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Dividing a pizza into equal parts – an easy job?

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Abstract *Theoretically seen dividing a pizza equally is not an easy task. For instance, with a normal knife (straight cuts) one has to hit the center so that the cut is a diameter. But there are alternatives (also for dividing equally between more than two persons) which have strong connections to elementary geometry and to integral calculus. This paper deals with these alternatives elucidating the so called "pizza theorem".*

Strictly speaking dividing a pizza into equal parts is not as easy as it may seem at first glance. Even if it is to be shared only between two people and the pizza is circularly shaped. After all, one has to hit the center so that the cutting line is the precise diameter. Cutting the pizza into roughly equal pieces will not be a problem at all in real life. There will normally be no conflict over who gets which piece. But what if the pieces are to be completely exact? Of course, such considerations are more theoretical than practical in nature, but they may provide useful mathematical and didactical input for teaching mathematics at different levels. In fact, in mathematics important questions are not always practical, but in some cases only theoretical.

1 The phenomenon of equally dividing a pizza ("pizza theorem")

There is a possibility using a pizza knife consisting of four straight blades with "center" P (P divides every blade into two parts, adjacent blades always have an angle of 45°) to carry out in reality a division of a pizza that is also theoretically exact. One can imagine this knife as a special cutter (the center P is put anywhere on the pizza) pressed with power onto the circular pizza so that afterwards there will be eight pieces (Fig. 1). The shape of these pieces is very similar to sectors of a circle but they are not really ones (except in the case that P hits the center of the circle), however we will call them "sectors" for simplicity reasons.

If the first person takes every second "sector" (e. g. the white ones in Fig. 1) and the second person the remaining ones (grey in the figure) then the pizza has been equally divided! A realization of the division process with a real pizza and a specially prepared "pizza knife" (cutter) can be seen in Fig. 2.

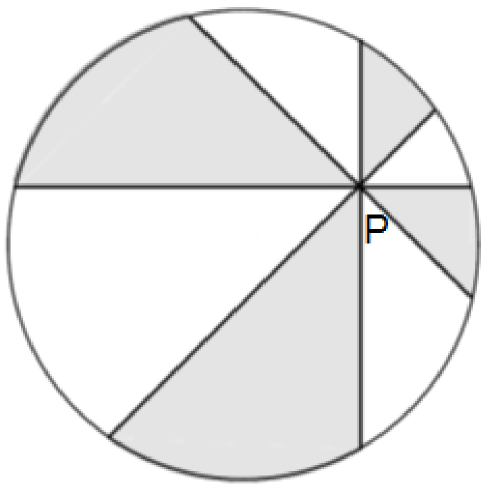

Fig. 1: Cutter on a pizza – schematically

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Fig. 2a: Cutter on a pizza – in reality

Fig. 2b: Pizza after division

This is probably surprising and does not fit in some sense to the symmetry conditions of the circle, hardly anybody would guess this intuitively. On the contrary, formulated as a question most people would negate that this procedure leads to a really equal (also theoretically) division. Nowadays with DGS (“Dynamic Geometry Software”) one can see the corresponding phenomena: Every DGS can measure areas and add up these measures (the areas are not real sectors but they can be separated in triangles and circular segments and these types of areas can easily be measured). One could produce a corresponding applet that works in that way (when moving P the area sums of the grey and white “sectors” respectively remain unchanged). Using hands on methods one can see this by cutting a circular piece of carton in the described way and weighing all the white and grey pieces at a time.

Remark: Kroll/Jäger 2010 propose to formulate this problem for teaching purposes as a story of distribution of an estate. I don’t think that this is a good idea because this formulation leads away from the mathematical problem and sounds rather artificially (treetops as circles etc.). On page 101 one can read (translated): “Nobody should be bothered by the fact that the problem is not realistic; students are not either.” Even if this should be correct (I really doubt that!) I consider this as quite problematic, formulations like this can easily lead to wrong and in some way “dangerous” beliefs about applied mathematics.

The problem of finding corresponding proofs – meant as a problem to be solved individually by students at school – is surely too difficult. But nevertheless this phenomenon has something fascinating and motivating to deal with it more intensively. What could a concrete teaching unit look like in which this phenomenon is dealt with? In which grade could this be done? Below we give answers to these questions.

Firstly we give mathematical analyses of our topic in two ways, on the one hand using elementary geometry and calculus on the other. Hereby we also deal with some references.

2 Elementary geometry

2.1 Presentations following S. Wagon and others

The following reference is somehow striking (at least if one is not familiar with this phenomenon). In the very short article of Carter/Wagon 1994 (1/2 page!) one can see a "proof without words", just a picture (Fig. 3), no explanations.

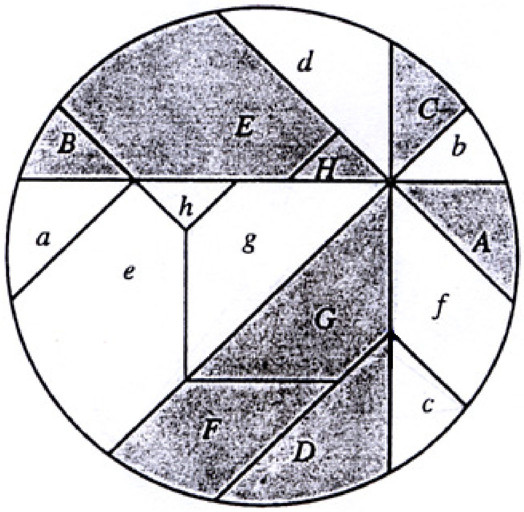

Fig. 3: Dissection following Carter and Wagon 1994

This picture shows (autonomously one would hardly come to similar results) that there seems to be not only equality of area sums but also that the grey and white areas can be dissected into pairwise congruent figures (equidecomposability). That means using the same "puzzle pieces" one can build up the white and the grey area. This is even more surprising than the mere equality of the area sums. The above mentioned word "striking" refers to two aspects: (1) The equidecomposability, in other words the possibility of puzzle pieces. (2) Is the corresponding proof really so easy that it is not necessary to use a single word of explanation?

Using corresponding letters (capital letters and small ones) for the areas Carter/Wagon indicate that the corresponding areas are congruent. But interpreting the picture itself (ideas behind, thoughts, reasoning, etc.) is not so easy. How can we find words? What happens in Fig. 3? How does the dissection emerge, how is the figure built up (construction)? Which lines are presumed to be parallel or equally long? Which angles are presumed? In the end: *why* are areas with corresponding letters congruent? The congruencies $A \cong a, B \cong b, C \cong c, D \cong d$ are easy to explain: In these cases probably a reflection took place on the horizontal and vertical diameter (not drawn in the figure); g is the reflection of G (with the common border as the axis of symmetry). But why do the other congruencies hold $(E \cong e, F \cong f, H \cong h)$? Relations that need not to be communicated in a mathematics journal ("proof without words") are from another point of view not self-evident. There is still a wide range of possible interpretations, how can the structure of reasoning for the congruencies be built up? The picture itself does not say how the authors thought, how the figure arose in their mind. Before I got to know other figures concerning our topic (see below) I gave this problem to student teachers in a geometry course: analyzing the above "proof without words". Some of the students came to important (partial) results but nobody could give a really correct and consistent analysis.

