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Good Models and Good Representations are a Support for Learners' Risk Assessment

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Abstract: When learners have to make sense of risky situations, they can use mathematical models and representations which facilitate successful risk assessment. Based on theoretical considerations on the benefits of specific models and specific representations in such contexts, we present empirical findings of a study which examined whether students use such models and representations in their risk assessment. We conclude that the availability of adequate models to learners may help them gain transparency when facing risks and thus foster their decision-making.

Keywords: risk, representations, models, assessment, decision-making.

Introduction

If learners are unable to model risk, dealing with risk is hindered – learners may view the loss of resources as wholly unpredictable. They will hardly have any criteria for assessing expectations and may develop excessive fear becoming risk averse – or – conversely – become risk prone and engage carelessly in high-risk endeavors.

Human life is characterized by risk: most decisions and actions have a risky side to them. The difference between the ways we deal with risks today and the way our ancestors dealt with them is that we now have theoretical support to analyze risks. Simply defining risk as the possibility of losing a (valuable) resource with a certain probability provides a framework that facilitates assessing risk. A typical example is a gamble: someone offers us to throw a coin and if heads turns up we gain, say 10 dollars. Yet if tails turns up we have to pay 5 dollars. What is the risk here? Should we accept the gamble? Observe that we lose 5 dollars with a probability of 0.5 (our risk) and gain 10 dollars also with a probability of 0.5 (our benefit). Thus it should be considered worthwhile to accept the gamble. We say that a situation (an experiment) is risky when at least one of the events with strictly positive probability produces a loss of resources (e.g. time, health, energy, money). Most decisions we make daily have risky sides, which we should be able to evaluate (Gigerenzer, 2014; Kurz-Milcke, Gigerenzer & Martignon, 2008; Martignon & Krauss, 2009). Risk analysis is today a highly developed sub-discipline of decision analysis and requires sophisticated mathematical treatment. Yet the basic elements for getting along in everyday life are simple and can be grasped even by young children.

Representations that foster risk assessment

In this section we illustrate two basic tools for risk assessment: distinguishing between absolute and relative risk and understanding conditional probabilities. We often hear that the use of a medicine may reduce the risk of a certain disease by, say, 50%. How should we interpret this information? Actually, there is something missing in this piece of information. In fact it communicates a relative risk without specifying the absolute risk.

Consider the following "true story": In the mid-1990s, the British press reported the results of a study that women who took this contraceptive pill increased their risk of thromboembolism by

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100%. *Thromboembolism* means blockage of a blood vessel by a clot and can lead to fatal strokes. Hearing the bad news, thousands of British women panicked and stopped taking the pill, which led to a wave of unwanted pregnancies. But what did the study in fact show? Out of every 7.000 women who did not take the pill, one had thromboembolism, and out of every 7.000 who took it, this number increased from one to two. An absolute risk is communicated with mention of the reference grade, whereas a relative risk only considers the gain or loss with respect to a specific element of this grade. Understanding the difference between absolute and relative risks should be part of the school curriculum.

Good representations for an elementary introduction of relative and absolute risk reduction in Primary School

The dynamical webpage www.eeps.com/riskicon presents convenient representation tools for instructing school students on the difference between absolute and relative risk. Figure 1 is extracted from that webpage. The simulation lets you explore risk reduction. A number of people have had bike accidents. Half of them were wearing helmets. The relative risk reduction when wearing a helmet is of 50%. Yet the absolute risk reduction is of 2 out of 10 to 1 out of 10. The sliders at the right hand side of the iconic illustration can be used to vary the size of the sample, the probability of injury when riding without a helmet and the risk reduction provided by the use of a helmet.

