{B}$ and it follows from the corollary to theorem 4.4 that C is a simple closed curve. Q.E.D.

We have seen that the closed unit interval $[0,1]$ is a Peano space. Consider all neighborhoods of every point in $[0,1]$. Since these neighborhoods are intervals, they form a basis of connected sets for $[0,1]$ such that the closure of every non-empty element of this basis is itself a Peano space. In view of the Hahn-Mazurkiewicz

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Theorem, we are inclined to believe that every Peano space has a basis with such properties. Indeed every Peano space has a basis of connected sets such that the closure of every non-empty element of the basis is compact and connected, but the closures in such a basis are not necessarily locally connected. However, with the aid of Seirpinski's property S we shall prove that in every Peano space there does exist a basis with these properties.

Definition 4.6 Suppose H is a subset of the space T. We shall say that H has property S if and only if, for every $\epsilon>0$, H can be expressed as the union of a finite number of connected sets, each having diameter less than ϵ. Clearly, then if H has property S, H is bounded.

Definition 4.7 Suppose H is a subset of the space T. H is termed totally bounded if and only if, for every $\epsilon>0$, there exists a set of points $x_1, x_2, \ldots, x_n$ of T which are finite in number and such that $H \subset \bigcup_{i=1}^{n} N(x_i, \epsilon)$.

Theorem 4.8 If a subset H of the space T has property S then H is totally bounded.

Proof. Suppose $H \subset T$ and H has property S. Letting $\epsilon>0$ we have $H = \bigcup_{i=1}^{n} C_i$ where each set C_i is connected and has diameter less than ϵ. Now, there is a point x_i of C_i for i = 1,2,...,n and $C_i \subset N(x_i, \epsilon)$ for all i. Thus, $H \subset \bigcup_{i=1}^{n} N(x_i, \epsilon)$, i.e., H is totally bounded. Q.E.D.

Theorem 4.9 Suppose T is a space. If A is the union of a finite number of subsets of T each having property S then A has property S.

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Proof. Suppose A is the union of a finite number of subsets K_i, $i = 1,2,\ldots,n$ of T such that each K_i has property S. Thus $A = \bigcup_{i=1}^{n} K_i$ and for every $\epsilon>0$ and every i, $K_i = \bigcup_{j=1}^{m} E_j^i$ where for each i and j, E_j^i is connected, $d(E_j^i)<\epsilon$ and $\bigcup_{j=1}^{m} E_j^i$ is finite for $i = 1,2,\ldots,n$. Therefore,

$$A = \bigcup_{i=1}^{n} K_i = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_j^i$$

and since the union of a finite set of finite sets if finite, A has property S. Q.E.D.

Theorem 4.10 A space T is locally connected if and only if for every point x of T and every open set G containing x there is an open set H containing x such that H is contained in a single component of G.

Proof. Suppose T is locally connected, x is any point of T and G is an open set containing x. By virtue of paragraph 1.25 there is an open connected set H containing x and contained in G. Now $x \in G$, hence x belongs to some component K of G. Therefore, $H \cap K \neq \emptyset$ which implies $H \subset K$ (1.24). It follows since components are either equal or disjoint that K is the only component of G which contains H.

To show the converse let G be an open set of T and let K_n be any component of G. If x is any point of K_n then $x \in T$, hence by supposition there is an open set H containing x and contained in a single component K_m of G. Since H is open, there is an $\epsilon>0$ such that $N(x,\epsilon) \subset H$, hence

$$N(x,\epsilon) \subset H \subset K_m.$$

Now $x \in K_m$ and $x \in K_n$, i.e., K_n and K_m are two components with a

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common point, hence $K_n = K_m$. Thus $N(x,\epsilon) \subset K_n$ and we see that for any point x of K_n there is a neighborhood of x contained in K_n, i.e., K_n is an open set. Since K_n is an arbitrary component of an arbitrary component of an arbitrary open set G we conclude that components of open sets are open, hence T is locally connected (1.25). Q.E.D.

<u>Theorem 4.11 Suppose K is a subset of the space T. If K has property S, then K is locally connected</u>.

<u>Proof</u>. Let x be any point of K and $\epsilon > 0$. To show that K is locally connected it shall be sufficient to show there is a $\delta > 0$ such that if $y \in N(x,\delta) \cap K$ then there exists a connected set C such that $x \cup y \subset C$ and $C \subset N(x,\epsilon) \cap K$ (1.25).

K has property S, hence $K = \bigcup_{i=1}^{n} G_i$ where each G_i is connected and has diameter less than $\frac{1}{2}\epsilon$. Since $x \in K$, there is some set G_r of $\bigcup_{i=1}^{n} G_i$ such that $x \in G_r$ and, therefore, $x \in \overline{G}_r$. Let

(1) $$x \in \overline{G}_i \text{ for } i = 1,\ldots,j$$

(2) $$x \notin \overline{G}_i \text{ for } i = j+1,\ldots,n.$$

If $j = n$ then $x \in \overline{G}_i$ for all i. Let $\delta = \frac{1}{2}\epsilon$ and let y be a point of $N(x,\delta) \cap K$. Thus $y \in K$, hence $y \in G_k$ for some k, and $y \in \overline{G}_k$. But $x \in \overline{G}_i$ for all i, hence $x \in \overline{G}_k$. Now then $x \cup y \subset \overline{G}_k$ and G_k is connected, hence $\overline{G}_k$ is connected. Since $G_k \subset G_k \cup x \subset \overline{G}_k$ we see that $G_k \cup x$ is connected and contains both x and y. Furthermore,

$$d(G_k \cup x) = d(\overline{G}_k) = d(G_k) < \tfrac{1}{2}\epsilon.$$

To show that $G_k \cup x \subset N(x,\epsilon) \cap K$, let $z \in G_k \cup x$. Thus $z \in K$ and

$$\rho(z,x) \leq d(G_k \cup x) < \tfrac{1}{2}\epsilon,$$

i.e., $z \in N(x,\frac{1}{2}\epsilon)$ and we have

$$z \in N(x,\tfrac{1}{2}\epsilon) \cap K \subset N(x,\epsilon) \cap K.$$

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Therefore, $G_k \cup x \subset N(x,\epsilon) \cap K$. Hence if $j = n$, K is locally connected.

If $j \neq n$ then (1) and (2) imply $x \in \mathcal{C}\overline{G}_i$ for $i = j+1,\ldots,n$. But each set $\mathcal{C}\overline{G}_i$ is open, hence there is a δ_i for $i = j+1,\ldots,n$, such that $N(x,\delta_i) \subset \mathcal{C}\overline{G}_i$, i.e., $N(x,\delta_i) \cap \overline{G}_i = \emptyset$, for $i = j+1,\ldots,n$. Let $\delta = \lfloor \delta_{j+1},\ldots,\delta_n \rfloor$ and $y \in N(x,\delta) \cap K$. Thus, $y \in \mathcal{C}\overline{G}_i$ for $i = j+1,\ldots,n$ hence y belongs to some G_s where $1 \leqq s \leqq j$. But $x \in \overline{G}_i$ for $i = 1,\ldots,n$ hence $x \cup y \subset \overline{G}_s$. As before, G_s is connected and since $G_s \subset G_s \cup x \subset \overline{G}_s$, $G_s \cup x$ is connected. Also $d(G_s \cup x) < \frac{1}{2}\epsilon$, and if we let $z \in G_s \cup x$ then $z \in K$ and $\rho(z,x) < \frac{1}{2}\epsilon$, i.e., $z \in N(x,\frac{1}{2}\epsilon)$. Hence we have $z \in N(x,\epsilon) \cap K$, i.e., $G_s \cup x \subset N(x,\epsilon) \cap K$ and again K is locally connected. Q.E.D.

Corollary. If a space T has property S, then T is locally connected.

Theorem 4.12 Suppose K is a compact subset of a space T. Then K has property S if and only if K is locally connected.

Proof. By theorem 4.11 if K has property S then K is locally connected. Thus, suppose K is locally connected. Let x be a point of K and $\epsilon > 0$. Consider $\frac{\epsilon}{3}$. The local connectedness of K implies there is a $\delta_x > 0$ such that if $y \in N(x,\delta_x) \cap K$ then there exists a connected set C_y containing x and y and $C_y \subset N(x,\frac{\epsilon}{3}) \cap K$. Let $D_x = \bigcup C_y$ where $\bigcup C_y$ is taken with respect to all points y of $N(x,\delta_x) \cap K$. Now each set C_y is connected and contains the point x hence $\bigcup C_y$ is connected, i.e., D_x is connected. Furthermore, since each $C_y \subset N(x,\frac{\epsilon}{3}) \cap K$, $D_x \subset N(x,\frac{\epsilon}{3}) \cap K$, thus $D_x \subset K$ and clearly, $d(D_x) \leqq \frac{2}{3}\epsilon < \epsilon$. Also $y \in N(x,\delta_x) \cap K$ implies $y \in \bigcup C_y = D_x$. Thus we have $N(x,\delta_x) \cap K \subset D_x$ for any x of K. Since T is open every point x

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of K has a neighborhood $N(x,\delta_x)$ contained in T. Therefore, $K\subset\bigcup_{x\in K}N(x,\delta_x)$, i.e., $\bigcup N(x,\delta_x)$ for all x of K is a covering of K. But K is compact, hence

$$K\subset\bigcup_{i=1}^{n}N(x_i,\delta_{x_i}).$$

Now then

$$\begin{aligned} K &= \left[\bigcup_{i=1}^{n}N(x_i,\delta_{x_i})\right]\cap K \\ &= \bigcup_{i=1}^{n}\left[N(x_i,\delta_{x_i})\cap K\right] \\ &\subset \bigcup_{i=1}^{n}D_{x_i}, \end{aligned}$$

i.e., $K\subset\bigcup_{i=1}^{n}D_{x_i}$ but $D_x\subset K$ for each x of K, hence $\bigcup_{i=1}^{n}D_{x_i}\subset K$ and we have $K=\bigcup_{i=1}^{n}D_{x_i}$. Furthermore, we have seen above that each D_{x_i} is connected and has diameter less than ϵ hence K has property S. Q.E.D.

<u>Corollary. If a space T is compact then T has property S if and only if T is locally connected</u>.

In view of this corollary, the following theorem is immediate.

<u>Theorem 4.13 Every Peano Space has property S</u>.

<u>Theorem 4.14 Suppose K is a subset of the metric space T and there is a set H of T such that $K\subset H\subset\overline{K}$. Then if K has property S, H has property S</u>.

<u>Proof</u>. Let $\epsilon>0$. Since K has property S, $K=\bigcup_{i=1}^{n}G_i$ where each set G_i is connected and has diameter less than ϵ. Now $\overline{K}=\overline{\bigcup_{i=1}^{n}G_i}=\bigcup_{i=1}^{n}\overline{G_i}$ and since each G_i is connected each $\overline{G_i}$ is connected. Also $d(\overline{G_i})=d(G_i)<\epsilon$. (Thus we have proved that if K has property S then $\overline{K}$ has property S). Since $K\subset H\subset\overline{K}$ we have

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$$\begin{aligned} H &= H \cap \overline{K} \\ &= H \cap \left(\bigcup_{i=1}^{n} \overline{G}_i \right) \\ &= \bigcup_{i=1}^{n} (H \cap \overline{G}_i). \end{aligned}$$

Furthermore, $G_i \subset \overline{G}_i$ and $G_i \subset K \subset H$ for each i hence $G_i \subset H \cap \overline{G}_i \subset \overline{G}_i$. Therefore, $H \cap \overline{G}_i$ is a connected set for each i (1.22) and since $d(H \cap \overline{G}_i) \leq d(\overline{G}_i) < \epsilon$ we see that H is the union of a finite number of connected sets $H \cap \overline{G}_i$, $i = 1, \ldots, n$, each having diameter less than ϵ hence H has property S. Q.E.D.

By virtue of theorems 4.12 and 4.14, and since $K \subset \overline{K} \subset \overline{K}$, the following theorem is now immediate.

Theorem 4.15. Let K be a subset of a metric space T. If K has property S then $\overline{K}$ has property S. Hence if K has property S, $\overline{K}$ is locally connected.

Recall that we are attempting to show that every Peano space P has a basis of connected sets such that the closure of every non-empty element of this basis is itself a Peano space. We have seen that the difficulty is not in finding a basis of non-empty connected sets whose closures are compact and connected, but in finding such a basis of sets whose closures are also locally connected. In view of theorem 4.15, we are now able to overcome this difficulty if we can show that there is a basis of connected sets each non-empty element of which has property S.

Definition 4.16 Suppose H and K are subsets of the metric space T and $\epsilon > 0$. We shall say that K is an ϵ growth of H if and only if the following conditions hold:

(a) If $x \in K$ then there is a connected subset A of K, such that $x \in A$, $A \cap H \neq \emptyset$ and $d(A) < \epsilon$.

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(b) There is a $\delta>0$, such that, if B is any connected subset of T satisfying $d(B)<\delta$ and $B\cap H \neq \emptyset$, then $B\subset K$.

<u>Theorem 4.17 Suppose H and K are subsets of a metric space T and there is an $\epsilon>0$ such that K is an ϵ growth of H. Then</u>

(i) <u>$H\subset K$</u>.

(ii) <u>If H is connected then K is connected.</u>

(iii) <u>K is contained in the ϵ neighborhood of the set H and thus $d(K)<d(H) + 2\epsilon$.</u>

(iv) <u>If $H = \emptyset$ then $K = \emptyset$</u>.

<u>Proof</u>. To prove (i) suppose $H\not\subset K$. Therefore, there is a point x of H such that $x\notin K$. Now the single point x is a connected subset of T and $d(x) = 0$. Thus $d(x)$ is less than any positive number, in particular, $d(x)<\delta$ where δ is the δ of condition (b) in 4.16. Since $x\cap H \neq \emptyset$ we should, by condition (b) of 4.16, have $x\subset K$. But $x\not\subset K$ hence we have a contradiction, Thus $H\subset K$.

To prove (ii) suppose H is connected but $K = E/F$. Since H is connected and $H\subset K$ by (i), H must be contained in either E or F (1.22). Without loss of generality let $H\subset E$. Now $F \neq \emptyset$ and $E\cap F = \emptyset$, hence there is a point $y\in F\subset K$ such that $y\notin E$. Since $y\in K$, by condition (a) of 4.16 above, there is a connected subset A of K such that $y\in A$, $A\cap H \neq \emptyset$ and $d(A)<\epsilon$. But A, being a connected subset of K must be contained in either E or F (1.22). Thus, it cannot meet both E and F at the same time, hence it cannot contain y and meet H at the same time. Therefore, we have a contradiction of (b) of 4.16, hence K is connected.

To prove (iii) let u and v be any points of K. By part (a) of 4.16 there is a connected subset A of K such that $u\in A$, $A\cap H \neq \emptyset$

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and $d(A) < \epsilon$. Similarly there is a connected subset B of K such that $v \in B$, $B \cap H \neq \emptyset$ and $d(B) < \epsilon$. Let p and q be any points of $A \cap H$ and $B \cap H$ respectively. Thus, we have $\rho(x,p) \leq d(A)$, $\rho(q,y) \leq d(B)$ and $\rho(p,q) \leq d(H)$. Hence

$$\begin{aligned} \rho(x,y) &\leq \rho(x,p) + \rho(p,q) + \rho(q,y) \\ &\leq d(A) + d(H) + d(B) \\ &< d(H) + 2\epsilon. \end{aligned}$$

But this is true for any points x and y of K hence $d(K) < d(H) + 2\epsilon$.

To prove (iv) suppose $H = \emptyset$ but $K \neq \emptyset$. Thus, there is a point x of K. Then by condition (a) of 4.16 there is a connected subset A of K such that $A \cap H \neq \emptyset$. But since $H = \emptyset$, H does not meet any set. Thus we have our contradiction, hence $K = \emptyset$. Q.E.D.

<u>Theorem 4.18 Suppose H is a subset of a metric space T. Then given any $\epsilon > 0$ there is a subset K of T such that K is an ϵ growth of H.</u>

<u>Proof</u>. If $H = \emptyset$ it is clear from 4.16 that $K = \emptyset$ is an ϵ growth of H. Suppose $H \neq \emptyset$. If a point p belongs to H let W(p) be the union of all connected subsets C of T such that $p \in C$ and $d(c) < \epsilon$. Let $K = \bigcup_{p \in H} W(p)$. Now if $x \in K$, then $x \in W(p)$ for some point p of H. Thus x belongs to some set C of W(p) where C is connected, $d(C) < \epsilon$ and $p \in C$ hence $C \cap H \neq \emptyset$. Also, $C \subset W(p) \subset K$ hence $C \subset K$ and we see that K satisfies condition (a) of 4.16. To show that K satisfies condition (b) let $\delta = \epsilon$. Suppose there is a connected set D, $D \cap H \neq \emptyset$ and $d(D) < \epsilon (= \delta)$. Thus there is a point $p \in D \cap H$, i.e., $p \in D$ and $p \in H$. It follows that since $d(D) < \epsilon$, $D \subset W(p) \subset K$ thus $D \subset K$. Since H and K satisfy both conditions (a) and (b) of definition 4.16, K is an ϵ growth of H. Q.E.D.

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Definition 4.19 Suppose H and K are subsets of a metric space T and $\epsilon>0$. We shall say that K is an ϵ-sequential growth of H if and only if there exists a sequence of positive numbers $\{\epsilon_i\}$ and a sequence of subsets $\{H_i\}$ of T such that

(a) H_1 is an ϵ_1 growth of H, H_2 is an ϵ_2 growth of H_1 and in general H_{i+1} is an ϵ_{i+1} growth of H_i for i = 1,2,....