Such an analysis can be given in several ways (focusing either on the idea of congruence or on geometric transformations) but it is not easy for student teachers to establish detailed and precise reasoning. Viewers of Fig. 3 mostly don't see arguments for the congruencies $E \cong e, F \cong f, H \cong h$ at a glance, so the title "proof without words" may seem to be not so appropriate (even for mathematics students at university).

A very similar version of the above Carter/Wagon dissection is given in Kohnhauser/Velleman/Wagon 1996, p. 118 (Fig. 4), the pieces h and H are half as big as in Fig. 3. But the crucial difference to Fig. 3 is given by the fact that in Fig. 4 there is a highly symmetric octagon *PQRSTUV* indicated which is very helpful for establishing a possible proof. The "genesis" of this octagon can be thought like this: The rectangle *PSTW* (its center is the center *M* of the circle) is rotated by 90° (center *M*) and the rotated points *Q*, *R*, *U*, *V* must lie on the “45° lines” through *P* and *W* (why?). The octagon *PQRSTUVW* is mapped onto itself under the 90° rotation with center *M*, and this can be a crucial hint when looking for some reasons for the mentioned congruencies. In my opinion it is more justified to call Fig. 4 a “proof without words” than Fig. 3 (here the octagon is missing and therefore viewers have no hint concerning the important 90° rotational symmetry).

The congruencies $A \cong a, B \cong b, C \cong c, D \cong d$, $G \cong g$ can be explained in a similar way as above (reflections). For the other congruencies one could argue using the highly symmetric octagon as follows: Because of the octagon symmetry (e. g. the sloped diagonals are of course “45° lines”) we can reason for $H \cong h$ (isosceles and rectangular triangles with the shorter octagon side as hypotenuse; in Fig. 3 the congruence $H \cong h$ is also apparent but how should one prove it?) The congruence $F \cong f$ could be proved like this: When rotating the octagon and *f* by −90° (rotation center *M*) the octagon maps onto itself and *f* maps onto *F* (why?), and therefore *f* and *F* are congruent (analogous: $E \cong e$). Due to the octagon one probably sees more easily (and can give reasons for it) that $f \mapsto F$ and $e \mapsto E$ under the mentioned rotation. Using Fig. 4 I have no experiences up to now how successful mathematics students are in explaining this “proof without words” but I suppose the success rate is higher than using Fig. 3.

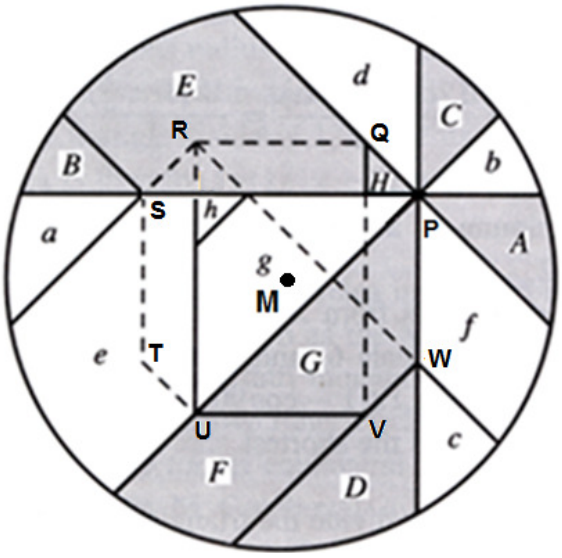

Fig. 4: Dissection – including an octagon

2.2 Presentations following P. Gallin

In Gallin 2011, p. 12 one can find a striking and very simple proof in which very many parts are needed and therefore this proof may seem a bit confusing at first glance. But one should not be “scared” by the huge number of pieces although it may take a while to fully understand the simplicity and brilliancy of the ideas behind. The corresponding figure (Fig. 5) could really be a “proof without words” even from the perspective of students, e.g. with the following text: *P* and the four “blades” are reflected on the point *M*, on the two coordinate axes, and on the angle bisectors of the coordinate system so that the reflected points yield an octagon (drawn thickly[2]). Now the outer and the inner part of this octagon may be considered separately and the equality of the area sums of the colored and white pieces respectively can be seen almost directly (even the dissection into pairwise congruent pieces).

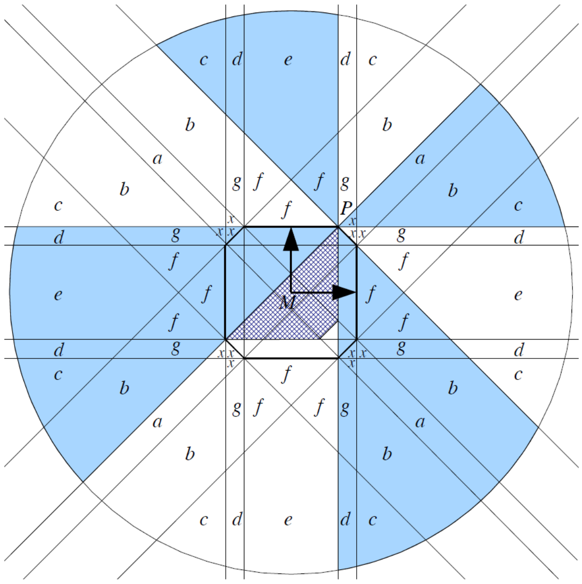

Fig. 5: Dissection following P. Gallin

[2] Due to these reflections a very high degree of symmetry is established – this we have neither in the initial situation nor in Fig. 3 and 4.

"Solution": Outer parts: Due to symmetry reasons (these need not be explained further in this situation) one finds congruent pieces in the colored and in the white area: In each area 2 pieces *a* and *e*; 4 pieces *b*, *c*, *d*, and *g*; 6 pieces *f* and *x*. Hence outside of the octagon the area equality is clear. Inner parts: The hatched trapezoid belongs to the white area and is congruent to the adjacent colored one and also the other parts within the octagon are easily recognized as consisting of congruent pieces (one mall trapezoid and four small triangles).