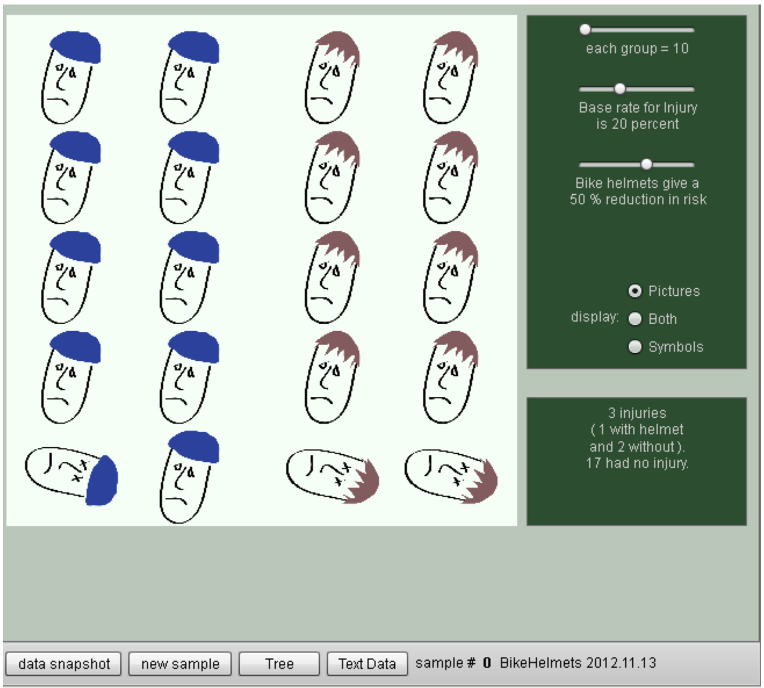

Figure 1. Absolute/relative risks with and without helmets

Another fundamental tool for reckoning with risks is conditional probability. Understanding conditional probabilities is essential when we are faced with decisions on our health: symptoms or tests tell us, with a certain probability, whether we suffer from a certain disease or not. How do we assess the probability of having a disease given that we have the corresponding symptom? This is a conditional probability. It is usually simpler to assess the probability that the symptom is present given that we have the disease (causal direction) than the other way round, namely to assess the probability of our having the disease when the symptom is present (diagnostic direction).

In order to compute, say, the validity of a test or symptom for diagnosing a disease we have to reason the Bayesian way. This means, that given the base rate or probability of a disease, its sensitivity (probability that a patient tests positive when she has the disease) and its specificity (probability that a patient tests negative given that she does not have the disease) we have to compute the probability of a patient having the disease given that she tests positive. Severe mistakes in this type of reasoning are made not just by lay people but also by experts which, according to the research in the Heuristics and Biases program of Kahneman and Tversky (1982), is mostly due to a base-rate neglect. Here the ABC Group led by Gerd Gigerenzer has proven how useful it is to resort to so called natural frequencies and imagining a population of, say, 1000 people (Gigerenzer & Hoffrage, 1995): In this imagined the number of persons out of these 1000 who have the disease corresponds exactly to the base rate, while the proportion of true positives out of the grade of people with the disease corresponds exactly to the sensitivity and the proportion of true negatives out of the grade of people without the disease corresponds exactly to the sensitivity. Thus if the base rate or prevalence of the disease is 10% then exactly 100 of the 1000 people imagined have the disease. If the sensitivity of the test is 8% then exactly 80 of the 100 people with the disease test positive, and if the specificity is 90% then exactly 891 out of 990 without the disease will test negative. In a strict mathematical sense the so-called natural frequencies correspond to expected frequencies: one expects that 100 out of the 1000 imagined people have – on average – the disease. Eliminating the hurdles of probabilities, like randomness and variability, expected frequencies transport the framework of the Bayesian task back to the simple arithmetic of proportions. Such a setting facilitates reasoning immensely (Gigerenzer & Hoffrage, 1995).

Good representations for an elementary introduction to Bayesian Reasoning in Primary School

Traditional developmental theory, as initiated by Piaget and Inhelder in the second half of the last century, suggests that children do not become proficient at making probabilistic inferences until age 7 (Piaget & Inhelder, 1975). Nevertheless, more recent research has shown that very young children are capable of performing simple probability calculations when task demands are reduced (Zhu & Gigerenzer, 2006). In general, many of Piaget's postulates on children's logical and probabilistic intuitions have had to be revisited and modified. For instance in their book "The early Growth of Logic in the Child" Inhelder and Piaget (1964 [1959]: 101) reported an experiment in which they showed 5- to 10-year-old children pictures, of which 16 were flowers and 8 of these 16 flowers were primroses. The children were asked several questions. One of them was "Are there more flowers or more primroses?" Only 47% of 5- to 7- years old children gave answers in accord with grade inclusion − that is that reflected un understanding that the flowers were more numerous because they included the primroses as a subset. Among 8-years olds, however, a majority (82 percent) gave answers consistent with grade inclusion. This kind of experiment, according to Inhelder and Piaget, indicated that children acquire logical intuitions even without instruction. Of course, this experiment has probabilistic connotations as well: flowers are more probable than primroses. It is important to note, in fact, that the step towards a proportional statement like "8 out of 16 flowers are primroses" is quite natural, once children learn to understand and formulate proportions. Here again, Piaget's pioneering results have been revised by recent experiments on children's early estimates of proportions for simple categorization.