(b) $K = \bigcup_{i=1}^{\infty} H_i$.

(c) $\sum_{i=1}^{\infty} \epsilon_i \leq \epsilon$.

Theorem 4.20 Suppose H and K are subsets of a metric space T and $\epsilon>0$. Then

(i) There exists and ϵ-sequential growth of H.

(ii) If K is an ϵ-sequential growth of H, then

(o_1) $H \subset K$

(o_2) If H is connected, then K is connected

(o_3) $K \subset N(H,\epsilon)$, hence $d(K) < d(H) + 2\epsilon$

(o_4) If $H = \emptyset$, then $K = \emptyset$.

Proof. Let $\epsilon_i = \frac{1}{2^i}\epsilon$, i = 1,2,.... First if $H = \emptyset$ we shall see that $K = \emptyset$ is an ϵ-sequential growth of H. By theorems 4.17 and 4.18, $H_1 = \emptyset$ is clearly an ϵ_1 growth of H, $H_2 = \emptyset$ is an ϵ_2 growth of H_1 and in general $H_{i+1} = \emptyset$ is an ϵ_{i+1} growth of H_i. Thus we have a sequence $\{\epsilon_i\}$ of positive numbers and a sequence of subsets $\{H_i\}$ of T such that (a) of definition 4.19 is satisfied. Letting $F = \emptyset$, we have $F = \bigcup_{i=1}^{\infty} H_i = \emptyset$ and since $\sum_{i=1}^{\infty}\epsilon_i \leq \epsilon$ we see that F is an ϵ-sequential growth of H thus (i) is satisfied. To show (ii) suppose K^* is an ϵ-sequential growth of $H = \emptyset$. Then there is a sequence of positive numbers $\{\epsilon_i\}$ and a sequence of

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subsets $\{H_i\}$ of T such that (a), (b) and (c) of definition 4.19 are satisfied. If we can verify (O_4) first, then (O_1), (O_2), and (O_3) will readily follow. Since $H = \emptyset$ we see from theorem 4.17 that $H_1 = \emptyset$. Similarly, since $H_1 = \emptyset$ we have $H_2 = \emptyset$. In general then $H_i = \emptyset$ for each $i = 1,2,\ldots$, thus $\bigcup_{i=1}^{\infty} H_i = \emptyset$. But $K^* = \bigcup_{i=1}^{\infty} H_i$ by definition 4.16, therefore, K^* must be the empty set, hence (O_1), (O_2), and (O_3) follow immediately.

Suppose $H \neq \emptyset$ and recall $\epsilon_i = \frac{1}{2^i}\epsilon$ for $i = 1,2,\ldots$. Since $H \subset T$, theorem 4.18 implies there is a subset H_1 of T such that H_1 is an ϵ_1 growth of H. Similarly, there is a subset H_2 of T such that H_2 is an ϵ_2 growth of H_1 and, in general, there is a subset H_{i+1} of T such that H_{i+1} is an ϵ_{i+1} growth of H_i. Thus we have a sequence of positive numbers $\{\epsilon_i\}$ and a sequence of subsets $\{H_i\}$ of T such that (a) of 4.19 is satisfied. Also, $\sum_{i=1}^{\infty}\epsilon_i = \frac{1}{2}\epsilon + \frac{1}{4}\epsilon + \ldots$ hence $\sum_{i=1}^{\infty}\epsilon_i \leq \epsilon$. Letting $G = \bigcup_{i=1}^{\infty} H_i$ we then see that G is an ϵ-sequential growth of H and we have proved (i). To show (ii) suppose K is any ϵ-sequential growth of H. Thus, there is a sequence of positive numbers $\{\epsilon_i\}$ and a sequence of subsets $\{H_i\}$ of T such that (a), (b) and (c) of 4.19 are satisfied. Suppose $H \not\subset K$, thus $H \not\subset \bigcup_{i=1}^{\infty} H_i = K$. But since H_1 is an ϵ growth of H, theorem 4.14 implies $H \subset H_1$ hence $H \subset \bigcup_{i=1}^{\infty} H_i = K$. Therefore, we have a contradiction, hence $H \subset K$ and (O_1) is verified. To verify (O_2) suppose H is connected but $K = E/F$. Now H is contained in either E or F, hence without loss of generality let $H \subset E$. Clearly, $H_1 \subset K$ and theorem 4.17 implies H_1 is connected. Therefore, H_1 is contained in either E or F but since $H \subset H_1$, H_1 must be contained in E. Continuing this process, we see that since each set H_i is connected

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and $H_i \subset H_{i+1}$ for all i we must have each set H_i contained in E. Thus $\bigcup_{i=1}^{\infty} H_i \subset E$ hence $K \subset E$ which contradicts the assumption that $K = E/F$. Therefore, K is connected. Now $H_1 \subset N(H,\epsilon_1)$ and similarly $H_2 \subset N(H_1,\epsilon_2) \subset N[N(H,\epsilon_1),\epsilon_2]$ (4.17). In general,

$$H_i \subset N(H_{i-1},\epsilon_i) \subset \ldots \subset N(H,\sum_{i=1}^{\infty}\epsilon_i).$$

Since this is true for all i, we have $K = \bigcup_{i=1}^{\infty} H_i \subset N(H,\sum_{i=1}^{\infty}\epsilon_i) \subset N(H,\epsilon)$ since $\sum_{i=1}^{\infty}\epsilon_i \leq \epsilon$, i.e., $K \subset N(H,\epsilon)$. It readily follows that $d(K) < d(H) + 2\epsilon$ and we have verified (O_3). Since we have shown earlier that if $H = \emptyset$ then $K = \emptyset$, the theorem is therefore proved. Q.E.D.

Theorem 4.21 Suppose T is a metric space with property S, H and K are subsets of T and $\epsilon>0$. If K is an ϵ-sequential growth of H, then K has property S and is open in T.

Proof. Suppose K is a ϵ-sequential growth of H. Then there are sequences $\{\epsilon_i\}$ and $\{H_i\}$ with the properties given in definition 4.19. Let x be any point in K. Since $K = \bigcup_{i=1}^{\infty} H_i$ there is a set H_j such that $x \in H_j$. Now T has property S, hence T is locally connected by the corollary to theorem 4.11. H_{j+1} is an ϵ_{j+1} growth of H_j. Let $\delta_{j+1} > 0$ where δ_{j+1} is the δ given in definition 4.16, and consider the open neighborhood $N(x,\frac{\delta_{j+1}}{3})$ in T. Let V be the component of $N(x,\frac{\delta_{j+1}}{3})$ containing x. Thus V is a component of an open set which is contained in the locally connected space T, hence V is open. Since $V \subset N(x,\frac{\delta_{j+1}}{3})$, $d(V) \leq \frac{2\delta_{j+1}}{3} < \delta_{j+1}$. Also $V \cap H_j \neq \emptyset$ since both sets contain the point x. Thus by definition 4.16 $x \in V \subset H_{j+1} \subset K$. But V is open, hence there is a $\lambda>0$ such that $N(x,\lambda) \subset V$, i.e., $x \in N(x,\lambda) \subset V \subset K$. And we see that for any point x

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of K there is a neighborhood of x contained in K, therefore, K is open.

We shall now show that K has property S. To do this it will be sufficient to show that for any $\beta > 0$, K can be expressed as the union of a finite number of connected sets each having diameter less than β. Clearly, there is an integer k such that $\sum_{i=k}^{\infty} \epsilon_i < \frac{\beta}{4}$ and by definition 4.16 there is a $\delta_{k+1} > 0$ such that if B is a connected subset of T, $d(B) < \delta_{k+1}$ and $B \cap H_k \neq \emptyset$ then $B \subset H_{k+1}$. Since T has property S, $T = \bigcup G_i$ where $\bigcup G_i$ is a finite collection of sets such that each set G_i is connected and has diameter less than the minimum of $\frac{\beta}{4}$ and δ_{k+1}. $H_k \subset T$ hence let $G_1, G_2, \ldots, G_n$ be those sets which intersect H_k. Now each of these sets $G_1, \ldots, G_n$ is connected, contained in T, meets H_k and has diameter less than δ_{k+1} hence $\bigcup_{i=1}^{n} G_i \subset H_{k+1}$ (4.16). But $H_k \subset T = \bigcup_{i=1}^{n} G_i$ and if $x \in H_k$ then x belongs to some of the sets G_i, namely, some of the sets $G_1, \ldots, G_n$. Therefore, $H_k \subset \bigcup_{i=1}^{n} G_i \subset H_{k+1}$. We shall now define a set of points W_i for each $i = 1, 2, \ldots, n$, in the following manner. A point $y \in W_j$ if and only if $y \in K$, $y \in C$ where C is a connected subset of K, $d(C) < \frac{\beta}{4}$ and $C \cap G_j \neq \emptyset$. Now $d(W_j) < \beta$ for $i \leqq j \leqq n$ because if $z \in W_j$ then $z \in K$, $z \in D$ where D is a connected subset of K, $d(D) < \frac{\beta}{4}$ and $D \cap G_j \neq \emptyset$. Assuming $y \in W_j$ as above let u and v be points of $C \cap G_j$ and $D \cap G_j$ respectively. Then $\rho(y,u) \leqq d(C) < \frac{\beta}{4}$, $\rho(u,v) \leqq d(G_j) < \frac{\beta}{4}$ and $\rho(v,z) \leqq d(D) < \frac{\beta}{4}$. Therefore, applying the triangle inequality twice we clearly have $\rho(y,z) \leqq \frac{3\beta}{4} < \beta$. Since y and z are arbitrary points of W_j we have $d(W_j) < \beta$, for $1 \leqq j \leqq n$. Also $G_j \subset W_j$ for $1 \leqq j \leqq n$, for if $r \in G_j$ then $r \in K$, $r \in G_j$ where G_j is a connected subset of K, $d(G_j) < \frac{\beta}{4}$ and $G_j \cap G_j \neq \emptyset$, thus by our

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definition of W_j, $r \in W_j$, i.e., $G_j \subset W_j$. Furthermore, each W_j is connected for $1 \leq j \leq n$. To show this suppose $x \in W_j$, thus $x \in C_x$ where C_x is a connected subset of K, $d(C_x) < \frac{\beta}{4}$ and $C_x \cap G_j \neq \emptyset$. Clearly, $W_j = \bigcup_{x \in W_j} C_x$ and since $G_j \subset W_j$, $W_j = (\bigcup_{x \in W_j} C_x) \cup G_j$. Now each set C_x is connected and meets the connected set G_j hence $(\bigcup_{x \in W_j} C_x) \cup G_j = W_j$ is connected (1.22).

We shall see that $K = \bigcup_{i=1}^{n} W_i$. Since it is easily see that $\bigcup_{i=1}^{n} W_i \subset K$, we need only show that $K \subset \bigcup_{i=1}^{n} W_i$. Let x be any point of K. Thus $x \in H_i$ for some i. If $i \leq k$ (where k is the same k previously used) then $x \in H_i \subset H_k \subset \bigcup_{i=1}^{n} G_i \subset \bigcup_{i=1}^{n} W_i$, i.e., $x \in \bigcup_{i=1}^{n} W_i$. Suppose $i > k$ and recall that H_i is an ϵ_i growth of H_{i-1}. Then by the definition of an ϵ growth (4.16), $x \in H_i$ implies there is a connected set L_i such that $L_i \subset H_i$, $x \in L_i$, $L_i \cap H_{i-1} \neq \emptyset$ and $d(L_i) < \epsilon_i$. Let $x_{i-1} \in L_i \cap H_{i-1}$. Again H_{i-1} is an ϵ_{i-1} growth of H_{i-2} hence $x_{i-1} \in H_{i-1}$ implies there is a connected set L_{i-1}, $L_{i-1} \subset H_{i-1}$, $x_{i-1} \in L_{i-1}$, $L_{i-1} \cap H_{i-2} \neq \emptyset$ and $d(L_{i-1}) < \epsilon_{i-1}$. Let $x_{i-2} \in L_{i-1} \cap H_{i-2}$. Continuing this process we have a connected set L_{k+1}, $L_{k+1} \subset H_{k+1}$, $x_{k+1} \in L_{k+1}$, $d(L_{k+1}) < \epsilon_{k+1}$ and $x_k \in L_{k+1} \cap H_k$. Let $L = \bigcup_{m=k+1}^{i} L_m$. Then $L_{k+1} \cap H_k \neq \emptyset$ implies $L \cap H_k \neq \emptyset$. Now recall from elementary point set theory that if we have two sets A_1 and A_2 such that $A_1 \cap A_2 \neq \emptyset$ then $d(A_1 \cup A_2) \leq d(A_1) + d(A_2)$. Furthermore, by induction, it can be shown that if sets $A_1, A_2, \ldots, A_n$ are such that $A_i \cap A_{i-1} \neq \emptyset$ for $i = 2, \ldots, n+1$ then $d(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} d(A_i)$. Applying this to our case, we see that we have sets $L_{k+1}, L_{k+2}, \ldots, L_i$ such that $L_{k+r} \cap L_{k+r-1} \neq \emptyset$ for each r, hence $d(L) = d(\bigcup_{m=k+1}^{i} L_m) \leq \sum_{m=k+1}^{i} d(L_m) = \epsilon_{k+1} + \epsilon_{k+2} + \cdots + \epsilon_i = \sum_{m=k}^{\infty} \epsilon_m < \frac{\beta}{4}$, i.e., $d(L) < \frac{\beta}{4}$. Since $x \in L_i$, $x \in L$. Now each L_m is connected, and, in general, we have $x_{m-1} \in L_m \cap L_{m-1}$ hence $L = \bigcup_{m=k+1}^{i} L_m$

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is connected. Also $L_m \subset H_m$ for $m = k+1, k+2, \ldots, i$, thus

$$\bigcup_{m=k+1}^{i} L_m \subset \bigcup_{m=k+1}^{i} H_m \subset K.$$

Therefore, $L \subset K$. Recall that $H_k \subset \bigcup_{i=1}^{n} G_i$ and that $L \cap H_k \neq \emptyset$, hence there is some set G_s, where $1 \leq s \leq n$, such that $L \cap G_s \neq \emptyset$. In summary, then $x \in L$ where L is a connected subset of K, $d(L) < \frac{\beta}{4}$ and $L \cap G_s \neq \emptyset$ for $1 \leq s \leq n$, thus $x \in W_s$ and we have $x \in \bigcup_{i=1}^{n} W_i$. We now see that any arbitrary point x of K is such that $x \in \bigcup_{i=1}^{n} W_i$. Therefore, $K = \bigcup_{i=1}^{n} W_i$, i.e., K is the union of a finite number of connected sets each having diameter less than an arbitrary positive number β hence K has property S. Q.E.D.

Theorem 4.22 Suppose T is a metric space having property S. Then T has a basis, every element of which is an open connected set having property S.

Proof. Let $V_n(p)$, $n = 1,2,\ldots$ be a countable basis for an arbitrary point p of T (1.18). Since each $V_n(p)$ is an open set containing p there is an $\epsilon_n > 0$ such that $N(p,\epsilon_n) \subset V_n(p)$ for $n = 1,2,\ldots$. Consider the neighborhood $N(p,\frac{1}{2}\epsilon_n)$. Clearly, $N(p,\frac{1}{2}\epsilon_n) \subset N(p,\epsilon_n)$. Let $W_n(p)$ be the component of $N(p,\frac{1}{2}\epsilon_n)$ containing the point p. Since T is locally connected (4.11) and $N(p,\frac{1}{2}\epsilon_n)$ is open we see that $W_n(p)$ is an open connected set such that

$$W_n(p) \subset N(p,\tfrac{1}{2}\epsilon_n) \subset V_n(p),$$

for $n = 1,2,\ldots$. In view of theorem 4.20 there is an ϵ-sequential growth $K_n(p)$ of $W_n(p)$ for each $n = 1,2,\ldots$ and for $\epsilon = \frac{1}{2}\epsilon_n$. Now since T has property S and each $W_n(p)$ is connected we see by theorem 4.21 that each $K_n(p)$ is an open connected subset of T

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having property S. Also for each n = 1,2,...

$$W_n(p) \subset K_n(p) \subset N(W_n(p), \tfrac{1}{2}\epsilon_n)$$

by theorem 4.20, but

$$N(W_n(p), \tfrac{1}{2}\epsilon_n) \subset N[N(p, \tfrac{1}{2}\epsilon_n), \tfrac{1}{2}\epsilon_n] \subset N(p, \epsilon_n) \subset V_n(p),$$

i.e., in particular, $K_n(p) \subset V_n(p)$ for n = 1,2,.... We readily see that the collection of sets $K_n(p)$, n = 1,2,... is a basis at p, for if H is any open subset of T containing p then there is a $V_j(p) \subset H$, but also $K_j(p) \subset V_j(p)$, i.e., there is a set $K_j(p)$, such that $K_j(p) \subset H$. Since p was an arbitrary point of T, the collection of sets $K_n(p)$ for n = 1,2,... and for all points p of T is a basis of the space T. Hence this is a basis of open connected sets each of which has property S and the theorem is proved. Q.E.D.

We have seen that every Peano space is a metric space having property S. Thus the preceeding theorem implies that every Peano space has a basis of open connected sets each of which has property S. We shall state this in a more formal manner in theorem 4.27, but first we briefly examine how the ideas of property S, local arcwise connectedness and local compactness are related.

Definition 4.23 Let T be a space. Then T is said to be <u>locally arcwise connected</u> if and only if, for any point p of T and any open set H of p in T, there exists an open set G of p in T such that if $x \in G$, $y \in G$ and $x \neq y$ then there exists an arc in H joining x and y.

