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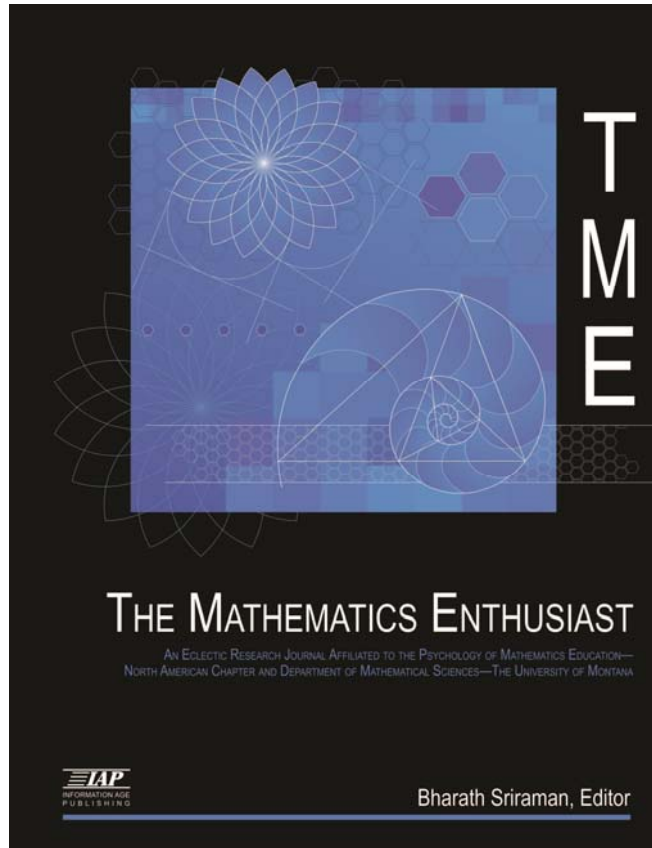
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The journal is published tri-annually.

Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning. The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-12 months.

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# THE MATHEMATICS ENTHUSIAST

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“ . . . if I had to reduce all of educational psychology to just one principle, I would say this: ‘The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.’”

– Ausubel

## Special Issue: The Mathematical Content Knowledge of Elementary Prospective Teachers

Guest Edited by *Eva Thanheiser & Christine Browning*

0. Foreword  
*Lynn Hart*.....pp. 197–200
1. Preface  
*Eva Thanheiser & Christine Browning*.....pp. 201–202
2. Prospective Elementary Teacher Mathematics Content Knowledge: An Introduction  
*Christine Browning, Eva Thanheiser, Alden J. Edson, Patrick Kimani, Dana Olanoff, Jennifer Tobias, & Ian Whitacre* .....pp. 203–216
3. Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Whole-Number Concepts and Operations  
*Eva Thanheiser, Ian Whitacre, & George Roy*.....pp. 217–266
4. Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Fractions  
*Dana Olanoff, Jane-Jane Lo, & Jennifer Tobias*.....pp. 267–310
5. Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Decimals  
*Signe Kastberg & Crystal Morton*.....pp. 311–332

6. Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Geometry and Measurement  
*Christine Browning, Alden J. Edson, Patrick Kimani, & Fatma Aslan-Tutak*.....pp. 333–383
  
7. Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Algebra  
*Krista Strand & Briana Mills*.....pp. 385–432
  
8. Prospective Elementary Mathematics Teacher Content Knowledge:  
What Do We Know, What Do We Not Know, and Where Do We Go?  
*Eva Thanbeiser, Christine Browning, Alden J. Edson, Jane-Jane Lo, Ian Whitacre,  
Dana Olanoff, & Crystal Morton* .....pp. 433–448
  
9. Appendix: References by Content Area .....pp. 449–459

## FOREWORD

The authors in this Special Issue of *The Mathematics Enthusiast* make an important contribution to the knowledge base in mathematics education. They examine a body of research on a significant issue. They review what we know and make suggestions about what we need to know. They move the field forward by taking the time to look back and learn.

Specifically, the authors examine the literature on prospective elementary teachers' (PTs) mathematical content knowledge in several domains: whole-number concepts and operations, fractions, decimals, algebra, and geometry and measurement. They situate their review in three time periods: historical (prior to 1998), current (1998–2011), and a view to the future (from 2012 on).

The warrants for doing this review are many, but the most concerning is the *ongoing* problem of the limited mathematical content knowledge of PTs. Coupled with the recent adoption in the United States of more rigorous mathematics in the *Common Core State Standards* (<http://www.corestandards.org>), the stakes are high. An example from a recent lesson I taught on measurement illuminates how dire the situation really is. *Area* is a topic that is addressed in some depth in the standards for third grade in the Common Core. I was teaching a model lesson on measurement to a group of somewhat high-achieving fourth graders. The standard asks that students *apply the area and perimeter formulas for rectangles in real world and mathematical problems* (Common Core State Standard MCC4.MD.3). An assumption of understanding from third grade is obvious, but I questioned if the understanding was there. I drew a simple  $5 \times 7$  rectangle on the board and asked the



students to name the figure and to tell me what the numbers meant. One young man suggested that “you can multiply and you get the area.” Another child disagreed and said, “No, that’s the perimeter.” After some discussion, the class agreed that you multiplied to get the area, but **not one student** could tell me *why* we multiply or *what the 5 and the 7 stood for*. I showed a video of the lesson to the PTs in my methods class and stopped at the point of asking the “why” questions. Only one PT raised her hand indicating she thought she could answer the questions. They all agreed if they were teaching the class they probably would have been satisfied if the students knew to multiply to calculate area. This is simply not acceptable. If we are to improve the mathematical content knowledge of elementary students, we must improve the mathematical content preparation of their teachers. We need to stop and take stock of what we know about this issue and what we need to know. The articles in this volume provide an important first step in this direction.

Another concern to me is who may actually read this work. There is no doubt it should become a dog-eared document for Ph.D. students working on research on mathematical content knowledge of PTs, or a wonderful resource for various mathematics education doctoral courses. My hope, however, is that it would become a regular part of course packets for graduate teaching assistants and part-time instructors teaching mathematics content courses for elementary teachers, and that it would actually find its way to the desks of faculty teaching those courses. Wouldn’t it be helpful if the instructor of a Geometry for Teachers course understood that there is solid research showing that many PTs aren’t able to articulate basic differences in quadrilaterals (see “Mathematical Content Knowledge for Teaching Elementary Mathematics: A Focus on Geometry and Measurement,” this volume) or that PTs generally do not understand the difference in

partitive and quotative division problems and are much more likely to use partitive models in division situations (“Mathematical Content Knowledge for Teaching Elementary Mathematics: A Focus on Whole-Number Concepts and Operations,” this volume)? We know that journals such as *Teaching Children Mathematics* provide superb examples of research into practice for elementary teachers that impact teaching. The work in this volume could have a similar effect on instructors of mathematics content courses for elementary teachers. Unless that happens, unless this knowledge is shared with all the stakeholders, much of what we have learned and that is so aptly reviewed in this volume will not have a real bearing on the mathematics education of prospective or practicing elementary teachers. I encourage you to read and to *share*.

Lynn C. Hart  
*Georgia State University*



## **PREFACE**

### **Special Issue: The Mathematical Content Knowledge of Elementary Prospective Teachers**

*Eva Thanheiser and Christine Browning*

Bharath Sriraman noted in his Editorial for Vol. 10, nos. 1–2 that the first issue of *The Mathematics Enthusiast* (known then as *The Montana Mathematics Enthusiast*) published in April 2004 was “the result of four idealistic elementary school teachers believing in the mission of this journal and writing about their attempts to reconcile the mathematics content they were learning in a mathematics for elementary school teachers course with existing mathematics education research found in practitioners’ journals as well as standards imposed by institutions’ framing policy” (p. 2). Ten years later we return to a similar focus.

We have searched the peer-reviewed published research for studies focused on examining and describing the mathematical content knowledge of prospective teachers between the years of 1978 to 2012 and summarized findings across various content areas: whole numbers and operations, fractions, decimals, geometry and measurement, and algebra. Each content area presents its findings within three time periods: a historical look at work prior to 1998, a current perspective of research from 1998 to 2011, and, finally, a view of the horizon of published research from 2011 and 2012. Specific details of the history of those involved in the writing of this collective summary work and the methods employed in the summary research, along with a brief description of the subsequent

articles, are presented in “Prospective Elementary Mathematics Teacher Content Knowledge: An Introduction.”

This Special Issue has taken many people, many years to complete. We are grateful to all of those who participated in any of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) Working Groups over the years of meetings. Many people made contributions to the overall task and we appreciate the time and work that you gave. We want to thank those who helped us in our work by reviewing earlier drafts of our papers: Dhimitraq Duni, Nicole M. Wessman-Enzinger, Sinan Kanbir, Tyler Lebsack, Megan L. Nickels, and Theodore J. Rupnow. Thanks to our super copyeditor, Hope Smith, who was very helpful and patient in determining appropriate formatting and styles for our paper. We also want to thank Drs. Lynn Hart and Ann Kajendar for their helpful comments, and a special thanks to Lynn for writing the Foreword.

**Prospective Elementary Mathematics Teacher Content Knowledge:  
An Introduction**

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*The first and fundamental requisite for every teacher is that he have thorough command of the subject matter which he teaches; that he have mastered it so well that he speaks with his own authority; only so can he hope to lead the pupil to the corresponding feeling of independent mastery.*

*– J. W. A. Young, 1920*

This Special Issue on the mathematical content knowledge of prospective elementary teachers (PTs) provides summaries of the extant peer-reviewed research literature from 1978 to 2012 on PTs' content knowledge across several mathematical topics, specifically whole number and operations, fractions, decimals, geometry and measurement, and algebra. Each topic-specific summary of the literature is presented in a self-contained paper, written by a subgroup of a larger Working Group that has

collaborated across several years, resulting in this Special Issue sharing the final work. The authors hope this summative look at prospective teacher content knowledge will be of interest to the mathematics education community and will be a useful resource when considering future research as well as designing mathematics content courses for prospective elementary teachers.

The following sections in this issue provide background information on our overarching framework for the mathematical content knowledge of prospective elementary teachers as well as our rationale for conducting the summary of research. We briefly describe the intent and history of the Working Group that conducted the summaries, followed by the methods utilized in the summary process. Finally, we provide a description of what follows in each subsequent paper and close with our intentions of how this Special Issue might be used by our readers.

### **Background**

The mathematical preparation of K–8 students is a challenge both in the United States and internationally. Studies from many countries report students coming away from their elementary education having memorized facts and procedures with varying degrees of success but not developing robust mathematical conceptions or flexibility in their reasoning (e.g., Reys et al., 1999; Stigler & Hiebert, 1999). Students who struggle in school mathematics have limited career options. Even those who perform well in mathematics courses are unlikely to enjoy mathematics or take an interest in science, technology, engineering, and mathematics careers if they experience the subject as dry and procedurally focused. Mathematics instruction can emphasize conceptual understanding and the engagement in mathematical practices. In order to positively influence the

direction of mathematics teaching and learning, our elementary teachers must be adequately prepared.

Research over the last few decades has shown that the work of teaching mathematics requires a different knowledge base than the mathematical knowledge required for other professions (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Conference Board of the Mathematical Sciences [CBMS], 2012). Ball and her colleagues identify this knowledge as *Mathematical Knowledge for Teaching* (MKT), which they define as the “mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” (Ball et al., 2008, p. 399). They developed a framework for MKT consisting of two domains, subject matter knowledge and pedagogical content knowledge (see Figure 1).

### **Domains of Mathematical Knowledge for Teaching**

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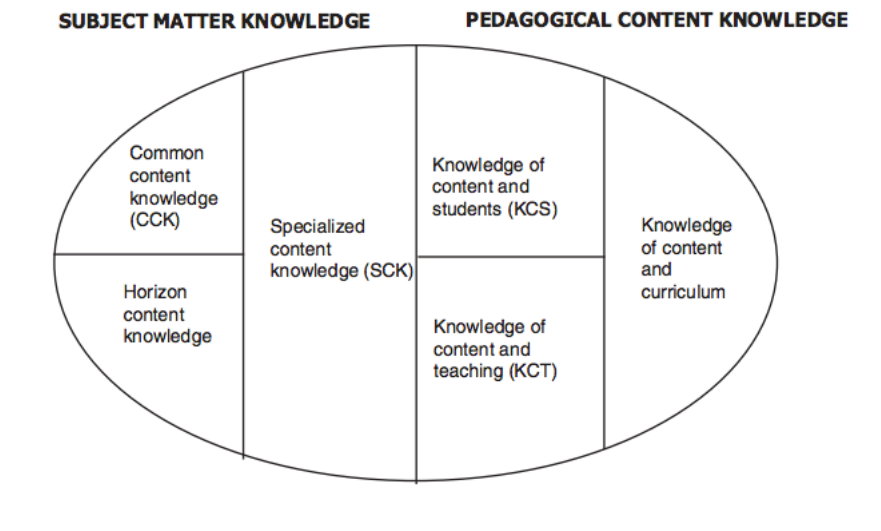


Figure 1. Mathematical knowledge for teaching framework (Ball et al., 2008, p. 403).



For these summary papers, we chose to focus primarily on what could be considered prospective teachers' subject matter knowledge, or *content knowledge*. Included in mathematics content knowledge are *Common Content Knowledge* (CCK), which is described as the mathematical knowledge that everyone should know; *Specialized Content Knowledge* (SCK), described as the mathematical knowledge that is special to the work of teaching; and *Horizon Content Knowledge*, which is an understanding of how mathematical topics fit together and make up a curriculum. In addition to these three types of knowledge, we also include *Knowledge of Content and Students* (KCS), which involves understanding students' thinking and difficulties with mathematics. While Ball and her colleagues include KCS in pedagogical content knowledge, we argue that understanding children's thinking can help in the development of PTs' specialized content knowledge, and thus should be included in a summary of PTs' content knowledge.

Unfortunately, PTs often do not come to teacher education with adequate subject matter knowledge (e.g., Ball, 1990). Even after having taken their college mathematics courses, many PTs do not reason mathematically in flexible or sophisticated ways (e.g., Yang, Reys, & Reys, 2009). These deficiencies in mathematics content knowledge are also seen in practicing elementary teachers in the United States and many other countries (e.g., Ma, 1999; Reys et al., 1999). Thus, there is evidence that PTs' mathematics content knowledge does not necessarily improve on its own once they become teachers. Rather, the responsibility falls on mathematics teacher educators to help PTs develop a strong base in their content knowledge during their college years.

Researchers, teacher educators, and organizations have noted the need to improve PTs' mathematics content knowledge and have called for efforts to that effect (Ball, 1990;

CBMS, 2001, 2012; Mathematics Teacher Preparation Content Workshop Program Steering Committee, 2001). Research is necessary to support these efforts. In particular, research concerning PTs' mathematical thinking in specific content areas would inform instruction. As the authors of *The Mathematical Education of Teachers* observe, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they do know" (CMBS, 2001, p. 17). Mathematics teacher educators need to understand the conceptions with which PTs enter their classrooms in order to design instruction that builds on those conceptions (Brown, Bransford, & Cocking, 1999).

By reviewing mathematics education research concerning PTs' content knowledge, we seek to establish what is known up to this point in time. Summarizing what is known also enables us to identify areas for further research. We envision these summary papers to be a starting point for future directions in research on PTs' content knowledge and the development of that knowledge, as well as providing information useful in the design and development of mathematics courses for PTs.

### **Brief History and Intent of the Working Group**

The current set of authors has participated in one or more Working Groups at the Psychology of Mathematics Education–North American (PME-NA) Chapter, the National Council of Teachers of Mathematics (NCTM), or the Association of Mathematics Teacher Educators (AMTE) meetings on a regular basis since 2007 and has presented at several of those meetings (e.g., NCTM 2007, PME-NA 2009, AMTE 2009, AMTE 2010), as well as at the International Congress on Mathematical Education (ICME 2012) and PME 2013. At PME-NA 2007, the Working Group agreed on the need for the construction of a research base for the study of prospective teacher content knowledge. This included a need to summarize

existing (completed and current) peer-reviewed research and to develop a research agenda. At PME-NA 2009, the Working Group grew in membership and allowed for the work of summarizing the existing literature to be divided into five content areas: whole number and operations, fractions, decimals, geometry and measurement, and algebra. The group was divided into five subgroups with each focusing on one of the five content areas. The ultimate goal was to have each group produce a summary paper of the existing literature on prospective teacher content knowledge. A secondary goal was to establish continued collaborations and to grow professionally through developing our pedagogy, especially as it relates to the teaching of content courses for prospective elementary teachers. To achieve the first goal we designed the following research questions:

1. What research has been conducted on elementary prospective teachers' content knowledge?
2. What have we learned about prospective teachers' content knowledge?

At AMTE 2010, the group met to refine the guidelines for creating these individual paper summaries, and at PME-NA 2010, a rough draft of the combined summaries was refined. Over the years, we continued this collaboration and worked together to refine the methodology and the parameters of the overarching study.

### **Methods**

As the Working Group met over the years, the scope of the research review was extended. Initially in 2007, the Working Group focused on a "current perspective" to provide an in-depth description of what is known about prospective elementary teachers' content knowledge from a review of peer-reviewed research articles of the last decade. Initially this time period was 1998–2008, which was eventually extended to include 2010,

as the work of the group continued over several years. At a subsequent PME-NA meeting, the Working Group decided to include a “historical look” to describe what was known in the specific content area prior to 1998. Lastly, the Working Group made the decision to update the current perspective to include recent peer-reviewed articles in 2011 and provide a “view of the horizon” to present future directions that built upon prior sections by examining peer-reviewed articles in 2012 and conference proceedings published in 2011–2012. While the actual review process did not follow a chronological timeline, in this Special Issue, each content group presents its summaries of the literature on mathematics content knowledge of prospective teachers in the following three time periods: *A Historical Look*, *A Current Perspective*, and *A View of the Horizon*. Thus, the description of the common methods will follow this chronological timeline as well.

### **A Historical Look**

For the time period prior to 1998, each content group conducted an ERIC search using general search terms such as *preservice*, *prospective*, *elementary*, *teacher*, *education*, and *content knowledge*, as well as specific content search terms such as *number*, *whole number*, *addition*, *subtraction*, *geometry*, and *algebra*. Combinations of search terms were entered into the ERIC database. Since all countries do not use the same grade-level classification system as the U.S., we decided to look at findings from studies of PTs preparing to teach children aged 3–14 to account for cases with combined middle and elementary certifications. The title and abstract of each research article resulting from the two searches were read to determine whether the article focused on elementary PTs’ mathematics content knowledge. If the title and abstract did not suffice to make a determination of fit, reviewers read the whole article.

### **A Current Perspective**

For the time period between 1998 and 2011, we conducted an ERIC search using the same keywords described in the historical perspective above. Likewise, a determination of fit for each article was made using the same process as described above. A list of 23 journals from which articles were found for summarizing was compiled.

Subsequently, it was brought to our attention that one article published between 1998 and 2011 in one of the 23 journals was not included in the results from our ERIC search. We then found that ERIC does not contain all years for all journals included in the database. Thus, each journal was then carefully reviewed for additional articles focusing on PTs' content knowledge between 1998 and 2011 to ensure all articles focusing on PTs' content knowledge in those identified journals were found. This review produced more articles that were published in a year not included in ERIC database or were not indexed with any of the previously listed search terms.

### **A View of the Horizon**

For the final time period, 2011–2012, we conducted an ERIC search for the year 2012 using the same process described earlier. In addition, we reviewed the conference proceedings from both the International Group and the North American Chapter of the Psychology of Mathematics Education (PME and PME-NA) for the years 2011 and 2012. For this review, each content group carefully searched for keywords in the titles and abstracts of all papers found in the proceedings. If the title and abstract did not suffice to make a determination of fit, reviewers read the whole paper.

### **Inclusion/Exclusion Criteria**

The Working Group established exclusion criteria across all content groups and excluded articles that had (a) a general description of content knowledge that lacked specific attention to three primary content areas (thus, our claims are restricted to these three content areas): numbers and operations (including whole numbers, fractions, decimals, and operations), geometry and measurement, and algebra; (b) a sole focus on perceptions about mathematics not connected to content knowledge needed for teaching (we make no claims about PTs' beliefs in this Special Issue); (c) a focus on describing classroom practice or activities with a lack of attention to research design methods; and (d) a primary focus on high school PTs, mathematics majors, or inservice elementary teachers (our claims are restricted to prospective elementary teachers). For each content group, at least two researchers met to discuss the inclusion/exclusion of articles in their related content area. All disagreements about inclusion/exclusion into the database were resolved through discussion.

### **Database**

The database included peer-reviewed research articles focusing on the mathematical content knowledge of elementary PTs in any of the content areas described earlier. The studies in our database listed the reference, content area, research questions, study type, research design, lens or approach used, selection criteria, description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions and implications.

## **Description of What Follows**

The results of the Working Group are summarized in each of the subsequent papers in this Special Issue. Each paper focuses on a different content topic (whole numbers and operations, fractions, decimals, geometry and measurement, and algebra) and is organized into the following categories: historical, current, and horizon. The papers are presented as literature reviews in terms of what is known regarding PTs' content knowledge.

### **A Focus on Whole-Number Concepts and Operations**

The research illustrates that prospective teachers' knowledge of this topic is largely dependent upon standard algorithms for solving a given type of task. In addition, they struggle with justifying why the algorithms work. The authors note that more research is needed regarding the types of conceptions PTs have when entering teacher education programs, as well as a need to document how their understanding develops.

### **A Focus on Fractions**

Research shows that PTs are often able to solve fraction problems algorithmically but not justify the algorithm or represent the situation with a correct model, such as a word problem or diagram. In addition, PTs' understanding of fractions, in general, tends to be limited, in that most think of fractions only in terms of a part-whole interpretation. More research is needed to better understand PTs' conceptions of fractions and ways to improve their understanding, as well as to document how their understanding develops.

### **A Focus on Decimals**

Though PTs may be able to successfully solve computational problems with decimals, they tend to lack a conceptual understanding of the structure of decimals. Historical and current research illustrates that PTs' difficulties stem from their

understanding of place value, incorrectly transferring whole number algorithms to decimals, and with their understanding of the density of decimals. Future research is needed to systematically examine how PTs' understanding of decimals develops.

### **A Focus on Geometry and Measurement**

Prior research has documented that PTs' understanding of geometry and measurement is limited largely to memorized procedures. Though the research literature does not address every topic within geometry and measurement, the research that has been done suggests that (a) the van Hiele levels, (b) dynamic geometry software, and (c) methods fostering PTs' understanding of concept images and definitions can be useful in improving PTs' conceptual understanding of these topics. Future research is needed to address ways to develop PTs' understanding of geometry and measurement as well as to address the topic gaps that still exist in the literature.

### **A Focus on Algebra**

The summary of the algebra research shows that PTs can readily use symbolic representations with variables, expressions, and equations; however, they have difficulties with interpreting and connecting various representations to each other or to a problem situation. In addition, PTs' computational methods are often inflexible, inefficient, or incorrect. Recent research suggests developing PTs' understanding by focusing instruction around justifying multiple representations and solution methods. However, more research is warranted regarding how this understanding develops, as most has focused on either PTs' incoming conceptions or analysis of pre/posttests.

A common theme throughout the papers is that PTs' understanding of each topic is limited to using algorithms, and difficulties lie in their ability to justify why the algorithms



work. In addition, previous research has focused chiefly on describing PTs' conceptions of these content topics, whereas recent research is moving toward documenting how PTs' mathematical understandings develop. More research is needed as each summary paper in this Special Issue illustrates that the research base regarding PTs' understanding is limited.

### **Final Thoughts**

The intent of this Special Issue is to summarize and share what research suggests regarding the mathematical content knowledge of prospective elementary teachers. Have they met the expectations described in Young's 1920 statement of mastering the subject so well that prospective teachers can lead their students "to the corresponding feeling of independent mastery"?

Given the summary information provided, we believe this Special Issue could be a resource in:

- graduate course work and seminars, prompting ideas for future research directions and topics;
- the design and development of current and future research on the mathematical content knowledge of PTs, providing a summary of the extant research literature through 2012 in the selected mathematical topics;
- the design of mathematics content courses for PTs, providing information about PTs' common misconceptions, as well as strategies and tools that may help PTs work through the misconceptions and develop a better understanding of the mathematics;

- the design of content-specific professional development for elementary teachers, where information on misconceptions and strategies and tools for learning may still provide useful information for improving the teachers' content knowledge.

Thus, we hope this Special Issue will be a useful reference for future research as well as strategies for practice related to the development of the mathematical content knowledge of prospective elementary teachers.

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Whole-Number Concepts and Operations**

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**ABSTRACT:** This report represents part of a recent effort to summarize the state of knowledge of prospective elementary teachers' (PTs') mathematics content knowledge and the development thereof. Extensive reviews of the research literature were conducted by a recent PME-NA Working Group across various content areas. This report focuses on whole number and operations. Research in this area is scarce. What we do know from the literature is that PTs' knowledge of whole number and operations is insufficient and in need of improvement. PTs reason about whole numbers and operations in ways that are tied to the standard algorithms. At the same time, they are hard-pressed to explain why these algorithms work. PTs tend to overgeneralize about operations and to overlook important distinctions. Some of the research reviewed helps us to understand the nuances of PTs' conceptions and can help to inform instruction. Further research is needed to (a) better understand PTs' conceptions when they enter our programs, and (b) better understand how PTs' conceptions develop.

**Keywords:** whole number, operation, number concepts, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

## Introduction

Consider a prospective elementary teacher (PT) solving  $527 - 135$ , using the standard algorithm and explaining regrouping as follows:

You put a 1 over next to the number and that gives you 10. . . I don't get how the 1 can become a 10. One and 10 are two different numbers. How can you subtract 1 from here and then add 10 over here? Where did the other 9 come from?

This PT clearly followed the correct procedure and arrived at the correct answer, but she was not able to provide an explanation for why this solution method results in a correct answer. Figure 1 shows her written work.

$$\begin{array}{r} 4 \cancel{5} 27 \\ - 135 \\ \hline 392 \end{array}$$

Figure 1. A PT's explanation of regrouping in  $527 - 135$  (Thanheiser, 2009, p. 251).

Now consider another PT's reflection describing her inability to explain regrouping:

I learned [at the beginning of my elementary mathematics methods class] that there was a lot more to the concept [of number and place value] than I was aware of. I am able to use math effectively in my everyday life, such as balancing my checkbook, but when I was presented with questions as to why I carry out such procedures as *carrying*<sup>1</sup> and *borrowing* in addition and subtraction, I was stuck. I could not explain why I followed any of these procedures or rules. I just knew how to do them. This came as a huge shock to me considering I did well in most of my math classes. I felt terrible that I could not explain simple addition and subtraction.

Both of these PTs have determined that they want to teach children, yet at this point neither of them would be able to conceptually help an elementary-aged child make sense of why regrouping works when using the standard algorithms taught in the United States.

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<sup>1</sup> Note that students in the United States often term regrouping in the context of addition *carrying* and regrouping in the context of subtraction *borrowing*.

Moreover, solving a problem using the algorithms is not sufficient knowledge for teaching mathematics to children. In the United States, the National Council of Teachers of Mathematics (NCTM, 2000a) and the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) call for children to develop a conceptual understanding (Hiebert & Lefevre, 1986) of the mathematics they encounter. Procedural fluency is one of several aspects of being mathematically proficient (National Research Council, 2001); the other four aspects are conceptual understanding, strategic competence, adaptive reasoning, and productive disposition. In order to be equipped to support students' development of mathematical proficiency, inservice teachers and PTs also need such an understanding of mathematics. Researchers have highlighted the need for teachers to have a deep and multifaceted understanding of the mathematics they teach (Hill, Ball, & Schilling, 2008; Ma, 1999). Less clear, however, is how improvement in teachers' knowledge can be accomplished.

At the core of elementary school mathematics is the teaching of number concepts and operations. NCTM (2000a) stressed that all pre K–12 students should “[a] understand numbers, ways of representing numbers, relationships among numbers, and number systems; [b] understand the meanings of operations and how they relate to one another; [and c] compute fluently and make reasonable estimates” (p. 32). A conceptual understanding of number and operations underlies learning of all future mathematics and other STEM subjects. “Number pervades all areas of mathematics. The other four Content Standards [other than Number and Operations] as well as all five Process Standards are grounded in number” (NCTM, 2000b, ¶1). In the CCSSM, “Number and Operation in Base

Ten” is one of the focal domains in each grade from K through 5, followed by “The Number System” in Grades 6–8 and “Number and Quantity” in high school.

Even with this strong focus on number throughout the K–12 curriculum, children in the United States and other countries “experience considerable difficulty constructing appropriate number concepts of multidigit numeration and appropriate procedures for multidigit arithmetic” (Verschaffel, Greer, & De Corte, 2007, p. 565). Rather than developing desirable number concepts and strategies, children often learn standard algorithms, which they view as involving concatenated single digits, rather than numbers of ones, tens, and hundreds (Fuson et al., 1997).

Research has also shown that elementary teachers and PTs in the United States and Australia continue to lack a conceptual understanding in this important area (Ball, 1988; Ma, 1999; Southwell & Penglase, 2005; Thanheiser, 2009, 2010). To be in a position to help PTs develop more sophisticated conceptions, mathematics educators need to (a) understand the conceptions with which PTs enter our classrooms, so that we can build on those conceptions (Bransford, Brown, & Cocking, 1999); and (b) understand how those conceptions can develop. As the authors of *The Mathematical Education of Teachers* stated, “The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know” (Conference Board of the Mathematical Sciences [CBMS], 2001, p. 17). In order to work from what PTs know, we must first find out what they know.

In our summary work, we examined the current knowledge in the field of mathematics education concerning PTs’ conceptions of whole numbers and operations and the development thereof. We present a summary of the research in three parts:

1. *A Historical Look*, which represents a summary of the research literature prior to 1998.
2. *A Current Perspective*, based on research articles published between 1998 and 2011.
3. *A View of the Horizon*, based on 2012 journal articles, as well as 2011 and 2012 proceedings of the International Group for Psychology of Mathematics Education (PME) and North American Chapter of the International Group for Psychology of Mathematics Education (PME-NA).

### **Methods**

The authors met as part of a larger Working Group (see introductory article to this Special Issue) focusing on summarizing the current knowledge of the field on PTs' content knowledge and the development thereof. The larger Working Group set the parameters for the search in general. In this section, we describe the methods that pertain to this particular article. We began by searching the ERIC database for combinations of the following search terms: *prospective, preservice, or pre-service* with any of *whole number, operation, place value, multidigit, algorithm, or number sense*.<sup>2</sup> Each combination of search terms was entered into the ERIC database. We searched separately for articles published prior to 1998 and for articles published from 1998 to 2011 in order to get an overview of the research that occurred during those periods.

All results were checked for a focus on PTs' content knowledge of whole numbers and operations. We read the title and abstract to determine whether each paper fit our

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<sup>2</sup> Other search terms, e.g., *elementary education* and *whole number*, yielded no additional relevant results.



criteria. If the title and abstract did not suffice to make a determination of fit, then we read the whole paper. We included all papers that met the following criteria:

- Focused on our target group of PTs.
  - We also included prospective middle school teachers because some certification programs focus on K–8, and not all countries follow the same school system. We excluded papers focusing on prospective high school teachers.
  - We included papers focusing on both pre- and inservice teachers (i.e., mixed groups) but excluded papers focusing only on inservice teachers.
- Focused on content knowledge of whole numbers and operations. We included papers that did not exclusively focus on whole numbers and operations, but we focused our summaries of these on the findings that speak to PTs' conceptions of whole numbers and operations. We excluded papers that focused on beliefs or general content knowledge.
- Published research studies in peer-reviewed research journals. Our larger Working Group (Thanheiser et al., 2010) identified 23 journals to include in our reviews for the section focusing on the years 1998–2011 (i.e., *A Current Perspective*). (See introductory article to this Special Issue for more details).

The section focusing on the years prior to 1998 (i.e., *A Historical Look*) followed the same methods. For the section looking forward (i.e., *A View of the Horizon*), we followed the same methods starting in 2012 for journal articles. We also searched the 2011 and 2012 proceedings of the annual conferences of PME and PME-NA for relevant papers. It was our assumption that new research is more likely to be presented at conferences since there is a

time delay between conducting research and publishing papers. For this search, we read all paper titles in the relevant category. For example, Chapter 6 of the 2012 PME-NA proceedings focuses on “Teacher Education and Knowledge—Preservice.” We read all paper titles in this chapter to identify candidates to include in our review, based on the same criteria for article content as described above.

Once the research articles were identified, we read each to make a final determination of whether it should be included in the review. Questions and disagreements were discussed and resolved. In the end, we identified a total of 28 articles that were relevant to our search—26 peer-reviewed journal articles and 2 conference proceedings. The pre-1998 historical search identified 7 relevant peer-reviewed journal articles. The 1998–2011 article search identified 18 relevant peer-reviewed journal articles. The search for *A View of the Horizon* yielded 1 relevant peer-reviewed journal article and 2 peer-reviewed conference proceedings. We then read and summarized each of the three groups of articles.

Within the groups of articles belonging to *A Historical Look* and *A Current Perspective*, we identified categories to help organize our summaries of the literature. These categories were not decided a priori; rather, they emerged in the course of our review through a process of constant comparative analysis (Strauss & Corbin, 1998). These analyses were focused within the group of articles and were influenced by the number and nature of the articles in the group. *A Historical Look* consists of seven articles, almost all of which focus on multiplication and division. This being the case, we made fine-grained distinctions regarding what content knowledge was investigated (e.g., *understanding of the long-division algorithm*). As a result, in some cases, there is only one article per category.

The articles belonging to *A Current Perspective* are more abundant and cover a broader range of topics than those belonging to *A Historical Look*. The grain size and focus of our categories reflect this. For example, *PTs' reasoning about alternative algorithms or nonstandard strategies* is broader than the categories identified in *A Historical Look*, and it includes four articles. The categories in *A Current Perspective* reflect the broadening range of recent research related to PTs' content knowledge. For example, PTs' reasoning about alternative algorithms or nonstandard strategies was not a focus of any of the articles in *A Historical Look*.

With only three articles in the section *A View of the Horizon*, it did not make sense to categorize them. We simply summarized each article.

## **Results and Discussion**

We first present *A Historical Look*, which represents a summary of the research literature prior to 1998. Next, we present *A Current Perspective*, based on research articles published between 1998 and 2011. Finally, we present *A View of the Horizon*, based on 2011 and 2012 PME and PME-NA proceedings and one article.

### **A Historical Look**

What was known about PTs' understanding of whole numbers and operations prior to 1998? It is important to look at articles published prior to 1998 in order to understand the history of research in this area. It enables us to characterize the state of the field prior to our current perspective. This review is based on research articles published in mathematics education journals before 1998. Only seven such research articles were found. A summary of articles is included in Table 1.

What was known relates primarily to multiplication and division. In particular, the following five categories were identified:

1. PTs' reasoning about division story problems (Simon, 1993; Tirosh & Graeber, 1991; Vest, 1978).
2. PTs' reasoning about the properties of multiplication and division (Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989).
3. PTs' understanding of the long-division algorithm (Simon, 1993).
4. PTs' understanding of divisibility and multiplicative structure (Zazkis & Campbell, 1996).
5. PTs' conceptions of zero (Wheeler, 1983).

Below, we report the results of our literature review. The results are organized according to the categories listed above.

Table 1

*Articles Written Prior to 1998 Dealing With PTs' Knowledge of Whole Numbers and Operation*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Graeber, Tirosh, & Glover	1989	129 PTs	Content or Methods course	USA	Survey for 129 Interview for 33 of the 129 PTs were asked to solve story problems for multiplication and division
Tirosh & Graeber	1991	80 PTs	Content course	USA	Survey; PTs were asked to (a) write expressions to match given story problems, (b) write story problems corresponding to given division expressions
Tirosh & Graeber	1989	136 PTs	Content or Methods course	USA	Survey; PTs were explicitly asked for misconceptions about multiplication and division and then asked to solve problems
Simon	1993	33 PTs	Methods course	USA	Open response written instrument for 33 PTs and interviews for 8 of the PTs – PTs were asked to write story problems for division and make sense of long division
Wheeler & Feghali	1983	52 PTs	Methods course	USA	Survey and interviews
Vest	1978	87 PTs	Content course	USA	Survey; PTs were asked to write story problems for division
Zazkis & Campbell	1996	21 PTs	Content course	USA	Interviews; The authors used a variety of tasks related to elementary number theory

**PTs' reasoning about division story problems.** Three studies investigated PTs' reasoning about the relationship between division and story problems. Studies by Vest (1978), Simon (1993), and Tirosh and Graeber (1991) all relate to PTs' reasoning about partitive and quotitive division story problems. Partitive problems involve the forming of equal-sized groups. In these story problems, a total amount or number of things is given, along with a desired number of equal groups. The question is how much or how many things should go in each group. Quotitive problems involve a predetermined group size. A total amount or number of things is given, along with a group size. The question that results from these situations is how many such groups can be formed. For example, in a context of children sharing candies, a partitive problem would give a total number of candies and a number of children and ask how many candies each child would receive, given that the candies are to be shared fairly. In the same context, a quotitive problem would give a number of candies that each child should receive and ask how many children can receive candy. It is important for children to be able to explore partitive and quotitive problems and to see both as related to the division operation (Carpenter, Fennema, Franke, Levi, & Empson, 1999).<sup>3</sup> For this reason, it is just as important for PTs as it is for practicing teachers (Carpenter, Fennema, Peterson, & Carey, 1988) to make sense of these problem types and to be able to clearly distinguish between them.