The important discovery in this context has been (Martignon & Krauss, 2007, Multmeier, 2012) that the representation of information plays a fundamental role in children's and adults' understanding of proportions and probabilities. Multmeier investigated children in second and in fourth grade and divided his samples out of second and fourth classes in two groups. Both groups had to solve six tasks like the following:

> *In a small and faraway fairyland village there are 10 inhabitants: 2 princesses and 8 mermaids. Of the 2 princesses, 1 wears a crown. Two of the 8 mermaids also wear a crown. If I tell you I met an inhabitant of the fairyland village and mention this inhabitant was wearing a crown, would you bet it was a princess?*

In one group of children, second graders (7 years) and fourth graders (9 years), had to solve the task based on the texts of the tasks only. In another group, children, second graders and fourth graders, had to solve the tasks based on texts and iconic visualizations, like the following, which corresponds to the task above, on princesses and mermaids:

Figure 2. Iconic representation of a fairyland village

The following graph exhibits the results of this experiment:

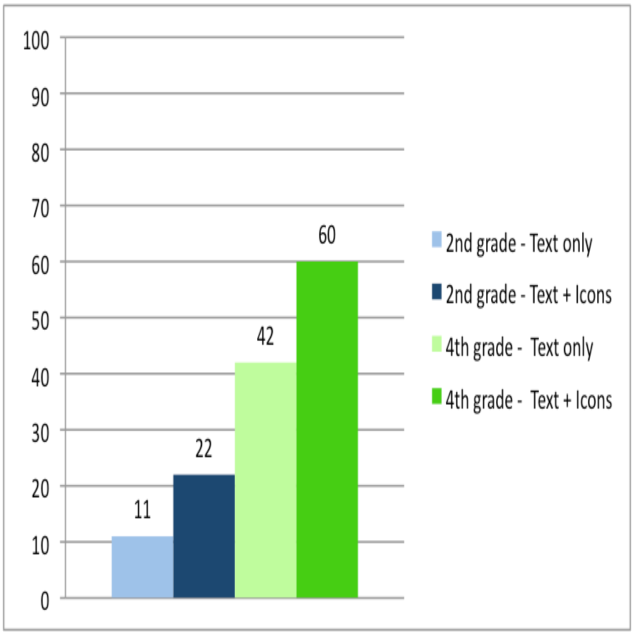

Figure 3. % correct of second graders (n = 91, 45 text only) and fourth graders (n = 85, 42 text only)

The results show that children in fourth grade have much stronger intuitions for proto-Bayesian thinking (i.e., reasoning on conditional frequencies and their inversion) than children in second grade yet they also show that iconic representations foster proto-probabilistic intuitions in both classes. They encourage instruction in fourth grade on proto-Bayesian reasoning by means of the systematic use of appealing iconic representations.

A dynamic webpage for instructing children in primary and secondary school

Selecting the village of mermaids and princesses as an inviting environment for teaching first steps of conditional probability and Bayesian Reasoning the dynamic webpage www.eeps.com/riskicon introduced above, also presents illustrations like the following:

Figure 4. Icon Array to represent a population of mermaids and princesses

In Figure 4 ten stylized inhabitants of a fairyland village are represented: 2 princesses and 8 mermaids. The user has a menu of possibilities: by using sliders he can vary the base rate of princesses among the fairy folk. She can also vary the "sensitivity" of "crown", i.e., the probability, that a princess wears a crown, and also the specificity of a crown, namely the probability that a mermaid does not wear a crown. She can click on the "tree" button to obtain the "double tree" describing this situation, say, of 20 fairy folk, 6 princesses, 4 with a crown and 14 mermaids, of whom 6 wear a crown:

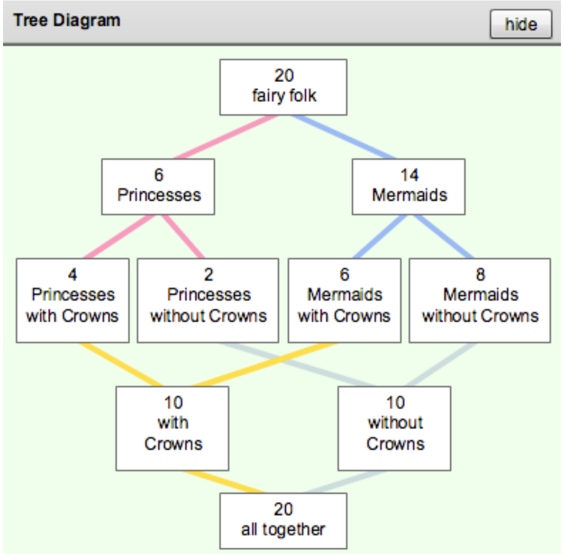

Figure 5. The double tree describing natural frequencies of a fairy folk environment

