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Approximation theorems for valuations on commutative rings

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APPROXIMATION THEOREMS FOR VALUATIONS ON COMMUTATIVE RINGS

By

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INTRODUCTION

This paper is the result of an investigation of the approximation theorems developed by M. Manis in Chapter III of his doctoral thesis ([2]). The results obtained in [2] were those needed for the author's development of Galois theory for rings. This study was made in an attempt to discover additional and more general cases in which these results apply. Particular emphasis was put on the so-called "inverse property" which can be considered the weakest form of an approximation theorem.

Sections I and II are adapted from Chapters I and II of [2] and contain the definitions and background material necessary for Sections III and IV. The arguments used are all taken from [2] or from lecture notes of a seminar given by M. Manis during the school year of 1966 and 1967.

In Section III, we introduce the concept of extending a valuation on a ring to an extension of the ring. Except for those dealing with the inverse property, the theorems of Section IV are limited to these extensions. Propositions 3.6 and 3.7 are included to show that every valuation on a ring can be extended to any integral extension of the ring; and hence, that these extensions occur with sufficient frequency to merit the consideration given them. The arguments in this section are taken from the same sources as those in Sections I and II with the exception of 3.6 which was adapted from a more general theorem on page 255 of [1] and was simplified to its present form by M. Manis in the course of this writing.

Part I of Section IV outlines the approximation theorems obtainable for valuations on a field and indicates the results desired for valuations on a ring.

Part II of Section IV considers the inverse property which somewhat replaces the multiplicative inverses inherent in a field. Propositions 4.9 and 4.10 and Examples 1, 2, and 3 are the result of an attempt to correlate the inverse property for two valuations with the relationship between the sets of elements from the ring for which the valuations assume the value zero.

Part III of Section IV shows that the conditions assumed for Part IV hold in the case of an integral extension.

Propositions 4.17 through 4.20 are the approximation theorems of Chapter III of [2]. These theorems are limited to sets of extensions of a single valuation. Proposition 4.21 concerns approximation properties 4.18 and 4.19 for sets of extensions of more than one valuation; that is, given conditions which make the previous theorems apply in the special case of extensions of a single valuation, then the inverse property and 4.18 "extend" to finite sets of distinct pairwise-independent extensions.

The material covered in this paper is but a beginning of a complete approximation theory for rings. Many cases and situations are still open to investigation.

SECTION I

VALUATIONS AND VALUATION PAIRS

Throughout this paper, we will use the following conventions: "Ring" will mean "commutative ring with identity", and subrings will always contain that identity. Ring homomorphisms will always take identity to identity. Prime ideals are always proper. The identity of a ring will be denoted by 1 and that of a group by e. Once it is introduced, notation will be assumed as standard wherever it does not cause ambiguity.

Definition 1.1. By a valuation semigroup G, we mean an abelian (multiplicative) group with a zero adjoined, linearly ordered by a relation "$<$" satisfying:

i) $a < b \Rightarrow ac < bc$ for all a,b,c in G, $c \neq 0$,

ii) $0 \cdot a = a \cdot 0 = 0 \leq b$ for all a,b in G.

Definition 1.2. A valuation V on a ring R is a homomorphism of the multiplicative semigroup of R onto a valuation semigroup satisfying:

$$V(x+y) \leq \max[V(x), V(y)] \quad \text{for all } x,y \text{ in } R.$$

We note that $V(1) = e$ and $V(0) = 0$ for all valuations. If R is a field and t a non-zero element of R, then $0 \neq e = V(1) = V(t)V(t^{-1})$, so $V^{-1}([0]) = [0]$. For this reason, in studying fields, one works with ordered groups rather than semigroups. The condition of 1.2 is the non-Archimedean condition in a field.

Proposition 1.3. Let V be a valuation on a ring R, and set

$$A_v = [x \text{ in } R \,/\, V(x) \leq e],$$
$$P_v = [x \text{ in } R \mid V(x) < e], \text{ and}$$
$$N_v = [x \text{ in } R \mid V(x) = 0].$$

Then A_v is a subring of R, P_v is a prime ideal of A_v, and N_v is a prime ideal of R. Further, if J is an ideal of R, $J \subset A_v$, and $A_v \neq R$, then $J \subset N_v$.

Proof: Note that $V(-1) = e$ since G is linearly ordered, V is a homomorphism, and $(-1)(-1) = 1$. Thus, $V(-x) = V(-1)V(x) = V(x)$ for all x in R. Thus we have $A_v = -A_v$, $P_v = -P_v$, and $N_v = -N_v$. The condition of 1.2 gives $(A_v + A_v) \subset A_v$, $(P_v + P_v) \subset P_v$, and $(N_v + N_v) \subset N_v$. If x is in A_v and y in P_v, then $V(x) \leq e$ and $V(y) < e$, so $V(xy) = V(x)V(y) < V(x)e = V(x) \leq e$; thus $A_vP_v \subset P_v$, and P_v is an ideal of A_v, a subring of R. If x is in R and y in N_v, then $V(xy) = V(x)V(y) = V(x)0 = 0$; so $RN_v \subset N_v$, and N_v is an ideal of R. If ab is in P_v, then $e > V(ab) = V(a)V(b)$, so either $e > V(a)$ or $e > V(b)$, so P_v is a prime ideal of A_v ($V(1) = e$ so 1 is not in P_v). If ab is in N_v, then $0 = V(ab) = V(a)V(b)$, so $V(a) = 0$ or $V(b) = 0$, so N_v is a prime ideal of R.

Finally, suppose $A_v \neq R$ and J is an ideal of R. If $J \not\subset N_v$, then $V(a) \neq 0$ for some a in J; but then

$V(a) = V(b)^{-1}$ for some b in R, and $V(c) > e$ for some c in R since $A_V \neq R$. But then abc is in J while $V(abc) = V(a)V(b)V(c) = eV(c) = V(c) > e$, so $J \not\subset A_V$.

<u>Definition 1.4</u>. By a valuation pair of a ring R, we mean a pair (A,P), where A is a subring of R and P is a prime ideal of A, such that x in $R\setminus A \Rightarrow xy$ in $A\setminus P$ for some y in P.

<u>Proposition 1.5</u>. (A,P) is a valuation pair of R iff there is a valuation V on R with $A = A_V$ and $P = P_V$. Furthermore, if V' is another valuation on R with $P = P_{V'}$ and $A = A_{V'} \neq R$, then there is an order-preserving isomorphism $\phi: G_{V'} \longrightarrow G_V$ with $\phi \circ V' = V$.

<u>Proof</u>: Let V be a valuation on R with $A = A_V$ and $P = P_V$. If x in $R\setminus A$, then $V(x) > e$, and $V(y) = V(x)^{-1}$ for some y in R. $e = V(x)V(x)^{-1} > eV(x)^{-1} = V(x)^{-1}$ so y is in P. Now $V(xy) = V(x)V(y) = V(x)V(x)^{-1} = e$ so xy is in $A\setminus P$. Thus by 1.3, (A,P) is a valuation pair of R.

Conversely, let (A,P) be a valuation pair of R. For x in R, define $V(x) = [z \text{ in } R \mid xz \text{ in } P]$, and let $G_V = G = [V(x) \mid x \text{ in } R]$.

<u>Claim 1</u>. $V(x) = V(1)$ iff x in $A\setminus P$.

<u>Subproof 1</u>: If x in $A\setminus P$, then $xP \subset P$ so $P \subset V(x)$.

$V(x) \cap (A \setminus P) = \emptyset$ since P is a prime ideal of A. If y is not in A, then there is a p in P with yp in $A \setminus P$. $x(yp) = (xy)p$ is in $A \setminus P$ so xy is not in A since P is an ideal of A. Thus, xy is not in P, so y is not in $V(x)$. Therefore, $V(x) \subset P$ so $V(x) = P = V(1)$.

Suppose $V(x) = V(1) = P$. If x is in P, then $x \cdot 1$ is in P so 1 is in $V(x) = V(1)$ so $1 \cdot 1$ is in P, a contradiction. If x is not in A, then xp is in $A \setminus P$ for some p in P so p is not in $V(x) = P$, a contradiction. Thus, x is in $A \setminus P$.

Claim 2. Let $V(x)V(y) = V(xy)$. Then this is a well-defined multiplication for G and makes G into an abelian group with zero ($= V(0)$) adjoined.

Subproof 2: Let $V(x) = V(a)$ and $V(y) = V(b)$. Then, t is in $V(xy)$ iff txy is in P iff tx is in $V(y)$ iff tx is in $V(b)$ iff txb is in P iff tb is in $V(x)$ iff tb is in $V(a)$ iff tba is in P iff t is in $V(ab)$. Thus, $V(xy) = V(ab)$ so $V(x)V(y) = V(a)V(b)$ and multiplication is well-defined. Furthermore, it is associative and commutative since multiplication in R is; $V(1)$ is clearly an identity and $V(0)$ a zero, and $V(1) \neq V(0)$ since 1 is in $V(0)$ but 1 is not in $V(1)$.

Finally, if $V(x) \neq V(0) = R$, then there is a y in R such that xy is not in P. If xy is in $A \setminus P$, then $V(xy) = V(1) = V(x)V(y)$ so $V(x)^{-1} = V(y)$. Otherwise, xy is not in A, so xyp is in $A \setminus P$ for some p in P; hence, $V(xyp) = V(1) = V(x)V(yp)$, and $V(x)^{-1} = V(yp)$. Thus, $G \setminus [V(0)]$

is an abelian group.