The octagon of Fig. 5 is in particular the same as in Fig. 4. The ingenious idea of P. Gallin is to do the reflections not only with *P* but also with the four blades of the cutter. This leads to a highly symmetric configuration in a situation that is in its origin (Fig. 1) not symmetric at all, Gallin has reached the highest possible level of symmetry by his method. The prize for that is a huge number of lines and area pieces but one can interpret the resulting figure in an easy way due to its symmetry. The congruencies of all the area pieces *a* (or *b* etc.) become completely evident, further reasoning for them is not needed. This feeling one does not have looking on Fig. 3 (and still in a weaker form with Fig. 4).

In the paper of P. Gallin (2011, p. 14f) there is also another proof (with elementary geometry) for the phenomenon of the equal area sums.

By the way, not only the area is divided equally also the boundary of the pizza, nobody needs to eat more of the perhaps pretty dry boundary which is often not so in favor because it tastes sometimes like ordinary bread. Also for this phenomenon (pizza boundary, arc length) there is a elementary proof (cf. Gallin 2011, p. 13f) that the sum of the four arc lengths of the colored pieces is equal with the corresponding sum of the white ones.

One can easily show that – having chords with a constant angle α – the sum of the arc lengths is also constant (in particular independent of the position of the intersection point *P* and the special position of the chords). The sum of the arc lengths only depends on the angle α (cf. Fig. 6a): $\left|\widehat{A_1C_1}\right| + \left|\widehat{B_1D_1}\right| = \left|\widehat{A_2C_2}\right| + \left|\widehat{B_2D_2}\right|$.

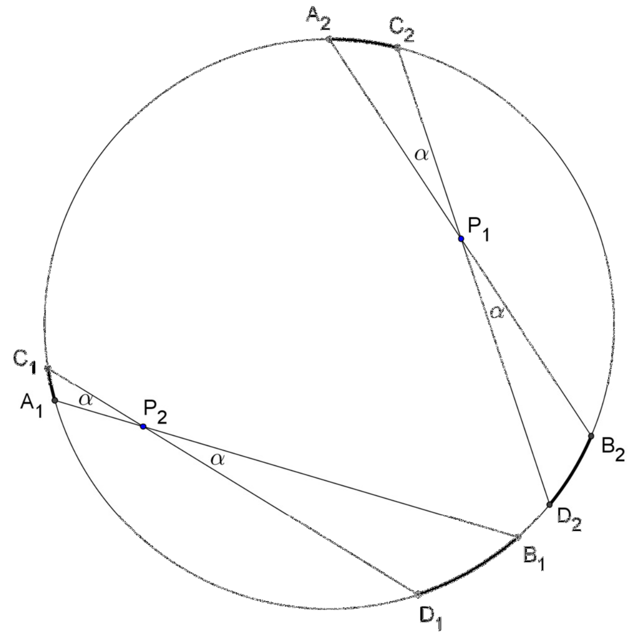

Fig. 6a: Equal sum of arc lengths

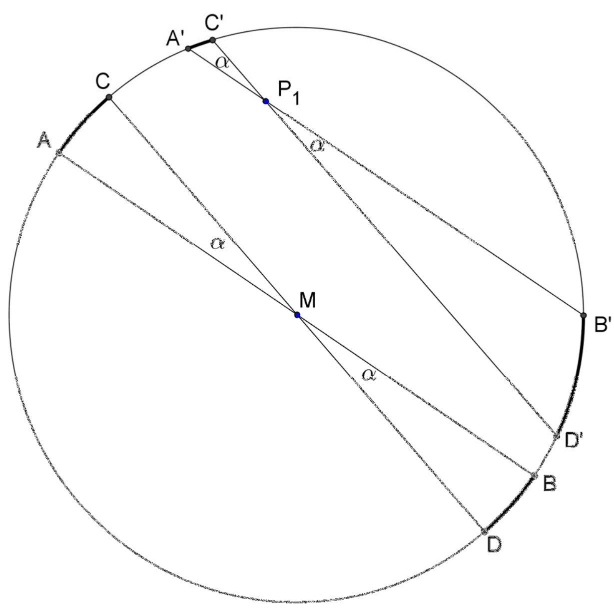

Fig. 6b: Pairwise parallel chords

A crucial idea is here: It suffices to show this for $P_2 = M$ and pairwise parallel chords (see Fig. 6b): $\left|\widehat{A'C'}\right| + \left|\widehat{B'D'}\right| = \left|\widehat{AC}\right| + \left|\widehat{BD}\right|$. It will turn out that the lengthening and the shortening in the transformations $\left|\widehat{AC}\right| \to \left|\widehat{A'C'}\right|$ and $\left|\widehat{BD}\right| \to \left|\widehat{B'D'}\right|$ are cancelling each other, and therefore in the end we have no change in the sum of the arc lengths. We have:

$$\left|\widehat{A'C'}\right| = \left|\widehat{AC}\right| + \left|\widehat{CC'}\right| - \left|\widehat{AA'}\right|$$

$$\left|\widehat{B'D'}\right| = \left|\widehat{BD}\right| + \left|\widehat{BB'}\right| - \left|\widehat{DD'}\right|$$

and because of $\left|\widehat{AA'}\right| = \left|\widehat{BB'}\right|$ and $\left|\widehat{CC'}\right| = \left|\widehat{DD'}\right|$ (this is clear due to the symmetry of the circle) we get by summation: $\left|\widehat{A'C'}\right| + \left|\widehat{B'D'}\right| = \left|\widehat{AC}\right| + \left|\widehat{BD}\right|$. Also the value of the sum is easy to see in the case that both chords pass through the center *M*, the value is $2\alpha r$ (α in radian measure).

Remarks:

- Considering the *area* one needs for constancy *all four* parts (sectors), in other words two "double sectors"; considering the *arc lengths* one needs for constancy only *two sectors* (one "double sector").
- An alternative way of reasoning we give below with calculus.

2.3 An important lemma of elementary geometry and a plausibility consideration

In the focus of the considerations above we had the phenomenon that the four colored "sectors" and the four white ones can be divided into pairwise congruent pieces (equidecomposability), this is more than the mere equality of the area measures. In the following we concentrate on the equality of the area measures using a completely different approach which is building a bridge to the next chapter (calculus).

Lemma: For every orthogonal pair of chords in a circle (segments *a*, *b*, *c*, *d* – see Fig: 7) following equation holds (*r* is the radius of the circle): $a^2 + b^2 + c^2 + d^2 = (2r)^2$

For this lemma there are several possibilities for proving it, a typical case for *problem solving*. A short and illustrative "proof without words" is given in Nelsen 2004.

This lemma plays an important role on the one hand in the following argumentation with *plausibility* (not a rigid proof) and on the other when we use methods of *integral calculus* (see below).