The double tree represents both directions, the direction from "princesses" to "crowns", which can be called the causal direction ("being a princess may cause wearing a crown"), and the diagnostic direction, from "crowns" to "princesses" ("wearing a crown can be a sign that one is a princess"). The double tree facilitates reasoning not just from cause to effect, but also in the inverse direction, from effect to cause, i.e., Bayesian reasoning with natural frequencies. The dynamic webpage also presents a HIV environment, designed for school students in ninth or tenth grade, where again the user can vary many parameters and visualize the corresponding double trees.

Reasoning with natural frequencies and double trees can be seen as a form of proto-Bayesian reasoning, which constitutes a good heuristic for Bayesian Reasoning with probabilities. First elements of proto-Bayesian reasoning can be conveyed to children as early as in fourth grade, not just with good iconic representations (mermaids and princesses, as in Figure 1) but also with concrete hands-on materials (Till, 2014).

In conclusion, we can think of many situations in which iconic representations foster the understanding and the assessment of risk not just of adults but of school students, young and old. We have also stressed the usefulness of dynamic representations with appealing materials.

Modelling and Comparing Risks

Modelling risks in several steps, as we shall see, makes these risks understandable. In analogy with the modelling cycle by Blum & Leiss (2005), the situational context of a risky situation first needs to be structured and/or simplified, in order to identify relevant characteristics of the situation, in order to determine the resource in danger and in order to establish criteria for describing and evaluating the potential loss of such a resource. This step does not yet require the use of mathematical models or quantification. Translating the situational model into a mathematical model is the subsequent step, which does require mathematical concepts, such as probabilities or expectations. Using a mathematical model for handling the assessment of a risk may lead to a mathematical result, which has to be interpreted against the background of the situation. The result interpreted in the context of the model then has to be validated. At this stage, additional criteria may come into play which could lead to the need of an improved or more refined mathematical model.

