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Classical quadrivium and Kepler's Harmonice mundi

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THE CLASSICAL QUADRIVIUM
AND
KEPLER'S <u>HARMONICE MUNDI</u>

by

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The four subjects of the medieval Quadrivium (arithmetic, geometry, music and astronomy) are traced from their classical origins in Pythagorean and Platonic traditions through detailed development and extensions by Kepler in the Renaissance.

Their traditional order of study is then found to be reversed in the historical order of their resolution in terms of harmonic analysis.

Particular attention is paid to the role of logarithmic conversion of musical ratios into differences (intervals) by the inner ear.

ACKNOWLEDGMENTS

I would like to thank Ernst Bindel, whose beautifully written books on music and mathematics helped turn a very important corner in my life, as well as Eugen and Hedwig Kuch, all of Stuttgart, for kindnesses too numerous to mention! In Missoula, I thank the members of my committee — John Ellis, William Manning, Joseph Mussulman, Howard Reinhardt, and Debra Shorrock — for their outstanding examples as teachers and/or performers, and for their great good will in permitting me to attempt such an interdisciplinary thesis. Finally, I thank Bryan Spellman for humoring me through the writing of it.

S. E.

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H.S. "Suppose you had pursued the profession of a painter. Do you feel that your career as a painter might have paralled that as a composer?"

A.S. "Yes, I'm sure it would have. So, I must say, technically I possessed some ability, at this time at least, and I'm afraid that I have partly lost it. For instance I had a good sense of space relations, of measurements. I was able to divide the line rather correctly in 3, 4, 5, 6, 7, even 11 parts, and they were quite near the real division. And I had also a good sense of other such measurements. At this time I was able to draw a circle which deviated very little from the objective [one] with a compass. I could draw really very well, but I think I lost this capacity. But I had the idea that this sense of measurement is one of the capacities of a composer, of an artist. It is probably the basis of correct balance and logic within, if you have a strict feeling of the sizes and their virtual relationships."

Arnold Schoenberg, in an interview with Halsey Stevens, recorded on Columbia M2S 709, The Music of Arnold Schoenberg, Volume III.

I. INTRODUCTION – THREE RIDDLES

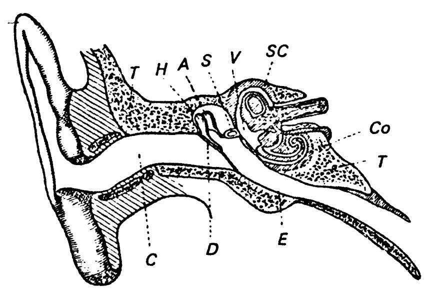
When asked his view, why the flute should be again enjoying such popularity after over a century of neglect, the eminent French flutist Jean Pierre Rampal (n.d., p. 2) replied

> "I think this is due to the need of the people after the last world war. They had a need for something well-balanced. Baroque music is ideal for this balance needed after a period which was so full of terrible things."

Harmonia was, after all, considered by the Greeks to be a daughter of Aphrodite and Ares (beauty and war). In Plato's dialogue The Republic, written in the aftermath of the terrible Peloponnesian War between Athens and Sparta, Socrates asks Glaucon (in Book III, St. II, pp. 410-411)

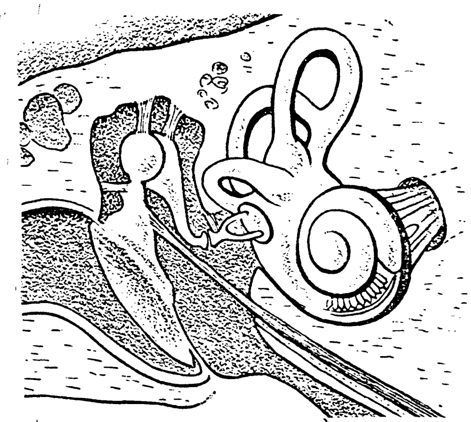
> ..."May we say that the purpose of those who established a joint education in music and gymnastic was not, as some people think, that they might tend the body with one and the soul with the other?"
> "What was it then?"
> "It is more likely ... that both music and gymnastic are meant especially for the soul."
> "How?"
> "Have you never noticed ... how a lifelong training in gymnastic without music affects the character, or what is the effect of the opposite training? ... I know ... exclusive devotion to gymnastic turns men out fiercer than need be, while the same devotion to music makes them softer than is good for them. ... It is the spirited element in their nature that produces the fierceness, and naturally enough. ... Then is not gentleness involved in the philosophic nature; but if it relaxes too much into gentleness, the temperament will be made too soft, while the right training will make it both gentle and orderly, will it not? ... Then seemingly for those two elements of the soul, the spirited and the philosophic, God, I should say, has given men the two arts, music and gymnastic. Only incidentally do they serve soul and body. Their purpose is to tune these two elements into harmony with one another by slackening or tightening, till the proper pitch be reached."

Anatomically, the organ of balance on which the body relies in gymnastic exercise is the set of semicircular canals in the inner ear – three curved pathways, lying in approximately mutually orthogonal planes, in which tiny "earstones" (otoliths) tumble against sensory hairs to make possible the analysis of spatial attitudes as well as momentary changes in spatial motions – accelerations and decelerations, due to muscular effort or the force of gravity. It is a riddle of nature that this organ is placed in immediate juxtaposition to the cochlea with its spiral arrangement of sensory hairs whereby we analyze the sounds of speech and music.



C = ear canal; D = ear drum; E = Eustachian tube; T = temporal bone; H = hammer; A = anvil; S = stirrup; V = vestibule; SC = semicircular canals; Co = cochlea.

Cross-section view of outer and inner ear from Winckel (1967).



Round view of hammer, anvil, stirrup, semicircular canals and cochlea, from Palmer and LaRusso (1965).

We are thus presented with three riddles: one spiritual (what is the nature of harmony as issue of beauty and war), one psychological (how does training in music and gymnastic fine-tune the soul), and one physical (what is the significance of the fusion of cochlea and semicircular canals).

Two routes seem open, by means of which to approach these inter-related riddles. One is m u s i c as it was understood by the Pythagoreans of pre-classical Greece, for their term μουσική was more comprehensive than that in Socratic times, embracing the λόγος, or meaningful word (whether spoken or thought), μήλος - melody of what we have come to perceive as music proper (vocal or instrumental), and χόρος of dancers in a rhythmically moving circle. Each aspect partook of the other two: ancient Greek speech was tonal (what is preserved today as accents being of the same nature as neumes — rudimentary indications of the rise-and-fall of speech melody) as well as metered (with vowels of long and short duration); the dancers at a comedy or tragedy also commented on the meaning of the action, and did so in song (the revival of which in the Italian Renaissance led to the birth of modern opera performance). Poetry, song, and dance were all subsumed under "music" as art of the Muses. (Cf. de Santillana [1961], Ch. 5.)

The other route is m a t h e m a t i c s , whose Greek name μάθεσις meant simply "learning." To the extent that any subject is universal today, in an age of super-specialization, it is mathematics. (Cf. Leonardo Da Vinci: "There is no certainty in science where one of the mathematical sciences cannot be applied.")

One possible answer to the riddle of fused cochlea and semicircular canals may be the inter-relatedness of song and dance in ancient music, another the inter-relatedness of analysis and geometry in modern mathematics.

In what follows, we shall attempt a middle road between the two, concentrating on the work of a scholar who stood at the threshhold between ancient and modern mentality, Johannes Kepler, whose life-work, the Harmonice Mundi, is as rooted in the Pythagorean tradition of music as it is anticipatory of the development of much later mathematics.

II. WORD ORIGINS — HARMONY AS RATIONAL ART

Liddell and Scott's *Greek-English Lexicon* (1879), p. 211, defines ἁρμονία as "a fitting, joining together, joint, cramp, like ἁρμός," which is defined as "a fitting or joining," whether belonging to "a limb, esp. the shoulder" or "in the fastening of a door;" hence it is cognate to English "arm" (via Old English "earm") as well as to "arms," "armature," "armillary," etc. (from Latin "arma") — cf. the etymological article on *ar-* in the appendix to the *American Heritage Dictionary* (1969), p. 1506. The sense of the Greek word seems to refer to the way in which parts fit together to form a whole, for we find in Liddell and Scott the further entries: ἁρμόζω "[I] fit together, join, ..., fit on clothes," whence in particular "armor" refers to the close-fittingness of a garment; ἁρμόδιος, as an adjective, meant "well-fitting, accordant, agreeable;" an ἁρμοστής was one who "joins, arranges, governs" as the "harmost" or governor of one of the islands; ἅρμοσις meant "a joining together, fitting, adapting," while most pertinent for our purposes ἁρμοσία meant "arrangement: tuning of an instrument." In Latin (cf. the article on *ar-* again) we find "artus" meaning "a joint" as a noun, "tight" as an adjective; we also find "ars" meaning "art" and "iners" meaning "unskilled, without art."

More problematically, the *American Heritage Dictionary* (loc. cit.) attempts to follow back the Indo-Germanic roots of the Greek ἄρθρον (meaning "a joint, esp. the socket of a joint," according to Liddell and Scott) to relate it to *ord-* words in Latin, such as "ordo" (from a conjectured Ind.-Germ. "*ōrdh-*") meaning "a row of threads in a loom," and all their

English derivates such as "ordain," "order," "ordinal," etc. On the other hand, it postulates a further relationship to Latin re- words, leading to English "arraign," "rate," "ratio" and "reason," as well as a relationship to Old Norse and Germanic "radh" and "rat," meaning "counsel," but most interestingly to Latin "rītus" and Greek (α)ριθμο- words from which we get "arithmetic" and "logarithm" (from a conjectured Ind.-Germ. rath) referring to the science of "number." Perhaps the most important of all these related word-meanings is "ratio," for it reveals most clearly a sense which is common to both families of words, (h)ar(m)- and (a)rat(h)-/(a)rit(h)-, the sense of proportion, or harmonious arrangement. Moreover, to the extent that harmony is a rational art, its principles may discovered by reason.

III. THE QUADRIVIUM IN PLATO'S REPUBLIC

Plato's Republic could be said (perhaps redundantly, in light of the foregoing) to be about the art of well-ordering. In Book V, philosophers and non-philosophers are contrasted as "lovers of reality" (truth, wisdom) as opposed to "lovers of belief" (St. II 480). In Book VI (St. II 484), he lets Socrates ask Glaucon rather archly:

> "As those are philosophers who are able to grasp that which is always invariable and unchanging, while those are not who cannot do this but are all abroad among all sorts of aspects of many objects, which of these ought to be leaders of the city?"
>
> ... Clearly the philosophers!

The question then arises how such philosophical leaders, or guardians, should most fittingly be trained. This is answered in Book VII. Four subjects are recommended by Socrates as training the intelligence in the pursuit of truth; these are not quite the four, as Socrates numbers them, which we normally think of as comprising the classical quadrivium, but – with a slight shift of emphasis – they are recognizably similar.

The first (St. II 525) is arithmetic, which Socrates shows "to lead towards truth ... in a pre-eminent degree." The second (St. II 527) is geometry, as the study of "knowledge ... of that which always is." (Interestingly, this comes closest, on the one hand, to agreeing with the definition of what a philosopher is primarily concerned with, according to Plato, hence the famous inscription over his academy door to let no one enter who was ignorant of geometry, while on the other hand it anticipates Felix Klein's famed Erlangen program [1872] to study geometry as the science of that which is invariant under transformations.) The geometry intended here is plane geometry, for when Glaucon attempts to suggest astronomy as third subject (St. II 527) Socrates rebukes him for omitting solid or spatial geometry (St. II 528), from which the study of motions in space, as astronomy, follows naturally (St. II 529). Finally we arrive (St. II 530) at "harmonics," or what we might call music theory. This Plato conceives as a further part of the study of motion, complementing astronomy, for he has Socrates say:

"Motion presents ... not one, but several forms, I imagine. The wise will perhaps be able to name the full list. But even I can distinguish two. ... One we have had, ... the other is its counterpart. ... Apparently, as the eyes are fixed on astronomy, so are the ears on harmonics, and these are sister sciences as the Pythagoreans say, and we, Glaucon, agree with them. Do we not?"

Glaucon, of course, agrees. Plato lets Socrates go on to say:

"Then ... since the subject is complicated, let us inquire of them what they say on these matters, and whether they have any other information to give us. And throughout we shall look after our special interests."

"What is that?"

"That those, whom we are to bring up, shall not attempt
to study anything in those sciences which is imperfect and
which does not always reach to that point at which all things
531 ought to arrive, as we have just been saying about astronomy.
Do you not know that the same sort of thing happens in harmonics? Men expend fruitless labour, just as they do in astronomy, in measuring audible tones and chords. ... But I shall not weary you with my simile by telling you of the blows they inflict with the plectrum and the accusations they bring, and of the strings' denials and blusterings. I leave that, and declare that I don't mean these people, but the Pythagoreans, whom we have just said we should question about harmonics. For they behave like astronomers; they try to find the numbers in audible consonances, and do not rise to problems, to examining what numbers are and what numbers are not consonant, and for what reasons."

"That would be a more than human inquiry." ...

"In any case, ... it is useful in the search for the beautiful and the good, but if it is pursued in any other way, it is useless."

Depending on one's inclination, one might count the above list item by item as containing five subjects to be studied; or one might lump the plane and the solid together as two aspects of geometry, and combine astronomy with harmonics as two forms of motion study, according to Plato, coming up with just three subjects. Perhaps it was the numinous quality of the number seven which caused the medieval scholastics to contrast the three subjects (grammar, logic, and rhetoric) of the Trivium with the four (arithmetic, geometry, astronomy and music) of the Quadrivium, in listing the seven liberal arts.

It should, perhaps, be remarked in passing that the Trivium subjects were in no wise considered "trivial." Both terms mean a "three(-fold) way," but the Trivium should be conceived as forming a perfect (equilateral) triangle of subjects ($\triangle$ or 人) while the latter refers to something like a side-street ($\dashv$), hence of side-interest. In terms of modern mathematics, the subjects of the old Trivium could be seen as ancestors of foundation questions, as presently studied in category theory (the grammar of functions and functors, etc.), topos and set theory (the basis of logic), and (something academically neglected) the art of presenting arguments most effectively (rhetoric as understood and defended by Pirsig [1974]).

IV. KEPLER'S <u>HARMONICE MUNDI</u>

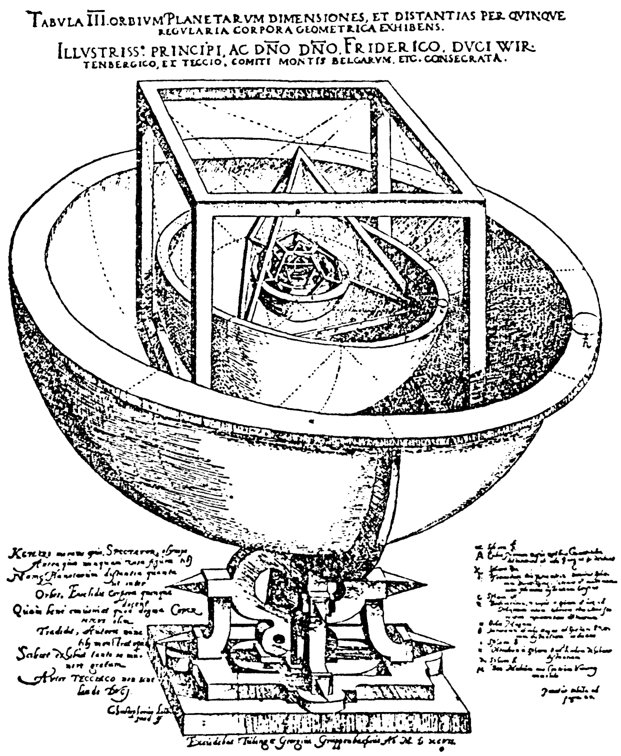
A. Textual Background

Plato's comments, through Socrates, on the "right" sort of study of harmonics which would lead to knowledge of truth, as distinct from beauty or goodness, were quoted at length because Johannes Kepler — whose poor eyesight made him listener not looker — set out some 2000 years after the time of Plato to pursue just that which Plato had advised against: a science of consonance, consonance that can be "known," not merely appreciated as beautiful or good! As a scientist, he makes the idea of "know-" or intelligibility central to all the arguments in the main work which he kept polishing all his life, the <u>Harmonice Mundi</u>, <u>Welt-Harmonik</u>, or <u>Harmony of the World</u>.

In the textual history supplied by Max Caspar, the first translator of the entire book into a modern language (1939), p. 13, we read how the world first learned of Copernicus' new Sun-centered theory in a <u>Narratio prima</u> by Georg Joachim Rheticus in 1540 (Copernicus' own book <u>De revolutionibus</u>' publication being delayed until 1543, the same year as Vesalius' equally revolutionary anatomy). In this "first narration," Rheticus takes up the theme of Pythagoreanism and the special significance of the number 6 (whose equality to 1+2+3, its divisor-sum, makes it a so-called "perfect" number), claiming that God had so arranged the world most perfectly

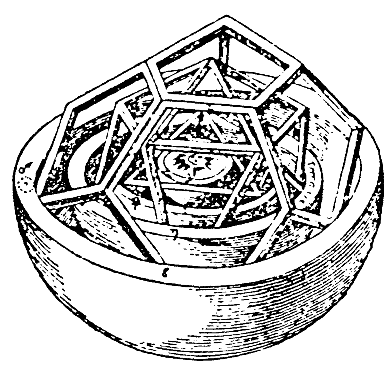
"... that a heavenly harmony is achieved by the six movable spheres, in that all these spheres follow one another in such a manner that no immeasurability arises in the distances from one planet to the next, but rather each one, geometrically enclosed, receives its place in such fashion that, if one wished to remove it from its place, the entire system would at once collapse." [My translation — S.E.]