With the aid of 1.25, part (2), the reader may easily verify that every locally arcwise connected space is locally connected.

Definition 4.24 Let T be a space. Then T is said to be <u>locally compact</u> if and only if, for any point p of T and any open

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set H of p, there is an open set G of p such that $G \subset H$ and $\overline{G}$ is compact.

It is readily seen that the definition of compactness 1.21 implies that every compact space is locally compact.

<u>Theorem 4.25 If T is a locally compact metric space having property S then T is locally arcwise connected</u>.

<u>Proof</u>. Suppose T is a locally compact metric space having property S. Let p be any point of T and let A be any open set containing p. Since T has property S, T is locally connected. Thus, if W is the component of A containing p, then W is an open connected set.

If we can show that W is arcwise connected we will have proved the theorem, for W is now an open set containing p and contained in A where A is any open set containing p. Furthermore, if W is arcwise connected then any two points of W may be joined by an arc contained in W, thus contained in A, hence by definition 4.23 T will be locally arcwise connected. We shall now show that W is indeed arcwise connected.

Let z be any point in W. Since T is locally compact there exists an open set H such that $z \in H$, $H \subset W$ and $\overline{H}$ is compact. Since T is a metric space there exists an open set G such that

$$z \in G \subset \overline{G} \subset H \subset W \quad (1.11).$$

Thus, $\overline{G} \subset \overline{H}$, i.e., $\overline{G}$ is a closed subset of the compact set $\overline{H}$, hence $\overline{G}$ is compact. Now T is a metric space having property S hence theorem 4.22 implies T has a basis every element of which is an open connected set having property S. Therefore, G is the union of some of the elements of this basis and since $z \in G$ there is a

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set $V(z)$ of the basis such that $z \in V \subset G$. Thus, in addition to being open, connected and having property S, the set $V(z)$ is such that

$$z \in V(z) \subset \overline{V}(z) \subset \overline{G} \subset W$$

and since $\overline{V}(z)$ is the closed subset of the compact set $\overline{G}$, $\overline{V}(z)$ compact. Furthermore, z was an arbitrary point of W hence the collection of sets $\bigcup_{z \in W} V(z)$ is an open covering of W. Let x and y be distinct points of W. Let P be the set of points of W which can be joined to x by a simple chain of sets in $\bigcup_{z \in W} V(z)$. Since $x \in P$, $P \neq \emptyset$. Let Q be the set of points of W which cannot be joined to x by a simple chain of sets in $\bigcup_{z \in W} V(z)$. Suppose $P \neq W$, therefore $Q \neq \emptyset$ and clearly $W = P \cup Q$ and $P \cap Q = \emptyset$. Let c be a point of $\overline{P} \cap Q$. Thus, c belongs to some set $V(c)$ of the open covering. Since $V(c)$ is open, there is a neighborhood of c contained in $V(c)$ and since $c \in \overline{P}$ this neighborhood contains a point d of P (1.11). Thus $d \in P$ and $d \in V(c)$. Now d can be joined to x by a chain of sets in $\bigcup_{z \in W} V(z)$. Let $V(z_h), \ldots, V(z_r)$ be such a chain. Let $V(z_i)$ be the first set of this chain intersecting $V(c)$. Therefore, $V(z_h), \ldots, V(z_i), V(c)$ is a simple chain of sets of $\bigcup_{z \in W} V(z)$ joining x and c. Hence $c \in P$, but we also had $c \in Q$. This is impossible since $P \cap Q = \emptyset$, and we see there is no point c of $\overline{P} \cap Q$, i.e., $\overline{P} \cap Q = \emptyset$. Suppose $a \in P \cap \overline{Q}$. Thus, $a \in V(a)$ where $V(a)$ is some set of the covering $\bigcup_{z \in W} V(z)$. $V(a)$ is open, hence there is a neighborhood of a contained in $V(a)$ but $a \in \overline{Q}$ hence this neighborhood contains a point b of Q, i.e., $b \in Q$ and $b \in V(a)$. Since $a \in P$ we let $V(z_k), \ldots, V(z_m)$ be a simple chain joining x and a. Let $V(z_j)$ be the first set of this chain intersecting $V(a)$.

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Therefore, $V(z_k),\ldots,V(z_j),V(a)$ is a simple chain of sets of $\bigcup_{z\in W}V(z)$ joining x and b. Thus $b \in P$. This is impossible since $b \in Q$, hence $a \notin P \cap \overline{Q}$, i.e., $P \cap \overline{Q} = \emptyset$. Finally then we see that $W = P/Q$ which is a contradiction since W is connected. Therefore, the assumption that $P \neq W$ is false, i.e., $P = W$ and there exists a simple chain $V(z_i)$, $i = 1,\ldots,n$ joining x and y. Let $M = \bigcup_{i=1}^{n} V(z_i)$.. Recall that any simple chain $\bigcup_{i=1}^{t} C_i$ is such that $C_i \cap C_{i+1} \neq \emptyset$ but $C_i \cap C_j = \emptyset$ for $|i-j| > 1$. Thus since each set $V(z_i)$ of the chain joining x and y is connected, $M = \bigcup_{i=1}^{n} V(z_i)$ is connected (1.22). Furthermore, we have seen that each $V(z_i)$ has property S hence $M = \bigcup_{i=1}^{n} V(z_i)$ has property S (4.9). It follows that $\overline{M}$ is connected and has property S (4.15), thus $\overline{M}$ is also locally connected (4.11). Clearly since each $\overline{V(z_i)}$ is compact $\bigcup_{i=1}^{n} \overline{V(z_i)}$ is compact. But $\overline{M} = \overline{\bigcup_{i=1}^{n} V(z_i)} = \bigcup_{i=1}^{n} \overline{V(z_i)}$, i.e., $\overline{M}$ is compact. We now see that $\overline{M}$ is a Peano space contained in W. By theorem 3.9 there is a simple arc joining x and y in $\overline{M}$ hence in W. But x and y were arbitrary points of W hence W is arcwise connected and the theorem is proved. Q.E.D.

Definition 4.26 We shall denote a subset G of a space T as a <u>region</u> in T if and only if G is both connected and open in T.

In theorem 4.25 the only requirement we made of W was that it be an open connected set containing an arbitrary point p of T, Thus the following corollary is immediate.

<u>Corollary. Suppose T is a locally compact metric space having property S. Then every region in T is arcwise connected.</u>

Since every compact space is locally compact and every Peano space is a compact metric space having property S, we see

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that theorem 4.25 applies to Peano spaces. We have now proved the following theorem.

Theorem 4.27 Suppose T is a Peano space. Then T has a basis each non-empty element of which is connected, has property S, and has a Peano space as its closure. Also T is arcwise connected (3.9), and locally arcwise connected (4.25) and every region in T is arcwise connected.

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CHAPTER V

CYCLIC ELEMENT THEORY

In this chapter we shall study the structure of Peano spaces with respect to their cut points. Thus, all spaces considered shall be Peano spaces, although this need not be the case for some of the theorems to hold.

<u>Theorem 5.1 Suppose a, b and p are distinct points of a Peano space T. Then, a and b lie in different components of T-p if and only if T-p has a separation G/H such that $a \in G$ and $b \in H$</u>.

<u>Proof</u>. Suppose T is a Peano space and a and b lie in different components of T-p. Let G be the component of T-p containing a. Since T is locally connected and G is a component of the open set T-p of T, G is open. Let $H = (T-p) \cap \complement G$. Now $b \in T-p$ and since $a \in G$, $b \in \complement G$. Therefore, $b \in H$. We shall show that $T-p = G/H$. Clearly, $T-p = G \cup H$, $G \cap H = \emptyset$, $G \neq \emptyset$ and $H \neq \emptyset$. Since $\complement G$ is closed and $\overline{H} \subset \overline{\complement G}$ we have

$$\overline{H} \cap G \subset \overline{\complement G} \cap G = \complement G \cap G = \emptyset$$

Furthermore, $G \subset \overline{G}$ and $G \subset T-p$, hence $G \subset \overline{G} \cap (T-p) \subset \overline{G}$. But G is connected hence $\overline{G}$ is connected which implies that $\overline{G} \cap (T-p)$ is connected (1.22). Now G is a component of T-p and $\overline{G} \cap (T-p) \subset T-p$. Thus since $G \subset \overline{G} \cap (T-p)$, $G = \overline{G} \cap (T-p)$ (1.24). Finally then

$$H \cap \overline{G} = (T-p) \cap \complement G \cap \overline{G} = \complement G \cap G = \emptyset$$

and $T-p = G/H$.

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To prove the converse, suppose a Peano space T is such that T-p = C/D where points a and b of T-p belong to C and D respectively. Let E be the component of T-p containing a and let F be the component of T-p containing b. Clearly since E is connected and $a \in E \cap C$ we must have $E \subset C$. Similarly F is connected and $a \in F \cap D$ hence $F \subset D$. But $C \cap D = \emptyset$ hence $E \cap F = \emptyset$, i.e., a and b belong to different components of T-p. Q.E.D.

Definition 5.2 Suppose a and b are points of a Peano space T. We shall say that a third point p of T *cuts between* a, b or *separates* a and b if a and b lie in different components of T-p.

Thus in view of theorem 5.1, a point p cuts between and b if and only if T-p has a separation G/H such that $a \in G$ and $b \in H$.

Definition 5.3 A Peano space T is termed *cyclic* if T has no cut points.

Definition 5.4 We shall say that a subset E of a Peano space T is *semi-connected* if and only if for every point x of T-E, the set E is a subset of some component of T-x. As a consequence of this definition the reader may easily verify the following statements.

(i) The empty set is semi-connected and T itself is semi-connected.

(ii) If a set E is connected, then E is semi-connected.

(iii) If E is semi-connected and $\mathcal{A}$ is any collection of semi-connected subsets of T such that if $A \in \mathcal{A}$ implies $E \cap A \neq \emptyset$, then the set $E \cup (\bigcup_{A \in \mathcal{A}} A)$ is also semi-connected.

(iv) If $\mathcal{A}$ is a collection of semi-connected sets and H is the common part of some (or all) of the sets of $\mathcal{A}$ then H is semi-connected.

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(v) If E is semi-connected and $E \subset F \subset \bar{E}$ then F is semi-connected. Thus we see that if E is semi-connected, then $\bar{E}$ is semi-connected.

Theorem 5.5 Suppose p and q are distinct points of a Peano space T and E is a semi-connected set containing p and q. If a point x cuts between p and q, then $x \in E$.

Proof. Suppose $x \notin E$. Since E is semi-connected E lies in some component B of $T-x$. Thus $p \cup q \subset B$, i.e., p and q lie in the same component. This is impossible since x cuts between p and q, hence $x \in E$. Q.E.D.

Definition 5.6 Suppose p and q are distinct points of a Peano space T. We shall let $K(p,q)$ be the set of all points of T that cut between p and q. Clearly $K(p,q)$ may be empty.

Theorem 5.7 The set $p \cup q \cup K(p,q)$ is semi-connected.

Proof. Let x be any point such that $x \notin p \cup q \cup K(p,q)$. Thus $x \notin K(p,q)$ hence x does not cut between p and q. It follows that p and q are in the same component C of $T-x$, and in view of (ii) of (5.4) that C is semi-connected. Thus by 5.5, $p \cup q \cup K(p,q) \subset C$, that is, $p \cup q \cup K(p,q)$ is semi-connected.

Definition 5.8 Let a and b the points of a Peano space T. We shall say that a is conjugate to b, written $a \bigcirc b$, if and only if no point of T cuts between a and b. Clearly every point is conjugate to itself and if $a \bigcirc b$ then $b \bigcirc a$.

In view of definitions 5.6 and 5.8, the following theorem is now immediate.

Theorem 5.9 If p and q are two points of a Peano space T

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<u>such that $p \mathrm{O} q$ then $p \cup q$ is a semi-connected set.</u>

Definition 5.10 Suppose $a_1,\ldots,a_n$ is a finite sequence of distinct points of T. These points taken in the given order, will be said to constitute a <u>O-chain</u> if $a_i \mathrm{O}\, a_{i+1}$ for $i = 1,\ldots,n-1$. Also, if $a_n \mathrm{O}\, a$, then we shall call the O-chain a <u>simple</u> <u>closed</u> <u>O-chain</u>. Otherwise, the O-chain will be termed open.

<u>Theorem 5.11 If $a_1,\ldots,a_n$ is a O-chain, then $\bigcup_{i=1}^{n} a_i$ is semi-connected.</u>

<u>Proof</u>. Since $a_1 \mathrm{O}\, a_2$, $a_1 \cup a_2$ is semi-connected by 5.9. Suppose $\bigcup_{i=1}^{n-1} a_i$ is semi-connected. Now $a_{n-1} \mathrm{O}\, a_n$, i.e., $a_{n-1} \cup a_n$ is semi-connected and since $(a_{n=1} \cup a_n) \cap \bigcup_{i=1}^{n-1} a_i \neq \emptyset$ we see that $\bigcup_{i=1}^{n} a_i$ is semi-connected. Q.E.D.

<u>Theorem 5.12 Suppose $a_1,\ldots,a_n$ is a simple closed O-chain. Then any two of the points $a_1,\ldots,a_n$ are conjugate.</u>

<u>Proof</u>. Clearly if we consider a cyclic permutation of the points $a_1,\ldots,a_n$, the new arrangement is again a simple closed O-chain. As examples, $a_{n-1},a_n,a_1,\ldots,a_{n-2}$ and $a_i,\ldots,a_n,a_1,\ldots,a_{i-1}$ are each again simple closed O-chains. Thus, to prove the theorem it is sufficient to show that $a_1 \mathrm{O}\, a_i$ for any $i = 2,\ldots,n-1$ since a similar proof would follow to show that $a_j \mathrm{O}\, a_i$ for any j and any i.

Suppose the theorem is false, that is, there exists a point x which cuts between a_1 and a_i. Now $a_1,\ldots,a_i$ is a O-chain and by 5.11, $\bigcup_{k=1}^{i} a_k$ is a semi-connected set containing a_1 and a_i. In view of theorem 5.5 $x \in \bigcup_{k=1}^{i} a_k$. Similarly, $a_i,\ldots,a_n,a_1$ is a O-chain and x belongs to the semi-connected set $\bigcup_{k=i}^{1} a_k$. In summary then x belongs to the two sets

$$\bigcup_{k=1}^{i} a_k = a_1 \cup a_2 \cup \ldots \cup a_i$$

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and

$$\bigcup_{k=i}^{i} a_k = a_i \cup a_{i+1} \cup \cdots \cup a_n \cup a_1.$$

But notice $(\bigcup_{k=1}^{i} a_k) \cap (\bigcup_{k=i}^{1} a_k) = a_1 \cup a_i$. Thus $x \in a_1 \cup a_i$, i.e., x coincides with one of the points a_1 or a_i. This is impossible since we assumed that x cuts between a_1 and a_i. Therefore, no such point x exists and we have $a_1 \mathrm{O}\, a_i$. Q.E.D.

Definition 5.13 Suppose a Peano space T contains a finite number of distinct points $p_1, \ldots, p_n$ and an equal number of sets $S_1, \ldots, S_n$ such that the following conditions hold.

(i) Each of the sets $S_1, \ldots, S_n$ is semi-connected.

(ii) $S_1 \cap S_2 = p_2$, $S_2 \cap S_3 = p_3, \ldots, S_{n-1} \cap S_n = p_n$ and $S_n \cap S_1 = p_1$.

(iii) $S_i \cap S_j = \emptyset$ if $i < |i-j| < n-1$.

If this is the case we shall say that the points $p_1, \ldots, p_n$ and the sets $S_1, \ldots, S_n$ constitute a <u>simple</u> <u>closed</u> <u>O-polygon</u>. We shall call the points $p_1, \ldots, p_n$ the <u>vertices</u> and the sets $S_1, \ldots, S_n$ the sides of the O-polygon. We shall denote the O-polygon by

$$(p_1, \ldots, p_n; S_1, \ldots, S_n).$$

The reader can easily verify that if the set of points $a_1, \ldots, a_n$ is a simple closed O-chain then we have the simple closed O-polygon $(a_1, \ldots, a_n; a_1 \cup a_2, a_2 \cup a_3, \ldots, a_{n-1} \cup a_n, a_n \cup a_1)$.

<u>Theorem 5.14 Any two vertices of a simple closed O-polygon are conjugate</u>.

<u>Proof</u>. Let $(p_1, \ldots, p_2; S_1, \ldots, S_n)$ be a simple closed O-polygon. We see by theorem 5.12 that it shall be sufficient to show that the vertices $p_1, \ldots, p_n$, in this order, form a simple

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closed O-chain, i.e., that $p_i O p_{i+1}$ for $i = 1,\ldots,n-1$ and that $p_n O p_1$. Again we see that the situation is unaffected by a cyclic permutation, hence we need only show that $p_1 O p_2$.

Suppose otherwise, that is, there is a point x which cuts between p_1 and p_2. Now S_i is semi-connected and $p_1 \cup p_2 \subset S_1$. Thus by theorem 5.5, $x \in S_1$. Furthermore, S_i is semi-connected for all i, hence by repeated use of 5.4, part (iii), $\bigcup_{i=2}^{n} S_i$ is semi-connected. Also, since $p_2 \in S_2$ and $p_1 \in S_n$ we have $p_1 \cup p_2 \subset \bigcup_{i=2}^{n} S_i$ and again by 5.5, $x \in \bigcup_{i=2}^{n} S_i$. But since $(\bigcup_{i=2}^{n} S_i) \cap S_1 \subset (S_1 \cap S_2) \cup (S_1 \cap S_n) = p_1 \cup p_2$, we must have x identical with one of the points p_1 and p_2. This, of course, is impossible hence our assumption that there is a point x which cuts between p_1 and p_2 is false. Thus $p_1 O p_2$. Q.E.D.