Vest (1978) surveyed 87 PTs enrolled in a content course in the southern part of the United States to investigate their preferences for the type of division story problem, partitive or quotitive.<sup>4</sup> When asked to write a division story problem, 59 of 87 PTs wrote a

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<sup>3</sup> Carpenter et al. used the language "partitive" and "measurement" problems.

<sup>4</sup> Vest used the language "partitioning" and "measurement" problems.

partitive problem, while only 6 of 87 wrote a quotitive problem. The remaining 22 responses were categorized as “Other.” In another task, participants were given a page from an elementary textbook, in which whole-number division was introduced through measurement situations, which are quotitive in nature. Participants were asked to write a story problem that they would use to introduce that page. Again, participants favored partitive problems. Of the 89 participants who responded to this task, 62 wrote a partitive division story problem, while only 12 wrote a quotitive problem. This is a striking finding. It does not merely show that PTs preferred partitive problems in general; it shows that they would inappropriately choose partitive problems to introduce a lesson on quotitive division.

The above results might indicate that PTs simply do not see a difference between partitive and quotitive problems. However, Vest (1978) found that the same PTs were able to distinguish between problems of the two types. Given the simple instruction to label problems according to whether they asked “How many sets?” or “How many in each set?” the study participants categorized an average of 95.4% of story problems correctly. The PTs also did not express an explicit preference for one type of problem over the other. Nonetheless, when PTs were asked to produce their own story problems, partitive problems were overwhelmingly more common. PTs’ apparent preference for partitive problems is a concern because they will need to support their students in coming to relate to division to *both* partitive and quotitive problems.

A study of Simon (1993) corroborated Vest’s (1978) findings. Simon’s study involved 33 PTs enrolled in a methods course in the United States. When asked to write division story problems involving given numbers, the majority of the participants also

wrote problems that reflected a partitive, rather than quotitive, meaning of division. Specifically, 74% of the problems created were partitive, and only 17% were quotitive. Simon found that most participants were able to relate partitive story problems to division of whole numbers. On the other hand, the quotitive meaning was more elusive. Many assumed the partitive meaning and had difficulty when it did not fit well.

Tirosh and Graeber (1991) investigated the effect of division problem type (e.g., partitive or quotitive) on PTs' performance. They surveyed 80 PTs who were enrolled in either a content or methods course for elementary education majors in the southeastern United States. When asked to write expressions to match given story problems, the participants were less successful on quotitive than on partitive problems. The PTs also performed worse on problems in which the divisor was greater than the dividend. Both effects were statistically significant. When asked to write story problems corresponding to given division expressions, when the divisor was a whole number, the majority of participants wrote a partitive division problem that correctly matched the given expression. There were three such items. The percentage of correct partitive story problems ranged from 63 to 78%. Only 1 to 3% of correct responses were quotitive problems. Given an expression in which the divisor was not a whole number (e.g.,  $4 \div 0.5$ ), participants did not attempt to write a partitive problem, and only 44% correctly wrote a quotitive story problem. Tirosh and Graeber concluded, "Many preservice teachers are familiar with the partitive interpretation of division but have limited access to the measurement [quotitive] interpretation" (p. 162).

**PTs' reasoning about properties of multiplication and division.** One study focused on understanding properties of multiplication and division. Graeber, Tirosh, and



Glover (1989) documented PTs' misconceptions related to these properties. The researchers surveyed 129 PTs who were enrolled in either a content or methods course for elementary education majors in the southeastern United States. They then interviewed 33 of these PTs. The authors found that the PTs had difficulty with story problems in which multiplication did not "make bigger" or division did not "make smaller." For example, they performed worse when solving multiplication tasks if the multiplier was a decimal, rather than a whole number. When solving story problems, which required them to determine the appropriate operation to use, participants' choices were often influenced by the relative sizes of the given numbers, as opposed to the relationships between quantities. For example, on the four given multiplication story problems that had a decimal operator less than 1, more than 25% of the PTs incorrectly wrote a division expression, rather than a multiplication expression (Graeber et al., 1989). Explicit, incorrect beliefs about division were more common. In interviews with 33 of the PTs, Graeber et al. found that 22 of them reversed the roles of dividend and divisor when given story problems in which the divisor was greater than the dividend. The authors reported, "All 22 claimed that in division the larger number should be divided by the smaller number" (p. 99).

Tirosh and Graeber (1989) surveyed 136 PTs enrolled in either a content or methods course for elementary education majors in the United States. When asked directly, 87% of the PTs in the study responded correctly to questions concerning whether a product would always be greater than the factors (Tirosh & Graeber, 1989). However, in practice, many of the PTs reasoned in ways that evinced the influence of the belief that multiplication makes bigger. In response to a set of four survey questions regarding the properties of division, 72% of participants answered at least one of the True/False

questions incorrectly. For example, 52 (i.e., 38%) of the participants who responded to the statement “In division problems, the quotient must be less than the dividend” incorrectly answered that the statement was true.

The misconceptions of multiplication and division that were identified concerned problems involving rational numbers. PTs’ generalizations that multiplication makes bigger and division makes smaller hold true for whole numbers, except in the special cases involving 0 and 1. So, PTs’ reasoning about multiplication and division seem to be strongly connected to their experiences with whole numbers. In the whole-number domain, their reasoning is essentially correct. Thus, if we restrict our view to reasoning about whole-number operations, PTs may appear to be equipped to support students’ learning. However, children’s learning of mathematics in the early grades should prepare them for continued learning as they mature. If PTs’ overgeneralize about multiplication and division, their future students may make the same overgeneralizations and face the same difficulties as PTs when it comes to operations involving rational numbers.

**PTs’ understanding of the long-division algorithm.** Simon’s (1993) study involving 33 PTs investigated their understand of the long-division algorithm. Given a dividend and divisor and a calculator to use, 76% of participants were unable to find the remainder. The PTs also had difficulty explaining the meaning of the remainder in a division calculation. Many of them related the remainder to a fraction or decimal in inappropriate ways. They knew the long-division algorithm but were unable to explain its steps conceptually, and their justifications appealed to the procedure itself. Simon reported that participants were unable to connect a meaning of division with symbolic representations of division calculations, regardless of whether the calculations were

performed by long division or with a calculator. Simon characterized “prospective teachers’ mathematical knowledge as procedural and sparsely connected” (p. 252).

**PTs’ understanding of divisibility and multiplicative structure.** One study, by Zazkis and Campbell (1996), investigated PTs’ understanding of divisibility and the multiplicative structure of natural numbers. This study involved 21 PTs enrolled in a mathematics content course in the United States. The authors used a variety of tasks related to elementary number theory in interviews with the PTs. They found that the PTs tended to reason about divisibility procedurally, in terms of performing the division operation itself, rather than on the basis of multiplicative composition of number. The authors reported, “A minority (6 out of 21) of the participants in this study group were able to consistently discuss and demonstrate an understanding of divisibility as a property of, or relation between, natural numbers” (p. 546).

Given  $M = 3^3 \times 5^2 \times 7$  and asked whether  $M$  was divisible by 7, participants thought that they needed to compute  $M$  and then divide by 7 to find out. The researchers observed that the PTs tended to be unsure of claims regarding divisibility in the absence of a specific quotient. For instance, even if a PT thought that  $M$  was divisible by 7, he or she was uncomfortable making such a claim without knowing what  $M$  divided by 7 actually equaled. The PTs also made reference to and use of divisibility rules, which were sometimes misremembered or misapplied, and they had difficulty reasoning about divisibility without the use of such rules. Participants also had difficulty in generating numbers with desired properties; they tended to guess and check, rather than to construct numbers in ways that would guarantee those properties.

**PTs' conceptions of zero.** One study reported on PTs' understanding of zero and of division by zero. The study, by Wheeler and Feghali (1983), involved 52 PTs enrolled in a methods course in the United States. The authors investigated the PTs' conceptions of zero, using a written instrument and individual interviews. The authors report that the PTs did not have an adequate understanding of zero. Most of the participants incorrectly answered items of the form  $a \div b$ , where  $b = 0$ . Most said that  $0 \div 0 = 0$ . In a classification task involving some cards with various images on them and some cards that were blank, most PTs rejected using blank cards as a category for classification. The participants were interested in the attributes of the images on the cards, and they viewed blank cards as being without attributes, rather than as having the attribute of being blank. The PTs described zero in a variety of ways, including as (a) a symbol, (b) a number, and (c) nothing. When asked directly whether zero was a number, most said that it was, but 15% of the participants disagreed. For example, one PT said, "Zero is not a number because it has no value" (p. 152).

**Summary of the historical look.** Our database search revealed seven research articles published in mathematics education journals prior to 1998 that addressed PTs' conceptions of whole numbers and operations. According to these reports, PTs favor the partitive over the quotitive meaning of division. They are more likely to write partitive story problems, except when the divisor is not a whole number (Simon, 1993; Tirosh & Graeber, 1991; Vest, 1978). PTs can recognize the difference between partitive and quotitive story problems, and they can find the solutions to problems of both types (Tirosh & Graeber, 1991; Vest, 1978); however, they perform worse on quotitive problems, and they perform worse on problems in which the divisor is greater than the dividend (Tirosh

& Graeber, 1991). When asked to make their beliefs about the properties of multiplication explicit, PTs tend to respond correctly (e.g., to indicate the multiplication does not always “make bigger”). However, their responses to various tasks reflect the influence of overgeneralizations about multiplication (Tirosh & Graeber, 1989).

When it comes to division, PTs often explicitly make incorrect claims, such as that the divisor must be less than the dividend (Graeber et al., 1989). In addition, PTs bring a range of procedural and conceptual knowledge to bear on division-related tasks; however, their knowledge of division is disconnected (Simon, 1993). Their understanding of the long-division algorithm tends to be procedural, and they have difficulty relating that procedure to real-world situations. PTs also reason procedurally about divisibility and often feel the need to perform calculations in order to answer questions regarding divisibility (Zazkis & Campbell, 1996). PTs have limited conceptions of zero. Some do not regard it as a legitimate number, and many PTs answer questions involving division by zero incorrectly.

**Reflections on the historical look.** The pre-1998 research literature characterized PTs’ knowledge as inadequate and partially incorrect. Descriptions emphasized PTs’ limited understandings and reliance on procedures. PTs were described as holding misconceptions, which led, at least some of the time, to incorrect answers. We learn from these reports that PTs’ knowledge of whole numbers and operations—especially multiplication and division—was in need of improvement. At the same time, this research literature is limited in its guidance regarding how to support PTs to develop more sophisticated mathematical understandings. The reports provide snapshots of PTs’ content knowledge, and these descriptions do not emphasize ways in which PTs may be able to

build on what they know to improve their understanding of whole numbers and operations.

The historical look also leaves us with many unanswered questions regarding specific content knowledge that was not addressed. The literature focused on multiplication and division and did not address addition or subtraction. It did not address PTs' conceptions of whole numbers themselves. In particular, their understanding of place value was not explored. Also, researchers did not report on PTs' understanding of number theory beyond divisibility. For instance, PTs' reasoning about oddness and evenness were not directly addressed. Perhaps the most noteworthy finding is simply how little the field knew about PTs' knowledge of whole numbers and operations prior to 1998.

### **A Current Perspective**

With number being such a pervasive topic in elementary school mathematics, surprisingly few papers have focused on PTs' conceptions of whole numbers and operations. Our search for research literature on PTs' understanding of whole numbers and operations, spanning the time from 1998 to 2011, resulted in 18 articles (see Table 2).

Table 2

*Articles Written About PTs' Understanding of Whole Numbers and Operations, Spanning the Time From 1998 to 2011*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Chapman	2007	20 PTs	Content course	Canada	Group tasks that allowed the PTs to reflect on the operations in order to develop a deeper understanding
Crespo & Nicol	2006	32 PTs	Methods course	Canada/ USA	Task involving division
Glidden	2008	381 PTs	Content course	USA	Tasks involving the order of operations
Green, Piel, & Flowers	2008	53/39 PTs	Child Development course	Canada/ USA	Survey
Kaasila, Pehkonen, & Hellinen	2010	269 PTs	Math Education course	Finland	Task involving nontraditionally posed division problem
Liljedahl, Chernoff, & Zazkis	2007	90 PTs	Content course		Tasks using a computer-based microworld
Harkness & Thomas	2008	71 PTs	Content course	USA	Case study of a student sharing an invented algorithm
Lo, Grant, & Flowers	2008	38 PTs	Content course	USA	Video of class sessions
McClain	2003	24 PTs	Methods course	USA	Survey
Menon	2003	77 PTs	Methods course	USA	Set of tasks involving two-digit multiplication

(continued)

Table 2—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Menon	2004	142 PTs	Methods course	USA	10-item number sense test
Menon	2009	64 PTs	Methods course	USA	Survey
Thanheiser	2009	15 PTs	Content course	USA	Interviews
Thanheiser	2010	33 PTs	Methods course	USA	Survey and interviews
Tsao	2005	12 PTs	Content course	USA	Interviews
Yackel, Underwood, & Elias	2007	45 PTs	Content course	USA	Video of class sessions
Yang	2007	15 PTs		Taiwan	Interviews
Zazkis	2005	116 PTs	Content course	Canada	Task involving prime numbers

Of the research papers reviewed, the following five categories emerged:

1. PTs' conceptions of number and the development thereof (McClain, 2003; Thanheiser, 2009, 2010; Yackel, Underwood, & Elias, 2007).
2. PTs' reasoning about alternative algorithms or nonstandard strategies (Harkness & Thomas, 2008; Kaasila, Pehkonen, & Hellinen, 2010; Lo, Grant, & Flowers, 2008; Menon, 2003, 2009).
3. PTs' number sense (Menon, 2004; Tsao, 2005; Yang, 2007; Zazkis, 2005).



4. PTs' conceptions of arithmetic operations and order of operations (Chapman, 2007; Crespo & Nicol, 2006; Glidden, 2008).
5. Addressing PTs' misconceptions through the use of manipulatives or computer microworlds (Green, Piel, & Flowers, 2008; Liljedahl, Chernoff, & Zazkis, 2007).

Below, we report the results of our literature review. The results are organized according to the categories listed above.

**PTs' conceptions of number and the development thereof.** Two research studies focused on PTs' conceptions of number (Thanheiser, 2009, 2010) and two research studies focused on the development thereof (McClain, 2003; Yackel et al., 2007). Thanheiser (2009) interviewed 15 PTs in the United States before their first content course for teachers. The interview data allowed for the identification and categorization of PTs' conceptions of multidigit whole numbers into four major groups: thinking in terms of (a) *reference units*, (b) *groups of ones*, (c) *concatenated-digits plus*, and (d) *concatenated-digits only*. See Table 3 for the definition and distribution of the conceptions among the PTs in that study.

Thanheiser (2009) found that two thirds of the PTs in that study saw the digits in a number incorrectly in terms of ones, at least some of the time. This conception prohibits the PTs from making sense of regrouping. And while the groups-of-ones conception is a correct conception, it also limits what a PT will be able to explain. While PTs may be able to correctly explain the regrouped 1 in Figure 1 as 100 *ones*, they may struggle to explain that the 1 represents 10 *tens* and thus combined with the 2 *tens* represents 12 *tens*. Thus, while five of the PTs held a correct conception, only three of those five held a conception that

enabled them to explain all aspects of regrouping, including why we “make the 1 a 10” when we move it over.

Table 3

*Definition and Distribution of Conceptions in the Context of the Standard Algorithm for the 15 U.S. PTs in Thanheiser’s (2009) Study (p. 263)*

Conception	# of PTs
<i>Reference units.</i> PTs with this conception reliably <sup>5</sup> conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 <i>hundreds</i> or 30 <i>tens</i> or 300 <i>ones</i> , the 8 as 8 <i>tens</i> or 80 <i>ones</i> , and the 9 as 9 <i>ones</i> . They can reconceive of 1 <i>hundred</i> as 10 <i>tens</i> , and so on.	3
<i>Groups of ones.</i> PTs with this conception reliably conceive of all digits correctly in terms of groups of <i>ones</i> (389 as 300 <i>ones</i> , 80 <i>ones</i> , and 9 <i>ones</i> ) but not in terms of reference units; they do not relate reference units (e.g., 10 <i>tens</i> to 1 <i>hundred</i> ).	2
<i>Concatenated-digits plus.</i> PTs with this conception conceive of <i>at least one</i> digit as an incorrect unit type, at least on occasion. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 <i>ones</i> but the 8 is only 8 <i>ones</i> ).	7
<i>Concatenated-digits only.</i> PTs holding this conception conceive of <i>all</i> digits in terms of <i>ones</i> (e.g., 548 as 5 <i>ones</i> , 4 <i>ones</i> , and 8 <i>ones</i> ).	3

Thanheiser (2009) also examined PTs’ conceptions in various contexts. One of these contexts was a time task. PTs were given an artifact of children’s mathematical thinking in which the child had incorrectly applied the standard subtraction algorithm in a time context (see Figure 2). Of the 15 PTs in the study, 9 initially thought that the child’s application of the standard algorithm was correct. Eight of those 9 PTs eventually changed their mind after calculating the time difference another way. However, only 8 of the 15 PTs

<sup>5</sup> *Reliably* in these definitions means that after the PTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

were able to explain why the application of this algorithm was incorrect (i.e., regrouping 100 rather than 60) and alter the algorithm to make it work for a time situation.

Last week, in a third-grade classroom, the children were given the following problem:

You were on a train that left Los Angeles at 2:53 p.m. and arrived in Phoenix at 7:08 p.m. How long were you on the train?

One student solved the problem as follows:

$$\begin{array}{r} 7:08 \\ - 2:53 \\ \hline 4:55 \end{array}$$

He explained, "I took the 3 from the 8; that is 5. Then I couldn't take the 5 from the 0, so I borrowed 1 from the 7 and put it by the zero. Then I took 5 from 10; that's 5. Then I took 2 from 6, and that's 4."

Figure 2. Time task (Thanheiser, 2009, p. 259).

In a different task, PTs were asked to relate hundreds and millions (i.e., how many hundreds are in a million?) and in that context were asked to relate tens and hundreds (i.e., 10 tens are a hundred) and hundreds and thousands (i.e., 10 hundreds are a thousand). Six of the 15 PTs, at least in some instances, claimed that  $100 \times 100 = 1,000$ . Thanheiser (2009) explained this mistake as possibly being based on an overgeneralization of the pattern  $10 \times 10 = 100$  (e.g., multiply a reference unit by itself to get the next larger one) resulting in  $100 \times 100 = 1,000$ . Thanheiser also noted that this notion would make it hard to see the regularity in our base-ten number system. In summary, Thanheiser found that PTs who held one of the concatenated-digits conceptions struggled when asked to explain why things worked, whereas PTs who held one of the correct conceptions were able to explain these things. This was true in the context of the standard algorithms, as well as in alternate contexts.

In a follow-up study, Thanheiser (2010) surveyed 33 PTs enrolled in a math methods course in the United States. In this investigation of PTs' interpretations of regrouped digits, Thanheiser (a) replicated the earlier results that most PTs held one of the concatenated-digits conceptions, even at the end of their teacher education programs; and (b) refined the *concatenated-digits plus* conception into three further categories:

1. Regrouped digits are consistently explained as 10, regardless of whether it is in the context of addition or subtraction.
2. Regrouped digits are explained consistently depending on context (i.e., 10 in subtraction, 1 in addition, or vice versa).
3. Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the tens' place in the context of addition as 10 or 1 in different tasks).

In this study, only 3 of 33 PTs were able to correctly explain the values of the regrouped digits in both addition and subtraction contexts. Of the remaining 30 PTs, 5 saw the values of all regrouped digits as 1, consistent with the concatenated-digits conception. The distribution of the remaining 25 PTs who fell into the concatenated-digits-plus category can be seen in Table 4.

Table 4

*Conceptions of the 33 PTs in the Context of Standard Algorithms (Detailed) in Thanheiser (2010)*

Conception Across Addition and Subtraction Tasks	Number of PTs
One of the two correct conceptions (reference units or groups of ones)	3
Concatenated digits plus Refined conception:	25
– Regrouped digits consistently explained as 10 (regardless of whether it is in the context of addition or subtraction) (7 PTs)	
– Regrouped digits explained consistently depending on context (i.e., 10 in subtraction, 1 in addition or vice versa) (10 PTs)	
– Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the ten’s place in the context of addition as 10 or 1 in different tasks) (8 PTs)	
Concatenated digits only	5

A surprising result in Thanheiser’s (2010) study was that eight PTs changed their explanation of the regrouped digits from one problem to the next. While they would interpret the regrouped 1 as 10 or 1 in one addition problem, they would interpret it differently in another (see Figure 3). For example, PTs may interpret the circled 1 in the first problem in Figure 3 as 1, but the circled 1 in the second problem in Figure 3 as 10, thus changing how they interpret the regrouped digit in the tens’ place in the context of addition.

$$\begin{array}{r}
 \textcircled{1} \\
 389 \\
 + 475 \\
 \hline
 864
 \end{array}
 \qquad
 \begin{array}{r}
 \textcircled{1} \\
 259 \\
 + \quad 38 \\
 \hline
 297
 \end{array}$$

Figure 3. Comparing regrouped digits in the tens' place in the context of addition (Thanheiser, 2010, p. 249).

Two studies focused on the development of PTs' conceptions of place value (McClain, 2003; Yackel et al., 2007). Both studies examined the PTs' development of conceptions by working with them in a context involving an alternate base (base eight). McClain (2003) asked 24 PTs enrolled in the second of two methods courses in the United States to work in the Candy Factory context (Cobb, Yackel, & Wood, 1992), in which eight candies were packed into a roll of candies and eight rolls were packed into a box of candies. While McClain asked PTs to work in the context of boxes, rolls, and pieces of candies, she did not ask PTs to use base-eight notation. In earlier work, she had found that the PTs were distracted by being asked to use base-eight notation and focused more on that than on the mathematics of quantifying, adding, and subtracting numbers. With the Candy Factory context, McClain found that PTs initially focused on pictures to represent numbers but then invented a notational form using B for boxes, R for rolls and P for pieces. McClain focused on grouping and regrouping to help the PTs understand place value and the multiplicative structure of the system. At the end of the sequence, PTs were asked to buy or sell candies to help them understand addition and subtraction. McClain found that PTs invented "nontraditional yet personally meaningful algorithms for addition and subtraction to symbolize their activity" (p. 298). The goal of this sequence was to help PTs develop a reference-units conception (cf. Thanheiser, 2009) and thus see a box not just as a box, but

simultaneously as eight rolls, as well as 64 candies, and then draw on that number concept to develop algorithms and a deeper understanding of the numbers and of the algorithms.

McClain (2003) examined the development of PTs' conceptions and compared it to the development of children's conceptions of base ten. She found that the PTs' development mirrored that of children. She stated, "This finding also has broader implications—that the broad base of research conducted in elementary classrooms can feed forward to inform efforts at supporting the development of PTs' content knowledge" (p. 301). As a result, PTs also came to the realization that in order to teach for conceptual understanding, they themselves would need to possess this type of understanding.

Yackel, Underwood, and Elias (2007) also used the Candy Factory context in base eight with 45 PTs in a content course in the United States. They examined how PTs learned to count in base eight and used that as an underpinning for operating on numbers in base eight. In contrast to McClain (2003), Yackel et al. did use base-eight language (e.g., they named a unit of eight as "one-e"). They spent a considerable amount of time in counting to lay the foundations for operating on numbers. One of their foci was to help PTs coordinate units of different rank (i.e., develop reference-units conceptions). They point out that the focus on counting not only helped the PTs make sense of the counting sequence and how it is learned by children, but it also helped the PTs make sense of early arithmetic. They note that it is often surprising to PTs as well as teacher educators how much sense making can happen in early arithmetic.

**PTs' reasoning about alternative algorithms or nonstandard strategies.** Five studies focused on exploring PTs' reasoning about alternative algorithms (Harkness & Thomas, 2008; Kaasila et al., 2010; Lo et al., 2008; Menon, 2003, 2009). Harkness and

Thomas (2008) worked with 71 PTs in three sections of a freshmen content course in the United States. They reported that PTs' understanding of invented algorithms is more procedural than conceptual. The PTs were presented with a case study of a student sharing an invented algorithm in front of her class and being told by her teacher that it is incorrect (Corwin, 1989). Then the PTs were asked to explore the validity of the invented algorithm (see Figure 4). Only 7 of 71 PTs were able to explain why the invented algorithm works. An additional 15 PTs showed some understanding but were not able to give a complete explanation. The remaining 49 PTs drew on procedural understanding to give explanations. For example, they used arguments such as that the 10 from the upper line was moved to the lower line.

34	34
<u>x23</u>	<u>x23</u>
102	92
<u>680</u>	<u>690</u>
782	782
"Downwards"	"Upwards"

Figure 4. Standard downwards and invented upwards algorithms (Harkness & Thomas, 2008, p. 129).

While the PTs struggled to make sense of the upwards method, they still empathized with the student in the case, either by relating to similar experiences in their past, highlighting that their current class allows alternative methods, or hoping that they will be able to allow for alternative methods in their own future classrooms. The PTs also disagreed with the teachers' choices in the case. Finally, the PTs highlighted how impressed they were by the child in the case. In addition, Harkness and Thomas found that it was



difficult to get PTs to attend to the details of the mathematics; if they expected PTs to write about the mathematics, the authors needed to explicitly ask them to do so.

Menon (2003) reported on PTs' responses to a set of tasks involving two-digit multiplication. A total of 77 PTs in two sections of a methods course in the United States were shown three different ways of performing two-digit multiplication. The PTs responded to each task individually and then discussed their ideas in small groups of four or five students. The first task concerned the standard algorithm as it relates to partial products. The partial products in  $65 \times 34$  were mislabeled, not taking place value into account, to draw the conclusion that the product consisted of 3 groups of 65 plus 4 groups of 65. PTs were asked whether they agreed with the description of the partial products. The instructor then pointed out that  $34 \times 65$  represented 34 groups of 65, so that 27 groups of 65 had not been accounted for. PTs were asked to respond. The author reported that, when responding individually, 39% of PTs said that nothing was missing from the product.

In Menon's (2003) second task,  $65 \times 34$  was computed from left to right. That is, the work showed 1,950 (i.e., the product of 30 and 65) on the first row and 260 (i.e., the product of 4 and 65) on the second row. PTs were asked whether this alternative method would work, and why or why not. The author reported that only 52% of PTs gave a correct explanation.

The third task involved yet another way of computing the same product. Three partial products were shown: 1,820, 240, and 150 (i.e.,  $(4 \times 5) + (30 \times 60)$ ,  $4 \times 60$ , and  $30 \times 5$ ). PTs were shown only the computed partial products. They were asked to determine the origins of these, to decide whether or not this algorithm was generalizable, and to justify their answers. The author reported that only 39% of the participants

produced a correct explanation. The author noted that the frequencies of correct responses from groups of PTs were considerably greater than for individuals. Thus, discussing the ideas in groups often led to a correct group response.

In another study, Menon (2009) surveyed PTs to investigate their understanding of multidigit multiplication. A written instrument was administered to 64 PTs enrolled in a middle school mathematics methods course in the United States. The author found that 95% of the PTs correctly computed  $456 \times 78$ . However, only 75% were able to write a correct word problem corresponding to this computation. The author gives two examples of incorrect responses. One was a division, rather than multiplication, story problem. The other showed lack of awareness of the distinct roles of multiplier and multiplicand: "There are 456 pencils, and 78 erasers in the classroom. If we multiply the 456 pencils and the 78 erasers, how many pencils and erasers will we have in total?" (p. 3). This PT seemed to rather directly translate the computation to a story involving pencils and erasers without taking into account what it would mean to multiply pencils by erasers. Evidently, the PT had in mind a meaning for multiplication as finding a total number of things, but the PT did not provide a rate in the story, and as a result the suggested multiplication was nonsensical. The vast majority (86%) of the PTs' explanations for the algorithm were largely procedural, and their ideas for helping a child learn to compute the product were likewise mostly procedural (72%). Menon described the PTs in this study as generally displaying the kind of understanding of multiplication that was required of them as students, noting that this understanding is inadequate for teaching multidigit multiplication.

Lo, Grant, and Flowers (2008) worked with 38 PTs enrolled in a content course in the United States. The authors found that the PTs' ability to develop and justify reasoning

strategies for multiplication develops slowly and presents several challenges. In their study, they describe a four-day lesson designed to help PTs develop a deeper understanding of multiplication. The researchers focused on both the development of and the justification of reasoning strategies. They found PTs struggled with both. Lo et al. hypothesize that the PTs struggled with the development of reasoning strategies because (a) the PTs lacked the multiplicative structure, and (b) the PTs lacked the understanding that there is more to a multiplication problem than finding the answer. One of the tasks they used was to ask students to multiply  $24 \times 38$  by starting with  $20 \times 40 = 800$  and adjusting the result. Lo et al. argue that PTs struggle with justifications for four reasons:

1. The PTs think justification is a description of the steps.
2. The PTs think justification is drawing a picture.
3. The PTs struggled in relating the picture to their reasoning, especially with the area model.
4. The PTs struggled coordinating the equal groups interpretation with their strategy.

Lo et al. (2008) also found that PTs struggled in recognizing the difference between procedural and conceptual descriptions of solutions to multiplication problems. It was not clear whether PTs needed more time or different kinds of experiences to continue to develop their understandings. As a result, the researchers suggest more research be conducted to investigate this. They also emphasized that we need to highlight the “ineffectiveness of memorizing and applying rules/procedures without understanding why they work” (p. 20).

Kaasila, Pehkonen, and Hellinen (2010) examined PTs'<sup>6</sup> understanding of a nontraditionally posed division problem. The participants were 269 Finnish PTs enrolled in a mathematics education course. The problem that the researchers posed was, "We know that  $498 \div 6 = 83$ . How could you conclude from this relationship without using long-division algorithm what  $491 \div 6 =$  is?" (p. 247). According to the authors, "This problem especially measures conceptual understanding, adaptive reasoning, and procedural fluency" (p. 247). Kaasila et al. found that 45% of the PTs were able to produce complete or almost correct solutions, and 30% produced complete and correct solutions. Of those PTs who answered correctly, almost all drew on both subtraction and division in their reasoning. Kaasila et al. (2010) identified four difficulties that the remaining 70% of the PTs had:

- (1) staying on the integer level (difficulties especially in conceptual understanding),
  - (2) inability to handle the remainder of the division (difficulties especially in procedural fluency),
  - (3) difficulties in understanding the relationships between different operations (problems especially in conceptual understanding), and
  - (4) inadequate reasoning strategies (difficulties especially in adaptive reasoning).
- (p. 257)

**PTs' number sense.** Four studies focused on number sense (Menon, 2004; Tsao, 2005; Yang, 2007; Zazkis, 2005). Tsao (2005) and Yang (2007) both found that PTs, especially the ones who struggled, relied on procedures rather than using number sense to solve problems. Tsao's study involved PTs enrolled in six sections of a mathematics content course in the United States. He found that the PTs were not ready to be immersed into a curriculum that reflects the vision of less emphasis on paper-and-pencil computation and more emphasis on number sense and mental arithmetic, as described in the NCTM

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<sup>6</sup> They also examined secondary students; however, in this report we leave out that part of the study.

Standards. Tsao compared six randomly selected high-ability PTs (scoring in the top 10% on a 25-item number sense test) and six randomly selected low-ability PTs (scoring in the bottom 10%). The data indicate that the high-ability students were more successful on each type of number sense item than the low-ability students. The items were intended to assess five components of number sense—number magnitude, use of benchmarks, decomposition/recomposition, relative effect of operations on numbers, and flexibility with numbers and operations. Compared to high-ability students, the low-ability students in this study (a) tended to use rule-based methods more frequently when answering interview items; and (b) preferred the use of standard, written computation algorithms rather than the use of “number sense based” strategies. The high-ability students tended to use benchmarks and to apply “number sense based” knowledge. Results also indicate that items including fractions were more difficult than whole number and decimal items for both groups of students.

Yang (2007) interviewed 15 PTs from a university in southern Taiwan. He examined strategies used by PTs when responding to number sense-related items. Yang defines number sense as consisting of the following four categories: (a) understanding the meanings of numbers, operations, and their relationships; (b) recognizing relative number size; (c) judging the reasonableness of a computational result by using strategies of estimation; and (d) developing and using benchmarks appropriately. Yang found that for each category, about two thirds of participants relied on rule-based methods to answer the questions. Thus, PTs, especially those in the low-ability group, tended to reason procedurally.

Menon (2004) worked with 142 PTs in four sections of a methods course in the United States. The PTs took a 10-item multiple-choice test designed to measure their number sense. The test consisted of items intended to measure their ability to (a) make mathematical judgments, (b) develop useful and effective strategies for numerical situations, and (c) understand number and operations related to fractions and decimals.<sup>7</sup> A student would be considered having number sense only if he or she provided both a correct response and a correct explanation to an item. Menon stated that a majority of the PTs were able to make mathematical judgments by being aware of the mathematical context while not blindly perform computations. However, Menon also noted that many of the PTs were unable to provide explanations describing the relationship between the numbers used to arrive at a solution.

Zazkis (2005) worked with 116 PTs enrolled in a content course for elementary teacher certification in Canada. After a unit on elementary number theory, the PTs were posed the question whether the product of  $151 \times 157$  was a prime number. Incorrect responses included: (a) two prime numbers multiplied together would result in another prime; (b) the last digit of 23,707 is 7, so the product is prime; and (c) the sum of the digits equals 19 and 19 is prime. Furthermore, Zazkis indicated that although 74 of the PTs correctly identified that the product was a composite number, only 52 of them were able to justify their reasoning using the definition of a prime or composite number. Zazkis summarized that the underlying feature of these shortcomings was PTs not understanding that the product of two whole numbers will have more than two factors.

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<sup>7</sup> We include only whole number items.

**PTs' conceptions of arithmetic operations and order of operations.** Three studies focused on understanding of operations (Chapman, 2007; Crespo & Nicol, 2006; Glidden, 2008). Chapman (2007) examined 20 PTs' understanding of arithmetic operations. The PTs were enrolled in an elementary mathematics content course in Canada. The PTs' initial knowledge of arithmetic operations was "inadequate to teach conceptually and in depth" (p. 347). The PTs' initial knowledge was based upon "procedural understanding of both the mathematical and semantic structure of a problem" (p. 347). Often PTs thought there was only one way to represent an operation. Chapman devised three group tasks that allowed the PTs to reflect on the operations in order to develop a deeper understanding. The first group task asked PTs to create word problems similar to given word problems and to compare different kinds of word problems. See Figure 5 for Part 1 of the task. After they worked on these problems individually, the PTs worked in groups to discuss their answers and then were asked to collaboratively create word problems that reflected the meaning for each of the four operations and then reflect on those.

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**PART 1:**

1. Create an original word problem for an elementary grade. Reflect on and describe what you thought about to create the problem.
  2. Create a word problem that is similar to the following problem.  
Golf balls come in packs of 4. A carton holds 25 packs. Marie, the owner of a sporting goods store, ordered 1600 golf balls. How many cartons did Marie order?  
Reflect on and describe what you thought about to create the problem.
  3. Compare and contrast this golf balls problem to the following problem:  
Jerry, a new elementary school teacher, wants his students to sit in groups on the carpet section of the classroom floor for certain activities. After doing some investigating, he found that the carpet area was 144 square feet and decided to allocate 6 square feet for one student. How many groups will Jerry be able to form if he wants 3 students in a group?
- 

Figure 5. Part 1 of Chapman's (2007) task (p. 343).

The second group task asked PTs to examine a given list of word problems representing various situations for each operation and asked them to analyze the word problems by modeling solutions and to reflect on similarities and differences. The third group task asked PTs to compare and contrast the problems given in the first group task, to create their own problems, to review an elementary textbook, and to choose one of the operations to create a lesson plan. The tasks were deemed effective as they allowed PTs to have:

- “relevant, practical, and meaningful examples and possibilities for thinking about the concepts”
- “allowed for simulation of real-world situations”
- “promoted reflection and discourse”
- “facilitated new understandings of familiar concepts.” (p. 384)

Crespo and Nicol (2006) focused on understanding division by zero. They examined 32 PTs enrolled in two methods courses (18 in course A in Canada, 14 in course B in the United States). In course A, PTs watched videos of children who stated that  $5 \div 0 = 0$ . In course B, PTs reacted to written artifacts stating the same thing. The authors’ stated reason for changing from video to written artifacts was to eliminate distractors from the mathematics, such as PTs focusing on the child’s emotional state or the interviewer’s actions. Crespo and Nicol found that initially almost all PTs in both courses thought that  $5 \div 0 = 0$ . They stated “the preservice teachers’ initial understandings of 0 and division by 0 were founded more on rule-based and flawed reasoning than on well-reasoned mathematical explanations and that they lacked the experience and inclination to understand or appreciate different ideas and approaches to this topic” (p. 94). Examining



the artifacts and discussing them helped the PTs make sense of division by zero, and at the end of the study, only two PTs remained who thought  $5 \div 0 = 0$ . The authors also noted that division by zero is often overlooked in prospective teacher education, and with such a high number of PTs entering with incorrect conceptions, we should include this topic into our courses.

Glidden (2008) focused on order of operations and found that PTs in a mathematics content course in the United States held superficial knowledge of the order of operations. He found that many PTs who performed multiplication before addition—correctly followed the order of operations—also performed addition before subtraction and multiplication before division. He hypothesized that they take the mnemonic PEMDAS (i.e., “Please Excuse My Dear Aunt Sally”) too literally. He also showed that almost 80% of the PTs used the incorrect order of operations to execute  $-3^2$ .