Claim 3. Define $V(x) < V(y)$ if $V(y) \subsetneq V(x)$. Then "$<$" is a linear ordering on G, and G is a valuation semigroup.

Subproof 3: Let x and y be in R and $V(x) \not\subset V(y)$. Then there is an a in $V(x) \setminus V(y)$; i.e., xa is in P and ya is not in P. If b is in $V(y) \setminus V(x)$, then yb is in P and xb is not in P; so there are t and t' in A with txb in $A \setminus P$ [i.e., t = 1 if xb is in A, otherwise t is in P since (A,P) is a valuation pair] and t'ya in $A \setminus P$. Then (txb)(t'ya) is in $A \setminus P$ since P is a prime ideal of A; but (txb)(t'ya) = (tt')(xa)(yb), tt' is in A, and xa and yb are in P, so (txb)(t'ya) is in P, a contradiction. Thus, b in V(y) implies b is in V(x), so $V(x) \not\subset V(y)$ implies $V(y) \subset V(x)$; i.e., $V(x) \neq V(y)$ implies $V(x) < V(y)$ or $V(y) < V(x)$.

Now if $V(x) < V(y)$, z in R, and $V(z) \neq V(0)$, then $V(y) \subsetneq V(x)$. t in $V(z)V(y) = V(zy) \Rightarrow$ tzy is in P $\Rightarrow$ tz is in $V(y) \subset V(x) \Rightarrow$ tzx is in P $\Rightarrow$ t is in $V(zx) = V(z)V(x)$, so $V(z)V(y) \subset V(z)V(x)$. $V(z) \neq V(0) \Rightarrow V(z)^{-1} = V(z')$ for some z' in R, so $V(zx) = V(zy) \Rightarrow V(x) = V(1)V(x) = V(zz')V(x) = V(z')V(zx) = V(z')V(zy) = V(z'z)V(y) = V(1)V(y) = V(y)$. Thus, $V(y) \subsetneq V(x) \Rightarrow V(z)V(y) \subsetneq V(z)V(x)$ for all $V(z) \neq V(0)$; i.e., $V(z)V(x) < V(z)V(y)$. Thus condition i) of 1.1 is satisfied. $0 \cdot V(x) = V(0)V(x) = V(0 \cdot x) = V(0)$ for all x in R, and $V(0) = R \Rightarrow V(y) \subset V(0)$ for all y in R so $V(0) \leq V(y)$ for all y in R. Thus, condition ii) is satisfied, and G is a valuation semigroup.

<u>Claim 4</u>. V is a valuation on R.

<u>Subproof 4</u>: V is obviously a homomorphism from R onto G by the definition of multiplication in G. Let $V(x) = \max[V(x), V(y)]$. Then $V(y) \leq V(x)$ so $V(x) \subset V(y)$. If t is in $V(x)$, then tx and ty are in P so $(tx+ty) = t(x+y)$ is in P so t is in $V(x+y)$; i.e., $V(x) \subset V(x+y)$ so $V(x+y) \leq V(x) = \max[V(x), V(y)]$. Thus, V is a valuation on R.

<u>Claim 5</u>. $A = A_V$ and $P = P_V$.

<u>Subproof 5</u>: If x is in P, then $P = V(1) \subset V(x)$. By Claim 1, $V(1) = V(x)$ iff x is in $A \setminus P$, so $V(1) \subsetneq V(x)$ so $P \subset P_V$. Let x not be in P. Then x in $A \setminus P \Rightarrow V(x) = V(1) \Rightarrow$ x is not in P_V, or x not in $A \Rightarrow$ there is a z in P with xz in $A \setminus P \Rightarrow P \not\subset V(x) \Rightarrow V(x) \subset P = V(1) \Rightarrow V(1) < V(x) \Rightarrow$ x is not in P_V. Thus, $P_V \subset P$. Therefore, $P = P_V$, and $A_V = [x \text{ in } R \mid V(x) = V(1)] \cup P_V = (A \setminus P) \cup P = A$. Thus, V is the valuation claimed in the proposition.

Now if V' is another valuation on R with $A = A_{V'} \neq R$ and $P = P_{V'}$, define $\phi: G_{V'} \longrightarrow G$ by $\phi(V'(x)) = V(x)$. Claim: ϕ is an order-preserving isomorphism.

<u>Subproof</u>: Note that by 1.3, $N_V = N_{V'}$ since $N_V \subset A_V = A_{V'} \neq R$ and $N_{V'} \subset A_{V'} = A_V \neq R$. Thus, $V'(x) = V'(0) = V'(y)$ iff $V(x) = V(0) = V(y)$. If $V'(x) =$

$V'(y) \neq 0$, then there is a z in R with xz in $A_{V'} \setminus P_{V'} = A_V \setminus P_V$, $V'(1) = V'(xz) = V'(x)V'(z) = V'(y)V'(z) = V'(yz)$ so yz is in $A_{V'} \setminus P_{V'} = A_V \setminus P_V$. Thus, $V(xz) = V(1) = V(yz)$, and $V(x) = V(x)V(1) = V(x)V(yz) = V(xz)V(y) = V(1)V(y) = V(y)$. Interchanging V and V' we obtain $V(x) = V(y) \Rightarrow V'(x) = V'(y)$, so $V'(x) = V'(y)$ iff $V(x) = V(y)$. Thus, ϕ is well-defined and "1-1". ϕ is obviously a homomorphism and "onto" by the way it is defined, so ϕ is an isomorphism.

Finally, $V'(x) < V'(y) \Rightarrow V'(y) \neq V'(0)$ so that there is a z in R such that $V'(yz) = V'(1) = e'$. $V(yz) = \phi(e') = e$. Thus, $V'(xz) = V'(x)V'(z) < V'(y)V'(z) = V'(yz) = e'$, so xz is in $P_{V'} = P_V$ and $V(xz) < e$. Thus, $V(x) = V(x)V(yz) = V(xz)V(y) < eV(y) = V(y)$, so $\phi(V'(x)) < \phi(V'(y))$ as claimed.

Thus, ϕ is the order-preserving isomorphism claimed in the proposition; and henceforth, we will speak of <u>the</u> valuation determined by a valuation pair (A,P).

<u>Corollary 1.6</u>. If (A,P) is a valuation pair of R, then

i) $R \setminus A$ is closed under multiplication;

ii) $R \setminus P$ is closed under multiplication;

iii) xy in $A \Rightarrow x$ in A or y in P;

iv) x^n in $A \Rightarrow x$ in A;

v) x^n in $A \setminus P \Rightarrow x$ in $A \setminus P$;

vi) $A = [x \text{ in } R \mid xP \subset P]$; and

vii) $A = R$ or $P = [x \text{ in } A \mid xy \text{ in } A \text{ for some } y \text{ not in } A]$.

<u>Proof</u>: Let V be the valuation associated with (A,P) in 1.5. Translating, we have

i) $V(x)V(y) > e$ if $V(x) > e$ and $V(y) > e$;

ii) $V(x)V(y) \geq e$ if $V(x) \geq e$ and $V(y) \geq e$;

iii) $V(x)V(y) \leq e \Rightarrow V(x) \leq e$ or $V(y) < e$;

iv) $V(x)^n \leq e \Rightarrow V(x) \leq e$;

v) $V(x)^n = e \Rightarrow V(x) = e$;

vi) $V(x) \leq e$ iff $V(x)V(y) < e$ for all $V(y) < e$;

vii) If $V(z) > e$ for some z in R, then $V(x) < e$ iff $V(x)V(t) \leq e$ for some $V(t) > e$.

<u>Proposition 1.7</u>. Let V be a valuation on a ring R, a,b in R with $V(a) \neq V(b)$. Then $V(a+b) = \max[V(a),V(b)]$.

<u>Proof</u>: Without loss of generality, we may assume $V(a) > V(b)$. Then $V(a) = V(a+b-b) \leq \max[V(a+b),V(b)] = V(a+b) \leq \max[V(a),V(b)] = V(a)$, so $V(a) = V(a+b)$.

<u>Corollary 1.8</u>. Let V be a valuation on a ring R and a_i in R for $i = 1,2,\cdots,n$. If $V(\sum_{i=1}^{n} a_i) < \max V(a_i)$, then $V(a_j) = \max V(a_i) = V(a_k)$ for some $j \neq k$.

<u>Proof</u>: Let $V(a_j) = \max V(a_i)$. Then since $V(\sum_{i=1}^{n} a_i) = V(\sum_{\substack{i=1\\i\neq j}}^{n} a_i + a_j) < \max\{V(\sum_{\substack{i=1\\i\neq j}}^{n} a_i),V(a_j)\}$, $V(\sum_{\substack{i=1\\i\neq j}}^{n} a_i) = V(a_j)$ by 1.7. But $V(\sum_{\substack{i=1\\i\neq j}}^{n} a_i) \leq \max_{i\neq j} V(a_i)$, so $\max_{i\neq j} V(a_i) \geq V(a_j) = \max V(a_i)$; that is, $V(a_j) = \max_{i\neq j} V(a_i) = V(a_k)$ for some $k \neq j$.

Corollary 1.9. Let V be a valuation on a ring R and a_i in R for $i = 1,2,\ldots,n,n+1,\ldots,k$ with $V(a_i) = 0$ for $n < i \leq k$. Then $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=1}^{n} a_i)$.

