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A Models and Modeling Approach to Risk and Uncertainty

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Abstract: In this article we describe potential contributions of a Models and Modeling Perspective to research focused on learners' developing conceptions about uncertainty and variation. In particular, we show how a particular class of realistic problem-solving tasks can illuminate how learners develop models to identify, describe, and predict emergent patterns of regularity in the behavior of various types of systems and in the data these systems generate. We begin by situating current design work in this area within a larger project to investigate idea development in the domain of data modeling over extended (course-length) periods. We give design principles and examples for key components in our research framework, and we provide illustrative examples of these components and their interactions around the themes of distance and measurement that arise centrally in our materials. Next, we show how our approach can support advances in research on risk perception and on the development of ideas around risk assessment and management. Specifically, we identify three key facets of our approach and materials that make them good candidates for contributing to risk-oriented design research in education. Within each of these facets, we suggest research questions that could be addressed, and we provide examples and conjectures based on prior and ongoing work. In particular, we return to the ideas of distance explored in our examples and show connections with important questions in research on learners' perception and reasoning about risk.

Keywords: modeling, mathematization, problem-solving, affective dimensions of knowledge.

Introduction

In this article we describe potential contributions of a Models and Modeling Perspective to research focused on learners' developing conceptions about uncertainty and variation. In particular, we show how a particular class of realistic problem-solving tasks can illuminate how learners develop models to identify, describe, and predict emergent patterns of regularity in the behavior of various types of systems and in the data these systems generate. We begin by situating our current design work in this area within a larger project to investigate learners' ideas in the domain of data modeling, as they develop over extended (course-length) periods. Our design is centered on a type of activities known as Model-Eliciting Activities (MEAs), which engage learners in a deep form of modeling to construct solutions to real-world problems. The mathematical concepts uncovered in these core activities are then explored through Model Development Sequences (MDSs), where the classroom group of learners unpack and extend their collective ideas, connecting them with more formal mathematical constructs and investigating the reach of these new conceptual tools. We give design principles and examples for key components in our research framework, and we provide illustrative examples of these components and their interactions around the themes of distance and measurement, which arise in student solutions to MEAs and can be explored further through MDS activities.

After describing our framework, we show how our approach can support advances in research on risk perception and on the development of ideas around risk assessment and management. Specifically,

we identify three key facets of MEAs that make them good candidates for contributing to risk-oriented design research in education:

1. MEAs engage learners in optimization processes involving constraints and tradeoffs.
2. MEAs prompt learners to draw upon a wide range of problem-solving resources.
3. MEAs move beyond simplified problem settings that solicit the application of mathematical concepts and procedures that have been previously taught. Instead, by giving learners computational tools and pushing them to develop new mathematical constructions, MEAs are supportive environments for the kind of mathematical work that occurs in authentic settings outside of school.

Within each of these facets we suggest research questions that could be addressed through our approach, and we provide examples and conjectures based on our prior and ongoing work. To illustrate the importance of connections among ideas throughout a course-length engagement with modeling of this type and *across* MDS sequences, we suggest connections between the ideas of distance that were explored in our examples with important questions in research on learners' perception and reasoning about risk.

Theoretical Framework: Research in the Models and Modeling Perspective

In this section we briefly outline the theoretical perspective that underlies our design work. For over thirty years, researchers adopting a Models and Modeling Perspective (M&MP) in mathematics education (Lesh, 2003a; 2003b; Lesh & Doerr, 2003) have engaged in research to understand the development of mathematical ideas. A fundamental principle underlying this work has been that learners' ideas develop in coherent conceptual entities, called *models*, described by Lesh & Doerr (2003) as:

...conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently. (p. 10)

Under appropriate conditions, learners' models can be evoked and expressed in "thought-revealing artifacts." These artifacts can become objects for reflection and discussion by both individual learners and collaborative groups, and they also present rich data sources for researchers. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models are observed to grow through relatively rapid cycles of development toward solutions that satisfy these specifications. Models are thus powerful elements both for creating educational activities and for conducting research into learning.

From the Models and Modeling Perspective, Knowledge is seen as:

- Involved in perception and intuitions as well as action and conscious thought.
- Situated and shaped by context.
- Socially-shared and shaped by the community.
- Connected.
- Systemic, distributed, and emergent.
- Expressed in a variety of external media.
- Not simply logical / mathematical in nature.
- Often tacit.
- Initially piecemeal, undifferentiated, un-integrated and unstable.
- Continually developing along a variety of interacting dimensions.

Figure 1. Features of the M&MP conception of knowledge

As a program of research, M&MP research developed explicitly to investigate the following kinds of questions:

- How can we characterize realistic problem-solving situations where solutions demand elementary-but-powerful mathematical constructs and conceptual systems?
- What kinds of “mathematical thinking” are emphasized in such situations?
- What does it mean to “understand” the most important of these ideas and abilities?
- How do such competencies develop, and what can be done to facilitate their development?
- How can we document and assess the most important (deeper, higher-order, or more powerful) conceptual achievements that are needed for full participation as citizens in increasingly complex societies and professions?
- How can we identify students who have exceptional potential that is not adequately measured by standardized tests?

These questions are tightly linked to the M&MP’s view on the nature of knowledge (see Fig. 1). Here, the M&MP builds on perspectives originating with Piaget and Vygotsky as well as with the American Pragmatists including Peirce, James, Meade, and the later Dewey, (*c.f.*, English et al, 2008; Lesh & Doerr, 2003).

Model-Eliciting Activities (MEAs)

Rooted in these perspectives and pursuing questions such as the ones listed above, M&MP research has sought to illuminate the nature of knowing and learning in authentic problem-solving settings. A key requirement of such settings is that they challenge learners to engage in original mathematical work (i.e., to produce mathematics constructions that are *new to them*), rather than merely applying mathematics learned from an authoritative source. Iterative design work to create such learning environments has led to the development of a genre of materials and activities, known as Model-Eliciting Activities (MEAs). In MEAs, students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—a *model*—comprising a conceptual structure for solving the given problem. These activities give students the opportunity to create, adapt, and extend scientific and mathematical models in interpreting, explaining, and predicting the behavior of real-world systems.