Definition 5.15 Suppose C is a subset of a Peano space T. We shall say that C is a <u>proper cyclic element</u> if and only if the following conditions hold.

(i) C is non-degenerate.

(ii) Any two points of C are conjugate.

(iii) If a point x is conjugate to two distinct points of C then $x \in C$.

<u>Theorem 5.16 If C is a proper cyclic element then C is semi-connected</u>.

<u>Proof</u>. Let p_o be a point of C. Then $p_o O p$ for all points p of C. Thus, in view of theorem 5.9 $p_o \cup p$ is a semi-connected set for all points p of C. Hence $\bigcup(p_o \cup p)$ for all p of C is semi-connected (5.4). But $C = \bigcup(p_o \cup p)$ for all p of C, hence C is semi-connected. Q.E.D.

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Theorem 5.17 Suppose C_1 and C_2 are proper cyclic elements. If $C_1 \cap C_2$ is non-degenerate then $C_1 = C_2$.

Proof. Suppose $C_1 \cap C_2 = a \cup b$ and let x_1 be any point of C_1. Then $x_1 O a$ and $x_1 O b$ since $a \cup b \subset C_1$. But $a \cup b \subset C_2$ also hence $x_1 \in C_2$. Thus, $C_1 \subset C_2$. Similarly $C_2 \subset C_1$, i.e., $C_1 = C_2$. Q.E.D.

Theorem 5.18 Suppose p and q are distinct points of a Peano space T and $p O q$. Then there is exactly one proper cyclic element that contains both p and q.

Proof. Let F be the set consisting of all points x of T such that $x O p$ and $x O q$. We shall show that F is a proper cyclic element containing p and q and, in fact, is the only one containing p and q. Since $p O q$ we have $p \cup q \subset F$. Thus, F is non-degenerate and contains p and q. Also, any two points of F are conjugate, for let x_1 and x_2 be any two points of F. Then, if $x_1 = x_2$ we have $x_1 O x_2$. Similarly, if x_1 is identical with either p or q, or if x_2 is identical with either p or q we again have $x_1 O x_2$. Hence, we exclude these cases and assume that p, q, x_1, and x_2 are distinct points. Since x_1 and x_2 belong to F we have $x_1 O p$, $x_1 O q$, $x_2 O p$ and $x_2 O p$. But the conjugacy relation is symetric (5.8), hence we have the simple closed O-chain $p O x_1 O q O x_2 O p$. Thus, in view of 5.12, $x_1 O x_2$, i.e., any two points of F are conjugate.

Let y be any point which is conjugate to two distinct points x and z of F. Thus

(1) $$p O x O y O z O q$$

and $p O q$. We shall show that $p O y O q$, i.e., $y \in F$. In view of theorem 5.12 to show this it shall be sufficient to show that y

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belongs to a simple closed O-chain containing both p and q. Now if y equals any one of the points p, q, x or z then clearly $y \in F$. Hence, we can assume that y is different from each of the points p,q,x and z. Furthermore, we know that $p \neq q$ and $x \neq z$. There are seven possibilities to consider.

(i) If $x = p$ and $z = q$, then from (1) we have the simple closed O-chain $p O y O q$.

(ii) If $x = p$ and $z \neq q$, then $z \neq p$ since $z \neq x$ and from (1) we have the simple closed O-chain $p O y O z O q$.

(iii) If $x = q$ and $z = p$, then from (1) we have the simple closed O-chain $q O y O p$.

(iv) If $x = q$ and $z \neq p$, then $z \neq q$ since $z \neq x$ and from (1) we have the simple closed O-chain $p O q O y O z$.

(v) If $x \neq p,q$, and $z = p$, then from (1) we have the simple closed O-chain $x O y O p O q$.

(vi) If $x \neq p,q$ and $z = q$, then from (1) we have the simple closed O-chain $p O x O y O q$.

(vii) If $x \neq p,q$ and $z \neq p,q$, then (1) itself is a simple closed O-chain.

Thus in every case, y belongs to a simple closed O-chain containing p and q, hence $p O y O q$, i.e., $y \in F$. Therefore, F satisfies all the requirements of a proper cyclic element and F contains p and q. Furthermore, F is the only proper cyclic element containing p and q for by theorem 5.17 if there were another it would be idential to F. Q.E.D.

<u>Theorem 5.19 Suppose x is a point of the Peano space T and and E is the set of all points of T that are conjugate to x. Then E is closed.</u>

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Proof. Let y be a point of $\mathcal{C}E$. We shall show that $\mathcal{C}E$ is open which implies that E is closed. Now there is a point z of T such that z cuts between x and y. Thus, in view of 5.2, x and y are in different components A and B of T-z. Now the single point z is closed, hence T-z is an open subset of the locally connected space T. It follows that since $B \subset T-z$, B is open (1.25). Clearly if w is any point of B then z cuts between w and x, i.e., w is not conjugate to x, hence $w \notin E$ and we have $W \in \mathcal{C}E$. Since for any point w of B we found that $w \in \mathcal{C}E$ we see that $B \subset \mathcal{C}E$. But B is open and $y \in B$, hence there is an $\epsilon > 0$ such that $N(y,\epsilon) \subset B \subset \mathcal{C}E$. That is, for any point y of $\mathcal{C}E$ there is a neighborhood of y contained in $\mathcal{C}E$, hence $\mathcal{C}E$ is open. Therefore, E is closed. Q.E.D.

Theorem 5.20 Suppose C is a proper cyclic element of a Peano space T. Then C is closed.

Proof. Let p_1 and p_2 be two distinct points of C. Let E be the set of all points x such that $p_1 O x O p_2$. Let E_1 be the set of all points x such that $p_1 O x$. Let E_2 be the set of all points x such that $p_2 O x$. In view of 5.15, C = E. But notice $E = E_1 \cap E_2$ and by theorem 5.19, E_1 and E_2 are closed sets. Thus, E is closed, i.e., C is closed. Q.E.D.

Theorem 5.21 Suppose C is a proper cyclic element of a Peano space T. Then C is arcwise connected. Furthermore, if H is any simple arc in T whose end points p_1 and p_2 are in C then $H \subset C$.

Proof. T is arcwise connected (3.9). Thus, it is clear since $C \subset T$ that any two points of C can be joined by a simple arc in T. We must, therefore, only show that such an arc lies entirely in C in order to prove the theorem.

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Let H be a simple arc whose end points p_1 and p_2 are in C. Suppose $H \not\subset C$. Then there exists a point x such that $x \in H$ and $x \notin C$. Consider the subarc xp_1 of H. C is closed (5.20), hence compact. Thus we have on xp_1 a first point q_1 of C. Let H_1 be the subarc of H with end points x and q_1. In a similar manner considering the subarc xp_2 of H we have on xp_2 a first point q_2 of C. Let H_2 be the subarc of H with end points x and q_2. We shall show that

(1) $$(q_1, x, q_2; H_1, H_2, C)$$

is a simple closed O-polygon. First q_1, x and q_2 are distinct points for clearly $q_1 \neq q_2$ and if x coincides with either q_1 or q_2 then $x \in C$ which would prove the theorem. Also, C is a proper cyclic element, hence is semi-connected (5.16). Now H_1 and H_2 being arcs are connected, hence by 5.4 they are both semi-connected. Furthermore, $H_1 \cap H_2 = x$, $H_2 \cap C = q_2$ and $C \cap H_1 = q_1$. Therefore, by 5.13, (1) is a simple closed O-polygon and in view of 5.14, $x O q_1$ and $x O q_2$. That is, $x \in C$. This contradicts the assumption that $x \notin C$, hence $x \in C$ and we have every point of H contained in C, i.e., $H \subset C$. Q.E.D.

<u>Theorem 5.22 Suppose C is a proper cyclic element of a Peano space T and D is any open connected subset of T. Then $C \cap D$ is connected</u>.

<u>Proof</u>. Clearly if $C \cap D = \emptyset$ or $C \cap D$ is a single point, the conclusion is proved. Therefore, suppose the set $C \cap D$ contains at least two points. Let x_1 and x_2 be points of $C \cap D$ such that $x_1 \neq x_2$. Since D is a region in the Peano space T, D is arcwise connected (4.27). Now $x_1 \cup x_2 \subset D$ so let H be a simple arc in D with

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end points x_1 and x_2. But $x_1 \cup x_2 \subset C$, hence theorem 5.21 implies $H \subset C$. Thus, $H \subset C \cap D$, i.e., any two distinct points of $C \cap D$ are joined by a simple arc lying entirely in $C \cap D$. Therefore, $C \cap D$ is arcwise connected, hence connected (3.10). Q.E.D.

Theorem 5.23 Suppose C is a proper cyclic element of a Peano space T and A is a component of T-C. Then $\bar{A}-A$ consists of exactly one point and this point belongs to C and is a cut point of T.

Proof. T-C is open (5.20). Thus, A is a component of an open set T-C in a locally connected space T, hence A is a region in T. This implies that $\bar{A} \neq A$ and we have $\bar{A}-A \neq \emptyset$. Furthermore, $\bar{A}-A \subset C$, for if not there is some point p of $\bar{A}-A$ such that $p \notin C$. Now then $p \in \bar{A}$, $p \in \mathcal{C}A$ and $p \notin C$, i.e., $p \in T-C$. Hence p must belong to some component Q of T-C. If Q = A, we would have $p \in A$ which is impossible since $p \in \mathcal{C}A$. Therefore, $Q \neq A$. It follows that $A \subset A \cup p \subset \bar{A}$ which implies that $A \cup p$ is connected (1.22). But $A \cup p \subset T-C$ and since $p \notin A$ we have $A \neq A \cup p$. Thus, the component A of T-C is a proper subset of a connected set $A \cup p$ in T-C. Since this is contrary to our definition of a component (1.24), we see that our assumption that such a point p existed is false, i.e., $\bar{A}-A \subset C$. In addition to this, if $\bar{A}-A$ consists of exactly one point x, then x is a cut point of T. To show this notice that since $x \notin A$ we have $A \subset T-x$. A is open in T, hence A is open in T-x (1.13). However, since $\bar{A}-A = x$, it follows that $\bar{A} = A \cup x$. Clearly then $A = (T-x) \cap \bar{A}$, i.e., A is closed in T-x (1.13). Hence $\mathcal{C}A$ is open in T-x and we see that T-x is the union of two non-empty disjoint

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sets A and $\mathcal{C}A$, each of which is open in $T-x$. It follows from 1.22 that $T-x$ is disconnected, i.e., x is a cut point of T.

We shall now show that $\overline{A}-A$ does consist of exactly one point thus proving the theorem. Suppose there are at least two distinct points p_1 and p_2 of $\overline{A}-A$. Since $\overline{A}-A \subset C$ we have $p_1 \cup p_2 \subset C \cap (\overline{A}-A)$. Now $p_1 \neq p_2$ hence $\rho(p_1,p_2) = r$ where $r > 0$. Consider the neighborhoods $N(p_1,\frac{1}{2}r)$ and $N(p_2,\frac{1}{2}r)$. These neighborhoods are clearly disjoint. Let G_1 be the component of $N(p_1,\frac{1}{2}r)$ containing p_1 and let G_2 be the component of $N(p_2,\frac{1}{2}r)$ containing p_2. Thus $p_1 \in G_1$, $p_2 \in G_2$ and $G_1 \cap G_2 = \emptyset$. Now G_1 and G_2 are components of the open subsets $N(p_1,\frac{1}{2}r)$ and $N(p_2,\frac{1}{2}r)$ of the locally connected space T (1.6). Thus, G_1 and G_2 are open connected sets (1.25). Let $G = G_1 \cup A \cup G_2$. G is open since G_1, A and G_2 are open. Also, G is connected. To show this notice that G_1, A and G_2 are each connected sets. Furthermore, since $p_1 \in \overline{A}-A$, $p_1 \in \overline{A}$. Thus $A \subset A \cup p_1 \subset \overline{A}$, i.e., $A \cup p_1$ is connected (1.22). Since $p_1 \in G_1$ and $p_1 \in A \cup p_1$, and both sets G_1 and $A \cup p_1$ are connected we have $G_1 \cup (A \cup p_1)$ connected. But $G_1 \cup (A \cup p_1) = G_1 \cup A$, i.e., $G_1 \cup A$ is connected. Similarly, $A \cup G_2$ is connected hence $G_1 \cup A \cup G_2 = G$ is connected (1.22). In summary then G is an open connected subset of T hence theorem 5.22 implies that $C \cap G$ is connected. This, however, is not the case for

$$\begin{aligned} C \cap G &= (C \cap G_1) \cup (C \cap A) \cup (C \cap G_2) \\ &= (C \cap G_1) \cup (C \cap G_2) \end{aligned}$$

since $C \cap A = \emptyset$. Also $C \cap G_1$ and $C \cap G_2$ are both non-empty sets, and $\overline{C \cap G_1} \subset \overline{G} \subset \mathcal{C}G_2 \subset \mathcal{C}(C \cap G_2)$, i.e., $(\overline{C \cap G_1}) \cap (C \cap G_2) = \emptyset$. Similarly, $(C \cap G_1) \cap (\overline{C \cap G_2}) = \emptyset$, and we see that $C \cap G = C \cap G_1 / C \cap G_2$. Thus we

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have a contradiction and our assumption that $\bar{A}-A$ contained at least two distinct points is false. Therefore, $\bar{A}-A$ consists of exactly one point x and as we have seen $x \in C$ and $T-x$ is disconnected. Q.E.D.

Theorem 5.24 Suppose C is a proper cyclic element of a Peano space T and $T-C \neq \emptyset$. Then T-C has at most a finite number of components A such that d(A) is greater than or equal to an assigned $\delta > 0$.

Proof. Suppose the conclusion of the theorem is false. Then there is a $\delta > 0$ such that infinitely many components $S_1, \ldots, S_n, \ldots$ of T-C each have diameter greater than or equal to δ. Since each component is connected and has diameter greater than zero, it must contain a non-countable number of points (1.22). Thus for each n there are two points p_n and q_n of S_n such that

$$\rho(p_n, q_n) > d(S_n) - \frac{1}{n} \geq \delta - \frac{1}{n}.$$

Hence we have two sequences of points $\{p_n\}$ and $\{q_n\}$ for n = 1,2,.... Since T is compact there is a subsequence $\{p_{k_n}\}$ of $\{p_n\}$ such that $\{p_{k_n}\}$ converges (1.21). Now the subsequence $\{q_{k_n}\}$ of $\{q_n\}$ may not converge, but a subsequence $\{q_{1_{k_n}}\}$ of $\{q_{k_n}\}$ will converge (1.21). Let $\{p_{1_{k_n}}\}$ be a subsequence of $\{p_{k_n}\}$. Since $\{p_{k_n}\}$ converges, $\{p_{1_{k_n}}\}$ converges. Let $y_n = p_{1_{k_n}}$ and $z_n = q_{1_{k_n}}$. Thus we have two converging sequences $\{y_n\}$ and $\{z_n\}$ such that

$$(1) \qquad \rho(y_n, z_n) > \delta - \frac{1}{1_{k_n}} \geq \delta - \frac{1}{n}.$$

If $y_n \to p$ and $z_n \to q$ we see that $p \neq q$. For suppose $2\epsilon < \delta - \frac{1}{n}$ where $\epsilon > 0$. Thus for n sufficiently large $\rho(y_n, p) < \epsilon$ and $\rho(z_n, q) < \epsilon$.

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But if $p = q$ we have

$$\begin{aligned}\rho(y_n, z_n) &\leq \rho(y_n, p) + \rho(p, z_n) \\ &= \rho(y_n, p) + \rho(q, z_n) \\ &< \epsilon + \epsilon,\end{aligned}$$

i.e., $\rho(y_n, z_n) < 2\epsilon < \delta - \frac{1}{n}$ which is a contradiction of (1), hence $p \neq q$. Therefore, $\rho(p,q) = r$ where $r > 0$. Consider the neighborhoods $N(p,\frac{1}{2}r)$ and $N(q,\frac{1}{2}r)$. Clearly they are disjoint open sets (1.6). Let P be the component of $N(p,\frac{1}{2}r)$ containing p and let Q be the component of $N(q,\frac{1}{2}r)$ containing q. Now $p \in P$, $q \in Q$ and P and Q are open connected sets since each is contained in an open set of the locally connected space T. Furthermore, $P \cap Q = \emptyset$. Since P is open there is a $\gamma > 0$ such that $N(p,\gamma) \subset P$. But $y_n \rightarrow p$, thus there is an n_1 such that if $n > n_1$ we will have all but a finite number of points of the sequence $\{y_n\}$ contained in $N(p,\gamma) \subset P$ (1.7). Therefore, $S_n \cap P \neq \emptyset$ for $n > n_1$. Similarly, there is an n_2 such that if $n > n_2$, $S_n \cap Q \neq \emptyset$. Let $n_o = \ulcorner n_1, n_2 \urcorner$. It follows that if $n > n_o$ we have $S_n \cap P \neq \emptyset$ and $S_n \cap Q \neq \emptyset$. Then since P and S_n are both connected, $P \cup S_n$ is connected. Now S_n is a component of T-C and $S_n \subset P \cup S_n$, hence we cannot have $P \cup S_n \subset T-C$ unless $P \subset S_n$ for $n > n_o$ (1.24). But the sets $S_1, \ldots, S_n, \ldots$ are disjoint, i.e., P can be contained in S_n for only one value of n. Therefore, $P \cup S_n \not\subset T-C$, i.e., $(P \cup S_n) \cap C \neq \emptyset$ for $n > n_o$. However, $S_n \subset T-C$, i.e., $S_n \cap C = \emptyset$ hence we must have $P \cap C \neq \emptyset$. Similarly, we have $Q \cap C \neq \emptyset$. Let $D = P \cup S_n \cup Q$. Clearly, D is open and if $n > n_o$, $P \cap S_n \neq \emptyset$ and $Q \cap S_n \neq \emptyset$ hence D is connected (if $n > n_o$). Now theorem 5.22 implies that $C \cap D$ is connected. But

$$C \cap D = (C \cap P) \cup (C \cap S_n) \cup (C \cap Q) = (C \cap P) \cup (C \cap Q).$$

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where $C\cap P \neq \emptyset$ and $C\cap Q \neq \emptyset$. Furthermore,

$$(\overline{C\cap P})\subset\overline{P}\subset\mathcal{C}Q\subset\mathcal{C}(C\cap Q)$$

hence $(\overline{C\cap P})\cap(C\cap Q) = \emptyset$. Similarly, $(C\cap P)\cap(\overline{C\cap Q}) = \emptyset$ and we see that $C\cap D = C\cap P/C\cap Q$. Therefore, we have a contradiction, hence our assumption that there existed a $\delta>0$ such that infinitely many components of T-C each had diameter greater than or equal to δ is false. Q.E.D.