Proof: $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=1}^{n} a_i + \sum_{i=n+1}^{k} a_i) \leq \max\left\{V(\sum_{i=1}^{n} a_i), V(\sum_{i=n+1}^{k} a_i)\right\} = V(\sum_{i=1}^{n} a_i)$. The last equality holds since $V(\sum_{i=n+1}^{k} a_i) = 0$ by 1.3. $V(\sum_{i=1}^{k} a_i) < V(\sum_{i=1}^{n} a_i)$ implies $V(\sum_{i=1}^{n} a_i) = V(\sum_{i=n+1}^{k} a_i) = 0$ by 1.7, but this contradicts the fact that zero is the least element of G_V, so the claimed equality holds.

Definition 1.10. For R a ring, let $T = T(R) = \left[(A,Q) \mid A \text{ is a subring of } R \text{ and } Q \text{ is a prime ideal of } A\right]$. For (A,Q) and (B,S) in T define $(A,Q) \leq (B,S)$ if $A \subset B$ and $Q = A \cap S$.

"$\leq$" is clearly an inductive partial order on T, so by Zorn's Lemma, T has maximal elements. We call maximal elements of T maximal pairs. Note that if (A,Q) is in T, then there is a maximal pair (B,S) with $(B,S) \geq (A,Q)$.

Proposition 1.11. (A,Q) is a maximal pair of R iff it is a valuation pair of R.

Proof: If (A,Q) is a valuation pair and $(A,Q) \leq (B,S)$, and if x is in $B\setminus A$, then xp is in $A\setminus Q$ for some p in $Q\subset S$; but x in B and p in S imply that xp is in S so that xp is in $(S\cap A)\setminus Q$ contradicting $(A,Q)\leq(B,S)$. Thus, $B\setminus A=\emptyset$, so $B=A$ and $S=Q$; i.e., (A,Q) is a maximal pair.

Conversely, let (A,Q) be a maximal pair of R, x in $R\setminus A$, $B=A[x]$, and $S=BQ$. Then S is an ideal of B with $Q\subset(S\cap A)$. If $Q=A\cap S$, then $A\setminus Q$ is a multiplicatively closed subset of B with $(A\setminus Q)\cap S=\emptyset$. Then by Krull's Lemma (see [1] page 253), there is a prime ideal S' of B with $S\subset S'$ and $(A\setminus Q)\cap S'=\emptyset$. That is, $Q=S'\cap A$ and $(B,S')\geq(A,Q)$. But since $A\neq B$, this is a contradiction; hence, $Q\subsetneq(S\cap A)$. Thus, there are p_i in Q and a' in $A\ \ Q$ with $\sum_{i=0}^{n}p_ix^i=a'$, so (*) $\sum_{i=1}^{n}p_ix^i=a'-p_o=a$ is in $A\setminus Q$. We can assume n is minimal for an expression of this form.

If $n=1$, we are done: p_1x is in $A\setminus Q$.

Suppose $n>1$. Let $y=\sum_{i=1}^{n}p_ix^{i-1}$. Then $xy=a$ is in $A\setminus Q$. If y is in $A\setminus Q$, then ya is in $A\setminus Q$ and $ya=\sum_{i=1}^{n}(p_ixy)x^{i-1}=\sum_{i=1}^{n}p_i'x^{i-1}$, an expression of form (*) with degree $n-1<n$, a contradiction of the minimality of n. Thus, y is not in $A\setminus Q$.

If y is not in A, then the same argument used for x

gives q_i in Q and b in $A \setminus Q$ with (**) $\sum_{i=1}^{m} q_i y^i = b$.

Again, we can assume that m is minimal for an expression of this type. Now either 1) $n \geq m$ or 2) $m > n$.

Case 1) If $n \geq m$, then $p_n b x^n = \sum_{i=1}^{m} p_n q_i (xy)^i x^{n-i}$.

a,b in $A \setminus Q \Rightarrow ab$ in $A \setminus Q$, and $ab = \sum_{i=1}^{n-1} p_i b x^i + p_n b x^n =$

$\sum_{i=1}^{n-1} p_i b x^i + \sum_{i=1}^{m} p_n q_i (xy)^i x^{n-i} = \sum_{i=1}^{n-1} p_i b x^i + \sum_{j=n-m}^{n-1} p_n q_{n-j} (xy)^{n-j} x^j =$

$\sum_{i=1}^{n-1} q_i' x^i$ [+ q_o' if n=m, but then $(ab - q_o')$ is in $A \setminus Q$].

This is of form (*) and degree $n-1 < n$, a contradiction; therefore, $m > n$.

Case 2) Using $q_m a y^m = \sum_{i=1}^{n} p_i q_m (xy)^i y^{m-i}$, we obtain $ab = \sum_{i=1}^{m-1} p_i'' y^i$ contradicting the minimality of m. Therefore, y is in A and y is not in $A \setminus Q$, so y is in Q; thus, n=1 and (A,Q) is a valuation pair of R.

SECTION II

DOMINANCE

<u>Definition 2.1</u>. If V and V' are valuations on a ring R; we say V' dominates V and write $V' \geq V$ if there is an order homomorphism ϕ of $G_V \rightarrow G_{V'}$ with $V' = \phi \circ V$. We say $V' = V$ if ϕ is an isomorphism.

<u>Proposition 2.2</u>. Let V and V' be valuations on R. Then $V' \geq V$ iff $N_V \subset P_{V'} \subset P_V \subset A_V \subset A_{V'}$.

<u>Proof</u>: Let $V' \geq V$.

1) If $V(a) \leq e$, then $V'(a) = \phi(V(a)) \leq \phi(e) = e'$ since ϕ preserves order; i.e., $A_V \subset A_{V'}$.

2) If $V'(a) < e'$, then $\phi(V(a)) = V'(a) < e' = \phi(e)$ so $V(a) \leq e$ but $V(a) = e \Rightarrow \phi(V(a)) = \phi(e) = e'$, so $V(a) < e$; i.e., $P_{V'} \subset P_V$.

3) If $V(a) = 0$, then $V'(a) = \phi(V(a)) = \phi(0) = 0$ so $N_V \subset N_{V'} \subset P_{V'}$.

Conversely, let $N_V \subset P_{V'} \subset P_V \subset A_V \subset A_{V'}$. Note: $N_V = N_{V'}$ by 1.3. Let $\phi(V(a)) = V'(a)$.

<u>Claim 1</u>. ϕ is well-defined.

<u>Subproof 1</u>: Let $V(a) = V(b)$.

i) If $V(a) = V(b) = 0$, then a, b are in $N_V = N_{V'}$ so $V'(a) = V'(b) = 0$.

ii) If $V(a) = V(b) \neq 0$, then there is a z in R

such that $V(az) = e = V(a)V(z) = V(b)V(z) = V(bz)$, i.e., az, bz in $(A_v \setminus P_v) \subset (A_{v'} \setminus P_{v'})$; but then $e' = V'(az) = V'(bz)$ so $V'(a) = V'(a)V'(bz) = V'(az)V'(b) = V'(b)$.

Thus $V(a) = V(b) \Longrightarrow V'(a) = V'(b)$, and $\emptyset$ is well-defined and clearly a homomorphism.

Claim 2. $\emptyset$ is order-preserving.

Subproof 2: Let $V(a) \leq V(b)$. If $V(a) = 0$, then $V'(a) = 0 \leq V'(b)$ since $N_v = N_{v'}$. If $V(a) \neq 0$, then $V(b) \neq 0$ so there is a z in R with $V(bz) = e$. $V(az) \leq V(bz) = e$ so az is in $A_v \subset A_{v'}$; and thus, $V'(az) \leq e' = \emptyset(e) = \emptyset(V(bz)) = V'(bz)$. Therefore, $V'(a) = V'(a)e' = V'(a)V'(bz) = V'(az)V'(b) \leq e'V'(b) = V'(b)$. Thus, $\emptyset$ is the order-homomorphism required in 2.1.

Note that $P_{v'}$ is a prime ideal of A_v since $P_{v'} \subset A_v \subset A_{v'}$ and $P_{v'}$ is a prime ideal of $A_{v'}$.

Proposition 2.3. If P and P' are prime ideals of A_v, $N_v \subset P \subset P_v$, and $N_v \subset P' \subset P_v$, then $P \subset P'$ or $P' \subset P$.

Proof: Let x be in $P \setminus P'$ and y be in $P' \setminus P$, then $V(x) \neq 0$ and $V(y) \neq 0$ since $N_v \subset P \cap P'$ so there are x', y' in R with $V(xx') = e = V(yy')$. Now $V(x) \leq V(y)$ or $V(y) \leq V(x)$.

Case 1) $V(x) \leq V(y)$ gives $V(xy') \leq V(yy') = e$ so xy' is in A_v. Now y is in P' so yxy' is in P'; but then, x in A_v and yy' in $(A_v \setminus P_v) \subset (A_v \setminus P')$ imply that x is in P' since P' is a prime ideal of A_v, which is a contradiction of x in $P\setminus P'$. Thus, $V(x)\nleq V(y)$.

Case 2) $V(y)\leq V(x)$. Interchanging x and y, x' and y', P and P' in the above argument, we obtain y in $P\cap(P'\setminus P)$, a contradiction. Thus, $V(y)\nleq V(x)\nleq V(y)$ which contradicts the linear order on G_V.

Thus $(P\setminus P') = \emptyset$ or $(P'\setminus P) = \emptyset$; i.e., $P'\subset P$ or $P\subset P'$.

Henceforth, we will use the sign ## for a contradiction.

<u>Proposition 2.4</u>. If V, V', and V'' are valuations on R, $V'\geq V$, and $V''\geq V$, then $V'\geq V''$ or $V''\geq V'$.