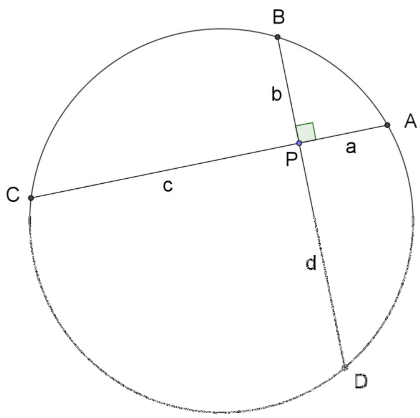

Fig. 7: Orthogonal chords

Plausibility consideration[3]

If two perpendicular blades of a "pizza knife" (intersection point *P*) are rotated (center *P*) by a small angle $\Delta\varphi$ then the area between the initial and the new position (grey in Fig. 8) is approximately

$$A \approx \frac{1}{2}\Delta\varphi\left(a^2+b^2+c^2+d^2\right) \qquad (1)$$

Explanation: We consider each of the four pieces as a real sector of a circle with angle $\Delta\varphi$ and the radii *a*, *b*, *c*, and *d*. The area of a sector with radius *r* and angle $\Delta\varphi$ is given by $b \cdot r/2 = (r\cdot\Delta\varphi)\cdot r/2 = r^2\cdot\Delta\varphi/2$.

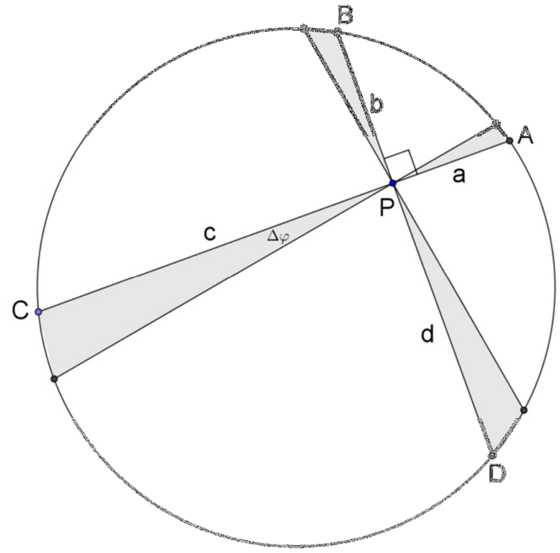

Fig. 8: Rotating further by $\Delta\varphi$

Here in this argumentation with plausibility we don't go into details why the approximate relation (1) also holds exactly (see "integral calculus" below), but one can see immediately: the real sectors with the radii *a* and *d* are a bit too big, the ones with the radii *b* and *c* a bit too small, therefore the approximation (1) will be rather accurate.

According to the above lemma $a^2+b^2+c^2+d^2=(2r)^2$ is constant and hence the *area* of the grey parts (Fig. 8) is *proportional* to the *angle of rotation* because the factor $\Delta\varphi/2$ is independent of the position of *P* and of the initial position of the two perpendicular blades. When we think of this rotation with $\Delta\varphi$ carried out many times we get the same proportionality, therefore there is no need to keep $\Delta\varphi$ small, it will work with any rotation angle φ. This proportionality can and should be affirmed with DGS. Such empirical results and findings with DGS of course cannot replace mathematical proofs but within arguments of plausibility they surely are a kind of affirmation that one is on the right way (if wanted one may look for a more rigid proof afterwards e.g. with integral calculus, see below). Hence the area of two perpendicular "double sectors" (as for instance the grey parts in Fig. 1 with $\varphi=\pi/4$) is $A=(1/2)\cdot\varphi\cdot(2r)^2=2r^2\cdot\varphi$, with $\varphi=\pi/4$ we get finally $A=r^2\pi/2$, that means that the grey parts together make exactly half the circle area.

Due to the mentioned *proportionality* (in some sense this is the mathematical core of this topic) we immediately get the following generalization: If the 90° quadrants are not only divided into two equal[4] parts of 45° – like above – but into say three 30° parts (in sum then we have 12 "sectors") then we may say with the same argumentation: With such a division one can divide a pizza equally between three persons, each person takes every third "sector". Again each person gets, in sum, two perpendicular pairs of opposite "sectors" ("double sectors"), the only difference: with an angle 30° instead of 45° (Fig. 9a: the first person gets the white double sectors, the second person gets the grey ones and the third person the black).

[3] I want to thank my friend and former colleague B. Schuppar (TU Dortmund, Germany) for useful hints.

[4] The "equality" is meant concerning angles.

The same procedure would work if we had n parts in every quadrant (in sum $4n$ parts), we would then have an equal division between n persons.

When we have an **even** number of "sectors" in each quadrant ($2k$, in sum then $8k$ "sectors") with this principle we have a possibility for an equal division of the pizza between $2k$ persons (each person gets 4 "sectors"). If we consider the $2k$ persons consecutively numbered (from 1 to $2k$) and if we think of the union of the "k even persons" and the "k odd persons" respectively then we again have an equal division into two subsets. In other words: If we number all the "sectors" from 1 to $8k$ (clockwise or counter clockwise) the area sum of the even ones is equal to the area sum of the odd ones!

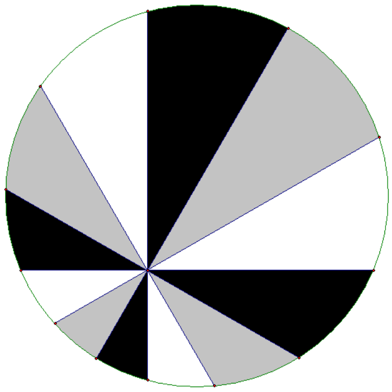

Fig. 9a: Equal division between three persons – three parts in every quadrant

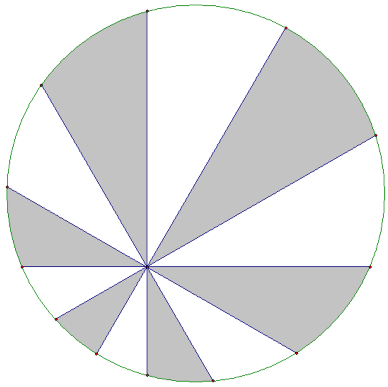

Fig. 9b: Equal division between two persons – three parts in every quadrant

But if every quadrant is divided into three parts (equal angles, see Fig. 9b; generally: an odd number of sectors in each quadrant) then our proof says nothing about the conjecture that still the area sum of the odd "sectors" is equal to the area sum of the even ones. In this case one does not have the easy situation of *pairs of perpendicular "double sectors" with equal color*. This case can be handled with calculus (see below). The phenomenon that the area sum is equal in both cases, adding all the odd "sectors" on the one hand and all the even ones on the other – regardless whether the quadrants are divided in an even or odd number of equal angles – , is called "pizza theorem" in the literature.