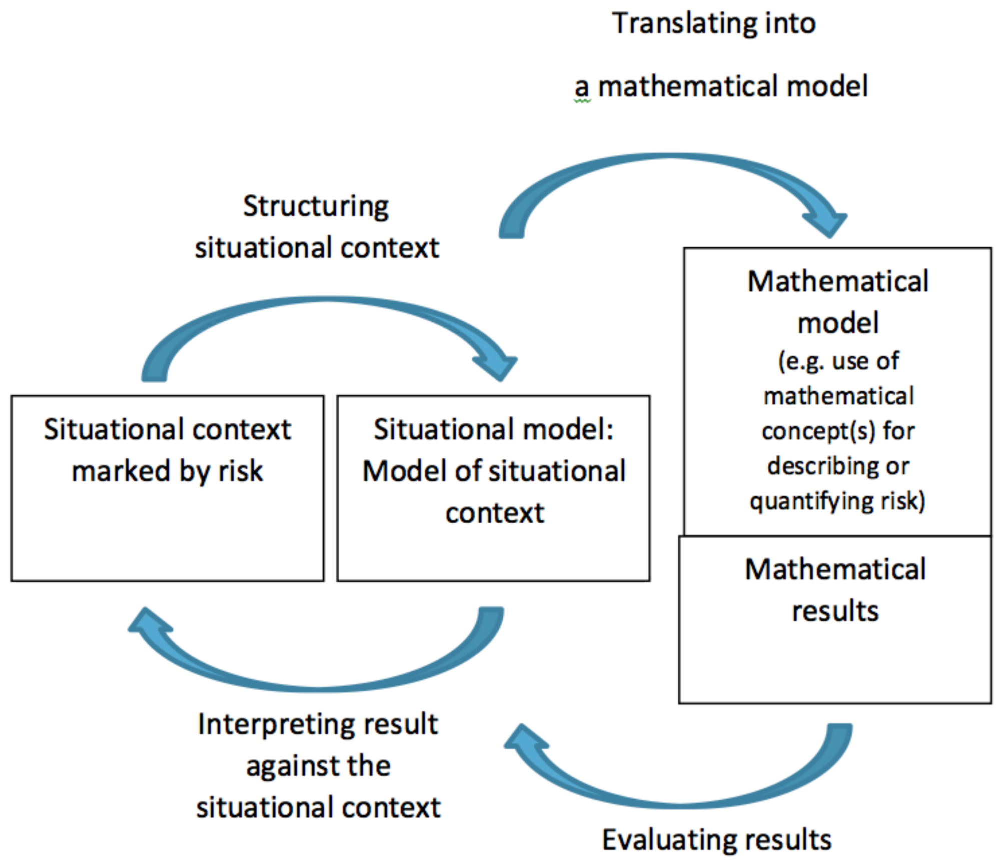

Figure 6. Modelling risky situations (see process models in Blum & Leiss, 2005)

Comparing Risks Playing Ludo

We will present here results of a study we carried through in Baden-Württemberg on the statistical perception and understanding of risk in school: we tested students of fourth grade in primary schools and ninth grade in "Real Schulen" (technical-track secondary schools), as well as undergraduates. We describe only one item of our test devoted to the Ludo game. This task allows us to perform an analysis of school students' use of representations and models for assessing risks.

We begin by introducing the Ludo Game:

Ludo: Players take it in turn to roll a single die. A player must first roll a six to be able to move a piece from the starting area onto the starting square. In each subsequent turn the player moves a piece forward 1 to 6 squares as indicated by the die. When a player rolls a 6 the player may bring a new piece onto the starting square, or may choose to move a piece already in play. Any throw of a six results in another turn. If a player cannot make a valid move she or he must pass the die to the next player. If a player's piece lands on a square containing an opponent's piece, the opponent's piece is captured and returns to the starting area. A piece may not land on square that already contain a piece of the same color. Once a piece has completed a circuit of the board it moves up the home column of its own color. The *player must throw the exact number to advance to the home square. The winner is the first player to get all four of their pieces onto the home square.*

We proposed the following task in fourth-classes and in ninth-classes in and near Stuttgart:

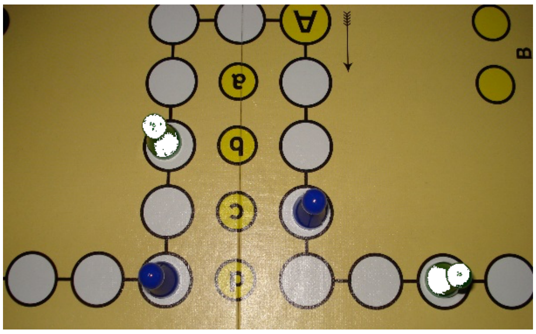

Figure 6. A risky situation for “White” in the Ludo game

The students had the instruction: *Consider two players Black and White. In one position, Black is two fields behind White, and at another position Black is three squares behind White. Assume it is White’s turn. He rolls a “one”. Which piece should White move? Which move is riskier?*

For an adequate modelling in this example, learners may argue that the probability of rolling a “three”, which is risky for the case that the piece on the left is moved, is smaller than the probability of rolling “two or four”, which is risky for the case when the piece on the right is moved. In an alternative adequate model, learners could consider the number of equally probable possibilities. It is hence interesting, whether learners of different age succeed in modelling the risks in the Ludo situation adequately. However, an additional perspective connected with the issue of modelling is of interest, namely how learners represent risks verbally, that is, how they communicate about risks. Even if it has to be taken into account that basic abilities in mathematics and statistics are often necessary for elaborated risk assessment, communicating about risks requires above all dealing with these abilities in a context-related way. In the realm of risk communication, two quality levels of students’ reasoning have been described (Engel, Kuntze, Martignon & Gundlach, 2010):

•Level 1: identifying risks verbally, describing the possibility of loss of a resource

•Level 2: Weighing/comparing risks, identifying decision alternatives and comparing them against the background of risk

Higher levels could also make sense, for instance the use of relative versus absolute frequencies when communicating about risks, the ability to translate from one to the other of these types of frequencies, the ability to establish the completeness of information provided by texts which inform about risks. We will use the example in Figure 6 (“Ludo task”) in order to illustrate the two Levels mentioned above. The resource at risk is, in the case of White, the pieces on the board, which should not be captured by the other player. Identifying this resource and describing the danger of losing it corresponds to Level 1 of risk communication. In case the risks associated with different decision alternatives of White are effectively compared, Level 2 is reached. These levels are treated independently from the decision made in the end. The quality of risk communication is the issue here.