Originally designed as environments for research into what it means to “understand” important concepts in the K-12 mathematics curriculum, MEAs were first and foremost intended to provide documentation and evidence to illuminate the development of ideas in classroom groups. Thus, MEAs were designed in such a way that students would clearly recognize the need to develop specific constructs – without dictating *how* they would think about relevant mathematical objects, relationships, operations, patterns, and regularities. In general, this approach is inspired by the way engineers are given design “specs,” which include brief descriptions of goals and available resources or constraints (such as time or money). As learning environments MEAs are also designed to optimize the chances that significant conceptual adaptations will occur during sufficiently brief periods of time so that the processes of conceptual change can be observed directly by researchers and teachers. Along with a range of particular MEA activities, the early M&MP community outlined six key design principles for MEAs to meet these goals (see, *e.g.*, Doerr & English, 2006; Lesh et. al., 2000; Hjalmarson & Lesh, 2007):

1. *Personal Meaningfulness*. Is the problem situation realistic, in the sense that a solution would be of genuine interest to a client? Is the problem space sufficiently open to ensure that different

groups of students are able to pursue diverse solution paths based in their own unique personal knowledge and experiences?

2. *Model Construction.* Does the problem truly require the new construction, modification, adaptation, or extension of a model in order to be solved? Does the problem engage with deep mathematical structures and regularities, rather than engaging mainly at the surface level?

3. *Self-Evaluation.* Are the problem’s criteria sufficiently clear that student groups can judge for themselves the usefulness or adequacy of proposed solutions?

4. *Model Generalizability.* Do the models that are created in the activity apply only to the specific situation of the problem, or are they likely to be generalizable to a broad range of situations?

5. *Model Documentation.* Will student responses to the problem explicitly reveal their characteristic ways of thinking about the situation? Will they provide clear evidence about the mathematical objects and relations they have engaged with in solving the problem?

6. *Simplest Prototype.* Is the problem situation as simple as it can be, while still meeting the other design principles? Does the experience of the MEA “stick” with students so that they are able to use it as a lens for viewing future problems that feature similar mathematical structures?

An Example MEA: The Darts Problem

To illustrate the genre of the MEA, we provide the example of the Darts problem. This problem plays an important role in our work with data modeling, statistics, and probability, as, among other things, it challenges students to invent notions associated with centrality, spread, and distance from an expected distribution. Students are usually introduced to the Darts problem after having engaged with several other MEAs that press them to create operational definitions for key constructs such as “worker productivity” or “volleyball-playing ability,” using data of various kinds to develop and apply quantitative measures of such constructs. The Darts problem pushes students to expand on this line of thinking, creating operational definitions of constructs for use in evaluating both performances (darts games) and performers (darts players).

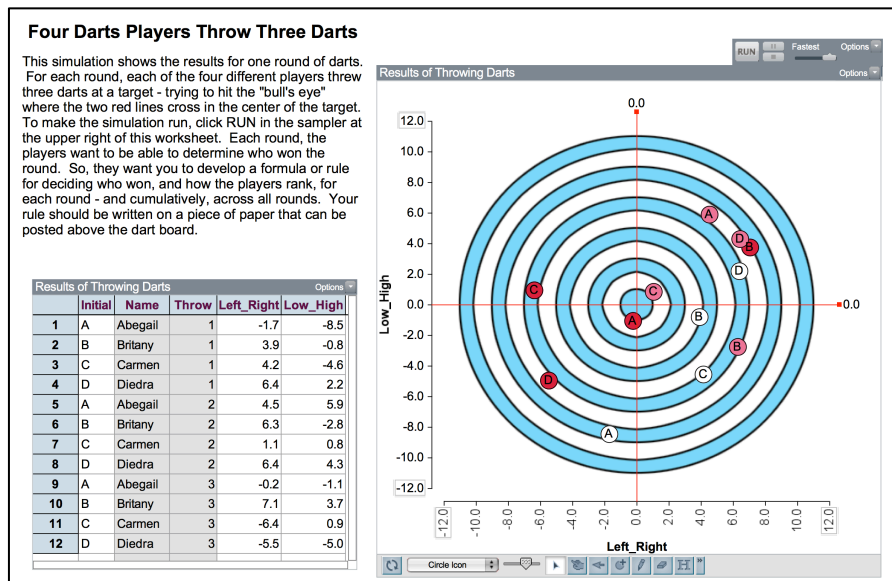


Figure 2a. Part one of the Darts MEA.

Darts, Part Two: Going for the Championship

The local pub is now preparing for a season of competitive darts, and you need to provide useful information to the captain, Laura, to help her field a team. Beyond the four players from Part One, a group of “regulars” at the pub will be trying out. Describe how results from several rounds of darts can be used to create a rating or description of players’ darts throwing skills. Your procedure should help Laura to choose a team based on the tryouts.

Figure 2b. Part two of the Darts MEA.

After breaking into groups, the students are presented with the two-part problem statement (Figures 2a and 2b, above). The Darts problem is somewhat unusual among MEAs in being itself embodied in software form, using a dynamic data exploration tool (here, TinkerPlots). In contrast, most MEAs are paper-based, with dynamic mathematics software being more heavily used in follow-on activities. The basic structure of students’ engagement with the Darts problem, however, is typical of MEAs. Student groups engage in cycles of thinking, characterized by identifying possible interpretation schemes for the problem context, testing these schemes, and revising or adapting them to accommodate new ideas or to address shortcomings. Through these cycles, their thinking evolves rapidly towards increasingly effective approaches to the problem. For instance, in the first 10 minutes of work, groups may focus on one or another feature of the problem (e.g., attending to one dart of the three thrown per player, or even, when attending to the coordinate values for a single throw, attending only to the greater of the two). As they grapple with and discuss the first part of the problem, however, groups generally move toward a definition of distance-from-the-bullseye. (This is often expressed either as Euclidean distance, as its square (adopted for convenience in calculation), as single-coordinate differences, or as a “taxicab” measure obtained by summing these coordinate distances (as absolute values of coordinate differences.))

While the idea of distance is sufficient for the task of ranking individual throws according to quality (with the rule that a lower distance corresponds to a better throw), it does not solve the problem of scoring a round. Recognizing this, some groups turn to a notion of an average (either based on their distance calculations or re-invented independently), while other groups attend to the rings drawn on the board, associating scores with each ring by drawing on intuitions associated with fair-rewards. To support this second strategy, groups may tap into ideas about probability and area, beginning to reflect on the likelihood that a “random throw” will land inside different target circles.

In tackling part 2 of the Darts problem, the groups’ focus generally moves from scoring *throws* and *rounds* to evaluating *players*. Here, groups begin to generate candidate attributes for players such as “consistency” “precision” and “accuracy.” In many cases, learners realize that their groupmates have slightly different attributes in mind for these terms. Many groups find that (at least) two are at work—one corresponding roughly to an idea of “centrality” and another corresponding to ideas of “spread.” Group discourse also begins to reveal questions about the nature of simulations and distributions, as students question whether the computationally simulated players “really” have different skill levels or attributes, and if so, how they might detect such differences with repeated trials of the simulation. Finally, additional concepts emerge as groups converge on a solution and begin to draft their letter to Laura, the “client.” For instance, they may reflect on which of the two attributes might be more “teachable” to players over the season, or they might become concerned about how to reduce the logistical complexity for judges attempting to choose between candidate players in noisy pub settings.