**Addressing PTs’ misconceptions through the use of manipulatives or computer microworlds.** One paper focused on addressing PTs’ misconceptions with the use of manipulatives (Green et al., 2008), and one paper focused on addressing understanding of factors, multiples, and primes using a computer microworld (Liljedahl et al., 2007). Green et al. worked with two sets of PTs in the context of a child development course. There were 53 PTs in the first study, which was conducted in Canada, and 39 PTs in the second study, which was conducted in the United States. Green et al. explored the use of manipulatives and found that manipulative-based instruction resulted in statistically significant decreases in arithmetic misconceptions and statistically significant increases in knowledge of the basic arithmetic operations. The authors reported that the use of manipulatives can effectively reverse most arithmetic misconceptions of PTs and that the

same activities used to reverse misconceptions can also improve the accuracy and depth of arithmetic knowledge. Thus, they conclude, manipulatives can and should be used effectively in PT classrooms.

Liljedahl et al. (2007) worked with 90 PTs enrolled in a content course. The authors engaged the PTs in tasks using a computer-based microworld called Number Worlds to encourage them to reason in new ways about basic concepts in elementary number theory. The microworld represents sets of numbers in grids. The user can determine the set to be represented and can also change the dimensions of the grid. The researchers stated that “about one-half” of the 90 PTs spent time in the computer lab using Number Worlds, and 17 of those who used Number Worlds participated in follow-up interviews. The authors reported that the PTs who used Number Worlds “thickened” their understandings of factors, multiples, and primes. They described new connections that the PTs made based on the visual representation of the microworld. The PTs noticed patterns that related to their previous understandings of factors, multiples, and primes, such as occurrences of multiples at regular intervals. They also developed new understandings that were grounded in visual features of the microworld, such as patterns in the distribution of primes.

**Summary of the current perspective.** Our review revealed the following five categories of current research examining PTs’ content knowledge of whole numbers and operations: (a) PTs’ understanding of whole-number concepts, and the development thereof; (b) PTs’ reasoning about alternative algorithms or nonstandard strategies; (c) PTs’ number sense; (d) PTs’ conceptions of arithmetic operations and the order of operations; and (e) addressing PTs’ misconceptions through the use of manipulatives or computer microworlds.

Many PTs realize that they need to understand mathematics conceptually in order to teach their future students for conceptual understanding (McClain, 2003). However, PTs tend to approach tasks procedurally because they lack the conceptual understanding required to do otherwise. For example, many PTs exhibit unsophisticated conceptions of digits in whole numbers, which then limits their understanding of regrouping when adding or subtracting (Thanheiser, 2009, 2010). Similarly with multiplication, PTs have difficulty explaining why algorithms work, and their reasoning is not easily improved (Lo et al., 2008). PTs may not recognize the difference between a procedural and conceptual description of a solution (Lo et al., 2008). Related to these difficulties is the finding that PTs tend to rely on procedures, rather than make use of number sense (Menon, 2004; Tsao, 2005; Yang, 2007). Many PTs are unable to describe relationships between numbers to arrive at a solution efficiently (Menon, 2004). Furthermore, many PTs experience difficulty understanding zero (Crespo & Nicol, 2006), and they have superficial understanding of the order of operations (Glidden, 2008). Overall, PTs, especially the ones who struggled, relied heavily on procedural knowledge.

**Reflection on the current perspective.** More research articles concerning PTs' knowledge of whole numbers and operations appeared between 1998 and 2011 than appeared prior to 1998. However, much remains to be learned about PTs' mathematical thinking in this area. As exemplified by the thinking of two PTs described in the introduction of this paper, researchers have found that PTs rely on memorized procedures involving whole numbers and operations. In addition, many PTs struggle to conceptually explain why the procedures work. Some research has examined how mathematics educators can help PTs develop more sophisticated conceptions, but there is still much

work to do for the mathematics education community to better understand how PTs' conceptions develop and how this development can be facilitated.

We note that several of the current research papers dealt with PTs' conceptions and/or the development thereof. This may suggest that mathematics educators are moving away from a focus on snapshot studies explicating what PTs do and do not know and toward attempting to understand PTs' conceptions and how their knowledge develops. The papers on alternative algorithms and nonstandard strategies address the need to help PTs develop the ability to make sense of children's mathematical thinking so that they will be prepared to do more than present standard procedures to their students. The papers on number sense show that PTs who exhibit better number sense are more able to make conceptual sense of problems. Thus, there is a need to promote PTs' number sense development. The papers on using manipulatives and computer microworlds identify tools that can help PTs make sense of mathematics. In the spirit of working from what PTs know (CBMS, 2001), these articles contribute to the literature on PTs' knowledge of whole numbers and operations. Thus, the current literature helps mathematics educators to be better equipped to support PTs' learning. However, many open questions remain.

### **A View of the Horizon**

Our review of journal articles published in 2012 and papers from PME and PME-NA proceedings for conference years 2011 and 2012 yielded only three relevant results (see Table 5).

Table 5

*Articles Published in 2012 and PME/PME-NA Proceedings From 2011 and 2012  
Dealing With PTs' Knowledge of Whole Numbers and Operation*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Feldman	2012	59 PTs	Content course	USA	Pre/post interviews with 6 PTs regarding their understanding of number theory, as well as pre/post surveys of 59 PTs
Thanheiser	2012	1 PT	Content course	USA	Two interviews in which one PT was asked to reason about and justify addition and subtraction algorithms in different bases
Whitacre & Nickerson	2012	7 PTs	Content course	USA	Interviews in which PTs were asked to perform mental computation and to justify their strategies

Thanheiser (2012) offers a case study of a PT's understanding of regrouping. The PT seemed to hold all the essential knowledge pieces needed to give a conceptual explanation for regrouping but was unable to do so. Thanheiser hypothesized that this may be due to the PT's lack of strategic knowledge (i.e., knowing when to draw on a piece of information). This point highlights the need to attend not only to conceptual understanding but also to strategic knowledge in PT content courses.

Whitacre and Nickerson (2012) report on PTs' reasoning in the area of whole-number mental computation. Building on the work of Yang (2007) that focused on the strategies that PTs tend to use, Whitacre and Nickerson investigated the mathematical justifications that U.S. PTs offer when using nonstandard mental computation strategies.

The authors describe PTs' justifications for both valid and invalid strategies. They draw distinctions between the mathematical ideas involved in the various justifications in order to clarify how PTs' strategies make sense to the PTs themselves. This analysis sheds light on PTs' reasoning when using nonstandard mental computation strategies.

In particular, Whitacre and Nickerson (2012) report the mathematical ideas used in PTs' justifications for four nonstandard addition strategies and four nonstandard subtraction strategies. This includes justifications for valid and invalid versions of subtrahend compensation. For example, two PTs computed  $125 - 49$  mentally to find the amount that a vendor would profit if he bought an item for \$49 and then sold it for \$125. Both PTs rounded 49 to 50, and both knew that  $125 - 50$  equaled 75. However, their thinking differed when it came to how to compensate for the initial rounding move. Trina reasoned that she should add 1 to 75 because by adding 1 to 49 she had "pretended [the vendor] used more money than he did," thus decreasing his profit. By contrast, Natalie reasoned that she had added 1 to "the problem" and now had to subtract 1 from the problem in order to compensate (p. 779). Thus, Trina distinguished the roles of minuend and subtrahend, and this enabled her to determine how to compensate correctly. By not making this distinction, Natalie drew the incorrect conclusion regarding how to compensate. These fine-grained distinctions in PTs' justifications reveal the reasoning underlying their strategies.

The only other report concerning PTs' knowledge of whole numbers and operations was a paper of Feldman (2012). Feldman gave a poster presentation, so the information in the proceedings paper is quite limited. He studied PTs' developing understanding of number theory during instruction on number theory in a mathematics content course in

the United States. Feldman used action-process-object-schema theory (Dubinsky, 1991) to analyze participants' interview responses and describe transitions between levels of understanding. He also mentions quantitative data that points to changes in PTs' understanding of number theory.

**Summary of the view of the horizon.** Although only three papers appeared in 2012 journals and recent PME and PME-NA proceedings, these reports do point to promising directions for research related to PTs' knowledge of whole numbers and operations. Each report involves analyses that move beyond pointing out deficits in PTs' content knowledge. Instead, these papers concern understanding PTs' reasoning in depth and studying the development of that reasoning. The report of Whitacre and Nickerson (2012) derives from Whitacre's (2012) dissertation, which focuses on PTs' number sense development. Note that in this work we did not search for or review dissertations. In the coming years, we hope that relevant dissertations, such as the works of Roy (2008) and others, will lead to valuable contributions to the research literature concerning PTs' knowledge of whole numbers and operations and the development thereof.

### **Conclusion**

We have summarized research literature concerning PTs' knowledge of whole numbers and operations in *A Historical Look*, *A Current Perspective*, and *A View of the Horizon*. Taking a step back to view the history of this research literature, we see a progression. Not only has more research been done and more learned in this area, but there is also evidence of a shift in emphasis. We know that there are inadequacies in PTs' knowledge, and these are cause for concern. Recently, researchers have become more interested in investigating the nuances of PTs' conceptions and the further development of

their conceptions. We see the emphasis on deficits and misconceptions giving way to insightful characterizations of how PTs reason when doing mathematics and how they can make use of what they know as they develop more sophisticated conceptions. We are optimistic about the future of research on PTs' knowledge of whole numbers and operations because the kind of research being done has the potential to illuminate our understanding of PTs' mathematical thinking and to better equip mathematics teacher educators to help PTs make sense of mathematics in new ways.

We conclude with a few suggestions regarding directions for future research:

- There is a need for more research like that of Thanheiser (2009, 2010) that provides insightful characterizations of PTs' conceptions, rather than evaluations of PTs' knowledge that emphasize what they do not know. Such findings can help mathematics teacher educators to better understand PTs' thinking and to envision how PTs' conceptions can develop over time.
- There is a need for more work like that of McClain (2003) and Yackel et al. (2007) that moves beyond snapshot studies of content knowledge to document PTs' learning process in an illuminating manner. Such studies have the potential to advance the field both theoretically and practically by helping mathematics teacher educators to better understand how to support productive learning in courses for PTs.

These and other suggestions are discussed in greater detail in "Mathematical Content Knowledge for Teaching Elementary Mathematics: What Do We Know, What Do We Not Know, and Where Do We Go?" in this Special Issue.



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**Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Fractions**

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**ABSTRACT:** This article presents a research summary of prospective elementary teachers' (PTs') mathematical content knowledge in the area of fractions. The authors conducted an extensive review of the research literature and present the findings across three time frames: a historical look (pre-1998), a current perspective (1998–2011), and a look at the horizon (2011–2013). We discuss 43 articles written across these time frames that focus on PTs' fraction knowledge. Consistent across these papers is that PTs' fraction knowledge is relatively strong when it comes to performing procedures, but that they generally lack flexibility in moving away from procedures and using "fraction number sense" and have trouble understanding the meanings behind the procedures or why procedures work. Across the time frames, the trend in the research has moved from looking almost entirely at PTs' understanding of fraction operations, particularly multiplication and division, to a more balanced study of both their knowledge of operations and fraction concepts. What is lacking in the majority of these studies are ways to help improve upon PTs' fraction content knowledge. Findings from this summary suggest the need for a broader study of fractions in both content and methods courses for PTs, as well as research into how PTs' fraction content knowledge develops.

**Keywords:** fractions, rational number, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

## Introduction

Elementary teachers need a “solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power” (Conference Board of the Mathematical Sciences [CBMS], 2001, p. xi). However, research studies on prospective teachers’ mathematics knowledge have shown that many possess a limited knowledge of mathematics in key content areas such as number (e.g., Ball, 1990a; Thanheiser, 2009; Tobias, 2013). This is particularly true in the case of fractions, which, along with ratio and proportion, Lamon (2007) calls, “the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites” (p. 629).

The National Mathematics Advisory Panel (2008) affirmed that “proficiency with fractions” is a major goal for K–8 mathematics education because “such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped” (p. xvii). Therefore, developing such proficiency in prospective elementary teachers (PTs) is a critical task for mathematics educators. As the authors of *The Mathematical Education of Teachers, Part 1* suggest, “The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know” (CBMS, 2001, p. 17). Thus, in order to design mathematics courses for prospective teachers that will help them to develop the “solid understanding of mathematics” called for by the Conference Board of Mathematical Sciences (2001), including a deep understanding of and “proficiency with fractions,” we must begin by determining what it is that PTs know. In this paper, we discuss the main findings from a research summary of existing studies on

prospective elementary teachers' fraction knowledge to identify directions for future research.

### **Theoretical Framework**

In looking at teacher knowledge, we begin by examining the work of Shulman (1986), who proposed three categories of content knowledge for teachers: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. For Shulman, *subject matter content knowledge* includes knowing a variety of ways in which “the basic concepts and principles of the discipline are organized to incorporate its facts” and “truth or falsehood, validity or invalidity, are established” (p. 9). *Pedagogical content knowledge* refers to the knowledge of useful forms of representations (e.g., analogies, illustrations, explanations) of subject-matter ideas that make it understandable to others, as well as an understanding of the conceptions and preconceptions students bring to the learning processes. The third type of knowledge, *curricular knowledge*, includes knowledge of a “full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances” (p. 10).

Shulman's ideas on pedagogical content knowledge sparked a huge interest in knowledge for teaching, eliciting over a thousand studies (Ball, Thames, & Phelps, 2008) throughout a number of content areas, with a large number of these studies focusing on teachers' knowledge of mathematics (e.g., Ball et al., 2008; Davis & Simmt, 2006; Hiebert, 1986; Ma, 1999). Deborah Ball and her colleagues introduced the term *mathematical*



*knowledge for teaching* (MKT) (e.g., Ball & Bass, 2002), which focused on the work that teachers do when teaching mathematics.

Building on Shulman's (1986) categories of knowledge, Ball, Thames, and Phelps (2008) introduced a framework for mathematical knowledge for teaching. This framework broke subject matter knowledge into three categories: *common content knowledge* (CCK), the mathematical knowledge that should be known by everyone; *specialized content knowledge* (SCK), the knowledge of mathematics content that is specific to the work of teachers; and *horizon content knowledge*, which involves understanding how different mathematical topics are related. Pedagogical content knowledge was similarly broken into *knowledge of content and students* (KCS), which dealt with understanding how students relate to different topics; *knowledge of content and teaching* (KCT), which involves the sequencing of topics and the use of representations; and *knowledge of the curriculum* as a whole. While a number of different frameworks look at mathematical knowledge for teaching, we chose to use this framework to ground our study, as it is widely recognizable in the mathematics education field.

### **Background and Research Questions**

This summary work was initiated at a PME-NA Working Group over a four-year period from 2007 to 2010 (Thanheiser et al., 2010). The members of the Working Group all taught specially designed mathematics courses for elementary school teachers in the United States and sought to improve their practice by building on PTs' current knowledge. The Working Group was formed with a goal of summarizing the prior research addressing PTs' content knowledge and its development with the idea that we could improve both our teaching and course design, as well as design further research to extend what we know

about PTs' mathematical knowledge. We broke into smaller groups by content area (whole-number concepts and operations, fractions, decimals, geometry and measurement, and algebra) and attempted to summarize the current research in each of these fields.

This paper reports a summary of the research that has been done to this point on prospective elementary teachers' knowledge of fractions. Our goals for the research summary were (a) to identify what we already know about PTs' knowledge of fractions in both the domains of common and specialized content knowledge, as well as knowledge of content and students; and (b) to identify the knowledge gap in the existing research base to help guide future research endeavors. We organize our summary into three categories: (a) a historical look at PTs' fraction understanding, (b) a look at a recent perspective on PTs' knowledge of fractions, and (c) a view of the horizon on what current and future work on PTs' knowledge of fractions may look like and what it should look like.

### **Background on Fractions in General**

Before we discuss what we know about prospective teachers' knowledge of fractions, we must look briefly at the topic of fractions in general, to gain an understanding of what knowledge of fractions would look like from a general perspective.

One research area that encompasses the study of fractions is that of rational number. A rational number is one that can be written in the form  $a/b$  where  $a$  and  $b$  are both integers, and  $b$  is not equal to 0; thus, the study of fractions is part of the study of rational numbers. Researchers (e.g., Ball, 1993; Kieren, 1976, 1993; Lamon, 2007, 2012) have tended to agree that in order to gain a deep understanding of rational numbers in general, one must be familiar with many different interpretations of fractions. While researchers have given slightly varying lists of these interpretations, Ball (1993)

summarizes that they have tended to agree that fractions “may be interpreted (a) in part-whole terms, where the whole unit may vary; (b) as a number on the number line; (c) as an operator (or scalar) that can shrink or stretch another quantity; (d) as a quotient of two integers; (e) as a rate; and (f) as a ratio” (p. 168), and that in order to have a deep understanding of rational number, students and teachers must be familiar with all of these representations, rather than merely the part-whole area models that are most commonly associated with fractions and most commonly taught in schools. Lamon (2007, 2012), in particular, has emphasized the need for students to be introduced to a variety of fraction interpretations, stating that “students whose instruction has concentrated on part-whole fractions have an impoverished understanding of rational numbers” (2012, p. 256). Thus, one of the important areas of prospective teachers’ knowledge of fractions is to have a deep understanding of all of the different interpretations of fraction.

Another research area that has looked at fractions deals with literature on multiplicative structures. Vergnaud (1988) includes rational numbers as part of what he calls the *multiplicative conceptual field*, which, he says, “consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide . . . [These include] fraction, ratio, rate, rational number, and multiplication and division” (p. 141). The basis of a conceptual field is that it contains a set of situations that are modeled by a similar action. Movement from the additive conceptual field to the field of multiplicative structures has been shown to be difficult for students and teachers (e.g., Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989). This difficulty is particularly due to a problem Taber (1999) calls the “multiplier effect.” Taber describes this effect in this way: “Students seem to select multiplication or division as the

operand that will solve the problem depending on their sense of whether the multiplicand is enlarged or reduced by the action of the problem” (p. 2). This problem was described by Fischbein, Deri, Nello, and Marino (1985) in their work with fifth, seventh, and ninth grade students. The students were given a variety of word problems dealing with multiplication and division of rational numbers and asked to write an equation that they would use to solve the problems. In general, when the students thought that the result of the problem should be smaller than the input, they chose to divide; when they thought their result should be larger, they chose to multiply, even though in many instances this was not the correct equation and did not lead to the correct answer.

One aspect of multiplicative structures that can be particularly difficult for students is the concept of division. Division is typically taught using two different interpretations. The *partitive* or *sharing* model involves dividing the total amount by the number of groups in order to find the number in each group (Greer, 1992). The *quotitive, measurement, or repeated subtraction* model of division involves separating the total number of things by the number in each group to find the number of groups possible (Greer, 1992).

The *partitive* model of division is typically taught to children first, and is called the “primitive” model of division by researchers (Fischbein et al., 1985; Tirosh & Graeber, 1989). This idea is introduced as division through “fair sharing” and can be modeled by giving one object to each person until there are none left. For example, the problem “I have 20 cookies and I want to share them among myself and 4 friends. How many cookies do we each get?” can be modeled by distributing a cookie to each person one at a time until each person has 4 cookies, and there are no cookies left.

The measurement type of division can be modeled by the process of repeated subtraction. The question “I have 20 cookies and I want to give 5 to each of my friends. How many friends can get cookies?” can be modeled by repeatedly taking out groups of 5 from the 20 objects until there are no cookies left, resulting in 4 groups.

Of the two models of division, the *measurement* model is much more easily translated into situations dealing with fractions. We can think of having  $5\frac{1}{2}$  pounds of candy, giving  $\frac{1}{2}$  of a pound to each person, and asking how many people get candy. This situation can be easily modeled by subtracting  $\frac{1}{2}$  from  $5\frac{1}{2}$  until there is nothing left, and we can see that there are 11 groups. Thus,  $5\frac{1}{2} \div \frac{1}{2} = 11$ . However, it gets more complicated when we try to translate the *partitive* model of division into fractional situations. “The fair sharing, or partitive model is a traditional teaching model for division of whole numbers, but it can act as a barrier in the representation of division of fractions” (Rizvi & Lawson, 2007, p. 378). When we look at division of fractions using this model, the original situation that we used with whole numbers does not make sense. We cannot talk about half or a third or three fifths of a person. The partitive situation can be modeled with a word problem, such as “I have  $5\frac{1}{2}$  pounds of candy. This is  $\frac{1}{2}$  of a serving of candy. How much candy is a whole serving?” We still know how much we started with and are trying to determine the size of one group, but the translation of the problem does not always make it seem like it is the same form.

In order to develop proficiency with fractions, one must not only be able to perform operations with them, but must also develop a fraction number sense, which means being able to think of fractions as numbers in a system. Lamon (2012) describes fraction number sense in this manner: “Students should develop an intuition that helps them make

appropriate connections, determine size, order, and equivalence, and judge whether answers are or are not reasonable” (p. 136). This makes being able to compare and order fractions an important component of teachers’ fraction knowledge.

Lamon (2012) suggests three different strategies for ordering fractions: *same-size parts*, *same number of parts*, and *compare to a benchmark*. These strategies are also suggested in the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010). In the same-size parts strategy, which is also referred to as the “common denominator” strategy, if two fractions have the same denominator, or size of parts, then they can be compared merely by looking at the numerators. For example,  $3/5 > 2/5$ , because 3 of something is more than 2 of the same thing. In the same number of parts, or “common numerator” strategy, if two fractions have the same numerators, or number of parts, we can compare them by looking at the size of the individual parts. For example,  $2/3 > 2/5$ , because if we break a whole into three equal-sized pieces and break an equivalent whole into five equal-sized pieces, then the thirds will be larger than the fifths.

The third fraction comparison strategy involves comparing two fractions to another “benchmark” fraction, such as  $1/2$ ,  $1/3$ , or 1. For example, in comparing  $3/7$  and  $6/11$ , we know that  $3/7 < 1/2$ , since 3 is less than half of 7, and  $6/11 > 1/2$ , since 6 is more than half of 11. Therefore since  $3/7 < 1/2 < 6/11$ , we can use the transitive property to determine that  $3/7 < 6/11$ .

Now that we have a better understanding of what knowledge of fractions might entail, we can move on to looking at what we know about PTs’ fraction knowledge in

particular. In order to do this, we need to do an extensive search of the literature to determine what we already know and to look at what we still need to learn.

### **Methods**

The first step of conducting this research summary on what we know about PTs' knowledge of fractions was to identify the existing literature. Thus, we began by looking for articles to fit into the *Current Perspective* section. To maintain the quality of the findings, we began by restricting our search to the peer-reviewed research articles published between 1998 and 2011 to cover the 12-year range prior to our Working Group's meetings. The Working Group chose this time period because it marked the beginning of a renewed interest on teacher knowledge since the publication of Ma's (1999) work that looked at elementary teachers' mathematical knowledge for teaching in the United States and China. This was particularly true in the area of division of fractions, which is the area where the majority of the U.S. teachers struggled.

With key words such as *preservice teachers*, *prospective teachers*, *fraction*, and *rational numbers*, we searched the ERIC, Google Scholar, Dissertation Abstracts, and Rational Number Reasoning databases ([gismo.fi.ncsu.edu/database](http://gismo.fi.ncsu.edu/database)) to find any papers that might fit into our study. The second step required our research team to locate these papers and skim through them to determine if they had a research question focusing on prospective elementary teachers' fraction knowledge. We ended up rejecting a number of papers, because they did not meet this criterion. For example, we found some papers in our searches that focused on prospective teachers' beliefs, rather than their knowledge. Others did not really encompass PTs' knowledge of fractions, but rather included a single example of one PT's thoughts on a problem that happened to have a fraction in it. We carried out

careful readings of these documents during the third step. To assist the comparison across these documents, we created a synthesis table with information such as “research questions,” “research design,” “descriptions of participants,” “content foci,” “data collection,” “data analysis,” “findings,” and “implications” for each. Each content group filled in a similar table with information from their respective content areas.

After our initial search, each content group summarized its findings and reported them at a Working Group meeting (Thanheiser et al., 2010). We shared our list of articles and discussed inclusion/exclusion criteria for the journals we had found, focusing on whether the journals published empirical studies and were peer-reviewed. We ended up compiling a list of 23 journals from which at least one group had found articles. We then carefully reviewed each journal for additional articles focusing on PTs’ content knowledge within the given time frame to make sure that we had identified all of the relevant articles for the *Current Perspective* section.

In our search for articles that focused on prospective teachers’ knowledge of fractions that were published prior to 1998, we chose to focus on articles that had been cited in later research. The rationale behind this is that these papers, while older, provided the basis for much of the later research on prospective teachers’ knowledge of fractions. In order to find these studies, we checked the reference sections of all of the articles that we found from our searches for *Current Perspective* articles. In addition, two of the authors of this article were in the process of writing dissertations that related to prospective teachers’ fraction knowledge, so they brought with them a number of articles from literature searches related to this work. While this process may not have identified all of the articles



written about PTs' fraction knowledge prior to 1998, we are confident that we have all the articles that provided the basis for future studies.

We conducted a review of recent research, 2011 through the beginning of 2013, to analyze the current and future trends in PTs' understanding of fractions. We conducted a journal search from our list of 23 journals for any articles published in 2012 and the first quarter of 2013. In addition, we manually searched for papers in conference proceedings from the International Group of Psychology of Mathematics Education (PME) and the Psychology of Mathematics Education–North America Chapter (PME-NA) from 2011 and 2012, because we recognized the time lag required for publication and were interested in the directions of future research. We added these articles to our synthesis table and began to organize the articles around different themes.

### **Results**

We organized our findings of 43 papers both into the time frames—pre-1998, 1998–2011, and 2011 and beyond; and around three main components of the theoretical framework outlined by Ball, Thames, and Phelps (2008)—Common Content Knowledge, Specialized Content Knowledge, and Knowledge of Content and Students—and also different instructional interventions designed to help improve this knowledge. We included sections on instructional interventions because we believe that the study of PTs' fraction knowledge encompasses not only what they know, but also how they come to know it. While Ball and her colleagues outlined other aspects of mathematical knowledge for teaching, these did not encompass what we would consider PTs' *content* knowledge, which is the focus of this article, and, thus, we did not frame our discussion around them.

### Historical Perspective (Prior to 1998)

In total, we found 12 articles from six different studies, which we felt provided the basis for subsequent work looking at PTs' fraction content knowledge. A summary of articles is included in Table 1.

Table 1

#### *Articles Written Prior to 1998 Dealing With PTs' Fraction Content Knowledge*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Ball	1990a	252 (217 elementary and 35 mathematics majors)	Point of entry into formal teacher education program	USA	Questionnaire and interviews/observations of a smaller group
Ball	1990b	19 (10 elementary and 9 secondary)	Prior to enrolling in their first education course	USA	Interviews with probing questions
Behr et al.	1997	30	Seniors in a methods course	USA	Videotaped interviews
Borko et al.	1992	1 as the focus (out of a larger group of 8)	During student teaching	USA	Observations of a teaching episode
Eisenhart et al.	1993	1 as the focus (out of a larger group of 8)	During senior year—student teaching and preparation	USA	Observations of teaching episodes
Graeber, Tirosh, & Glover	1989	129	Enrolled in either a content or a methods course	USA	Written test and interviews with 33 of the students

(continued)

Table 1—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Khoury & Zazkis	1994	124 (100 elementary and 24 secondary mathematics)	After some mathematics content work	USA	Written assessment and clinical interviews
Simon	1993	33	Enrolled in a methods course	USA	Written test and interviews with 8 students
Tirosh & Graeber	1989	136	Enrolled in either a content or a methods course	USA	Written test and interviews with approximately half of the students ( $n = 71$ )
Tirosh & Graeber	1990a	21 selected based on pretest data	11 in a content course, 10 in a methods course	USA	Pre- and posttests and interviews with probing questions
Tirosh & Graeber	1990b	136	Enrolled in either a content or a methods course	USA	Written test and interviews with over 85 students
Tirosh & Graeber	1991	80	Enrolled in either a content or a methods course	USA	2 written tests and interviews with 33 of the students

While two of these articles dealt directly with the subject of fractions (Behr *et al.*, 1997; Khoury & Zazkis, 1994), the majority of them focused more on PTs' conceptions of multiplicative structures in general, particularly in the case of multiplication and division, with only portions of these studies focusing on using these operations specifically with fractions. The focus of these papers dealt with the misconceptions that PTs had about multiplication and division in general (e.g., Graeber, Tirosh, & Glover, 1989), PTs' difficulty representing fraction division (e.g., Ball, 1990a, 1990b; Simon, 1993), and the difficulty that

one PT had in explaining fraction division to students (Borko et al., 1992; Eisenhart et al., 1993). These articles could all be described as falling under either the CCK or SCK areas of mathematical knowledge for teaching.

**Prospective teachers' common fraction knowledge.** All of the studies in this time period except for two (Behr et al., 1997; Khoury & Zazkis, 1994) focused on some aspects of PTs' common content knowledge of fractions. These articles focused primarily on aspects of fraction division. Across all of the articles, the vast majority of prospective teachers were able to perform the traditional invert-and-multiply procedure for dividing fractions. However, none of the PTs across the studies were able to explain why this algorithm worked. Ball (1990a) writes: "Although almost all the prospective teachers were able to calculate  $1\frac{3}{4} \div \frac{1}{2}$  correctly, strikingly few were able to represent the meaning underlying the procedure they had learned" (p. 458). This is possibly because PTs do not see the need to understand why they perform the procedures that they do, as long as they work. However, this belief persists into student teaching, when it becomes necessary for some PTs to explain the meanings behind the procedures (Borko et al., 1992; Eisenhart et al., 1993). One student teacher, Ms. Daniels, did not find it necessary to find an explanation of the invert-and-multiply rule either for herself or for the student, even after being unable to answer a question posed to her by a student during a student teaching lesson (Borko et al., 1992).

Tirosh and Graeber and their colleagues' studies (Graeber et al., 1989; Tirosh & Graeber, 1989, 1990a, 1990b, 1991) focus mainly on looking at whether PTs have the same misconceptions about multiplication and division that Fischbein and his colleagues (1985) found in children. They found that PTs do show evidence of Taber's (1999) "multiplier

effect,” believing that multiplication always makes bigger and division always makes the result smaller (Graeber et al., 1989). Thus, in deciding whether to use multiplication or division to solve a given word problem, they chose multiplication when they believed the answer would be larger than the initial quantities, and division when they believed the answer would be smaller (Tirosh & Graeber, 1991). These misconceptions persisted in interviews when PTs were asked to perform a division problem where the quotient was larger than the dividend; rather than changing their beliefs, they instead determined that they had made a mistake in computation (Tirosh & Glover, 1990b). Thus, like children and adolescents, PTs seem to have a tendency to overgeneralize rules for whole number operations and apply them to fraction operations. Without deliberate attention to attempt to fix these misconceptions, we believe this cycle of both students and teachers struggling with these ideas will continue.

**Prospective teachers’ specialized fraction knowledge.** One key aspect of teaching is being able to design problems for students. While not necessarily a component of common content knowledge, the ability to create realistic problems, especially those in context, is an important part of a teacher’s SCK. Ball (1990a, 1990b), Simon (1993), and Tirosh and Graeber (1991) all found that prospective teachers had great difficulty writing word problems that represented division by a fraction. When asked to do this, most PTs either were unable to come up with a problem at all, or suggested a problem that represented a number expression different from what was asked. For example, when asked to create a division problem for  $\frac{3}{4} \div \frac{1}{4}$ , the most common error among the students in Simon’s (1993) study was providing a problem for  $\frac{3}{4} \times \frac{1}{4}$ . Students in Ball’s (1990a, 1990b) study often gave problems that represented  $1 \frac{3}{4} \div 2$ , when asked to create one for

$1\frac{3}{4} \div \frac{1}{2}$ . When discussing their inability to write problems involving fractions, many PTs attributed their problems to the fact that the problems involved fractions, saying, “You don’t think in fractions; you think more in whole numbers” (Ball, 1990a, p. 455). Both Ball and Simon attribute the difficulties that the PTs had more to a lack of a full understanding of division, which was exacerbated by introducing fractions into the problems. Simon (1993) and Tirosh and Graeber (1991) did find that the students who used a measurement model of division were more successful than those who attempted to use a partitive model.

Behr, Khoury, Harel, Post, and Lesh (1997) and Khoury and Zazkis (1994) looked at other aspects of PTs’ specialized content knowledge. We classify these as specialized knowledge because they go beyond the traditional knowledge that everyone should have. The former was interested in PTs’ ability to look at an operator model of fractions, rather than the traditional part-whole model. The latter looked at PTs’ abilities to think about fractions and decimals in different bases, to delve into their understandings of place value and how they relate to fractions.

In Behr and colleagues’ (1997) study, the researchers investigated PTs’ ability to deal with the operator concept of a fraction when finding  $\frac{3}{4}$  of 8 four-stick bundles. The PTs were asked to do this in more than one way if they could, but the majority of them applied only one solution strategy, which usually focused on what the authors call a duplicator/partition-reducer (DPR) strategy. This strategy revolves around the partitive method of division, which other studies (e.g., Simon, 1993; Tirosh & Graeber, 1991) have found PTs to favor. The second strategy, called stretcher/shrinker (SS), which corresponds to the measurement model of division, was less prevalent.

While all of the 100 prospective elementary teachers in Khoury and Zazkis' (1994) study were able to conclude that  $0.2_{\text{three}}$  and  $0.2_{\text{five}}$  were unequal, only 26 correctly said that  $1/2_{\text{three}}$  was equal to  $1/2_{\text{five}}$ . Thus, these PTs believed that fractions changed their numeric values under different symbolic representations, rather than realizing that  $1/2$  was half of a whole and  $1_{\text{three}} = 1_{\text{five}}$ . These studies (Behr et al., 1997; Khoury & Zazkis, 1994) show that in addition to PTs having an understanding of fraction operations that is not very robust, they also struggle in understanding different interpretations of fractions in general.

**Improving prospective teachers' fraction knowledge.** While the majority of these early studies do not give suggestions on improving PTs' fraction knowledge, Tirosh and Graeber (1990a) do suggest evoking what they call "cognitive conflict" in order to help PTs with the misconception that division always makes smaller. In interviews with PTs who held this misconception, the prospective teachers were asked to talk about what division meant and think about the terms *dividend*, *divisor*, and *quotient*. The researchers also provided examples, such as  $4 \div 1/2$ , which were meant to help PTs question the idea that division always made smaller. Following these interviews, the majority of PTs were able to clear up many of the misconceptions that they held about division, as their pretest performance improved on the posttest.

From our search of literature on PTs' fraction knowledge from research prior to 1998, we find that the majority of studies focus on the understandings or misunderstandings that PTs have with relating multiplication and division to fractions. In general, we found that PTs are familiar and mostly comfortable with performing the algorithms when working with fractions, but struggle when asked to explain why the algorithms work (e.g., Ball, 1990a; Borko et al., 1992), or to create word problems that

represent division by a fraction (e.g., Ball, 1990a, 1990b; Simon, 1993). These are both types of tasks that will be necessary for PTs in their work as teachers; thus, helping PTs to improve upon their procedural understandings is an important step in preparing them for the future. In addition, PTs tend to overgeneralize rules for whole numbers, such as “multiplication makes bigger,” and attempt to apply them to operations dealing with fractions as well (e.g., Graeber et al., 1989). Creating cognitive conflict about these misconceptions seems to be a way to help PTs question their own faulty understandings and clear up their misconceptions (Tirosh & Graeber, 1990a). As we continue our review into more current articles, the focus shifts somewhat from looking at mostly fraction operations, to a more rounded view of PTs’ understandings of fractions.

### **Current Perspective (1998–2011)**

We found 17 journal articles published during the period of 1998–2011 that are included in this review. These studies were conducted in several different countries with groups of prospective teachers ranging in size from 4 to 344. For summary purposes, we have listed the articles in Table 2.



Table 2

*Articles Written From 1998–2011 Dealing With PTs’ Fraction Content Knowledge*

Authors	Year	Number of PTs Studied	PTs’ Level	Country	Methodology
Chinnappan	2000	8	First year of education program	Australia	Interview consisting of training and problem solving
Domoney	2002	4	Student teachers in the teacher training program	Great Britain	Task-based interviews
Green, Piel, & Flowers	2008	50 in study 1; 39 in study 2	Study 1: Enrolled in child development course; Study 2 unclear	USA	Pretest, treatment, and posttest
Isiksal & Cakiroglu	2011	17	Final year of their program	Turkey	(1) Questionnaire on PCK; (2) a follow-up interview on multiplication of fractions
Li & Kulm	2008	46	Math methods course/middle school math and science interdisciplinary program	USA	(1) survey for general pedagogical knowledge; (2) a math test for MKT; (3) an assignment on curriculum planning
Lin	2010	48	Integrated content and methods course	USA	Pretest, treatment and control groups, posttest
Luo	2009	127	Mathematics methods course	USA	Written test
Luo, Lo, & Leu	2011	89 USA; 85 Taiwan	Mathematics methods course	Taiwan and USA	Written test
Menon	2009	64	Mathematics methods course	USA	Written test
Newton	2008	85	Mathematics content course for PTs	USA	Pre- and posttests

(continued)

Table 2—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Rizvi	2004	17	Completed mathematics curriculum studies courses	Australia	Pre-interview, treatment, post-interview
Rizvi & Lawson	2007	17	Primary/lower secondary Bachelor of Education students	Australia	Pretest A, pretest B, treatment, posttest A, posttest B
Son & Crespo	2009	17 elementary, 17 secondary	Mathematics methods course	USA	Written test
Tirosh	2000	30	Mathematics methods course	Israel	Questionnaire, instruction, midterm assignment, final assignment
Toluk-Ucar	2009	50 experimental; 45 control	Mathematics methods course	Turkey	Written test, questionnaire as pre/posttests, math journals
Yang, Reys, & Reys	2008	280	Unclear	Taiwan	Written test
Young & Zientek	2011	344	Enrolled in one of three different mathematics courses required for PTs	USA	Pre/post written tests

As in the historical section, we classified the articles into one or more of the following four categories based on their research questions—prospective teachers' common fraction knowledge, prospective teachers' specialized fraction knowledge, prospective teachers' knowledge of common fraction errors and non-traditional strategies, and improving prospective teachers' fraction knowledge—which we summarize below.