<u>Proof</u>: $P_{V'}\subset P_{V''}$ or $P_{V''}\subset P_{V'}$ by 2.3. Without loss of generality, we may assume $P_{V'}\subset P_{V''}$. If x is not in $A_{V'}$, then x is not in A_V so there is a y in P_V with xy in $A_V\setminus P_V\subset A_{V'}\setminus P_{V'}$ and xy in $A_V\setminus P_V\subset A_{V''}\setminus P_{V''}$ so $V(xy) = e$, $V'(xy) = e'$, and $V''(xy) = e''$. Now $V'(x) > e' \Rightarrow V'(y) < e'$, i.e. y in $P_{V'}\subset P_{V''} \Rightarrow V''(y) < e'' \Rightarrow V''(x) > e'' \Rightarrow x$ not in $A_{V''}$. Thus ${}^{\setminus}A_{V'}\subset {}^{\setminus}A_{V''}$ so $A_{V''}\subset A_{V'}$ and we have $N_{V''} = N_{V'}\subset P_{V'}\subset P_{V''}\subset A_{V''}\subset A_{V'}$; i.e., $V'\geq V''$.

Thus $[V' / V'$ a valuation on R and $V'\geq V$ for a fixed valuation $V]$ is linearly ordered by "$\leq$".

<u>Definition 2.5</u>. A subgroup H of a valuation semigroup G is said to be isolated if 0 is not in H and whenever a,b,c are in G with $a\leq b\leq c$ and a,c in H then b is in H.

<u>Proposition 2.6</u>. The isolated subgroups of a valuation semigroup G are linearly ordered by inclusion.

<u>Proof</u>: Let H and H' be isolated subgroups of G and suppose that a is in $H \setminus H'$ and b is in $H' \setminus H$. Then a,b in G implies that $a \leq b$ or $b \leq a$.

Case 1) $a \leq b$.

i) If $e \leq a$, then $e \leq a \leq b$, e,b in H' give a in H', ##.

ii) If $a \leq e$ and $b \leq e$, then $a \leq b \leq e$, a,e in H give b in H, ##.

iii) If $a \leq e \leq b$, then $b^{-1} \leq e$. If $a \leq b^{-1} \leq e$, then a,e in H give b^{-1} in H, ##. If $b^{-1} \leq a \leq e$, then b^{-1},e in H' give a in H', ##.

Interchanging a and b, H and H', we likewise obtain a contradiction for case 2); but case 1) or case 2) must hold for a,b in G, so $H \subset H'$ or $H' \subset H$.

<u>Proposition 2.7</u>. Let V be a valuation on R and $G = G_V$ its valuation semigroup. Then there is a "1-1" order-preserving correspondence between $I(G) = I = \{H \mid H \text{ is an isolated subgroup of } G\}$ and $D(V) = D = \{V' \mid V' \text{ a valuation on } R \text{ with } V' \geq V\}$.

<u>Proof</u>: For V' in D, let $f(V') = \phi^{-1}(e')$ where ϕ is the order-homomorphism in the definition of $V' \geq V$.

<u>Claim 1</u>. $f: D \longrightarrow I$; i.e., $\phi^{-1}(e')$ is in I.

<u>Subproof 1</u>: If a,b,c are in G, $a \leq b \leq c$, and $\emptyset(a) = \emptyset(c) = e'$, then $e' = \emptyset(a) \leq \emptyset(b) \leq \emptyset(c) = e'$ since $\emptyset$ is order-preserving. Also, $e' = \emptyset(aa^{-1}) = \emptyset(a)\emptyset(a^{-1}) = \emptyset(a^{-1})$ and $e' = \emptyset(a)\emptyset(c) = \emptyset(ac)$ so b, a^{-1}, ac are in $\emptyset^{-1}(e')$ so $\emptyset^{-1}(e')$ is an isolated subgroup of G and hence in I.

Also f is obviously well-defined and "1-1" since $V' = V''$ implies $V'(x) = e'$ iff $V''(x) = e''$.

<u>Claim 2</u>. For H in I, there is an order-homomorphism $\emptyset_H = \emptyset$ of G onto a valuation semigroup $G_{\emptyset o V}$ with $\emptyset^{-1}(e) = H$.

<u>Subproof 2</u>: Set $\emptyset(a) = aH$ for all a in G. Then since G abelian implies H normal in G, $\emptyset(G) = ((G \setminus [0])/H) \cup [0]$, with the usual coset multiplication, is an abelian group with zero adjoined and $H \neq 0 \cdot H = 0$.

Define: $aH < bH$ if $aH \neq bH$ and $a < b$.

If $aH \neq bH$ and $a < b$ ($\Rightarrow ab^{-1} < e$), then $ah' \geq bh''$ for some h', h'' in H gives $e > ab^{-1} \geq h'^{-1}h''$ and $e, h'^{-1}h''$ in H so ab^{-1} in H since H isolated so $aH = bH$, ##. Thus, $ah' < bh''$ for all h', h'' in H so "$<$" is well-defined on $\emptyset(G)$ and linear since if aH, bH are in $\emptyset(G)$ and $aH \neq bH$, then $a \neq b$ so $a < b$ or $b < a$.

It is easily checked that $\emptyset(G)$ with this definition of "$<$" satisfies conditions i) and ii) of 1.1. Thus, $\emptyset(G)$ is a valuation semigroup, and $\emptyset$ is obviously an order-homomorphism onto. $\emptyset(a) = e = H$ iff a in H so

$\phi^{-1}(e) = H$. Further, $\phi \circ V = V_H$ is clearly a valuation on R with $V_H \geq V$. Thus, given H in I, there is a valuation V_H on R with $f(V_H) = H$; i.e., f is onto.

Claim 3. Let V', V'' be in D with $V'' \geq V'$. Then $f(V') \subset f(V'')$.

Subproof 3: There are order-homomorphisms ϕ, ϕ', ϕ'' such that $\phi' : G \longrightarrow G_{V'}$, $\phi'' : G \longrightarrow G_{V''}$, and $\phi : G_{V'} \longrightarrow G_{V''}$. $\phi \circ \phi'(V(x)) = \phi(V'(x)) = V''(x) = \phi''(V(x))$ for all x in R, i.e. all V(x) in G. Thus $\phi \circ \phi' = \phi''$ so $\phi''^{-1} = \phi'^{-1} \circ \phi^{-1}$ so $\phi''^{-1}(e'') = \phi'^{-1}(\phi^{-1}(e'')) \supset \phi'^{-1}(e')$; i.e., $f(V'') \supset f(V')$.

Thus, f is the claimed "1-1" order-preserving correspondence between I and D.

SECTION III

EXTENSIONS

Throughout this section, let V be a fixed valuation on a ring K, and let R be an extension of K.

<u>Definition 3.1</u>. A valuation W on R is called an extension of V to R if there is an order-isomorphism $\emptyset$ of G_v into G_w with $\emptyset(V(x)) = W(x)$ for all x in K.

<u>Proposition 3.2</u>. Let W be a valuation on R. Then the following are equivalent.

i) W is an extension of V to R.

ii) $(A_w,P_w) \geq (A_v,P_v)$ and $N_v = N_w \cap K$.

iii) $(A_w,P_w) \geq (A_v,P_v)$ and $W|_K$ is a valuation on K.

<u>Proof</u>: i) $\Longrightarrow$ ii)

If x is in A_v, $W(x) = \emptyset(V(x)) \leq \emptyset(V(e)) = e$ since $\emptyset$ is order-preserving, so $A_v \subset A_w$. If x is in K, $W(x) = \emptyset(V(x)) < e$ iff $V(x) < e$ since $\emptyset$ is an order-isomorphism, so $P_v = P_w \cap K = P_w \cap A_v$. Thus, $(A_w,P_w) \geq (A_v,P_v)$. Also, $N_w \cap K$ is an ideal of K and $(N_w \cap K) \subset (P_w \cap K) = P_v \subset A_v$ so $(N_w \cap K) \subset N_v$ by 1.3. If $V(x) = 0$, $W(x) = \emptyset(V(x)) = \emptyset(0) = 0$, so $N_v \subset (N_w \cap K)$; and hence, $N_w \cap K = N_v$.

ii) $\Longrightarrow$ iii)

If x is in K with $W(x) \neq 0$, then x is not in N_v so there is a y in K with xy in $A_v \setminus P_v \subset A_w \setminus P_w$. $W(xy) =$

$e = W(x)W(y)$ so $W(y) = W(x)^{-1}$. Thus, $W(K)$ is a valuation semigroup contained in G_w; i.e., $W|_K$ is a valuation on K.

iii) $\Rightarrow$ i)

$$(A_{w|_K}, P_{w|_K}) = (A_w \cap K, P_w \cap K) \geq (A_v, P_v) \quad \text{so}$$

$(A_w \cap K, P_w \cap K) = (A_v, P_v)$ since (A_v, P_v) is a maximal element of $T(K)$. Thus, by 1.5, there is an order-isomorphism $\emptyset$ of G_v onto $G_{w|_K}$ with $W|_K(x) = \emptyset(V(x))$ for all x in K, and $G_{w|_K} \subset G_w$.

Henceforth, if W is an extension of V to R, we will identify G_v with $G_{w|_K}$ and thus consider $G_v \subset G_w$.

<u>Proposition 3.3</u>. V has extensions to R iff $K \cap RN_v = N_v$.

<u>Proof</u>: If V has an extension W to R, then $N_v \subset N_w$ so $(K \cap RN_v) \subset (K \cap RN_w) = K \cap N_w = N_v$. Thus, $K \cap RN_v = N_v$.