In this way, students iteratively develop solutions to the problem in the time allotted—usually 60 minutes for this MEA. In some implementations, the class then gathers together for a structured “poster session” event. Here, one member in each 3-person group hosts a poster presentation showing the results of their group. The other two students use a Quality Assurance Guide to assess the quality of the results produced by other groups in the class. These forms are submitted to the teacher and contribute to assessment in various ways, providing evidence for the modeling and conceptual achievements of both individuals and groups.

Model Development Sequences (MDSs)

Moving beyond individual MEAs, recent M&MP research has also investigated ways in which MEAs can be integrated within larger instructional sequences: Model Development Sequences, or “MDSs” (Lesh et. al., 2003). MDSs offer classroom groups opportunities to unpack, analyze, and extend the models they have produced in MEAs, as well as to connect their ideas with formal constructs and conventional terminology. This unpacking work helps to ensure the lasting retention of concepts at the level of generality required to apply them flexibly in novel situations. MDS activities also set the stage for the critical connection between conceptual development (the centerpiece and focus of MEAs), and procedural knowledge that is required for students to achieve well-rounded competence in any subject area. In our examples here, we will highlight how the key theme of *distance* that emerges in students’ MEA work can be developed in MDS activities.

Within an MDS, *reflection tools* support students in stepping back from their modeling processes and reviewing this work as critical observers of both individual and group modeling behavior. In the design of our course materials, we consider these tasks core to the learning process. In general, M&MP research expects that when students interpret situations mathematically, the interpretation systems they engage are not purely logical or analytical in nature. Rather, they also involve attitudes, values, beliefs, dispositions, and metacognitive processes. Moreover, the M&MP does not treat group roles or group functioning as if these were fixed student attributes that determined their behaviors. Instead, students are expected to develop a suite of problem-solving *personae* that they learn to apply purposively as the situation demands.

In *product classification* and *toolkit inventory* activities, students continue the work of abstraction, identifying links among their solutions to different MEAs and between these solutions and the “big ideas” of the course. *Model exploration activities* (MXAs) provide a model-rich environment for introducing more conventional terminology, concepts, and skills, which students need in order to formulate sophisticated models and present them to a mathematical community. These may use a combination of pointed YouTube videos and interactive simulations in dynamic mathematics software. (In our work to construct a course-sized repository of materials to fuel MDSs there are to date approximately 50 of these YouTube videos with accompanying simulations in dynamic geometry and dynamic statistics software. These are currently collected under the ProfRLesh channel.) Finally, *model adaptation activities* (MAAs) allow students to transfer ideas and techniques developed in MEAs to situations calling for similar performances. These MAA activities also provide smaller-timescale modeling scenarios that exercise concepts students have explored in other components of the MDS. They may be pursued individually or in small groups, depending on the nature of the task and the teacher’s instructional or assessment goals.

Example MDS Activities Associated with the Darts MEA

Model Exploration Activity (MXA)

As mentioned above, most student groups develop one of several distance formulas in order to evaluate the quality of dart *throws* in Part 1 of the Darts MEA, and most also operationally define an attribute of a darts *round* that is analogous to a construct expressing the *center* of a dataset. They then examine the variation of this attribute over many rounds to describe an attribute of the *player*. A MXA that helps to unpack and extend this work is shown in Figure 3 below, implemented using the NetLogo software (Wilensky, 1999). Here the NetLogo environment produces visualizations of “level curves” of the sums of distances to chosen points. Students can add, remove, and rearrange these points, and they can select among various definitions of distance. Screens 3a, 3b, and 3c show the different results obtained by using three points and varying the distance definition (3a uses the square of Euclidean distance, 3b the Euclidean distance, and 3c the “taxicab” or “Manhattan” distance). Screen 3d shows the effects of adding points (there are five) and zooming out in scale. The numerical values for distances and the coloring update in real time as points are moved, giving students a multidimensional sense of the way different distance functions operate. Moreover, because NetLogo is a “glass box” environment, it is a relatively simple matter to modify distance formulae or add new ones, change the coloring visualization, and so forth.

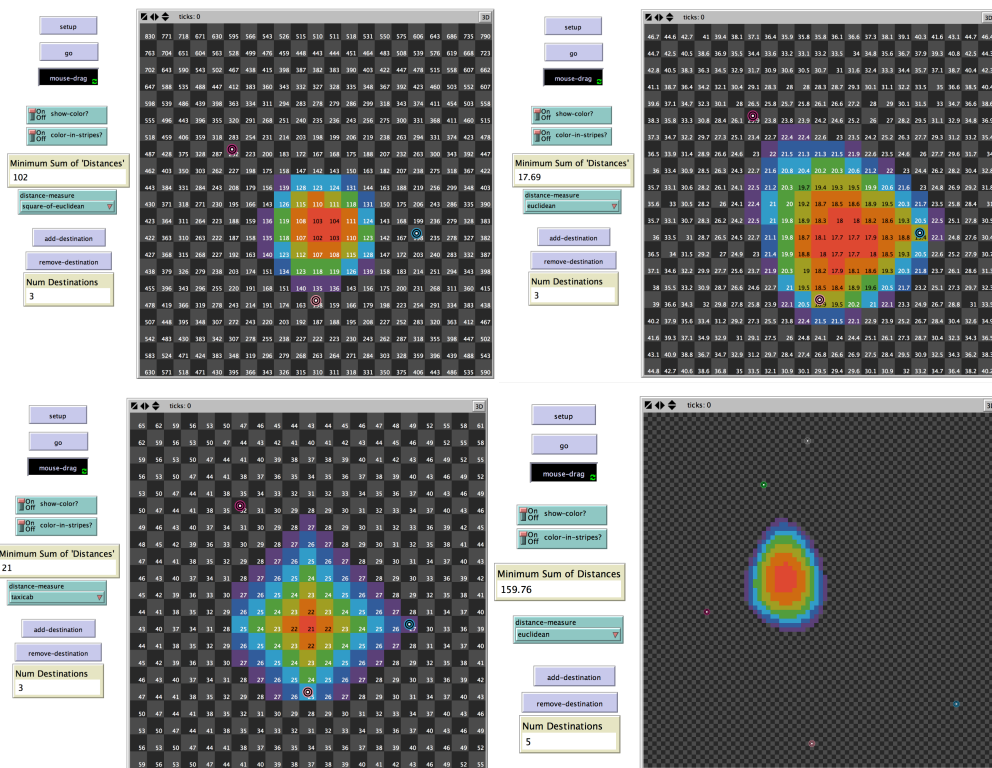


Figure 3. A dynamic environment for visually exploring the effects of different distance formulae. Fig 3a (upper left) Square of Euclidean distance; Fig 3b (upper right) Euclidean distance; Fig 3c (lower left) taxicab distance; Fig 3d (lower right) Euclidean distance, with five points, zoomed out.