Definition 5.25 A subset H of a space T is termed a retract of T if and only if there exists a continuous mapping f of T onto H such that $f(x) = x$ for every point $x\in H$. Such a mapping f is called a retraction of T onto H.

Theorem 5.26 If C is a proper cyclic element of a Peano space T, then C is a retract of T.

Proof. If $C = T$ then the identity mapping is clearly a retraction of T onto C, hence we assume $T-C \neq \emptyset$. We shall define a mapping $f:T\rightarrow C$ in the following manner. Let $f(x) = x$ if $x\in C$. If $y\in T-C$, let A_y be the component of T-C containing y. By theorem 5.23, $\overline{A}_y-A_y$ is a single point $z\in C$. Let $f(y) = z$. Clearly then $f(T) = C$, i.e., f maps T onto C. To show that f is a retraction of T onto C we must yet show that f is continuous.

Suppose f is not continuous. Since T is compact and f is not continuous there is a sequence $\{x_n\}$ and a point x_o of T such that

(1) $$x_n\rightarrow x_o \text{ but } f(x_n)\rightarrow y_o \neq f(x_o).$$

There are three cases to consider.

(i) Suppose $x_o\in T-C$. Then $x_o\in A_{x_o}$. A_{x_o} is contained in the open set T-C, hence A_{x_o} is open (1.25). Thus, there is an

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$\epsilon > 0$ such that $N(x_o, \epsilon) \subset A_{x_o}$. Furthermore, all but a finite number of the points x_n are contained in $N(x_o, \epsilon)$, hence in A_{x_o}. That is, there exists an m such that if $n > m$, $x_n \in A_{x_o}$. It follows that $A_{x_n} = A_{x_o}$ for $n > m$. Hence $f(x_n) = \bar{A}_{x_n} - A_{x_n} = \bar{A}_{x_o} - A_{x_o} = f(x_o)$, i.e., $f(x_n) = f(x_o)$.

(ii) Suppose $x_o \in C$ and $x_n \in C$ for infinitely many values of n. Then $f(x_o) = x_o$, and $f(x_n) = x_n$ for the x_n in C. Since $x_n \in C$ for infinitely many values of n, there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that each point $x_{k_n} \in C$, hence $f(x_{k_m}) = x_{k_n}$ for each point x_{k_n}. From (1) we see that $x_n \to x_o$ hence $x_{k_n} \to x_o$, i.e., $f(x_{k_n}) \to x_o$. But $f(x_o) = x_o$ hence $f(x_{k_n}) \to f(x_o)$. Again from (1) we see that $f(x_n) \to y_o$ and it follows that $f(x_{k_n}) \to y_o$. Therefore, $f(x_o) = y_o$ which contradicts (1).

(iii) Suppose $x_o \in C$ and $x_n \in T-C$ for n greater than a certain m. Then if $n > m$, $f(x_n) = \bar{A}_{x_n} - A_{x_n}$. Again $f(x_o) = x_o$ and since from (1) $f(x_o) \neq y_o$, we have $x_o \neq y_o$. We also know from (1) that $x_n \to x_o$ and $f(x_n) \to y_o$. Thus if n is sufficiently large

$$\rho(f(x_n), x_n) > \frac{\rho(x_o, y_o)}{2}.$$

This implies that $d(A_{x_n}) > \frac{\rho(x_o, y_o)}{2}$ since $x_n \in A_{x_n} \subset \bar{A}_{x_n}$ and $f(x_n) = \bar{A}_{x_n} - A_{x_n} \subset \bar{A}_{x_n}$ for n sufficiently large and since $d(A_{x_n}) = d(\bar{A}_{x_n})$. In view of theorem 5.24 there must be a component A of T-C which occurs an infinite number of times in the sequence $A_{x_1}, \ldots, A_{x_n}, \ldots$. Let $p = \bar{A} - A$. Thus $x_n \in A$ for infinitely many values of n, hence $f(x_n) = p$ for infinitely many values of n. We now have $x_n \to x_o$, $f(x_n) \to p$ and $f(x_n) \to y_o$, i.e., $y_o = p$. Since $x_n \in A$ for infinitely many values of n there is a subsequence

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$\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \in A$ for each n. But $x_n \to x_o$ implies that $x_{k_n} \to x_o$, i.e., $x_o \in \bar{A}$ (1.9). But $x_o \in C$ by assumption and since $C \cap A = \emptyset$, $x_o \in \complement A$, we have $x_o \in \bar{A}-A = p$. Thus $x_o = p = y_o$. This is again a contradiction since $x_o \in C$ implies $f(x_o) = x_o$ and (1) states that $f(x_o) \neq y_o$, i.e., $x_o \neq y_o$. Therefore, in each case we found that the assumption that there existed a point x_o and a sequence $\{x_n\}$ such that $x_n \to x_o$ but $f(x_n) \to y_o \neq f(x_o)$ was false. Hence f is continuous. Q.E.D.

Theorem 5.27 Suppose C is a proper cyclic element of a Peano space T and A is a component of T-C. According to theorem 5.23, $\bar{A}$-A is a single point $p \in C$. Now let G be any connected set of T that intersects both A and C. Then $p \in G$.

Proof. Clearly, $G = [A \cap G] \cup [(T-A) \cap G]$. Since G intersects both A and C, and $A \subset T-C$ we have $A \cap G \neq \emptyset$ and $(T-A) \cap G \neq \emptyset$. Also A is open as we have seen in 5.23 hence T-A is closed, i.e., $T-A = \overline{T-A}$ and since $A \cap (T-A) = \emptyset$ we have $A \cap (\overline{T-A}) = \emptyset$. Thus

$$[A \cap G] \cap [\overline{(T-A) \cap G}] \subset A \cap G \cap (\overline{T-A}) \cap \bar{G} = \emptyset,$$

hence also $[A \cap G] \cap [(T-A) \cap C] = \emptyset$. Now G is connected by hypothesis, so we must have $[\overline{A \cap G}] \cap [(\overline{T-A) \cap G}] \neq \emptyset$ in order to preserve this connectedness of G. Furthermore, we found in theorem 5.23 that $\bar{A} = A \cup p$. Thus it follows that

$$\begin{aligned} \emptyset \neq [\overline{A \cap G}] \cap [(T-A) \cap G] &\subset \bar{A} \cap [(T-A) \cap G] \\ &= [A \cup p] \cap [(T-A) \cap G] \\ &= [A \cap (T-A) \cap G] \cup [p \cap (T-A) \cap G] \\ &= p \cap G, \end{aligned}$$

i.e., $p \cap G \neq \emptyset$ hence $p \in G$. Q.E.D.

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For further reference, Rado's Length and Area contains a similar discussion of sections 5.2 through 5.27.

<u>Theorem 5.28 Suppose C is a proper cyclic element of a Peano space T. Then</u>

(i) <u>If H is any connected subset of T, then $H \cap C$ is connected</u>.

(ii) <u>C is a cyclic Peano space</u>.

<u>Proof</u>. First we notice that if either $H = T$ or $H \cap C = \emptyset$ the statement is immediate. So we can assume that $T-C \neq \emptyset$ and $H \cap C \neq \emptyset$. By theorem 5.26 there exists a retraction f of T onto C. We shall use the same retraction f defined in that theorem. Since f is continuous and H is connected, the set f(H) in C is connected (1.27). We shall show that $f(H) = H \cap C$, thus proving the connectedness of $H \cap C$. Now $H \cap C \subset C$ and $f(x) = x$ for $x \in C$. Therefore,

$$H \cap C = f(H \cap C) \subset f(H).$$

To show that $f(H) \subset H \cap C$ let w be any point in f(H). Thus there is a point r of H such that $f(r) = w$. We now have $w \in f(H) \subset C$. There are two cases to consider.

(1) If $r \in C$ then $f(r) = r$, i.e., $r = w$ hence $w \in H$ and we have $w \in H \cap C$.

(2) If $r \in T-C$ then $r \in A_r$ where A_r is the component of T-C containing r. Recall that we defined f so that $w = \bar{A}_r - A_r$. Now we have $H \cap C \neq \emptyset$ and since $r \in H$ and $r \in A_r$ we have $H \cap A_r \neq \emptyset$. Thus by theorem 5.27, $w \in H$, i.e., $w \in H \cap C$.

Therefore, in both cases $w \in f(H)$ implies $w \in H \cap C$, i.e., $f(H) \subset H \cap C$ and hence $f(H) = H \cap C$.

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To prove (ii) notice that f maps T continuously onto C hence we see that C is a Peano space. Let p be any point of C, and consider the set C-p. We shall show that C-p is connected, thus in view of 5.3, C is cyclic. Let a be a fixed point of C-p. Since T is cyclic, $a \bigcirc x$ in T for all points x of C-p, hence a and x lie in the same component E of T-p. Clearly, then $p \notin E$ and $C-p \subset E$, and since $C-p \subset C$ it follows that $C-p = E \cap C$. But by virtue of (i) of this theorem, $E \cap C$ is connected, i.e., C-p is connected. Q.E.D.

As a conclusion to this paper we are going to prove the cyclic connectivity theorem that any two points of a cyclic Peano space T lie on a simple closed curve in T. We now have sufficient background to prove some introductory theorems leading to this result.

Theorem 5.29 Suppose A and B are non-degenerate disjoint closed subsets of a cyclic Peano space T. Then there exists two disjoint arcs P and Q in T, such that each one has one end point in A and the other end point in B and no interior point in $A \cup B$.

Proof. Let us define a subset E of T to consist of the set A and all points w of T-A such that

(i) There exists an arc F in T with one end point in A, the other end point in B, and no interior point in $A \cup B$.

(ii) There is an arc G in T-F such that w is one end point of G and G intersects A in exactly one point, which is its other end point.

Clearly $E \neq \emptyset$ since $A \subset E$. If we can show that E is both open and closed in T then we must have E = T in order to preserve the connectedness of T. For, otherwise, T will be the union of two

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non-empty, disjoint open set E and $\mathcal{C}E$, hence T would be disconnected. It follows that if E = T then $B \subset E$, thus clearly the required arcs exist.

First E is open. For let z be any point of E and we have two cases to consider.

(1) Suppose $z \in A$. Since T is cyclic and z is a closed set, T-z is a region in T. Thus T-z is arcwise connected (4.27). Furthermore, since $A \cap B = \emptyset$ and $z \in A$ we have $B \subset T-z$. Let x and y be any two points of A and B respectively such that $x \neq z$ and $y \neq z$. Then there exists an arc F_1 in T-z joining x and y. Clearly, F_1, A and B are each compact sets such that $F_1 \cap A \neq \emptyset$ and $F_1 \cap B \neq \emptyset$. Therefore, there are first points of A and B on F_1. Denoting these points by p in A and q in B, we have an arc F in T-z joining p and q, and F satisfies (i) above. Since F is closed and $F \subset T-z$, z belongs to the open set $\mathcal{C}F$. Let V be the component of $\mathcal{C}F$ containing z. T is locally connected, hence V is a region in T (1.25). It follows that there is an $\epsilon > 0$ such that $N(z,\epsilon) \subset V$. If $V \subset A$ then $V \subset E$ hence for any point z of E such that $z \in A$ there is an $\epsilon > 0$ such that $N(z,\epsilon) \subset E$, i.e., E is open. If $V \not\subset A$ then there is a point w of V-A. Thus $z \cup w \subset V$ and since V is a region in T, V is arcwise connected. Let G_1 be an arc in V joining w and z. But A and G_1 are compact, $w \in V-A$ and $z \in A$ hence there is a first point r of A in G_1. Let G be the arc wr. Thus we have $G \subset V \subset \mathcal{C}F$, G has one end point r in A and no other point in A and the other end point is w. Hence G satisfies (ii) and we have $w \in E$. Thus if $w \in V-A$ then $w \in E$ and it follows that whether $V \subset A$ or $V \not\subset A$ we have $V \subset E$ and again $N(z,\epsilon) \subset V \subset E$, i.e., E is open.

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(2) Suppose $z \notin A$, i.e,, $z \in E-A$. Now by the same reasoning as in (1) we find there are arcs F and G satisfying (i) and (ii) respectively. Thus $G = tz$ where G intersects A in only one point t and $z \in E-A$, i.e., $z \in T-A$. Since $G \subset T-F$, $z \notin F \cup A$. Therefore, $z \in \mathcal{C}(F \cup A)$. Now F and A are both closed sets, hence $F \cup A$ is closed, i.e., $\mathcal{C}(F \cup A)$ is open. Let M be the component of $\mathcal{C}(F \cup A)$ containing z. Thus M is an open connected set (1.25). Let w be any point of M. Since M is a region, M is arcwise connected hence there is an arc in M joining w and z. Now $z \in G \cap wz$ hence $G \cap wz \neq \emptyset$. Let s be the first point of $G(=tz)$ on wz going in the order from w to z. Thus we can obtain the arc tsw. Clearly $tsw \subset T-F$ and tsw has w as one end point and intersects A in only one point, namely, the other end point t. Therefore, tsw satisfies (ii), hence $w \in E$. But w was an arbitrary point of M thus $M \subset E$. Now $z \in M$ and M is open hence there is an $\epsilon > 0$ such that $N(z,\epsilon) \subset M \subset E$. That is, any arbitrary point z of E is in a neighborhood which is contained in E. Thus E is open in either case (1) or (2).

We shall now show that E is also closed. (See Figure 1, page 93). Suppose E is not closed. Thus there exists some point w of T-E which is a limit point of E, i.e., $w \in \bar{E}$ but $w \notin E$ (1.10). Since the single point w is a closed set, and T is cyclic, we see that T-w is a region in T. Therefore, T-w is arcwise connected. Now A and B are non-degenerate sets by hypothesis, hence there is some arc F_1 in T-w such that F_1 has one end point in A and one end point in B and no other points in either A or B, i.e., F_1 satisfies (i). Since $A \subset E$ and $w \in T-E$ we have $w \notin A$. Also $F_1 \subset T-w$ hence $w \notin F_1$, thus $w \notin A \cup F_1$, i.e., $w \in \mathcal{C}(A \cup F_1)$. $\mathcal{C}(A \cup F_1)$ is clearly an

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open set hence there is a $\delta > 0$ such that $N(w,\delta) \subset \mathcal{C}(A \cup F_1)$. Consider the neighborhood $N(w,\frac{1}{2}\delta)$. By theorem 4.27, T has a basis, each non-empty element of which is open, connected and has a Peano space as its closure. Now $N(w,\frac{1}{2}\delta)$ is open hence can be expressed as the union of some of the elements of such a basis. Let H be the element of the basis in $N(w,\frac{1}{2}\delta)$ such that $w \in H$. Then

$$w \in H \subset N(w,\tfrac{1}{2}\delta) \subset \overline{N(w,\tfrac{1}{2}\delta)} \subset N(w,\delta) \subset \mathcal{C}(A \cup F_1).$$