Conversely, suppose $K \cap RN_v = N_v$. Then let $Q = P_v + RN_v$ and $B = A_v + RN_v$. Now Q is an ideal of B, and $A_v = B \cap K$ and $P_v = Q \cap K = Q \cap A_v$ so $Q \cap (A_v \setminus P_v) = \emptyset$. Thus, by Krull's Lemma, there is a prime ideal Q' of B with $Q \subset Q'$ and $Q' \cap (A_v \setminus P_v) = \emptyset$; i.e., $(B, Q') \geq (A_v, P_v)$. If (A_w, P_w) is any valuation pair of R with $(A_w, P_w) \geq (B, Q')$, then $(A_w, P_w) \geq (A_v, P_v)$. $N_v \subset RN_v \subset B \subset A_w$ so $RN_v \subset N_w$ by 1.3; i.e., $K \cap RN_v = N_v \subset (N_w \cap K)$. $(N_w \cap K) \subset (P_w \cap K) = P_v$ so $(N_w \cap K) \subset N_v$; hence, $N_w \cap K = N_v$, and W is an extension of V to R by 3.2.

<u>Definition 3.4</u>. If W extends V to R, we write $\{(G_w \setminus [0])/(G_v \setminus [0])\} \cup [0]$ as G_w/G_v. We say that G_w/G_v is torsion iff for each a in G_w there is an integer $n > 0$ with a^n in G_v.

<u>Proposition 3.5</u>. Let V and V' be valuations on K with $V' \geq V$. Then

i) V has extensions to R iff V' has extensions to R.

ii) If W is an extension of V to R, then there is an extension W' of V' to R with $W' \geq W$; and further, if G_w/G_v is torsion, then the W' is unique.

<u>Proof</u>: $N_v = N_{v'}$ by 2.2 so i) is clear by 3.3.

ii) Let H be the isolated subgroup of G_v corresponding to V' (confer 2.7). Let $S = [a \text{ in } G_w \mid \text{there are } b,c \text{ in } H \text{ with } b \leq a \leq c]$. Then S is clearly an isolated subgroup of G_w with $S \cap G_v = H$. Now let W' be the valuation corresponding to S so that $W' \geq W$. Note that in proving 2.7, we also proved that $G_v/H \cong G_{v'}$ and $G_w/S \cong G_{w'}$. Define $\phi: G_v/H \to G_w/S$ by $\phi(aH) = aS$ for all a in G_v. $aH = bH \Rightarrow ab^{-1}$ in $H \Rightarrow ab^{-1}$ in $S \Rightarrow aS = bS$ so ϕ is well-defined. If a,b are in G_v and $aS = bS$, then ab^{-1} is in $S \cap G_v = H$ so $aH = bH$ so ϕ is "1-1". ϕ is onto $\phi(G_v/H)$ and clearly a homomorphism with the usual coset multiplication, so $\emptyset$ is an isomorphism onto $\emptyset(G_v/H)$. If $aH < bH$, then $aH \neq bH$ and $a < b$ so $aS \neq bS$ and $a < b$; i.e., $aS < bS$. Hence, $\emptyset$ is order-preserving. Thus, $G_{v'} \cong G_v/H \cong \emptyset(G_v/H) \subset G_w/S \cong G_{w'}$, and the map required in 3.1 is the obvious one; so W' extends V'.

<u>Claim 1</u>. If W" is an extension of V' to R with $W" \geq W$ and S' is the isolated subgroup of G_w corresponding to W", then $S \subset S'$.

<u>Subproof 1</u>: By the above argument, W" extends the valuation $V" \geq V$ corresponding to $S' \cap G_v$. By 3.2, $(A_{v"}, P_{v"}) = (A_{w"} \cap K, P_{w"} \cap K) = (A_{v'}, P_{v'})$ so by 1.5, $G_{v'} \cong G_{v"}$; that is, V' = V". Thus, by 2.7, $S' \cap G_v = H$. Now let x be in S. Then there are a,b in $H = S' \cap G_v$ with $a \leq x \leq b$, so x is in S' since S' is isolated.

<u>Claim 2</u>. If G_w/G_v is torsion, then $S' \subset S$.

<u>Subproof 2</u>: Let x be in S'; then x is in G_w so there is an integer $n > 0$ with x^n in G_v; i.e., x^n is in $S' \cap G_v = H$. 1) If $x \geq e$, then $x^n \geq x \geq e$ and x^n, e are in H, so x is in S. 2) If $x \leq e$, then $x^n \leq x \leq e$, so x is in S.

Thus, if G_w/G_v is torsion, then S = S' so W" = W' by 2.7; i.e., W' is unique.

<u>Proposition 3.6</u>. If (A,P) is a valuation pair of a ring R, then A is integrally closed in R.

<u>Proof</u>: Let $\bar{A}$ be the integral closure of A in R.

Then $A \subset \bar{A}$ clearly. Let c be in $\bar{A}$, then there are a_i in A and $n > 0$ such that $c^n = \sum_{i=0}^{n-1} a_i c^i$. Let V be the valuation on R associated with (A,P). Now if c is not in A, then $V(c) > e$ so $V(a_i c^i) \leq V(c^i) < V(c^n)$ for $i < n$. But then $V(c^n) = V(\sum_{i=0}^{n-1} a_i c^i) \leq \max_{i<n} [V(a_i c^i)] < V(c^n)$, ##. Thus, c is in A so $\bar{A} \subset A$; and hence, $A = \bar{A}$.

<u>Proposition 3.7</u>. Suppose that R is integral over K. If V is a valuation on K and W a valuation on R with $(A_W, P_W) \geq (A_V, P_V)$, then W extends V to R.

<u>Proof</u>: $K \cap N_W \subset K \cap P_W = P_V \subset A_V$ and $K \cap N_W$ is an ideal of K, so $K \cap N_W \subset N_V$ by 1.3.

Let t be in N_V and x in R. Since R is integral over K, there are a_i in K and $n > 0$ with $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$. Then $t^n \cdot 0 = 0 = (tx)^n + \sum_{i=0}^{n-1} a_i t^{n-i} (tx)^i$. But for $i < n$, $a_i t^{n-i}$ is in $N_V \subset A_W$; that is, tx is integral over A_W. Since A_W is integrally closed in R, tx is in A_W. Thus, $RN_V \subset A_W$ so $RN_V \subset N_W$ by 1.3. Therefore, $N_V \subset RN_V \cap K \subset N_W \cap K$, and W extends V by 3.2.

Thus, every valuation V on K has extensions if R is integral over K.

Proof: Choose y_i in F* with $V_i(y_i) < V_i(x_i)$ and use 4.2.

Proposition 4.4. If $x_1, \ldots, x_n$ are in F*, then there is an a in F with $V_i(a) = V_i(x_i)$.

Proof: The a in 4.3 works since $V_i(a-x_i) < V_i(x_i) \leq \max[V_i(a), V_i(x_i)]$ implies $V_i(a) = V_i(x_i)$ by 1.7.

Proposition 4.5. Let $x_1, \ldots, x_n$ be in F*; then there is an a in F with $V_i(a) < V_i(x_i)$.

Proof: Choose y_i in F* with $V_i(y_i) < V_i(x_i)$ for each i and use 4.4 on the y_i's.

Proposition 4.6. Let L be the set of all valuations on F and let x be in F*, then there is a y in F with $V(xy) = e$ for all V in L.

Proof: Let $y = x^{-1}$.

Altough this last proposition is quite trivial in the case of fields, we experience considerable difficulty in obtaining a similar result for rings. Part II is directed to this problem.

Proof: Choose y_i in F* with $V_i(y_i) < V_i(x_i)$ and use 4.2.

Proposition 4.4. If $x_1,\ldots,x_n$ are in F*, then there is an a in F with $V_i(a) = V_i(x_i)$.

Proof: The a in 4.3 works since $V_i(a-x_i) < V_i(x_i) \leq \max[V_i(a), V_i(x_i)]$ implies $V_i(a) = V_i(x_i)$ by 1.7.

Proposition 4.5. Let $x_1,\ldots,x_n$ be in F*; then there is an a in F with $V_i(a) < V_i(x_i)$.

Proof: Choose y_i in F* with $V_i(y_i) < V_i(x_i)$ for each i and use 4.4 on the y_i's.

Proposition 4.6. Let L be the set of all valuations on F and let x be in F*, then there is a y in F with $V(xy) = e$ for all V in L.

Proof: Let $y = x^{-1}$.

Altough this last proposition is quite trivial in the case of fields, we experience considerable difficulty in obtaining a similar result for rings. Part II is directed to this problem.

PART II

THE INVERSE PROPERTY

Definition 4.7. We say that a set L of valuations on a ring R has the inverse property if for every x in R there is an x' in R such that $V(xx') = e$ whenever V is in L and $V(x) \neq 0$. L is said to have the strong inverse property if for every x in R there is an x' in R with $V(xx'-1) < e$ whenever V is in L and $V(x) \neq 0$.

Proposition 4.8. Let L be a set of valuations on R which has the inverse property and L' a set of valuations on R such that for every V' in L' there is a V in L with $V' \geq V$. Then $L \cup L'$ has the inverse property; in particular, L' has the inverse property.

Proof: Let x, x' be in R with $V(xx') = e$ whenever V is in L with $V(x) \neq 0$. Let V' be in L' and suppose $V' \geq V$, V in L, and $V'(x) \neq 0$. Then $V(x) \neq 0$ by 2.2 so $V(xx') = e$. Then xx' is in $A_V \setminus P_V \subset A_{V'} \setminus P_{V'}$, so $V'(xx') = e$.