Though learners are given freedom to explore in this environment, developing intuitions and a feel for how distance measures are affected by changes in the data, there are several directions that teachers may wish students to explore and discuss more systematically. One such direction involves investigating what happens when all data points are collinear. In this case, the points can be interpreted

as scalar values along the shared line, and the different distance-minimizing centers can be connected with the familiar notions of the mean and median of a dataset. Another involves studying the effects of *parity* on the different distance definitions – how it sometimes matters whether there are an odd or even number of points in the dataset. Finally, one can explore questions of weighting by making two or more points coincident and investigating the impact on the various centers. In general, moving to the two-dimensional setting helps to ground many of the definitions that have been learned with one-dimensional (scalar) data in a broader context.

Toolkit Inventory Activity

While completing the Darts MEA and reviewing their peers' solutions, students may collectively identify such modeling toolkit entries as “simulating many trials” or “eliminating outliers.” In terms of the “big idea” of distance, questions about how to identify or define outliers arise naturally from learners' impulses to offer darts players “do-overs” under some conditions or to discard certain throws as not being “accurate representations” of the players' skill level. Indeed, a somewhat unusual but by no means rare strategy for scoring a round is to discard the best and worst throws, basing the score solely on the “median” throw. While this strategy may drop out in favor of other approaches, the idea of discarding data may reappear in different guises, and the question of its fairness can draw out debate on a variety of important topics, including the nature and identification of outliers. This can be a very rich topic for classroom discussion, as it integrates a variety of key ideas including notions of representativeness, rarity, variation between and within individuals, central tendency, and distance.

Model Adaptation Activity (MAA)

Again building on the idea of generalizing distance from a landmark outcome (the bullseye in the Darts MEA), students can be asked to consider other simulations in which an outcome is a complex event, as, in the example shown below, 30 throws of a six-sided die. Though this situation is in some senses quite different from the Darts problem, students find that powerful ideas from that setting can guide their work in this new context.

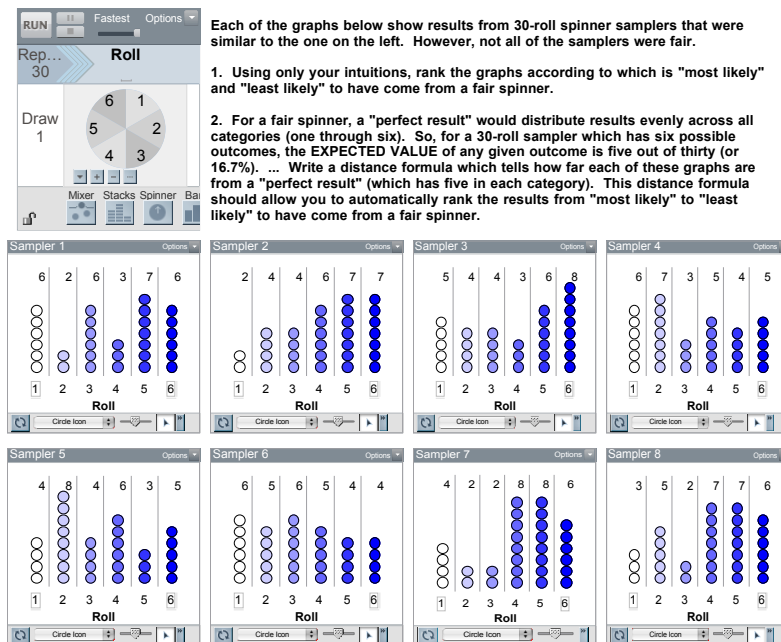


Figure 4. Model Adaptation Activity (MAA). Students generalize the idea of an invented distance from the Darts MEA to measure the distance of these distributions from an “expected” distribution.

Here, students are first asked to rank the outcomes intuitively according to their sense of which is more likely to occur. Discussions reveal some agreement (usually at the extremes of “likely” and “unlikely” distributions), with other points of dispute (usually among the middle-likelihood distributions). On one hand, the agreement helps to confirm the intuitive nature of applying the idea of a distance and the construct of an expected value in this setting. On the other hand, the disputes drive the class toward establishing an explicit, quantitative definition of this distance to settle arguments. After developing and coming to agreement on a distance concept and working through the process of operationalizing it as a formula, students can express that formula computationally in the TinkerPlots environment and run a large number of simulated trials of fair dice. Representing these trials according to their distance measure (or, if there are several candidate distance measures, according to each of these), students estimate the probability that sample’s distribution will be a given distance, or further, from the expected distribution. (See Figure 5, below.) That is, students can observe that in, say, 10,000 trials using a fair spinner, only about 100 trials yielded a distance measure of greater than a threshold value. This would provide an empirical estimate of the *probability* of meeting or exceeding this threshold (of approximately .01).

Furthermore, the classroom group can use this simulation approach to reflect back on the distance measures themselves. For instance, they can confirm whether their measure does in fact correlate to likelihood (i.e., whether larger distances from the expected value are (uniformly) rarer). They can also make judgments about whether one definition or another does a better job of “separating” outcomes in different ranges. Distance measures can then more formally and confidently be connected with the probability of being that far or further from the expected value (essentially a one-tailed *p*-value). Finally, students can either learn that the procedure they have invented is essentially the *chi-square* measure, or they can compare their approach to that standard test. Importantly, this experience of the chi-square concept does not require a blind appeal to the authority of statistical tables or results from mathematical analysis beyond the level of middle- or high-school students, and it appropriately grounds such tools (when they are encountered) in intuitive notions of distance and expectation.