This idea of an arrangement of the six known planets by means of a system of geometrically nested spheres enclosing their respective orbits — a system that would "explain" why there should be exactly six such planets — was taken up by the then 25-year-old Kepler in 1596 in a youthful work entitled _Prodromus dissertationum cosmographicarum, continens mysterium cosmographicum_. The "cosmographic mystery" at the heart of the work was his system of nested spheres, a fore-runner of later armillary spheres. A static concept, it was designed solely to explain why there should be six planets, and how they are spaced (as nearly as this was known). That there should be six spheres needed, and how they should be spaced — both of these things at once — he explained by appealing to the five regular or "Platonic" solids, knowledge of which was attributed to the Pythagoreans (6th to 5th cent. B.C.), but first published as the culmination of Euclid's _Elements_ (ca. 330-320 B.C.): there are six spheres because there are five such solids to separate them, and the distances between them depend on the order of the solids (see illustrations next page). Outermost, we find the sphere of Saturn, in which is inscribed a regular cube, touching its corners; within that, touching its face-middles, we find the sphere of Jupiter; this is separated from the sphere of Mars by a similarly in- and circumscribed regular tetrahedron; a similarly in- and circumscribed dodecahedron separates the Mars from the Earth sphere; and a similarly in- and circumscribed icosahedron separates the Earth from the Venus sphere; only in the case of Mercury did Kepler have to make an exception to his procedural



Model of the Solar System
from Kepler's Youth - Work
"Mysterium Cosmographicum"
(showing spheres of Saturn,
Jupiter, Mars and Earth)

[Illustrations reproduced



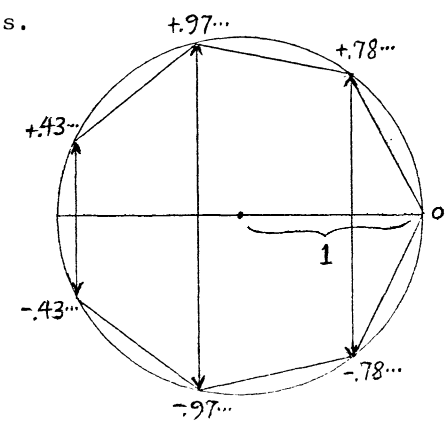
from Bindel (1971) pp. 32 and 34]

Enlargement of Central Portion
(showing spheres of Mars, Earth,
Venus and Mercury, with central Sun)

rule: a regular octahedron is inscribed in the Venus sphere, but the Mercury sphere passes not tangent to the octahedron's face-middles, circumscribed by it, but rather through the octahedron's edge-middles, partially penetrating it. This, of course, was fudging, and Kepler knew it. He remained proud of this youth-work, and convinced of its basic true intent, circulating as many copies as he could afford to print among the nobility of Europe, much as Galileo circulated copies of his telescope. The commoner Kepler never received a telescope from Galileo — couldn't even get Galileo to send him clear reports of what he'd seen (moons of Jupiter, phases of Venus) for Galileo would only send him messages cryptically encoded in Latin anagrams! But near-sighted as he was, he would have made a poor observer in any case. Instead, destiny called Kepler to the court of Emperor Rudolph in Prague, together with the very skilled Danish observer Tycho Brahe, and it was Kepler's understanding of Brahe's observational data that first revealed the true distance relationships within the solar system. At age 50, Kepler redid his youth-work on a much more ambitious scheme, following the order of subjects of Plato's Republic, but pushing the investigation further at each stage, contrary to Plato's advice, to ask which things "are ... and are not consonant, and for what reasons."

B. Knowledge of Individual Figures

The first book of the Harmonice Mundi treats plane geometry, but its interest is exclusively in regular polygons, divisions of the circle into so-and-so-many equal parts. For those figures which were known to be constructible with straightedge and compass since classical times, Kepler merely reviews what is already known: the triangle, square, pentagon, hexagon, octagon, decagon, 12-, 15-, and 20-gons. (He does not at first make clear how this series is to be continued, but implicitly it is 3, 4, 5, or 15 times 2^n for any $n = 0, 1, 2, 3, \cdots$.) For those figures not known to be so constructible he attempts to investigate in one case, the regular heptagon, why their construction should be elusive, coming up with various equations for the side of such a polygon inscribed in a unit circle, e.g. "7j - 14iij + 7v - 1vij" on p. 49 of (1939), corresponding to the modern equation $7x - 14x^3 + 7x^5 - x^7 = 0$, whose roots are $x = \pm 2\sin(\frac{n}{7}\cdot 180°)$. $2\sin(\frac{180}{7}) \doteq 0.867767478$ is the side of the regular heptagon inscribed in a circle of unit radius, and the seven distinct values $\{0, \pm 0.433883739, \pm 0.781831483, \pm 0.974927912\}$ assumed by $\pm\sin(\frac{n}{7}\cdot 180°)$ give the vertical distances to its seven corners as directed lengths of half-sides or half-chords.



Such an equation has come to be known as "cyclotomic," for it is "circle-splitting" in the evident sense. Kepler would have been able to find approximate numerical values for these roots, but he is geometer enough to know that that is of no use in determining the construction, say, of the ratio of the heptagon side to the radius of its circumscribed circle.

> "No, since this proportion is not given to me by any geometrical construction, I shall wait, for the time being, until someone comes and shows how I can produce it." (loc. cit.)

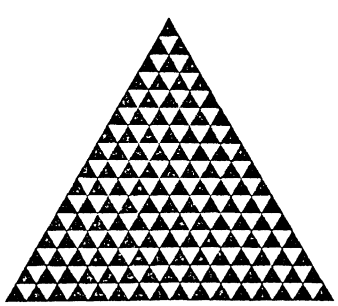
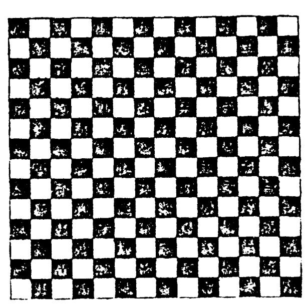
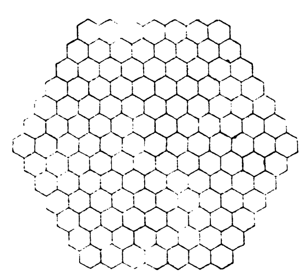
In Theorem XLVII, p. 58, he says finally that the situation is the same (undecided – probably unconstructible, but we must wait until someone comes to show us) for all figures with an odd number of sides greater than 5, with sole known exception of the regular 15-gon, which is readily constructible. He is aware of close, but false, approximate constructions, e.g. that on p. 52 which approximates a heptagon side as half the side of an inscribed equilateral triangle (which would give a side length value of $\frac{1}{2}\sqrt{3} \doteq 0.866025404$), and rejects them. More interestingly, he is aware of angle trisection procedures by means of transcendental curves such as the conchoid of Nichomedes, p. 56, which would permit the exact construction of a heptagon side – cf. Morley and Morley (1954), §90; these, too, he rejects, since the conchoid e.g. would require placement of a mark on one's straightedge, thus for Kepler "begging the question" of knowing how to construct it, p. 50.

The key to his thought throughout the work is contained early on in Definitions VII - IX, p. 20: A figure (regular circle division) is said to be "know-" or "intelligible" if and only if it can be "demonstrated" (produced) by a (finite) chain, however long, of construction steps using only an unmarked straightedge and compass. Only such figures, he felt, have ratios fully accessible to the human ratio, hence the significance of their constructibility for Kepler's thought. But why should the figures be regular divisions of a circle? Why the emphasis on cyclotomy?

C. Formation of Space-Filling Harmonies

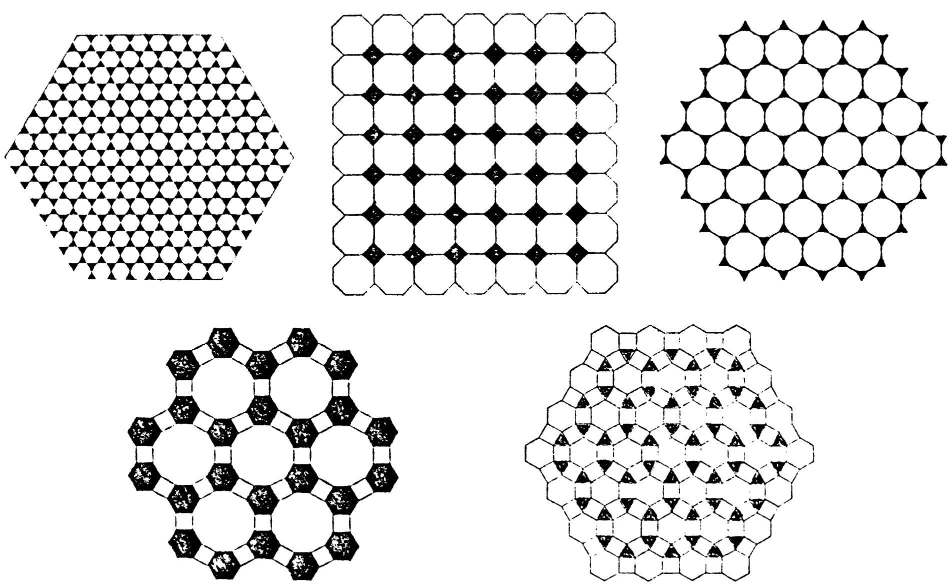
"Why divisions?" is easily answered — Kepler is consistently pursuing what is of ratio nature, representing each number n as ratio n : 1. "Why circles?" That we see next! The whole work is, after all, entitled the <u>Harmony of the World</u>, and Book II begins to make good the promise of this title. As the first book dealt with the demonstrability of such figures, so the second deals with what he calls their "congruences," how they fit together to fill out the plane as tilings or mosaics, appealing to the Greek word ʽαρμόττειν (Attic variant of ʽαρμόζειν), p. 63, "to fit together" as being like the Latin "congruere," so that a fitting-together of such regular polygons into space-filling tesselations constitutes quite literally a kind of harmony, Latin "congruentia" corresponding to Greek ʽαρμονία.

While a single circle can be divided regularly into any number of parts, the whole plane or surface of a sphere cannot. The plane can be covered in most regular fashion (all pieces, tiles, or faces alike and all their meetings at corners alike) in only three ways: by equilateral triangles (six per corner), regular squares (four per corner), or regular hexagons (three per corner).

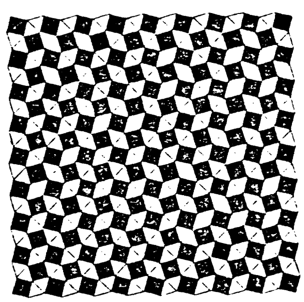




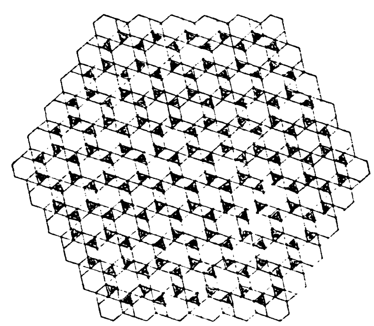
In 3-dimensional space, the curved surface of a sphere can be covered in this most regular fashion in five different ways, corresponding to the so-called "Platonic solids" (after Plato's treatment of them in the Timaeus), the culminating figures of the 13th book of Euclid which Kepler used in his youth-work — triangles: three (tetrahedron), four (octahedron), or five (i-cosahedron) per corner; squares: three per corner (cube); or pentagons: three per corner (dodecahedron).

If one relaxes the requirement that all faces or tiles be alike, but still insists that same numbers of same things come together at every corner in the same manner (and that the plane or sphere not have any one distinguished direction or axis), then seven more so-called semi-regular tesselations of the plane become possible, five of which may be thought of as derived from the three regular ones by progressive "truncation" of corners, creating 6-, 8-, and 12-sided figures out of the former 3-, 4-, and 6-sided ones.

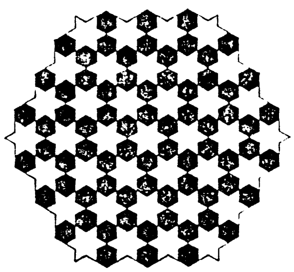


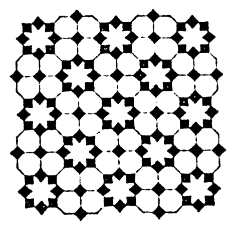
The other two mix triangles and squares, or triangles and hexagons, five at each corner — an odd number, entailing some sort of imbalance at the corners, which is off-set or brought into larger balance again in the mosaic as a whole by other means. In the case of the one with triangles and squares, there is an evident alternation between two kinds of handednesses: half of the squares are tilted slightly clockwise, and the other half slight counterclockwise, each of one kind surrounded by four of the other kind. In the case of the one with triangles and hexagons, the entire mosaic is either oriented in a clockwise sense (as here), or is its mirror twin.

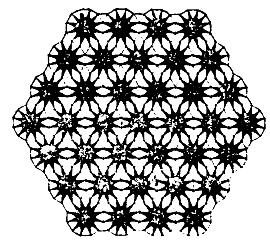


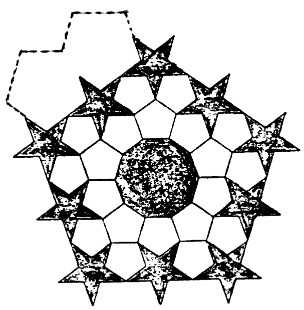


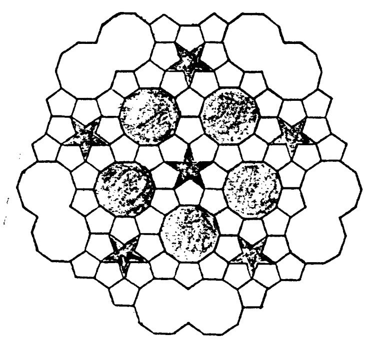
If one tries to fit pentagons together in the plane, then two of them leave a 144° gap, just right to be filled in by a decagon, while three of them leave a 36° gap, just right for a pentagram star corner. Kepler found all sorts of ways to fill the plane semi-regularly with 6-, 8-, or 12-pointed stars, but when he tried to imitate them with 5- or 10-pointed ones he found he kept being forced to merge some of them in pairs, or even multiple pairs as in the case of the 10-pointed decagram stars.

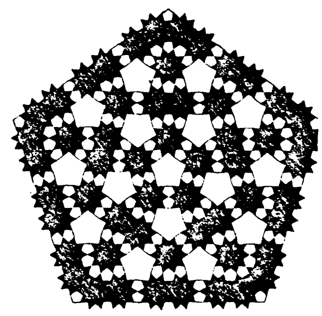




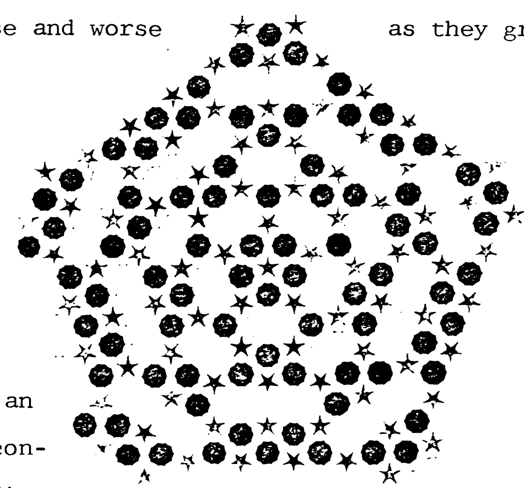








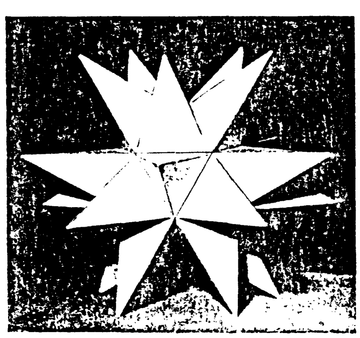
While that centered on a decagon leads to pairs of fused pentagons of double size, and that centered on a pentagon to multiple pairs of fused decagram stars, getting worse and worse as they grow outwards,

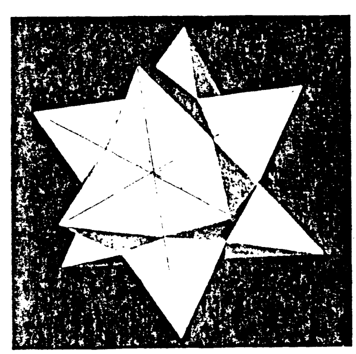


the one ↗ centered on a pentagram star admits an indefinite continuation, discovered by Wolfgang Dessecker (see Bindel [1964], pp. 24-25), if not known to Kepler, which never uses any figure more complicated than a single pair of (symmetrically) overlapping decagons; yet it does have to use these pairs, and it does have a distinguished central figure.

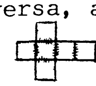
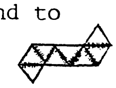
A pair of regular heptagons placed side to side leave open corners of $102\frac{5}{7}^\circ$ which cannot be filled in by any third regular figures, while three heptagons side to side at a corner overlap by $25\frac{5}{7}^\circ$ so, Kepler reasoned, there can be no way to cover either a flat plane or a spherically curved surface with them (surfaces of negative curvature, whereon a covering would be possible, being unknown at the time). Heptagons are not harmony-producing, and by similar arguments neither are 9-, 11-, or 13-gons. The 15-gon can start to produce a harmony in the plane by surrounding itself with a wreath of alternating triangles and pentagons, but 15 being an odd number the alternation cannot succeed. Like one of the foolish virgins, the 15-gon arrives too late after the doors to harmony have been shut by the 7- to 13-gons; cf. p. 27* of (1939).

Kepler concludes this book by enumerating the seventeen semiregular sphere-coverings (the so-called Archimedean solids), to which he adds two new ones using pentagram star faces, three or five to a corner — the stellated icosa- and dodecahedra.





If figures with five-fold symmetry were only limitedly successful at forming space-filling harmonies in the plane, on a spherical surface in solid space three kinds succeed: the regular pentagon itself (on the Platonic dodecahedron and four of the Archimedean solids, including the pattern now found on soccer balls), the decagon (on two other Archimedean solids), and the pentagram star (on Kepler's two stellated solids).