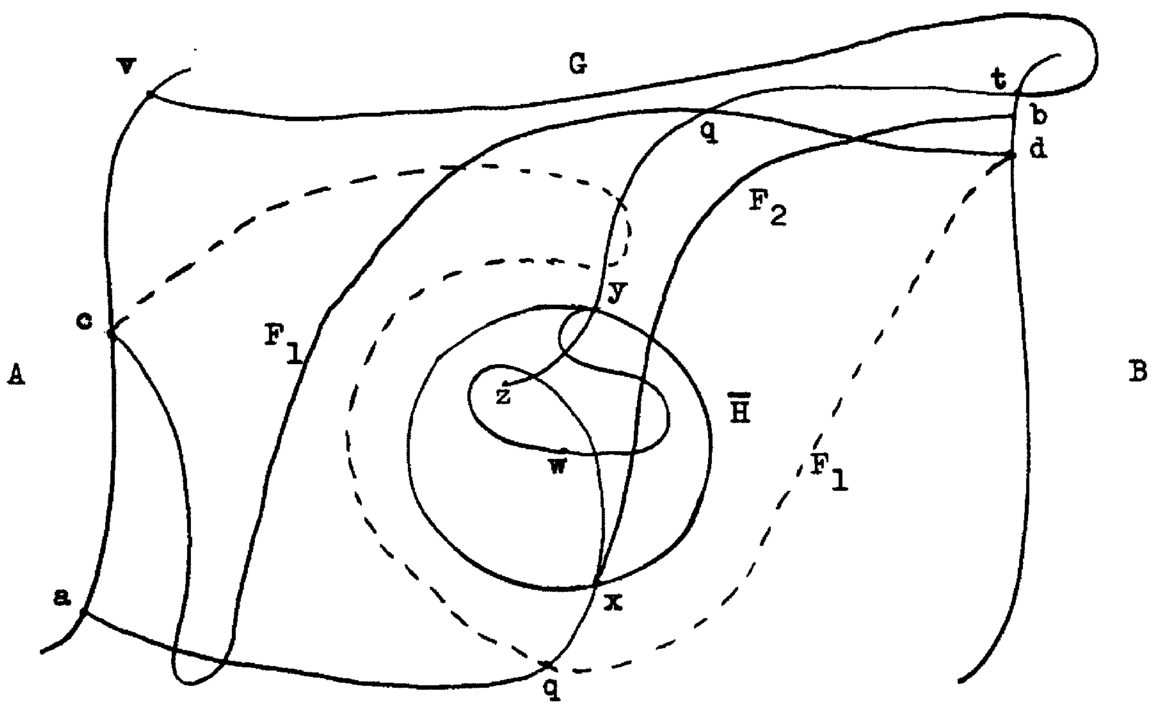

Figure 1

Furthermore, $\overline{H} \subset \overline{N(w,\frac{1}{2}\delta)} \subset \mathcal{C}(A \cup F_1)$ hence $\overline{H} \cap (A \cup F_1) = \emptyset$. Since $w \in H$ and w is a limit point of E, any neighborhood of w will contain points of E. Therefore, clearly $H \cap E \neq \emptyset$. Let $z \in H \cap E$. Then $z \in E$ and since $\overline{H} \cap (A \cup F_1) = \emptyset$, $z \notin (A \cup F_1)$, thus $z \notin A$. It follows, by the same reasoning as used earlier in the theorem, that since T is arcwise connected there exists an arc $G = vz$ satisfying (ii)., i.e., $G \cap A = v$, and there exists an arc F_2 satisfying (i). Now $z \in H$ hence $z \in \overline{H}$ thus $G \cap \overline{H} \neq \emptyset$. Let y be the first point of G in $\overline{H}$

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going in order from v to z. Now $y \in \bar{H}$ and $w \in \bar{H}$. But $\bar{H}$ is a Peano space hence we have an arc $yw \subset \bar{H}$. Consider the arc vyw. We readily see that $vyw \cap F_1 \neq \emptyset$ and $vyw \cap F_2 \neq \emptyset$. For if either of these intersections were the empty set then vyw would satisfy (ii) hence w would be a point of E. This is impossible since by assumption $w \notin E$, hence the intersections are both non-empty. On the other hand, we know that $yw \cap F_1 = \emptyset$ since $yw \subset \bar{H} \subset C(A \cup F_1)$, so we must have $vy \cap F_1 \neq \emptyset$. Similarly, $vyw \cap F_2 \neq \emptyset$ implies $yw \cap F_2 \neq \emptyset$ since $vy \subset vz = G \subset T - F_2$, i.e., $vy \cap F_2 = \emptyset$. Now $yw \subset \bar{H}$ hence the relation $yw \cap F_2 \neq \emptyset$ implies $\bar{H} \cap F_2 \neq \emptyset$. Let $F_2 = ab$ where $a \in A$, $b \in B$ and $ab \cap (A \cup B) = a \cup b$. Let x be the first point of $F_2(=ab)$ in $\bar{H}$, going in order from a to b. Thus we have the arc ax. Since $x \cup w \subset \bar{H}$ there is an arc $xw \subset \bar{H}$ (since $\bar{H}$ is arcwise connected). We see that the arc axw is such that $axw \cap F_1 \neq \emptyset$. For if $axw \cap F_1 = \emptyset$, then axw would satisfy (ii) and we would have the contradiction that $w \in E$. But since $xw \subset \bar{H} \subset C(A \cup F_1)$ we have $xw \cap F_1 = \emptyset$ thus $ax \cap F_1 \neq \emptyset$. Recall that F_1 satisfies (i) so if we let $F_1 = cd$ we have $c \in A$, $d \in B$ and $cd \cap (A \cup B) = c \cup d$. Also recall that $vy \cap F_1 \neq \emptyset$ and $ax \cap F_1 \neq \emptyset$. Let q be the last point of $F_1(=cd)$ which lies on $vy \cup ax$, going in order from c to d. Thus $q \in F_1$ and $q \in (vy \cup ax)$. But notice $vy \cap ax = \emptyset$ since $ax \subset ab = F_2$, $vy \subset vz = G$ and $G \subset T - F_2$, i.e., $G \cap F_2 = \emptyset$. Thus we have two cases to consider, either $q \in vy$ or else $q \in ax$.

(1) Suppose $q \in vy$. Thus $q \notin ax$. Now $q \neq v$, for if we had $q = v$ then since $v \in A$ we would have $q \in A$. But we know $q \in cd = F_1$ where $cd \cap A = c$. Therefore, we would have $q = v = c$. Since q is the last point of cd which lies on $vy \cup ax$ we see that

$$(qd - q) \cap (vy \cup ax) = \emptyset$$

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hence $(qd-q)\cap ax = \emptyset$. But since $q = c$ this implies $(cd-d)\cap ax = \emptyset$. Furthermore, $q \notin ax$ hence $c \notin ax$, i.e., $ax \cap c = \emptyset$. It follows that $cd \cap ax = \emptyset$, i.e., $F_1 \cap ax = \emptyset$ which is impossible since we know that $F_1 \cap ax \neq \emptyset$. Thus $q \neq v$. By similar reasoning $q \neq c$. Let $D = vq \cup qd$. Since $d \in (vq \cup qd) \cap B$ we have $(vq \cup qd) \cap B \neq \emptyset$. Let t be the first point of D which lies on the compact set B and let $F = vt$ and $G^* = axw$. Clearly F is an arc joining v of A and t of B and $F \cap (A \cup B) = v \cup t$. Thus F satisifes (i). Furthermore, G^* has one end point w in $T-E \subset T-A$ and since $\overline{\overline{H}} \cap A = \emptyset$, we have

$$\begin{aligned} G^* \cap A = axw \cap A &= (ax \cup xw) \cap A \\ &\subset (ab \cup \overline{\overline{H}}) \cap A \\ &= (ab \cap A) \cup (\overline{\overline{H}} \cap A) \\ &= a, \end{aligned}$$

i.e., $G^* \cap A = a$. Therefore, if $G^* \subset T-F$ then G^* satisfies (ii), hence $w \in E$, in contradiction with our assumption that $w \notin E$. We shall show that this is the case. Since F_2 and G satisfy (i) and (ii), $F_2 \cap G = \emptyset$. But $vq \subset G$ and $ax \subset F_2$, hence $vq \cap ax = \emptyset$. Also, since q is the last point of cd which lies in $vy \cup ax$ and $q \in vy$ we have $qd \cap ax = \emptyset$. Thus $(vq \cup qd) \cap ax = \emptyset$ and since $F \subset (vq \cup qd)$ we have $F \cap ax = \emptyset$. Now $q \in F_1$ and $\overline{\overline{H}} \cap (F_1 \cup A) = \emptyset$ implies $\overline{\overline{H}} \cap F_1 = \emptyset$. Thus $q \notin \overline{\overline{H}}$. But $q \in vy$ where vy has only one point in $\overline{\overline{H}}$, namely y, hence $q \neq y$. Therefore, $vq \cap \overline{\overline{H}} = \emptyset$ and since $xw \subset \overline{\overline{H}}$ we have $vq \cap xw = \emptyset$. Also $qd \subset F_1$, hence $qd \cap xw = \emptyset$, and we have $(vq \cup qd) \cap xw = \emptyset$. This implies that $F \cap xw = \emptyset$. Finally then we have $F \cap (ax \cup xw) = \emptyset$, i.e., $F \cap G^* = \emptyset$, hence $G^* \subset T-F$.

(2) Suppose $q \in ax$. (In the rest of the proof since we are assuming $q \in ax$, we are considering a second possibility for F_1 as

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indicated by the dotted line in the figure). First $q \neq d$, for suppose otherwise, i.e., suppose $q = d$. Then $q \in F_2$ and since $d \in B$ we have $q \in B$. Thus $q \in F_2 \cap B$. But $F_2 \cap B = b$, hence $q = b$. This is impossible since $aq \subset ax \subset ab$ and if $q = b$ clearly $x = b = q = d$, i.e., $x = d$. Then since $d \in F_1$, $x \in F_1$. But we know $x \in \overline{H}$ hence $F_1 \cap \overline{H} \neq \emptyset$ which contradicts the fact that $F_1 \cap \overline{H} = \emptyset$. Therefore we can assume that $q \neq d$. Let $F^* = aqd$ and $G^{**} = vyw$. Clearly F^* satisfies (i) and G^{**} meets A in exactly one point; namely v. Furthermore, G^{**} has its other end point w in T-A. Thus, if $G^{**} \subset T\text{-}F$, G^{**} satisfies (ii) and again we will have $w \in E$. Now $aq \subset F_2$, $vy \subset G$ and $F_2 \cap G = \emptyset$, hence $aq \cap vy = \emptyset$. Also $aq \subset ax$, $q \in F$, $x \in \overline{H}$ and $F_1 \cap \overline{H} = \emptyset$, hence $q \neq x$. Thus, $x \notin aq$ and we have $aq \cap \overline{H} = \emptyset$. But since $yw \subset \overline{H}$, it follows that $aq \cap yw = \emptyset$, thus $aq \cap vyw = \emptyset$, i.e., $aq \cap G^{**} = \emptyset$. Since we defined q to be the last point of F_1 on $vy \cup ax$ we see that $(qd\text{-}q) \cap (vy \cup ax) = \emptyset$. Thus $(qd\text{-}q) \cap vy = \emptyset$ and since $q \in ax$ implies $q \notin vy$, we have $qd \cap vy = \emptyset$. Also qd F_1 and yw $\overline{H}$, hence qd yw $= \emptyset$, thus qd vyw $= qd \cap G^{**} = \emptyset$. Finally then $F^* \cap G^{**} = aqd \cap G^{**} = \emptyset$, i.e., $G^{**} \subset T\text{-}F^*$ and G^{**} satisfies (ii), i.e., $w \in E$.

Now in every case we found that $w \in E$. But the assumption that E was not closed implied $w \in T\text{-}E$, hence we have a contradiction, i.e., E is closed. Q.E.D.

<u>Theorem 5.30 Suppose x is any point of a Peano space T. Then x is a cut point of some Peano space D contained in T if and only if x is an interior point of an arc in T.</u>

<u>Proof</u>. Suppose x is a cut point of a Peano space D and $D \subset T$. Then D-x has a separation A/B. Let c be a point of A and

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d be a point of B. Since D is arcwise connected and c and d are in D, there is an arc cd contained in D. Clearly $c \neq x$, $d \neq x$ and cd is connected. If $cd \subset D-x$ then either $cd \subset A$ or else $cd \subset B$ (1.22). But $cd \not\subset A$ since $d \in B$ and $A \cap B = \emptyset$. Similarly, $cd \not\subset B$ since $c \in A$. Thus $cd \not\subset D-x$. However, $cd \subset D$, hence x is an interior point of the arc cd and $cd \subset D \subset T$.

To show the converse suppose x is an interior point of an arc ab in T. Thus $x \in ab$, $x \neq a$ and $x \neq b$. Since ab is a Peano space (3.3) and contains only two non-cut points, namely its end points a and b, we see that x must be a cut point of ab. Therefore, x is a cut point of a Peano space in T. Q.E.D.

<u>Theorem 5.31 Suppose T is a non-degenerate Peano space, p is a non-cut point of T and H is an open set containing p. Then there is an open set G containing p such that $p \in G \subset \overline{G} \subset H$ and T-H is contained in a single component of T-G.</u>

<u>Proof</u>. Let H be an open set containing p. Since T is non-degenerate there exists a point q of T such that $p \neq q$. Thus $\rho(p,q) = \delta$ where $\delta > 0$. H is open, hence there is an $\epsilon > 0$ such that $N(p,\epsilon) \subset H$. Let $\lambda = \lfloor \frac{1}{2}\epsilon, \frac{1}{2}\delta \rfloor$ where $\lambda > 0$. Then $p \in N(p,\lambda)$ and $N(p,\lambda)$ is open. Therefore

$$p \in \overline{N(p,\lambda)} \subset \overline{N(p,\tfrac{1}{2}\epsilon)} \subset N(p,\epsilon) \subset H$$

and clearly $q \notin \overline{N(p,\lambda)}$. Let $W = N(p,\lambda)$. Then W is open, $\overline{W} \subset H$ and $q \in T-\overline{W}$ hence $T-\overline{W} \neq \emptyset$. Since $\overline{W} \subset H$ we have $T-H \subset T-\overline{W}$ where $T-\overline{W}$ is an open set. Thus the components of $T-\overline{W}$ are open (1.25) and, furthermore, form an open covering of T-H. But T-H is closed, hence compact, and it follows from 1.21 that a finite number of these components $A_1, A_2, \ldots, A_n$ cover T-H. Then we have $T-H \subset \bigcup_{i=1}^{n} A_i$ where

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each A_i is non-empty, open, connected and contained in $T-\overline{W}$. Since $p\in\overline{W}$, $p\notin T-\overline{W}$ hence $p\notin T-H$. This implies that $p\notin A_i$ for $i = 1,\ldots,n$, so we have $A_i\subset T-p$ for $i = 1,\ldots,n$. Now p is a non-cut point of T and p is also a closed set hence T-p is a region in T. Therefore, T-p is arcwise connected and there exists an arc F_j for $j = 2,3,\ldots,n$, joining a point q_1 of A_1 with a point q_j of A_j. Clearly $F_j\subset T-p$ for $j = 2,\ldots,n$, and each F_j is closed. Letting $F = \bigcup_{j=2}^{n}F_j$ we see that F is closed (1.10). Also $F\subset T-p$, hence $p\notin F$. Since each F_j is connected and has the common point q_1 it follows from 1.22 that F is connected. We know that $p\in W$ hence we now have $p\in W-F$. Let $G = W-F$ and let $E = F\cup(\bigcup_{i=1}^{n}A_i)$. Clearly $q_i\in F\cap A_i$ for $i = 1,\ldots,n$ and since F and each A_i are connected sets we see that E is connected (1.22). We shall show that $E\subset T-G$. Since $G\subset W-F$ implies $G\subset CF$ we have $F\subset CG = T-G$. Also $A_i\subset T-\overline{W}$ for each i, but $T-\overline{W}\subset T-W\subset T-G$ since $G\subset W$. Therefore, $\bigcup_{i=1}^{n}A_i\subset T-G$ and it follows that $E\subset T-G$. Recall that $T-H\subset\bigcup_{i=1}^{n}A_i$. Thus $T-H\subset E\subset T-G$. Finally, then since E is connected, no component of T-G can be a proper subset of E. And since components are disjoint or equal, E must itself be a component of T-G or else must lie in one component of T-G. In either case T-H is contained in a single component of T-G. Furthermore, since $p\in G$ and $G\subset W$, we have $p\in G\subset\overline{G}\subset\overline{W}\subset H$ and since $G = W-F$, G is open, hence the theorem is proved. Q.E.D.

<u>Theorem 5.32 Suppose p is a point of a non-degenerate cyclic Peano space T and H is any open set containing p such that $\overline{H}$ is a Peano space and p is a non-cut point of $\overline{H}$. Then there is a proper cyclic element of $\overline{H}$ which contains p</u>.

<u>Proof</u>. If H = T the theorem is trivially true, Thus, we

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assume H is a proper subset of T. Furthermore, $p \neq H$, for if $p = H$ then since p is closed, H is closed. But H is an open set by hypothesis hence T would be the union of two non-empty disjoint open sets H and $\mathcal{C}H$, i.e., T would be disconnected which is impossible. Therefore, we have $p \neq H$ hence $H-p \neq \emptyset$, i.e., there is a point q such that $q \neq p$ and $q \in H \subset \overline{H}$. Thus $\overline{H}$ is a non-degenerate Peano space. Now if it can be shown that there exists a point r of $\overline{H}$ such that $r \neq p$ and $r \bigcirc p$ in $\overline{H}$ then by theorem 5.18 there will exist a proper cyclic element of H containing r and p, in particular containing p, and the theorem will be proved. We shall show that such a point does exist.

Since T is a metric space and H is an open set contained in T and containing p there is an open set W containing p such that $p \in \overline{W} \subset H$ (1.11). Thus $W \subset H \subset \overline{H}$ and since H is open and $H \neq T$, we have $H \neq \overline{H}$, hence $W \neq \overline{H}$. Therefore, $\overline{H}-W \neq \emptyset$. Now p is a non-cut point of the Peano space $\overline{H}$ and W is an open set containing p and contained in $\overline{H}$, hence by theorem 5.31 there is a set G containing p such that G is open in $\overline{H}$, $G \subset \overline{G} \subset W$ and $\overline{H}-W$ is contained in a single component A of $\overline{H}-G$. Since G is open in $\overline{H}$, $\overline{H}-G$ is closed in $\overline{H}$. Therefore, A is closed in $\overline{H}$ (1.24). Let t be a point in A. Clearly $p \neq t$ since $A \subset \overline{H}-G$ and $p \in G$. Now both p and t belong to $\overline{H}$ and $\overline{H}$ is arcwise connected, hence there is an arc pt in $\overline{H}$ and $pt \cap A \neq \emptyset$. Thus we may let y be the first point of pt in A. Clearly $p \neq y$ and we have an arc py such that $py \cap A = y$ and $y \in \overline{H}$. We shall show that $p \bigcirc y$ in $\overline{H}$. Suppose otherwise, i.e., there exists a point z of $\overline{H}$ such that z cuts between p and y. Therefore p and y lie in different components of $\overline{H}-z$.