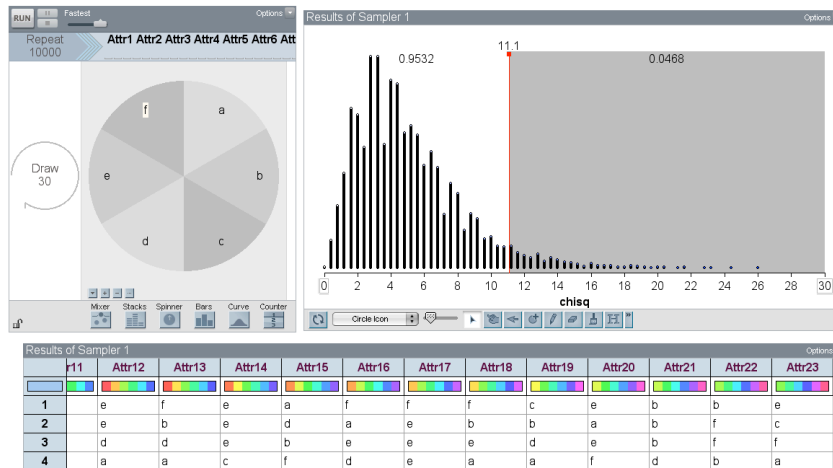


Figure 5. TinkerPlots simulation of 10,000 trials of 30 samples, evaluated according to the class’s chosen definition of “distance from the expected distribution”

Three Facets of MEAs that Make them Propitious Environments for Research on Risk

Having now given an account of M&MP research in general and our approach to MEAs and MDSs in particular, we are in a position to discuss the application of this design research methodology

to study questions associated with perceptions and reasoning about risk. In particular, we describe three facets of MEA research that resonate with risk-related education research are as follows:

1. *They engage learners in optimization processes involving constraints and tradeoffs.* MEAs present learners with problem settings that require them to optimize solutions and processes, working within constraints, making trade-offs, and accounting for second-order effects. Risk can play a key role in such situations, introducing the need to balance prospective gains against potential losses. Learners' invented strategies can provide a rich source of information about their ideas and attitudes about both.
2. *They prompt learners to draw upon a wide range of problem-solving resources.* MEAs promote reasoning and the use of knowledge resources in a broad sense, including not only logical/analytical thinking but also affective/intuitive thinking. As such, the view of modeling that emerges from such activities includes the feelings, attitudes, values, and beliefs of learners. Recent research has become increasingly tuned to the role of feelings and intuitions both in conditioning perceptions of risk and in generating the human meaning behind abstract statistics and probabilities; thus, these dimensions of problem solving may be of particular interest (see, e.g. Slovic, 2000, 2010; Borovcnik, 2011).
3. *They move beyond simplified problem settings that solicit the application of mathematical concepts and procedures that have been previously taught.* MEAs involve learners in the interpretation of phenomena and situations that are not pre-classified as examples of constructs taught in a particular topic or textbook area. Thus, they can offer excellent opportunities for exploring the development of ideas associated with a variety of conceptions of uncertainty that are interconnected in real-world settings but that are taught and learned in different domains and at different points in school curricula. Because MEAs involve learners in authentic creation of mathematical models, they offer designers the opportunity to engage learners with phenomena that may evoke very different conceptions of indeterminacy, ranging from randomness and error on one hand to complexity and emergent phenomena on the other. Perceiving and managing risk may take on very different aspects in these different settings.

We deal in more depth with each of these facets in the sections below. For each facet, we discuss potential contributions of M&MP research and we outline research questions and conjectures that can be pursued by the design research approach we have outlined above.

Facet 1. MEAs engage learners in optimization processes involving constraints and tradeoffs

In MEAs learners are presented with challenges that require them to optimize processes or outcomes, work within constraints, make trade-offs, and account for second-order effects. Many problems involving risk are similar in the sense that they require balancing the opportunity for some kind of gain with the potential for some kind of loss. Such modeling situations are conceptually rich in part because they demand a deep and flexible understanding of the systems that are involved. To understand these features, learners must envision the *parameter space* of a system—the range of its behaviors as its key parameters change. Essentially, this replaces 'snapshot' evaluations of a system's behavior with a sense of multi-dimensional covariation. Moreover, when the consistency of a system's behavior and/or the measurement of that behavior are themselves problematized, multiple notions of uncertainty and variation are introduced and must be coordinated. MEAs can place students in situations that require them to develop models of systems under these kinds of conditions; as a result, they offer extremely rich environments to study learners' ways of thinking and the broader development of ideas about these topics

Risk- and uncertainty-related research questions that could be pursued via MEA designs include questions like the following:

- When faced with realistic problem settings, what kinds of uncertainty and variation are most salient to learners and enter most strongly into their calculations and their proposed solutions?
- Do learners develop different coping or mitigation strategies for different categories of variability in their data?
- How does the frame of recommending a course of action for someone else (e.g., the “client” of the MEA) affect learners’ assessments of risk and the weight they assign to possible outcomes?

Finally, in a course-length engagement with MEAs and MDSs, there are interactions with big ideas and constructs developed elsewhere. Taking as an example the idea of *distance*, as discussed above, we might ask whether and how learners draw on such multi-dimensional notions of distance as resources for expressing and operationalizing “riskiness.” Similarly, we could attend to learners’ perceptions of weaknesses or limitations in using the distance construct as a definition of riskiness, and we could use these responses to illuminate their reasoning processes.

Facet 2. MEAs prompt learners to draw upon a wide range of problem-solving resources

Research in modeling indicates that learners develop *problem-solving personae* over an extended period of engaging with problems like those found in MEAs (Hamilton, Lesh, Lester, & Yoon, 2007). Investigations using the Reflection Tools that are part of MDS designs described above increasingly suggest that while these problem-solving personae may have a logical or technical core, they also involve “soft” aspects of knowledge, including attitudes, feelings, and beliefs (both about oneself and the domain). Here, we find a strong potential overlap with recent studies on the psychology of risk perception, which emphasize the impact of non-logical processes on learners’ conceptions of risks and probabilities. Particularly interesting in this regard is evidence supporting *dual process* theories of probabilistic thinking and responses to risk (e.g., Kahneman, Slovic, & Tversky, 1982; Kahneman & Frederick, 2002). Such research posits the existence of “heuristics” that support intuitive, “System 1” responses to real-world situations. More generally, the role of feelings in human responses to risk and uncertainty is increasingly important in this domain, as led by work of Slovic and colleagues (e.g., Slovic, 2000, 2010). Their research has established an “affect heuristic” (Finucane et al, 2000; Slovic et al, 2002), by which “feelings serve as an important cue for risk/benefit judgments and decisions.” (Slovic, 2010, xxi).

In spite of evidence that people do in fact give great weight to their intuitive, experiential sense of risk, as compared with more deliberative and analytical sense-making processes, Slovic and colleagues do *not* argue in favor of extirpating the intuitive in favor of the analytical. On the contrary, their research suggests that intuitive and experiential factors play a key role in grasping the *meaning* of probabilistic situations, so as to weigh risks appropriately (Slovic, 2002). That is, this line of research suggests “we, as a species, think best when we allow numbers and narratives, abstract information and experiential discourse, to interact, to work together” (Slovic, 2010, p. 79). At the same time, there are many situations in which humans can be misled by their feelings. A reasonable approach to this dilemma would seem to involve supporting learners in building *both* a better toolkit for analytical thought *and* more effective ways of associating or formulating stories, scenarios, and images for data, as resources for affective responses (see also Borovcnik, 2011).