Proposition 4.9. Let V, V' be valuations on R with $P_V \subset P_{V'}$. Then $L = [V, V']$ satisfies the inverse property iff $A_{V'} \subset A_V \cup N_{V'}$.

Proof: If $A_{V'} \subset A_V \cup N_{V'}$, then $A_{V'} \setminus P_{V'} \subset (A_V \cup N_{V'}) \setminus P_{V'}$ $= A_V \setminus P_{V'} \subset A_V \setminus P_V$ and $N_V \subset N_{V'}$ by 1.3. If x is in R and

$V'(x) \neq 0$, then there is an x' in R with xx' in $A_{V'}\setminus P_{V'} \subset A_V\setminus P_V$ so $V'(xx') = e$ and $V(xx') = e$.

If L satisfies the inverse property and x is in $\setminus(A_V \cup N_{V'})$, then $V'(x) \neq 0$ and $V(x) > e$ so there is an x' in $P_V \subset P_{V'}$ with $V(xx') = e$ and $V'(xx') = e$. x' in $P_{V'} \Rightarrow V'(x') < e \Rightarrow V'(x) > e \Rightarrow x$ in $\setminus A_{V'}$. Thus, $\setminus(A_V \cup N_{V'}) \subset \setminus A_{V'}$, so $A_{V'} \subset (A_V \cup N_{V'})$.

Example 1. Let Q be the rational numbers and $Q_p = \left[ab^{-1} \mid b \neq 0,\ (a,b) = 1, \text{ and } (b,p) = 1\right]$. Let $R = Q[x]$, $A_V = Q_p[x]$, $P_V = pQ_p[x]$, $A_{V'} = Q_p + xR$, and $P_{V'} = pQ_p + xR$. Then (A_V, P_V) and $(A_{V'}, P_{V'})$ are valuation pairs of R, $P_V \subset P_{V'}$, and $N_{V'} = xR$, all of which the reader can check for himself. $t = (1 + xp^{-1})$ is in $A_{V'} \setminus (A_V \cup xR)$ so $[V, V']$ does not satisfy the inverse property. Specifically, it is not satisfied for t since t in $A_{V'}\setminus P_{V'} \Rightarrow V'(t) = e$ and t not in $A_V \Rightarrow V(t) > e$; and if $V(tt') = e$, then $V(t') < e \Rightarrow t'$ is in $P_V \subset P_{V'} \Rightarrow V'(t') < e \Rightarrow V'(tt') < e$.

Notice that in Example 1, "x" is in $N_{V'}\setminus N_V$ so that $N_{V'} \neq N_V$. This observation led to the conjecture that perhaps if $N_V = N_{V'}$, then $[V, V']$ satisfies the inverse property. This is not always true as Example 2 will show.

For V a valuation on a ring R, R/N_V is a domain and $\overline{V}(x + N_V) = V(x)$ defines a valuation on R/N_V with $G_{\overline{V}} = G_V$. Letting F be the quotient field of R/N_V and W defined by $W(ab^{-1}) = \overline{V}(ac)$ where $\overline{V}(bc) = e$, then W

is an extension of $\overline{V}$ to F with $G_W = G_{\overline{V}} = G_V$. The details of these statements are easily checked. Thus, if V and V' are valuations on R with $N_V = N_{V'}$, then we can consider $N_V = N_{V'} = \{0\}$ since $R/N_V = R/N_{V'}$ and $\overline{V}(x + N_V) = 0$ iff x is in $N_V = N_{V'}$.

Proposition 4.10. Let V and V' be valuations on a domain R with $N_V = N_{V'} = \{0\}$ and F be the quotient field of R. Then the following are equivalent:

i) $L = \{V, V'\}$ satisfies the inverse property.

ii) $F = \{xy^{-1} \mid x \text{ is in } R \text{ and } y \text{ is in } S\}$ where $S = (A_V \setminus P_V) \cap (A_{V'} \setminus P_{V'})$.

iii) J a principle ideal of R with $J \cap S = \emptyset$ implies that $J = \{0\}$.

Proof: i) $\Rightarrow$ ii). Let L satisfy the inverse property and let t be in F. Then $t = xy^{-1}$ for some x,y in R, $y \neq 0$. y in R, $y \neq 0$ imply that there is a z in R with $V(yz) = e$ and $V'(yz) = e$ by the inverse property. Thus, $t = xy^{-1} = xz(yz)^{-1}$, and xz is in R and yz is in S.

ii) $\Rightarrow$ i). If x is in F, then there are y,z in R with $x = yz^{-1}$ and $V(z) = e = V'(z)$. If W and W' are the extensions to F noted preceeding the proposition, then $W(x) = W(yz^{-1}) = V(y)V(z)^{-1} = V(y)$ and $W'(x) = W'(yz^{-1}) = V'(y)V'(z)^{-1} = V'(y)$; i.e., if x is in F, then there is a y in R with $W(x) = V(y)$ and $W'(x) = V'(y)$. Thus, if

t is in R, then t^{-1} is in F so there is a y in R with $W(t^{-1}) = V(y)$ and $W'(t^{-1}) = V'(y)$ so $e = W(t)W(t^{-1}) = V(t)V(y)$ and $e = W'(t)W'(t^{-1}) = V'(t)V'(y)$.

i) $\Rightarrow$ iii). L has the inverse property implies that for x in R, $x \neq 0$, there is a y in R such that xy is in S. Thus, $xR \cap S = \emptyset$ iff $x = 0$.

iii) $\Rightarrow$ i). x in R, $x \neq 0$, $\Rightarrow xR \neq [0] \Rightarrow xR \cap S \neq \emptyset$; i.e., there is a y in R with xy in S.

<u>Example 2</u>. Let Z be the integers. Let $R = Z[x,x^{-1}]$, $A_v = Z[x]$, $P_v = xZ[x]$, $A_{v'} = Z[x^{-1}]$, and $P_{v'} = x^{-1}Z[x^{-1}]$. The reader can check that (A_v,P_v) and $(A_{v'},P_{v'})$ are valuation pairs of R, $N_v = N_{v'} = [0]$, and $(A_v \setminus P_v) \cap (A_{v'} \setminus P_{v'}) = Z$. $(1 + x)R \cap Z = \emptyset$ but $(1 + x)R \neq [0]$ so $[V,V']$ does not satisfy the inverse property. Also note that $F \neq [zy^{-1} \mid z \text{ is in } R \text{ and } y \text{ is in } Z]$, since $\frac{1}{1+x}$ cannot be written as zy^{-1} where z is in R and y is in Z. Also if $t = 1 + x$, then t is in $A_v \setminus P_v$ and t is not in $A_{v'}$, so $V(t) = e$ and $V'(t) > e$. Therefore, if $V'(tt') = e$, then $V'(t') < e$, i.e., t' is in $P_{v'}$; but then, t' is not in A_v so $V(tt') = V(t') > e$.

<u>Example 3</u>. Let p and q be distinct prime integers. Let $R = Z[x,x^{-1}]$, $A_v = Z[x] + pR$, $P_v = xA_v + pR$, $A_{v'} = Z[x^{-1}] + qR$, and $P_{v'} = x^{-1}A_{v'} + qR$. Then $N_v = pR$ and $N_{v'} = qR$; but $[V,V']$ satisfies the inverse property. If t is in $R \setminus (N_v \cup N_{v'})$, then $t = \sum_{i=0}^{n} a_i x^{i-k}$, a_j is not in

pZ for some j and a_r is not in qZ for some r. Let J = $\min\left[j \mid a_j \text{ is not in } pZ\right]$ and $M = \max\left[r \mid a_r \text{ is not in } qZ\right]$. Then $t(qx^{k-J} + px^{k-M})$ is in $(A_V \setminus P_V) \cap (A_{V'} \setminus P_{V'})$. The details are left to the reader. Thus, the fact that $\left[V, V'\right]$ satisfies the inverse property does not imply that $N_V = N_{V'}$.

PART III
ALGEBRAIC EXTENSIONS

Throughout Part III, R is assumed to be an extension of a ring K, V a valuation on K with extensions to R, and L a set of valuations on R which extend V.

Proposition 4.11. Let J be an ideal of R with $J \subset \cap\left[N_W \mid W \text{ in } L\right]$ and $J \cap K = N_V$. If R/J is algebraic over K/N_V, then L satisfies the inverse property.

Proof: Note that $W(t) = 0$ for all t in J and W in L. If $x + J$ is in R/J, then there are a_i in K and t in J with a_r not in J ($V(a_r) \neq 0$) and $\sum_{i=0}^{r} a_i x^i = t$. Let $s = \min\left[i \mid V(a_i) \neq 0\right]$. Then for W in L, $0 = W(t) = W(\sum_{i=0}^{r} a_i x^i) = W(\sum_{i=s}^{r} a_i x^i) = W(x^s) W(\sum_{i=s}^{r} a_i x^{i-s})$. Thus, if $W(x) \neq 0$, then $W(\sum_{i=s}^{r} a_i x^{i-s}) = 0 = W(\sum_{i=s+1}^{r} a_i x^{i-s} + a_s) <$

$\max\left[W(\sum_{i=s+1}^{r} a_i x^{i-s}), W(a_s)\right]$, so by 1.7, $W(\sum_{i=s+1}^{r} a_i x^{i-s}) =$ $W(a_s) = W(x)W(\sum_{i=s+1}^{r} a_i x^{i-s-1})$. Choose a' in K with $V(a'a_s) = e$. Then with $x' = a'(\sum_{i=s+1}^{r} a_i x^{i-s-1})$, $W(xx') =$ $W(a'a_s) = V(a'a_s) = e$ whenever W is in L with $W(x) \neq 0$.