The full list of harmony-producing figures which Kepler provides, p. 82, includes seven regular polygons — the 3-, 4-, 5-, 6-, 8-, 10-, and 12-gons — and four stars — the $\frac{5}{2}$-, $\frac{8}{3}$-, $\frac{10}{3}$-, and $\frac{12}{5}$- gonal penta-, octa-, deca-, and dodecagrams. His case for including the $\frac{10}{3}$ star is weak, as the plane tiling it creates requires fusion of increasingly many pairs; there have been, however, further star-faced polyhedra discovered since Kepler's time which incorporate it satisfactorily (Wenninger [1971]) so his intuition on this has been borne out. More problematic is his inclusion of the regular 20-gon, on the grounds that it admits a complete wreath of alternating squares and pentagons. If this were grounds for admission into the select company of "harmonious figures," then he should have also included the 24-gon, which admits a wreath of alternating triangles and octagons, all of which are "demonstrable" and should have been welcome; but then he would also have had to have included the "non-demonstrable" hepta- and enneagons, for the former are capable of completing a wreath around a 42-gon, and the latter around an 18-gon, both in alternation with triangles (discovered by the present author some fifteen years ago, and apparently unmentioned elsewhere in the literature). If the list of harmony-producing figures is to be stretched to twelve members, then a worthier inclusion would be the double triangle or Star of David as $\frac{6}{2}$ - star (hexagram), occurring e.g. in the stellation of the octahedron as a pair of interpenetrating tetrahedra. Then the twelve would decompose as seven polygons plus five star polygrams, a decomposition elsewhere in geometry, e.g. in the way any 12-edged network to be assembled into a cube or octahedron will always require seven edges to be glued and five folded or vice versa, and to be met again in the realm of dodecaphonic music.  

If allowed this revision (about which more later) of the "harmonious twelve," we see that there is excellent agreement between the notions of those regular figures which are knowable (accessible to the human ratio) as individuals in the sense of Book I and those which go on to cooperate socially, as it were, forming space-filling harmonies in the sense of Book II. It is in the nature of the circle, as opposed to the plane and sphere, that while infinitely many regular divisions of the former are possible, even infinitely many of them knowable (all powers of 2), only finitely many of these can qualify to fill out regular divisions of the latter; the doors to spatial harmony are shut at the 5-gon, among the odd-sided ones, and at the 12-gon among the evens. This leaves the 15-gon "out in the cold," as we have seen. On this, Kepler concludes Book II, p. 84, by saying

"For as its demonstration is no proper one, but only accidental, so is its congruence not a complete one, but only one which makes a beginning and does not enclose the entire figure. This is to be considered below in the IIIrd Book with regard to the origin and application of the semitone."

D. Theory of Musical Proportions

This IIIrd Book is entitled "On the Origin of the Harmonious Proportions and the Nature and Differences of Musical Things." We are prepared to take the third step in our study of the quadrivium by the prefatory quote from Proclus' introduction to Euclid, p. 85 of (1939) to the effect that mathematics

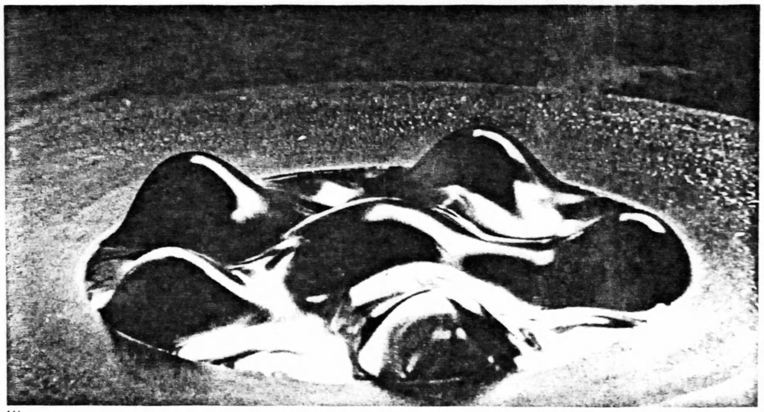
> "spreads before us the well-ordering of the virtues, doing so in one fashion in numbers, differently in figures, and differently again in musical harmonies."

But as we noted on p. 9 above, Kepler has a different intent than Plato and the Pythagorean tradition. After a brief excursion on the Pythagoreans' love of whole numbers, and their derivation of numinous properties of the sums 1+2+3 = 6 (perfect number) and 1+2+3+4 = 10 (tetraktys) to which we shall return later, Kepler rejects these considerations, pp. 92-94, as being too abstract and not taking the judgment of the ear into consideration, siding rather with Ptolemy, who did so, but preferring a middle ground. Whereas Ptolemy went so far as to admit the proportions 6:7 and 7:8 as harmonious (what Kepler describes as "Ut Ri Fa" instead of "Ut Re Mi Fa," i.e. G A♯/B♭⁻ C as opposed to G A B♮ C), Kepler finds this offensive to his ear and rejects it, too, but accepts Ptolemy's proportions of 8:9 and 9:10 as harmonious for passing tones, melodically, reserving the honor of full harmoniousness in standing chords, as intervals, to those numbers found to be knowable and congruence-producing in Books I and II. (Perhaps his notation of both the sums 1,2,3 etc. and proportions 6,7 etc. alike with commas, rather than + and : signs, kept him from noticing a more cogent reason to think the Pythagorean considerations inappropriate for the theory he is about to develop — they add and subtract to form partitions, where he multiplies and divides to form ratios.

The reason for the study of regular divisions of the circle in Book I is revealed clearly, at last, in the first chapter of Book III: We are to think of such circles as like monochord strings bent round, vibrating, and study the extents to which the parts are consonant or dissonant with the whole. How Kepler would have loved to have seen a Chladni plate in action — a circular plate of steel, supported in the center, dusted with a very light powder, bowed with a well-rosined violin bow at some point and touched with a finger tip at another: Where the bow strikes, there is vibration, hence a scattering of the powder; where the finger touches, there is stillness, hence a gathering of the powder. The angular distance between bow and finger, and the intensity of the bowing, determine a rhythmic activity throughout the plate, resulting in kaleidoscopic patterns of motion and rest, standing waves, made visible by the medium. Exquisite divisions of a circular drop of water, for example, can be achieved, vibrating in response to sound waves, as caught e.g. by the camera of Hans Jenny (1967, 1974) — see the photographs reproduced on the next page from (1974), p. 113, which stroboscopically "freeze" a tiny drop of water momentarily sculpted into plastic shapes with 5-, 7-, or (crossed!) 4-fold symmetry. It is something like this which Kepler is anticipating, some 350 yrs. before the physical and photographic means to demonstrate it. (Cf. Goethe's famous aphorism: "Architecture is frozen music.")

From Hans Jenny, <u>Cymatics</u> (1974), Fig. 144 - 146:

A drop of water, ca. 2 cm in diameter, is made to vibrate in response to sound. The stroboscopic camera catches one moment's action; a moment later the present high points will be low, the low ones high, producing intricate 10-, 14-, and 8-sided patterns, seen from overhead, but the side observer sees the lively up and down motions of 5- 7-, and 4-gons.



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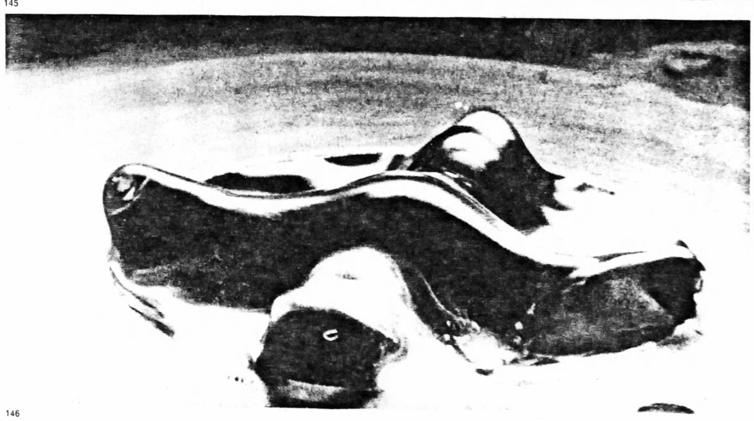

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Compare Jenny's language (1974), p. 100 ...

"The more one studies these things, the more one realizes that sound is the creative principle. It must be regarded as primordial. No single phenomenal category can be claimed as the aboriginal principle. We cannot say, in the beginning was number, or in the beginning was symmetry, etc. These are categorial properties which are implicit in what brings forth and what is brought forth. By using them in description we approach the heart of the matter. They are not themselves the creative power. This power is inherent in tone, in sound. Tone and sound are, so to speak, the entelechies which are active here."

... with Kepler's, p. 93:

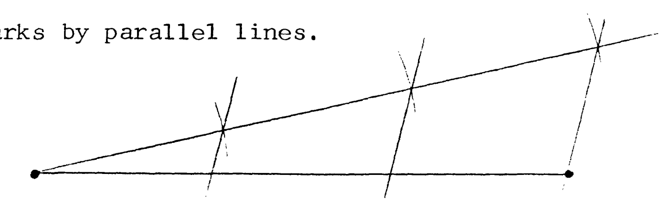
"Neither can it suffice the theoretician that the numbers 1, 2, 3 are the symbols of basic principles, of which all natural things consist. For an interval is not a natural thing, but a geometric one. If then these numbers did not count something which was closely related to intervals, a philosopher could lend no credence to this cause; he would have to be suspicious of it as cause."

Jenny rejects the "in the beginning was number" or "symmetry" approach to harmonical forms as merely descriptive, without the power to create. Similarly, we saw Kepler recognize the algebraic approach through cyclotomic polynomials as also merely descriptive, without the power to construct. But there are also common convictions to be found comparing Plato with Kepler. Just as Plato berates those who "expend fruitless labour ... in measuring audible tones and chords" (see p. 7 above), so Kepler points out, p. 93, that one can tune strings to any proportion, but that "as soulless things these offer no judgment, merely following without resistance the hand of the unskilled theoretician." Yet both seem to agree that one must start with observations before attempting to interpret them theoretically, whether in astronomy (the need for observation here was to be filled by Brahe, in Kepler's case) or in music. The principal difference between them, then, is in the direction the interpretation should take from there. Plato warns against becoming all too mired in the observations of the consonances (Kepler agrees), counselling the would-be harmonist to "try to find the numbers" therein (which Kepler does) but not to attempt to examine "what numbers are and what numbers are not consonant" in themselves. Kepler attempts nothing less than a qualitative grading of the consonance or musical harmonic properties of the numbers themselves, as realized in the form of a resounding circle, as notes on a vibrating monochord in the round!

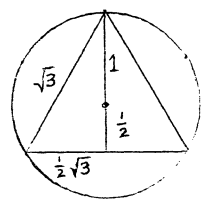
There are two important parts of, or aspects to, Kepler's difference with Plato and the Pythagorean tradition (with which he is otherwise in profoundest sympathy, to the point of wondering whether the soul of Pythagoras might not have migrated to himself — see Max Caspar's quotation of a Kepler letter to Herwart in the introductory pages 23*-24*). Firstly, he is in possession of an adequate theory of irrational numbers, permitting gradation of degrees of irrationality. Secondly, he is determined not merely to start with accurate observations, but to return to them again and again until they are satisfactorily accounted for (Max Caspar on this, p. 17*: "As much as his lively spirit was inclined to a priori speculations, even so was it clear and self-evident to him that the testable results of his deductions would have to be checked against positive facts").

Regarding degrees of irrationality: The school of Pythogoras (ca. 6th cent. B.C.) is credited with two major discoveries (whether, and to what extent, these are indebted to Babylonian precursors being left here moot) — the fact that the simplest musical intervals are based upon rational relationships of the lengths of the sounding bodies to one another, and hence reciprocally to their rates of vibration, and the fact that there exist geometric quantities (arising through the "Pythagorean theorem") which cannot be rationally expressed; the former are rational in that their proportions may be simplified to lowest terms as a fraction, or quotient, of two relatively prime whole numbers (numbers having no common factor larger than 1), while the latter are irrational in that the assumption that such a ratio or quotient of relatively prime integers exists leads to a contradiction (if $\sqrt{2}$, for example, is assumed to be = p/q for some pair of relatively prime whole numbers p and q, then it is easily shown that p and q must have a common factor of 2 after all, contrary to assumption).

So long as we think classically of our monochord laid out straight, as a stretched string, we are in the position of being able to divide an arbitrary line segment easily into any number of equal parts by straightedge and compass: Simply carry off that many parts on some other line crossing the first at one end, join the other ends, and transfer the compass marks by parallel lines.



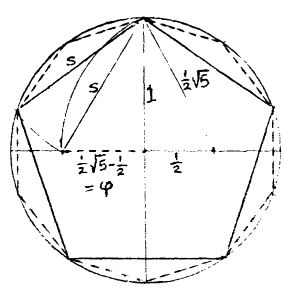
But as soon as we take Kepler's intuitive leap to bend the sounding medium around into a circle (how it is to be done, p. 96, he says "would lead too far here"!) the situation changes completely! Trisecting an arbitrary line-segment is no problem; trisecting an arbitrary angle, on the other hand, is a very big problem, not settled clearly until the time of Galois in the early 19th century (to which we shall return). An entire circle (angle of 360°) may be readily trisected to form an inscribed equilateral triangle, but the measurements of this triangle involve the irrational quantity $\sqrt{3}$.



When we "try to find the numbers in the audible consonances" of musical octave - fifths, Pythagorean-style, we come up with the rational relationship of 3:1; when we think of the same musical interval in Kepler's terms we find a figure with side length $\sqrt{3}$ inscribed in a unit circle. Rationality vs. irrationality, in the Pythagorean sense, is not at issue for Kepler — all of the regular polygons will involve irrationalities of varying degrees. The problem is to grade these irrationalities by a new kind of rationality-criterion: accessibility to the <u>human</u> ratio, as degrees of knowability, intelligibility. For, p. 94,

> "since it is a spiritual being which has so fashioned human souls that they take delight in such an inverval (in this lies the true definition of consonance and dissonance), so too must the differences of one interval from another and the causes why these intervals are harmonious be of spiritual and intelligible nature, i.e. this nature must consist in the fact that the determining parts of the consonant intervals are properly knowable, those of the dissonant ones improperly knowable or unknowable."

Kepler's idea, then, is that the figures he deems "knowable" are in some sense connatural with the human spirit, hence accessible to it, while those that are not do not produce that delight of recognition wherein, he claims, true consonance lies. The intervals he associates with the regular triangle and the doubling process (musical fifths and fourths, octave transpositions) involve irrationalities of first degree, such as $\sqrt{2}$ and $\sqrt{3}$; these were the harmonies used in medieval Europe. The intervals associated with the regular pentagon and its doubles (musical thirds and sixths) involve what he considers to be irrationalities of second degree, i.e. two-layered ones, the side length of a pentagon inscribed in a unit circle being $\sqrt{\frac{5-\sqrt{5}}{2}}$; these were the harmonies that arose with the revival of interest in Greek art and geometry during the European renaissance. That same pentagon side s can be written in factored form as $\sqrt{5} \cdot \sqrt{\frac{\sqrt{5}-1}{2}}$, where $\frac{\sqrt{5}-1}{2} \doteq 0.618033989$ is the ratio of "golden section" φ, in turn the side length of a regular decagon inscribed in the same unit circle.

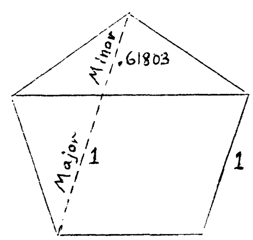


Both the triangle and the pentagon, therefore, consist of parts that are knowable, and properly so, though of differing degrees; hence the historical precedence of musical fifths and fourths over thirds and sixths, as perceived consonances. If one constructs first a regular triangle, and then a pentagon starting at each of its corners, a regular 15-gon results — not properly, on its own merits, but "accidentally" as Kepler says; hence he accepts the ratio 15 : 16 as leading tone Mi Fa melodically, but denies its ability to stand on its own as a consonant interval.

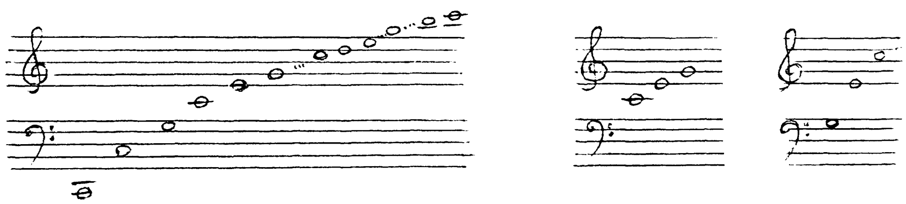
There is a slight problem with this. As the observant reader may have noticed on p. 30 above, the decagon side-length's simple one-layer-deep radical expression should have classified it as more directly knowable than the pentagon, yet if one is given a decagon then a pentagon follows from it immediately by omitting every other corner; hence the pentagon's degree of knowability should be no greater than that of the decagon, yet the expression for its side length is two layers deep. Kepler, in fact, rates them both as being of his second degree, 15- and 20-gon both third. In his appended notes, p. 369, Max Caspar points out that the areas of triangle and square have one layer deep radical expressions, those of pentagon and decagon two layers, and those of 15- and 20-gons both three. This is apt to strike the reader as rather ad hoc, but it will turn out to be searching in very appropriate directions, once the light which algebra can shed on these questions is understood (v. note, p. 69).

Meanwhile, it may be appreciated that the most important thing which Kepler's own work contributed toward a qualitative understanding of European musical harmony was to have selected, on an intuitive basis that proved to be essentially correct, a middle road between earlier Greek tuning theories built exclusively upon powers of 2 and 3 (the Pythagorean ratios) and those admitting powers of both 5 and 7 (due to Plato's friend Archytas and the Alexandrian Ptolemy) some three and six hundred years later. By choosing to include powers of 2, 3, and 5, but exclude those of 7, Kepler's work accounts well for the actual course of harmonic history in the west.