**Prospective teachers' common fraction knowledge.** Six studies collected data on PT's conceptual and procedural knowledge of fractions. Domoney (2008) investigated

whether student teachers who were trained to teach lower-primary age students in Great Britain had the same limited conceptions of fraction, dominated by part-whole constructs. Chinnappan (2000) investigated PTs' ability to transfer their understanding of fractions to a computer environment called JavaBar. Yang, Reys, and Reys (2008) found that while the PTs from Taiwan were fluent in their procedural knowledge when comparing fractions, most of them were not able to use number sense to compare fractions, even when doing so would be more efficient. Young and Zientek (2001) investigated PTs' understanding of fraction operations through four specific problem types: (a) addition with common denominators, (b) addition with relatively prime denominators, (c) multiplication with relative prime denominators, and (d) division of reciprocal fractions. Luo, Lo, and Leu (2011) compared PTs from Taiwan and the U.S. on a variety of fundamental fraction knowledge topics, including part-whole, quotient constructs in different reorientations, as well as their concepts of equivalence and meanings of fraction operations. Newton (2008) conducted a comprehensive survey of PTs' fraction knowledge that included both routine and non-routine problems covering different types of fraction questions typically found in middle school textbooks.

Generally speaking, these studies found PTs were procedurally proficient in fraction addition and subtraction (Newton, 2008; Young & Zientek, 2001). However, their procedures were rule-based and lacked flexibility. For example, 72 out of the 85 PTs in Newton's (2008) study changed both  $\frac{2}{4}$  and  $\frac{3}{6}$  to the equivalent fractions of the same denominator to solve the problem  $\frac{2}{4} - \frac{3}{6}$  rather than renaming both to  $\frac{1}{2}$ . This lack of flexibility extended into PTs' work on fraction multiplication as well, as many of the 344 U.S. PTs in Young and Zientek's (2011) study converted fractions into the same

denominator when performing fraction multiplication, even though it was not necessary. A good portion of the prospective teachers had difficulty working with fraction multiplication and fraction division procedures in general (Newton, 2008; Young & Zientek, 2001). For example, on a pretest given at the beginning of their mathematics content course, 49 PTs ( $n = 85$ ) had at least one computation error with multiplication and 45 had at least one with division problems (Newton, 2008). These numbers dropped to 44 and 17, respectively, on a posttest. Although PTs seemed to improve in their fraction division knowledge, some of the fraction multiplication problems persisted despite the semester-long instruction. This was largely due to the wrongful application of the “cross-multiply” procedures, (e.g., they perform  $a/b \times c/d = ad/cb$ ). The same “cross-multiply” pattern also appeared as the most common fraction division procedure error (Newton, 2008; Young & Zientek, 2011).

The dominating rule-based reasoning also showed up in studies examining PTs' ability to compare fractions (Chinnapan, 2000; Domoney, 2002; Yang et al., 2008). In each of these studies, most of the PTs chose procedural methods when comparing fractions, even when applying number sense would have been more efficient. For example, less than half of the 280 Taiwanese PTs used a benchmark of 1 to solve the following fraction comparison problem: “Vicky and Mary each have a ribbon. Vicky used  $30/31$  of a meter for her ribbon, and Mary used  $36/37$  of a meter for hers. Who used more tape for their ribbon? Why?” (Yang et al., 2008). Instead, they relied on changing the fractions to decimals, or finding common denominators, which required more difficult calculations than using number sense and also caused nine of the PTs to get an incorrect answer because of a miscalculation.

PTs' performance on conceptual items and items that require deeper understanding of operations was less than satisfactory. Studies conducted by Luo, Lo, and Leu (2011) with PTs in the U.S. and by Domoney (2002) with PTs from the UK found a strong preference for the part-whole meanings of fraction over other meanings such as quotient and ratio. PTs from these two countries also had difficulties working with number lines. For example, when asked to locate the number  $\frac{3}{5}$  on the number line of 5 units long, with 0–5 being labeled, one PT placed  $\frac{3}{5}$  on the unit labeled "3" (Domoney, 2002). In addition, none of the four UK PTs interviewed in this study were able to come up with two fractions that summed to 5 on the number lines. However, this difficulty with number lines did not show up in Luo et al.'s study with PTs from Taiwan. The PTs in this study were also found to be strong with the quotient meanings of fraction. This points to possible differences in different countries' methods of teaching fractions. Researching these instructional differences could lead to improved performance in other countries as well, especially with the increased focus on using number lines to represent fractions and their operations in the U.S. *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010).

Finally, Newton (2008) found that PTs' low performance on problem solving, transfer, and flexibility did not improve much after instruction. For example, 40% of the 85 prospective teachers did not appear to recognize the importance of the equal wholes when performing fraction addition when combining one glass of chocolate milk that contains  $\frac{1}{3}$  of the glass of chocolate syrup, and another glass that is twice as large with  $\frac{1}{4}$  of the glass of chocolate syrup. It is doubtful that PTs with such understanding of fractions could support their elementary students' learning of fractions in a meaningful way.

**Prospective teachers' specialized fraction knowledge.** Several studies examined PTs' ability to create diagrams or word problems for given fraction expressions (Li & Kulm, 2008; Luo, 2009; Menon, 2009; Rizvi, 2004; Rizvi & Lawson, 2007; Toluk-Ucar, 2009). The findings of these studies, based on PTs from Australia, Taiwan, Turkey, and the U.S., suggest the majority of PTs are not proficient in this area. This echoes earlier studies (e.g., Ball, 1990a, 1990b; Simon, 1993), which also report PTs' difficulties in creating fraction word problems. These studies have identified a variety of misconceptions behind the poor performance. For example, Luo's (2009) study focused on PTs' ability to represent fraction multiplication expressions. She found that the majority of the PTs used a "multiplication as repeated addition" model that can be problematic when they are not sure how to add a quantity a fraction of a time. Rizvi and Lawson (2007) found a pattern of declining performance from whole number division problems to fraction division problems, when representing division problems either with word problems or diagrams. Toluk-Ucar (2009) found that many Turkish PTs were unable to identify the unit to which each fraction in an expression referred, so when asked to create a word problem for  $3/4 - 1/2$ , they instead wrote one for  $3/4 - 3/8$  (note:  $3/8$  is  $1/2$  of  $3/4$ ).

Another type of specialized knowledge is the ability to provide student-accessible justifications to why given rules and procedures work. Li and Kulm (2008) asked 46 prospective middle school teachers in the U.S. how they would explain to students why  $2/3 \div 2 = 1/3$  or  $2/3 \div 1/6 = 4$ . About 26% of the participants use pictorial representations to explain the division procedures, and 22% explained using the "flip and multiply" procedure by describing *how* it should be performed. Most of the other PTs were unable to

explain either problem, and none of the 46 participants were able to provide an explanation of *why* “flip and multiply” worked.

**Prospective teachers’ knowledge of common fraction errors and non-traditional strategies.** When teachers enter the classroom, they need to have an understanding of student thinking in addition to understanding mathematics content (Ball et al., 2008). With this understanding, teachers can establish classrooms where discussions focus on the validity of students’ responses. Knowing how prospective teachers interpret student responses before they enter a classroom can provide a foundation for the types of activities needed in teacher education programs.

Tirosh (2000) set out to investigate PTs’ abilities to identify common student mistakes and the possible source of these mistakes, when evaluating fraction division expressions and solving fraction division word problems. She found that while the majority of PTs were fluent in evaluating fraction division expressions, and most were able to identify at least one common student mistake, they were not able to do so with the word problems. In this paper, Tirosh also discussed several class activities specially designed to help strengthen the PTs’ fraction knowledge for teaching. One of her activities was later adapted by Li and Kulm (2008) and Son and Crespo (2009) to investigate PTs’ KCS. They both asked PTs to evaluate the validity and efficiency of a non-traditional division method:  $a/b \div c/d = (a \div c)/(b \div d)$ . Only 2 out of 46 participating PTs in Li and Kulm’s study stated that this division method was correct. Son and Crespo developed a framework of six levels of reasoning to classify the PTs’ responses that was based on validity, generalizability, and efficiency. Eleven out of the 17 elementary PTs were classified at one of the three lowest levels on this scale, because they did not think the division method described above was

generalizable. Those who were classified with lower level reasoning tended to use teacher-focused approaches to respond to their students. That is, they would tell or show directly whether the method worked, and they provided little opportunity for students to explain their reasoning.

Isiksal and Cakiroglu (2011) conducted a study to examine PTs' knowledge of student misconceptions and sources of these misconceptions on fraction multiplication. Based on a written test and semistructured interviews with 17 Turkish PTs, they identified five main categories of misconceptions suggested by PTs for students' errors: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge of fraction operations, misunderstandings of the symbolism with fractions, and misunderstanding the problems. The first three were consistent with findings from Tirosh (2000), while the last two were new findings from this study. For example, one PT pointed out that students may not be able to answer the word problem "Elif bought a bottle of milk. She gave  $\frac{1}{2}$  of it, which was  $1\frac{3}{4}$  lt, to her grandmother. How much did the bottle of milk contain originally?" because they did not understand the key point that half of something is  $1\frac{3}{4}$ . This PT's description of student error was classified under "misunderstanding the problem."

**Improving prospective teachers' fraction knowledge.** Several studies have examined the effects of specially designed mathematics courses (Newton, 2008), or special instructional strategies on prospective teachers' knowledge of fractions, for example, the use of manipulatives (Green, Piel, & Flowers, 2008), Web-based instruction (Lin, 2010), and problem-posing activities (Toluk-Ucar, 2009).



Lin (2010) and Toluk-Ucar (2009) used an experimental design to investigate the effect of certain treatments on improving PTs' fraction knowledge. The treatment in Lin's study consisted of 6 weeks (18 hours) of Web-based instruction that included modules from the National Library of Virtual Manipulatives (<http://nlvm.usu.edu/en/nav/vlibrary.html>) and the National Council of Teachers of Mathematics' *Illuminations*. The treatment in Toluk-Ucar's study included a 6-hour fraction unit over 3 weeks, where problem posing was used as the primary teaching approach. PTs were given different fractions and asked to pose problems where these fractions were answers, and then to justify the validity of their problems to the rest of the class. The PTs were encouraged to use different representations to support their arguments.

While PTs in all of these studies showed significant improvements over the semester course or after the instructional interventions, many PTs still leave their mathematics or methods courses with various deficiencies and misconceptions. For example, 40% of the prospective teachers in Newton's study (2008) did not appear to recognize the importance of the equal wholes when performing fraction addition. This finding suggests that PTs with such an understanding of fractions may need further professional development in order to be able to support their elementary students' learning of fractions in a meaningful way that meets the expectations of the *Common Core State Standards*.

### **A View of the Horizon**

In searching for articles that represented the future trends in research on PTs' fraction content knowledge, we looked at journal articles from 2012 and the first quarter of 2013, as well as conference proceedings from PME and PME-NA for 2011 and 2012. We

found a total of 14 articles that focused on PTs' fraction conceptions, which are listed in Table 3.

Table 3

*Articles Written Between 2011 and Early 2013 Dealing With PTs' Fraction Content Knowledge*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Caglayan & Olive	2011	10	Enrolled in an Algebra for Teachers course	USA	Interview
Harvey	2012	13	Graduate students, 5 were in their final month of their teacher education program and had completed a math education course; 8 were in their first month and had not yet completed this course	New Zealand	(1) written questionnaire, then (2) participated in a teaching experiment either individually or in pairs
Ho & Lai	2012	92	First year of the program	Australia	Ten-item test
Kajander & Holm	2011	Over 600	Enrolled in a mathematics methods course	Canada	Pre/posttest
Lin et al.	2013	49 from U.S.; 47 from China	U.S.–third year of program; China–third year of program	China and USA	Test adapted from Cramer, Post, and delMas (2002) given during first week of fall semester
Lo & Grant	2012	16	3 had completed their first required mathematics course; 13 had not yet taken the course	USA	Interviews

(continued)

Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Lo & Luo	2012	45	Enrolled in a mathematics methods course	Taiwan	Interview and written questionnaire
McAllister & Beaver	2012	First phase: >100; Second phase: 72	Enrolled in mathematics content courses at two universities: the second phase included 3 groups of students enrolled in a content course that covers fractions and 1 group enrolled in a content course focusing on geometry (having completed the fraction course).	USA	Given 8 fraction operation problems. First phase: asked to solve the problem and write word problems. Second phase: asked only to write story problems
Mochon & Escobar	2011	21	Last semester of formal course work	Mexico	Questionnaire and classroom observations
Muir & Livy	2012	279	Enrolled in a first-year course	Australia	Mathematical Competency, Skills, and Knowledge test
Rosli, Gonzalez, & Capraro	2012	3	Had completed most required coursework	USA	Interviews
Tobias	2013	33	Enrolled in the first mathematics content course (all at least sophomores)	USA	Classroom teaching experiment
Utley & Reeder	2012	42	Enrolled in an intermediate methods course	USA	Pre/posttest
Whitacre & Nickerson	2011	7	Enrolled in a first mathematics content course	USA	Pre/post interview

Seven of the articles focused on fraction concepts (i.e., comparison, equivalence), five on fraction operations, and two focused on both concepts and operations. This is a shift

from previous research in which the majority of articles focused on fraction operations and only a small number on fraction concepts. With the more recent publications, research with PTs is starting to include a more comprehensive analysis of their fraction content knowledge.

**Prospective teachers' common fraction knowledge.** Eight of the studies focused on PTs' common content knowledge of fractions. The focus of these studies varied widely, including fraction comparison (Whitacre & Nickerson, 2011), converting fractions to decimals (Muir & Livy, 2012), fraction meanings (Lo & Grant, 2012; Mochon & Escobar, 2011; Utley & Reeder, 2012), and fraction operations such as multiplication (Caglayan & Olive, 2011) and division (Kajander & Holm, 2011; Lin, Becker, Byun, Yang, & Huang, 2013). In addition, the studies utilized a variety of methods, including one-on-one interviews (Caglayan & Olive, 2011; Lo & Grant, 2012), questionnaires (Lo & Grant, 2012; Mochon & Escobar, 2011), and pre/posttests (Kajander & Holm, 2011; Lin et al., 2013; Muir & Livy, 2012; Utley & Reeder, 2012; Whitacre & Nickerson, 2011).

Studies found that PTs' fraction conceptions are still largely procedurally based (Caglayan & Olive, 2011; Kajander & Holm, 2011; Lin et al., 2013; Muir & Livy, 2012; Whitacre & Nickerson, 2011). For example, Whitacre and Nickerson (2011) found during pre-interviews that PTs tended to favor standard comparison strategies such as using common denominators when solving comparison problems, even when the numbers were cumbersome to work with, but they became more flexible after completing targeted instruction designed to help them reason about fraction size in different ways. For the seven PTs who were asked to solve nine fraction comparison problems, over 73% of their comparisons involved standard strategies on the pretest, compared to 44.4% on the

posttest. In addition, Caglayan and Olive (2011) found that when representing fraction multiplication with pattern blocks, PTs could solve the problem but struggled with representing multiplication using the blocks in a meaningful way. When solving  $1/2 \times 1/3$ , some PTs drew out  $1/2$  and  $1/3$  separately with a multiplication sign in between, as opposed to drawing  $1/2$  of  $1/3$ . Likewise, Kajander and Holm (2011) gave more than 600 PTs a pre/posttest analyzing their knowledge of solving  $1\ 3/4 \div 1/2$  and their ability to justify their solution. They found that most PTs relied on procedures to solve the problem and explained the procedural process as their justification.

Others note that while PTs' procedural knowledge is often stronger than their conceptual understandings, it is still not always correct (Lin et al., 2013; Muir & Livy, 2012). Muir and Livy (2012) found that PTs had difficulty when converting fractions to decimals. Only 15% of the 279 PTs in their study could convert  $3/7$  into a decimal to four places. Errors included rounding incorrectly and dividing 7 by 3 instead of 3 by 7. In a cross-cultural study that included 96 PTs from both the United States and Taiwan, Lin et al. (2013) found that although PTs from both countries were similarly successful when solving fraction division problems, they equally had difficulties explaining fraction division concepts.

Prospective teachers also tended to focus on the part-whole meaning of fractions (Lo & Grant, 2012; Mochon & Escobar, 2011; Utley & Reeder, 2012). Lo and Grant (2012) found that when PTs were asked a series of questions ranging in difficulty, they struggled more when questions could no longer be answered using the part-whole meaning of fractions. For example, when given the picture below (see Figure 1) and asked to find what fraction was represented by D, with the largest outer square representing one unit, more

PTs used guess-and-check strategies than on other questions, because they had no other recourse.

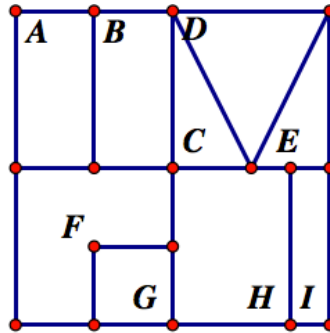


Figure 1. What fraction of the outer square is D? (Lo & Grant, 2012, p. 171)

Lo and Grant (2012) found that questions requiring fraction concepts such as partitioning were conceptually harder for PTs to answer. Predominantly focusing on the part-whole meaning can also affect PTs' ability to understand fractions as quantities (Utley & Reeder, 2012). In a methods course, Utley and Reeder (2012) studied 42 PTs on the topics of fraction benchmarks, sequences, comparison, ordering, and part-whole understanding. They found that PTs struggled with finding the whole, especially when the given fraction was greater than 1. For example, when given a picture of an amount larger than 1 and asked to draw what 1 would look like, only 42.9% of the PTs were able to do this correctly.

**Prospective teachers' specialized content knowledge.** Four studies focused on PTs' common content knowledge but also added a component of analyzing PTs' specialized content knowledge (Ho & Lai, 2012; Lo & Luo, 2012; McAllister & Beaver, 2012; Rosli et al., 2011). The SCK component required PTs to write word problems for fraction operations and draw pictorial representations of fraction situations. All of the studies dealt with

fraction operations with the exception of the work from Rosli et al. (2011) that focused on unitizing.

Underlying the difficulties PTs had with representing operations in a context or in pictorial form was their struggle with understanding the unit (McAllister & Beaver, 2012; Rosli et al., 2011). For example, in a study with three PTs during one-on-one interviews investigating their knowledge of units and unitizing, Rosli et al. (2011) found that PTs had difficulties distinguishing between *how much* and *how many*. When asked how much pizza each person would get when 4 pizzas were shared among 5 people, PTs would answer 4 slices, rather than  $\frac{4}{5}$  of one pizza. In addition, the PTs struggled with using composite units and being flexible in their thinking. McAllister and Beaver (2012) found in a survey with over 100 PTs that an error that caused PTs to struggle to write appropriate word problems stemmed from their incorrect use of units. When asked to write a word problem for  $\frac{2}{3} + \frac{4}{5}$ , one PT posed the question, “Two thirds of the kindergarten class and four fifths of the eighth-grade class mixed together. What fraction of the two classes was mixed?” (McAllister & Beaver, 2012, p. 93). Within this problem, the whole number of students in each of the two classes is unknown; thus, the problem has no answer. In addition, if this problem were solved using  $\frac{2}{3} + \frac{4}{5}$ , the answer would be greater than 1, and it is impossible to talk about more than 100% of a class.

Other studies found that PTs have difficulty understanding fraction operations beyond a procedure (Ho & Lai, 2012; Lo & Luo, 2012). In a study with 92 PTs in Australia, Ho and Lai (2012) found that when given the problem  $\frac{1}{3} \times \frac{3}{4}$ , 82.6% of the PTs could solve the problem correctly, but 67.1% of the PTs who provided “justifications” provided an explanation of just the procedure. However, of the 35 PTs who were able to use a

context to solve a problem, all were able to provide a pictorial representation for the situation. Lo and Luo (2012) also found similar results in that Taiwanese PTs were able to solve fraction division problems but struggled with writing word problems to represent the situation. When PTs were asked to illustrate a fraction division situation, 40% and 35% of the models they generated were area and linear, respectively. Only 5% of the pictures represented division with a set model.

**Improving prospective teachers' fraction knowledge.** Two studies gave examples of ways to improve upon PTs' fraction content knowledge. Harvey (2012) suggested using manipulatives as a way to help improve PTs' common content knowledge specifically in the areas of equivalence and comparison. During one-on-one or pair instruction with 13 PTs, Harvey found that an elastic strip that was subdivided into 10, 20, or 25 parts was helpful in developing their understanding of fractions and comparison strategies. For example, three of the PTs who were unable to use a benchmark strategy during a pre-questionnaire were able to do so after instruction using the elastic strip. One PT was able to compare  $\frac{8}{17}$  and  $\frac{10}{17}$  by using a benchmark of  $\frac{1}{2}$  to determine that  $\frac{8}{17}$  is less than a half and  $\frac{10}{17}$  is greater than a half. Other PTs used similar methods in determining when fractions were greater than or less than a half. The researchers noted that elastic strips can be useful tools in aiding PTs to keep track of the size of a unit as well as develop their image of number lines.

Whitacre and Nickerson (2011) also found improvements in PTs' number sense and ability to compare fractions after targeted instruction. They designed a sequence of tasks in such a way as to build on and extend PTs' procedural understandings to help them develop a list of fraction comparison strategies, along with agreed-upon names and examples, on



which they could draw to solve problems. After completing the tasks, the students in the study improved both in their abilities to correctly solve fraction comparison problems, and in the flexibility of their comparison strategies, getting an average of almost two more questions correct out of nine, and using an average of 2.71 more valid-correct strategies in order to solve the problems.

**Prospective teachers' fraction development.** Recent reports have also begun to document the ways in which PTs develop an understanding of fractions in a whole classroom setting (Tobias, 2013). Tobias describes how language can confound PTs' understanding of wholes for fractions both less than and greater than 1. For example, when asked to share 4 pizzas equally among 5 people, PTs had difficulties naming the solution of  $\frac{4}{5}$  in terms of the whole. Some correctly determined the answer to be  $\frac{4}{5}$  of one pizza, whereas others defined  $\frac{4}{5}$  to be out of the 5 pizzas, so the question of  $\frac{4}{5}$  "of what" became important. When developing an understanding of topics, such as fraction language, Tobias notes that PTs' fraction understanding does not develop linearly in that knowledge of one topic may not be fully developed before they start to learn another. For example, PTs started developing the idea that solutions depend on a whole before they developed an understanding of defining an "of what" for fractions, even though the latter idea was introduced to the class first. Likewise, the idea of developing language in terms of what the denominator represents was introduced before the class developed a full understanding of the previous two ideas. Thus, classroom instruction may need to focus on multiple fraction concepts before PTs can fully develop an understanding of one idea.

## **Conclusions**

We began this summary with the intention of determining what we know from research about PTs' knowledge of fractions in the domains of common and specialized content knowledge, and knowledge of content and students. In general, the research we examined indicated that PTs' common content knowledge is relatively strong when it comes to performing procedures, but that they generally lack flexibility in moving away from procedures and using "fraction number sense" (e.g., Newton, 2008; Yang et al., 2008). They also have trouble understanding the meanings behind the procedures or why procedures work (e.g., Borko et al., 1992). PTs seem to favor the part-whole interpretation of fractions, but have trouble with other fraction interpretations such as the operator model (e.g., Behr et al., 1997) and number line models (e.g., Domoney, 2002; Luo et al., 2011).

While prospective teachers' CCK is often adequate to good, many of them have trouble in the areas requiring specialized content knowledge. PTs struggled with representing fractional situations using diagrams and in word problems. Difficulties arose for a number of reasons, including PTs' preference for particular models of multiplication (Luo, 2009) and division (Ball, 1990a, 1990b), which did not lend themselves as easily to working with fractions. PTs also had trouble identifying the unit when trying to represent fraction models (Newton, 2008; Rosli et al., 2011), and with language around fraction ideas, confusing the number of pieces with the fractional part of the whole when these were different things (Rosli et al., 2011; Tobias, 2013).

While knowledge of content and students was not the focus of much of the research on PTs' knowledge of fractions, the studies that were conducted showed that PTs were able

to predict some errors that students might make when dealing with fractions; however, they generally attributed these errors to mistakes in following procedures, rather than conceptual errors (Tirosh, 2000). This aligns with findings that PTs' own knowledge is based mostly on following procedures. PTs also had difficulties interpreting non-standard algorithms (Li & Kulm, 2008; Son & Crespo, 2009). This indicates that they may have trouble interpreting their students' solutions to problems.

While the majority of the studies discuss problems that prospective teachers have in working with fractions, few studies have discussed ways to improve PTs' fraction knowledge. Some have suggested special courses (Newton, 2008; Whitacre & Nickerson, 2012) and targeted work with manipulatives, which has seemed to help (Green et al., 2008; Harvey, 2012), but, overall, we do not have enough information on this issue, and we suggest that future research look more at ways to improve PTs' fraction understandings.

In looking at trends in the research on PTs' fraction knowledge, we note that past research has focused primarily on their understanding of fraction operations, predominantly multiplication and division. This is currently starting to shift to include concepts, such as examining PTs' fraction number sense. A trend in all three time frames is that PTs' common content knowledge and/or specialized content knowledge is the focus of the majority of studies. Few have analyzed how to improve PTs' understanding with fractions. Thus, this is still a gap in the research that needs to be filled. In addition, most past research has incorporated quantitative methods that include pre/posttests and/or qualitative methods that include one-on-one interviews. Though this trend is still continuing, there is also research to suggest that future studies will address how PTs learn as they participate in whole-class settings or groups.

Research indicates the need for mathematics courses for PTs to include additional topics such as analyzing student thinking, focusing on standard and non-standard algorithms for solving problems, highlighting concepts that may impact operations such as the role of units, and addressing multiple concepts for PTs to develop one idea. Based on the gaps in the research literature, we suggest future research include more studies on the use of manipulatives with PTs, the role of language with fractions, an understanding of why PTs may have more difficulty with number lines or a linear model over area, and more studies focusing on international comparisons across cultures. By taking into account what we know about prospective teachers' fraction understanding, we can continue to improve our content and methods courses. By also understanding the gaps that still exist, we can design research studies to address these needs. Together these can be used to help us as mathematics educators improve in developing PTs' understanding of the mathematics they are to teach.

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Decimals**

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**ABSTRACT:** In the last 25 years a small collection of reports of studies focused on gaining insight into PTs' knowledge of decimals has been published. Three themes are used to frame findings from papers published prior to 1998. Additional findings from papers published between 1998 and 2011 are discussed. Direction for future research that can contribute to the development of curriculum and instruction in mathematics teacher education is shared.

**Keywords:** decimals, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

## **Introduction**

The historical evolution of decimals as a representation of quantity rests largely on the development of place value and the use of zero in the numeration system. Far more difficult than using the notational system is understanding the quantities represented with the system (Irwin, 2001) in context. Of particular difficulty are decimal fractions (decimals), rational numbers “which originate by subdivision of each unit interval into 10, then 100, 1000, etc., equal segments” (Courant & Robbins, 1996, p. 61). Research on children’s conceptions of decimals illustrates a series of conceptual hurdles involved in interpreting and using the notational system (Resnick et al., 1989; Sackur-Grisvald & Leonard, 1985). Because children build their understandings of decimals from their existing or coemergent understandings of multidigit whole numbers and fractions, they tend to over-apply concepts for these more familiar objects when the numerals being discussed are decimals. Findings from studies of children’s understandings encouraged researchers to begin to explore prospective teachers’ (PTs’) understandings of decimal notations (Putt, 1995; Thipkong & Davis, 1991). Such studies unearthed parallels between categories of reasoning used by children and reasoning used by PTs, encouraging researchers to identify teachers’ misconceptions as a source of children’s faulty reasoning.

Research on PTs’ knowledge of decimal fractions has focused on exploring how decimals are interpreted and used in computation, and how mathematics educators might challenge existing beliefs about the use of decimal fractions. In this report, we focus primarily on terminating decimals that are included in primary school curriculum. A very small collection of reports focused on PTs’ knowledge of decimals has been published over

the last 25 years, but findings point to the importance of place value in PTs' understanding and application of decimals.

### **Approaches and Orientations**

In the sections that follow, we have summarized historical influences in the study of PTs' knowledge of decimals, findings of published peer-reviewed papers from 1998 to 2011, and additional insights drawn from more recent work. Our approach to identification of articles was consistent with the method described in the introductory article of this Special Issue. In addition, our perspective on decimal understanding influenced our interpretations of the articles. We share this perspective to enable readers to gain insight into our interpretations.

Our view of decimal is informed by explorations of PTs' understandings (D'Ambrosio & Kastberg, 2012; Kastberg & D'Ambrosio, 2011) of decimals using a framework including units, relationships between units, and additivity. As Courant and Robbins (1996) suggest, decimal units in the place value system involve repeatedly "subdividing" an individual unit into 10 parts. So if we begin with 1, then subdividing this unit into 10 parts produces 10 subunits 0.1. This action creates the opportunity for the development of relationships between 1 and 0.1, namely, that 1 is 10 times 0.1 and 0.1 is one tenth of 1. While this example involves adjacent units in the set of place value units { ..., 10, 1, 0.1, 0.01, ... }, any two units in the set can be thought of as related multiplicatively. Finally, sums of multiples of the units can be used to create new decimals, an idea that is represented in expanded notation. For example, if we compare 0.606 and 0.66 using the additive structure, we can see that  $0.606 = 0.6 + 0.006$  and  $0.66 = 0.6 + 0.06$ . This understanding and understanding of multiples of the units 0.001 and 0.01 allow us to

quickly determine that 0.606 is less than 0.66. Understanding decimals as linear combinations of place value units allows us to compose and decompose decimals to quickly compare them. While there are certainly other views of decimals, it was this view that we held and used to make sense of the findings reported in the research.

The limited number of existing studies encouraged us to create a “conceptual review” (Kennedy, 2007, p. 139) of the historical research rather than a systematic review. Such a review focuses on “gaining new insights into an issue” (p. 139) rather than providing an answer to a specific research question. Our approach was iterative in that we each read a portion of the papers and developed central ideas that we drew from the papers. We then shared the ideas we drew from our readings, read the balance of the papers, and again met to revisit initial perspectives on the papers. A final set of three themes emerged and were refined as we developed our perspective on the papers over time. Because there were only a few papers published since 1998, they were treated more as individual cases informed by drawing from the existing literature and extending the insights researchers had historically provided.

### **A Historical Look: Decimal Fraction Prior to 1998**

In this overview, we discuss the themes we found in research exploring PTs’ difficulties with decimals: PTs’ interpretations of decimals, PTs’ use of concepts and associated beliefs, and changing PTs’ concept through cognitive conflict.

#### **Interpretation of Decimals**

Prior to 1998, nine research reports were published whose focus was PTs’ difficulties with decimals (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Khoury & Zazkis, 1994; Putt, 1995; Thipkong & Davis, 1991; Tirosh & Graeber, 1989, 1990a, 1990b;

Zazkis & Khoury, 1993). The reports document difficulties PTs have with decimal tasks, including comparing and ordering decimals as well as representing decimals (Thipkong & Davis, 1991), and suggest that the origin of such difficulties stems from the ways in which PTs interpret decimal notation. Authors identified PTs' misconceptions and hypothesized about the origins of these misconceptions. For example, Putt (1995) asked PTs to order a collection of decimals between zero and one (0.606, 0.0666, 0.6, 0.66 and 0.060). In studies of children's approaches to decimals, ordering decimals had been shown to be more cognitively demanding than simply comparing two decimals (Sackur-Grisvald & Leonard, 1985). The PTs in Putt's study also found ordering the collection difficult, but Putt noted that the errors suggested a varied collection of reasons for the PTs' difficulty. Among these reasons were the longer-is-larger and shorter-is-larger misconceptions originally identified in research on children's approaches to decimals (Resnick et al., 1989; Sackur-Grisvald & Leonard, 1985). Learners who use the longer-is-larger misconception apply whole number reasoning to decimals and would identify 0.125 as greater than 0.25 since 0.125 is longer. The shorter-is-larger misconception stems from the application of an early understanding of place value. Positions to the right of the radix point decrease in value, so learners who identify 0.1 as greater than 0.12 do so since tenths are greater than hundredths. In addition, Putt's interviews with participants revealed that that some PTs interpreted decimals as negative numbers, so, when asked to compare a decimal and zero, these PTs' chose zero as larger than a decimal.

The origin of PTs' difficulties seems to be a lack of understanding of place value units and the relationships between units. Strategies used by PTs illustrate that they made comparisons using procedures learned in elementary school, such as appending zeros and

treating the quantities like whole numbers, or converting each decimal to a fraction and finding a common denominator that allowed the numerators to be compared as whole numbers. Putt (1995) suggested that some PTs had difficulty understanding that 0.7 and 0.70 are equivalent. In particular, they seemed to struggle to interpret decimals as composites of multiples of units. These difficulties shed light on PTs' interpretation of decimals.

Khoury and Zazkis (1994; Zazkis & Khoury, 1993) proposed that explorations of PTs' concepts rather than their ability to apply rules could be conducted using quantities in bases other than ten, for example, converting  $12.34_{\text{five}}$  to base ten. They reasoned that such tasks would encourage the participants to use a general place-value structure rather than rules or procedures. Zazkis and Khoury found that PTs related the fractional part of a number to the base in the number in non-standard ways. For example, in  $12.34_{\text{five}}$ , some PTs suggested that the 3 was in the 0.5 position and the 4 was in the 0.05 position, reasoning that is aligned with the consistent use of 1 in decimal notation for tenths (0.1) and hundredths (0.01). Other PTs ignored the fractional part of the number, noting that decimals exist only in base ten (Zazkis & Khoury, 1993). The digits after the decimal were unchanged, while the integer part of the number was converted using a conventional strategy.

Khoury and Zazkis (1994) investigated PTs' "concepts of invariance of fractional number under different symbolic representation" (p. 203). This work explored the students' ability to reason in situations where the quantities were different, but the representations were similar ("Is  $(0.2)_{\text{three}} = (0.2)_{\text{five}}$ ?" [p. 193]) and when the quantities were the same and the representations were similar ("Is the number 'one-half' in base

three equal to the number ‘one-half’ in base five?” [p. 193]). Sixty-three of the 100 elementary PTs correctly answered the first problem using place-value charts and computations such as “ $(0.2)_{\text{three}} = 2 \times 1/3$ ” to generate fractions in base ten they could compare (p. 194). While these students provided correct answers, their reasoning during interviews often revealed attention to place-value syntax rather than quantity value. Some students overgeneralized reasoning derived from their experience with base-ten place-value units to reason about values of the positions to the right of the radix point. The values were identified as  $1/5$ ,  $1/50$ ,  $1/500$  (p. 195), a finding consistent with reasoning the authors identified in their prior work (Zazkis & Khoury, 1993). Investigating one half in different bases (“Is the number ‘one-half’ in base three equal to the number ‘one-half’ in base five?” [p. 197]) was far more difficult for the students, with only 26% of elementary PTs concluding the two representations for the second task referred to the same quantity. Drawing from the computational strategies used by the students, the authors concluded that PTs’ “knowledge of place value and rational numbers is more syntactical than conceptual” (p. 203).

Thipkong and Davis (1991) used line and area models to assess PTs’ understanding of decimals. PTs were asked to place given decimals on a number line and represent decimals with an area model, in the form of a square as a unit. They were also asked to reverse this reasoning and identify decimals from positions on number lines and identify the decimal being represented using a square as a unit. PTs had the greatest difficulty when units on the number line were subdivided into subunits other than 10. For example, when asked to mark 1.4 on a number line with subunits of 8, 42% of PTs represented 0.4 as 4 of the 8 subunits in a unit. PTs performed better when they were asked to represent 1.4 using



an area model with 8 subunits in the unit. Only 17% counted 4 subunits as 0.4. More than 80% of PTs were successful in representing more familiar decimals, such as 0.5. These findings suggest that PTs can reason about and represent familiar decimals using models with subunits other than 10, but may struggle with less familiar decimals. The authors suggest that models may be useful in supporting PTs to build “relationships of the parts of the unit to the unit” (p. 98).

Findings from this collection of studies suggest that while some PTs master computational strategies that allow them to compare decimals, convert between bases, and represent familiar decimals, others have difficulty. Sources of this difficulty seem to be in building meaning for and interpreting decimal notation. Notations are designed to represent different linear combinations of quantities using a system of units, yet explorations of place-value systems in bases other than ten reveal that PTs attend to the patterns in the base-ten notation rather than the quantities they are meant to represent. For such students, since 0.1 and 0.01 are units in the base-ten system, 0.5 and 0.05 are incorrectly assumed to be units in the base-five system. Research contains evidence that use of the number line and area models to represent decimals were effective tools in revealing overgeneralizations based on subunits of 10 used in non-base-ten systems. In addition, these findings point to the importance of exploring units and relationships in contexts where base ten is not used, such as time.