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This implies $z \neq p$ and $z \neq y$. Also $z \in py$ for if not then since py is a connected set it must lie in one component of $\overline{H}-z$ which is impossible since p and y lie in different components of $\overline{H}-z$. It is clear that $z \cap A \subset py \cap A = y$, i.e., $z \cap A \subset y$. But $z \neq y$, hence $z \cap A = \emptyset$, i.e., $z \notin A$. But $A \subset \overline{H}$ hence $A \subset \overline{H}-z$. Now since A is connected and $y \in A$, y and A must lie in the same component of $\overline{H}-z$. Thus p and A lie in different components of $\overline{H}-z$, i.e., z cuts between p and all of the points of A. Let F be any arc in $\overline{H}-z$ joining p to a point of A. Suppose $z \notin F$. Then $p \cup F \cup A$ is clearly a connected set contained in $\overline{H}-z$. Thus $p \cup F \cup A$ must be contained in the same component of $\overline{H}-z$. This is impossible since p and A lie in different components of $\overline{H}-z$. Therefore, $z \in F$. Recall that W is open, hence $T-W$ is closed and since T is compact, $T-W$ is compact. We have seen that $\overline{H}-W \neq \emptyset$ hence we may let b be a point of $\overline{H}-W$. Clearly then $\overline{H} \subset T$ implies $\overline{H}-W \subset T-W$ and $b \in T-W$. Furthermore, $\overline{H}-W \subset A$, hence $b \in A$ which implies $b \neq z$ since $z \notin A$. Thus $b \in T-z$. Now the single point z is closed and T is cyclic hence $T-z$ is a region in T. Therefore, $T-z$ is arcwise connected so we have an arc pb in $T-z$. Since $b \in T-W$ we have $pb \cap (T-W) \neq \emptyset$. Let q be the first point of pb in the compact set $T-W$. Therefore, pq is an arc such that $pq \cap (T-W) = q$, i.e., $q \in \mathcal{B}W$. It follows that $(pq-q) \cap (T-W) = \emptyset$, i.e., $(pq-q) \subset W$. But

$$pq = \overline{(pq-q)} \subset \overline{W} \subset H \subset \overline{H},$$

thus $q \in \overline{H}$ hence $q \in \overline{H}-W \subset A$, i.e., $q \in A$. Finally then we have an arc pq joining the point p and a point q of A. And since $pq \subset \overline{H}-z$, $z \notin pq$. But this is a contradiction since z cuts between p and all of the points of A. It contradicts our assumption that p was not

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conjugate to y. Hence $p \,O\, y$ where $y \in \overline{H}$ and $y \neq p$. Thus by theorem 5.18 there is a proper cyclic element of $\overline{H}$ containing p and y, and in particular containing p. Q.E.D.

Theorem 5.33 If p is a point of a non-degenerate cyclic Peano space T, then p is an interior point of some arc in T.

Proof. Let us assume to the contrary that no arc of T has p as an interior point. In view of theorem 5.30 this assumption is equivalent to assuming that p is a non-cut point of every non-degenerate Peano subspace of T which contains p. We shall arrive at our contradiction by showing that there is an arc in T having p as an interior point.

Since T is a non-degenerate connected set, it contains a non-enumerable number of distinct points (1.22). We are going to define a series of arcs and sets inductively and we begin by letting $\epsilon_n = \frac{1}{n}$ for $n = 1,2,\ldots.$ Let a_o and b_o be distinct points of T-p. Consider the open neighborhood $N(p,\epsilon_1)$. Clearly, $T-(a_o \cup b_o)$ is open hence $[N(p,\epsilon_1)] \cap [T-(a_o \cup b_o)]$ is an open set. Thus there is a $\delta > 0$ and an open neighborhood $N(p,\delta)$ such that $N(p,\delta) \subset [N(p,\epsilon_1)] \cap [T-(a_o \cup b_o)]$. Consider the open neighborhood $N(p,\frac{1}{2}\delta)$. Now T is a Peano space, hence T has a basis every non-empty element of which has a Peano space as its closure (4.27). Being an open set $N(p,\frac{1}{2}\delta)$ can be expressed as the union of some of the elements of this basis. Therefore, there is a set H_1 of the basis such that

$$p \in H_1 \subset N(p,\tfrac{1}{2}\delta) \subset N(p,\delta) \subset [N(p,\epsilon_1)] \cap [T-(a_o \cup b_o)].$$

Furthermore, $\overline{H}_1$ is a Peano space and

$$p \in \overline{H}_1 \subset \overline{N(p,\tfrac{1}{2}\delta)} \subset N(p,\delta) \subset [N(p,\epsilon_1)] \cap [T-(a_o \cup b_o)].$$

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By assumption p is a non-cut point of every non-degenerate Peano space containing it. Hence, p is a non-cut point of $\overline{H}_1$. Thus applying theorem 5.32 there is a proper cyclic element E_1 of $\overline{H}_1$ such that $p \in E_1$. Furthermore, E_1 is closed in $\overline{H}_1$ (5.20). It follows from (1.13) that E_1 is closed in T. Since

$$E_1 \subset \overline{H}_1 \subset [N(p,\epsilon_1)] \cap [T-(a_o \cup b_o)]$$

we have $E_1 \cap (a_o \cup b_o) = \emptyset$, where E_1 and $a_o \cup b_o$ are both non-degenerate closed subsets of the cyclic Peano space T. Thus by theorem 5.29 there exists two arcs $a_o a_1$ and $b_o b_1$ in T such that $a_o a_1 \cap E_1 = a_1$, $b_o b_1 \cap E_1 = b_1$ and $a_o a_1 \cap b_o b_1 = \emptyset$, hence $a_1 \neq b_1$. Since E_1 is a proper cyclic element of the Peano space $\overline{H}_1$ we see by theorem 5.28 that E_1 itself is a cyclic Peano space. Recall that $p \in E_1$. Suppose $p = a_1$. Since E_1 is a Peano space there is an arc $a_1 b_1$ in E_1 and $a_o a_1$ $a_1 b_1$ is clearly the union of two arcs whose only common point is a_1, hence $a_o a_1 \cup a_1 b_1$ is itself an arc. Similarly $a_o a_1 \cup a_1 b_1 \cup b_1 b_o$ is an arc, but more than that, it is an arc with $p(=a_1)$ as an interior point which is impossible by assumption. Thus $p \neq a_1$ and similarly, $p \neq b_1$.

By theorem 4.27 we see that E_1 has a basis every non-empty element of which has a Peano space as its closure. Consider the open neighborhood $N(p,\epsilon_2)$. Clearly $E_1-(a_1 \cup b_1)$ is an open set in E_1, hence $[N(p,\epsilon_2)] \cap [E_1-(a_1 \cup b_1)]$ is open in E_1. Thus there exists $\gamma > 0$ such that

$$N(p,\gamma) \subset [N(p,\epsilon_2)] \cap [E_1-(a_1 \cup b_1)]$$

and $N(p,\gamma)$ is open in E_1. Consider $N(p,\frac{1}{2}\gamma)$. $N(p,\frac{1}{2}\gamma)$ is open in E_1 hence may be expressed as the union of some of the sets in the basis of E_1. Therefore, there is an element H_2 of this basis

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such that

$$p \in H_2 \subset N(p, \tfrac{1}{2}\gamma) \subset N(p, \gamma) \subset [N(p, \epsilon_2)] \cap [E_1 - (a_1 \cup b_1)]$$

where $\overline{H}_2$ is a Peano space and

$$p \in \overline{H}_2 \subset \overline{N(p, \tfrac{1}{2}\gamma)} \subset N(p, \gamma) \subset [N(p, \epsilon_2)] \cap [E_1 - (a_1 \cup b_1)].$$

Furthermore, by assumption, p is a non-cut point of $\overline{H}_2$ and clearly H_2 is open in E_1. Thus by theorem 5.32 there is a proper cyclic element E_2 of $\overline{H}_2$ such that $p \in E_2$ and E_2 is closed in $\overline{H}_2$ (5.20). Furthermore, H_2 is closed in E_1 and $E_2 \subset \overline{H}_2 \subset E_1$, thus E_2 is closed in E_1 (1.13). Recall that $a_1 \cup b_1 \subset E_1$ and clearly the set $a_1 \cup b_1$ is closed in E_1. Since $E_2 \subset \overline{H}_2 \subset [N(p, \epsilon_2)] \cap [E_1 - (a_1 \cup b_1)]$ we have $E_2 \cap (a_1 \cup b_1) = \emptyset$. Also we have seen that E_1 is itself a cyclic Peano space. Thus applying theorem 5.29 there exists two arcs a_1a_2 and b_1b_2 in E_1 such that $a_1a_2 \cap E_2 = a_2$, $b_1b_2 \cap E_2 = b_2$ and $a_1a_2 \cap b_1b_2 = \emptyset$. Since E_2 is a proper cyclic element of the Peano space $\overline{H}_2$ we see by theorem 5.28 that E_2 is itself a cyclic Peano space. Recall that $p \in E_2$. Suppose $p = a_2$. Clearly there is an arc a_2b_2 in E_2 and $a_1a_2 \cup a_2b_2$ is also an arc since it is the union of two arcs whose only common point is a_2. Similarly, $a_1a_2 \cup a_2b_2 \cup b_2b_1$ is an arc, but more than that, it is an arc with $p(=a_2)$ as an interior point which is impossible by assumption. Thus $p \neq a_2$ and, similarly, $p \neq b_2$.

Now $a_oa_1 \cup a_1a_2$ is the union of two arcs whose only common point is an end point a_1, hence $a_oa_1 \cup a_1a_2$ is an arc and we denote it as a_oa_2. Similarly, we obtain an arc b_ob_2. Recall that $a_oa_1 \cap E_1 = a_1$ and $E_2 \subset E_1 - (a_1 \cup b_1)$, hence $a_oa_1 \cap E_2 = \emptyset$. Also $a_1a_2 \cap E_2 = a_2$ hence we have $a_oa_2 \cap E_2 = a_2$. Similarly, $b_ob_2 \cap E_2 = b_2$ and clearly $a_oa_1 \subset a_oa_2$ and $b_ob_1 \subset b_ob_2$.

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Continuing this process for n = 1,2,... we obtain a sequence of cyclic Peano spaces $\{E_n\}$ and arcs a_oa_n and b_ob_n such that $a_oa_n \cap E_n = a_n$, $b_ob_n \cap E_n = b_n$, $p \notin a_oa_n$, $p \notin b_ob_n$, $a_oa_n \subset a_oa_{n+1}$, $b_ob_n \subset b_ob_{n+1}$, $a_oa_{n-1} \cup a_{n-1}a_n = a_oa_n$ and $b_ob_{n-1} \cup b_{n-1}b_n = b_ob_n$. Let

$$A = \bigcup_{n=1}^{\infty} a_{n-1}a_n = a_oa_1 \cup a_1a_2 \cup \cdots$$

and

$$B = \bigcup_{n=1}^{\infty} b_{n-1}b_n = b_ob_1 \cup b_1b_2 \cup \cdots .$$

A is connected since $a_{n-1}a_n$ is connected for each n and $a_{n-1}a_n \cap a_na_{n+1} = a_n \neq \emptyset$. Similary, B is connected. Thus $\overline{A}$ and $\overline{B}$ are each connected, and, furthermore, they are each compact. Since $a_{n-1}a_n \subset E_{n-1} \subset \overline{H}_{n-1} \subset N(p,\epsilon_{n-1})$ we have $\rho(a_n,p) < \epsilon_{n-1} = \frac{1}{n-1}$ for all n. Clearly then $\rho(a_n,p) \to 0$ as $n \to \infty$ and since $a_n \in A$ for all n we have $p \in \overline{A}$ (1.9). Similarly, $p \in \overline{B}$ and it follows that $A \cup p \subset \overline{A}$ and $B \cup p \subset \overline{B}$. We shall show that $A \cup p = \overline{A}$ and $B \cup p = \overline{B}$. Let q be a point of $\overline{A}$. Thus there is a sequence $\{t_n\}$ of distinct points of A such that $t_n \to q$ (1.9). If infinitely many points of $\{t_n\}$ are in a_oa_1, then since a_oa_1 is compact there is a subsequence $\{t_{h_n}\}$ of $\{t_n\}$ which converges to some point r of a_oa_1. But since $t_n \to q$ we have $t_{h_n} \to q$ (1.8), i.e., q = r, hence $q \in a_oa_1 \subset A$, i.e., $q \in A$. Thus we can assume only a finite number of points of $\{t_n\}$ are in a_oa_1. Let $\lambda > 0$ and choose an integer k so that $\frac{1}{k} < \lambda$. Now continuing this line of reasoning we can assume only a finite number of points of $\{t_n\}$ are in $a_{k-1}a_k$, i.e., there is a number n_o such that if $n > n_o$ then $t_n \notin a_oa_1 \cup \cdots \cup a_{k-1}a_k$. Thus if $s > k$, and $s-1 \geq k$,

$$t_n \in a_{s-1}a_s \subset E_{s-1} \subset E_k \subset \overline{H}_k \subset N(p,\tfrac{1}{k}) \subset N(p,\lambda),$$

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therefore $t_n \in N(p,\lambda)$ for all $n > n_o$. But since k can be made arbitrarily large and there still will exist an appropriate number n_o, we see that λ can be made arbitrarily small. Thus as $n \to \infty$ we have $\rho(t_n,p) \to 0$, i.e., $t_n \to p$. However, since $t_n \to q$ this implies $p = q$, thus $q \in A \cup p$. Therefore, if q is an arbitrary point of $\overline{A}$ we have $q \in A \cup p$, i.e., $\overline{A} \subset A \cup p$, hence $\overline{A} = A \cup p$. Similarly, $\overline{B} = B \cup p$.

Also $A \cap B = \emptyset$, for if not then there is a point w of $A \cap B$. Therefore, there are specific numbers m and n such that $w \in a_{n-1}a_n$ and $w \in b_{m-1}b_m$. If $m = n$ then by the way we have defined our arcs we have $a_{n-1}a_n \cap b_{m-1}b_m = \emptyset$. This is impossible since

$$w \in a_{n-1}a_n \cap b_{m-1}b_m,$$

thus $m \neq n$. Suppose $m > n$. Then since $b_{m-1}b_m \subset E_{m-1} \subset E_n$ and $a_{n-1}a_n \cap E_n = a_n$ we must have $a_{n-1}a_n \cap b_{m-1}b_m \subset a_n$.

(i) If $m = n+1$, then $a_{n-1}a_n \cap b_nb_{n+1} \subset a_n$. But $a_n \notin b_nb_{n+1}$ since $a_na_{n+1} \cap b_nb_{n+1} = \emptyset$, hence $a_{n-1}a_n \cap b_nb_{n+1} = \emptyset$, i.e., $a_{n-1}a_n \cap b_{m-1}b_m = \emptyset$, and we have a contradiction.

(ii) If $m > n+1$, i.e., $m-1 \geq n+1$, then

$$b_{m-1}b_m \subset E_{m-1} \subset E_{n+1} \subset [N(p,\epsilon_{n+1})] \cap [E_n-(a_n \cup b_n)],$$

i.e., $b_{m-1}b_m \subset E_n-(a_n \cup b_n)$, hence $a_n \notin b_{m-1}b_m$. Thus again we have the contradiction that $a_{n-1}a_n \cap b_{m-1}b_m = \emptyset$. In a similar manner, the reader may easily verify that the assumption that $m < n$ leads to a contradiction. Therefore, we conclude that no such point w exists, hence $A \cap B = \emptyset$. Clearly then since $\overline{A} = A \cup p$ and $\overline{B} = B \cup p$ we have $\overline{A} \cap \overline{B} = p$.

Now since $\overline{A}$ and $\overline{B}$ are both compact connected sets, if it can be shown that each has at most two non-cut points then by theorem 3.5

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each set is a simple arc. We shall see that this is the case. First $\overline{A}-p = (A\cup p)-p = A$ and A is connected, hence p is a non-cut point of $\overline{A}$. Similarly, p is a non-cut point of $\overline{B}$. Clearly $\overline{A}-a_o$ may be expressed as follows,

$$\overline{A}-a_o = (A\cup p)-a_o = (a_oa_1-a_o)\cup a_1a_2\cup a_2a_3\cup\ldots\cup p$$
$$= (a_oa_1-a_o)\cup a_1a_2\cup a_1a_3\cup\ldots\cup a_1a_n\cup\ldots\cup p.$$

Now a_oa_1 is an arc with a_o and a_1 as its end points, hence a_o is a non-cut point of a_oa_1. Therefore, $a_oa_1-a_o$ is connected. Also, it is clear that $a_1a_2\cup a_1a_3\cup\ldots\cup a_1a_n\cup\ldots$ is a connected set meeting $a_oa_1-a_o$ in exactly one point; namely a_1, hence

$$(a_oa_1-a_o)\cup a_1a_2\cup\ldots\cup a_1a_n\cup\ldots$$

is a connected set. But

$$\overline{(a_oa_1-a_o)\cup a_1a_2\cup\ldots\cup a_1a_n\cup\ldots} = A\cup p = \overline{A}$$

and since $\overline{(a_oa_1-a_o)\cup a_1a_2\cup\ldots\cup a_1a_n\cup\ldots}\subset(a_oa_1-a_o)\cup a_1a_2\cup\ldots \cup a_1a_n\cup\ldots\cup p\subset A\cup p$ we see that $(a_oa_1-a_o)\cup a_1a_2\cup\ldots\cup a_1a_n\cup\ldots\cup p$ is connected. Thus, $\overline{A}-a_o$ is connected, i.e., a_o is a non-cut point of $\overline{A}$. We must now show that p and a_o are the only non-cut points of $\overline{A}$. Let v be any point of $\overline{A}$ such that $v\neq p$ and $v\neq a_o$. There is some i such that $v\in a_{i-1}a_i$. There are two distinct cases to consider.