By making this choice, pp. 129 ff., Kepler is forced to break with the tradition of the medieval church modes (although he dutifully describes them all briefly) and the older Greek tonalities ("diatonic," "chromatic," and "enharmonic" which he avoids altogether, despite a description of the Greater Perfect System elsewhere, pp. 144 ff.) and begin instead to lay the groundwork for an understanding of what have since become the two principal European modes, major and minor, relating them explicitly to the proportions of the regular pentagon. The diagonal chord of a pentagon is to a side length as 1.61803 : 1, and one diagonal divides another into 1 + .61803, satisfying 1.61803 : 1 = 1 : .61803



(the whole being to the larger part as the larger is to the smaller, one definition of "golden section"). This was first encountered numerically on p. 26 in Book I; now in Book III we meet it again on p. 107 in simplest approximations in the number series 1,1,2 of unison and octave and 1,2,3 of octave and fifth, thinking of pairs of successive members of these series being set into ratio with one another 1 : 1, 1 : 2, etc. If these two simplest series-segments are joined end to end, overlapping in common terms, they give rise to the well-known series of Fibonacci numbers 1,1,2,3,··· formed via adding 1 + 1 = 2, 1 + 2 = 3, whose next members would be 2 + 3 = 5, 3 + 5 = 8, etc., yielding successive ratios 2 : 3, 3 : 5, 5 : 8, etc., growing ever closer to the "golden" pentagon ratio 0.618···:1, or 1 : 1.618···, alternately over- and under-approximating it as 0.666···, 0.6, 0.625, 0.615···, 0.619···, and so on.



1 2 3 4 5 6 8 9 10 12 15 16   4 5 6   3 5 8

Thinking in Pythagorean terms, we must imagine a series of monochord strings, the first vibrating as a whole, the second stopped half way, the third in thirds, etc., but otherwise originally of equal length and tension. The series of pitches formed thereby produce an octave ratio between the first and second string (say C and C'), a fifth between the second and third (C' and G'), and so on. The major third (E") as 5th tone was classically justified as arithmetic mean between the root and fifth as 4th and 6th tones. Thinking instead in Fibonacci terms, we see that it can also be regarded as major/minor division of the interval from the fourth below to octave above as 3rd and 8th tones, like pentagon proportions, and we call the ratio 3 : 5 (G' to E") "major" and that of 5 : 8 (E" to C"') "minor" as two different sizes of intervals of a sixth, the complements with respect to the octave of the ratios of 5 : 6

and 4 : 5 as "minor" and "major" thirds. The "perfect" intervals of the octave, fifth and fourth, as ratios 1:2, 2:3 and 3:4, based upon arbitrary powers of 2 and the 1st power of 3, come in only one size apiece; medieval music theory was grounded upon them. As soon as the 5th tone is admitted we arrive at thirds and sixths in two different sizes. Even the two sizes of seconds and sevenths as ratios 9:10 and 15:16, 5:9 and 8:15, respectively, are all seen to involve 5 or multiples of it, as well as bringing the 2nd power of 3 into play (9th tone used in passing). Kepler relates the notions of major and minor modalities to major and minor divisions of the pentagon on pp. 165-166, likening them to the division of humanity into male and female genders. The tendency toward pairing noted in Book II would seem to belong here too, thematically, but Kepler does not mention it further.

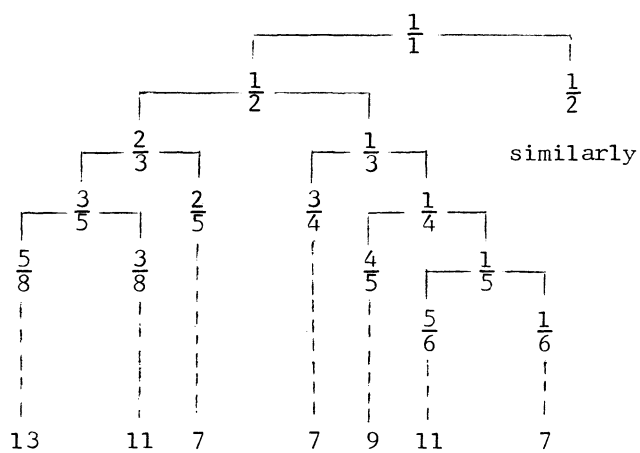
The harmonious relationship of the number 2 to the number 5 (witness the family of 2-seed-leaved or dicotyledonous plants and their almost universally 5-fold-symmetric flowers – Bindel [1962], p. 198, notes that Kepler intended to write about them someday but never did) but to no other odd number shows up also in the following manner, according to Kepler. He takes it to be axiomatic, pp. 96-97, that for all odd numbers greater than 5 and their doubles all numbers relatively prime to them are dissonant to them. To illustrate this, he finds it sufficient to examine the numbers from 1 up to half the number in question, producing the table of dissonant parts shown on the next page. As an axiom, it needs no formal justification; informally, he tells us he needs it to mark off the consonances he wishes.

"The parts	dissonate with the whole
1. 2. 3	7
1. 2. — 4	9
1. 2. 3. 4. 5	11
1. 2. 3. 4. 5. 6	13
1. 2. — 4. — — 7	15
1. 2. 3. 4. 5. 6. 7. 8	17
1. — — — 5. — 7. —	18
1. 2. 3. 4. 5. 6. 7. 8. 9	19

and so on indefinitely."

With this axiom (it is no. III) he seeks to prove a theorem (no. V), albeit with what strikes us as rather ad hoc argumentation, to the effect that although demonstrable (constructible) star figures form parts of a circle that are just as consonant with the whole as the sides of demonstrable polygons, nevertheless certain of their sides are to be excepted as dissonant. These are found as follows: Take the number of the star side, say 9, and divide the number of the whole, say 20, by 2 repeatedly until you arrive at a number that is less than half of the side. The side in question is deemed consonant if and only if the resulting ratio is one of the admissible harmonious divisions of a monochord string. In the example, then, we form successive halves of 20 until we reach $2\frac{1}{2} < 4\frac{1}{2}$; then we test the ratio $2\frac{1}{2} : 4\frac{1}{2}$ and find it equal to $5 : 9$, which is not one of the admissible divisions, so although the 20-gon is constructible the $\frac{20}{9}$ star is deemed dissonant. The $\frac{5}{2}$ star, similarly tested, leads to the admissible ratio of $5 : 8$, hence is harmonious, as is the $\frac{10}{3}$ star by virtue of the harmoniousness of the ratio $5 : 6$, and so on. (The proposed $\frac{6}{2}$ "Star of David" to be added to Kepler's list is harmonious since $3 : 4$ is.)

The list of admissible divisions of a string is arrived at on p. 111 in just as ad hoc a manner, as shown on the next page, by starting with the unison ratio, then at every stage taking the given ratio p/q and testing the sum p+q; if it is a number of form $2^a 3^b 5^c$ (where a = 0,1,2,··· but b and c are either 0 or 1) then he forms two new ratios 1/(p+q) and (p+q-1)/(p+q) and repeats the test, but if it is not of that form then he writes the offending sum after a dashed line to show the process has ended.



In this manner he finds a total of seven admissible divisions.[†] Actually, he tells us, he found them first by ear and searched a long time for a satisfying explanation, calling what he wrote earlier in his Mysterium Cosmographicum "fantasy" by comparison. The observation of what is harmonious is made first by the ear; then and only then is mathematics brought to bear by way of explanation.

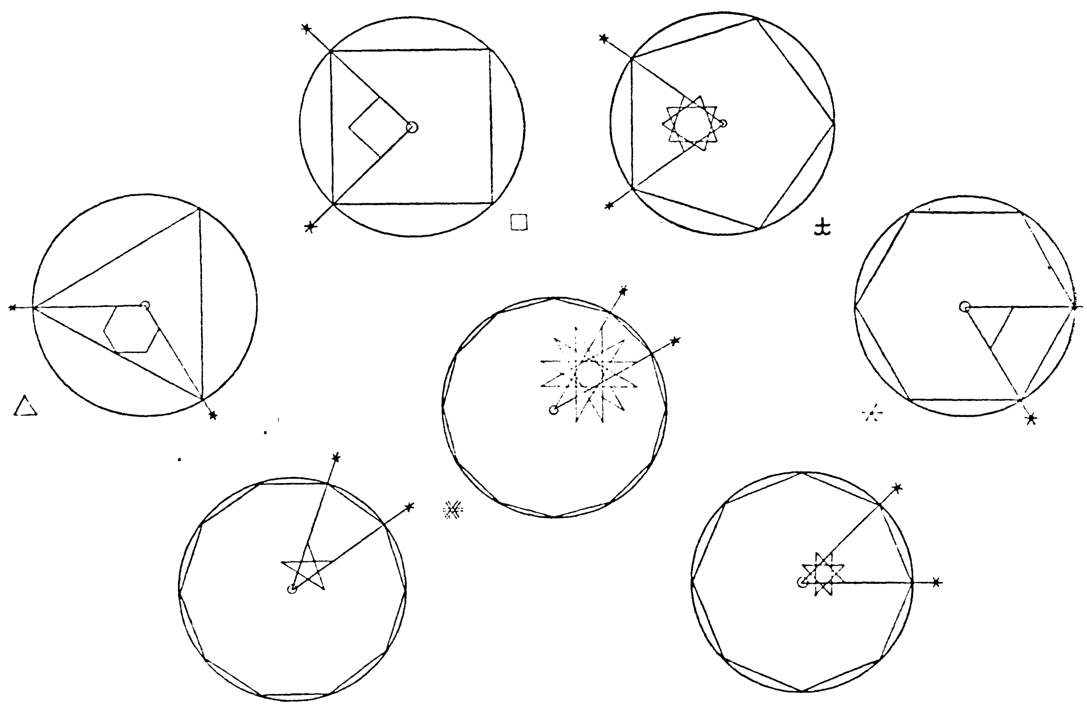
[†](I.e. into whole, halves, thirds, quarters, fifths, sixths, and eighths.)

E. Effective Aspects in Astrology

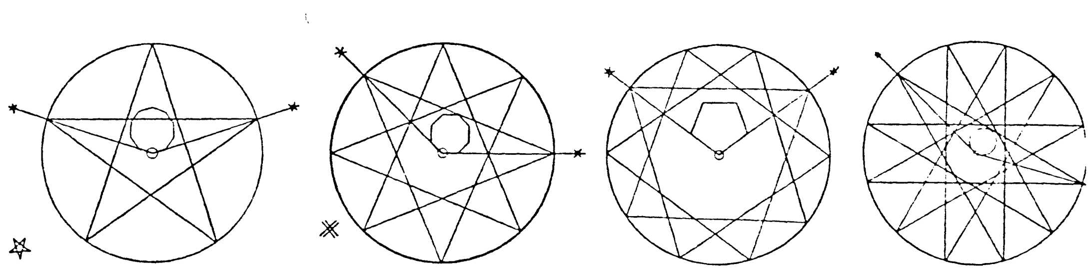
If the III^rd Book had to do with proportions of side- or chord-lengths of regular figures inscribed in circles, representing somehow-wrapped-around monochord strings as harmoniously sounding divisions, so the IV^th Book treats much these same figures but now as they are experienced from the center point, angularly, Both the Earth and the individual human being are conceived as possessing a soul, and the soul extends circularly out from both, surrounding them, and responding to harmonious angular separations between the various planets as they move about the Zodiac, provided these angles are "knowable," for therein Kepler sees the effectiveness of harmony.

That Kepler ever received a call to the court of the Emperor Rudolph in Prague was probably due much more to the success of the political and agricultural predictions made in farmers' almanacs which he edited than to the ambitious constructions of his Mysterium. The emperor, after all, retained an alchemist attempting to make gold (from which, alas, Kepler was to be paid). Kepler, for his part, knew that he had been lucky in some predictions (such as the year of a Turkish invasion), but in other matters he knew he had made extensive observations and based his statements on experience (cf. his account of the weather during the winter of 1609, compared with day-to-day astrological aspects, as given in [1971], §138). The idea of a world- as well as human soul bent round the Zodiac he took from Plato's Timaeus (31b - 47e), whence also the "Platonic solids." This is not the place to discuss the merits of an astrological world-view; the interested reader is referred to the work of Jung and Pauli (1955).

Concerning this book of the <u>Harmonice</u>, it is sufficient for our purposes to note that Kepler finds again (allegedly on the independent practical experience in the field) twelve aspects (angles between planets, as seen from the Earth) to which wind and weather and the affairs of men seem to respond to a significant extent. The twelve that he cites (plus conjunction as an implicit thirteenth) correspond essentially to the original list of harmonious figures in Book I. There are seven aspects derived from centriangles of regular polygons: 120° from the triangle, 90° from the square, 72° from the pentagon, 60° from the hexagon, 45° from the octagon, 36° from the decagon, and 30° from the dodecagon.



There are four aspects derived from star polygrams: 144° from the $\frac{5}{2}$-star or pentagram, 135° from the $\frac{8}{3}$-star or octagram, 108° from the $\frac{10}{3}$-star or decagram, and 150° from the $\frac{12}{5}$-star or dodecagram.



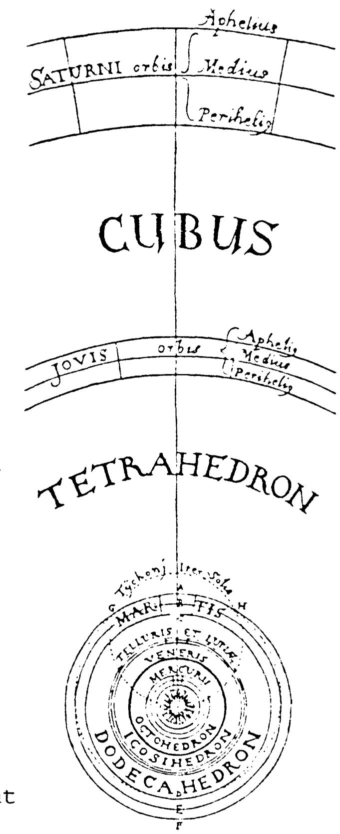
Alone of the list in Book I the 20-gon contributes no effective aspect. As twelfth aspect, Kepler takes the opposition of two planets, corresponding to a digon, but the reader may be willing to accept the proposed "Star of David" or hexagram as pair of triangles in opposition to one another — indeed the outstanding example of conjunctions and oppositions which Kepler studied elsewhere was the 60-year cycle of Jupiter and Saturn, describing successive corners of such a hexagram every 10 years, that being 5/6 of Jupiter's 12-year period and 1/3 of Saturn's 30.

These twelve also correspond to twelve of the fourteen ratios cited in Book III (see p. 36 above), if one takes them as fractions of 180°. Only the ratios of 5 : 8 and 3 : 8, corresponding to angles 112.5° and 67.5°, do not form effective aspects, according to Kepler (because of the half degrees?).

F. Ordering of the Solar System

The V^th^ and final chapter or book of Kepler's mature work is subtitled "The most perfect Harmony in the Heavenly Motions and the Eccentricities, Orbital Diameters, and Revolutionary Periods arising therefrom." In his youth-work, he had once modelled the orbits of the 6 known planets as equators of spheres in- and circumscribed about the 5 Platonic solids, hoping to explain thereby both their number and their spacing. He had already felt slightly uneasy about having to place the innermost Mercury sphere through the edge-middles of the innermost octohedral solid, rather than tangent to its face-middles as with all the other larger inscribed spheres. When improved observational data refined the knowledge of Mercury's actual orbit and forced Kepler to conclude that it was not circular but elliptical, with a definite eccentricity, his first attempt to revise the model was to imagine each sphere's wall as having a particular thickness proportional to the eccentricity of the orbit of the planet which it modelled: considerable for Mercury and Mars, moderate for Earth, Jupiter and Saturn, very slight for Venus (cf. illustration on p. 11 — the little circle on the rim of the Jupiter and Saturn sphere indicates the eccentricity of that planet's orbit). But this was hardly an improvement in the model!

Instead, Kepler was eventually forced to abandon the old statuesque geometry and work through countless pages of Tychonic data before finally arriving at a new dynamic conception of geometry in motion. A last remembrance of his youthful construction is shown on p. 287 of (1939), indicating



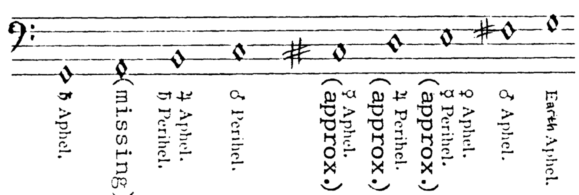
for each planet by three circles the extent of its orbit at Perihelion (nearest the Sun), at mean distance, and Aphelion (farthest from Sun). Tycho himself believed neither the old Ptolemaic system nor the new Copernican one entirely, but preferred to place the Earth in the center and let the Sun circle about it, with all other planets circling the Sun; this is indicated in Kepler's drawing by a dashed circle labelled "Tychoni Iter Solis," centered on the orbit of the Earth and Moon (Telluris et Lunae). The names of the 5 Platonic solids are printed between the orbits, where Kepler had thought them placed as a youth.