### **Use of Concept and Associated Beliefs**

Perhaps the most complete investigation of PTs’ understandings of decimals and their use is the collection of investigations conducted by Graeber and Tirosh (1988; Graeber et al., 1989; Tirosh & Graeber, 1989, 1990a, 1990b). Following the work of

Fishbein et al. (1985), the authors conjectured that PTs might hold misconceptions regarding multiplication and division that are held by children. Fishbein and his colleagues identified a collection of generalizations, primitive models, children used to predict the results of multiplication and division, such as multiplication makes larger or division makes smaller. The work of Graeber and Tirosh exploring PTs' conceptions of multiplication and division is relevant because it includes a discussion of how PTs' primitive models of multiplication influence their performance writing expressions for word problems. In particular, the authors found that "nonintegral operators, especially operators less than 1, proved troublesome to preservice teachers" (Graeber & Tirosh, 1988, p. 264), a finding confirmed later by Thipkong and Davis (1991). Performance on word problems, meant to be modeled with multiplication or division strategies, was impacted by the presence of decimals. For example, 41% of the 129 students studied incorrectly modeled the following problem.

One kilogram of detergent is used in making 15 kilograms of soap. How much soap can be made from .75 kilograms of detergent? (Graeber & Tirosh, 1988, p. 264)

The most common incorrect expression, given by 17% of the 129 PTs, was  $15 \div .75$ . The authors concluded that the source of the students' difficulty was not the "presence of the decimal," but rather "the role (operator) the decimal plays in these word problems" (p. 265). In particular, when the decimal in the word problem conformed to the primitive model, the PTs performed very well.

The authors found that the PTs were perfectly capable of performing operations with decimals and were generally able to identify statements such as "In a multiplication problem, the product is greater than either factor" (p. 270) as false. Despite the knowledge held in writing expressions from word problems or writing word problems for particular

expressions, the PTs enacted their implicit beliefs, including that “multiplication makes larger” and “division makes smaller.” This finding has implications for teacher educators as they work with PTs. Exploring generalizations such as “In a multiplication problem, the product is greater than either factor” (p. 270) with subsets of the real numbers may encourage PTs to revise whole-number reasoning to build ideas about operations with decimals. While PTs identify this statement as false, their reasoning from interview data reveals that counterexamples they use to support their reasoning are drawn from their experiences with whole numbers and algorithms. PTs used procedures to reason that you cannot “divide by a decimal” (p. 273) because you change it to a whole number by moving the decimal point before dividing. The authors suggest that the use of the algorithm “may support their misbeliefs about the relative size of the quotient and the dividend” (p. 274) in a division problem.

Tirosh and Graeber (1989) noted that one source of difficulty with division involved the primacy of the partitive model. Decimal quantities as divisors violate the primitive model that dictates that, in division, one is partitioning a whole rather than finding the number of units of a given size in a given whole. So, while understanding the decimals was admittedly difficult for the PTs (Graeber et al., 1989), the primitive models of operations and the sets of numbers that were allowed to perform various roles in a computation were central to the difficulties in performance when decimals were involved.

### **Changing the Concept Through Cognitive Conflict**

Tirosh and Graeber (Graeber & Tirosh, 1988; Tirosh & Graeber, 1990a) recommended various instructional techniques and activities meant to help PTs connect their explicit reasoning and implicit beliefs. The method highlighted as having the most

promise involved the use of “conflict teaching” (Bell, 1983, cited in Tirosh & Graeber, 1990a). The authors used the technique to encourage participants’ conscious consideration of the statement “In a division problem, the quotient must be less than the dividend” (Graeber & Tirosh, 1988, p. 275) in light of computational evidence to the contrary. All but 1 of the 21 participants interviewed for this study “realized that a conflict existed between their belief about the relative size of the dividend and the quotient and their computation with decimals” (pp. 275–276). The realization impacted the participants’ performance providing correct expressions for word problems involving decimals. In particular, the authors share, “When the conflict approach is carefully applied, pre-service teachers may form a more accurate conception about the relative size of the quotient and the dividend and improve their performance in writing expressions for multiplication and division word problems” (Tirosh & Graeber, 1990a, p. 107).

The one-to-one interviews the authors used to change PTs’ beliefs about the impact of decimals on the product or quotient were viewed as inefficient. Instead, Tirosh and Graeber (1990a) suggested that modifications or whole-class activities based on building connections between algorithms, beliefs about operations and subsets involved in operations, and word problems could draw on the strengths of the conflict approach.

### **A Current Perspective: Decimal Fraction from 1998 to 2011**

Between 1998 and 2011 there were three reports of studies whose focus was the development of PTs’ knowledge of decimals (Stacey et al., 2001; Widjaja, Stacey, & Steinle, 2008, 2011). Widjaja and her colleagues (2008, 2011) explored the density of rational numbers and the representation of negative decimals to gain insights into misconceptions about decimals that might be hidden in more familiar contexts. The work of Stacey et al.

(2001) is the only study that explores the relationship between PTs' performance on decimal comparison tasks and their identification of decimal comparison tasks that would be difficult for children.

Noting that the whole numbers do not have the density property, Widjaja et al. (2008) justified their exploration PTs' notions of the density property of the rational numbers as a mechanism to unearth misconceptions about decimals. The density property is described by the authors as "the property that between any two decimals, there are infinitely many other decimals" (p. 118). Based on a pre- and posttest, the authors described four incorrect strategies used to identify decimals between two given decimal quantities. Using whole-number reasoning, some students noted that there were no decimals between given decimals, for example, 3.14 and 3.15. Reasoning with only one additional decimal place, other PTs suggested there were a finite number of possibilities between decimals such as 3.14 and 3.15. These students developed lists of possibilities, such as 3.141, 3.142, ..., 3.149 (p. 125). Another subset of PTs relied on the "rounding rule" (Stacey, 2005; Steinle & Stacey, 2004), viewing decimals such as 0.799 and 0.80 as the same—"0.80 is the result of rounding 0.799" (p. 125)—so there are no decimals in between them. Some PTs subtracted the two given decimals to find the number of decimals in between. Widjaja et al. (2008) attribute PTs' challenges with the density of decimals to the lack of opportunity to work with decimals that are not the same length. In addition, PTs must understand that the discreteness of whole numbers does not apply to decimals.

Widjaja et al. (2011) returned to interpretations of decimals, this time focusing on PTs' placement of negative decimals on a number line. The significance of this work lies in the power it has to provide insight into PTs' understanding of decimals as a number

system. Widjaja et al. identified two misconceptions that involved “interpreting the negative part of the number line” (p. 86) that explain incorrect responses given by PTs who were asked to identify the position of decimals, including numbers such as  $-0.35$  and  $-1.65$  on a number line with  $-1$ ,  $0$ , and  $1$  marked (subunits of  $10$ ). To explain the PTs’ misconceptions, Widjaja et al. described two rays students used: the “positive number ray,” beginning at zero and continuing to the right, and a “negative number ray,” also beginning at zero (some students identified zero as negative zero, and continued to the right with negative integers identified [p. 86]). The first misconception involved students using the “separate negative number ray misconception” (p. 86). Students with this misconception overlap the positive and negative number rays so that the positive number ray is laid on top of the negative number ray. This action creates values  $-0$ ,  $-1$ , and  $-2$ . The value  $0$  on the positive number ray is coincident with  $-2$  on the negative number ray. When these rays, both extending to the right, are overlapped, the number line begins with negative zero and the sequence of integers  $-1$ ,  $0$ ,  $1$ ,  $2$  are marked. Using this approach,  $-0.5$  would be viewed as less than  $-1.2$ , since  $-0.5$  is between  $-0$  and  $-1$  and  $-1.2$  is between  $-1$  and  $0$ . The second misconception involved students “translating positive intervals” to positions “between integers” in the negative region (p. 88). Students with the “translating positive intervals misconception” (p. 88) can correctly locate positive and negative integers on a number line. It is negative decimals that give them difficulty. Students translating positive intervals “know that  $1.2$  is to the left of  $1.3$  and assume that the same relationship holds for negative numbers so that  $-1.2$  is to the left of  $-1.3$ ” (p. 88). For these students,  $-1.2$  is interpreted as positioned at  $-0.8$ . The authors identified variants of this thinking that result from placing the translated positive intervals in different positions on the number line. This exploration

of the landscape of the number line adds a dimension to the work of Thipkong and Davis (1991), suggesting not only that some students find decimals between zero and 1 difficult to represent, but also that understanding number lines can be a challenge. PTs' approaches illustrate the importance of models in understanding decimals, but also suggest that meanings for models, such as number lines, must be developed with PTs rather than assumed to exist in the minds of PTs.

Only one study examined PTs' knowledge of decimals and application of this knowledge to tasks common in teaching (Stacey et al., 2001). PTs' knowledge of decimal concepts and an awareness of common misconceptions are essential components of mathematical knowledge for teaching. Stacey et al. (2001) asked 553 PTs and 25 practicing teachers to complete a collection of decimal comparisons. The Decimal Comparison Task (DCT), created by Stacey and colleagues, contains groups of decimal comparisons designed to identify known misconceptions (Moloney & Stacey, 1997; Stacey & Steinle, 1998). Participants in the Stacey et al. (2001) study were asked to look at pairs of decimals and indicate which one was "larger." PTs were also asked to identify comparison items that were likely to be difficult for children and provide possible rationales for the cause of children's difficulties. The researchers sought to broaden their understanding of PTs' ideas about the difficulty of the decimals for children and the ability of PTs to recognize gaps in their own understanding.

PTs demonstrated a moderate awareness of their own difficulties with decimals, and this awareness made them more sensitive to possible difficulties for children. Correlations existed between the errors made by PTs and difficulties they identified for children. Fifty-seven percent of PTs' errors on the comparison tasks correlated with

possible difficulties for children identified by PTs. Four common misconceptions were identified by PTs: “length, comparison with zero, presence of a zero digit, and similarity” (p. 217). PTs were aware that children might think that longer decimals have a larger value than shorter ones. In addition, they seemed to know that comparing a decimal to zero or to a decimal containing a zero between non-zero digits can be challenging. PTs further were aware that decimals that had a collection of digits in common were harder to compare (e.g., 8.245 and 8.24563 [p. 216]). PTs showed very little awareness of the shorter is larger misconception.

PTs identified two aspects of lengths of decimals that could make decimal comparison tasks difficult for children. These aspects were long decimals and decimals of unequal length (p. 218), difficulties likely connected to the longer is larger misconception. Regarding comparisons with zero, PTs commented that children may apply whole number thinking and conclude that zero is larger or may think that decimals are negative numbers. The presence of zero in the tenths place could make decimal comparison more difficult for students, but PTs did not elaborate on this reasoning. PTs provided no discussion of zero making the value smaller, but their perception of children’s difficulty with the presence of a zero digit is mostly likely connected to the longer-is-larger misconception. Similar decimal numbers were identified by PTs as a potential source of difficulty for children because they may not know or recognize the effect of the additional digits in the third and fourth decimal place. Within these four categories, PTs expressed an awareness of the longer-is-larger misconception, but there were fewer comments about the shorter-is-larger misconception, a misconception more common in older students.



### **The Horizon, Future Directions, and Considerations**

Research at the horizon is much more difficult to find. No reports of studies focused on PTs' understanding of decimals were published in recent proceedings reviewed by the research team. Rather than speculate regarding why such evidence was missing, in this section we focus on future directions for researchers exploring PTs' knowledge of decimals and considerations they should attend to as they plan research agendas.

As mathematics educators move forward in their exploration of PTs' understanding and use of decimals, three questions shape suggestions for future research. How do PTs develop decimal concepts? How can mathematics educators support the development of PTs' decimal concepts? How do PTs use their concept of decimals in activities that approximate the work of teaching?

Current findings illustrate how PTs interpret, represent, and use decimals. What is less clear is how these concepts develop. Of significant importance to mathematics educators planning and developing mathematics courses for PTs has been the identification of difficulties with decimals, yet also of importance is an understanding of how these concepts might develop. Studies of children's development are of use in building ideas about adult development (McClain, 2003), yet adults' prior experiences with decimals and their facility with them can allow them to share correct answers that reveal little about underlying concepts. As the reports discussed have shown, it is difficult to develop tasks and activities that take seriously the existing constraints of PTs' productive computational and procedural approaches to tasks. Adult tasks should be built (a) with PTs' existing ways of operating in mind, and (b) to create cognitive conflict. Such tasks will stand in contrast to situations in which PTs are told by teachers and researchers that they may not use their

most productive procedural approaches (e.g., converting decimals to fractions with a common denominator to compare them) on given tasks. Mathematics educators must challenge themselves to explore PTs' development of decimal understanding that does not rely on PTs' compliance with authority. Tasks that cannot be solved effectively, efficiently, or correctly with procedures will challenge PTs to develop new understandings of decimals.

Efforts to challenge PTs' existing understandings have proven to be labor-intensive, as suggested in the work of Tirosh and Graeber (1990a). Thus far, studies have included collections of paper-and-pencil tasks given to a group of PTs and interviews of a subset of the participants to explore reasoning underlying approaches identified. While these studies have provided tasks that can serve as the basis for instructional materials, developmental research (Gravemeijer, 1994) with the goal of creating instructional materials and pedagogical approaches is needed. Mathematics content courses for PTs must provide opportunities to build decimal concepts, while honoring and identifying the power of efficient approaches PTs use, such as appending zeros to compare decimals.

Only one research study included explorations of PTs' use of decimals in approximations of practice (Grossman, 2011). Approximations of practice allow PTs to create insights through activities they will have to perform as practicing teachers. Stacey et al. (2001) found that although PTs correctly identified the longer-is-larger misconception as a challenge for children, they were less likely to identify the shorter-is-larger misconception. These and other findings from Stacey et al. stand alone. Research exploring how PTs make sense of decimal tasks and, in turn, what they identify as

challenging for learners, provides insight into the evolution of their mathematics, including their understandings of children's mathematics (Steffe, 1994).

As the horizon of PTs' decimal understanding is constructed by research findings, we also suggest two additional considerations that should be attended to by researchers. First, we question what concepts should be studied. Programs of study in mathematics teacher education are necessarily limited in scope. Time and content limits are significant considerations mathematics teacher educators attend to as they plan opportunities for PTs to learn. In order to serve mathematics teacher educators charged with supporting PTs, it is critical that researchers move beyond asking what can be studied to what should be studied to serve the important goals of teacher education. For example, Wajaja and her colleagues' (2008, 2011) work exploring PTs' representations of negative decimals initially may seem less significant to mathematics teacher educators working with PTs. Yet, this work not only builds from the existing literature, but also contributes new insights regarding difficulties with positive decimal quantities. Widjaja, Stacey, and Steinle (2011) illustrate that understanding the "twisted geography" (p. 80) of the number line for negative decimals unearthed further evidence that "decimals with a zero integer part (e.g. 0.35) are conceptually harder than decimals with a non-zero integer part, and so would benefit from special attention in teaching" (p. 90). Many concepts could be explored by researchers. Deciding which concepts demand the most attention from the research community should involve discussions with mathematics teacher education colleagues and the identification of the grand challenges in conceptual development they see as central to their teaching of PTs.

Second, we encourage researchers who study concept development to provide insights regarding how such concepts might efficiently be developed with PTs. Tirosh and Graeber (1990a), in their studies of cognitive conflict as a method of developing understandings of decimals, recognized the inefficiency of their individual interviews as a method for teaching PTs and suggested alternatives. Activity sequences designed as part of research that takes many instructional hours to implement, but only supports the development of one concept, must be critically examined. Researchers should attend to the institutional constraints mathematics teacher educators face as they work with PTs and should provide insights into how research findings could be translated into practices that efficiently and productively support concept development. One such example is the examination of non-terminating repeating decimals (Burroughs & Yopp, 2010; Weller, Arnon, & Dubinsky, 2009, 2011). While our review focused on terminating decimals, these studies provide insights that allow mathematics educators to speculate about reasoning PTs might use to build from the set of points everywhere dense to a “correspondence between all the points on the number axis and all the finite and infinite decimal fractions” (Courant & Robbins, 1996, p. 63). The challenge for mathematics teacher educators may be whether the collection of ideas associated with such reasoning is critical for PTs and to their future work with children. This example illustrates the importance of creating not only new knowledge about PTs’ development and use of concepts, but the significance of such findings for mathematics teacher education.

In this review spanning 25 years, we found a very small collection of reports focusing on PTs’ decimal content knowledge. Yet mathematics teacher educators planning instruction for PTs need research results to inform their practices. Of critical importance

are studies that (a) result in understandings of how PTs develop decimal concepts, including understanding notations and representations of decimal quantities; (b) generate instructional activities and methods that mathematics teacher educators can implement with PTs to create opportunities to understand decimal concepts and representation; and (c) explore and describe how PTs use decimal concepts in tasks that approximate the work of teaching. Such studies support the development of curriculum and instructional methods that expand opportunities for PTs to learn content critical to their future work as teachers.

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Geometry and Measurement**

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**ABSTRACT:** This paper summarizes the extant peer-reviewed research on PTs' understanding of geometry and measurement, focusing on a wide variety of topics within these content domains. When looking across the 26 studies reviewed, findings span a variety of content topics, providing little depth in either the geometry or measurement content domain. However, collective findings do indicate PTs' overall conceptions in geometry and measurement to be limited and weak, with PTs relying on memorized procedural processes. Some evidence indicates that cognitive development, along with spatial visualization skills, plays a greater role in learning geometry than memory skills. In addition, the van Hiele levels of geometric learning provide a helpful framework to think about the development of geometric ideas. Direction of future research is elaborated to address ways to develop PTs' understanding of geometry and measurement. Gaps that still exist in the research literature regarding PTs' mathematical content knowledge in geometry and measurement are identified.

**Keywords:** geometry, measurement mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education



## **Our Beginnings and Theoretical Perspective**

The mathematical content knowledge required for teaching elementary mathematics is not insignificant. Elementary teachers are responsible for laying a mathematical foundation for their students on which they can build their current and future understanding of mathematical content. The quality of this foundation relies to a great extent on the quality of the teachers' own mathematical knowledge. "However, the nature of the knowledge required for successful teaching of mathematics is poorly specified, and the evidence concerning the mathematical knowledge that is needed to improve instructional quality is surprisingly sparse" (Kirby, 2005, p. 2).

Recently, there has been an emphasis in the mathematics education community to describe the needed and desired mathematical content knowledge for teaching, with various descriptions emerging from research (e.g., Hill, Rowan, & Ball, 2005; Ma, 1999; National Research Council, 2001; Shulman, 1986). Hill, Ball, and Shilling (2008) provide a framework for distinguishing the different types of knowledge included in a construct of mathematical knowledge for teaching. This framework distinguishes between subject matter knowledge and pedagogical content knowledge, building on the work of Shulman (1986). This framework serves as a theoretical lens for our summary work.

As mathematics teacher educators, our interest is in examining and summarizing peer-reviewed research related to the understanding of subject matter (mathematical content) knowledge described in the Hill et al. (2008) framework. Further, we are interested in research about elementary prospective teachers (PTs), as the development of the mathematical content knowledge for teaching is initiated in teacher preparation. As elementary teachers lay a learning foundation for mathematics with elementary students,

mathematics teacher educators should lay a similar learning foundation for mathematical content knowledge for teaching with PTs.

### **Children's Understanding of Geometry and Measurement**

The focus of this paper is on mathematical content knowledge for teaching elementary mathematics with particular attention to geometry and measurement. Before discussing what we know about prospective elementary teachers' knowledge of geometry and measurement, we briefly articulate research on children's understanding of these topics.

Children's experiences with geometry start even before school. Geometric thinking levels proposed by van Hiele (1999) indicate that elementary school students' geometric thinking starts from recognizing shapes based on their appearance and proceeds to identifying properties of shapes. Clements and Battista (1992) emphasized school geometry's role as mathematizing objects, relationships, and transformations, in addition to developing skills to construct visual representations via spatial reasoning. Furthermore, van Hiele geometric thinking theory emphasizes the importance of *experience* in geometry learning. Simply growing older does not ensure a growth in geometric understanding; children need to experience and engage in many various activities that allow them to explore and construct geometric ideas (Battista, 2007).

Stephan and Clements (2003) addressed children's understanding of measurement. The authors defined measurement as "assigning a number to continuous quantities" (p. 301) and stressed that as children keep learning about numbers and counting, they get more into measurement. In a sense, measurement is an amalgam of understanding of numbers and geometry. Stephan and Clements presented six categories that emerged from

research on learning linear measurement: partitioning, unit iteration, transitivity, conservation, accumulation of distance, and relations between number and measurement. The authors highlighted particular difficulties children had in partitioning and unit iteration for area measurement and angle measurement, along with challenges in structuring an array and in conservation of area measurement. The authors found children's difficulties in linear measurement transfer into learning area measurement. In the case of learning angle measurement, the authors stated children's difficulty of defining the attribute (angle) adds onto partitioning and unit iteration difficulties.

Thus, in the elementary school years, the type and number of experiences in which schoolteachers engage children to reason about and make sense of geometry and measurement will greatly affect their future learning experiences in higher grades (National Council of Teachers of Mathematics [NCTM], 2006). Since the experiences of children are key elements of learning geometry and measurement, the knowledge of teachers who shape those experiences is very important. However, "teachers are expected to teach geometry when they are likely to have done little geometry themselves since they were in secondary school, and possibly little even then" (Jones, 2000, p. 110). Baturu and Nason (1996) corroborate this concern that PTs who were lacking in knowledge in measurement might transfer it to their students, noting that "The impoverished nature of the students' [PTs'] area measurement subject matter knowledge would extremely limit their ability to help their learners develop integrated and meaningful understandings of mathematical concepts and processes" (p. 263).

As the research suggests that PTs' content knowledge is critical in the development of children's understanding of geometry and measurement, knowledge of what PTs

understand themselves about these areas should be of importance to those who are involved in the content preparation of future elementary teachers. Thus, this summary paper reports on the research conducted (as of 2012) that examines PTs' content knowledge of geometry and measurement. Our goals for the research summary were to (a) identify what we know about PTs' knowledge of geometry and measurement, and (b) identify the gaps in the existing research literature to highlight topics that warrant further research.

### **Research Methods and Analysis**

The authors of this paper were part of a larger group of mathematics teacher educators who participated in a series of Working Groups at the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) (e.g., Thanheiser et al., 2010; for more, see the introductory paper of this Special Issue). We were charged with providing a description of what is known about PTs' geometry and measurement content knowledge from peer-reviewed research articles published prior to 1998—a historical look; an in-depth description of what is known about PTs' geometry and measurement content knowledge from 1998 to 2011—a current perspective; and, finally, a view of the horizon from 2011 to 2012 that builds on the previous time periods. Although the charge spans these three time periods, the work of the group started with the current perspective. For this perspective, common methods were established for each subgroup that focused on different mathematical content and are reported in the introductory paper of this Special Issue. This section reports on the methods for the historical look, methods' modifications made by our subgroup for the current perspective differing from the larger group, and the methods for the view of the horizon.

## **Methods for the Historical Look**

As we began to search for peer-reviewed research journal articles published prior to 1998, we first decided to draw upon any of the cited references from current-perspective articles that focused on elementary PTs' geometry and measurement content knowledge. These studies were included in a list of potential studies for the historical look. Second, a search using the Education Resources Information Center (ERIC) database was conducted to find any additional studies. The ERIC search included various combinations of keywords such as *preservice, prospective, elementary, teacher, education, and content knowledge*, specific content terms such as *geometry, measurement, length, area, volume, and angle*, and the prior-to-1998 publication date requirement. This produced a total of 62 studies that were added to the list of potential studies.

Each of the potential studies was reviewed to determine if the study was published in a peer-reviewed research journal. In this process, titles and abstracts were first used to determine if the focus of the study was on elementary PTs' geometry and measurement content knowledge. If a determination could be made that it clearly was not relevant to elementary PTs' geometry and measurement content knowledge, it was not included in our database of accepted studies. If there were any questions, possibilities, or doubts that an article focused on elementary PTs' geometry and measurement content knowledge, it went through an independent review that identified the research questions, study type and research design, location of study, lens and/or approach used, selection and description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions/implications. If the location of study [country of the population of PTs] was not described or referenced in the study itself, we assumed that the location was the country of

the authors' institution. This information was used to make a determination as to whether an article was excluded or included in our database. If there were any questions or discrepancies from the independent reviews regarding inclusion/exclusion, a mutual consensus was established by subgroup members. Examples of excluded articles were studies that focused on

(a) a general description of content knowledge that lacked specific attention to geometry or measurement, (b) a selection of inservice teachers or college students majoring in mathematics as opposed to mathematics education, (c) a sole focus on perceptions about mathematics not connected to content knowledge needed for teaching, and (d) a focus on describing classroom practice or activities with a lack of attention to research design methods. (Browning, Edson, Kimani, & Aslan-Tutak, 2011, p. 453)

Finally, the research literature cited in any studies included in the database of accepted articles for the historical section was used to determine potential new articles for the database. A total of nine studies were found for the historical look.

### **Modifications for the Current Perspective**

As discussed earlier, a thorough description of the methods for the current perspective are detailed in the introductory paper of this Special Issue. Modifications of the methods for the current perspective section included potential studies suggested by mathematics teacher educators outside of the Working Group. Due to the limited number of studies found in our search, expert mathematics education researchers focusing on elementary PTs' geometry and measurement content knowledge, not part of the Working Group, with several outside of the United States, were contacted to see if they were aware of any additional peer-reviewed research publications, especially in those journals outside of the United States. This produced two additional articles that were included in our

database of accepted studies for the current perspective section, with a total of 12 articles reviewed.

### **Methods for the View of the Horizon**

The view of the horizon section includes peer-reviewed research articles published in 2012, as well as 2011 and 2012 conference proceedings from PME and PME-NA. We examined the proceedings from both the North American Chapter and the International Group of PME to examine current research in geometry and measurement for PTs, to compare with our previous summaries and to note the most recent issues and trends in this area of research. The methods for this section followed a similar process for the other two time periods. Titles and abstracts of research reports, brief research reports, and posters were reviewed to determine whether there was a possibility for inclusion in our work. The inclusion of posters is a modification that differs from other content groups of the larger Working Group. All potential studies were independently reviewed. A total of five papers from the conference proceedings published in 2011 were accepted in our database; no related proceedings papers or research articles were found for 2012.

### **Analysis**

In order to summarize findings across all the studies reported in this paper, we examined the study types, research design, and research questions and characterized each study that dealt with elementary PTs' geometry and measurement content knowledge. All studies reported research results found "in the moment," indicating the *status* of the knowledge of PTs at that time in the study. There were no longitudinal studies, examining the development of content knowledge over an extended period of time. Yet, within this overarching type of study, we found comparison studies that examined *associations and/or*

*differences* between two entities, aspects, relationships, etc., and then nested within these comparison studies, we found those that experimented with and/or described the *impact of a treatment* in a mathematics content or methods course or lesson. Italicized text emphasizes the key features of the questions in the studies for these three groups. It is important to note that all three groups reference descriptions of elementary PTs' geometry and measurement content knowledge; however, the associations/differences and impact of some treatment categories also contain one or two of these foci in the work, namely, examining to see if there are or are not any connections between two things and describing the outcomes of testing (typically) an instructional intervention.

Classification of a study into one of these groups is to give insight into the various types of research questions that have been investigated in the areas of geometry and measurement. We chose this classification scheme as differences in question type or what the researchers were investigating stood out to us as we read through and summarized the research. As there are a huge variety of topics within the content areas of geometry and measurement and a relatively small amount of studies summarized, using topic themes as a means of classifying the summaries was not possible; many topic themes would have included only one study. We realize there may have been other ways in which to collectively summarize the data, but we chose to systematically examine and highlight the research focus and present findings, allowing the readers of the summaries to sort findings in a manner appropriate for their own future research.

### **Historical Look: What Was Known About the Geometry and Measurement Content Knowledge of Prospective K–8 Mathematics Teachers Prior to 1998?**

A total of nine studies published prior to 1998 focused on geometry and measurement content knowledge of prospective K–8 mathematics teachers (Table 1). The



studies are individually and collectively described below. The collective descriptions are framed around the three broad categories described in the previous methods section, based upon the type of research questions investigated.

Table 1

*Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published Prior to 1998*

	Author, Year	Content	Location of Study
Status	Baturo & Nason, 1996; Enochs & Gabel, 1984; Mayberry, 1983; Reinke, 1997	Perimeter, area, volume, surface area, and van Hiele levels of geometric thought	Australia and USA
Associations and/or Differences	Battista, Wheatley, & Talsma, 1982, 1989; Bright, 1979	Spatial ability, formal reasoning, geometric problem solving, and embedded figures	USA
Impact of a Treatment	Bright, 1985; Gabel & Enoch, 1987	Estimation of angle and length measurements and spatial ability and volume	Australia and USA

### **The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement**

Four studies (Baturo & Nason, 1996; Enoch & Gabel, 1984; Mayberry, 1983; Reinke, 1997) focused on what our group labeled the *status* of elementary PTs' content knowledge of geometry and measurement.

Mayberry (1983) investigated elementary PTs' van Hiele levels of geometric thinking related to seven concepts: squares, right triangles, isosceles triangles, circles, parallel lines, similarity, and congruence. The results of the study support van Hiele's

(1959) implication that “a student cannot function adequately at a level without having had experiences that enable the student to think intuitively at each preceding level” (p. 67). The results also support the implication that “if the language of instruction is at a higher level than a student’s thought processes are, the student will not understand the instruction” (p. 67). General findings of the study suggest that elementary PTs were at different levels for different concepts and were not ready for a formal deductive geometry course.

In an exploratory study focused on the meaning of volume, Enochs and Gabel (1984) were interested in identifying PTs’ misconceptions of volume and surface area. To this end, the researchers developed, validated, and established reliability of the *Surface Area/Volume Misconception Inventory* (SAVMI) questionnaire instrument. A total of 125 PTs who were enrolled in a science education course for elementary education completed this questionnaire. Interviews were conducted with a small subset of participants, asking them to “think aloud” as they solved different problems and wrote down their calculations. Findings of this exploratory study indicated that “a large percentage of elementary education majors do not understand the concepts of volume and are unable to distinguish volume from surface area” (p. 679). Errors based on misconceptions included concept/definition of volume or surface area; formula memorizing mode; confusion between length, area, and volume; unit memorizing mode; conversion of  $\text{cm}^3$  to ml; multiplication of units not correct or units incorrect; and wrong arithmetic. The researchers report that PTs were “found to solve problems using a ‘memorizing mode’ rather than basing their answers on the concept itself” (p. 679). Although the researchers do not indicate how volume and surface area should be taught, they do suggest that an exclusive formula approach is not beneficial for students.

Research related to area and perimeter concepts was conducted by Baturu and Nason (1996) and Reinke (1997), with findings from both studies suggesting struggles in understanding these concepts, some similar to those found in Enochs and Gabel's work (1984). Baturu and Nason investigated 13 teacher education students' subject matter knowledge of area measurement concepts and processes during the first year of their primary prospective program at the Queensland University of Technology. Based upon the work of Ball and McDiarmid, collectively and individually (Ball, 1990, 1991; Ball & McDiarmid, 1989; McDiarmid, 1988), Baturu and Nason viewed subject matter knowledge to be comprised of substantive knowledge, knowledge about the nature and discourse of mathematics, and knowledge about mathematics in culture and society. Results from structured, clinical interviews of area measurement tasks indicate that the PTs' knowledge was "rather impoverished in nature," namely, that their substantive knowledge was incorrect, incomplete, and unconnected, while having a limited ability in transferring from one form of representation to another. PTs had limited meanings for their rule-driven processes for finding area, as these rules were not connected to concrete experiences. For example, they could not explain why one must divide by 2 in the area formula for a triangle. Their knowledge of the nature and discourse of mathematics as well as about mathematics in culture and society appeared to be based on limited assumptions such as:

(1) mathematics is mainly an arbitrary collection of facts and rules . . . ; (2) most mathematical ideas have little or no relationship to real objects and therefore can only be represented symbolically; and (3) the primary purpose of learning area measurement was the utilitarian one of being able to calculate areas of regular shapes. (p. 262)

Baturu and Nason suggest that teaching mathematics without meaning promoted a low self-esteem for many of these PTs, as they failed to remember isolated facts and rules and

attributed their failure to low mathematical ability. These negative dispositions could possibly remain with the PTs and hinder their effectiveness in teaching mathematics to children.

Reinke (1997) investigated elementary PTs' solution strategies for finding the perimeter and area of a shaded geometric figure. A total of 76 PTs, enrolled in a second semester of an elementary mathematics content course, participated in the study. Findings indicated that the most common incorrect strategy by PTs was determining perimeter using the same method for area, suggesting PTs were confused about linear measurement and area measurement. Reinke supports the notion that PTs have been taught to rely on procedural learning and lack comfort with conceptual learning in mathematics, suggesting that PTs need more exposure to problems promoting conceptual understanding.

**Summary.** Findings across these four studies suggest that PTs enter their mathematics content preparation programs with limited geometry and measurement experiences, experiences chiefly focused on manipulation of formulas. Work using a van Hiele model for geometric learning indicated PTs were at different levels for different concepts, they tended to be at lower levels of geometric understanding, and they were not ready for a formal deductive geometry course (Mayberry, 1983). Other research studies conducted during this time period that focused on the status of PTs' content knowledge cite specific issues. For measurement with perimeter, area, and volume, PTs tend to not understand the concepts behind the measure formulas and confuse the measures, finding surface area instead of volume, or area instead of perimeter, as they rely solely on their memory of disconnected rules and formulas (Baturro & Nason, 1996; Enochs & Gabel, 1984; Reinke, 1997).

## **Examining Associations and Differences**

Three studies (Battista, Wheatley, & Talsma, 1982, 1989; Bright, 1979) had at least one research question that focused on examining associations and differences related to what PTs understood about specific topics in geometry and measurement.

Bright's (1979) work with 145 PTs involved the identification of embedded figures in complex drawings, finding shapes within shapes. Analyzed data were taken from PTs' work with either two triangle figures or two quadrilateral figures, all having embedded shapes within. His findings suggest non-overlapping figures were easier to identify, that non-overlapping figures are generally identified first, and that PTs could identify embedded triangles more easily than quadrilaterals. Noted limitations to the study included the limited types of data analyses and that interviews were not conducted to verify students' thinking on the task. Bright found that only about half of the PTs completely and correctly solved one of the four drawings. Bright indicated that "it is therefore unlikely that as future teachers these people can be expected to teach such problem-solving techniques effectively to students" (p. 326), a somewhat dismal implication.

Battista, Wheatley, and Talsma (1982) investigated the interaction of spatial ability and cognitive development to examine their impact on mathematics learning, specifically that of geometry concepts. Participants for their study were 82 PTs enrolled in an informal geometry course. Instruments for data collection included the Purdue Spatial Visualization Test: Rotations (PSVT) (Guay, 1977) and a modified Longeot Test of cognitive development. Data were summarized on 82 of the enrolled students and included four measures: pre- and posttest means on the PSVT ( $S_1$  and  $S_2$ ), mean on the modified Longeot

test taken at the end of the semester (C), and the course grade score, which was the total of the student's scores on three course exams (G).

The spatial visualization scores significantly improved by the end of the semester, suggesting to the researchers that the type of activities used in the course may have helped with this improvement. However, research on whether instruction can improve spatial ability was inconclusive at that time. Further, missing from the article was any description of the type of activities used in the course. Examining multiple correlations of course grade (G) on C and  $S_1$  supported the importance of both cognitive development and spatial visualization in learning geometric concepts. The data further suggested that cognitive development is a better predictor of the course grade in geometry than the spatial visualization ability.

In a second study by Battista, Wheatley, and Talsma (1989), they explored the connections between spatial visualization, formal reasoning, and geometric problem-solving abilities of elementary PTs. They worked from research that suggested learning mathematics may depend upon fundamental or "primary" mental abilities; students lacking those primary abilities may not be able to use certain problem-solving processes (Kulm & Bussman, 1980). Building on their previous work described above, Battista and colleagues investigated the relationship between the two primary abilities of spatial visualization and formal reasoning and the strategies used by PTs in geometric problem solving. Using similar instruments from their 1982 study, in addition to a problem-solving strategies test, Battista and colleagues collected data from 83 students enrolled in a geometry course for elementary PTs. From their findings, the researchers suggested an implication for instruction that relates to strategy use and strategy effectiveness, where PTs "should be

taught to identify those strategies that they are most likely to use effectively” (p. 28). “In particular, it seems that all but the brightest of prospective elementary teachers would benefit from learning to use a drawing strategy or some other strategy that would replace the use of pure visualization” (p. 29).

**Summary.** All three studies examined PTs’ spatial visualization in some way and how those skills connected with some other ability. The work of Battista and colleagues (1982, 1989) connects cognitive development and spatial visualization to geometric problem-solving ability and to the learning of geometry concepts in general, with Bright (1979) finding spatial visualization skills connected to identifying embedded figures in complex drawings. Bright also found the visualization skills of the PTs developed over time.

### **Describing the Impact of a Treatment**

Two studies (Bright, 1985; Gabel & Enochs, 1987) had at least one research question that explored the impact of a treatment related to what PTs understood about specific topics in geometry and measurement.