3 Calculus

Another possible treatment of this topic is given by an application of the **basic idea of integral calculus: Integrals are limits of product sums** $\sum_i f(x_i)\cdot\Delta x_i$. This useful and important basic concept for integrals – in many contexts – has been described quite often in the didactical literature (see e.g. Blum/Kirsch 1996, p. 62ff). With integrals we not only can calculate areas by using "slim stripes" (this geometric interpretation surely plays a big role but it should not be the only one) but they are useful in many other contexts. Often integral calculus at school is restricted to the application of the fundamental theorem of calculus $\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a)$: Calculating integrals by using antiderivatives. Teachers and school text books often want to come to this result as quickly as possible in order to have plenty of possible calculation problems to solve for the students. The mentioned *theorem* is often even *degraded* to a *definition* like $\int_a^b f(x)\,\mathrm{d}x := F(b)-F(a)$ which in the sense of Hans Freudenthal can be seen as an "*antididactic inversion*". From the perspective of many teachers and school text book authors one may save several troubles and efforts (explaining what an integral *is*) but the prize to pay for that "advantage" seems clearly too high: students never get to know what an *integral* really *is*. A famous formula for the calculation of integrals – rightly named *fundamental theorem*! – is degraded to a *definition* in order to save time and possible troubles! If an explanation is more complicated just make a definition out of it? I think this must not be the kind of teaching that we should aim at!

In the following we also deal with calculating areas, not by slim rectangular stripes but by slim circular sectors, however the principle of product sums is the same. Of course the following cannot be expected as autonomous students' work but has to be explained by the teacher.

Different to Kroll/Jäger 2010 I propose to omit all the formal and technical aspects concerning calculating the area of the particular "sectors" because when adding the four areas one does not need any technical calculations by using the mentioned lemma of elementary geometry. By using integral calculus also the formal gap in the plausibility consideration in 2.3 is closed.

3.1 Reasoning with the Leibniz sector formula

Using the mentioned lemma of elementary geometry $a^2+b^2+c^2+d^2=(2r)^2$ and an application of integral calculus – also known as "Leibniz sector formula" – one can easily prove with calculus that the white and grey "sectors" in Fig. 1 have equal area sums (half the area of the circle).

Many different types of areas can approximately be seen as sums of slim circular sectors especially the above "sectors" that are very similar to real sectors. We are thinking of a fixed center *P* and a "radius ray" with variable length *r* (depending on the angle of rotation φ; "polar coordinates" but this explicit name is not necessary) and we are interested in the covered area of the "sector" that corresponds to the increase of φ by $\Delta\varphi$ (Fig. 10). The sector with radius |*PU*| is too small, the one with radius |*PV*| is too big for the area of the "sector" with the edges *P*, *U*, and *V* (this "sector" indicates the area increment when the corresponding angle increases from φ to $\varphi+\Delta\varphi$).

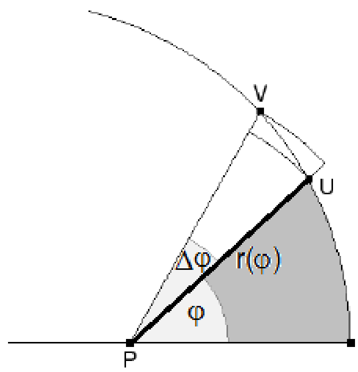

Fig. 10: Leibniz sector formula

For $\alpha\le\varphi\le\beta$ the covered area is divided in slim circular sectors. Adding up all these small sector areas yields a product sum which in the limit becomes an integral:

$\frac{1}{2}\sum_i r^2(\varphi_i)\cdot\Delta\varphi_i \;\rightarrow\; \frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,\mathrm{d}\varphi$ where $[\alpha,\beta]$ is the domain of integration (concerning the associated angle φ, "Leibniz sector formula").

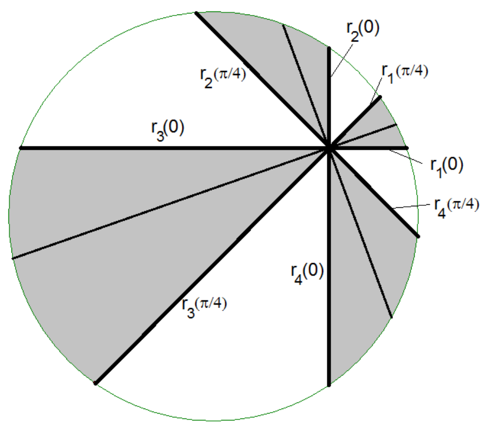

Fig. 11: Leibniz sector formula in the pizza theorem

Applied to a white or grey "sector" in the pizza theorem we get $A=\frac{1}{2}\int_0^{\pi/4} r^2(\varphi)\,\mathrm{d}\varphi$ because the angle in each such "sector" is $\pi/4 = 45°$. We restrict to the grey "sectors" and draw thickly the "initial radii" $r_1(0), r_2(0), r_3(0), r_4(0)$ and radii in the end position – meant after the rotation by 45° – $r_1(\pi/4), r_2(\pi/4), r_3(\pi/4), r_4(\pi/4)$; we also draw a "position in between" of these radii (Fig. 11).

For every angle $0\le\varphi\le\pi/4$ we have: The corresponding radii are perpendicular to each other!

We need not calculate the particular areas of these "sectors" because of the above lemma from elementary geometry with r_1, r_2, r_3, r_4 instead of a,b,c,d: $r_1^2+r_2^2+r_3^2+r_4^2=(2r)^2$. With this we get:

$$A_{\text{grey}} = A_1 + A_2 + A_3 + A_4 = \frac{1}{2}\int_0^{\pi/4} r_1^2(\varphi)\,\mathrm{d}\varphi + \frac{1}{2}\int_0^{\pi/4} r_2^2(\varphi)\,\mathrm{d}\varphi + \frac{1}{2}\int_0^{\pi/4} r_3^2(\varphi)\,\mathrm{d}\varphi + \frac{1}{2}\int_0^{\pi/4} r_4^2(\varphi)\,\mathrm{d}\varphi$$
$$= \frac{1}{2}\int_0^{\pi/4} \underbrace{\left(r_1^2(\varphi)+r_2^2(\varphi)+r_3^2(\varphi)+r_4^2(\varphi)\right)}_{=(2r)^2}\,\mathrm{d}\varphi = \frac{1}{2}\cdot(2r)^2\cdot\frac{\pi}{4} = \frac{r^2\pi}{2} \tag{2}$$

Remarks:

- Hereby the formal gap in 2.3 (plausibility consideration) is closed: Looking at (2) one can see immediately that the approximate formula (1) even holds exactly.

- Analogously here in the analytical perspective we easily see the mathematical core, the above mentioned proportionality: instead of $\pi/4$ this would work with every other angle φ of perpendicular "double sectors", one would get $A=\frac{1}{2}\cdot(2r)^2\cdot\varphi = 2r^2\cdot\varphi$ as above in the plausibility considerations (using elementary geometry) in 2.3.