Of course, understanding the affect heuristic and its relation to other dynamics of risk perception and reasoning is an ongoing challenge. Here, the design of MEAs can offer researchers a range of tools for getting at related questions. For instance:

- How does the social setting of a small problem solving group change the nature of risk perception in its members? Does in-the-moment social discourse provide resources that can be used by the group to blend System 1 and System 2 responses?
- How durable is affect-based risk perception? Does it leave traces in the solution strategies formulated by groups and/or in their communication strategy for explaining their solutions to the “client” of the MEA?

Finally, connecting again with the distance theme, when the likelihoods of threats and risks (or, benefits and rewards) are described with reference to distance, does this promote new and/or unanticipated ways of thinking about contingency? A spatial metaphor may, for example, suggest to learners notions of intervening regions or “buffers” affected by and depleting the threat or benefit.

Researching Facet 2: Early design of an MEA and MDS on Probability and Risk Perceptions

We have not yet incorporated activities specifically focused on risk perception and reasoning with our existing materials on data modeling and statistics. However, we have begun the design process of creating an MEA and MDS to study related issues, and we describe some of our conjectures here. The process of designing MEAs is itself an iterative modeling process, and so we expect that our thinking in this area will develop significantly as we proceed.

A preliminary statement of a prototype-MEA, the Charity Benefit problem, is shown below:

Charity Casino Night Benefit

In this activity, your task is to design two different chance-based gambling booths for a charity benefit event. At charity benefits, the attendees are in principle happy for the “house” to win more than the players, but they still want the games to offer them a reasonable chance of winning. In fact, they may be more likely to play the games where they think they are more likely to win.

The organizers of the benefit expect approximately 500 people to attend the event. As you design your two booths, be sure to explain to the benefit organizers:

- how each game is played,
- how much the organizers should charge people to play each game, and
- how much each game should pay out to winners.

NOTE: In writing your game’s rules and describing a player’s winnings, be sure to clarify how the player’s stake (what they paid to play) is treated when they win. For example, if the game costs \$1 to play and winners receive \$2, does this mean that they receive \$2 *plus* the \$1 they staked, or \$2 *including* the \$1 they staked?

Write a letter to the benefit organizers including the description of your game and any guidance you can give them about what they can expect to earn from your two booths in the one-night charity benefit event.

Figure 6. The Charity Casino Night problem.

In building toward this problem, we explored several questions with small groups of 7th-9th grade learners from a weekend enrichment program. The first question was whether the students would find it interesting to invent chance-based gambles. We found this to be both an engaging and a revealing activity, particularly when the task was phrased in terms of creating gambles that would fool prospective players about the likelihood of winning. Our most recent activity to test the viability of this type of design task was as follows: Each group worked independently to design a pair of chance-based gambles. When all groups had completed their designs, one group was chosen to present. An opposing group (the

“challengers”) was offered their choice of the two, and the students themselves (the “authors”) were then forced to take the other gamble. The challengers and the authors then played their gambles until one side won. Following the challenge, the authors explained their design, with an emphasis on demonstrating which side “should have won” based on the relative probabilities of winning the two gambles.

In the students’ designs, “analytical” strategies (such as creating gambles whose probabilities were difficult to compute) were deeply mixed with “psychological” or “rhetorical” strategies, in which the authors attempted to manipulate their opponents’ perceptions of relative probabilities through various means. Figures 7-9, below, show three groups’ challenges. In Figure 7, the group hoped that the phrasing of Game 1 would trigger their opponents to assess the probability of winning as $1/36$ rather than $1/6$. In Figure 8, the group used contrasting descriptions (descriptions of winning in Game 1 versus descriptions of losing in Game 2) to attempt to manipulate their imagined opponents’ assessment of probabilities. In Figure 9, the group was hoping their opponents would focus on the number of winning balls, instead of the proportion of winning outcomes to total outcomes. (In fact, the students’ calculation of the probability for game 2 was incorrect, so Game 1 actually had slightly more favorable odds.)

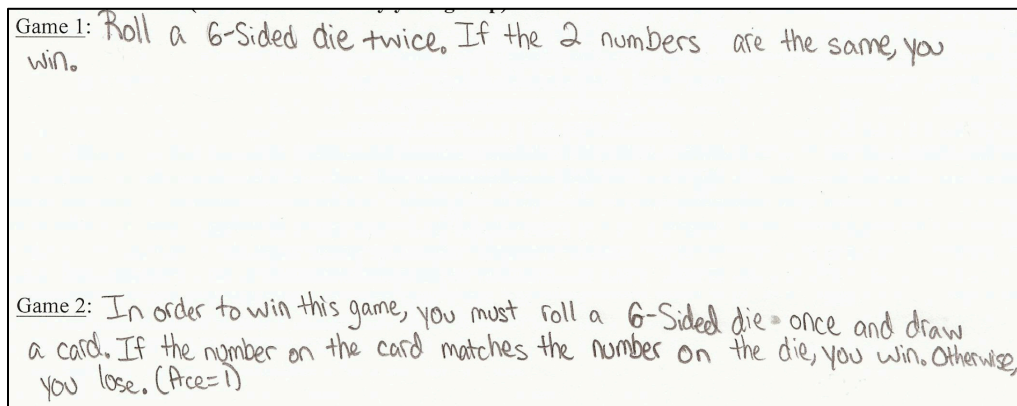


Figure 7. Students are attempting to cue the idea of a $1/36$ probability for double sixes in Game 1.

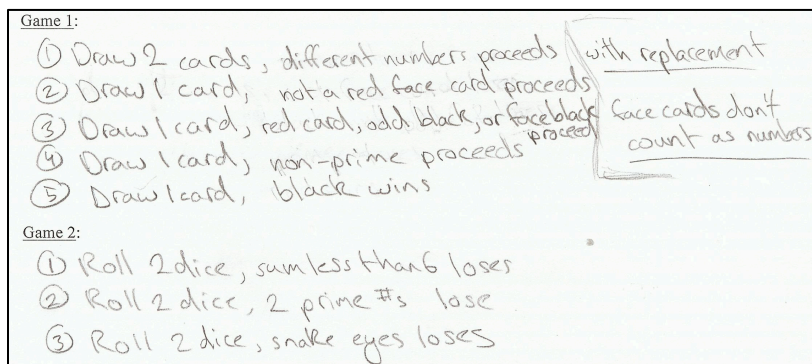


Figure 8. By describing winning conditions in Game 1 and losing conditions in Game 2, this group hoped to lure opponents into choosing the less-favorable gamble.