In 1985, Bright conducted a study to determine the effectiveness of the computer game *Golf Classic* (Kraus, 1982) on PTs’ estimation of length and angle measurements. (Bright’s study also included a probability game, but findings only from the geometry computer game will be discussed in this summary.) Bright, Harvey, and Wheeler (1982) had conducted work with *Geogolf*, a non-computer instructional game, with tenth graders and found that the game effectively taught the students to estimate length and angle measurements. Bright wanted to determine if a computer version of the game would be just as effective in developing estimation skills for length and angle measure with PTs. During a 5-week period of time, each PT ( $n = 78$ ) was randomly assigned to play one of the

two computer games (focused on geometry or probability). PTs played the game twice during this time period, once alone and once with someone else assigned to play the same game. Each time, the game was played for 20 minutes for a total game time of 40 minutes. Pre- and post-measures were taken for both length and angle estimation skills. Findings from these measures showed the computer game, Golf Classic, to have a marginal effect at improving angle estimation skills, with patterns in the performance of length estimation inconsistent. Bright believes his study's findings suggest "expectations should not be too high when attempts are made to translate effective non-computer instructional techniques into computer formats" (p. 522). This raises questions as to how the time length of 40 minutes was determined as sufficient time with the computer game to develop angle and length estimation skills as compared to the length of time on task for the tenth graders when playing the non-computer game.

Gabel and Enochs (1987) examined the research question of "whether spatial-visual skills are related to learning the volume concept, and whether a particular mode of presentation for teaching volume is preferable for students of different spatial ability" (p. 592). In this experimental study, elementary PTs in five sections of an introductory science class in a large Midwestern university used four different instructional sequences for length, area, and volume: length-area-volume ( $n = 30$ ), length-volume-area ( $n = 25$ ), volume-area-length ( $n = 38$ ), and area-volume-length ( $n = 37$ ). Three sections of students were randomly assigned the "length-last" treatment and two sections assigned the "length-first" treatment. Within each section, PTs were assigned "volume-before-area" and "volume-after-area" treatments. To answer the research question, four instruments were administered in this study: the cube-comparison test (French, Ekstrom, & Price, 1963),



surface development test (French et al., 1963), computational volume pretest (Bilbo & Milkent, 1978), and an adapted version of the volume test (Bilbo & Milkent, 1978).

Findings of the experimental study (Gabel & Enochs, 1987) indicated that spatial orientation is a key factor for volume test performance. Further, “the sequence in which the metric system is taught to PTs is an important factor in teaching the metric system if the visual-orientation ability of the student is considered” (p. 596). For elementary PTs of low visual orientation, teaching volume before area and length is beneficial, whereas those with high visualization skills can “use them to logically construct volume from area and height” (p. 596). Findings indicated that the order in which length, area, and volume were presented to PTs did not have a significant effect on how well they performed on the volume test. However, the researchers found that volume-area-length is preferable for students of low spatial orientation, whereas students of high spatial orientation prefer length-area-volume. It is important to note that elementary PTs likely experienced length, area, and volume sequence in school mathematics and that the study was limited by examining only volume of the metric system. In addition, the researchers emphasized they did not compare the effectiveness of the instruction on PTs’ understanding of length, area, and volume, and that “other sequences might be preferable for teaching these other concepts [length and area], and this needs to be considered in teaching the entire unit on the metric system” (p. 597).

**Summary.** The two treatments under consideration were the use of computer games in instruction and the sequence of instruction for measurement topics in a geometry course. Results indicated that PTs’ use of computer technology software, Golf Classic, positively impacted their estimation of angle measurements, a similar finding of the

software used with Grade 10 high school students. In terms of sequencing the measurement topics of length, area, and volume, results indicated that the various sequence tests did not have a significant effect; however, the researchers noted that the sequence did matter in terms of PTs' spatial orientation. PTs' with low spatial orientation prefer the volume-area-length sequence, as challenges occurred when constructing volume concepts from area and height.

Key findings across the nine studies support the importance of PTs developing their spatial abilities as related to geometric problem solving, finding embedded shapes, and developing concepts of measure; but spatial ability alone is not sufficient for success in geometric learning. PTs need to move away from focusing on memorization of formulas and focus on making meaning of concepts. Most findings, in general, support the importance of having numerous geometric experiences to advance geometric understanding, as noted by van Hiele (1959); it appears the importance of experience is true for both children and for PTs.

### **Current Perspective: What Was Known About the Geometry and Measurement Content Knowledge of Prospective K–8 Mathematics Teachers From 1998 to 2011?**

Twelve studies focused on geometry and measurement in the 1998 to 2011 timeframe (Table 2). Topics explored in these studies include shape and shape properties; measurement topics of area, perimeter, and volume; use of dynamic geometry environments; and van Hiele levels of understanding. We note the range of topics is fairly similar to those in the previous historical section. Differences include a lack of any studies examining measurement estimation skills, and the technology focus has shifted from computer games to dynamic learning environments. Again, we present the studies

individually and collectively, where the collective framework revolves around the research focus of the study, described in our analysis section.

Table 2

*Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published From 1998 to 2011*

	Author, Year	Content	Location of Study
Status	Fujita, 2011; Fujita & Jones, 2007; Gutierrez & Jaime, 1999; Menon, 1998; Pickreign, 2007; Zevenbergen, 2005	Quadrilaterals, triangle altitudes, volume, perimeter, and area	Australia, UK, USA, Scotland, and Spain
Associations and/or Differences	Halat, 2008; Pitta-Pantazi & Christou, 2009; Lin, Luo, Lo, & Yang, 2011; Tsamir & Pitta-Pantazi, 2008	Geometry thinking using the van Hiele levels of understanding, intuitive rules theory applied to geometric tasks (median, bisector, perimeter and area), and the relationship between cognitive styles and mathematical performance in measurement and spatial tasks	Cyprus, USA, Taiwan, Turkey
Impact of a Treatment	Cunningham & Roberts 2010; Gerretson, 2004; Zevenbergen, 2005	Altitudes of triangles and diagonals of polygons, the impact of using dynamic geometry software on understanding of similarity, and the impact of various learning dispositions on the understanding of volume	Australia and USA

## **The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement**

Six studies focused on the status of what PTs understand about specific topics in geometry and measurement.

Menon (1998) investigated PTs' understanding of perimeter and area. The participants of the study were 54 students who had completed one semester of their teacher preparation program prior to their enrollment in an elementary mathematics methods course taught by the researcher. Data came in the form of the participants' responses to four tasks. The study found that PTs' conceptions of mathematical ideas, particularly in geometry, were not fully developed, with most of the participants lacking the ability to articulate complete descriptions of rectangle and rhombus. In addition, Menon stated, "Yet, even with an apparently better foundation in mathematics, the students seemed to have poor conceptual understanding (*in perimeter and area*)" (p. 365). In sum, Menon decried the lack of conceptual understanding despite satisfactory performance on paper-and-pencil assessments, as the implication was that these PTs were less likely to offer their students opportunities to explore problems that require conceptual reasoning.

Further research on PTs' knowledge of rectangles and rhombi was conducted by Pickreign (2007). In particular, this study examined what is revealed about PTs' understanding of the properties and relationships among parallelograms through their articulation of the meaning of rectangle and rhombus. Participants of the study were 40 PTs taking the first course in a two-course sequence randomly selected from four sections of the course taught by the same instructor. Data came from the PTs' written personal definitions of rectangle and rhombus.

Nine of the 40 participants (22.5%) gave a definition of a rectangle that was classified as complete—inclusive of squares and excluding any parallelogram that did not have a right angle. Only 1 of the 40 (2.5%) defined a rhombus in a way that was inclusive of squares and excluded parallelograms that did not have equal adjacent sides. Pickreign added, “It is not the complete definitions that are most interesting, nor even how few of the participants got them correct; it is the misconceptions regarding these shapes that seem to be indicated by the other responses” (p. 3). Pickreign concluded that irrespective of the experiences the students in this study had with rectangles and rhombi, PTs lacked the ability to articulate these two types of quadrilaterals.

PTs’ conceptual understanding of quadrilaterals in general was studied by Fujita and Jones (2007). The research reported was part of a larger study focusing on PT education in the UK and Japan. The researchers explored the nature of PTs’ *personal figural concepts* and *formal figural concepts*, building on the work of Tall and Vinner (1981) with respect to concept image and concept definition and Fischbein’s (1993) figural concept. The study examined data from 158 PTs in order to investigate the gap between their formal and personal concept images of quadrilaterals. The results indicated that PTs rely on their personal concept images of shapes to construct definitions rather than examining and using properties of shapes.

Fujita (2011) continued to investigate learners’ understanding of quadrilaterals by developing a questionnaire that focused on inclusion relations. The questionnaire was piloted with 19 PTs in the UK, and then with 85 Japanese lower secondary school students. Results from the PTs’ answers to geometry questions revealed that the majority of them hold a prototype definition for quadrilaterals based on limited personal figural concepts of

the shape, and they have difficulty in understanding the inclusion relationships between quadrilaterals. For example, even though PTs stated a definition of parallelogram, they could not use it to show that a square is a parallelogram. The author suggested that participants' literal use of definitions may cause deficiencies in understanding the inclusion relationships. Fujita suggested carefully integrating visual and conceptual aspects of quadrilaterals to create an effective learning environment to help overcome the prototype definition phenomenon. Further, "a careful use of dynamic geometry environments . . . might encourage learners to develop their dynamic images of shapes and to pay attention to what properties are changed/unchanged between the different shapes (Leung, 2008).

Similar to the work of Fujita and Jones (2007) and Fujita (2011), Gutierrez and Jaime (1999) used a theoretical framework of *concept image* and *concept definition* (Vinner, 1991; Vinner & Hershkowitz, 1980, 1983) to investigate primary PTs' understanding of the concept of altitude of a triangle. The study identified the students' reasoning process and the effect of variables such as the students' "previous knowledge, the presence of a formal definition in the test, or the influence of learning activities that dealt with altitudes of triangles as part of the content of a course on mathematics education" (p. 259). The researchers reported evidence of PTs holding onto certain concept images that are not helpful. Specific student misconceptions included poor concept images, with students (a) relying more on visual cues for defining shapes, (b) believing that altitudes of triangles must exist within the shape, (c) mixing definitions of medians and altitudes, and (d) mixing perpendicular bisectors and altitudes. In sum, this study found PTs had poor concept images that are comparable to those of primary or secondary students and offered that this

situation can provide a platform for the PTs to examine their and their classmates' concept images and concurrently learn what types of concept images children are likely to have.

Zevenbergen (2005) explored the understandings of volume among primary PTs. This study set out to critically explore the reactions and learnings of PTs in a course in which discipline knowledge was taught in tandem with pedagogical content. The participants in this study were 98 PTs enrolled in a third-year course in which students were expected to "develop a strong understanding of mathematics discipline knowledge" (p. 8). Data came from the PTs' responses to a quiz item requesting the amount of concrete needed to fill a barbeque area with dimensions of  $8.5\text{m} \times 3.2\text{m} \times 30\text{cm}$ . PTs were asked to express their answer "in the way you would if you were to phone the concrete company to place the order" (p. 8). Follow-up interviews were conducted with 30 of the PTs.

From the quiz data, the researcher reported "only 32 out of 98 students were able to calculate a result and transfer the result into an appropriately communicable form (i.e., approximately 8 cubic meters) and concluded the data suggested "students have the esoteric knowledge of school mathematics but have not transferred it to the practical context, and that there has been a prioritizing of school mathematical knowledge over practical mathematical knowledge (or numeracy)" (p. 10). The interviews were performed after quizzes were corrected and aimed to offer insights into the students' thinking as well as an appreciation of the diversity of the responses. Data from the interviews offered more evidence of incomplete concepts about volume among the PTs. Zevenbergen stated, "The interview data suggested that there was heavy reliance on procedural knowledge, that is, algorithmic methods in which lock-step strategies were used to solve the task. These

strategies suggest that the students relied on particular ways of knowing in mathematics" (p. 11).

**Summary.** The researchers described PTs' understandings of geometry and measurement as not fully developed, based on unproductive concept images and/or concept definitions, and lacking in their ability to articulate their reasoning with geometry and measurement. The prospective teachers in these studies relied on visual examinations to define shapes, relied on procedural knowledge, and lacked conceptual understandings of geometry and measurement concepts. Their understandings were compared to those of primary grade or secondary school students, which raises questions about their preparedness to teach geometry and measurement concepts with fidelity to the standards expected for elementary grades. For example, given prospective teachers' superficial understandings of geometry and measurement concepts and their deficient concept images and concept definitions, it is difficult to imagine their ability to see the structure of the geometry and measurement concepts they will be asked to develop in their students, especially under the new standards in the United States that ask for making connections among mathematics concepts.

### **Examining Associations and Differences**

Four studies included research questions that examined relationships and/or differences related to PTs' geometry content knowledge; these are described below.

Halat (2008) administered a van Hiele Geometry Test (VHGT), based upon the work of Usiskin (1982), to compare two groups of PTs' (elementary and secondary) geometric thinking levels while investigating for differences in terms of gender. The researcher used data from 281 Turkish PTs (125 elementary and 156 secondary). There were 68 female



and 57 male elementary PTs who took the test after completion of a geometry course at a Turkish university. Also, 72 female and 84 male secondary PTs answered van Hiele test questions after they completed advanced level mathematics and geometry courses. Halat found

no statistically significant difference in regard to the reasoning stages between the pre-service elementary school and secondary mathematics teachers, and that although there was a difference with reference to van Hiele levels between male and female pre-service secondary mathematics teachers favoring males, there was no sex-related difference found between male and female pre-service elementary school teachers. (p. 1)

Further work using the van Hiele levels of understanding was conducted by Lin, Luo, Lo, and Yang (2011) involving a comparative study to investigate and compare the geometry knowledge and levels of PTs from the United States and Taiwan. Data were collected from 48 U.S. PTs and 40 Taiwanese PTs, with both groups enrolled in a mathematics methods course. Two instruments (the VHGT and the Entering Geometry Test [EGT] also created by Usiskin, 1982), were used to collect data regarding PTs' knowledge and their levels of geometric thinking. The 20-item multiple-choice EGT was used to measure the PTs' content knowledge. The 25-item multiple-choice VHGT is divided into five levels with five questions in each level that focuses "not only on content knowledge but also on the sophistication levels of geometric thought including proof" (p. 9).

The PTs' performance on the EGT showed a statistically significant difference between the two groups, suggesting that Taiwanese PTs entered their teacher education program with a better understanding of geometry than their U.S. counterparts. The Taiwanese PTs also outperformed the U.S. students on each item on the EGT. The VHGT data also showed significant differences between the U.S. and Taiwanese students. While 77.5% of Taiwanese PTs achieved at least the third van Hiele level, only 27% of their U.S.

counterparts achieved at least level three on the VHGT. However, unlike on the EGT where Taiwanese students outperformed the U.S. students on every item, they did not outperform their U.S. counterparts on every VHGT item. The data indicated no significant associations between the EGT and VHGT scores for Taiwanese students, while there was evidence of a positive weak relationship among the U.S. participants.

The authors noted that despite the importance of teachers' mathematical content knowledge, it is not known what minimal van Hiele level of understanding elementary teachers should achieve so they can provide a sufficient quality of geometric teaching for their students. They argue that a satisfactory level of achievement for PTs needs to be justified prior to making suggestions for change in geometry expectations for elementary teacher preparation.

Shifting from a focus on levels of geometric understanding to analyzing errors in PTs' geometric thinking, Tsamir and Pitta-Pantazi (2008) focused on the intuitive rules theory posited by Stavy and Tirosh (2000) with 98 PTs in a mathematics education course from the University of Cyprus. The intuitive rules theory was designed by Stavy and Tirosh for analyzing and predicting inappropriate responses to a wide variety of mathematical and scientific tasks. Tsamir and Pitta-Pantazi used the framework to help interpret errors made by the PTs in solving a variety of geometric tasks, specifically tasks related to geometric ideas of median, bisector, perimeter, and area.

The intuitive rule *more A–more B* was identified in tasks in which there are two objects or systems where one quality or quantity A, fulfills the condition  $A1 > A2$  and this inequality is either perceptually or directly given, or alternatively, it can be logically derived through the schemes of conservation or proportion. However, participants are asked to compare the two objects or systems with regard to another quantity B, for which the two given objects or systems fulfill either  $B1 = B2$  or  $B1 < B2$ . A common incorrect response to such tasks, regardless of the content domain, takes the form: “ $B1 > B2$  because  $A1 > A2$ , or more A–more B. (pp. 72–73)

Tsamir and Pitta-Pantazi (2008) found that PTs' solutions were consistent with the intuitive, more A-more B or same A-same B lines of reasoning, where most PTs based their arguments on their visual grasp of the data in illustrations provided in the tasks. Further, as the authors found comparable findings to the Cypriot data in an earlier study done in Israel with secondary school mathematics PTs, they have provided extended data regarding PTs' ways of thinking about perimeter and area. Tsamir and Pitta-Pantazi suggest that similar findings across two different countries provide the mathematics education community a better picture of intuitive pitfalls hidden in these topics and suggest possible reasons for the PTs' difficulties.

Pitta-Pantazi and Christou (2009) investigated the relationship between cognitive styles and mathematical performance in measurement and spatial tasks of 116 elementary prospective kindergarten teachers with varying mathematical ability. Given that there are many different ways to define cognitive styles (Riding & Cheema, 1991), the researchers used two dimensions of cognitive styles that grouped most definitions: Verbal-Imagery and Wholistic-Analytic. The first dimension, Verbal-Imagery, refers "principally to mental representations, i.e., to the way individuals represent knowledge in mental pictures or words" (p. 132). The second dimension, Wholistic-Analytic, refers to "individuals' typical methods for organizing and processing information, either in parts or as a whole" (p. 133).

This study used two cognitive style tests, Verbal-Imagery Cognitive Style test (VICS test) and the Extended Cognitive Style Analysis Wholistic-Analytic test (CSA-WA) (Peterson, 2005), and a mathematics test with six spatial and six measurement tasks. The findings of the study suggest that there are "no performance differences between spatial and measurement tasks across the various cognitive styles of the participants . . . however,

... the impact of cognitive style is significant for some groups of participants (the low achievers)” for the measurement pictorial tasks (p. 146). Low-achieving thinkers who consider information they read, see, or listen to in words (verbalizers) and those who deconstruct information to its components (analytic thinkers) performed much better than those who use mental pictures (imagers) in all pictorial measurement tasks. Therefore, the results from the study suggest “verbalisers and analytic low achievers perform best when given an instructional format enhanced with graphical features” (p. 146) and raised the importance that classroom material should be presented in various formats.

**Summary.** PTs’ geometry thinking was examined using the van Hiele levels of understanding, with some studies comparing PTs’ understanding across different groups. While Halat (2008) found no significant differences in van Hiele levels of understanding between Turkish elementary and secondary PTs, significant differences were found between Taiwanese and U.S. PTs, with a majority of Taiwanese PTs reaching level 3 understanding based on the VHGT compared to only 27% of U.S. PTs (Lin et al., 2011). These results suggest some possible international differences in the content preparation of PTs prior to their entrance in a teacher preparation program. Other work made use of Stavy and Tirosh’s (2000) intuitive rules theory. Tsamir and Pitta-Pantazi (2008) compared findings of elementary PTs’ thinking of area and perimeter with triangles to that of secondary mathematics PTs, finding the intuitive reasoning of more A–more B or same A–same B prevalent in both groups. Pitta-Pantazi and Christou (2009) examined the relationship between cognitive styles and mathematical performance in measurement and spatial tasks finding that for low achievers, the cognitive style was significant on the measurement pictorial task, with the low performance from those PTs who used mental

pictures (imagers). A study of Battista et al. (1989), summarized in the earlier historical look, found similar results, suggesting that the majority of PTs would benefit from replacing their singular visualization strategy (similar to imagers) by learning to use some other problem-solving strategy, such as drawing.

### **Describing the Impact of a Treatment**

There are three research peer-reviewed studies published in journals that had questions of an investigative nature, exploring the impact of a treatment. Each of these studies is described in detail below.

Gerretson (2004) examined whether there was a difference in elementary PTs' performance on similarity tasks when using dynamic geometry software as compared to a paper-and-pencil learning environment using traditional tools (e.g., compass, ruler). There were 52 PTs who were enrolled in an introductory course that addressed content, methods, material development, and assessment in mathematics teaching. Using a pre- and posttest control group experiment using randomized blocks controlling for initial performance, she found a statistically significant difference in learning environments between the two treatment groups. "Fundamentally, software users outperformed non-software users even when prior knowledge variability was taken into consideration" (p. 18). Analysis suggested elementary PTs using a paper-and-pencil learning environment encountered more difficulties particularly situated around similarity properties of unfamiliar shapes, whereas PTs using dynamic geometry software had "acquired a greater knowledge base to access, network, and apply" (p. 19).

In an exploratory study, Zevenbergen (2005) investigated the impact of various learning dispositions emphasized within a mathematics course module on volume. These

dispositions included developing mathematical meaning of volume as opposed to using only algorithmic methods, measurement and spatial sense, and the capacity within the PTs to identify errors in children's mathematical thinking. However, despite the various methods used in the course to develop these dispositions, there were a "worrying number of students" who had not achieved them (p. 21), with some students quite resistant to alter their thinking about how to learn mathematics. Responses to interviews of students in the course highlighted the power of the teaching practicum, with PTs rejecting the nature of the work done in their mathematics course due to their experiences in the schools. "Ideally, it would be useful to expose students to schools and classrooms that demonstrate the values embedded within teacher education courses if such courses are to effectively change teaching practice" (p. 21).

Cunningham and Roberts (2010) used the theoretical framework of *concept image* and *concept definition* (Vinner & Hershkowitz, 1980) to explore the impact of a treatment lesson involving instructional strategies designed to assist the development of PTs' concept images and concept definitions related to altitudes of triangles and diagonals of polygons. They used a one-group pre- and posttest design with 57 primary school PTs enrolled in a content course. For this study, the researchers investigated instructional strategies that included the use of graphic organizers (e.g., Frayer, Frederick, & Klausmeier, 1969) along with the concept attainment model (Eggen, Kauchak, & Harder, 1979; Joyce, Weil, & Calhoun, 2004) in the development of definitions. Pretest results showed PTs' weak conceptual understanding. A combination of the teaching strategies resulted in some posttest improvement in their understanding of triangle altitudes and diagonals of polygons. The researchers posited, "This study advocates that teaching challenging

geometry concepts to PTs needs careful attention so that the mismatch between concept definitions and their concept images may be minimized” (p. 10). This study concluded that owing to the weak conceptual understanding for some PTs, there is a need for mathematics teacher educators to utilize more than “passive” or traditional teaching approaches, going beyond memorizing concept definitions.

**Summary.** These studies explored the impact of instruction using graphic organizers and concept attainment strategies on the understanding of altitudes of triangles and diagonals of polygons, the impact of using dynamic geometry software on the understanding of similarity, and the impact of various learning dispositions on the understanding of volume. While these studies reported some gains in the development of PTs’ conceptual understandings of geometry and measurement concepts, the gains were somewhat tempered by the PTs’ perceptions of the nature of mathematics as a body of knowledge that can be developed through the memorization of formulas. These perceptions may be due, in part, to traditional teaching approaches that focus on memorization of formulas. Also highlighted in these results was the challenge of implementing change in teacher education. For example, the PTs’ teaching practicum experiences need to support the productive ways of reasoning developed in prospective teacher training; otherwise, the gains achieved during teacher training are eroded.

Similar to key findings from the historical look, status research in the current perspective still shows that PTs’ understanding of geometry and measurement is not fully developed, with deficient concept images and concept definitions, and PTs performing at low van Hiele levels of understanding. Research suggests that PTs who are low-achieving need to tap into other cognitive styles beyond mental imagery or visualization, such as

those that make more use of reading or listening to information (verbal) or those styles that focus on deconstructing given information into components (analytic). Further, the finding that teaching and learning geometry and measurement concepts need to move away from a focus on memorizing formulas remains constant in this timeframe as well. Teachers need to consider the use of alternative instruction strategies that engage PTs more in developing their own geometric definitions through problem-solving experiences that may also improve their concept images. With geometry dynamic software becoming more accessible, learning experiences can be more exploratory and investigative and can enhance geometric understanding, as Gerretson's (2004) work shows.

**View of the Horizon: What Is Known About the Geometry and Measurement Content Knowledge of Prospective K-8 Mathematics Teachers Since 2011**

For the view of the horizon, peer-reviewed research articles published in 2012 as well as relevant research from 2011 and 2012 conference proceedings from PME and PME-NA were examined. A total of five studies were found, all published in 2011, that focused on geometry and measurement content knowledge of prospective mathematics teachers (Table 3). No related proceedings or research articles were found for 2012. (As writing for this current Special Issue took place during the majority of 2013, searches for related research ended in December 2012.)



Table 3

*Peer-Reviewed Research Articles on PTs' Content Knowledge of Geometry and Measurement Published Since 2011*

	Author, Year	Content	Location of Study
Status	İymen, Pakmak, & Paksu, 2011; Patton & Parker, 2011	Parallelogram and geometric terms	Turkey and USA
Associations and/or Differences	Köse & Özen, 2011	Problem solving in paper-pencil and dynamic geometry environments	Turkey
Impact of a Treatment	Morgan & Sack, 2011; Schnorenberg & Chamberlin, 2011	Progression through the van Hiele levels of geometric thinking based on the use of triangles and area and volume	USA

### **The Status of Prospective Teachers' Content Knowledge of Geometry and Measurement**

Two papers from PME and PME-NA conference proceedings (İymen, Pakmak, & Paksu, 2011; Patton & Parker, 2011) had at least one research question that focused on the status of PTs' content knowledge of geometry and measurement. As both papers are based on poster presentations, details are minimal and brief. However, both examine knowledge of shape and geometric terms, and findings still show PTs struggling with definitions of geometric shapes.

İymen, Pakmak, and Paksu (2011) investigated PTs' geometry content knowledge with a focus on their understanding of parallelogram. Forty-five PTs were interviewed using a parallelogram task, with 82% responding correctly. However, the interview

revealed weaknesses in the PTs' understanding of the shape and relationships between parallelograms and trapezoids.

Rather than assessing PTs' knowledge of key measurement and geometry terms, Patton and Parker (2011) investigated if PTs were, perhaps, not able to apply their knowledge of vocabulary in a measurement application test. Fifty-two PTs were given the test consisting of 12 multiple-choice questions in the first month of a mathematics methods course. The results indicated 33% of PTs scored at the mastery level (scoring 90–100%), 33% passed (scoring 75–90%), and 33% failed (scoring below 75%). A follow-up vocabulary test of 24 sixth-grade-level terms was given in the following semester. Scores indicated that approximately 60% of the PTs scored mastery, 35% passed, and 5% failed. No additional information was provided to gain further insight into these results.

### **Examining Associations and Differences**

Köse and Özen's (2011) PME conference proceeding paper had at least one research question that focused on examining associations and/or differences related to what is understood about specific topics in geometry and measurement. Their qualitative work with a sample of three PTs compared the PTs' problem-solving process in a paper-and-pencil situation with that done in a dynamic geometry environment (DGE). They found that the PTs attempted to solve a given problem using similar processes, yet could not find a solution in the paper-and-pencil environment. Further, in DGE, all students used different problem-solving processes. (No clear reference was made to the correctness of the PTs' solutions in the proceedings summary of the poster.) It was found that the PTs have essentially two stages in problem solving, that of constructions and investigations, with the PTs having no difficulties at the construction stage. With the investigation stage, PTs used a

helical process of seeking a relationship, finding a relationship, testing the relationship, seeking for a new relationship, and justifying. In general, the findings suggest that future work in DGE is promising, in that students were more willing to seek, find, and test relationships prior to justifying, as compared to students using a paper-and-pencil approach.

### **Describing the Impact of a Treatment**

Two papers from PME and PME-NA conference proceedings (Morgan & Sack, 2011; Schnorenberg & Chamberlin, 2011) had at least one research question that explored the impact of a treatment related to what PTs understand about specific topics in geometry and measurement. The treatments explored in these two studies included the use of “giant triangles” in the development of a variety of geometric ideas, and the effect of the use of differentiated instruction on PTs’ understanding of area and volume.

Morgan and Sack (2011) present a learning trajectory, based upon the van Hiele levels of geometric thinking, that make use of “giant triangles,” flexible manipulatives that are 1-meter in edge length, which can be assembled to make a variety of polyhedra with triangular faces. (The use of the triangles was thought of as an “instructional treatment” and thus placed in this category of studies.) The trajectory presented describes activities that took place during a single 160-minute class session in the semester-long course. The activities are intended to move PTs through the van Hiele levels, visual to descriptive to relational. The PTs in this study were enrolled in an elementary/middle school mathematics methods course at a mid-Southwestern university in the United States, federally designated a Minority Serving Institution. The authors state that “substantial, deep and interconnected mathematics” is made available quickly and effectively using the

triangles. It is further stated that “no entry-level content knowledge is required and transfer from prior content courses has generally not been observed” (p. 255). We interpret this was meant to be a positive finding, namely, that weak conceptual understanding did not interfere with the learners in engaging with the triangle activities and making sense of the geometry concepts involved. However, it raises a question as to why PTs retain limited knowledge from high school geometry experiences and why the understandings they do retain tend to be weak and fragmented. A third finding presented suggested that “high levels of student engagement and collaboration are achieved associated with hands-on play and figuring out activities, in a positive affective social context” (p. 255). And finally, the authors indicated that the “use of these manipulatives may avoid some of the affective pitfalls that occur when introducing challenging mathematical problems” (p. 255).

Schnorenberg and Chamberlin (2011) investigated how differentiated instruction impacts elementary PTs’ mathematical understanding of area and volume. In this lesson experiment, instruction was differentiated by the use of several formative assessments, flexible heterogeneous and homogeneous groups, various activities with multiple modalities (e.g., visual, audio, kinesthetic mediums), and tracking of student progress on learning goals. Specifically, two groups of PTs were formed based upon their pre-assessment results. Each group focused on a series of activities designed for either area or volume. To examine the impact on PTs’ understanding of area and volume in a geometry and measurement course for elementary teachers, data sources on students’ work of pre-assessments, group activities, and post-assessments were collected for nine elementary PTs. In addition, audio recordings captured each group’s discussion and video recordings

captured the instructors' teaching. Although analysis was ongoing in this study, the findings indicated that PTs improved their knowledge and skills related to area and volume. For example, PTs gained understanding in area as covering a two-dimensional shape and volume as filling a three-dimensional shape, understanding that surface area and volume are independent, and understanding that measuring objects with different unit sizes may lead to different measures. Thus, in conclusion, the researchers state that "differentiating by area and volume in this lesson enhanced the students' understandings, allowed us to maximize the use of class time, and possibly provided a model of differentiated instruction for the students" (p. 1499).

**Summary.** The summary of research from a view of the horizon, based upon minimal insights and research findings garnered from conference proceedings and posters, suggests PTs continue to struggle with meanings of content vocabulary terms from specific geometry and measurement topics. However, these same findings did provide some evidence that future work in dynamic geometry environments, learning trajectories grounded in van Hiele levels of geometry thinking, and differentiated instruction can positively improve PTs' content knowledge of geometry and measurement.

### **Discussion and Conclusion**

The *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000), the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM, 2006), and the recent *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) all include content expectations specific to geometry and measurement; thus, minimally, PTs would need to have a solid understanding of these

same expectations. In addition, two reports released from the Conference Board of the Mathematical Sciences (CBMS) discuss recommendations for the mathematical preparation of teachers at all grade levels. *The Mathematical Education of Teachers* (CBMS Report I; CBMS, 2001) based recommendations at that time upon the PSSM. Similarly, *The Mathematical Education of Teachers II* (CBMS Report II; CBMS, 2010) uses the CCSSM “as a framework for outlining the mathematical ideas that elementary teachers, both prospective and practicing, should study and know” (p. 25).

Table 4 shows a correlation between the recommendations from CBMS Reports I and II and the research focusing on elementary PTs’ geometry and measurement content knowledge. We initially sorted research articles into the CBMS Report II recommendations. Only 15 of the 26 studies’ content emphases could be matched to content recommendations from the CBMS Report II. We were curious if perhaps some of the remaining studies focused their research on recommendations found in the earlier CBMS Report. So we sorted these remaining studies using CBMS Report I recommendations. Five more studies were then classified. We noticed that the remaining six research studies focused on general PTs’ geometry and measurement content, such as examining van Hiele levels of understanding for geometry. This new category is also shown in Table 4.

Table 4

*Correlation of Recommendations from CBMS Reports I and II and Geometry and Measurement Research*

	Historical Look	Current Perspective	View of the Horizon
Geometry			
Understanding geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; describing and reasoning about spatial locations (including the coordinate plane). (CBMS Report II)		Gutierrez & Jaime (1999); Cunningham & Roberts (2010)	Morgan & Sack (2011)
Classifying shapes into categories and reasoning to explain relationships among the categories. (CBMS Report II)		Fujita & Jones (2007); Pickreign (2007); Fujita (2011)	İymen, Pakmak, & Paksu (2011)
Basic shapes, their properties, and relationships among them: developing an understanding of angles, transformations (reflections, rotations, and translations), congruence and similarity. (CBMS Report I)			
Reason about proportional relationships in scaling shapes up and down. (CBMS Report II)		Gerretson (2004)	Morgan & Sack (2011)
Visualization skills: becoming familiar with projections, cross-sections, and decompositions of common two- and three-dimensional shapes; representing three-dimensional objects in two dimensions and constructing three-dimensional objects from two-dimensional representations. (CBMS Report I)	Bright (1979); Battista, Wheatley, & Talsma (1982, 1989)		
Communicating geometric ideas: learning technical vocabulary and understanding the role of mathematical definition. (CBMS Report I)			Patton & Parker (2011)

(continued)

Table 4—continued

	Historical Look	Current Perspective	View of the Horizon
General understanding of geometry	Mayberry (1983)	Halat (2008); Lin, Luo, Lo, & Yang (2011)	Köse & Özen (2011)
<b>Measurement</b>			
The general principles of measurement, the process of iterations, and the central role of units: that measurement requires a choice of measurable attribute, that measurement is comparison with a unit and how the size of a unit affects measurements, and the iteration, additivity, and invariance used in determining measurements. (CBMS Report II)	Bright (1985)		
The process of measurement: understanding the idea of a unit and the need to select a unit appropriate to the attribute being measured, knowing the standard (English and metric) systems of units, understanding that measurements are approximate and that different units affect precision, being able to compare units and convert measurements from one unit to another. (CBMS Report I)			
How the number line connects measurement with number through length. (CBMS Report II)			

(continued)



Table 4—continued

	Historical Look	Current Perspective	View of the Horizon
Understanding what area and volume are and by giving rationales for area and volume formulas that can be obtained by finitely many compositions and decompositions of unit squares or unit cubes, including formulas for the areas of rectangles, triangles, and parallelograms, and volumes of rectangular prisms. (CBMS Report II)	Enochs & Gabel (1984); Gabel & Enochs (1987); Baturu & Nason (1996); Reinke (1997)	Menon (1998); Zevenbergen (2005)	Schnorenberg & Chamberlin (2011); Morgan & Sack (2011)
Length, area, and volume: seeing rectangles as arrays of squares, rectangular solids as arrays of cubes; recognizing the behavior of measure (length, area, and volume) under uniform dilations; devising area formulas for basic shapes; understanding the independence of perimeter and area, of surface area and volume. (CBMS Report I)			
General understanding of measurement		Tsamir & Pitta-Pantazi (2008); Pitta-Pantazi & Christou (2009)	

Given the content recommendations from the CBMS Reports I and II, what have we learned from our summary of research? What have we learned about PTs' understanding of these topics across the years that would give us insights into the nature of this understanding? Examining the table, we do note gaps in the research literature of topics identified for the preparation of elementary mathematics teachers. There was no peer-reviewed published research found that specifically addressed the general principles of measurement or how the number line connects measurement with number. Not all components within each recommendation were addressed, leaving much to be investigated

regarding what we know about PTs' understanding of geometry and measurement. Yet many cells in the table show related work to the recommendations.

The total of 26 studies spans across a wide variety of topics within the content areas of geometry and measurement. Although it is encouraging that a variety of topics exist in the research literature, concentrated research effort is needed for targeted topics in order to have a better picture of PTs' understanding in geometry and measurement. Across the historical look, the current perspective, and the view of the horizon, PTs' general understanding of core ideas in geometry and measurement is limited and weak (Baturó & Nason, 1996; Cunningham & Roberts, 2010; Enochs & Gabel, 1984; Fujita, 2011; Fujita & Jones, 2007; Gutierrez & Jaime, 1999; Lin et al., 2001; Menon, 1998; Pickreign, 2007; Reinke, 1997; Zevenbergen, 2005), with PTs relying on procedural processes, recalled from the depth of their memory (Baturó & Nason, 1996; Enochs & Gabel, 1984).

Work from Battista et al. (1982, 1989) suggests cognitive development, along with spatial visualization skills, plays a greater role in learning geometry than memory skills, as many PTs purport. The van Hiele levels of geometric learning (van Hiele, 1959) provide a framework for helping those teaching geometry to think about the development of geometric ideas through stages and to provide experiences for the learners that engage them in thinking and reasoning (Mayberry, 1983; Morgan & Sack, 2011). Also, the use of dynamic geometry environments (DGE) might foster a more dynamic image of shapes and allow for visual and conceptual aspects of shapes to meaningfully coalesce when forming concept images and definitions, thus helping create a more effective learning environment for learners. Gerretson's (2004) study supports this suggestion, with PTs acquiring a greater knowledge base when using a DGE. Köse and Özen (2011) also found PTs engaging

in different problem-solving strategies in a DGE as compared to using paper and pencil and not being able to find solutions in that environment.