3.2 The boundaries of the pizza "sectors"

We have mentioned and proved already (with elementary geometry) that not only the pizza area but also the pizza boundary is divided equally by this method, this should be investigated now with calculus. The basic idea behind is the well known phenomenon that the circumference of a circle is the rate of change of the area with respect to the radius. For a proper understanding of the principle of the derivative it is very important to understand the mentioned phenomenon with regards to contents, not only formally by $\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\pi\right)=2r\pi$ (cf. Blum/Kirsch 1996, p. 61, Hefendehl-Hebeker 1998, p. 198f). Analogous considerations yield the insight that the surface area of a sphere is the rate of change of the sphere's volume with respect to the radius. These interesting issues can enrich the mathematics teaching of teacher students and students at school if not the syntax but semantics, contents, and understanding are in the focus of the teaching process.

We have already mentioned that – with arbitrary radii – the area of two "perpendicular double sectors" (as in Fig. 11 the four grey "sectors") is independent of the point's P position and of the position (concerning rotation) of the blades. The area only depends on the opening angle φ of the "sectors" (proportionality). Hence the same holds for the *difference* of two such areas, that is for the area sum $\Delta A = \Delta A_1 + \Delta A_2 + \Delta A_3 + \Delta A_4$ of the four grey parts of the annulus when increasing the radius from r to $r+\Delta r$ (Fig. 12). Since

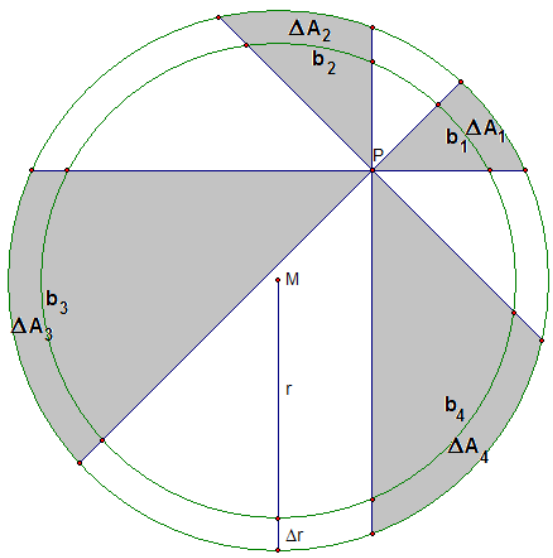

Fig. 12: Equal division of the pizza boundary

$\Delta A \approx (b_1 + b_2 + b_3 + b_4) \cdot \Delta r$ (the annulus everywhere has the same width Δr, hence these parts of the annulus together can approximately be thought of as a rectangle with length $b_1 + b_2 + b_3 + b_4$ and width Δr) does not change under translation or rotation of the cutter (see above) the same holds for $\frac{\Delta A}{\Delta r}$ and in the limit also for $\lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = \frac{\mathrm{d}A}{\mathrm{d}r} = b_1 + b_2 + b_3 + b_4$ (sum of the four arc lengths of the grey "perpendicular double sectors"). That means the sum of the four arc lengths $b_1 + b_2 + b_3 + b_4$ on the pizza boundary is independent of the point's P position and of the rotation position of the blades. This sum itself therefore is proportional to the opening angle φ. By this we have shown in an elementary way that for an opening angle of $\varphi = 45°$ we get exactly half of the circumference of the circle – an equal division of the pizza boundary between two persons (analogous if we have *n* persons and an opening angle $\varphi = 90° / n$ of the "sectors").

3.3 The pizza theorem when each quadrant is divided in an arbitrary number of "sectors" (odd or even)

With the considerations so far we have shown in two ways (using elementary geometry on the one hand and calculus on the other) that the pizza theorem holds in the case of dividing each quadrant in an even number of pieces ("sectors"): The area sum of the grey "sectors" equals the area sum of the white ones (in each case we have half the area of the circle). In the following we will show that this is the case in general (also when we have an odd number of pieces in every quadrant).

One could ask here: Why did we not use this method (general case) from the very beginning? The answer is easy: The mathematical core of the above reasoning (the mentioned proportionality) is very important for properly understanding the phenomenon, but this proportionality does not appear in the following considerations. A second reason could be that in the one or another situation the general case may be not so interesting, that one wants to reduce complexity and deal only with our initial special case of four cutting blades. For this case we wanted to provide possibilities.

We have to prove that the sum of all the areas of the odd "sectors" is exactly half of the circle area. We will have to calculate sector areas by corresponding integrals (in the above version we actually did not do real calculations). We will not determine areas of single "sectors" but of two opposite ones, so called "double sectors" (see Kroll/Jäger 2010).

First we have to deal with two important items that we will need:

- $\sum_{k=0}^{m-1} \sin\left((2k+1)\cdot\frac{\pi}{m}\right) = 0$ and $\sum_{k=0}^{m-1} \sin\left(2k\cdot\frac{\pi}{m}\right) = 0$ (3)

  Proof: If we write $\frac{2\pi}{2m}$ instead of $\frac{\pi}{m}$ these trigonometric equations become quite clear: $\frac{2\pi}{2m}$ is the central angle of a regular (2*m*)-gon (*even* number of vertices!), e.g. drawn in the unit circle with one vertex at (1|0) and the opposite one at (−1|0) (Fig. 13, regular octagon). $(2k+1)\cdot\frac{\pi}{m}$ are the odd multiples of this central angle and the corresponding vertices of the regular (2*m*)-gon lie symmetrically with respect to the x-axis. Thus in sum the corresponding sin values cancel each other (analogous in the case of $2k\cdot\frac{\pi}{m}$, the even multiples of the central angle).