Research Interest

Against the background outlined in the previous section, our preliminary study focused on the following research questions: *How do students assess risks in the context of the Ludo task? Do they succeed in modelling the risks in this situation and how do they communicate about these risks?*

Design of the Study

In order to investigate these research issues, the Ludo task was presented to primary students, secondary students and undergraduate students. This study was carried out in the framework of the project "RIKO-STAT". We report about results from 385 fourth-graders from 21 classes at primary schools) and from 549 ninth-graders from 21 classes at technical-track secondary schools.

The choice of the Ludo task is due to its connection to risk and risk reduction. Observe that the risk of White losing a piece is reduced by ½ if the piece at the left hand side advances one field. Here ½ is the relative risk reduction, while the absolute risk reduction is of 1 piece. Besides the Ludo game belongs to the typical table games played by young students in Germany.

Results

In the following, we will first present some answers from students of different ages. This will illustrate not just our coding of the students' answers but also provide some qualitative insight into their reasoning. We will then turn to overview results on code frequencies in these subsamples. We begin by presenting Noemi's answer. She is a primary student (4th grade) who gives only the short answer displayed in Figure 7.

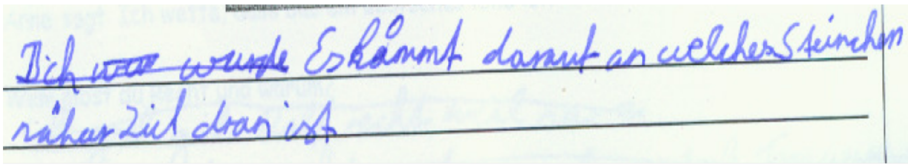

Figure 7. Noemi's answer (4th grade).

Noemi writes: "It depends upon which piece is closer to its final goal". In her answer, Noemi does not describe the risk, instead she mentions the criterion of being closer to the goal as the only basis for making the decision. This is obviously a generally meaningful aspect of the Ludo situation. However, in this case, this criterion is not sufficient for guaranteeing a good model of the risks at stake here. For this reason, this answer was coded as non-adequate modelling. Figure 8 now shows the answer given by Lara (9th grade).

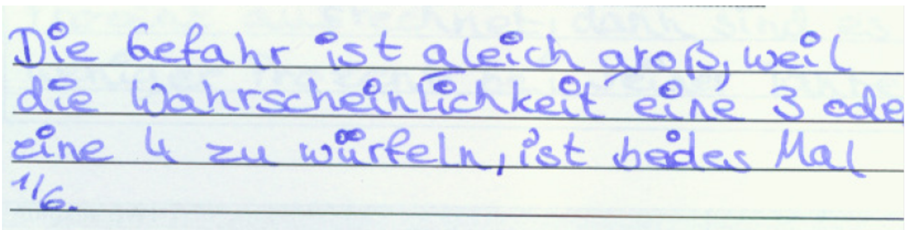

Figure 8. Answer by Lara (9th grade)

Lara says that the danger is the same, because the probability of rolling a 3 or rolling a 4 is 1/6 in each case. From Lara's answer, we can see that she knows that the probability to roll a 3 or a 4 (she meant a 2 or a 4) is each time 1/6. However, she appears not to have remarked that in this case, the probability of rolling a 2 or a 4 has to be compared with the probability of throwing a 3 in order to compare the risks in the probability model she has chosen. Lara mentions a "danger", but she does not explain in detail what the danger consists of, with respect of the possible decisions. Describing the risks explicitly might have helped her to see and distinguish the possibilities of losing one piece with "2 or "4". This could have supported a modelling adequate to the problem situation. Figure 9 now exhibits an answer, in which the risk is verbalized rather indirectly.

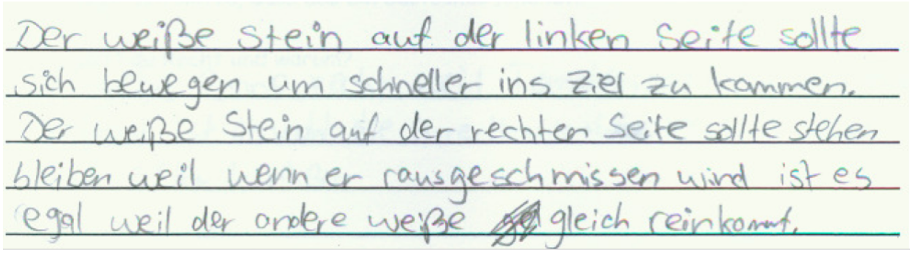
Der weiße Stein auf der linken Seite sollte sich bewegen um schneller ins Ziel zu kommen. Der weiße Stein auf der rechten Seite sollte stehen bleiben weil wenn er rausgeschmissen wird ist es egal weil der andere weiße gleich reinkommt.