In the hard-won conception of his maturity, each planet moves in an elliptical orbit, with one focal point at the Sun. A line joining that planet to the Sun (radius vector) sweeps out equal areas in equal times, fastest at perihelion and slowest at aphelion. Finally, if the orbital diameters (or radii) of any 2 planets are compared with the lengths of the corresponding revolutionary periods, then cubes (3rd powers) of the former are proportional to the squares (2nd powers of the latter — a purely algebraic statement, having no pictorial counterpart in terms of visualizable cubes or squares. For example, if we compare Jupiter's distance from the Sun and length of year with those of the Earth, we find it is 5.20 times as far away and takes 11.86 times as long to complete one revolution; in modern terms, $5.20^3 \approx 11.86^2$, so that the ratio of $5.20^3 : 11.86^2$ is approximately equal to the ratio of 1 astronomical unit (Earth-Sun distance of 93 million miles) to 1 year (365 Earth days). These three laws of planetary motion, for which every modern text on astronomy praises Kepler, take up a scant paragraph of Kepler's own work (p. 289, bottom half of page). What interests Kepler is the following:

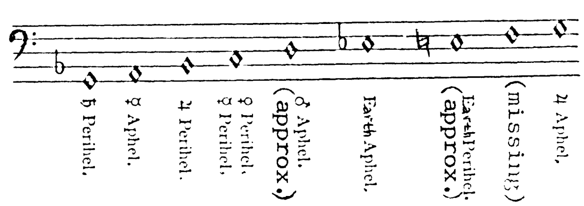
Rather than conceiving of each planet's eccentricity pictorially as a relative thick- or thinness of a static spherical shell wall, Kepler thinks of the planet in motion, speeding up as it comes slightly nearer the Sun, slowing down again as it recedes in its elliptical path. For each planet he reckons angular distance travelled, as seen from the Sun, at aphelion and at perihelion, and relates each ratio of slowest to fastest motion to a musical interval (p. 301 of [1939]).

	PLANET	APPARENT DAILY MOTION at aphel.	at perihel.	OWN RATIO	MUS. INTERVAL	PAIRWISE RATIOS converg.	diverg.
♄	Saturn	a = 1'46"	b = 2'15"	$\frac{a}{b} = \frac{4}{5}$	Maj. 3rd	$\frac{b}{c} = \frac{1}{2}$	$\frac{a}{d} \approx \frac{1}{3}$
♃	Jupiter	c = 4'30"	d = 5'30"	$\frac{c}{d} \approx \frac{5}{6}$	Min. 3rd	$\frac{d}{e} = \frac{5}{24}$	$\frac{c}{f} \approx \frac{1}{8}$
♂	Mars	e = 26'14"	f = 38'1"	$\frac{e}{f} \approx \frac{2}{3}$	Perf. 5th	$\frac{f}{g} = \frac{2}{3}$	$\frac{e}{h} \approx \frac{5}{12}$
	Earth	g = 57'3"	h = 61'18"	$\frac{g}{h} \approx \frac{15}{16}$	Semitone	$\frac{h}{i} \approx \frac{5}{8}$	$\frac{g}{k} \approx \frac{3}{5}$
♀	Venus	i = 94'50"	k = 97'37"	$\frac{i}{k} \approx \frac{24}{25}$	Diesis	$\frac{k}{\ell} \approx \frac{3}{5}$	$\frac{i}{m} \approx \frac{1}{4}$
☿	Mercury	ℓ = 164'0"	m = 384'0"	$\frac{\ell}{m} \approx \frac{5}{12}$	Octave + Min. 3rd		

Only the first (outermost) planet, Saturn, exhibits a musical ratio of its own extreme daily motions which is virtually exact (to within 2" of arc — a should be 1'48"); the others could be improved by taking c = 4'35", e = 25'21", g = 57'28", k = 98'47", and m = 394'0", as he notes (p. 301). "But" he writes (on p. 302) "if one compares the extreme motions of pairs of planets, then at once the Sun of Harmony appears in all its glory, whether one considers the divergent [outer at aphel., inner at perihel.] or convergent [outer at perihel., inner at aphel.] extremes." Actually, the divergent extremes and inner two convergent ones are also only near approximations, though nearer than any of the planets by themselves; but the outer three convergent extremes are virtually exact. More importantly, <u>all ten pairs are good approximations to familiar musical intervals</u>! Even more: <u>taken together, they form major and minor scales</u>!! Following the Guidonian "gamma-ut" tradition of letting Γ, A, B, ... represent do, re, mi, ..., Kepler lets Saturn's lowest note (slowest motion, shortest apparent arc) be represented as a G; then if that Saturn G is taken as the aphelion value a = 1'46" he obtains a major scale (with extra C# but missing A)

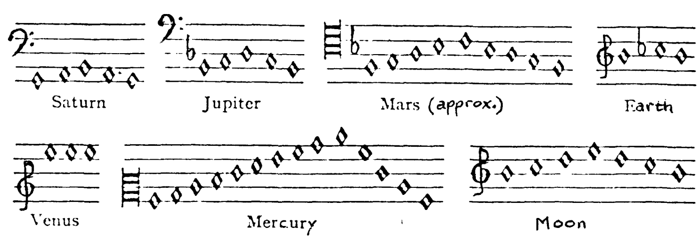


but if it is taken as the perihelion value b = 2'15" he obtains a minor one (with extra E♮ but missing F).



In each case there are two doublings (Jup. at aphel. is octave of Sat. at perihel., and Merc. at perihel. is double octave of Ven. at perihel.) and two omitted (in the major scale, Earth at perihel. would be a quartertone between G# and A, Ven. at perihel. a quartertone between E and F♮ — in the minor scale, Mars at perihel. would be a G# on pitch but foreign to the scale, Ven. at aphel. a semiquartertone below pitch C).

Returning each planet's pitch to its proper relative octave, but letting each vary over the interval found by comparison of its own ap- and perihelion motion, Kepler obtains finally the following ranges of the "voices" in the heavenly "motet" perceived in this way (pp. 309-310), including a perfect 4th contributed by the Moon's $\frac{3}{4}$ motion ratio at apo- and perigee (farthest and nearest the Earth):



Kepler then contemplates the kaleidoscopic effect of all of the 6 planetary harmonies shifting in rhythms that are essentially irrational to one another, wondering if the same combination ever occurs twice in the history of the universe, but contenting himself to pick out a few of the possibilities in which all 6 could join in a single consonance, describing chords in e minor, C major, E♭ major, and c minor. In E♭, the plaintive "G-A♭-G" line of the Earth's melody becomes "Mi-Fa-Mi," which Kepler interprets as an allusion to the seemingly endless round of "Miseria" and "Famina" of life during the 30 years' war. (The latter two keys are possible since Venus is always hovering between E and D# or Eb, and could be interpreted either way.) For hunger miserably he does – the emperor's alchemist never succeeds in making gold, and Kepler is eventually forced to leave Prague for Linz.

In 1621, two years after publication of the Harmonice Mundi, Kepler has to risk a dangerous trip back to Württemberg to defend his mother against accusations of witchery (she dies a year later at age 75). In 1625, the Counter-Reformation forces him to leave catholic Linz in Austria for the more tolerant Regensburg near the Swiss border of southwest Germany, after several years of wandering, dying there in 1630.

In 1633, Galileo Galilei is tried and convicted for teaching the heresy of a Sun-centered universe, dying in 1642.

In 1643 Isaac Newton is born.

Note:

Professors Willie Ruff and John Rodgers, members of the music and geology faculties of Yale University, respectively, have created an electronically synthesized realization of Kepler's "motet," including rhythmic pulses to represent the otherwise inaudible subsonic contributions of Uranus, Neptune, and Pluto, writing up their results in the American Scientist, Vol. 67, No. 3 (May-June 1979), pp. 286-292. The recording, LP 1571 (Kepler's birth year), The Harmony of the World, introduces each planet individually, then combines them at various speeds, finally playing the full "motet" over Kepler's lifetime and over one full Pluto year (248 Earth years).

G. The Newtonian Reformulation

Each of the three Keplerian laws of planetary motion changes under the hand of Newton:

The first law, that all planets move in elliptical orbits, paths that are algebraically quadratic in nature, becomes deducible on the assumption of a gravitational force that diminishes inversely proportionally to the square of the distance between the two attracting bodies. The elegance of the deduction is mitigated by the fact that once three or more bodies are involved the orbits become essentially ineffable, forever shifting.

The second law is modified to recognize the Sun as having much the greatest mass of any body in the solar system, which places it near the ideal focal point of a given planet's elliptical orbit; but the actual center (were there just two bodies involved) is the average center of mass of Sun and planet together, like a large adult and small child on a teeter-totter, with the fulcrum nearer the larger person. Even with just two bodies, the Sun is no longer at rest in the center of the Copernican system but ever moving to stay in balance with each of its planetary children. (The common center of the Sun-Jupiter system lies in the outer atmosphere of the Sun, so that the Sun must move by an amount equal to its own radius just to off-set that one other body.)

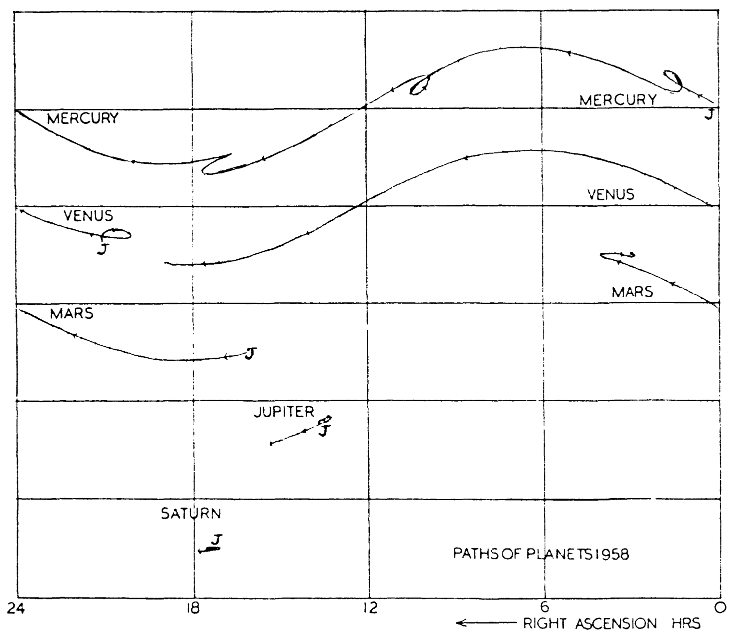
The third law is changed in more subtle manner. If r and R are the radii of a given pair of planets' orbits, and t and T the times taken respectively to revolve once around them, then Kepler would have formulated his law in modern notation as

$r^3/R^3 = t^2/T^2$ (setting $R = T = 1$ for the Earth's distance and time, we saw that this meant $5.20^3 \approx 11.86^2$ for Jupiter). Since distances are being compared to distances, and times to times, it does not matter what the units of measure are – they cancel, leaving pure numbers, musical ratios, a unique "sound" emitted in spirit by each member of the solar system. As reformulated by Newton, this becomes $r^3/t^2 = R^3/T^2$, a single physical constant holding for all members of the solar system at once, but one whose numerical size is meaningless, dependent on arbitrary choices of units involved. It is the old medieval debate between the Platonic realists and the Aristotelian nominalists! In terms of the three riddles posed in the introduction, the difference between the two formulations of this law might also by characterized by saying that Kepler's thinking remained within the cochlea, expressing everything in terms of musical ratios, while Newton's thinking took into consideration what the semicircular canals told of gravity. Or again: In Kepler's musical view of the solar system as singing a motet by the dynamics of its movements, the Earth "holds" the middle or tenor voice, while the other six planets move "against" that "holding" as contra-tenors, altus (higher, faster) or bassus (lower, slower), the way counterpoint was conceived from the middle voice outward, above and below, by composers of his day, while Newton's reformulation parallels his century's re-thinking of harmony as rooted in the bass, as though by gravity.

V. THE LARGER HISTORY OF THE QUADRIVIUM

A. Ptolemaic Epicycles

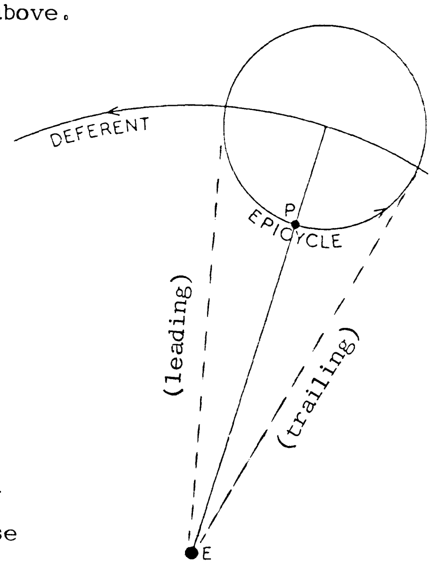
The paths of the planets (from the Greek πλάνης, a wanderer) were observed very carefully by both the ancient Chinese and Babylonian civilizations, who kept faithful records of which asterisms they appeared against from season to season. But that was a pointillist approach: isolated positions here and there.

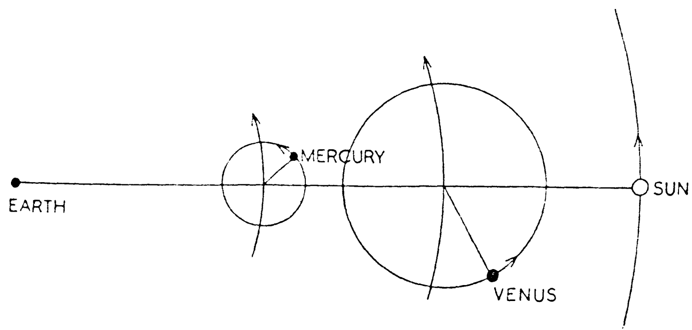


To appreciate the difficulty of describing the actual continuous loop-the-loop paths, consider the above illustration (reproduced from Tricker [1967], p. 44) showing the geocentric appearances of Mercury through Saturn from the year 1959 (January 1st positions being marked with a J). The vertical lines mark hours of right ascension (0 corresponding to the Sun's position at time of the spring equinox, at present between the constellations of Pisces and Aquarius; 6 corresponds to its northernmost position at midsummer, 12 autumnal equinox, and 18 southernmost position at midwinter.) The horizontal lines are repeated copies of the celestial equator (the projection outward onto the sky of the Earth's equatorial circle). Each planet follows essentially the Sun's path, climbing $23\frac{1}{2}°$ north in summer, then falling $23\frac{1}{2}°$ south of the equator in winter; but each planet embroiders this path from time to time with zig-zag and loop-like figures of the most varied nature. The center of the backward or retrograde motion which forms the loop always coincides with the period of brightest luminosity of the planet, calling even greater attention to the phenomenon. Comparable to the full moon, the planets rise at sunset and set at sunrise during their loop-motions, remaining visible all night long. Mars at such times is a brilliant object, outshining all other natural lights in the nighttime sky except the Moon and Venus, while at the time of its apparent fastest forward motion it appears only as a medium-bright star, made all the dimmer by rising or setting shortly before or after the Sun. The times of these brilliant periods were known to the Egyptians; isolated intermediate places were recorded by the Chinese and Babylonians; but it remained for the Greeks to penetrate the phenomenon with mathematical understanding.

As put forward by Claudius Ptolemaeus in Alexandria (Egypt) during the 2nd century A.D., the essential forward motion of each planet was accounted for by thinking it to move along a <u>deferent</u> circle of appropriate size (small for the swifter-moving ones, large for the slower). The retrograde loops were accounted for by picturing the planet to be departing from the simple circular motion of the deferent by following instead a smaller second circle atop the first, called an epicycle (or "on-circle"); its center in turn followed the deferent. Finally, the varied zig-zag and loop-the-loop forms were accounted for by thinking of the epicycles as tilted with respect to the deferents. The deferents were all more or less in the plane of the Sun's path (the ecliptic), tilted at about 23½° to the plane of the Earth's equator; slight further tilts of the epicycles caused their motions to sometimes appear back-and-forth when seen edge-on, sometimes as looping upward or downward when seen from below or above.

The absolute sizes of the deferents were not known, but recognized as relating to the times taken to revolve around them (the exact relationship is Kepler's 3rd law). The relative sizes of deferent and epicycle were determined by direct observation, measuring how much an actual planet seems to lead or trail a steadily moving ideal point. Those inner planets (Mercury and Venus) moving faster than the Sun had the centers of their epicycles affixed to a radial line-of-sight from Earth to Sun, while the outer planets (Mars, Jupiter, Saturn) had epicycles which could





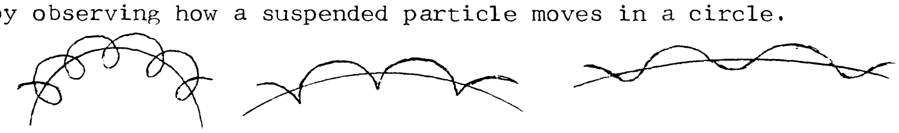
move freely along their deferents, thus accounting for the observation that Mercury and Venus never wandered more than 36° or 45° respectively to one side or other of the Sun, while the other three could appear at any angular distance from it around the zodiac. All this tends to strike the modern mentality as so much clockwork without material gears, and without a driving mechanism, but it does describe the appearances purely geometrically!

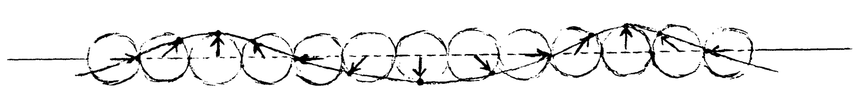
When the versatile English science writer R. A. R. Tricker was approached by his publisher (Cambridge U. Press) to put out a modern book on constructing Ptolemaic epicycles, he balked at first; but as he got into the work, its true nature and intent became clear to him, and he grew enthusiastic. Indeed, it was his remarks in the preface to the resulting book (1967) that formed a seed-point in the present author's mind, about which many other experiences began to crystallize. Tricker wrote:

"It is hardly to be expected that such elementary work would expose problems of interest to current thought, yet in putting the book together the author found his own appreciation of certain aspects developed further as a result. In particular he had hardly realised before the essential role played by the Ptolemaic theory in the development of science. In common with many other writers he had tended to regard it as an obstacle to progress which had to be removed rather than as a contribution to the final end. The achievement of Copernicus consisted essentially in transferring an annual component in the motion of all planets from them to the earth, thus replacing five independent movements by a single motion. However, ... there is no annual component to be directly discerned in the movement of the planets. There is, in fact, only one annual movement to be observed in the sky, and that is the apparent movement of the sun itself, or of the earth, according to the point of view. The annual components in the motions of the planets only become apparent after the harmonic analysis, provided by Ptolemy's theory, has shown them up."