<u>Proposition 4.12</u>. Let $J = \bigcap\left[N_W \mid W \text{ in } L\right]$ and suppose R/J is algebraic over $K/(K \cap J) = K/N_V$. Then G_W/G_V is torsion for all W in L.

<u>Proof</u>: Let x be in R and W in L. If $W(x) = 0$, there is nothing to show, so suppose $W(x) \neq 0$. Then there are a_i in K, t in J, and a_r not in J such that $\sum_{i=0}^{r} a_i x^i = t$. Since $W(a_r x^r) \neq 0$, we have $0 = W(t) =$ $W(\sum_{i=0}^{r} a_i x^i) < \max\left[W(a_i x^i)\right]$, so by 1.8, $W(a_i x^i) = \max\left[W(a_i x^i)\right]$ $= W(a_j x^j) \neq 0$ for some $i \neq j$.

Assume $i > j$, and let $W(x)^{-1} = W(x')$ and $W(a_i)^{-1} =$ $W(a')$; then $W(x^{i-j}) = W(a_i x^i)W(x')^j W(a') = W(a_j x^j)W(x')^j W(a_i)^{-1}$ $= W(a_j)W(a')$ is in G_V.

<u>Proposition 4.13</u>. Let W be in L, $W' \geq W$, and $V' =$ $W'|_K$. If G_W/G_V is torsion, then so is $G_{W'}/G_{V'}$.

<u>Proof</u>: Let $\phi: G_W \longrightarrow G_{W'}$ be the homomorphism such that

$W' = \emptyset \circ W$. Then $V' = \emptyset \circ V$. If $\emptyset(x)$ is in $G_{W'}$, then x^n is in G_V for some $n > 0$ so $\emptyset(x^n)$ is in $G_{V'}$.

<u>Proposition 4.14</u>. If W is in L and G_W/G_V is torsion, then $W(R) = [e,0]$ iff $V(K) = [e,0]$.

<u>Proof</u>: $V(K) \subset W(R)$ so "$\Rightarrow$" is clear. $W(x^n)$ is in $[e,0]$ for some $n > 0$ only if $W(x)$ is in $[e,0]$ so "$\Leftarrow$" is also clear.

<u>Note 4.15</u>. If R is integral over K and J is any ideal of R, then R/J is integral (and hence algebraic) over $K/(K \cap J)$. Clear.

PART IV

APPROXIMATION THEOREMS

In Part IV, we assume that R is an extension of K, V is a valuation on K, and L is a set of extensions of V to R with the inverse property and such that G_W/G_V is torsion for each W in L. In some of the results, we also require $P_W \not\subset P_{W'}$ if W, W' are in L and $W \neq W'$. The following proposition indicates the effect of this additional restriction.

<u>Proposition 4.16</u>. Let W and W' be distinct elements of L with $P_W \subset P_{W'}$. Then P_V is an ideal of K, and R

is not integral over K.

<u>Proof</u>: If P_V is an ideal of K, then P_W and $P_{W'}$ are ideals of R by 4.14. Then $A_W = A_{W'} = R$, and if R were integral over K, we would also have $P_W = P_{W'}$ (see [4] page 259), contradicting P_W and $P_{W'}$ distinct.

It remains only to show that if P_V is not an ideal of K, then $P_W \not\subset P_{W'}$.

If P_V is not an ideal of K, than P_W and $P_{W'}$ are not ideals of R, so by 1.6, $A_W \neq A_{W'}$.

Case 1) $A_W \setminus A_{W'} \neq \emptyset$. Let y be in $A_W \setminus A_{W'}$. Then $W(y) \leq e < W'(y)$. Since $G_{W'}/G_V$ is torsion, there is an integer $n > 0$ and an a in K with $W'(y^n) = V(a)$. Then $W'(y) = W'(y^{n+1}a') > e$ while $W(y^{n+1}a') = W(y^{n+1})W(a') < e$ since $V(a') < e$. Thus, $y^{n+1}a'$ is in $P_W \setminus P_{W'}$.

Case 2) $A_{W'} \setminus A_W \neq \emptyset$. By Case 1), there is a y in R with $W(y) > e > W'(y)$. Then $W(1 + y) = W(y) > e$ while $W'(1 + y) = W'(1) = e$, so $W((1+y)') < e$ while $W'((1+y)') = e$. Thus, $(1+y)'$ is in $P_W \setminus P_{W'}$.

<u>Proposition 4.17</u>. Let $W_1, \ldots, W_n$ be distinct elements of L with $P_{W_i} \not\subset P_{W_1}$ if $i \neq 1$. Then there is an x in R with $W_1(x) \geq e$ and $W_i(x) < e$ for $i \neq 1$. Further, if P_V is not an ideal of K, one can require $W_1(x) > e$.

<u>Proof</u>: Case 1) P_V an ideal of K. Then P_{W_i} is

a prime ideal of R, $i = i,\ldots,n$. Choose x_i in $P_{w_i} \setminus P_{w_1}$, $i = 2,3,\ldots,n$ and let $x = \prod_{i=2}^{n} x_i$.

Case 2) P_v not an ideal of K. Proof by induction on n.

For $n = 2$, choose y in $P_{w_2} \setminus P_{w_1}$. Then $W_1(y) \geq e > W_2(y)$. Since G_{w_2}/G_v is torsion and $G_v \neq [e,0]$, there is an $n > 0$ and an a in $K \setminus N_v$ with $e > W_2(a) > W_2(y^n)$. Then with $x = a'y^n$ we have $W_1(x) \geq W_1(a') > e$ while $W_2(aa') = e$ $W_2(x)$.

Now assume 4.17 holds for $r = n-1$, $n > 2$. For $i = 2,3$, choose y_i in R with $W_1(y_i) > e$ and $W_j(y_i) < e$ if $j \neq 1$ and $j \neq i$. If $W_i(y_i) \leq e$, let $x_i = y_i$; otherwise let $x_i = (1+y_i)'y_i$.

Claim. $W_1(x_i) \geq e$, $W_i(x_i) \leq e$, $W_j(x_i) < e$ if $i \neq j \neq 1$.

Subproof: This is automatic if $x_i = y_i$. Otherwise, $W_1(1+y_i) = W_1(y_i) > e$ and $W_1(x_i) = e$; $W_i(1+y_i) = W_i(y_i) >$ e and $W_i(x_i) = e$; $i \neq j \neq 1$, $W_j(1+y_i) = W_j(1) = e$ and $W_j(x_i) = W_j(y_i) < e$.

Thus, we have $W_1(x_2x_3) \geq e$ and $W_i(x_2x_3) < e$ if $i \neq 1$. Let $z = x_2x_3$. Again since G_{w_i}/G_v is torsion and $G_v \neq [e,0]$, there is an $n > 0$ and an a in $K \setminus N_v$ with $e > W_i(a) > W_i(z^n)$ for all $i \neq 1$, and $x = a'z^n$ has $W_1(x) > e$ and $W_i(x) < e$ for all $i \neq 1$.

Proposition 4.18. Assume P_v is not an ideal of K and $W_1,\ldots,W_n$ in L are pairwise independent. Then if

a_i is in $G_{w_i} \setminus [0]$ for $i = 2,3,\ldots,n$, there is an x in R with $W_1(x) \geq e$ and $W_i(x) < a_i$ for $i \neq 1$.

<u>Proof</u>: Since G_{w_i}/G_v is torsion for $i = 2,3,\ldots,n$, there are $n_i > 0$ with $a_i^{n_i}$ in $G_v \setminus [0]$. Let $0 < a < \min[e, a_i^{n_i} \;\; i=2,\ldots,n]$. It suffices to show that there is an x in R with $W_1(x) \geq e$ and $W_i(x) < a$ for $i = 2,\ldots,n$.

Let $H = [a \text{ in } G_v \mid \text{there is an } x \text{ in } R \text{ with } W_1(x) \geq e$ and $W_i(x) < \min(a,a^{-1}) \text{ for } i \neq 1]$. Then e is in H by 4.17, and it is easily checked that H is an isolated subgroup of G_v. The proposition will be established if $H = G_v \setminus [0]$, or equivalently, that if V' is the valuation determined by H, then $V'(K) = [e,0] = G_v/H$.

Since $V' \geq V$ and G_{w_i}/G_v is torsion for each i, by 3.5 there is a unique $W_i' \geq W_i$ which extends V', $i=1,\ldots,n$. Since the W_i are independent, either $W_i'(R) = [e,0]$ for some i so that $V'(K) = [e,0]$ by 4.14 and the proposition is established, or the W_i' are distinct.

Assume the W_i' are distinct. By 4.8 and 4.13, 4.17 applies to $W_1',\ldots,W_n'$. Thus there is an x in R with $W_1'(x) > e$ and $W_i'(x) < e$, $i = 2,\ldots,n$.

By 4.13, there is an integer $r > 0$ and a b in K with $W_i'(x^r) < W_i'(b) = V'(b) < e$ for $i=2,3,\ldots,n$. Let $y = x^r$, then $W_i(y)H < V(b)H < H$; so $W_i(y) < V(b) < e < V(b)^{-1}$; so $W_i(y) < \min[V(b),V(b)^{-1}]$ $i = 2,3,\ldots,n$. But $W_1'(y) \geq e$ gives $W_1(y)H \geq H$; so $W_1(y) \geq e$. This is a contradiction since then $V(b)$ is in H so that $V'(b) = e$. Thus, $V'(K) = [e, \bar{0}]$.