Figure 9. Answer by Dennis (9th grade)

Dennis' answer is: "The white piece on the left should move in order to reach the goal sooner. The white piece on the right should stay because if it gets caught this plays no role as the other white piece will reach the goal immediately." Dennis thus describes risks of pieces being caught only for the alternative that the piece at the right hand side is moved. We may, yet, assume that he is generally aware of this type of risk. Moreover, Dennis compares the risks associated with two decision alternatives for White. It is however probable that Dennis interprets the grey fields a, b, c, and d in the middle as the goal of the white pieces. This interpretation is unfortunately wrong. In fact, the task description specifies that the pieces shown are the last ones in the game and so two other pieces must already have reached the goal. Moreover, merely arguing in terms of the distance to the goal is in general not sufficient, because the probabilities of loss of resources have to be properly considered. The answer was hence coded as not exhibiting adequate modelling. The following answer by Katja (9th grade), illustrated in in Figure10, uses relationships between frequencies of possible outcomes when rolling the die.

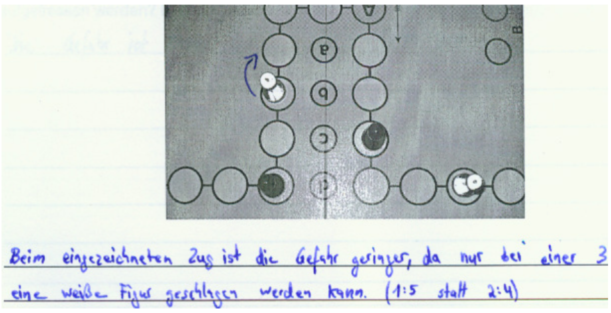

Figure 10. Answer by Katja (9th grade)

Katja's answer was: "The white piece on the left should move in order to reach the goal sooner. The white piece on the right should stay because if it gets caught this plays no role as the other white piece will reach the goal immediately." Katja's answer shows a very short verbal identification of the risk, including comparisons between decision alternatives. Moreover, she uses an adequate modelling of the situation, by counting positive and negative possibilities and comparing their proportions ("1:5 instead of 2:4") In this case, the use of the model appears to be closely connected with reasoning on the comparison of the decision alternatives. As Figure 11 now shows, even primary students can be able to model risks adequately and to describe risks in order to compare them reasonably.

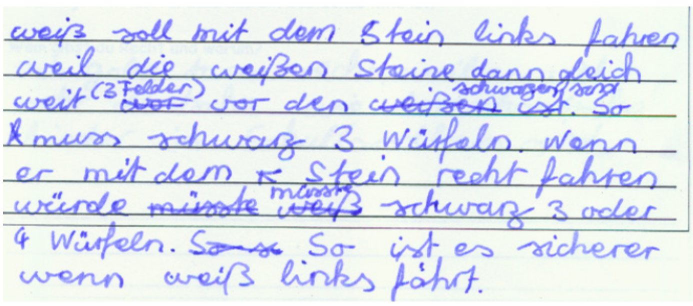
weiß soll mit dem Stein links fahren weil die weißen Steine dann gleich weit (3 Felder) ~~weg~~ vor den ~~weißen~~ schwarzen sind. So muss schwarz 3 würfeln. Wenn er mit dem Stein recht fahren würde ~~müsste weiß~~ müsste schwarz 3 oder 4 würfeln. ~~So~~ So ist es sicherer wenn weiß links fährt.

Figure 11: Answer by Lasse (4th grade)

Lasse answers: "White should move the piece on the left, as the white pieces are then equally far (3 fields) from both black pieces. Thus Black must roll a 3. If he moved the piece on the right hand side, Black would have had to roll a 3 or a 4. Thus it is safer for White to move the piece on the left hand side."

Lasse (4th grade) models mainly by considering all possible cases in a structured way, by asserting that they are basically equally probable. The small mistake ("3 or 4" instead of "2 or 4") obviously does not imply that Lasse did not model the problem adequately. Looking at the way he communicates about risks, we have here a case in which even if risks are neither identified verbally nor described in detail, it is obvious that Lasse compares different decision alternatives against the background of risk considerations. Lasse's answer was hence coded as having reached Level 2 of risk-related communication.