And with that we meet the first historical instance of what is known in modern mathematics as <u>harmonic analysis</u>: the resolution of a complicated but periodic phenomenon into simple cyclic components. Its history begins where Kepler's work ends, with astronomy; and there it was to lie dormant for 1600 years before reawakening in another field which gave it its name, musical harmony (parallelling the next-to-last book of Kepler's work), as analyzed by the 18th century French composer and theorist Jean-Philippe Rameau.

To see the connection between epicycles and wave forms, we need only let the radius of the deferent circle become increasingly large; then the epicyclic path undergoes a gradual transformation from looped to cusped to undulant form, approaching sine wave shape in the limit as the deferent circle approaches a straight line (circle of infinite radius). This process may be used to describe the propagation of water waves by observing how a suspended particle moves in a circle.





B. The Discovery of Overtones and Exotic Tunings

Some periods seem ripe for certain discoveries, so that they are made independently by several different people at about the same time. One often-cited example is the discovery of non-Euclidean geometry at the turn of the 19th century. Another, less well-known, is the discovery of overtones at the turn of the 18th.

The first published account, giving experimental evidence for the existence of overtones, was due to Joseph Sauveur: Principes d'acoustique et de musique, included in the Histoire de l'Académie Royale des sciences, Paris, 1700/01. A much earlier observation of at least the first partial is found as a remark by Descartes in his 1618 Compendium Musicae: "We never hear any sound without its upper octave somehow seeming to strike the ear," but he did not develop the idea.

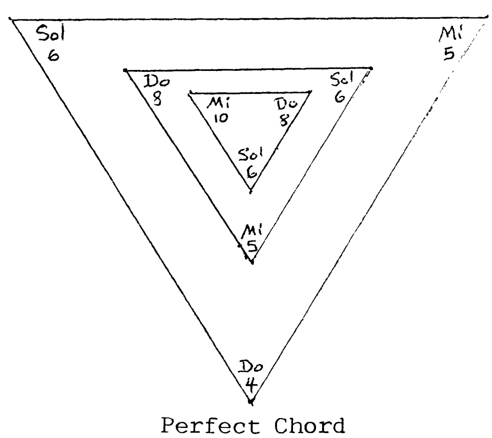
Rameau was well aware of Descartes' writings on music, to the extent of borrowing entire passages from their French translation (out of the Latin) by Father Poisson, as well as those of other theorists such as Saint-Lambert, merely changing the odd word or two per sentence — very strange behavior for one of the most original minds of his century! Yet Rameau seems to have been entirely unaware of the physical work of his countryman Sauveur, at least at the time of the publishing of the first edition of his famous Traité de l'Harmonie reduite à ses Principes naturels in 1722. By the time of writing the Nouveau système of 1726 he has begun to read and appreciate Sauveur's work, and in the Génération harmonique of 1737 he discusses in detail how the two theories bear one another out.

The Traité adopts the notion of harmony as founded upon the lowest sounding tone the way Copernicus' De Revolutionibus adopts the description of planetary motions as centered on the largest body: it is merely a reckoning convenience. Rameau is engaged in setting forth the practice of figured bass; Copernicus was originally concerned with computing many Easter dates (based on rhythms of Sun, Earth, and Moon). Only gradually did a sense of physical reality creep into their work (the bass *is* the fundament, the Sun *is* the center).

A chord such as C-E-G could have its root C (Do) in the bass, or its third E (Mi), or its fifth G (Sol), referred to as "perfect position," "first" and "second inversion," respectively. In the Traité (p. 41 of its English translation [1971]) these three positions are shown schematically in an equilateral triangle, a figure that would have gladdened the heart of any medieval theorist:

Second Inversion

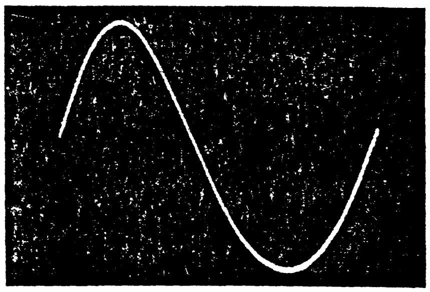
First Inversion



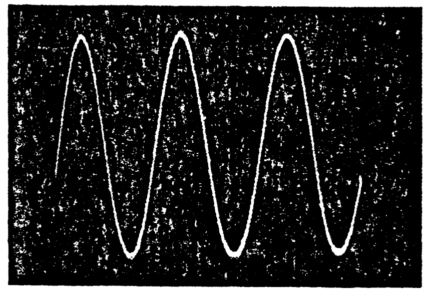
Perfect Chord

The numbers 4,5,6,8,10 are naively identified with octave doublings of the root 1, fifth 3, and third 5. A chord was presumed to contain the third and fifth degree (scale step) above the first note in its bass unless otherwise specified. A perfect chord had both, so required no notation; a chord in first inversion (E-G-C) had the third above the bass but sixth in place of the fifth, so was known as a "sixth chord," denoted 6; one in second inversion (G-C-E) had notes six and four degrees above the bass note instead of five and three, so was known as a "six-four chord," denoted $\substack{6 \\ 4}$. While the placement of the root position chord at the base of the triangle and its name "perfect" suggest a preferred status, the overall symmetry of the diagram stresses an essential similarity of the three chords. Indeed, Rameau writes "no matter what corner is chosen as the base, we shall always find a consonant chord. We shall find Do, Mi, and Sol in each chord, and the differences among these chords will arise only from the different arrangement of these three notes or sounds" (loc. cit.).

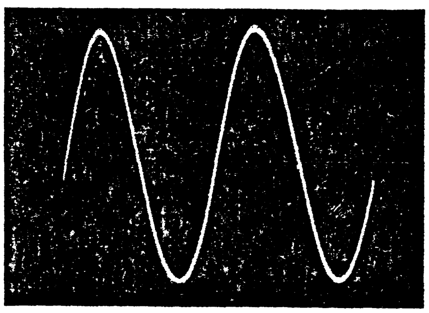
The confusion between the numbering of chord and scale notes on the one hand, and the different sound impressions of the inversions on the other, are both clarified when the concept of overtones is available. The Pythagoreans had known that a monochord string yields a succession of different pitch-levels when stopped at different proportions of its whole length, but it was apparently Sauveur who first realized that one and the same string could do all these things at once (vibrate in all these different modes), producing a coloristic effect. With the aid of a modern oscilloscope it can be shown that part of the characteristic of a flute sound is the relative strength of its fundamental, while an oboe has strong second harmonic or overtone (the octave) and a clarinet a strong third harmonic (the octave-fifth) — see the illustrations on this page and the next from the article "Musical Tones" by Hugh Lineback, Scientific American, May 1951.



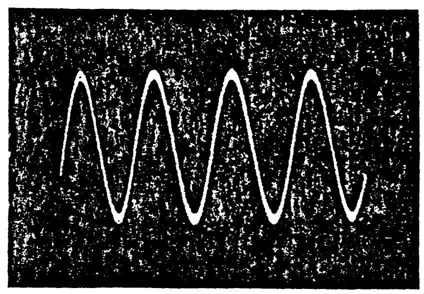
Fundamental



Third harmonic

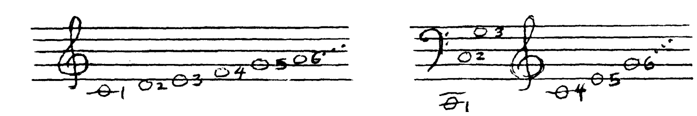


Second harmonic

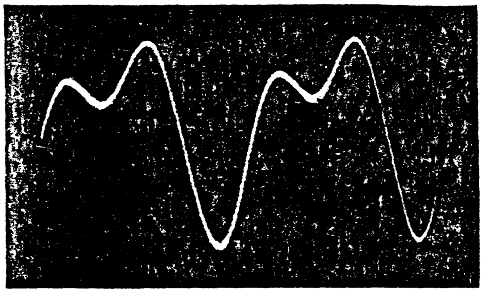


Fourth harmonic

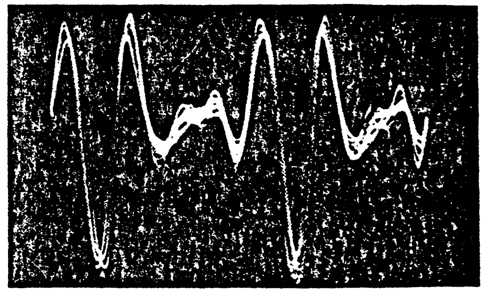
Scale-degree numberings follow steps, while chord-tones follow harmonics:



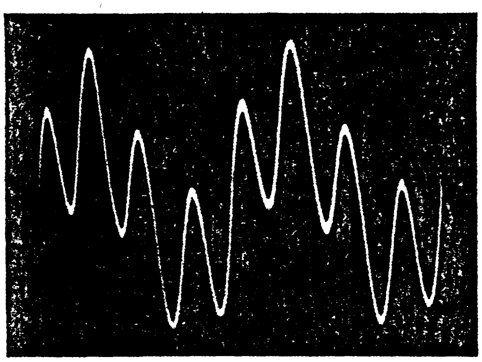
If we now examine the rest of the chord-tones (the series of harmonics) above the note in the root of, say, a C major chord in root or perfect position, first and second inversions, we see that the root position is "perfect" in that all of



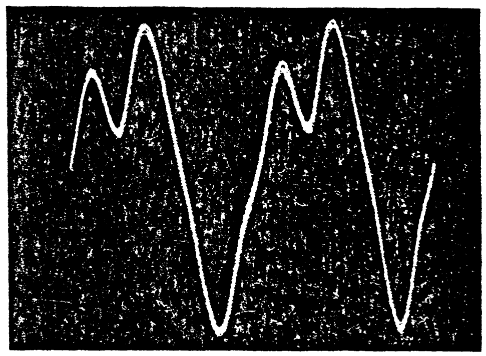
Combination of fundamental and second harmonic



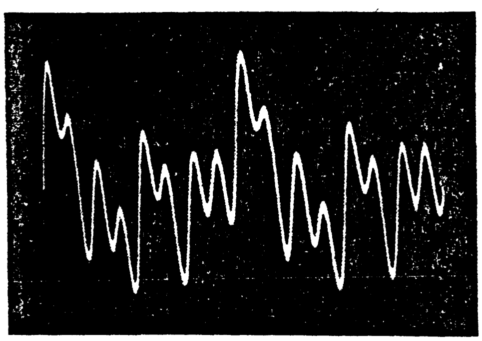
Tone of oboe



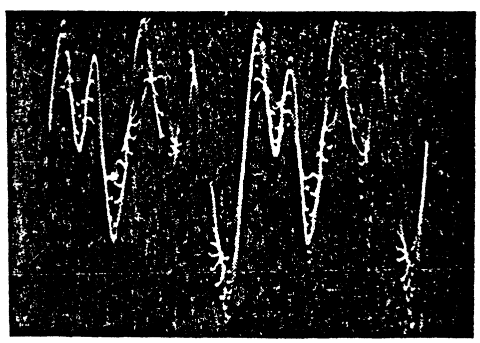
Combination of fundamental and fourth harmonic



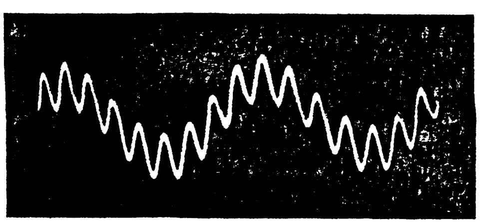
Tone of French horn



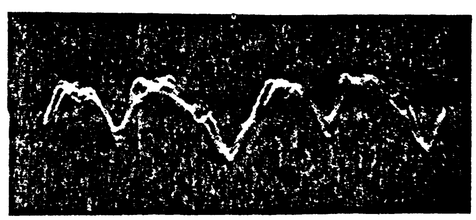
Combination of all even harmonics



Tone of trumpet

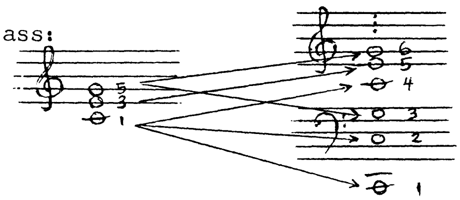


Combination of fundamental and eighth harmonic

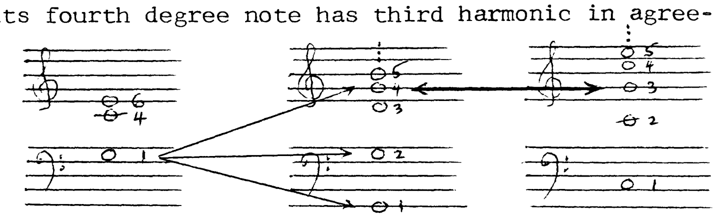


Tone of flute with strings

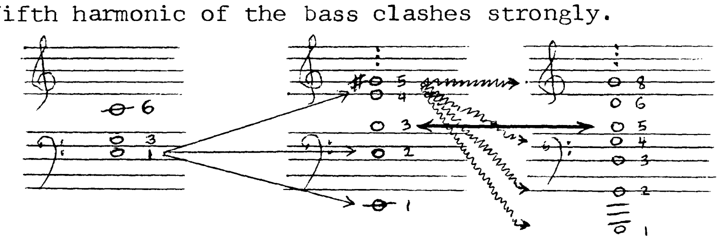
its chord steps are in perfect agreement with the overtone series of its bass:



A chord in second inversion (a $^{6}_{4}$ chord) has only its bass in common with the overtone series of that bass, but the overtone series of its fourth degree note has third harmonic in agreement.



A chord in first inversion (a 6 chord), however, has third harmonic of the bass agreeing with fifth harmonic of the third, while the fifth harmonic of the bass clashes strongly.



Out of the relative agreement or disagreement of the harmonics of the lowest and next-to-lowest notes of a chord follows the impression it makes on the musical ear: The $^{6}_{4}$ chord is one of mild suspense, and is typically struck by the orchestra just before an instrumental soloist launches into a cadenza. The 6 chord's more pungent character is appropriate to interrupting the dramatic action of an opera or oratorio, as typically struck by the harpsichord before a singer declaims a short speech or sings a short recitative.

One other innovation of the early 17th century must be mentioned before we can proceed with further discussion of musical intervals and scales: the creation of tables of logarithms, again apparently independently by two different people at nearly the same time. In Edinburgh, 1614, Sir John Napier, Lord of Merchiston, published his Mirifici Logarithmorum Canonis Descriptio, showing how problems of multiplication and division could be reduced to addition and subtraction, and those of powers and roots to multiplication and division, all by means of certain "artificial numbers," exponents to a base $e = 2.718\cdots$ possessing many wonderful properties. Meanwhile, on the continent, a Swiss mathematician Jobst Bürgi in charge of the little astronomical observatory of the Margrave of Hesse developed equivalent tables of what he called "red and black numbers" published in his Arithmetische und Geometrische Progress at Prague in 1620; these were to a base near 1 (1.0001). The Londoner Henry Briggs advanced the theory of Napierian logarithms, but proposed use of base 10 for common reckoning, while Kepler, in Prague, invented the familiar notation of "putting out" (whence the name exponent) the red numbers in small print to the upper right hand corner of the base to express the same sense or value (λόγος) as the other numbers (αρίθμοι) which Bürgi printed in black. Briefly, if $a = b+c$ or $b\times c$, then we can solve for b as easily as for c by subtraction or division, respectively; but if $a = b^c$, then we must extract b as $\sqrt[c]{a}$ (c^{th} root of a) but express c as $\log_b a$ (logarithm base b of a). To the base 2, for example, the product $4\times8 = 32$ becomes $2^2\times2^3 = 2^{2+3} = 2^5$, a sum $2 + 3 = 5$, where the 2, 3, and 5 are logarithms base 2 of 4, 8, and 32, respectively. The difficulty, of course, lay in establishing the values of powers with fractional exponents; $2^{.5} = 2^{\frac{1}{2}} = \sqrt{2}$ since $\sqrt{2}\times\sqrt{2} = 2^{\frac{1}{2}}\times2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$, but $2^{.49}$ and $2^{.51}$? What should they equal? This was the accomplishment of Napier and Bürgi!

Without logarithms, Kepler could not have computed the note-values of the planets' daily angular motions in his "motet." As a sample computation, using the commonly available logarithms base 10, let us see how close Mercury's daily motion at aphelion comes to being proportional to a C# (Kepler says it is only approximate — cf. the upper chart on p. 43) if Saturn's daily motion at aphelion is taken as a G: A C# should be 6 equal-tempered chromatic steps (twelfths of an octave) up from a G. 1'46" = 1.7666···' and 164'0" = 164.000···'. The former is between $1 = 10^0$ and $10 = 10^1$, so its logarithm base 10 must be between 0 and 1; similarly, the latter is between $100 = 10^2$ and $1000 = 10^3$, so its logarithm base 10 must be between 2 and 3. In fact (to 4 place accuracy, by table look-up), $\log_{10}(164.000/1.7667) = \log_{10}164.000 - \log_{10}1.7667 = 2.2148 - 0.2472 = 1.9676$, so $\log_2$ of $164.000/1.7667 = 1.9676/\log_{10}2 = 1.9676/0.3010 = 6.5369$. The integer part of this, 6, tells us that the "pitch" of Mercury is six octaves (six doublings) above that of Saturn, while its fractional part 0.5369 tells us that it is 53.69% of the way up the 12 chromatic steps of the next octave, and $0.5369\times12 = 6.4428$, so we recognize that the tone in question is rather high, nearly half way (6.5) between 6 and 7 steps above a G, i.e. nearly half way between a C# and a D. In general, to convert any ratio (of vibration rates) p/q to the number of chromatic steps (half-steps, semitones, in equal 12-tone temperament) of the corresponding musical interval, divide $\log p - \log q$ by $\log 2$ (to whatever base is available) and multiply by 12; or, in case the cents system is preferred, multiply by 1200, taking 100 cents = 1 chromatic step (twelfth of an octave). The G to C# interval computed above would measure 644.28 cents, 44.28 cents higher than an equal-tempered tritone of 600 cents.