Proposition 4.19. (Approximation Theorem) Suppose P_V is not an ideal of K and $W_1, \ldots, W_n$ are in L and are pairwise independent. Then if a_i is in $G_{W_i} \setminus [0]$, for $i=1,\ldots,n$, there is an x in R with $W_i(x) = a_i$ for $i=1,\ldots,n$.

Proof: For each i, choose z_i in R with $W_i(z_i) = a_i$. Choose x_i in R with $W_i(x_i) > e$; and for $j \neq i$, $W_j(x_i) < \min[a_j W_j(z_i'), e]$ if $W_j(z_i) \neq 0$ and with $W_j(x_i) < e$ if $W_j(z_i) = 0$. (This can be done by 4.18.) Let $t_i = x_i(1+x_i)'$. Then $W_i(t_i) = e$, and $W_j(t_i) = W_j(x_i)$ if $i \neq j$.

Now $W_i(t_i z_i) = W_i(z_i) = a_i$, and if $i \neq j$, $W_j(t_i z_i) =$

$$W_j(t_i)W_j(z_i) = \begin{cases} 0 & \text{if } W_j(z_i) = 1 \\ W_j(x_i)W_j(z_i) < a_j & \text{if } W_j(z_i) \neq 0. \end{cases}$$

Thus, $W_j(t_i z_i) = \max_k W_j(t_k z_k)$ only if $i=j$, so by 1.8, $W_j(\sum_{i=1}^{n} t_i z_i) = W_j(t_j z_j) = a_j$ for $j = 1,2,\ldots,n$.

Proposition 4.20. (Strong Approximation Theorem) Suppose L has the strong inverse property and $W_1, \ldots, W_n$ in L are pairwise independent. If a_i in R have $W_i(a_i) \neq 0$ $i = 1,2,\ldots,n$, then there is an x in R with $W_i(x) = W_i(a_i) > W_i(x-a_i)$ $i = 1,2,\ldots,n$.

Proof: Case 1) P_V an ideal of K. Then the P_{W_i} are maximal ideals of R so $P_{w_i} \not\subset P_{w_j}$ if $i \neq j$ and 4.17 applies. For each i, choose x_i in R with $W_i(x_i) = e$ and $W_j(x_i) = 0$ if $i \neq j$. Choose x_i' in $A_{w_i} \setminus P_{w_i}$ with $x_i x_i' = 1+t_i$ for some t_i in P_{w_i}. Then $W_j(x_i x_i' a_i) = 0$ if $i \neq j$ while $W_i(x_i x_i' a_i - a_i) = W_i(a_i t_i) = 0 < W_i(a_i) = W_i(x_i x_i' a_i) = e$.

Let $x = \sum_{i=1}^{n} x_i x_i' a_i$, then $W_i(x-a_i) = W_i(x_i x_i' a_i - a_i + \sum_{j \neq i} x_j x_j' a_j) = 0$.

Case 2) P_v not an ideal of K. Choose a_i' so that $W_j(a_i a_i') = e$ whenever $W_j(a_i) \neq 0$. For each i, choose x_i in R with $W_i(x_i) > e$; $W_j(x_i) < \min[W_j(a_j) W_j(a_i'), e]$ if $W_j(a_i) \neq 0$, and $W_j(x_i) < e$ if $W_j(a_i) = 0$. Choose y_i in R with $W_j(y_i) = W_j(1+x_i)^{-1}$ if $W_j(1+x_i) \neq 0$ and so that $W_i(y_i(1+x_i)-1) < e$.

Then $y_i(1+x_i) = 1+t_i$ where $W_i(t_i) < e$; $(x_i y_i - 1)(1+x_i) = x_i y_i (1+x_i) - 1 - x_i = x_i t_i - 1$; so $W_i(x_i y_i - 1) W_i(1+x_i) \leq \max[W_i(x_i t_i), W_i(1)] < W_i(x_i) = W_i(1+x_i)$; so $W_i(x_i y_i - 1) < e$ and $W_i(x_i y_i a_i - a_i) < W_i(a_i)$.

Also if $i \neq j$, $W_j(y_i) = W_j(1+x_i)^{-1} = W_j(1)^{-1} = e$ so $W_j(x_i y_i a_i) = W_j(x_i) W_j(a_i) < W_j(a_j)$.

Now if $x = \sum_{j=1}^{n} x_j y_j a_j$, we have $W_i(x-a_i) = W_i((x_i y_i a_i - a_i) + \sum_{j \neq i} x_j y_j a_j) \leq \max[W_i(x_i y_i a_i - a_i), W_i(x_j y_j a_j)\ i \neq j] < W_i(a_i)$.

<u>Proposition 4.21</u>. Let R be an integral extension of K, B a set of pairwise-independent valuations on K, and E a set of valuations on R with W in E $\Rightarrow W|_K$ is in B. For each V in B, suppose that P_V is not an ideal of K. If finite subsets of B satisfy the inverse property and the property of Proposition 4.18 (and hence 4.19) and if $W_1, \ldots, W_m$ are distinct, pairwise-independent elements of E, then $[W_1, \ldots, W_m]$ satisfies the inverse property and 4.18 (and hence 4.19).

<u>Proof</u>: Separate the W_i's into classes $W_{11}, W_{21}, \ldots, W_{n_1 1}; W_{12}, W_{22}, \ldots, W_{n_2 2}; \ldots ; W_{1r}, \ldots, W_{n_r r}$ such that $W_{ij}|_K = W_{ks}|_K$ iff $j = s$. For $j=1,\ldots,r$, let $W_{ij}|_K = V_j$. Note that $G_{W_{ij}}/G_{V_j}$ is torsion for all i and j by 4.12. For each j we have $[W_{1j}, \ldots, W_{n_j j}]$ satisfies the inverse property, 4.17, 4.18, and 4.19; so if $r=1$, we are done.

Assume $r>1$. If x is in R, then by 4.11 there is a y_1 in R with $W_{i1}(xy_1) = e$ for $i=1,\ldots,n_1$. Let $t = xy_1$. By 4.12 there is an $n>0$ with $W_{ij}(t^n)$ in G_{V_j} (let $n = \prod n_{ij}$ where n_{ij} works for ij). By 4.19 there is a z in K with $V_1(z) = e$ and for $j \neq 1$, $V_j(z) < \min_{i=1}^{n_j} [W_{ij}(t^n)^{-1}]$

$W_{ij}(t^n) \neq 0\big]$. Thus, $W_{i1}(zt^n) = e$; and for $j \neq 1$, $W_{ij}(zt^n)$ $= V_j(z)W_{ij}(t^n) < W_{ij}(t^n)^{-1}W_{ij}(t^n) = e$ when $W_{ij}(t^n) \neq 0$, and $W_{ij}(zt^n) = 0 < e$ when $W_{ij}(t^n) = 0$. Therefore, letting $t_1 = y_1{}^n x^{n-1} z$, we have $W_{ij}(xt_1) = e$ if $j=1$ and $W_{ij}(xt_1) < e$ if $j \neq 1$. Thus, for each $k=1,\ldots,r$, there is a t_k in R with $W_{ij}(xt_k) = e$ if $k=j$ and $W_{ij}(xt_k) < e$ if $k \neq j$, so by 1.8 $W_{ij}(x(\sum_{k=1}^{r} t_k)) = e$ for all ij; i.e., the inverse property is satisfied.

Let a_{ij} be in $G_{W_{ij}} \setminus [0]$. By 4.18 there is an x in R with $W_{11}(x) \geq e$ and $W_{i1}(x) < \min[e, a_{i1}]$ for $i=2,3,\ldots,n_1$. By the torsion property, there is an $n > 0$ with $W_{ij}(x^n)$ in G_{V_j} and $(a_{ij})^n$ in G_{V_j} $[0]$ for all i and j. By 4.19 there is a y in K with $V_1(y) = e$ and for $j \neq 1$, $V_j(y) < \min_{i=1}^{n_j}\left[W_{ij}(x^n)^{-1}(a_{ij})^n,\ W_{ij}(x^n)^{-1} \,\middle|\, W_{ij}(x) \neq 0\right]$. Thus if $j \neq 1$, $W_{ij}(yx^n) = 0 < a_{ij}$ when $W_{ij}(x) = 0$; and when $W_{ij}(x) \neq 0$, $W_{ij}(yx^n) = V_j(y)W_{ij}(x^n) < \min_{k=1}^{n_j}\left[(a_{kj})^n,\ W_{kj}(x^n)^{-1}W_{ij}(x^n)\right] \leq \min\left[(a_{ij})^n, e\right]$ so that 1) if $a_{ij} < e$, then $(a_{ij})^n < a_{ij} < e$ and $W_{ij}(yx^n) < \min\left[(a_{ij})^n, e\right] = (a_{ij})^n < a_{ij}$; or 2) if $a_{ij} \geq e$, then $W_{ij}(yx^n) < \min\left[(a_{ij})^n, e\right] = e \leq a_{ij}$. Hence if $j \neq 1$, $W_{ij}(yx^n) < a_{ij}$.

Now we have:

$$W_{11}(yx^n) = V_1(y)W_{11}(x)^n = W_{11}(x)^n \geq e;$$

for $i=2,3,\ldots,n_1$,

$$W_{i1}(yx^n) = V_1(y)W_{i1}(x)^n = W_{i1}(x)^n < W_{i1}(x) < \min\left[e, a_{i1}\right] \leq a_{i1};$$

and for $j \neq 1$,

$W_{ij}(yx^n) < a_{ij}$. That is, 4.18 is satisfied.

Thus, $\left[W_1,\ldots,W_m\right]$ satisfies the inverse property and 4.18 (and hence 4.19).

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