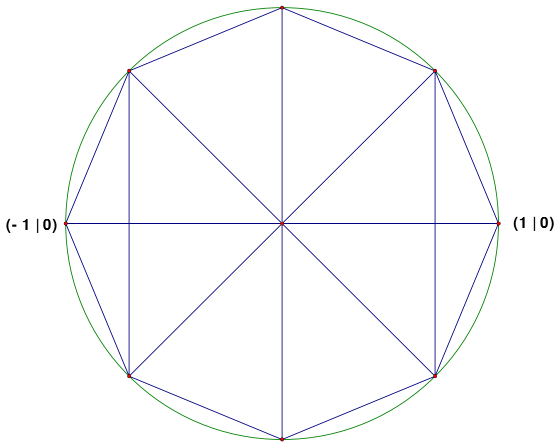

Fig. 13: Regular polygon with an even number of vertices

- Let $r_1(\varphi) := |PS|$ and $r_1'(\varphi) := |PR|$ be the distances from P to the perimeter of the circle (Fig. 14). One can read off immediately $f := |QP| = e\cdot\cos\varphi$ and $h := |MQ| = e\cdot\sin\varphi$. Because of symmetry reasons we have $c := |QS| = |QR|$ and with Pythagoras we get $c^2 = r^2 - h^2 = r^2 - e^2\cdot\sin^2(\varphi)$. Due to $r_1' = c + f$ and $r_1 = c - f$ we can write

$$(r_1')^2 + r_1^2 = 2\left(c^2 + f^2\right)$$
$$= 2\left(r^2 - e^2\cdot\sin^2(\varphi) + e^2\cdot\cos^2(\varphi)\right)$$

and finally: $(r_1')^2 + r_1^2 = 2\left(r^2 + e^2\cdot\cos(2\varphi)\right)$

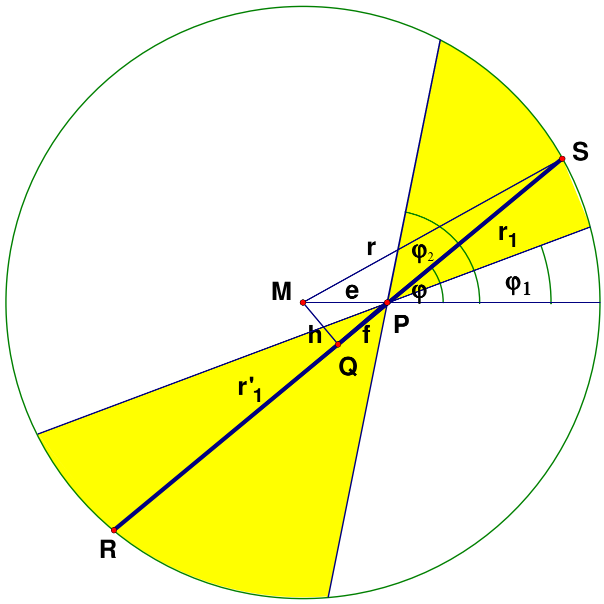

Fig. 14: Leibniz sector formula in a "double sector"

With this knowledge it is not difficult to calculate the area of a double sector by using the Leibniz sector formula. We get (see Fig. 14):

$$\frac{1}{2}\int_{\varphi_1}^{\varphi_2}\left[r_1^2 + (r_1')^2\right]\mathrm{d}\varphi = \int_{\varphi_1}^{\varphi_2}\left[r^2 + e^2\cdot\cos(2\varphi)\right]\mathrm{d}\varphi =$$

$$r^2(\varphi_2 - \varphi_1) + \frac{1}{2}e^2\left(\sin(2\varphi_2) - \sin(2\varphi_1)\right) \qquad (4)$$

We apply this formula to the particular double sectors, e.g. in Fig. 15 we have altogether n = 6 such double sectors (the odd ones are white). This number n is even at any rate (n = 2m) independent of the fact whether each quadrant is divided into an even or odd number of "sectors".

In the i-th "double sector" the "polar angle" ranges from $\varphi_1 = (i-1)\cdot\frac{\pi}{\underbrace{n}_{2m}}$ to $\varphi_2 = i\cdot\frac{\pi}{\underbrace{n}_{2m}}$.

Thus from (4) we get for $|DS_i|$, the area of the i-th "double sector":

$$|DS_i| = r^2\cdot\frac{\pi}{n} + \frac{1}{2}e^2\left(\sin\left((2i)\cdot\frac{\pi}{\underbrace{n}_{2m}}\right) - \sin\left((2i-2)\cdot\frac{\pi}{\underbrace{n}_{2m}}\right)\right)$$

$$= r^2\cdot\frac{\pi}{n} + \frac{1}{2}e^2\left(\sin\left(i\cdot\frac{\pi}{m}\right) - \sin\left((i-1)\cdot\frac{\pi}{m}\right)\right)$$

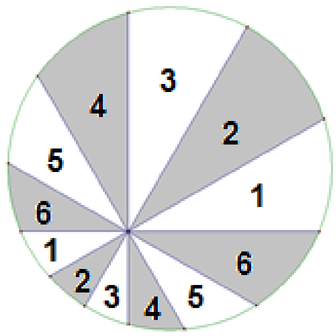

Fig. 15: Pizza theorem with three "sectors" in each quadrant

Now we calculate the area sum $\sum_{k=0}^{m-1}|DS_{2k+1}|$ of the odd "sectors" with the numbers $i = 2k+1$ and do the summation of the particular parts separately: We have $m = n/2$ times the summand $r^2 \cdot \frac{\pi}{n}$ which yields $r^2 \cdot \frac{\pi}{2}$, that is exactly the half of the circle area. The other trigonometric sums yield 0 according to (3), and by this the general pizza theorem is proved.

Finally another pizza theorem (http://mathworld.wolfram.com/PizzaTheorem.html), easier to prove: The volume of a pizza with radius z and thickness a is given by $\text{Pi}\, z\, z\, a$.

4 Teaching aspects

All in all this is an ambitious topic with many possible connections to other mathematical fields. It seems to be a good opportunity to foster elementary geometry in mathematics teaching by dealing with an interesting phenomenon (equal division of a pizza). In German speaking countries in my opinion there should be payed more attention to elementary geometry in lower secondary schools. Here one may think of matters like symmetry, reflections, rotations, division in pairwise congruent pieces (equidecomposability); these learning matters can be strengthened, extended, and linked when dealing with this topic. In the teaching of differential and integral calculus in school (high school, college) often syntax and special techniques of calculation (things that often could be done by computers) are given priority while semantic aspects and real understanding are neglected. But within the presented topic the idea of an integral as a limit of a sum of products (areas as sums of slim circular sectors) is constitutive, and this is a really semantic aspect. New terms like "polar coordinates" or "Leibniz sector formula" etc. are not important. So far to competencies with regards to contents. Competencies with regards to processes that can be fostered by the suggested topic are: cross-linking matters, problem solving, exploring situations, verifying phenomena (using Dynamic Geometry Software – DGS), argueing and reasoning.

How and in which grades could this phenomenon be dealt with at school?

In lower grades (say grade 7 to grade 9) a similar problem with a square instead of a circle may be dealt with as a problem that students are supposed to solve individually. Maybe some smaller hints from the teacher are necessary?