We can now address the problem, raised on p. 22, of how the octave, both in European and Chinese music, came to be divided into 12 parts, and why this 12 is naturally partitioned as 7 + 5. Taking any given pitch as a fundamental, vibrating at a unit rate, the 2nd harmonic will simply be an octave of the fundamental, but the 3rd harmonic will be a new pitch. If we keep on cubing, taking 3rd harmonics of 3rd harmonics, how many new pitches do we generate? Do we ever return to some higher octave of the fundamental? Numerically, this is equivalent to asking whether there exist integers m and n such that $2^m = 3^n$, to which the answer must clearly be no, since the former will forever remain even and the latter odd. Failing that, we ask whether there are integers m and n such that $2^m \approx 3^n$, or that $1 \approx 3^n/2^m$, or $2^{m-n} = (2/1)^{m-n} \approx 3^n/2^n = (3/2)^n$, i.e. such that m-n musical octaves approximate n musical fifths (ratios 2/1 and 3/2). Yes, $m = 19$ and $n = 12$ yields 7 octaves approximately equal to 12 fifths, so one fifth must be approximately equal to 7/12 of an octave; i.e. the ratio $3/2 \approx 2^{7/12}$ (in fact, $3/2 = 1.5 > 2^{7/12} \approx 1.4983$, larger by almost 2 cents, so fifths must be tuned slightly flat on a modern piano and octaves slightly sharp). The complementary interval of a fourth is thus $4/3 \approx 2^{5/12}$, and the octave as fifth + fourth is $2^1 = 2^{7/12+5/12}$. Musical addition of intervals is at the exponential level; the human ear hears not the physical vibration rates themselves but their logarithms!

Supposing neither the musical fifth or fourth were considered of primary harmonic importance, but the major third ... what equal-tempered division of the octave would approximate the ratio 5/4? This is equivalent to asking for a rational approximation of the logarithm of 5/4 base 2, or $0.321928\cdots$, and the theory of finding rational approximations to irrational numbers is well understood. One expresses the given irrational number as a so-called continued fraction $1/(p+1/(q+1/\cdots))$, and then truncates it to find $1/p$ as first approximation, $1/(p+(1/q)) = 1/((pq+1)/q) = q/(pq+1)$ as second improved approximation, and so forth. The p's and q's etc. are found as follows: Invert the given number $0.321928\cdots$ to find $1/0.321928\cdots = 3.106283\cdots$ and take its integer part $3 = p$, then take the fractional part $0.106283\cdots$ and repeat the process; $1/0.106283\cdots = 9.408778\cdots$, $9 = q$; $1/0.408778\cdots = 2.446310\cdots$, $2 = r$; etc. etc., giving $0.321928\cdots$ expanded as $1/(3+1/(9+1/(2+\cdots)))$. To unravel this expression and find the rational convergents $1/p$ etc., we set up the simple computational scheme in which for

		3	9	2	...
1	0	1	9	19	...
0	1	3	28	59	...

a number < 1 we take initial entries $\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$, then use the continued fraction entries to fill each row in as follows: $1+0\times3=1$, $0+1\times9=9$, $1+9\times2=19$, etc., and similarly $0+1\times3=3$, $1+3\times9=28$, $3+28\times2=59$, etc. In the case of a major third, this tells us that the 5/4 ratio is musically 1/3 of an octave in first approximation (the 5/4 ratio is 386.31 cents, while $2^{1/3}$ is 400 cents, 13.69 cents too high). More closely, it is 9/28 of an octave (9/28 of 1200 is 385.71 cents, 0.60 cents too low). Still more closely, it is 19/59 of an octave (19/59 of 1200 is 386.44 cents, 0.13 cents too high), and so on, alternately over- and under-approximating the given value ever more closely. (Cf. Schechter [1980].)

Looking at the complementary interval 8/5 (always twice the reciprocal of the original ratio) of a minor sixth, we first compute $\log_2(8/5) = (\log_{10}8 - \log_{10}5)/\log_{10}2 = 0.678071\cdots$, and recognize this as just $1 - 0.321928\cdots$. Then we expand $0.678071\cdots$ as a continued fraction, finding $1/0.678071\cdots = 1.474769\cdots$, $1 = p$; $1/0.474769\cdots = 2.106283\cdots$, $2 = q$; $1/0.106283\cdots = 9.408778\cdots$, $9 = r$; etc., then set up

$$\begin{array}{cc|cccc} & & 1 & 2 & 9 & \cdots \\ \hline 1 & 0 & 1 & 2 & 19 & \cdots \\ 0 & 1 & 1 & 3 & 28 & \cdots \end{array}$$

to find, not surprisingly, just the complements of the ratios we found above.

In this way, we can show that a minor third ratio of 6/5 is extremely accurately approximated by 5/19 of an octave (true interval measures 315.64 cents, whereas 5/19 of 1200 is 315.79 cents, 0.15 cents too high). An equal 19-tone temperament of the octave would be a fairly natural extension of our present 12-tone system, necessitating the addition of one more black key next to each of the five present black keys plus one more between each of the two pairs of white keys not at present separated by a black key: 7 white keys + 12 black keys in all, permitting distinction between D♭ and C#, etc.

We can also work the other way: Suppose we start with the Javanese <u>sléndro</u> scale (see Kunst [1949]), which is approximately equal-tempered pentatonic, and ask what overtone ratio the scale step $2^{1/5}$ might be close to. First find $\log_{10} 2 = 0.301029\cdots$, 1/5 of which is $0.060205\cdots$, so $2^{1/5} = 10^{.060205} = 1.148698\cdots$ (from table), then expand as $1/(1 + 1/(6 + 1/(1 + \cdots)))$ and set up

$$\begin{array}{cc|cccc} & & 1 & 6 & 1 & \cdots \\ \hline 0 & 1 & 1 & 7 & 8 & \cdots \\ 1 & 0 & 1 & 6 & 7 & \cdots \end{array}$$

which tells us that the natural 7th overtone (which western ears hear as a "blues" note "in the cracks" between B♭ and A above a fundamental C) provides a good approximation to such an equal-tempered pentatonic step down to its lower neighbor the sixth harmonic (266.87 cents, as opposed to ideal 1/5 of 1200 = 240), and a better one up to its higher neighbor the eighth harmonic (231.17 cents — only 8.83 cents too low instead of 26.87 cents too high). (Note the change in initial entries $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ for a number > 1.)

Similarly, we can ask what natural overtones are heard in the equal-tempered 7-tone scale sometimes used in Thai music, expanding $2^{1/7} = 1.104089\cdots$ as $1/(1 + 1/(9 + 1/(1 + \cdots)))$ and finding

$$\begin{array}{cc|cccc} & & 1 & 9 & 1 & \cdots \\ \hline 0 & 1 & 1 & 10 & 11 & \cdots \\ 1 & 0 & 1 & 9 & 10 & \cdots \end{array}$$

it approximated by the steps between the 10th overtone and its lower (182.40 cents) and upper (165.00 cents) neighbors, as compared with an ideal 1/7 of 1200 = 171.43 cents (10.97 cents too high and 6.43 cents too low, respectively). This is interesting, since diatonic 7-tone scales in East Asia have, for the most part, a fourth degree that departs sharply from both the equal-tempered ($2^{5/12} = 1.3348\cdots$) and mean-tempered ($4/3 = 1.3333\cdots$) ones, being closer to $11/8 = 1.375$, with Do : Re :

Mi : Fa : Sol ≈ 8 : 9 : 10 : 11 : 12, locally approximating equal 7-tone temperament.

Returning to Kepler's image of the monochord string bent round in a circle, we may ask finally whether there might be some psychological reason why some cultures (Greek and Chinese) with a strong tradition of interest in geometric constructions should find octaves divided into 12 parts musically pleasing, inasmuch as these favor subdivisions into 3, 4, and 6 equal parts, while other cultures (Javan and Thai) without that geometric tradition favor use of the natural 7th and 11th overtones leading to equal 5- and 7-part divisions, recalling that 3-, 4- and 6-gons can tile the plane regularly (are "harmony-forming" in the sense of Kepler's Book II) while 5- and 7-gons cannot. (The music of India would be hard to discuss in this context, since it partakes of both traditions, akin to Southeast Asian cultures religiously and iconographically, but with Greek art and science imposed on it from the time of the Alexandrian invasion; we will not attempt to do so. It should be mentioned, however, that it was exposure to Javanese music at the Paris World Exhibition of 1889 that induced Debussy to experiment with whole-tone scales — cf. p. 116 of Lockspeiser [1978]).

Any answer to such a question would have to deal with matters of comparative culture-based epistemology quite beyond the scope of this thesis, yet it may perhaps be permitted to point in certain directions: Favoring lower vs. higher overtones may correspond to religious emphasis on this vs. higher worlds, preference in dwelling on the (rationally) know- vs. non-knowable.

C. Gauss and Galois

Looking at the cyclotomic polynomials (cf. pp. 13-14) for the regular 7-, 9-, 11-, and 13-gons (discovered, incidentally, by Bürgi), Kepler felt that such equations were unsolvable, could not be factored by ordinary algebraic means, that their roots were "ineffable" — could not even be named, much less constructed by ordinary means. Only the trigonometric solutions in terms of sin(180/7)° etc. seemed to exist, but he could not prove it. That had to wait for others.

One of the others who had to come was Carl Friedrich Gauss, who in 1796 (at the age of 19) decided to become a mathematician when he discovered a proof that the only regular polygons having a prime number of sides which could be constructed with straightedge and compass were divisions of a circle into $2^{2^n}+1$ parts, provided that number is prime. From a purely number-theoretic view, numbers of this form had been studied by Fermat who thought they were prime for all values of n, but Euler showed that $2^{2^5}+1$ is composite.

n	$2^{2^n}+1$
0	3
1	5
2	17
3	257
4	65537
(5	4294967297 = 641×6700417)

Extensive computer sweeps for n into the thousands have been tried, but none > 4 found for which $2^{2^n}+1$ is prime.

To be constructible, any other odd-number-sided polygon must be a product of at most 1st powers of such Fermat primes, such as 3×5 = 15, while the constructible even-sided ones are these times arbitrary powers of 2. Ironically, Kepler held a key to this in the beginning of his IIIrd book when, describing the Pythagorean tetraktys on p. 91 as sum of numbers from 1 to 4, totalling 10, he illustrates a saying of Proclus concerning the quality of the number 10

> "All-embracing Mother, surrounding on all sides,
> Who knowest not of change, untiring, sublime"

by omitting the central number 1 from the usual triangular display, replacing it with a 0, and thus arrives inadvertently at the beginning of what could be construed as the Pascal triangle of binomial coefficients

```
   1
  1 1
 1 0 1
1 1 1 1
```

in which the entries are replaced by 1's when odd, 0's when even (i.e. they are read as members of the finite field of integers "modulo 2," as set forth by Gauss in his <u>Disquitiones arithmeticae</u>).

```
   1
  1 1
 1 2 1
1 3 3 1
```

If entire rows of this modular triangle are then read in base 2 arithmetic as coefficients of successive powers of 2 [as when expanding the binomial $(1+2)^n$ for $n = 0, 1, 2,$ and 3, but all done "mod 2"],

the zeroth row reads $1\cdot2^0 = 1 = 1$

the first row $1\cdot2^0 + 1\cdot2^1 = 1+2 = 3$

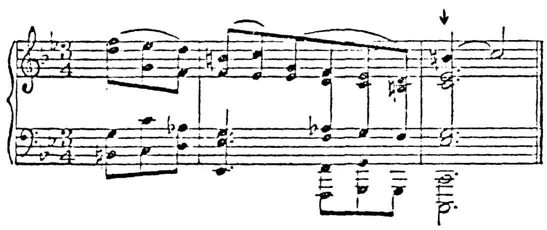
the second $1\cdot2^0 + 0\cdot2^1 + 1\cdot2^2 = 1+0+4 = 5$

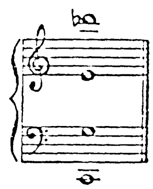
and third $1\cdot2^0 + 1\cdot2^1 + 1\cdot2^2 + 1\cdot2^3 = 1+2+4+8 = 15 = 3\cdot5$,

the fourth $1\cdot2^0 + 0\cdot2^1 + 0\cdot2^2 + 0\cdot2^3 + 1\cdot2^4$ giving $1+0+0+0+16$ which is 17, the next constructible prime, followed by $3\cdot17$ in the fifth row, $5\cdot17$ in the sixth, $3\cdot5\cdot17$ in the seventh, and the next new prime 257 in the eighth, and so on, through 65537 in the sixteenth row and the product of all of these in the 31^st^ row (the 32^nd^ row, as 5^th^ power of 2, yields the 5^th^ Fermat number, which fails to be prime). Rows 0 through 31 of the Pascal triangle, read first mod 2 then base 2, provide an exhaustive list of all knowable harmony-producing numbers, "proper" ones in rows 1, 2, 4, 8, and 16 (powers of 2), "improper" ones elsewhere. How Kepler would have loved this!

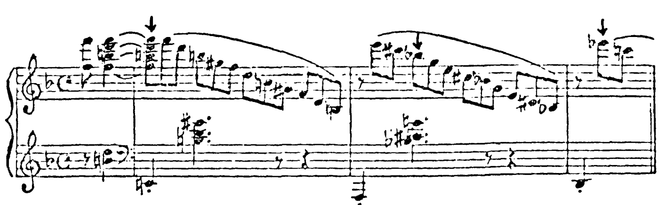
Do these larger constructible primes have musical uses? Kepler freely admitted the leading tone below the 16th note or quadruple octave — why not the leading tone above it? As Coxeter (1962/1968) points out, pp. 317-318,

"The interval fifteen, from a low C to a high B, has thrilled audiences for two hundred years in the unearthly grandeur of the appoggiatura that ends the *St. Matthew Passion*."





"The interval seventeen, from a low C to a high D♭ is not as ugly as we might at first expect, especially if we use the sustaining pedal and hold it long enough for some of the intermediate harmonics to assert themselves, giving the effect of a *minor ninth* chord such as Brahms used in each of three consecutive bars in the development section of the first movement of his *First Sonata for Cello and Piano*. Beethoven used the same interval at least once in the first movement of his *Ninth Symphony*. So perhaps the 'rule of small numbers' is really a 'rule of cyclotomic numbers.'"



(Coxeter uses "cyclotomic" here in the sense of "*constructibly* circle-splitting," not just "circle-splitting.")

In fact, if we expand $2^{1/12}$ by continued fractions as in section B preceding, we find that the first rational approximants to the basis of 12-tone tuning are 17/16 and 18/17 (104.96 and 98.95 cents respectively, 4.96 cents too high and 1.05 cents too low compared to one chromatic step of 100 cents). How Kepler would have loved this, too!

The more general question of what geometric constructions are possible to be carried out with straightedge and compass was settled by Évariste Galois who proved in 1832 (on the eve of his tragic death in a duel at age 21) that only those numbers are Euclideanly constructible which can be expressed by nested square roots†; technically, they must be quantities whose minimal polynomials can be factored over an extension field of degree a power of 2, hence Gauss's result as a special case. The minimal polynomial for heptagon and 13-sided polygon side/radius ratios are of degree 3, and that for the regular 11-gon of degree 5, ruled out by Galois. Likewise ruled out are extractions of cube roots (the Delian problem) and trisection of a general angle, not to mention the famous "squaring of the circle" (finding a square of area equal to that of a given circle) since any expression with π as a root would have to be transcendental.

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†(This is the result to which we referred earlier on p. 31, saying that ordering degrees of "knowability" of certain polygons by the number of layers' depth of certain square roots was an intuitively correct approach on Kepler's part, anticipating Galois.

D. Fourier, Cantor, and the Infinite

The forward progress of history has taken us backward through the Platonic quadrivium from astronomy to music to geometry to the realm of number. The last stage, the foundations of arithmetic in modern analysis, took its inspiration also from the vibrating monochord string and led into study of the infinite.

While the details are too technical to discuss here, it is interesting to note that the founder of harmonic analysis, Jean Baptiste Fourier, was present as a young man on the Napoleonic expedition to Egypt, and witnessed the beauties of decorative geometric patterns on tomb friezes. Like his 20th century counterpart Andreas Speiser (1922/1956), he was moved by this experience to the study of abstract symmetry patterns in mathematics (groups and group characters). In Fourier's case, this led to the expansion of periodic functions in trigonometric series form, which have come to be known as Fourier series in his honor. The essential idea is the same as it was at the time of Ptolemy: the resolution of complicated but periodically repeating phenomena into their simple cyclic components (in this case, circular trigonometric functions). See Mackey (1980) for a historical survey of "harmonic analysis as the exploitation of symmetry."

The name of Georg Cantor is associated with the development of the theory of transfinite cardinals through the English translation of his work on this topic (1915/1955), but even among mathematicians few people are aware of the problems that led him to their study. In a series of five papers written from 1870-1871 (summarized by Dauben [1971]) Cantor set out to prove that if a real function f(x) was represented by a trigonometric series which converged for all x, then the series was necessarily unique — i.e. the coefficients (how much is contributed by each term of the series, corresponding to how large the radius of each little wheel-upon-a-wheel should be to obtain the overall wave-form)are all well-determined.