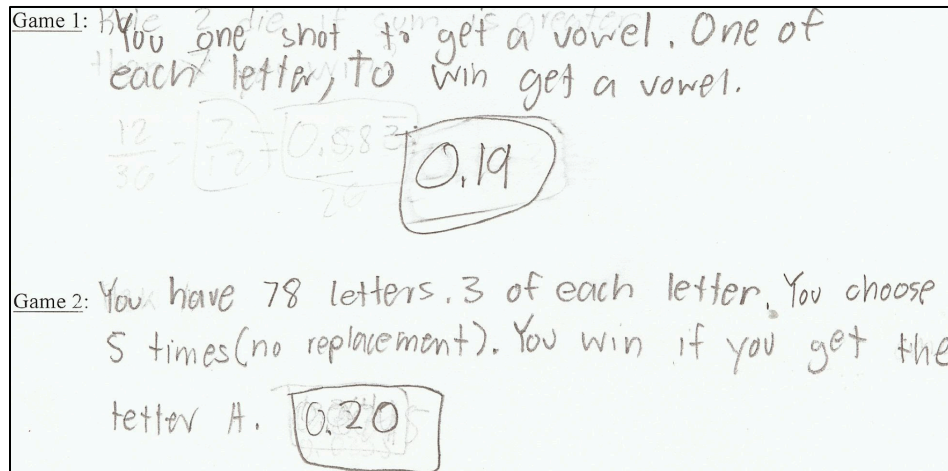


Figure 9. There are more “winning balls” in Game 1, but more chances to draw in Game 2.

This early experiment has suggested the potential of designs in which students attempt to manipulate the presentation of a gamble to make it more likely for another person to mis-perceive the odds. To capitalize on this feature in the Charity Benefit MEA, we plan in our next iteration to convert the “poster session” of the MEA into a simulation of the charity event itself. To do this, we will use a networked participatory simulation or “PartSim” (Colella, 2000; Colella, Borovoy, & Resnick, 1998; Wilensky & Stroup, 1999a; Klopfer, Yoon & Perry, 2005), implemented using the HubNet module (Wilensky & Stroup, 1999b) of NetLogo. In this simulation one group member will “man” the group’s booths while their groupmates circulate in the virtual charity casino, spending an allotment of tokens on the available games.

In other work related to social policy and risk of flooding (Brady et. al, in preparation), we have seen the power of such PartSims to provide learners with a “feeling of risk” (Slovic, 2010), even when the associated computer visualizations are quite simple and do not provide an immersive “virtual reality” experience. In the Charity Event PartSim, we aim to provide two kinds of risk sensation: one, for the two roving players, as they make selections similar to those that gamblers at the event would do; and a second, for the booth-manning team member, who can monitor many more instances of game play and watch the winnings for the booths, compared to the expectations they have stated in their designs. Students will circulate among roles, experiencing both. If our flood modeling experience is a good indicator, students in both roles will feel substantial and “realistic” levels of tension about the results, even though no actual currency is at stake in the gains or losses of the PartSim.

Patterns that emerge in students’ interactions with the PartSim will also create realistic feedback to the groups about the relative attractiveness of their games. Games with more attractive winning propositions may be played more often, and this will give them a greater opportunity to return revenue as expected by the theoretical probabilities that underlie their gambles. On the other hand, the “casino” environment will also introduce complications and complexities that should make the analysis of the experience and the data that results into rich contexts for reflection and discussion. For instance, games that happen to return atypical results in their early plays or games that return variable winnings may experience changes in popularity that the group can reason about in terms of such concepts as information cascades on the one hand or the attractiveness of certain kinds of winnings profiles on the other. In any case, the tabulated results on the net revenues earned for the charity will be a dataset worthy of group reflection, as will be the various experiences and perceptions of the learners-as-players.

Facet 3. MEAs move beyond simplified problem settings that solicit the application of mathematical concepts and procedures that have been previously taught.

M&MP research argues that models are *conceptual systems* that support and organize perception itself. In a wide range of settings, when we engage with the world our models enable us to *see* phenomena in particular ways: they make selected attributes of situations salient to us, and they orient us toward potential actions and judgments. Our *mathematical* models do this work in ways that emphasize structural features of situations. Among other things, this means that in realistic problem settings, experts distinguish themselves from non-experts not only by what they *do* but also by what they *perceive*.

This perspective also implies that authentic modeling is a fundamentally *interpretive* process. MEAs, like real-world problem settings, are not framed as occasions to apply a particular procedure or construct. Instead, they admit a variety of possible approaches, and most adequate solutions are characterized by bringing together ideas from different subject areas. In general, a modeler's entire store of prior knowledge and experiences acts as an interpretation system through which she or he is able to make sense of new phenomena. And in the cyclic modeling process of MEAs, learners build upon their past experiences and prior knowledge to develop new ways of seeing problem situations. Again, this is one of the primary reasons why solutions to MEAs often incorporate ideas from across the experience base of problem solvers. They may draw on ideas and techniques from multiple domains or textbook topic areas, and they may merge logical analyses and calculations with value-based judgments, feelings, and beliefs.

We argue that research is needed to investigate fundamental questions about students' reasoning and what it means to understand core ideas in probability, uncertainty, and risk. Given this, we believe it is essential to study learners' modeling processes in settings that feature open and authentic inquiry in the sense described above. The openness and authenticity of the challenge is critical to the value of modeling activities and can be overlooked in discussions of modeling. For instance, in spite of an encouraging emphasis on "modeling" in the Common Core State Standards in Mathematics (CCSSM, 2010), it is unfortunately possible to see the description of modeling there as describing mere *applications* of already-learned mathematical constructs to simplified real-world situations. As an example, in the CCSSM practice standard, "Model with Mathematics" we read:

Mathematically proficient students can *apply the mathematics they know* to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as *writing an addition equation to describe a situation*. In middle grades, a student might *apply proportional reasoning to plan a school event* or analyze a problem in the community. By high school, a student might *use geometry to solve a design problem* or *use a function to describe how one quantity of interest depends on another*. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. (CCSSM, 2010, emphasis [with italics] added)

The distinction between modeling and application is essential, both to research and to teaching. When problems are offered to learners as mere applications of particular mathematical principles or concepts, students do not experience authentic, fundamental challenges of interpretation. In contrast, when learners are presented with realistically complex problems (before instruction), they may engage in modeling as we have described it: a fully interpretive struggle. Student thinking in these contexts can thus illuminate connections and relations between topics normally treated as separate. We argue that this is particularly true with probability and uncertainty, and we believe that this is one of the reasons to expect that MEA-based research can be illuminating in this area.