Yet even if PTs' content knowledge is strengthened, does it ensure successful teaching? Our research review did not examine work related to teaching geometry and measurement. However, Zevenbergen (2005) noted the importance of classrooms that demonstrate the values of the teacher preparation program in order to help sustain the dispositions developed in the program, as well as dispositions that put an emphasis on conceptual understanding and a developmental approach to learning that doesn't rush to a procedural rule.

Pickreign (2007) questions if there is sufficient time in a teacher preparation program for PTs to have the needed experiences to advance their learning to satisfactory levels of understanding. Further, Lin et al. (2011) note there is no definitive van Hiele level of understanding set as an expectation for all elementary PTs. So, what is a "satisfactory level of understanding"? If we do set minimal expectations, we return to our general question of how do we help PTs attain these expectations in the geometry and measurement courses? Fujita's work (Fujita, 2011; Fujita & Jones, 2007), based upon a synthesis of learning theories from van Hiele (1959), Tall and Vinner's (1981) concept definition, Fischbein's (1993) figural concepts, personal and formal figural concepts (Fujita & Jones, 2007), dynamic figural concepts (Walcott, Mohr, & Kastberg, 2009) and Hershkowitz's (1990) prototype phenomenon of geometrical figures, provided some suggestions for learning opportunities, specifically for understanding inclusion relations for quadrilaterals, to help PTs move beyond simply memorizing procedures and relying solely on personal figural concepts. It involves helping learners identify their

misconceptions, clarifying definitions of shapes, applying relationships between shapes that they understand to other situations, and using definitions to further reflect on properties of shapes. Others (Cunningham & Roberts, 2010; Gutierrez & Jaime, 1999) have similar findings from their collective work, highlighting the importance of the appropriate development of concept images and concept definitions. As the CBMS (2001) observes,

The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they do know—the mathematical ideas they hold, the skills they possess, and the context in which these are understood—so they can move from where they are to where they need to go. . . . And this is where the mathematics courses for elementary school teachers must begin. (p. 17)

This quote from CBMS highlights how readers can use information from our summary of PTs' geometry and measurement content knowledge based upon peer-reviewed research published over the past 20 years. With respect to the design of curriculum, we see a need for well-designed, engaging geometric and measurement experiences that (a) further the content understanding of PTs, moving them beyond a focus on procedural and memorization skills; (b) further develop PTs' spatial visualization but also help PTs develop other geometric problem-solving skills, such as drawing; (c) focus on developing PTs' concept definitions of shapes and their properties; (d) still engage the PTs at beginning van Hiele levels of understanding, rather than assuming all PTs can initially engage with thinking at advanced levels; and (e) incorporate the use of dynamic geometry software to further develop reasoning skills.

Our summary work also has indicated areas of more and needed future research, such as research focusing on (a) *how* PTs develop their content knowledge using technology; (b) determining a satisfactory level of geometry (and measurement) understanding for PTs; and (c) addressing the gaps to the content expectations from the

MET I and II documents (CBMS, 2001, 2010) as shown in Table 4, with measurement showing the greatest need for further study. (See the final paper of this Special Issue for several areas of future research common across the other content areas summarized.) Such future work that builds upon what we know regarding the geometry and measurement content knowledge of PTs can help us strengthen our existing content preparation programs to develop the independent mathematical thinkers future elementary teachers need to be.

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:  
A Focus on Algebra**

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**ABSTRACT:** As part of a recent effort to summarize research-based knowledge of prospective elementary school teachers' (PTs) mathematical content knowledge, this paper summarizes research literature on PTs' knowledge of algebra, focusing on the range of years from 1998 through 2012. The 21 papers included in this summary focus on a broad range of topics within algebra, such as (a) producing, representing, and justifying generalizations; (b) interpreting and using algebraic symbols; (c) solving algebraic word problems; and (d) understanding functions. Looking across this body of research, three themes are identified: (1) PTs generally have strong procedural skills and can make mathematically sound generalizations of many different types of patterns; (2) however, PTs tend to struggle to (a) interpret and effectively use algebraic symbols, even those that they have produced themselves; (b) interpret graphical representations; and (c) make connections between representations; and (3) PTs have limited algebraic problem-solving strategies, often relying, inflexibly, on inefficient and/or incorrect computational methods. Suggestions for future research directions are discussed.

**Keywords:** algebra, algebra content knowledge, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

## **Background and Introduction**

In recent decades, algebra has become infamous as a gatekeeper of success in school mathematics (Cai et al., 2005; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Stephens, 2008). Moses and Cobb (2001) underscored the importance of algebra by making a comparison between people who lack an education in algebra today to “the people who couldn’t read and write in the Industrial Age” (p. 14). Unfortunately, though, it has been documented repeatedly that many students struggle when they reach algebra in middle school or high school (e.g., Kenney & Silver, 1997).

In response to this phenomenon, members of the mathematics education community have called for the inclusion of algebra content in the elementary school curriculum, with the goal of removing the abrupt, often derailing transition from arithmetic to algebra by infusing algebraic ideas into instruction in the elementary and intermediate grades (Kaput, 1998). For example, the National Council of Teachers of Mathematics (NCTM, 2000) suggests incorporating algebra into elementary level curricula:

By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school. (p. 37)

Research also lends support for the notion of including algebraic ideas in elementary school curricula (e.g., Britt & Irwin, 2008; Schliemann et al., 2003). In her research brief on algebra, Kieran (2007) concludes that the current body of research “emphasizes that arithmetic can be conceptualized in algebraic ways” and “this emphasis can be capitalized on to encourage young students to make algebraic generalizations without necessarily using algebraic notation” (p. 1). Thus, algebra can be infused into arithmetic instruction in a way that is appropriate for elementary-aged children.

Accordingly, algebra topics have been included in recent standards documents as an essential component of the elementary mathematics curriculum. The NCTM's *Principles and Standards for School Mathematics* (2000) states that students in all grades should develop their understanding of the following algebraic ideas:

- understanding patterns, relations, and functions;
- representing and analyzing mathematical situations and structures using algebraic symbols;
- using mathematical models to represent and understand quantitative relationships; and
- analyzing change in various contexts. (p. 37)

More recently, in the *Common Core State Standards* (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010), the Operations and Algebraic Thinking content domain begins in kindergarten and continues through fifth grade, progressing from a focus on understanding properties of, and having flexibility with, the four basic operations, toward a focus on generalizing, describing, and justifying patterns and relationships, and interpreting symbolic expressions.

In light of these standards, it is evident that elementary school teachers are responsible for facilitating their students' development in algebraic concepts, and, therefore, they need to have a deep understanding of the foundations of algebra themselves (Hill, Rowan, & Ball, 2005; Ma, 1999). Moreover, members of the mathematics education community support the notion "that there is a powerful relationship between what a

teacher knows, how she knows it, and what she can do in the context of instruction” (Hill, Blunk, et al., 2008, p. 498).

Thus, the mathematical education of PTs in algebra is of critical importance to the quality of the mathematical education of children. This is reflected in the recently updated recommendations of the Mathematical Education of Teachers II (METII; Conference Board of the Mathematical Sciences, 2012) report, which states that kindergarten through Grade 5 teachers need to be able to “[recognize] the foundations of algebra in elementary mathematics” (p. 26). As an example of this, the report gives the following as an illustrative activity for the mathematical preparation of elementary teachers in algebra: “Explain how to solve equations such as  $283 + 19 = x + 18$  by ‘thinking relationally’ (e.g., by recognizing that because 19 is 1 more than 18,  $x$  should be 1 more than 283 to make both sides equal) rather than by applying standard algebraic methods” (p. 26). Further, the METII recommends that half of PTs’ mathematical preparation should focus on “number and operations, treated algebraically with attention to properties of operations,” with the other half focused on “additional ideas of algebra (e.g., expressions, equations, sequences, proportional relationships, and linear relationships)” (p. 31), along with geometry and measurement and data.

With these recommendations in mind, it is also critical that the mathematical instruction of PTs is built on their currently held knowledge (Bransford, Brown, & Cocking, 1999). Thus, we need to know what PTs’ currently held knowledge of algebra is, how it changes, and how it develops, so that mathematics educators can appropriately tailor instruction. Accordingly, summarizing the current state of research on PTs’ knowledge of algebra is the goal of this article. In particular, the following questions guide our summary:

1. What research has been conducted on elementary PTs' knowledge of algebra?
2. What is known from this research about elementary PTs' algebra content knowledge?
3. What about elementary PTs' algebra content knowledge remains as-of-yet unstudied?

We address these questions in three sections, organized chronologically according to date of publication:

- First, a historical look, in which we summarize findings of research published in peer-reviewed journals prior to 1998;
- Then, a current perspective, in which we summarize findings of research published in peer-reviewed journals between 1998 and 2011;
- Finally, a view of the horizon, in which we summarize findings of research published in peer-reviewed journals in 2012, along with findings of research published in the 2011 or 2012 proceedings of the meetings of the International Group for the Psychology of Mathematics Education (PME), or in the 2011 or 2012 proceedings of the meetings of the North American Chapter of the Psychology of Mathematics Education (PME-NA).

### **Methods**

Our search for studies to include in this summary closely followed the general methods guiding all content areas for the larger summary project of which this algebra-specific summary is a part (see introductory article to this Special Issue). In particular, we conducted many searches of the ERIC database using combinations of the following search terms: *elementary education, elementary, education, preservice teacher, pre-service teacher,*



*prospective teacher, algebraic thinking, algebra, function, symbolic, equation, commutative, associative, distributive, rate of change, patterns, factoring, inequalities, generalization, generalized arithmetic, and graphs.*

As we conducted these searches, we realized that there were unique circumstances regarding the search for published literature on PTs' content knowledge of algebra, in particular, that warranted special methodological considerations. Specifically, (a) we needed to establish a definition of *algebra* for this summary, that aligned with the overall purposes of this project; and (b) after conducting our initial searches of the database, we agreed on one exclusionary criterion in addition to those of the larger summary project. We describe both of these considerations below.

### **Definition of Algebra**

Although algebra is now a major component of mathematical standards for grades K–12, what algebra is, exactly, is debated within the mathematics education community. (See Stephens, 2008, for a succinct review of definitions of algebra.) For the purposes of this summary, we put together an inclusive definition of algebra with an eye toward elementary and middle grades content. Drawing on the ideas of many mathematics educators (e.g., Kaput, 1998; Kieran, 1992; National Research Council, 2001), we conceptualized algebra to be content focused on pattern generalization, arithmetical generalization, algebraic symbolization, functions, proportional reasoning, or problem solving when the problems are not amenable to arithmetic strategies. Additionally, we chose to include studies in which the focus was on properties of the arithmetic operations; however, we excluded studies in which the focus was on PTs' understanding a specific type of number (e.g., decimal numbers) in the context of an operation, because these studies are

well-addressed in other articles in this issue. We feel that this conceptualization of algebra satisfies the criteria of being (a) broad and inclusive so as not to unnecessarily exclude studies from the summary, and (b) appropriate for discussing the content knowledge of prospective teachers of children ages 3 through 14.

### **Inclusion/Exclusion Criteria**

This project includes all research published in peer-reviewed journals on the Common and/or Specialized Content Knowledge (Ball, Thames, & Phelps, 2008; Hill, Ball & Schilling, 2008) of elementary PTs. However, as explicated in the introductory article to this Special Issue, our search of the literature adhered to these four exclusionary criteria:

1. We excluded studies that lacked specific attention to algebra.
2. We excluded studies that focused primarily on PTs' perceptions about mathematics or beliefs.
3. We excluded research that focused on describing classroom practice or activities for PT education courses with lack of attention to research design methods.
4. We excluded studies in which the populations primarily consisted of high school PTs, mathematics majors, or inservice elementary teachers.

Additionally, we added to the exclusionary criteria to also exclude studies that focused solely on secondary-level content knowledge of middle grades PTs. To come to this decision, we first revisited the reasoning for including studies on middle grades PTs in the larger summary project. As stated in the introductory article to this Special Issue, "We decided to look at findings from studies of PTs preparing to teach children aged 3–14 to account for cases with combined middle and elementary certifications." The intent of the overall project, then, was to include studies that have a population of middle grades PTs for

the purpose of avoiding inadvertently excluding relevant research on elementary PTs. We decided, though, that some studies that focused on the algebra content knowledge of solely middle grades PTs had a distinctly secondary feel and therefore did not fit with the aim of the larger project to summarize research on elementary PTs. As an example, we excluded a study that focused on prospective middle grades teachers' knowledge of rational functions. Using the *Common Core State Standards* (CCSS) (NGA & CCSSO, 2010) as a guide for determining what counted as secondary level (i.e., high school level) algebra, our fifth exclusionary criterion was as follows:

5. We excluded studies in which both (a) the population was entirely middle grades PTs, and (b) the mathematical knowledge studied was at a secondary school level.

#### **A Historical Look: Prior to 1998**

Using the criteria outlined above, we identified only one research study that was published prior to 1998 on PTs' content knowledge of algebra (see Table 1 below). Moreover, a supplementary search through the reference sections of research published between 1998 and 2011 did not yield any additional studies.

Table 1

*Information on the Article Published Before 1998 on the Topic of PTs' Algebra Content Knowledge*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Schmidt & Bednarz	1997	66 elementary, 65 secondary, 33 special education	PTs at the beginning of the certification program	Canada	Written survey

The one study we found (Schmidt & Bednarz, 1997) explored Canadian elementary ( $n = 66$ ), secondary ( $n = 65$ ), and special education ( $n = 33$ ) PTs' strategies for solving arithmetic and algebraic word problems.<sup>8</sup> The authors define arithmetic problems as those that are amenable to arithmetic solutions, where the solver can work from known information to find the unknown. They define algebraic problems as those that are not amenable to working from known values to find an unknown value, where the solver must directly work with unknown quantities. In the study, PTs' solution strategies for solving algebraic and arithmetic problems were categorized as either algebraic or arithmetic, regardless of which type of problem was being solved. A solution was considered algebraic if it satisfied both of the following criteria: (a) the solution contains at least one equation wherein known and unknown values are related to each other, and (b) the answer is found via transformation of the equation(s) and operating on the unknowns without choosing specific values for the unknowns. All other solutions to either type of problem were

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<sup>8</sup> Schmidt and Bednarz (1997) use the terms *connected* and *disconnected* problems instead of *arithmetic* and *algebraic* problems, respectively. We choose to use the terms *arithmetic* and *algebraic* to align with the terms used in more recent research discussed later in this article.

considered arithmetic. Additionally, arithmetic solutions were categorized as either *guess-and-check* or *manipulating-the-structure*. An example of each type of solution, for each type of problem, appears in Table 2 below.

Table 2

*Examples of Arithmetic and Algebraic Problem Types, and Examples of Algebraic and Arithmetic Solution Strategies for Each Problem Type (van Dooren, Verschaffel, & Onghena, 2003)<sup>9</sup>*

	Arithmetic Problem	Algebraic Problem
	A primary school with 345 students has a sports day. The students can choose between in-line skating, swimming and a bicycle ride. Twice as many students choose in-line skating as bicycling, and there are 30 fewer students who choose swimming than in-line skating. 120 students want to go swimming. How many chose in-line skating and bicycling?	372 people are working in a large company. There are 4 times as many laborers as clerks, and 18 clerks more than managers. How many laborers, clerks, and managers are there in the company?
<b>Algebraic Solutions</b>	$i = \text{in-line skating}$ $345 = i + i/2 + i - 30$ $690 = 2i + I + 2i - 60$ $750 = 5i$ $150 = i$  In-line skating: 150, swimming: 120, bicycle riding: 75	Number of managers = $x$  $x + (x+18) + 4(x+18) = 372$ $6x = 372 - 90$ $6x = 282$ $x = 47$  There were 65 clerks ( $47 + 18$ ), and 260 laborers ( $4 \times 65$ ).

(continued)

<sup>9</sup> The article by Schmidt and Bednarz (1997) is written in French. In order to avoid translation issues, the examples in this table are taken from the van Dooren, Verschaffel, and Onghena (2003) study, which uses the same problem type and strategy type categorizations as Schmidt and Bednarz's study does, but is written in English.

Table 2—continued

	Arithmetic Problem	Algebraic Problem
<b>Arithmetic Solutions</b>	<b>“Manipulating the Structure”</b>	
	<p>Let us assume that the number of students who swim equals the number of in-line skaters. The total augments by 30 to 375. This total is divided into 5 groups:            2 groups in-line skaters            2 groups swimmers (in fact 30 fewer)            1 group bicycle riders            Each group consists of 75 students.            Thus 75 students went for the bicycle ride, 150 chose in-line skating and 120 swimming.</p>	<p>Let us suppose for a moment that there are 18 managers more (i.e. as many managers as clerks). Then the total of people = 390. This total consists of 6 equal parts:            4 parts laborers            1 part clerks            1 part managers            Each part consists of <math>390/6 = 65</math> people. So there are 65 clerks, 260 laborers, and 47 managers (65-18).</p>
	<b>“Generating Numbers”/“Guess and Check”</b>	
	120 students went swimming	Suppose the number of clerks is 80,
	$120 + 30 = 150$ students went in-line skating	then there are 62 managers and 320 laborers, which gives 462 people total Too many.
	$150/2 = 75$ students went for the bike ride	Suppose the number of clerks is 60...(Student continues in this way until finding correct numbers.)

Analyses of a written survey that included three arithmetic problems and three algebraic problems suggest that, of the three groups of PTs (elementary, special education, and secondary mathematics), elementary level PTs were the most flexible in their strategy selection, with 58.5% using one of the arithmetic strategies to solve arithmetic problems, and 61.5% using algebraic strategies to solve algebraic problems (Schmidt & Bednarz, 1997). However, follow-up interviews suggest that all groups of PTs displayed difficulties surrounding arithmetic and algebraic problem solving. Most notably, algebraic symbols

(i.e., variables used to translate from a story problem to algebraic equations) were used to stand in for information in the problem (e.g., “ $M$ ” for the amount of money that Marie has), yet the equations did not necessarily correctly describe relationships between quantities. For example, to describe a situation in which Marie had 15,000 more of something than Chantal, one preservice teacher wrote  $M + 15,000 = Chantal$ , which represents the inverse relationship. This difficulty has been identified in previous research with other populations (cf. the Students and Professors problem in Clement, Narode, & Rosnick, 1981).

Moreover, Schmidt and Bednarz (1997) note that PTs struggled to use their equations to solve problems. For example, one PT assigned variables to all of the unknown quantities in the problem, yet when the PT reached an answer, there was still an  $x$  in it. The authors also describe another PT who used variables to stand for unknown numbers but then substituted numbers in place of those variables in order to solve. In this way, even though the PT used algebraic symbolism, the individual seemed to be thinking arithmetically with a “guess-and-check” strategy. In light of these difficulties, the authors express concern regarding the elementary PTs’ preparedness to help students transition from arithmetic to algebraic reasoning.

Apart from this study, there were no studies published about elementary PTs’ content knowledge in algebra prior to 1998. Fortunately, the roughly 14 years that followed (1998 to 2011) saw an increase in the number of studies on elementary PTs’ content knowledge of algebra. This research is summarized in the next section.

### **A Current Perspective: 1998 to 2011**

Our initial search yielded 18 potential articles to be included in our summary of peer-reviewed research papers published on PTs’ content knowledge of algebra between

1998 and 2011. Sixteen of these papers met our inclusion criteria (see Table 3). Because this collection of papers spans a wide range of topics within algebra, we grouped the research into four non-mutually-exclusive content-themed sections, with the understanding that the research could be grouped differently and that the content foci of the different sections clearly overlap. Accordingly, we present findings of the current research in the following four sections: (a) producing, representing, and justifying generalizations; (b) interpreting and using algebraic symbols (in contexts other than generalization tasks); (c) solving algebraic word problems; and (d) understanding functions.

Table 3

*Articles Published Between 1998 and 2011 on the Topic of PTs' Algebra Content Knowledge, in Alphabetical Order by First Author*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Berk, Taber, Carrino, Gorowara, & Poetzl	2009	148	First-year PTs in the second of three mathematics courses for prospective K-8 teachers	USA	Data from written pretest, posttest, delayed posttest, and individual interviews
Billings & Klanderma	2000	19	Undergraduate juniors (3rd year students) in a combination content/methods course; K-8 teachers	USA	Copies of student work, field notes
Briscoe & Stout	2001	106	Undergraduate seniors (final semester of school before student teaching) in a methods course	USA	Transcripts of video presentations, class discussion, and copies of documents produced by students (lab report and overhead transparencies)

(continued)



Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Meel	1999	29	PTs at the end of their teaching certification program	USA	Written assessment given prior to unit, in the middle of the course
Nillas	2010	5	Elementary and special education majors; point in program is unclear	USA	Data from 3 written test items
Otto, Everett, & Luera	2008	72	Undergraduate science majors in a required capstone course	USA	Copies of student work, instructors' notes, observations of class activity
Pomerantsev & Korosteleva	2003	119	Elementary and middle grades PTs in various content courses for future educators	USA	Written survey of 5 questions
Prediger	2010	45	Second year middle school PTs, at the beginning of an instructional unit	Germany	Written survey during class, class observation and video data
Richardson, Berenson, & Staley	2009	25	PTs in a methods course, immediately before student teaching	USA	Audio recordings, observational notes, all student work (teaching experiment methodology)
Rivera & Becker	2007	42	PTs in an introductory course for elementary mathematics teachers	USA	Clinical interview data
Schmidt & Bednarz	2002	8, specifically selected based on prior written survey	Undergraduate PTs in an introductory course to a teacher education program	Canada	Data from pair interviews

(continued)

Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Stylianou, Smith, & Kaput	2005	28	PTs in a mathematics course for elementary teachers; point in program is unclear	USA	Video of group, task-based pretest and posttest interviews, and video of all class sessions
van Dooren, Verschaffel, & Onghena	2003	45 first year, 52 third year	Comparison of PTs in the beginning and end of teacher education program	Belgium	Data from a written survey
You & Quinn	2010	104	Last stages of a study of PTs in a teacher education program	USA	Data from a 15-item written survey
Zazkis & Lildejahl	2002a	20	PTs in a core course for elementary PTs; after topic was covered	Canada	Clinical interview data
Zazkis & Lildejahl	2002b	36	Unclear	Canada	Student journals of mathematical investigations and follow-up interview data

### Producing, Representing, and Justifying Generalizations

In this section, we focus on research that explores PTs' generalizations of patterns or generalizations of physical phenomena. Our search of the literature yielded six such papers. Within this small collection of research, there are studies that focus on a range of aspects of PTs' generalizations, including producing and representing generalizations, connecting those representations, and producing justifications.

The findings of all six studies support the notion that there are many situations in which PTs can produce correct<sup>10</sup> generalizations. With respect to generalizing visual patterns, the findings of Richardson, Berenson, and Staley's (2009) teaching experiment were that 23 of 25 PTs, working in pairs, found a correct explicit rule to describe the perimeter of the  $n$ th figure in a train of squares, equilateral triangles, or regular hexagons (see Figure 1 for an example). Similarly, 35 of the 42 PTs working individually during clinical interviews in Rivera and Becker's (2007) study produced a correct generalization for the number of dots in a growing pattern of dots arranged in a square (i.e.,  $n^2$  dots for the  $n$ th figure, as in Figure 2).

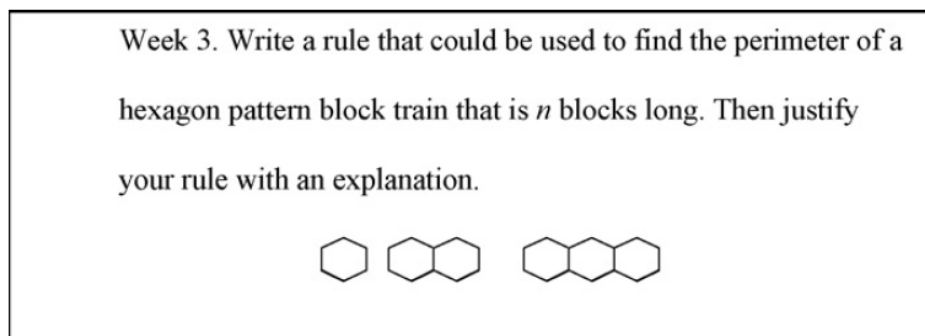
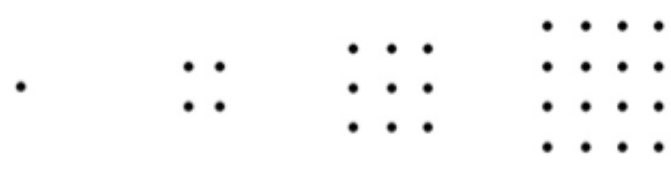


Figure 1. A task from Richardson and colleagues' (2009) teaching experiment (p. 190).

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<sup>10</sup> We use the term *correct* to reflect the way generalizations are discussed in the research we reviewed. This research was grounded in the idea that there exist generalizations of patterns that are more correct, natural, and/or mathematically sound than others.

Consider the manner in which the dots below are used to form squares.



Number of arrays	1	2	3	4
Number of dots	1	4	9	16

a. Draw the next two squares and determine the total number of dots you need to form each of them.

b. How many dots do you need to form a square with  $n$  arrays?

Figure 2. A task from Rivera and Becker's (2007) study (p. 144).<sup>11</sup>

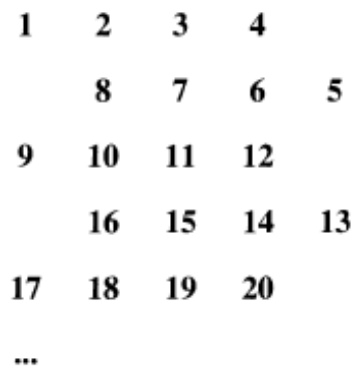
With respect to generalizations of physical phenomena, 32 of the 49 small groups of three or four PTs across two studies produced correct verbal and symbolic generalizations in the form of an equation, of an observed relationship modeled by a Class 1 lever<sup>12</sup> (Briscoe & Stout, 2001; Otto et al., 2008).

With respect to generalizations of arithmetic sequences of integers, findings from clinical interviews conducted in Zazkis and Liljedahl's (2002a) study of 20 Canadian PTs enrolled in a core mathematics course for elementary PTs, suggest that most PTs can easily

<sup>11</sup> In this task, Rivera and Becker (2007) use the word *array* in a way that is different from our understanding of the word. For the pattern in Figure 2, we think of an array as an entire square of dots. Rivera and Becker's use of the word *array* seems to be synonymous with the word *row* or *column*.

<sup>12</sup> In a Class 1 lever, the product of a mass ( $M_1$ ) and its distance from the fulcrum ( $D_1$ ) on one end of the lever is equal to the product of a second mass ( $M_2$ ) and its distance from the fulcrum ( $D_2$ ) on the other end of the lever;  $M_1D_1 = M_2D_2$ , or equivalently,  $M_1/D_2 = M_2/D_1$ .

recognize the underlying structure of arithmetic sequences of multiples (e.g., 7, 14, 21, 28, etc.). For example, PTs were able to produce the 712th element of the sequence by multiplying 712 by the common difference between consecutive elements. Findings of another study of Zazkis and Liljedahl's (2002b) exemplified that PTs can make correct generalizations, usually expressed verbally, in response to a rich visual number pattern (see Figure 3). The 36 Canadian PTs in the study were asked to journal about their mathematical investigations of the numerical pattern for 2 weeks, for periods of at least 30 minutes every other day. Based on this journal data, and follow-up interviews with four of the participants, Zazkis and Liljedahl concluded, "Participants engaged in detecting sameness and differences, in classifying and labeling, in seeking algorithms, in conjecturing and argumentation, in establishing numerical relationships among components or, more generally, in generalizing about data and mathematical relationships" (p. 399)—all demonstrations of algebraic thinking through generalization.



Task prompt: In general, given any whole number, how can one predict where it will appear in this pattern? Explain the strategy that you propose.

*Figure 3.* A visual arrangement of a sequence of numbers from a task used in one of Zazkis and Liljedahl's studies (2002b, p. 383).

In contrast to these examples of successful generalizations, however, research has also identified problem situations with which PTs struggle to produce a correct generalization, represented symbolically or otherwise. One such situation is generalizing arithmetic sequences of non-multiple integers (e.g., 8, 15, 22, 29, etc.). Of the 20 PTs in Zazkis and Liljedahl's (2002a) study, 9 indicated that they believed multiples of the common difference between consecutive elements would generate new elements in a non-multiple sequence. For example, the common difference between the elements of the sequence 8, 15, 22, 29, etc., is 7, so the PT might indicate that 7 times 712 is an example of a member of the sequence. Two of the 20 PTs realized that multiples of the common difference were not elements of non-multiple arithmetic sequences, yet they indicated that any non-multiple was a potential element of the sequence. For example, a PT might acknowledge that 70 is not an element of the sequence 8, 15, 22, 29, etc., because 70 is a multiple of 7, but the PTs might also indicate that 75 might be in the sequence because it is not a multiple of 7. Zazkis and Liljedahl conclude that these PTs tend to interpret non-multiple arithmetic sequences as being "sporadic" (p. 116), or lacking any discernable pattern.

Even PTs with seemingly mathematically mature responses to Zazkis and Liljedahl's (2002a) generalization tasks sometimes seemed to lack an understanding of the underlying multiplicative structure of the sequences. For example, one PT correctly generalized the sequence 15, 28, 41, 54, etc., by stating that "the constant difference in the sequence is 13, and any number of the sequence is going to be a multiple of 13 plus 15" (p. 109), yet when he was asked if 1,302 was in the sequence, the PT indicated, incorrectly, that it was not, since "1,300 is . . . a multiple of 13, and that 1,302 is . . . only 2 away from that" (Zazkis &

Liljedahl, 2002a, p. 109). Zazkis and Liljedahl suggest that the PT incorrectly rejected 1302 as an element of the sequence because he was thinking of the sequence formulaically, instead of having a more developed understanding of the invariant multiplicative structure of non-multiple sequences, such as “multiples adjusted” (p. 110). Zazkis and Liljedahl conclude that an individual PTs’ *additive* and *multiplicative schemes* (Vergnaud, 2004) seem to develop dynamically through the identification of differences and invariants in problem situations.

In the other aforementioned study of Zazkis and Liljedahl’s (2002b), the authors found a common tendency among PTs to be satisfied with disjunctive generalizations, instead of looking for generalizations that captured the structure of a pattern as a whole. The authors found that many<sup>13</sup> PTs in their study searched for patterns in a piecemeal fashion, column by column, instead of recognizing the invariant unit-of-repeat of the pattern, which was 8. Moreover, those PTs that did search for a unit-of-repeat often focused on less mathematically salient units-of-repeat in the pattern, such as 4, 40, or 100, instead of 8. Although some of the PTs’ findings were potentially algebraically useful, the PTs often failed to recognize that potential.

Research (Richardson et al., 2009; Rivera & Becker, 2007) also suggests that PTs struggle to justify their own symbolic generalizations, regardless of the type of pattern they are generalizing. For example, in Richardson and colleagues’ (2009) teaching experiment, 23 of 25 U.S. PTs found a correct explicit rule to describe the perimeter of at least one of the pattern block trains (squares, triangles, or hexagons), but they struggled to justify their

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<sup>13</sup> Exact counts are not given in this article, as the authors’ focus is on describing and exemplifying ways in which PTs approached the generalization task.

generalizations. Based on their teaching experiment data, the authors describe a five-level framework that characterizes PTs' levels of success with justifying a generalization of a train of regular polygons (see Figure 4). At the lowest level, PTs generalize a recursive rule with no justification of the coefficient or  $y$ -intercept, relying on observations of numerical growth. As the PTs attain higher levels of justification, they are able to justify various portions of an explicit formula until they reach level 4, where a PT is able to successfully generalize a rule and justify the coefficient and the  $y$ -intercept using the model. The framework suggests that PTs gradually develop the ability to justify formulas for linear figural patterns through the process of working in small groups on pattern justification tasks. The authors propose that four features of the tasks used in their teaching experiment contributed to the PTs' development through the levels: (a) the linear and geometric (in a visual sense) nature of the patterns, (b) the use of physical pattern block manipulatives, (c) the isomorphism between the tasks, and (d) the use of tasks that promotes discourse among small groups of PTs, creating communities of ideas.

0	Generalizes a recursive rule with no justification of the coefficient and $y$ -intercept
1	Generalizes an explicit rule with no justification of the coefficient and $y$ -intercept
2	Generalizes an explicit rule with partial or faulty justification of the coefficient and $y$ -intercept
3	Justifies the coefficient and $y$ -intercept and generalizes an explicit rule inconsistently or inefficiently
4	Generalizes an explicit rule and justifies the coefficient and $y$ -intercept

*Figure 4.* Richardson and colleagues' (2009) five-stage framework for PTs' generalizations of linear figural patterns (p. 197).

In another study of PTs' justifications of geometric patterns, Rivera and Becker (2007) focus on the relationship between justification and the stage of generalizing



wherein explanatory inferences are made.<sup>14</sup> They suggest that the findings of their study point to the relationship between the types of representational cues (either figural or numerical) of the pattern that the PTs use to produce a hypothesized formula, and the PTs' ability to justify the formula. Specifically, PTs relying on visual cues – while generally not able to produce as many strategies as PTs relying on numerical cues – were more often able to justify the viability of their generalization. For example, PTs, who generalized a pattern by focusing on the common difference between the numbers generated by successive figures in the pattern (numerical cues) seemed to be unable to link the generalization back to the geometric pattern. As another example, PTs who used the trial and error until they found a formula that worked with the numbers generated by the pattern, struggled to justify why their generalizations worked with the geometric pattern. By contrast, PTs who generalized geometric patterns by focusing on the figural cues of the pattern were more successful in justifying their generalizations because they were more readily able to connect them to the underlying structure of the pattern.

Findings from Zazkis and Liljedahl's (2002b) analyses of elementary PTs' generalizations in response to the numerical pattern task in Figure 3 above suggest that PTs may struggle to make connections between their symbolic and verbal representations of their own generalizations. Zazkis and Liljedahl noted that, in the rare instances when PTs produced correct verbal and symbolic generalizations, there was often no evidence to suggest that the PTs saw connections between them. For example, one PT in their study

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<sup>14</sup> Rivera and Becker (2007) use the term *abduction* to describe PTs' explanatory inferences, and they conceptualize generalization as an abduction–induction process. We use the broad term *generalization* here to create cohesion with the rest of our summary. For a detailed theoretical discussion of the roles of abduction, induction, and deduction in the process of generalizing, we recommend referring to Rivera and Becker's report.

symbolized her generalization of a pattern as  $1 + 8r$ , yet the same PT later seemed excited to realize that the numbers in the same pattern were “one more than the multiples of eight” (p. 393). Zazkis and Liljedahl point out that this demonstrates a lack of understanding on the PT’s part of the meaning of the algebraic symbols she herself had generated. In light of this, Zazkis and Liljedahl suggest, “Neither the presence of algebraic notation should be taken as an indicator of algebraic thinking, nor the lack of algebraic notation should be judged as an inability to think algebraically.”

Indeed, looking across the studies on PTs’ generalizations, it seems that connections between representations lies at the heart of many PTs’ difficulties related to generalizations. On the whole, the research summarized above lends support to the idea that PTs are usually able to successfully generalize patterns, either verbally or symbolically (Briscoe & Stout, 2001; Otto et al., 2008; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b). However, PTs’ struggles interpreting or connecting representations are well-documented. In particular, PTs struggle to connect their symbolic generalizations back to the original patterns (Richardson et al., 2009; Zazkis & Liljedahl, 2002b), particularly when the generalization was produced only from numerical cues instead of visual cues (Rivera & Becker, 2007). PTs also struggle to connect their own verbal generalizations to their own symbolic generalizations, or to leverage their own observations of patterns into more complete or correct generalizations (Zazkis & Liljedahl, 2002a, 2002b).

Further research is needed to document how PTs develop in their ability to interpret and connect representations of their generalizations. Only one of the six above studies (Richardson et al., 2009) compiled a developmental framework for PTs’

justifications of their generalizations, yet this framework is limited, because it is developed out of data taken from observations of small groups of students from only one class, and it applies only to generalizations of linear visual patterns. The other studies were either studies of PTs at one time point, or were more exploratory, documenting examples of PTs' knowledge of generalization. Thus, further research on the ways in which PTs overcome their struggles connecting, interpreting, and justifying generalizations, across various situations, is needed.

### **Interpreting and Using Algebraic Symbols**

In this section, we discuss two studies that focus on PTs' interpretations and procedural use of algebraic symbols (e.g., expressions that include variables, and the equal sign) in contexts other than generalization tasks. Findings from these two studies are summarized below.

Using a framework of three meanings for the equal sign, Prediger's (2010) study explored the collective development of a class of second-year middle school-level PTs in Germany. The three meanings of the equal sign are as follows: An *operational* understanding of the equal sign is defined as a conceptualization of the symbol as a signal to "do something" or as something that separates a problem from its answer. A *relational* meaning, on the other hand, is a conceptualization of the equal sign as a symmetric indicator of equality or a formal equivalence describing equivalent terms. Finally, *specification* refers to the equal sign as a symbol to indicate a definition. Ideally, according to Prediger, PTs will have a flexible understanding of the equal sign that includes all three interpretations.