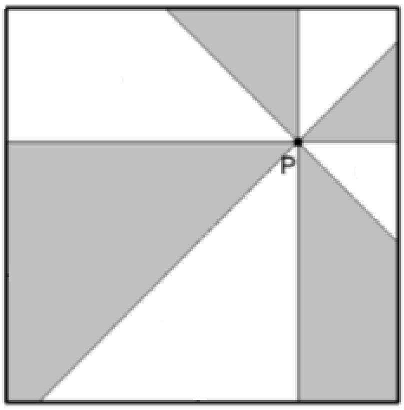

Fig. 16: Square and cutter

In a square the knife (cutter) is placed like in Fig. 16: two blades parallel to the edges and the other two parallel to the diagonals. Find arguments to show that the area sums of the white and grey pieces respectively are equal (cf. Kroll/Jäger 2010, 103f).

Remark: If P is not the center of the square (in the case of the center everything is clear) then P lies in a special quadrant. Without loss of generality we may assume that P lies in the right quadrant above.

Here there are several possibilities to see that the area sums are equal. When having in mind individual work of students it is always a big advantage if there are several ways to solve a problem because in this case different approaches can lead to the solution. A possible hint for a short and elegant solution could be: "construct a special line segment (dashed) – the upper edge reflected on the horizontal line segment through *P* (Fig. 17)".

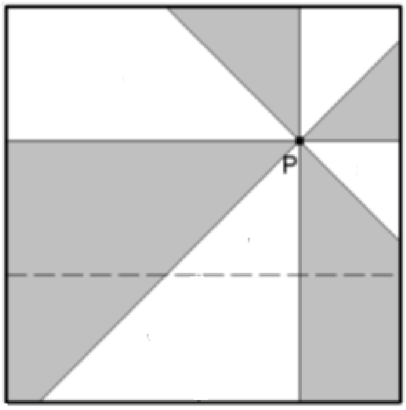

Fig. 17: Square and cutter – first hint

With this hint the following is clear by symmetry: above the dashed line the white parts equal the grey parts, therefore one has to think only of the parts below this line. In the case that students cannot find a solution individually despite of this hint there can be another one: Draw another line segment which leads to equal areas III and IV. Then the problem is reduced to finding reasons why I and II have equal areas (Fig. 18). If lower grade students succeed in reasoning this fact it can be seen as a good performance (here 45° angles and the resulting symmetries can play an important role).

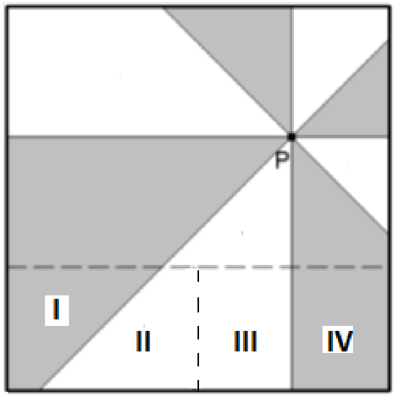

Fig. 18: Square and cutter – second hint

Concerning the original problem (circular pizza) in secondary schools students could gather practical experiences on the one hand (cutting circles of carton in the described way and weighing all the grey and white pieces[5]) and on the other the phenomenon could be explored by using DGS as a means of measuring the corresponding areas.

Also dealing with the dissection following P. Gallin (Fig. 5) seems to fit well to secondary school (grade 7 to 9), maybe even as problem to be solved autonomously by groups of students. Fig. 5 probably has the character of a "proof without words" even for many students of grade 7 – 9. I do think it could work with these students because one does not need more ingenious ideas, the helpful octagon is drawn in the figure, it is also explained how it arose. Also not so capable students can have the idea to simply count the area pieces *a*, . . ., *g*, *x* (their congruency is here immediately clear) in the white and in the grey zone outside the octagon. Also inside the octagon the way of reasoning is evident. The proof for the equality of the pizza boundaries following P. Gallin (using elementary geometry, see Fig. 6a,b) is also suitable for secondary school students. If the problem reduction Fig. 6a → Fig. 6b is done by the teacher (students in secondary schools will probably not see this possibility by themselves) then the further handling of the problem could be done autonomously in groups (maybe small hints from the teacher are necessary). For better supporting ideas and concepts and for visualizing the use of DGS (as a means of measuring) could be recommended, best in advance of the proof.

In high school or college (when the basics of calculus are taught) the phenomenon could be presented by the teacher, independent students' activities seem not to fit here. When realizing this ambitious program the main goal would be the important idea of the integral as a limit of product sums (here: slim circular sectors). In school the aspects of 3.1 would suffice, the ones of 3.2 (boundaries of the pizza pieces) and 3.3 (general case) are not necessary there. The important lemma of elementary

[5] If the grey pieces together have the same weight as the white pieces together then it is clear that the area sums are equal, too.

geometry can provide aspects of crosslinking several mathematical fields (calculus, elementary geometry), also individual students' work is well possible (problem solving, maybe in groups).

Also at university – teacher education – Fig. 5 (dissection following P. Gallin) could be dealt with, students of a geometry course could analyze the figure autonomously. But at this level students could also analyze Fig. 4 (including the helpful octagon; Fig. 3 – without this octagon – is with regards to my experiences not so good for that purpose). For preparing such an analysis of Fig. 4 one could pose the following problem (then working with the octagon is easier):

A rectangle *ABCD* is rotated (center *M*) by 90° (→ $A'B'C'D'$) so that the octagon $A'BCB'C'DAD'$ is established (see Fig. 19). Explain exactly why e.g. the line segments $A'B, D'C, B'C$ are "45° lines" (for reasons of clearer arrangement only one diagonal $D'C$ is shown, but the others are such "45° lines" as well).

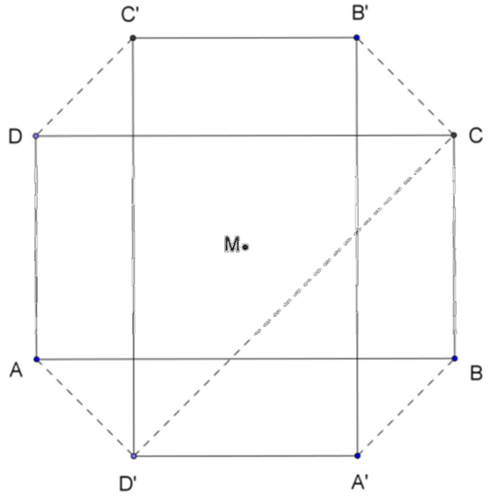

Fig. 19: 90° rotation of a rectangle

It is clear that in calculus courses at university (teacher education) all here presented aspects can be covered. But I have made the experience that the Leibniz sector formula is very rarely presented is such courses. I regret that, not because I want students to solve many integrals in polar coordinates, I regret it primarily because I think having another geometric idea of product sums (not only slim stripes and rectangles, but also slim circular sectors) is a valuable enrichment of the idea of integrals in mathematics teacher education. As I got to know it for the first time I asked myself: Why did nobody during my studying time at university (in calculus courses) tell us about this simple and important idea? Dealing with the general case (cf. 3.3) is in my opinion even at university not obligatory.

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