The difficulty was that some functions have exceptional points where their behavior is momentarily not defined. How many such exceptional points could there be, and still guarantee uniqueness of the Fourier expansion? For finitely many exceptional points, Cantor managed a relatively easy proof in 1870; but it was not until the nature of the problem had forced him to recognize two different sizes or cardinalities of infinite sets that he was able to extend the proof to the infinite case in 1871. If the set of exceptional points was countable (pairable with the set of counting numbers 1, 2, 3, ..., or with any other set pairable with them, such as the set of all rational numbers) then uniqueness still held; if the set was uncountable (e.g. a Cantor set, or set including some interval), then, & only then, uniqueness no longer held. His creation of the distinction between ℵ (the cardinality of the integers) and C (the cardinality of the continuum) aroused considerable anger on the part of some leading mathematicians of his day, notably Kronecker, who considered the proper activity of a mathematician to be investigative, not creative; but others such as Hilbert realized the gain and championed him.

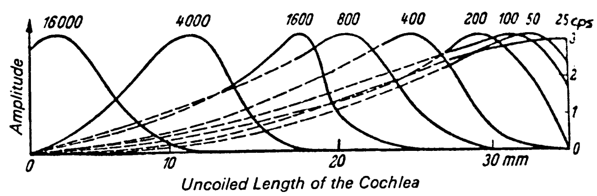
VI. MUSEMATHEMATICS — LISTENING AS LOGARITHMIC ACT

In the "Sirens" chapter of James Joyce's Ulysses (1961), p. 278, we find Leopold Bloom in the bar of the Ormond Hotel, attempting to pen a letter to a secret lady-friend, Martha, while assorted arias are being sung around him. "Grandest number in the whole opera," remarks one by-stander. "It is," agrees Bloom, and continues thinking to himself:

> "Numbers it is. All music when you come to think. Two multiplied by two divided by half is twice one. Vibrations: chords those are. One plus two plus six is seven. Do anything you like with figures juggling. Always find out this equal to that ... Musemathematics. And you think you're listening to the ethereal. But suppose you said it like: Martha, seven times nine minus x is thirty-five thousand. Fall quite flat. It's on account of the sounds it is."

What is being expressed here, humorously, is recognition that music and mathematics are almost universally perceived as being closely related, yet paradoxically music is generally a source of pleasure and mathematics a source of pain. To express the singer's air vibrations in mathematical equations would not win the love of fair lady; they must sound, and be listened to, for musical effect. What happens when musical sounds are listened to?

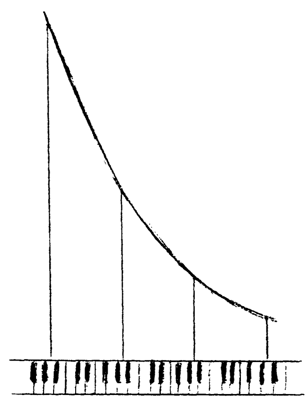
Very little is actually known about the workings of the inner ear, due to its protected location in the body. In Helmholtz' day (1862), it was thought that particular hairs of the basilar membrane in the cochlea were attuned to particular frequencies of sound, viewing the inner ear somewhat in the manner of an Aeolian harp. This is now known not to be true. If the skin of the forearm is exposed to sources of gradually in- and decreasing warmth over several inches, the perception is not of gradually in- and decreasing warmth but of a concentration of that warmth at the place of greatest stimulation, as though the source were localized there. Similarly, the entire basilar membrane of the cochlea is now known to respond to every frequency of in-falling sound, to varying degrees, giving a sense of localization of each sound of definite musical pitch at a particular place of greatest stimulation. Response-amplitudes of sample frequencies from 16000 to 25 cycles per second at upper and lower limits of human hearing are shown here over the ca. 35 mm. length of the cochlea, uncoiled (from Winckel [1967]).



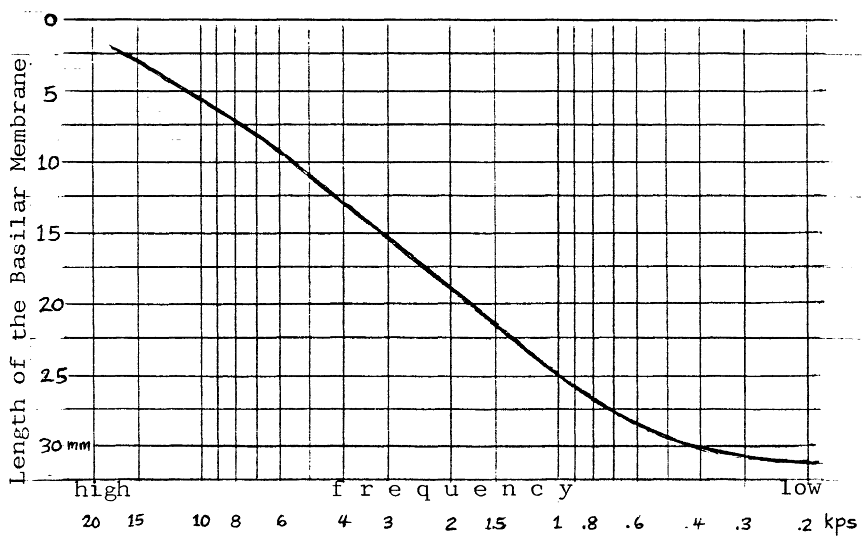
The bulging of the basilar membrane as a function of frequency.

It will be seen from this that the response is not even over this range, does not accord to a single mathematical law; but on closer examination it may be seen to follow two different laws, approximately, over different parts of that range. The peaks at 100, 200, 300 (interpolated), and 400 cps are roughly evenly spaced, as are the peaks at 400, 800, 1600, and 3200 (interpolated). The former are 100 times 1,2,3,4 while the latter are 200 times 2 to the 1st, 2nd, 3rd, or 4th power, i.e. times 2,4,8, or 16. 1,2,3,4 are the l o g a r i t h m s base 2 of 2,4,8,16, hence while the former are said to be l i n e a r l y spaced, the latter are spaced l o g a r i t h m i c a l l y.

Musicians are familiar with logarithmic spacing (if unawares) for they have all seen piano and organ keyboards. Behind the keyboard, lengths of the piano's strings or organ's pipes vary directly as the wave-lengths of the pitches to be sounded (at least in first approximation – in practice string and pipe thicknesses are changed in different registers for reasons of housing and color control, which complicates matters), hence inversely as their frequencies, giving rise to an (approximately) exponential curve followed by the string-ends or pipe-tops. Successive concert A's vibrating at 220, 440, 880, and 1760 cps. are produced by elements one half, quarter, eighth, and sixteenth the length, respectively, of that producing A 110.



Yet on the keyboard, each octave (each frequency-doubling) is the same hand's breadth apart. Instead of multiplying or dividing by 2 for each octave, we move another hand's breadth. The process of passing from spacing according to $2^1, 2^2, 2^3, 2^4$ exponentially to 1, 2, 3, 4 linearly is called t a k i n g l o g a r i t h m s b a s e 2, hence any scale on which exponentially-spaced numbers appear to be linearly spaced is called a logarithmic scale. Piano and organ keyboards are one example, a slide rule is another. On all such scales, once the powers of one base (say 2, corresponding to the comfortable octave reach of a normal hand) have been evenly spaced, so have those of every other (the powers of 3, say, corresponding to octave-fifths A, E', B", F#'", all lie equally within the reach of a hand of Rachmaninoffian dimensions). Since this is the case, it is customary to label logarithmic scales by powers of 10. If we use such a scale to plot the frequencies (in kilocycles, or thousands of cycles, per second) logarithmically against positions of peak responses (in mm.) linearly, we can see graphically how the data from p. 73 fit different laws over different portions of our hearing range. (Adapted from Winckel [1967].)



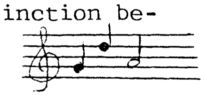
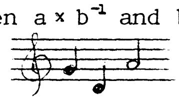
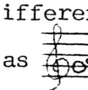
Throughout the mid-range of such a graph, the data plot to form essentially a straight line, indicating that the location of peak response in the cochlea varies as the logarithm of the stimulus. Only below .6 or .5 kps (600 or 500 cps) does the graph-plot become curved here (but would straighten out if linear scales were used on both axes, location in the cochlea then varying with the stimulus).

Because pitch-location in the inner ear follows a logarithmic law throughout most of the range of human hearing, it was grouped together with many other phenomena such as subjective perceptions of strength of other stimuli (e.g. loudness on a decibel scale) as examples of the Weber-Fechner law of sense-perception in the 19th century (perception varies as the logarithm of the stimulus — the stimulus must become physically 2, 4, or 8 times as intense to be perceived psychologically as increasing 1, 2, or 3-fold). Because this is not a perfect law with regard to exact locations of peak responses in the cochlea (failing for low frequencies, and to a lesser extent for high ones, whence the danger of performing low notes too flat and high ones too sharp), and because the analogous laws applied to other senses all rest on difficult-to-quantify subjective impressions, 20th century researchers have tended to de-emphasize them as a class of phenomena, yet they remain realities to the performing artist.

Many professional scientists have some musical background, especially those leaning toward theoretical mathematics, but few professional musicians have mathematical or other scientific backgrounds — this is another paradoxic aspect of the close-relatedness of the two disciplines. One happy exception is Ernest Ansermet, long-time conductor of the Orchèstre de la Suisse Romande, under whose baton many works of Stravinsky were premiered. Besides his musical training, Ansermet was conversant with the language of electrical engineering and acoustics, i.e. with the mathematics of wave phenomena, and in 1961 he attempted to synthesize his musical and mathematical experiences in a 2-volume work entitled Les Fondements de la Musique dans la Conscience Humaine. Like Kepler before him, he is concerned with how the intellect, or sentient soul, gains knowledge of musical intervals, so that in his discussion of the Weber-Fechner law he speaks of "logarithmes noétiques" — logarithms as essential to the auditive act, as instruments of cognition.

The basic properties of logarithms are that they convert

- products to sums, $\log(a\times b) = \log a + \log b$,
- quotients to differences, $\log(a/b) = \log a - \log b$,
- powers to products, $\log(a^b) = (\log a)\times b$ or $b \log a$, and
- roots to quotients, $\log(\sqrt[b]{a}) = \log(a^{1/b}) = (\log a)/b$ or $\frac{1}{b}\log a$.

Accordingly, if a single musical interval is expressed as a ratio of vibration frequencies a/b, then to say that it is perceived logarithmically is to say that it is converted into a difference, $\log a - \log b$, making it akin to more familiar kinds of intervals (e.g. a time interval "from start to finish" is measured as a difference, clock-time-at-finish minus clock-time-at-start). The composition of two intervals, on the other hand, is a product of frequency-ratios, and the logarithm of this is a sum (e.g. a fifth plus a fourth compose as $\frac{3}{2}\times\frac{4}{3} = \frac{4}{2}$ or $\frac{2}{1}$, an octave, which if we take $\log_2(\frac{3}{2}) \approx \frac{7}{12}$ and $\log_2(\frac{4}{3}) \approx \frac{5}{12}$ becomes $2^{7/12}\times2^{5/12} = 2^{7/12+5/12} = 2^1$ as we saw on p. 62 — a sum of exponents, i.e. logarithms base 2), unless the first interval is taken to be rising and the second to be falling (e.g. $\frac{3}{2} \times [\frac{4}{3}]^{-1} = \frac{3}{2} / \frac{4}{3} = \frac{9}{8}$, a major second) in which case we again have a logarithm of a quotient converted to a difference of logarithms ($\frac{7}{12} - \frac{5}{12} = \frac{1}{6}$, where $2^{1/6} \approx \frac{9}{8}$), giving us an intervallic expression for the net rise in frequency similar to expressions for such things as profit = income - expense. (Notice that in the former case the commutivity of a/b as $a \times \frac{1}{b} = \frac{1}{b} \times a$ is without interpretation since the vibrations with this frequency-ratio are taking place simultaneously, whereas in the latter case the distinction between $a \times b^{-1}$ and $b^{-1} \times a$ has melodic meaning, e.g. as  and , moving from G to A by different melodic routes rather than sounding them simultaneously as .) Finally, if only the difference in amplitude or volume is of concern, then by the Weber-Fechner law this is a physical ratio, say v/w, perceived psychologically as log v - log w, a difference of logarithms.

These are the three aspects of music as ancient art of the Muses, in modern mathematical guise: The first, intervallic harmony proper, requires use of logarithms to translate into something like a spatial or temporal interval, for while it lives in space as ratio of periods or wave-lengths and reciprocally in time as ratio of frequencies, it lives also in the tone colors of individual instruments, and personalities of human voices, enabling them to be recognized even when played back on a recording — this is the λόγος of music. The second, melody, obviously lives in time as μήλος, while the third, dynamic crescendo and decrescendo, imitates the spatial advance and retreat of the χόρος, or gives rhythmic pulses to their dance.

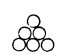
Fourier analysis may be said to concern itself with the first and third of these aspects, as it resolves the complicated motion of the ear drum (or loudspeaker or ambient air) into a sum of cyclic components $a_1 \sin b_1 t + a_2 \sin b_2 t + a_3 \sin b_3 t + \cdots$ of amplitude a_i and frequency $b_i/2\pi$, picking out the individual voices in a chord and noting their relative strengths (something possible only with aural colors, not visual – we have no sense comparable to a spectrograph to resolve light mixtures into constituent parts).

This brings us to the difference between the physical process of addition, whether of finitely many voices in a chordal harmony or of infinitely many in a tonal color, and the psychologically converted process of addition of logarithms of physical factors, whether denoting volume or pitch. There is no way mathematically to simplify $\log(a \pm b)$ – addition is the simplest process of combination – nor is there any way to pass mathematically from the logarithm of the sine of some quantity to the logarithm of that quantity, since $\sin b = \frac{e^{ib} - e^{-ib}}{2i}$ (where $e = 2.718\cdots$ is the base of so-called "natural logarithms" and $i = \sqrt{-1}$ the unit of "imaginary numbers") so that the logarithm of a sine is again a logarithm of a sum or difference, not admitting any further simplification.

We noted on p. 24 that Kepler rejected Pythagorean partitions such as $1 + 2 + 3 = 6$ and $1 + 2 + 3 + 4 = 10$ as irrelevant to the study of harmony since, for this, not sums or differences but products and quotients – ratios – were of the essence, noting also that his use of commas rather than +'s or ×'s between terms or factors tended to obscure this distinction. We now realize the deeper significance of the distinction between sums or differences _per_ _se_ and products or ratios converted logarithmically into sums or differences: _When confronted with a physical sound wave_, mathematically of form $\sum_{i=1}^{\infty} a_i \sin b_i t$, _the inner ear may well convert frequency ratios_ $(b_i/2\pi)/(b_j/2\pi) = b_i/b_j$ _to differences_ $\log b_i - \log b_j$ _and amplitude ratios_ a_i/a_j _to differences_ $\log a_i - \log a_j$ _in its analysis of intervals between pitch- and volume-levels_, thus converting these melodic and choric or dynamic aspects of music into forms analogous to interval relationships in time and space. _But the tone-color aspect of wave addition is already logically a sum, thus neither requiring nor admitting further logarithmic conversion._

VII. CONCLUSION

The qualitative number theory of the Pythagoreans was of two kinds, multiplicative and additive. They approached harmonic and melodic intervals in music multiplicatively as proportions of lengths of strings (under equal tension) or pipes (of same diameter), viewed as ideal lines in space which could be split into any number of equal parts. According to Plato, which of such proportions produced consonances and which did not was for the ear alone to decide. Only when Kepler made the imaginative leap to bend straight monochord strings around into zodiacal circles some $1\frac{1}{2}$ to 2 thousand years later did the splitting of circles — cyclotomy — introduce a basis on which to classify harmonies (equating the sensation of consonance to the soul with constructability by compass and straightedge and knowability to the intellect), culminating in the work of Galois.

The additive theory of partitions which the Pythagoreans studied in the form of figurate numbers via close-packings of circles in the plane (such as ○ for 1,  for 1 + 2,  for 1 + 2 + 3, etc.) had no immediate application to music. When Kepler extended the study of tilings as a "social" aspect of harmony, seeing which regular polygons could cooperate with which others to fill a surface and which could not, a point of view was introduced from which one could consider harmonic vibration-states of the plane, as experimented with subsequently by Chladni and Jenny. But these, too, amounted to regular divisions, albeit with certain interesting restrictions (only 12 polygons were found to be fully social or harmony-forming, on plane or sphere).

Truly additive phenomena in the theory of harmony were not recognized until the early 18th century, when Rameau intuitively postulated and Sauveur experimentally demonstrated the existence of partial vibration states capable of simultaneous support and responsible for our perceptions of tone color. To recognize them, the classical spatial view of vibration in terms of wave length had to be complemented by the Renaissance temporal view in terms of frequency, much as Kepler felt the need to enliven his earlier static model of the solar system to a dynamic one that "sang" by virtue of relative speeds. One and the same string or pipe, of fixed length, can be subdivided at different frequencies over different partial lengths, and the overtones produced in this manner combined additively to yield that musical quality which e.g. distinguishes the sound of flutes from that of oboes or clarinets (by favoring 1st, 2nd, or 3rd partials) and makes the performance of violins from some manufacturers preferable to that of others.

Having arrived at this distinction between the group aspect of harmony, produced multiplicatively as ratios or proportions (though heard logarithmically as differences or intervals), and the individual coloristic aspect given additively as cumulative effect of simultaneous partials, we may brave an answer to the first and hardest of the three riddles posed in the introduction, the spiritual one concerning understanding harmony as issue of beauty and war: Wars are commonly waged as boundary disputes over contested lands; each party wants peace on its own terms, and if the sundry parties manage to unite at all, then it is only to form a partition, mathematically a disjoint union. Such is the nature of additive thought, attempting to work from the parts to the whole. Beauty, on the other hand, is perceived when each part bears a fitting relationship to the whole, when the factors contribute to produce something jointly, and rationality prevails, working from the whole to the parts with some over-all design in mind. It is fruitful and multiplies. In the color quality of what is truly additive in our perception of music we recognize characteristics of individuals; in the intervallic blend of what is multiplicative we recognize what makes group cooperation possible. In the balance between these two (mythologically: what issues from their interaction) we recognize harmony, finally, as the challenge to free individuals to unite cooperatively in the formation of a society — a philharmonic society — out of love for the commonweal.

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