(i) Suppose v is an interior point of $a_{i-1}a_i$. Under this assumption the reader may easily verify that

$$(A\cup p)-v = \bigcup_{j<i}a_{j-1}a_j\cup(a_{i-1}v-v)/\bigcup_{i<j}a_{j-1}a_j\cup(va_i-v)$$

i.e., $(A\cup p)-v$ is not connected.

(ii) Suppose $v = a_{i-1}$. Again it is easily seen that

$$(A\cup p)-v = \bigcup_{j<i-1}a_{j-1}a_j\cup(a_{i-2}a_{i-1}-a_i)/\bigcup_{i<j}a_{j-1}a_j\cup(a_{i-1}a_i-a_{i-1})$$

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hence, $(A \cup p)-v$ is disconnected.

Similarly, if $v = a_i$ we see that $(A \cup p)-v$ is disconnected, thus, in every case an arbitrary point v of $\bar{A}$ such that $v \neq p$ and $v \neq a_o$ is found to be a cut point of $\bar{A}$. Similarly, p and b_o are the only non-cut points of $\bar{B}$. We now see that $\bar{A}$ and $\bar{B}$ are each simple arcs with end points p and a_o of $\bar{A}$ and p and b_o of $\bar{B}$. Furthermore, $\bar{A}$ and $\bar{B}$ meet in exactly one point p which is an end point of each set. Thus $\bar{A} \cup \bar{B}$ is itself a simple arc with end points a_o and b_o. But $p \in \bar{A} \cup \bar{B}$, $p \neq a_o$ and $p \neq b_o$, hence p is an interior point of an arc in T. This contradicts our first assumption, hence the theorem is proved. Q.E.D.

Theorem 5.34 (Cyclic Connectivity Theorem) Suppose T is a Peano space. Then T is cyclic if and only for every two points a and b of T there is a simple closed curve in T containing a and b.

Proof. First suppose given any two points a and b of T there is a simple closed curve C in T containing a and b. Let d be any point of T different from a and b, thus $a, b \in T-d$. There are two cases to consider.

(i) Suppose $d \in C$. By theorem 4.4, C may be expressed as two independent arcs $(ab)_1$ and $(ab)_2$. Suppose the notation is chosen so that $d \in (ab)_1$, thus $d \in (ab)_2$. Since $(ab)_2$ is a connected set it belongs to one component of T-d. But $a \in (ab)_2$ and $b \in (ab)_2$, hence a and b are in the same component of T-d. Since d was arbitrary this implies no point of T cuts between a and b. Hence $a \bigcirc b$ for any two points a and b of T. Therefore, T is cyclic.

(ii) Suppose $d \notin C$. Since C is a connected set it is

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contained in a single component of T-d. Thus a and b lie in a single component of T-d, hence $a \bigcirc b$. Again then since a and b were arbitrary points of T we see that T is cyclic.

To prove the converse, suppose that T is a cyclic Peano space and that a and b are any two distinct points of T. We must show there a simple closed curve in T containing a and b. By theorem 5.33, a is an interior point of some arc xy in T. Since T is cyclic and the single point a is closed, we readily see that T-a is a region in T. Now, T-a is arcwise connected and $x, y \neq a$ hence there is an arc $\widehat{xy}$ in T-a joining the points x and y. Clearly $a \in \widehat{xy}$, hence $xy \neq \widehat{xy}$. Consider the subarc ax of xy. Since ax is closed, it is compact. Furthermore, $x \in ax$ and $x \in \widehat{xy}$, hence $ax \cap \widehat{xy} \neq \emptyset$. Let u be the first point of ax on $\widehat{xy}$. Thus, $u \in ax \cap \widehat{xy}$ and we clearly have $au \cap \widehat{xy} = t$ where au is a subarc of ax. Also, $u \neq a$ since $u \in \widehat{xy} \subset T-a$. By similar reasoning, the subarc ay of xy is compact and meets $\widehat{xy}$ hence we may let w be the first point of ay on $\widehat{xy}$. Thus, $aw \cap \widehat{xy} = w$ where $a \neq w$. Now then $u \in \widehat{xy}$, $w \in \widehat{xy}$ and $u \neq w$ since $au \cap aw \subset ax \cap ay = a$.

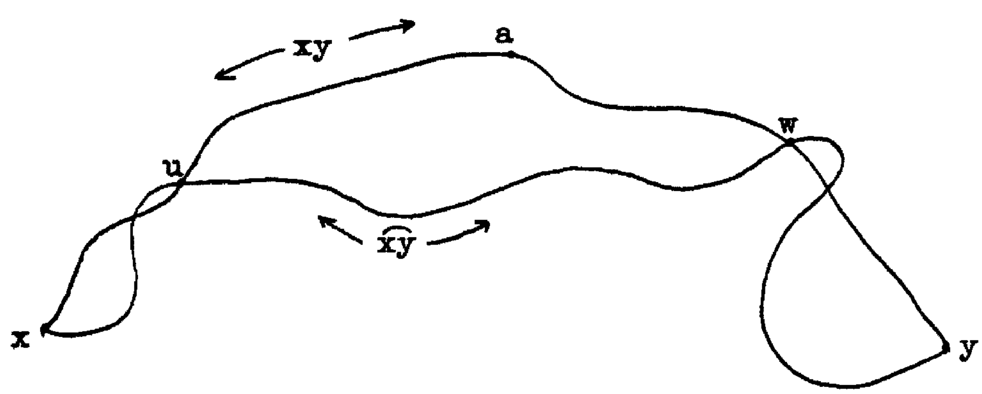

Figure 2

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Consider the subarc $\widehat{uw}$ of $\widehat{xy}$. Clearly $\widehat{uw} \cap aw \subset \widehat{xy} \cap aw = w$ and since $w \in uw$ and $w \in aw$ we have $\widehat{uw} \cap aw = w$. Similarly, since

$$\widehat{uw} \cap au \subset \widehat{xy} \cap au = u$$

and $u \in \widehat{uw}$ and $u \in au$ we have $\widehat{uw} \cap au = u$. It is also clear that $au \cap aw = a$. We see that au and aw are simple arcs meeting at one point; namely a. But a is an end point of both au and aw, hence $au \cup aw$ is a simple arc with end points u and w and clearly

$$(au \cup aw) \cap \widehat{uw} = (au \cap \widehat{uw}) \cup (aw \cap \widehat{uw})$$
$$= u \cup w.$$

Thus $(au \cup aw)$ and $\widehat{uw}$ are independent arcs from u to w, hence by the corollary to theorem 4.4 we see that $(au \cup aw) \cup \widehat{uw}$ is a simple closed curve. Let $C_a = (au \cup aw) \cup \widehat{uw}$. Now $a \in C_a$, hence a lies on a simple closed curve in T. By similar reasoning there is a simple closed curve C_b containing b.

It is immediate that if either $a \in C_b$ or $b \in C_a$ then a and b lie on the same simple closed curve and the theorem is proved. Thus excluding this possibility there are three cases to consider.

(i) Suppose $C_a \cap C_b = \emptyset$. Then C_a and C_b are non-degenerate, disjoint, compact (hence closed), connected subsets of the cyclic Peano space T (4.5). In view of theorem 5.29, there exists disjoint arcs pq and rs each having one end point in C_a and one end point in C_b. That is, $p \in C_a$, $q \in C_b$ and $pq \cap (C_a \cup C_b) = p \cup q$. Similarly, $r \in C_a$, $s \in C_b$ and $rs \cap (C_a \cup C_b) = r \cup s$.

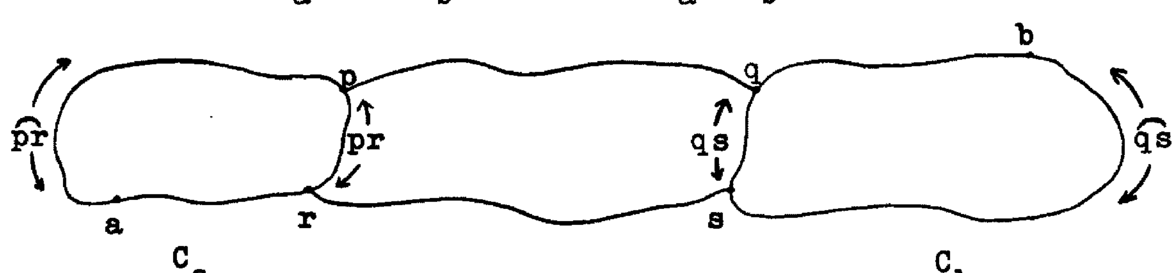

Figure 3

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It is easily seen that we now have arcs pr and $\widehat{pr}$ such that $pr \cup \widehat{pr} = C_a$ and also we have arcs qs and $\widehat{qs}$ such that $qs \cup \widehat{qs} = C_b$. Suppose the notation is chosen so that $a \in \widehat{pr}$ and $b \in \widehat{qs}$ (a similar discussion would follow if $a \in pr$ and $b \in \widehat{qs}$ or else $a \in \widehat{pr}$ and $b \in qs$). We know that $pq \cap rs = \emptyset$ and clearly $pq \cap \widehat{pr} = p$ and $pq \cap \widehat{qs} = q$. Similarly, it is readily seen that $rs \cap \widehat{pr} = r$ and $rs \cap \widehat{qs} = s$. Furthermore, $\widehat{pr} \cap \widehat{qs} \subset C_a \cap C_b = \emptyset$, i.e., $\widehat{pr} \cap \widehat{qs} = \emptyset$. It follows that $\widehat{pr} \cup pq$ is the union of two simple arcs which intersect in exactly one point; namely p, which is an end point of each. Hence, $\widehat{pr} \cup pq$ is itself a simple arc. Similarly, $(\widehat{pr} \cup pq) \cup \widehat{qs}$ is a simple arc. Now then we have two independent arcs $(\widehat{pr} \cup pq \cup \widehat{qs})$ and rs from r to s, hence by the corollary to theorem 4.4,

$$(\widehat{pr} \cup pq \cup \widehat{qs} \cup rs) = A$$

is a simple closed curve. Furthermore, $a \in \widehat{pr}$ and $b \in \widehat{qs}$, hence a and b lie on A.

(ii) Suppose $C_a \cap C_b = z$, i.e., C_a and C_b meet in a single point.

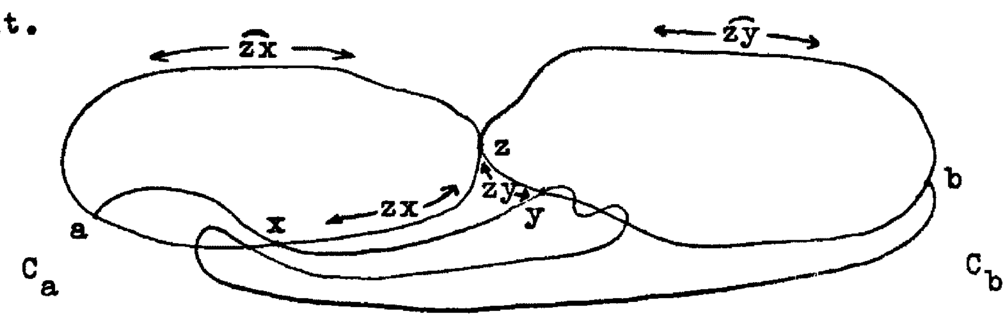

Figure 4

Clearly $z \neq a$ and $z \neq b$. Since T is cyclic and z is a closed set, $T-z$ is a region in T and $a,b \in T-z$. Thus, there is an arc ab in $T-z$ joining a and b. Now C_b is compact and since $b \in ab \cap C_b$ we have $ab \cap C_b \neq \emptyset$. Therefore, there is a first point y of C_b on ab.

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Hence, $y \in ab$, $y \in C_b$ and $ay \cap C_b = y$. Furthermore, $y \neq z$ since $ab \subset T-z$, hence $y \notin C_a$. By similar reasoning, there is a first point x of C_a on the arc ay going in order from y to a, such that $x \in ab$, $x \in C_a$, $x \neq z$ and $x \notin C_b$. Thus we have a subarc xy of ab such that $xy \cap C_a = x$ and $xy \cap C_b = y$. Since $xy \subset ab \subset T-z$ we have $xy \subset T-z$. It is clear (see the figure in (ii)) that we now have subarcs zx, $\widehat{zx}$ of C_a and zy, $\widehat{zy}$ of C_b. Suppose the notation is chosen so that $a \in \widehat{zx}$ and $b \in \widehat{zy}$ (a similar discussion would follow if $a \in zx$ and $b \in \widehat{zy}$ or $a \in \widehat{zx}$ and $b \in zy$). Clearly, $\widehat{zx} \cap xy = x$ and since x is an end point of both $\widehat{zx}$ and xy we see that $\widehat{zx} \cup xy$ is a simple arc. Furthermore, $(\widehat{zx} \cup xy) \cap \widehat{zy} = z \cup y$, i.e., the two simple arcs $(\widehat{zx} \cup xy)$ and $\widehat{zy}$ meet in two points; namely, the end points of each. Hence $\widehat{zx} \cup xy \cup \widehat{zy} = B$ is a simple closed curve. But $a \in \widehat{zx}$ and $b \in \widehat{zy}$, hence a and b both lie on B.

(iii) Suppose $C_a \cap C_b$ contains more than one point.

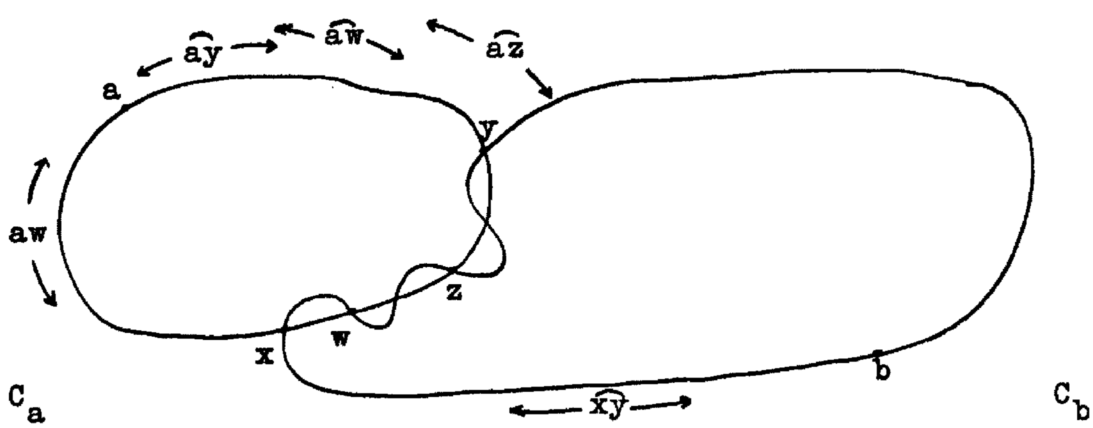

Figure 5

Thus, there is a point w of $C_a \cap C_b$ such that $w \neq a$, $w \neq b$ and $w \neq z$ where $z \in C_a \cap C_b$. Now then we have subarcs aw and $\widehat{aw}$ of C_a. Let $z \in \widehat{aw}$ (since a similar discussion would follow if $z \in aw$). Thus $z \notin aw$ since $aw \cap \widehat{aw} = a \cup w$ and $z \neq a$ and $z \neq w$. Let x be the

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first point of aw on C_b going in order from a to w. Thus we have a subarc ax of aw such that $x \in aw$, $x \in C_b$, $x \neq a$ and $ax \cap C_b = x$. Now also there is a subarc $\widehat{az}$ of C_a such that $\widehat{az} \subset \widehat{aw}$. Let y be the first point of $\widehat{az}$ belonging to C_b. Thus $y \neq a$, $y \in C_a$ and $y \in C_b$. Consider the arc $\widehat{ay}$. Clearly, $\widehat{ay} \subset \widehat{az} \subset \widehat{aw}$ and $\widehat{ay} \cap C_b = y$. We now have

$$ax \cap \widehat{ay} \subset aw \cap \widehat{az} \subset aw \cap \widehat{aw} = a \cup w,$$

i.e., $ax \cap \widehat{ay} \subset a \cup w$. But $w \notin \widehat{az}$, hence $ax \cap \widehat{ay} = a$, where the point a is an end point for each arc ax and $\widehat{ay}$. Thus, $ax \cup \widehat{ay}$ is a simple arc joining x and y and clearly $(ax \cup \widehat{ay}) \cap C_b = x \cup y$. Furthermore, there exists subarcs xy and $\widehat{xy}$ of C_b and we choose the notation so $b \in \widehat{xy}$. Now since arcs $(ax \cup \widehat{ay})$ and $\widehat{xy}$ meets in exactly two points, x and y, which are the end points of each arc we see that $ax \cup \widehat{ay} \cup \widehat{xy} = D$ is a simple closed curve. But $a \in ax$ and $b \in \widehat{xy}$, hence a and b lie on D.

Thus, in every case, there is a simple closed curve containing a and b. Q.E.D.

For a similar presentation of theorems 5.28 through 5.34, the reader may refer to Hall and Spencer's Elementary Topology.

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