In fact, there are dimensions of research in risk and uncertainty that can be specifically illuminated by this form of learning activity. In closing this section, we outline one such topic: the distinction between sources and types of variation. In conducting authentic inquiry or making sense of novel situations, modelers often engage with various forms of uncertainty and may interpret them as arising from a variety of causes or mechanisms—for example, as due to *error*, to *chance*, or to *complexity*. Their tacit or explicit classification of these sources of uncertainty and variation may frame their modeling activity to highlight dimensions of the phenomena they describe, for example focusing on issues of measurement, on the action of unpredictable or stochastic systems, or on the operation of systems involving feedback loops, emergence, or sensitive dependence on contextual conditions.

Importantly, many rich or multi-layered phenomena admit more than one valid interpretation. Moreover, even when one source or type of variation is dominant, the quantitative data collected from such settings themselves do not always offer distinctive cues to guide correct classification. For instance, the distributions of Figures 10 and 11, below, arise from situations that are more or less clearly dominated by issues of measurement (10a), probability (10b), and complexity (11). Yet the data distributions themselves are all visually similar and do not betray the types of mechanisms that generated them. As with “distribution,” so too with other core ideas involved in modeling risk and uncertainty, which also cut across these domains (e.g., variation, independence, sample space, or expected value).

Given that the data themselves may not determine the “correct” model or interpretation of the source of uncertainty in the situation, students have the opportunity to engage in authentic forms of argumentation, negotiation and inquiry. In such settings, students’ different ways of thinking about phenomena and the different resources they draw on for interpretation can offer researchers insights about possible connections among ideas.

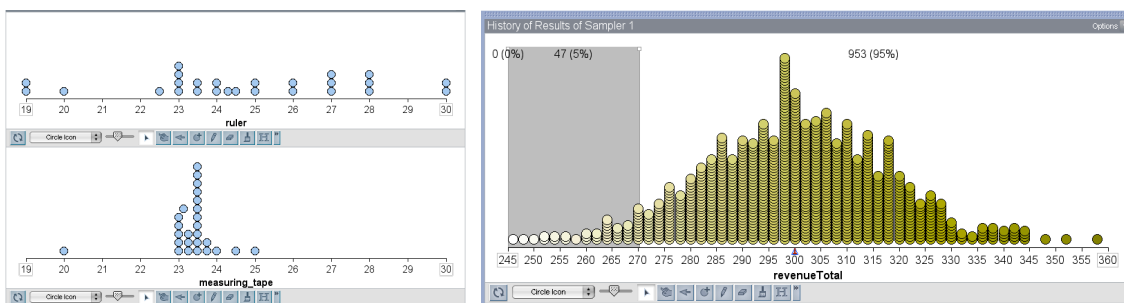


Figure 10. Measurement (left). Distributions in measures of the circumference of the teacher’s head using a ruler and a measuring tape (after Konold & Lehrer, 2008). Probability and sampling (right). Revenue outcomes from one booth in 1,000 simulated charity events in the extended gamble activity.

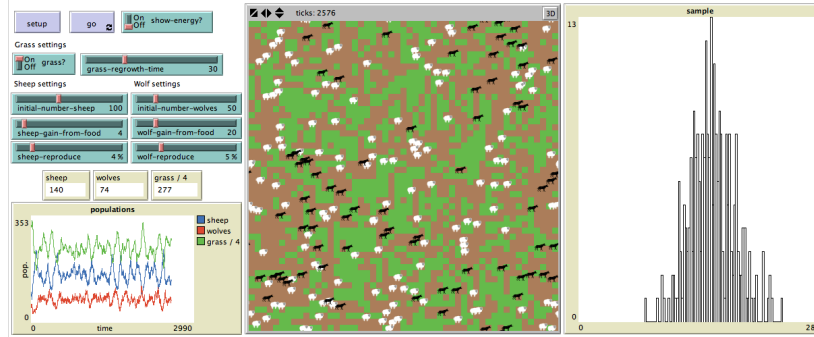


Figure 11. Complexity. Periodic sampling of the sheep population in a NetLogo simulated ecosystem.

Furthermore, in the course of developing solutions to modeling problems, learners are forced to take a range of *stances* toward the uncertainty they encounter in phenomena, depending on their immediate goal and focus. These stances include:

- *Conceptualizing the situation.* Here, the modeler invokes notions of uncertainty, chance, or complexity in making sense of the primary phenomenon (i.e., answering the question, “What kind of thing is it that we are describing?”).
- *Gaining data or information about the situation.* In making or analyzing measurements of any phenomena, modelers grapple with error and imprecision. Further some forms of measurement may come to seem epistemologically problematic (i.e., these actions introduce a kind of “Heisenberg” principle, the urgency of which is heightened by the framing in terms of risk).
- *Describing and interpreting a collection of data about the situation.* In understanding situations of uncertainty, we engage such concepts as our “expectation,” or our assessments of “best case and worst case scenarios.” Different domains offer different tools for addressing these questions, even if most involve some form of statistical thinking. We have shown above how distance can act as such a tool, one that derives from students’ experiences of geometry and physical measurement.
- *Combining and weighing these perspectives to formulate a strategy or decision in response to the situation at hand.* Returning to ideas of variation, managing risk may primarily involve *controlling* and *minimizing* variation, or *describing its cause* and *estimating its consequences* on some other process or decision. In such cases, substantially *different* approaches and lines of inquiry may suggest themselves depending on whether the modeler views the variation in the system as arising from error (e.g., “noise,”), from inherent randomness, or from complexity (e.g., “feedback” or “emergence”).

In MEAs, as learners assume any of these stances toward the phenomena they are modeling, or as they shift among them, their work offers us different perspectives on their emerging thinking. Moreover, these stances toward a problem do not necessarily form a sequence: learners may cycle through them multiple times, with the thinking from each stance deeply affecting the others.

Conclusion

We have a great deal to learn about idea development in authentic problem settings where probabilistic reasoning, uncertainty, and risk are foregrounded. In this article we have argued that Model-Eliciting Activities and Model Development Sequences can offer promising settings for investigating key questions in this area. In closing, we note that in this area as in many others, learners’

knowledge is constituted as much by *connections* forged between big ideas in the domain and between these ideas and prototype situations as by an “intrinsic” understanding of these big ideas in isolation. The process of learning may therefore be expected to be multi-dimensional and non-linear. Thus, research into idea development in this area needs to identify both (a) local operational definitions of what it means for students to have learned big ideas in the domain, and (b) longer-timescale accounts of students’ growing appreciation of the significance and interrelatedness of these big ideas. We hope that the focus on learning processes across multiple MEAs and MDS units can contribute insights to this ongoing effort.

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