The overview results shown in Figure 12 suggest that it was not easy for the students of both age groups to model risk adequately.

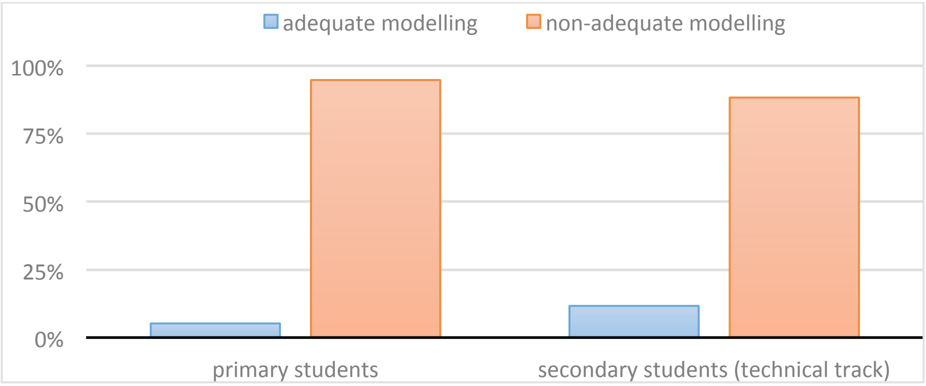

Figure 12: Relative frequencies of codes related to modelling

The relative frequencies show that even 9th-graders (technical track secondary schools) did not exhibit relevant conceptual knowledge to the extent that would have enabled them to model risks successfully.

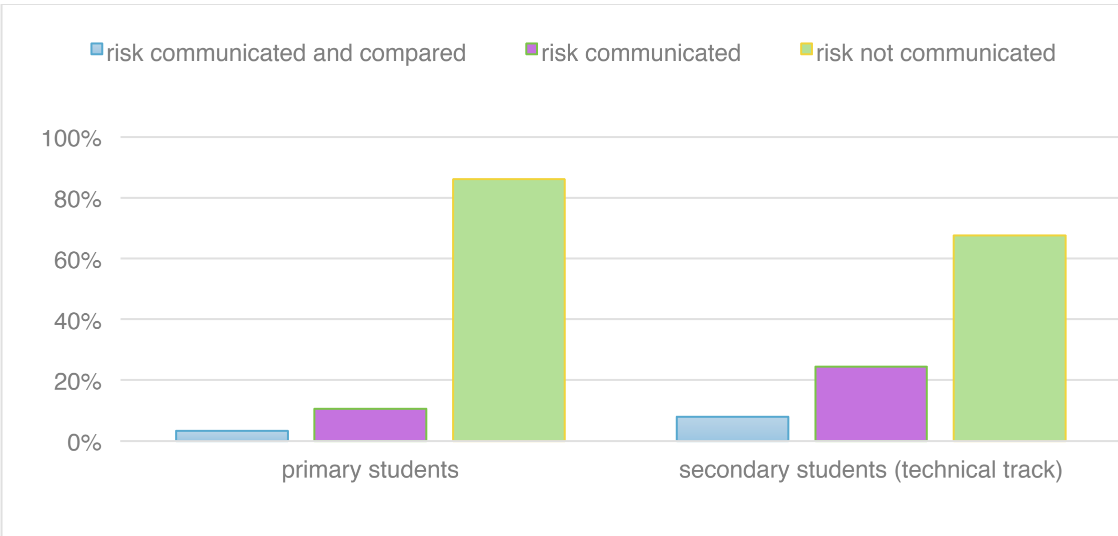

Figure 13. Relative frequencies of codes related to communicating risks

The descriptive results of the code frequencies in Figure13 indicate that a majority of students from both groups did not give verbal descriptions of the risks. Only a minority of the students gave comparisons of risks and different decision alternatives.

Discussion and Conclusions

The results of this study suggest that school students are in need of elementary strategies for modelling risky situations as well as they need strategies for verbalizing and communicating about risks. To express by means of language what the danger of losing a valuable resource amounts to, and in which cases such a loss of a resource may happen is an extremely useful strategy. It does not only support an understanding of a problem situation, but it can contribute substantially to an orientation and to a structured consideration of the risky situation. Mathematical models may subsequently emerge from these strategies for communication about risk. Our proposal here is to prepare school students in the understanding of risks and on the communication of risks by means of the systematic use of transparent, iconic representations.

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