At the beginning of an instructional unit in Prediger's (2010) study, a class of 45 middle school PTs were asked to determine if and why given chains of equal signs were mathematically correct or incorrect, when presented as examples of children's work (see Figure 5). From written data collected during the first task, Prediger created four representative profiles for PTs' understandings of the equal sign. PTs that fit the first profile<sup>15</sup> reproduced Emily's misconception, stating that Lisa's use of the equal sign was incorrect. PTs that fit the second profile did not seem to try to understand Emily's perspective in their response to the task. PTs that fit the third and fourth profiles seemed to have a mathematically correct understanding of the equal sign, but struggled to fully understand or appropriately respond to Emily's confusion about Lisa's use of the equal sign.

Building off these representative profiles, Prediger (2010) implemented three in-class tasks, in addition to the one above, designed to help PTs progress in their understanding of the equal sign. The first task presented the PTs with a video in which a student understood the equal sign operationally. The teacher in the video explained to the student that mathematicians "get nervous where there is not the same [amount] on the left and the right side [of the equal sign]," to which the student responded that she is "not a mathematician" (Prediger, 2010, p. 88). The PTs were asked to discuss this video, and Prediger suggests that, as a result of this discussion, the PTs explicated the notion of both an operational use and relational use of the equal sign. Moreover, PTs raised the questions "Why can't we allow different notations in different contexts? Why don't we allow chain

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<sup>15</sup> Prediger (2010) did not include any counts to indicate the number of PTs that fit each of the four profiles.

notation in arithmetic contexts, and forbid them in algebraic contexts?" (p. 88). To help PTs explore these questions, Prediger designed the second task, wherein the PTs were presented with a pool of examples of the equal sign being used in a variety of ways, and they were asked to consider the purpose of the equal sign in each situation and determine how they were similar or different from the others. In the final task, the PTs were asked to look for changes in the meaning of the equal sign in a sample step-by-step solution to a problem involving perimeter, area, and derivatives.

Situation: In order to study strategies for flexible mental arithmetic, students in grade 5 were asked to solve the following task:  
Lisa calculates  $24 \times 7$  by decomposing:

$$24 \times 7 = 20 \times 7 + 4 \times 7 = 140 + 28 = 168$$

- i) Did she calculate correctly? How would you have done it?
- ii) Calculate  $54 \times 6$  like Lisa did.

Emily (age 10) is skeptical: "Lisa calculates wrong. 24 times 7 does not equal 20! And what is that after the 20?" Due to her difficulties with the unfamiliar symbolic representation, Emily does not continue with the task although she usually uses the same strategy of decomposing  $24 \times 7$  into  $20 \times 7$  and  $4 \times 7$ .

Questions posed to PTs:

- a) What does Emily mean?
- b) Which view is right?

*Figure 5. A response-analysis-type question about the equal sign from Prediger's study (2010, p. 76).*

At the end of the course, students were able to choose to complete four of five problems on a final exam. Twenty-seven of the 41 PTs completed a problem that involved an analysis of the uses of the equal sign, and of those, 24 were successful in their analysis, according to Prediger (2010), suggesting that the PTs likely deepened their understanding of the equal sign. Moreover, Prediger's analyses of PTs' responses on an end-of-unit

assessment suggest that the PTs were better able to discern when to use operational or relational definitions of the equal sign after the instructional sequence. Prediger attributes these changes in understanding to the fact that the in-class tasks designed were to help make PTs' implicit knowledge about the equals sign explicit, coupled with tasks that involved comparing and interpreting well-chosen examples of equations presented as artifacts of children's thinking.

Beyond the equal sign, PTs' struggles working procedurally with algebraic expressions and equations are documented in Pomerantsev and Korosteleva's (2003) study. Through a written survey of five questions (see Figure 6) given to two large classes of PTs ( $n = 119$ ) in the U.S., the authors investigated elementary and middle-grades PTs' abilities to recognize the structure of certain algebraic expressions (questions 1, 4, and 5) and to apply rules for cancellation of a common factor (questions 2 and 3). Question 1 was multiple-choice, and the remaining four questions were free-response.

Question 1. What is the name of the expression  $4x^2 - 9y^2$ ?

Choices: (a) difference of squares, (b) difference of products, (c) square of difference.

Question 2. If possible, cancel out the common factor in the expression  $\frac{2+x}{2}$ .

Question 3. If possible, simplify the expression  $\frac{2x+2}{2x}$ .

Question 4. Use the statement "If  $x^2 = 25$ , then  $x = \pm 5$ " to solve the equation  $(x+1)^2 = 25$ .

Question 5. Use the statement "If  $2x + 3 = 5$ , then  $x = 1$ " to solve the equation  $2(y+1) + 3 = 5$ .

*Figure 6.* Questions about algebraic expressions, used in Pomerantsev and Korosteleva's (2003) study (p. 2).

While more than 85% of the PTs correctly identified the type of expression in Question 1 as "difference of squares," the reasons they gave on the survey to justify their

answers were often superficial (e.g., “I’ve heard ‘difference of square’ most often in past math classes”). Additionally, few students answered Questions 2 through 5 correctly (see Table 4), suggesting that PTs have weak procedural skills with respect to symbolic expressions and equations. In light of these findings, the authors suggest further research, with the goal of developing effective methods for teaching algebra to elementary school teachers.

Table 4

*Percentage of Correct Responses by Question on Pomerantsev and Korosteleva’s (2003) Written Survey*

		Q1	Q2	Q3	Q4	Q5
Class 1	(n = 47)	95.8	44.7	29.8	10.7	21.3
Class 2	(n = 72)	87.5	44.4	22.2	5.6	18.1

The two studies summarized in this section (Pomerantsev & Korosteleva, 2003; Prediger, 2010) address different aspects of PTs’ understandings of symbols. The results of Prediger’s (2010) equal sign study is encouraging in that it demonstrates a possible route for broadening and improving PTs’ understandings of the equal sign via carefully designed tasks based on artifacts of children’s thinking. However, it is limited in that the article’s emphasis is more on providing a theory-supporting example of how a class of PTs might develop, rather than on reporting in-depth rigorous research, and it follows only one class of PTs. The findings of Pomerantsev and Korosteleva’s (2003) study is limited as well. The study documents struggles that PTs have in interpreting and using algebraic expressions, but the data are from a single point in time, on a single written survey. Thus, both studies

can be thought of as jumping-off points for the work yet to be done: in-depth qualitative research focused on documenting the details of PTs' understandings of algebraic symbols and the equal sign, and how they develop in those understandings.

### **Solving Algebraic Word Problems**

In the years since Schmidt and Bednarz' (1997) study (described in *A Historical Look* section above), two published papers have focused on elementary PTs' strategies for solving arithmetic and algebraic word problems (Schmidt & Bednarz, 2002; van Dooren et al., 2003). One additional paper focuses on PTs' flexibility of strategy choice when solving proportional reasoning word problems (Berk, Taber, Carrino Gorowara, & Poetzl, 2009). The two former studies explicitly build off of the study by Schmidt and Bednarz (1997). Specifically, van Dooren and colleagues' (2003) study is a near replication of Schmidt and Bednarz' (1997) study, with two differences: (a) their population of PTs are in Belgium, whereas the PTs in the earlier study were in Canada; and (b) the study conducted in Belgium compares PTs at the beginning of their teacher training program to PTs at the end of their program (using a cross-sectional design), whereas the study conducted in Canada includes only PTs at the beginning of their program. Schmidt and Bednarz' (2002) study builds on the findings of their 1997 study, exploring (a) what links between algebraic and arithmetic types of reasoning PTs make (or do not make) in the contexts of algebraic problem solving, (b) characterizations of the types of reasoning PTs use, and (c) potential difficulties in creating a bridge between arithmetic and algebraic reasoning.

van Dooren and colleagues (2003) presented 97 PTs with various arithmetic and algebraic word problems (Schmidt & Bednarz, 1997) via a paper-and-pencil survey. As in Schmidt and Bednarz' (1997) study, PTs' strategies for solving algebraic and arithmetic



problems were categorized as either “algebraic” or “arithmetic,” regardless of which type of problem was being solved. A solution was considered algebraic if it satisfied both of the following criteria: (a) the solution contained at least one equation wherein known and unknown values are related to each other, and (b) the answer is found via transformation of the equation(s) and operating on the unknowns. All other solutions (to either type of problem) were considered arithmetic.

Results of van Dooren and colleagues’ (2003) study differ from the findings of Schmidt and Bednarz’ (1997) study in that that the PTs in the more recent study can be categorized into two groups according to their preferred solution strategies to six algebraic and six arithmetic word problems: (a) those who almost exclusively use arithmetic solution methods regardless of the type of problem, and (b) those who are more flexible in their strategy preference. Specifically, elementary PTs solved 78.8%<sup>16</sup> of the arithmetic problems using arithmetic strategies. By contrast, however, elementary PTs solved 42.5% of the algebraic problems using arithmetic strategies (e.g., “guess-and-check” or “manipulating the structure”), 40.1% of algebraic problems were solved using algebraic strategies, and the rest of the problems (17.5%) were not answered. The authors point out that this finding suggests an opportunity to leverage the existence of the two groups of reasoners in teacher preparation courses, building connections between arithmetic and algebra. For example, they suggest first highlighting the existence of these two groups of reasoners in a PT classroom as a way to start a meaningful, explicit discussion among the PTs about children’s transitions from arithmetic to algebraic thinking.

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<sup>16</sup> Van Dooren et al. (2003) largely reported percentages instead of exact counts.

Notably, preferences for strategy selection between first- and third-year students did not differ statistically in the study, although the ability to solve problems correctly was greater for third-year students than for first-year students, largely due to increased proficiency with the “manipulating the structure” strategy (van Dooren et al., 2003). However, just as with the combined group of first- and third-year PTs, there was a subgroup of third-year PTs who tried to use arithmetic strategies to solve algebraic problems, usually with little success in finding a correct solution. The authors expressed their concern about the readiness of this particular subgroup of PTs to prepare elementary school children with the skills that will help them later transition to algebra at the secondary level.

Schmidt and Bednarz’ (2002) exploratory study compared PTs’ algebraic and arithmetic reasoning to illuminate the nature of the similarities and differences between them. In the study, eight Canadian PTs were interviewed in pairs, drawn from a larger pool of preschool or elementary PTs, special education PTs, and secondary mathematics PTs. The pairs were specifically selected to include one PT who tended to reason arithmetically regardless of the type of problem being solved, and one PT who tended to reason algebraically regardless of the type of problem being solved, based on their responses on a preliminary written survey consisting of eight arithmetic and algebraic problems. One of the two elementary PTs included in the interviews answered all four algebraic problems using an arithmetic guess-and-check strategy, and the other elementary PT included in the interviews answered all four algebraic problems by using an arithmetic manipulate-the-structure strategy.

Schmidt and Bednarz' (2002) analyses of the interviews illuminated the nature of the similarities and differences between manipulating-the-structure arithmetic reasoning and algebraic reasoning, and between guess-and-check arithmetic reasoning and algebraic reasoning. In particular, manipulating-the-structure and algebraic reasoners are similar in that they can both successfully solve algebraic problems when there is a known quantity, yet these two types of reasoners differ when there are no known quantities given in the problem statement. For example, consider the following problem: "Luc has \$3.50 less than Michel. Luc doubles his money while Michel increases his amount by \$1.10. Luc then has \$.40 less than Michel. How much did they have to begin with?" (Schmidt & Bednarz, 2002, p. 85). Because this problem does not give a specific amount of money for either Luc or Michel, manipulating-the-structure reasoners had difficulty solving the problem, whereas algebraic reasoners are able to successfully solve the problem by using the given relationships.

An apparent similarity between guess-and-check reasoners and algebraic reasoners is that both rely on manipulating quantities—known or unknown quantities, respectively—yet these two types of reasoners differ in that guess-and-check reasoners are limited by their local, sequential treatment of the elements of a problem via calculations. Algebraic reasoners, on the other hand, tend to account for all of the elements of the problem from the outset of their strategy.

When comparing all three types of reasoners (manipulating-the-structure arithmetic, guess-and-check arithmetic, and algebraic), only the manipulating-the-structure reasoners consistently demonstrated the ability to use the problem context to work with all the elements of a problem; algebraic reasoners tended to verify that their symbolic

procedure was correct but often did not check their reasoning against the problem context. That said, the authors stress that one type of arithmetic reasoning is not definitively superior over the other, and they suggest that future research could further explore whether there is a type of arithmetic reasoning that best prepares students to progress to algebra at the secondary level.

The findings of a study on U.S. elementary PTs' flexibility with solving proportional reasoning word problems echoes the findings of the above research on PTs' strategies for solving word problems, in that PTs tend to enter their training programs with limited flexibility in strategy use (Berk et al., 2009). Specifically, analyses of 148 PTs' solutions of four different types of proportional reasoning problems suggest that PTs either (a) have a limited number of strategies for solving proportional word problems, or (b) are unable to choose strategically among the strategies that they know. Although the PTs in the study demonstrated reasonable proficiency in solving proportional reasoning word problems successfully and accurately, many used cumbersome and/or inefficient solution strategies.

However, Berk and colleagues' (2009) study supports the notion that PTs' problem-solving flexibility in the domain of proportional word problems can improve through the PTs' exposure, discussion, and careful consideration of others' solution strategies. Results of quantitative analyses suggest that PTs achieved gains in strategy flexibility after an instructional intervention on multiple-solution strategies for proportional word problems. Notably, this increase in flexibility occurred along with an improvement in the accuracy of PTs' solutions, and the PTs retained their flexibility at a 6-month follow-up.

Despite the fact that these three studies were conducted in three different countries (Belgium, Canada, and the U.S.), they complement each other in their emphasis on the need

for PTs to have flexibility in solution strategies with respect to algebraic and arithmetic problem solving (Berk et al., 2009; Schmidt & Bednarz, 2002; van Dooren et al., 2003). Moreover, considered together, two studies (Berk et al., 2009; van Dooren et al., 2003) lend tentative support for the idea that PTs can increase their flexibility with strategies and/or their success in solving problems correctly. In order to gain further insight into the ways in which PTs might develop in their problem-solving abilities, more research is needed. Further research is also needed to gain insight into the strengths and weaknesses of PTs' uses of particular strategy types, as Schmidt and Bednarz (2002) have begun to explore.

### **Understanding Functions**

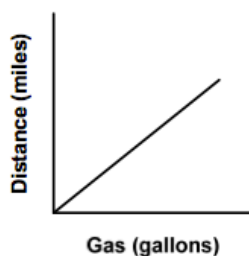
In this section, we discuss the findings of the five studies that explicitly focus on PTs' understandings of functions. Findings from these studies are summarized below.

Research on linear functions suggests that PTs generally have strong procedural skills (Nillas, 2010; You & Quinn, 2010). Quantitative analyses of 104 U.S. PTs' responses to a 15-item survey suggest that PTs perform well on questions designed to test procedural skills related to linear functions, such as calculating a slope (You & Quinn, 2010). This finding is further supported by a qualitative study of the responses of five U.S. PTs on three written test items (Nillas, 2010).

Results of both studies (Nillas, 2010; You & Quinn, 2010), however, suggest that many PTs struggle to (a) interpret the slopes of the graphs of linear functions in real-world contexts, and (b) flexibly translate between multiple representations of a function. PTs seem to have particular difficulty flexibly translating between symbolic and visual representations, and between a representation of a function and its real-world context (You & Quinn, 2010). Nillas (2010) asked PTs to interpret the meaning of an increase in the

slope of a line on a non-scaled graph that showed gallons of gas consumed by Jake’s car versus distance traveled by his car (see Figure 7). While two of the five PTs in the study offered correct responses (e.g., “Jake’s car would burn less gas per mile . . . having better gas mileage”), three of the five PTs gave the reverse interpretation, stating the car was “using more gas for less distance traveled” or “a gallon of gas takes Jake a shorter distance than before” after the increase in steepness. Moreover, one PT stated that “the cost of gas has increased,” despite the fact that no information about the cost of gas was given in the problem.

**Test Item 1. Gas Mileage (Linear).** The graph below shows gallons of gas consumed by Jake’s car compared to distance traveled.



- What would it mean in this situation if the line was steeper than shown and the scales on the graph stayed the same?
- Put a sensible scale on the axes, and write a question that can be answered from your graph. Show where the answer can be found on your graph.

*Figure 7.* Graph-related task, as posed on a written test in Nillas’s (2010) study (p. 24).

The notion that PTs have difficulty interpreting graphs of functions is further supported by research focusing specifically on PTs’ interpretations of graphs involving speed or motion (Billings & Klanderma, 2000; Stylianou, Smith, & Kaput, 2005). Analyses of responses on a pretest given to 28 elementary PTs in the U.S. suggest that many PTs hold

one of two well-documented misconceptions about graphs involving motion:<sup>17</sup> (a) a graph-as-picture misconception, and (b) a slope/height misconception. PTs with the graph-as-picture misconception interpret graphs as though they are aerial pictures of the actual paths traveled. PTs with the slope/height misconception conflate slope with height so that a positive slope is interpreted as an up-hill path, and a negative slope is interpreted as a down-hill path (Stylianou et al., 2005).

Billings and Klanderma's (2000) analyses of the written work and in-class observations of 19 U.S. grade K–8 PTs in a junior-level class focused on algebra reported similar findings; they identified four cognitive difficulties that PTs in their study seemed to have when creating or interpreting graphs where one variable is speed: (a) confusing the concepts of instantaneous speed and average speed; (b) confusing the variables of speed and distance in various ways, for example, graphing a line segment with increasing slope on a time-versus-speed graph to show constant speed; (c) failing to identify the slope of a line segment in a distance-versus-time graph as speed; and (d) difficulty creating an appropriate scale for the axes of a graph involving speed.

To address these cognitive difficulties and misinterpretations of graphs, Stylianou, Smith, and Kaput (2005) suggest the use of specific motion-detection technology, such as Calculator-Based-Rangers (CBRs), during PT education courses. In their study, PTs participated in a 2-week classroom-based, exploratory study wherein PTs completed activities focused on making and interpreting graphical representations of motion using CBRs as data collection devices and as graphing calculators. Based on analyses of a pre- and

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<sup>17</sup> For a detailed review of this literature, see Leinhardt, Zaslavsky, and Stein (1990).

posttest, the authors note that the in-class activities appeared to help PTs overcome some of their misconceptions about graphs involving motion (specifically, the graph-as-picture and slope/height misconceptions). Most notably, the PTs seemed to improve in their abilities to interpret graphs as representations of a situation. The PTs also more frequently used graphs as problem-solving aids and as conscription devices to facilitate communication with one another. The researchers attributed these apparent gains in student understanding to the use of the CBRs, coupled with the rich peer-to-peer discussion about graphs that occurred throughout the activities.

One research study (Meel, 1999) explored PTs' definitions of functions via a written survey. In the study were 29 U.S. elementary or early childhood PTs who had chosen to specialize in mathematics and were near the end of their teaching certification program. Results of the study showed that each of the six statements in Figure 8 was indicated as a true definition of function by at least 75% of 29 PTs, with the highest number of PTs (26 out of 29) correctly marking statement B as true. Moreover, statement A was selected as the "best" definition of function by the highest number of PTs (15 out of 29). However, when asked to "define the mathematical concept: function" (p. 4) later in the survey, most PTs produced definitions similar to statement E. Statement E reflects a limited, historical "function-as-formula" understanding of function, which Meel (1999) suggests impedes PTs' understandings of functions.



- (A) A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set.
- (B) A function is a dependence relation between two variables ( $y$  depends on  $x$ ).
- (C) A function is a rule which connects the value of  $x$  with the value of  $y$ .
- (D) A function is a computational process which produces some value of one variable ( $y$ ) from any given value of another variable ( $x$ ).
- (E) A function is a formula, algebraic expression, or equation which expresses a certain relation between factors.
- (F) A function is a collection of numbers in a certain order which can be expressed in a graph.

Figure 8. Statements about functions from an item on a survey used in Meel's study (1999, pp. 3–4).

This collection of studies on PTs' understandings of function lends support for the idea that PTs typically have strong procedural skills with respect to linear functions (Nillas, 2010; You & Quinn, 2010), yet they tend to hold a formulaic understanding of function (Meel, 1999), and they tend to exhibit a wide range of struggles with respect to connecting and interpreting representations of functions, with the most-documented struggles relating to graphical representations of functions and story contexts (Billings & Klanderma, 2000; Nillas, 2010; Stylianou et al., 2005; You & Quinn, 2010).

One study lends support to the idea that hand-held graphing technology can be used to help PTs overcome some of their struggles interpreting graphical representations of functions. This is the only study to document pre/post change in PTs' understandings of function, and it is one of only two studies (Billings & Klanderma, 2000; Stylianou et al., 2005) that look at PTs' understandings of function at more than one time point. Further,

neither of these studies attempt to explain how PTs develop in their understandings of functions. Insight into how PTs learn to overcome their struggles with various representations of functions is a necessary next step in this line of research.

### **A View of the Horizon**

In our search of recent literature, we did not find any research published in 2012 in peer-reviewed journals that focused on elementary or middle-grades PTs' content knowledge of algebra. We did, however, identify four such papers in our search of the 2011 and 2012 proceedings of PME and PME-NA (Callahan & Hillen, 2012; Jacobson & Izsák, 2012; Milinkovic, 2012; Mills, 2012; see Table 5 below).

Within these proceedings papers, we found that research is continuing within the topics of the equal sign and proportional reasoning. An exploratory, case-based study suggests that current frameworks of understandings of the equal sign might not be adequate for capturing the understandings of the population of elementary PTs (Mills, 2012). Based on the findings from a brief interview study of one PT, Mills concludes that the PT holds a predominantly operational view of the equal sign, yet the PT also demonstrates some relational understanding of the equal sign. In light of this case, Mills suggests that frameworks for understandings of the equal sign might need to be reworked in order to capture the understandings of PTs that seem to think of the equal sign as both operational and relational.

Table 5

*Articles Published in 2012, or in the 2011/2012 PME/PME-NA Proceedings, on the Topic of PTs' Algebra Content Knowledge, in Alphabetical Order by First Author*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Callahan & Hillen	2012	22	Undergraduate PTs enrolled in a math content course	USA	Transcripts of video of whole-class discussions, field notes, copies of PTs' written classwork, audio recordings of teacher-researchers' weekly discussions
Jacobson & Izsák	2012	28	PTs enrolled in a middle-grades math methods course	USA	Whole-class video data, transcripts of task-based interviews of four pairs of students, written pre- and posttests of all students
Milinkovic	2012	121	Undergraduate PTs in their fourth year of study	Serbia	Written survey data
Mills	2012	1	Undergraduate PT enrolled in her first content course	USA	Task-based interview data

Within the topic of proportional reasoning, a study of 28 middle-grades mathematics PTs enrolled in a methods course suggests that PTs often try to set up and use proportional equations for non-proportional problem scenarios, even when the PTs seem to have a strong understanding of the non-proportional covariance situation (Jacobson &

Izsák, 2012). Analyses of whole-class video, transcripts of task-based interviews of four pairs of students from the class, and written pre- and posttests of all students suggest that PTs who correctly explain relationships between quantities that are not proportional still attempt to use proportion equations to represent the relationships. The authors point out that “these results suggest a sobering assessment of the pedagogical challenge faced by teacher educators,” (p. 635) given that encouraging PTs in understanding the scenarios presented in proportional and non-proportional covariation situations might not help with their ability to judiciously apply or not apply proportion equations to those situations.

In addition to the above-described continuing lines of research on PTs’ understandings of algebra, new lines of research are emerging within the areas of generalized arithmetic and properties of operations. The findings of a study of 22 prospective middle school teachers suggest that PTs struggle to understand a given visual representation of even and odd numbers (Callahan & Hillen, 2012). For example, although PTs were able to describe even numbers as divisible by 2, or having no remainder after dividing by 2, or being a multiple of 2, they struggled to make sense of a geometric representation wherein even numbers were represented as 2-by-whole-number rectangles. In other recent research, the findings of a survey-based study of 121 PTs’ knowledge of representations of multiplication and the Commutative Law of Multiplication suggest that PTs’ choices of representations are linked to problem abstractness (Milinkovic, 2012). Specifically, PTs tended to draw grouping (also commonly referred to as repeated addition) representations for questions using concrete numbers, whereas they tended to draw area representations for questions involving expressions using variables. The author concludes that problem abstractness (in this case, concrete numbers vs. variables) affects

PTs' representational choices, suggesting further research to confirm and explore the significance of these findings.

### **Conclusion**

Looking across the findings of all research on PTs' content knowledge of algebra, we see three overarching themes:

1. Within the content domain of algebra, PTs generally have strong procedural skills and can make mathematically sound generalizations of many different types of patterns (Briscoe & Stout, 2001; Otto et al., 2008; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b);
2. However, PTs tend to struggle to (a) interpret and effectively use algebraic symbols, even those that they have produced themselves (Mills, 2012; Pomerantsev & Korosteleva, 2003; Prediger, 2010; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b); (b) interpret graphical representations (Billings & Klanderma, 2000; Nillas, 2010; Stylianou et al., 2005; You & Quinn, 2010); and (c) make connections between representations (Billings & Klanderma, 2000; Nillas, 2010; Pomerantsev & Korosteleva, 2003; Prediger, 2010; Richardson et al., 2009; Rivera & Becker, 2007; Stylianou et al., 2005; You & Quinn, 2010; Zazkis & Liljedahl, 2002a, 2002b);
3. Moreover, PTs generally have limited algebraic problem-solving strategies, often relying, inflexibly, on inefficient and/or incorrect computational methods (Berk et al., 2009; Schmidt & Bednarz, 1997, 2002; van Dooren et al., 2003).

Fortunately, though, there is emerging research to suggest that PTs' algebraic thinking and understandings in various areas can develop by focusing on

justification through connections between representations (Richardson et al., 2009; Rivera & Becker, 2007), analysis of children's artifacts (Prediger, 2010), consideration and analyses of multiple solution methods (Berk et al., 2009; van Dooren et al., 2003), and work with hand-held graphing technology (Stylianou et al., 2005).

Notably absent from the themes above are research-based conclusions about how PTs develop in their understandings of various topics within algebra. The few studies that followed the development of PTs' content knowledge of algebra were limited in scope and generalizability. In light of this, we suggest that more research is needed on PTs' development in understanding, interpreting, and connecting representations of various topics within algebra, such as generalizations, functions, and word problems. It is clear from the current research that graphical, symbolic, and contextual representations (and the connections between them) can be points of struggle for many PTs. Accordingly, we also recommend that methodologically rigorous research be devoted to exploring and developing pedagogical innovations for teaching algebra to PTs.

Further, given the recommendations of the METII (Conference Board of the Mathematical Sciences, 2012) and the CCSS (NGA & CCSSO, 2010) that the foundations of algebra should be laid in the elementary grades, we point to the need for continued research that focuses on how PTs come to understand the connections between arithmetic and algebra, including (but not limited to) properties of operations, and judicious and flexible strategy selection in problem solving.

Our summary of the existing body of literature on PTs' understandings of algebra suggests that implications for teacher education courses are tentative and somewhat

scattered. While there are some research-based recommendations for teacher education courses—such as making use of motion-sensor and graphing technology, having PTs analyze children’s artifacts, identifying and leveraging various problem solving strategies of PTs, and encouraging PTs to justify their ideas through connections between representations during visually- or contextually-based tasks—the picture is far from complete. Clear and comprehensive research-based guidance for the development of PT-centered mathematical preparation in algebra for our future educators remains to be established.

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**Prospective Elementary Mathematics Teacher Content Knowledge:  
What Do We Know, What Do We Not Know, and Where Do We Go?**

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**ABSTRACT:** In this Special Issue, the authors reviewed 112 research studies from 1978 to 2012 on prospective elementary teachers' content knowledge in five content areas: whole numbers and operations, fractions, decimals, geometry and measurement, and algebra. Looking across these studies, this final paper identifies the trends and common themes in terms of the counts and types of studies and commonalities among findings. Analyses of the counts show that the number of articles published each year focusing on prospective teacher (PT) content knowledge is increasing. Most articles across the content areas show that PTs tend to rely on procedures rather than concepts. However, the focus of most articles is identifying PTs' misconceptions rather than understanding PTs' conceptions and the development thereof. Both the limitations of the reviews and the directions for future research studies are elaborated.

**Keywords:** mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

## Introduction

The collection of papers in this Special Issue is the result of a PME-NA Working Group titled “Preservice Elementary School Teachers’ Content Knowledge in Mathematics” (Thanheiser et al., 2009; Thanheiser et al., 2010; Thanheiser, Lo, Kastberg, Canda, & Eddy, 2007) that met three times (2007, 2009, and 2010) and continued to collaborate after those years. All of the authors of this volume are mathematics educators teaching content and methods courses to prospective elementary teachers (PTs) and are involved in research related to PTs’ content knowledge in various content areas. The goal of the group was to provide a summary of the research (as of 2012) conducted on PTs’ mathematical content knowledge needed for teaching and to inform the research community on (a) what we currently know, (b) what we do not know yet, and (c) what we need to know.

The collection of papers in this Special Issue represents a summary of PTs’ mathematical content knowledge for teaching mathematics to children up to age 14 (see the *Common Core State Standards* [CCSS], National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), with emphasis on number and operations (treated in three papers: whole numbers, fractions, and decimals), geometry and measurement, and algebra. For each of these listed areas, an individual paper summarizes the current state of the research literature. The papers provide an insight into areas well researched (e.g., division of fractions) and areas that need more work (e.g., fraction number sense) to round out our understanding of PTs’ mathematical content knowledge for teaching.

This final paper of the Special Issue is based on a focused collection of findings spanning across the five content area papers. We acknowledge that it provides a somewhat

incomplete perspective on what we know about PTs' content knowledge and development due to the following limitations: (a) the exclusion of mathematics outside the scope of our Working Group, (b) the exclusion of Standards of Mathematical Practice, and (c) the limitations of the methodology of the Working Group (described in the introduction of this Special Issue).

### **Descriptive Themes of the Summarized Research: Counts and Types**

In this Special Issue, we summarized a total of 112 peer-reviewed research articles published in journals reporting on prospective teachers' content knowledge, spanning the years 1978 to 2012. We categorized the research articles into three sections: *A Historical Look* (pre 1998), *A Current Perspective* (1998–2011), and *A View of the Horizon* (2011–2012). We incorporated a review of an additional 18 papers published in PME and PME-NA conference proceedings in the years 2011 and 2012 to allow us to see what is on the horizon; however, those 18 papers are not included in the summary totals we are reporting in this section as they are conference papers and did not appear in peer reviewed journals. Thus, the total numbers reported in this section refer to peer-reviewed research articles from journals.

### **Number of Research Articles Published Increased Over Time**

The number of published research articles across the content areas can be seen in Table 1. Before 1998, we found a total of 38 research articles focusing on PTs' content knowledge; the number increased to 68 in the timespan from 1998 to 2011. The count of published research articles for 2012 suggests a decline in research on mathematical content knowledge of PTs; however, if we include the counts of papers from the

proceedings (parenthetical counts in the table),<sup>18</sup> the View of the Horizon promises a possible increase in publications for the next decade. Across two of the three time periods, we note that the content area of fractions has the highest frequency of publications, suggesting perhaps that the challenges faced when PTs are learning fraction content prompts more research attention. When we view the counts by individual years (see Figure 1), we see that 1989 marks an increase in research focused on PTs' content knowledge, followed by a second increase in 2007. It is interesting to reflect on particular events in mathematics education that occurred during and near those particular years, such as Shulman's (1986) introduction of pedagogical content knowledge and the launch of the *Curriculum and Evaluation Standards for School Mathematics* by the National Council of Teachers of Mathematics (1989). These events are followed by the debut issue of the *Journal of Mathematics Teacher Education* (1998), with the initial articulation of mathematical knowledge for teaching (Ball & Bass, 2002; Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005) setting the stage for the second increase.

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<sup>18</sup> We include papers from the conference proceedings here as they may evolve into publications in the coming years.

Table 1

*Peer-Reviewed Research Articles Reporting on PTs' Mathematical Content Knowledge for Teaching*

	A Historical Look (Pre 1998)	A Current Perspective (1998–2011)	A View of the Horizon (2012)	Total
Whole Number	7	18	1 (2)	26
Fractions	12	17	5 (7)	34
Decimals	9	5	0	14
Geometry & Measurement	9	12	0 (5)	21
Algebra	1	16	0 (4)	17
TOTAL	38	68	6 (18)	112

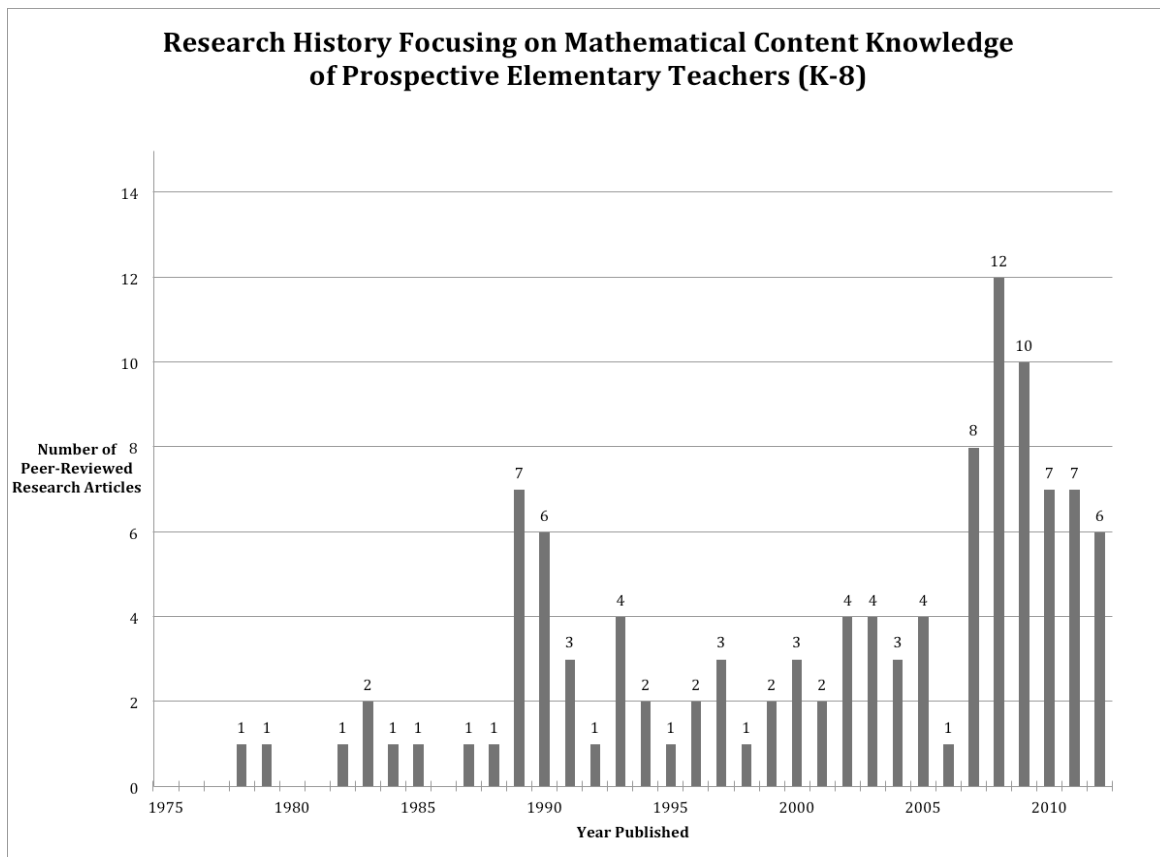


Figure 1: The number of peer-reviewed journal articles from 1978 to 2012.



While the number of research articles increases over time, the relative number of research articles focusing on PTs’ mathematical content knowledge for teaching is still small. In addition to examining the counts of studies across topics, we also considered the frequency of the research in different geographical locations.

**Number of International Versus U.S. Studies**

In our review of 112 articles, 72 presented research conducted in the United States, while 40 were based in other countries (see Table 2). Thus, while most of the reviewed studies were done in the United States, about a third of them are international and show that PTs’ mathematical content knowledge is of interest around the world. More than half of the international studies were conducted in four countries: nine studies in Austria, seven in Canada, and five each in Turkey and Taiwan. While we do not attempt to claim that our review examined research in all international journals, the common concerns that arose through the summary work were found to exist across studies conducted in the United States as well as in the other included countries.

Table 2

*The Number of Peer-Reviewed Articles Published Focusing on PTs Outside the United States*

	Total	International
Whole Number	26	7 (27%)
Fractions	34	13 (38%)
Decimals	14	5 (35%)
Geometry & Measurement	21	9 (42%)
Algebra	17	6 (35%)
TOTAL	112	40 (35%)

### **Static Studies of Knowledge Versus Motion Studies of Learning**

Of the 112 studies surveyed, 104 (93%) focused on assessing PTs at a certain point in time, or several points in time (*static studies of knowledge*), while only eight (7%) of the studies (two in whole number, one in fractions, and five in algebra) focused on closely examining PTs' learning (*motion studies of learning*) (see Table 3). We use the phrase *static studies of knowledge* to describe studies that focus on multiple data captures with assessments on PTs' mathematical understanding at specific moments in time, but do not focus on the development of learning mathematical ideas. The work of Kaasila, Pehkonen, and Hellinen (2010) described by the whole numbers and operations group in this Special Issue presents an example of this type of static study. PTs enrolled in a mathematics methods course in Finland were given an item related to the division of whole numbers that they were asked to solve without using the traditional division algorithm. Responses were analyzed for evidence of and difficulties in conceptual understanding, adaptive reasoning, and procedural fluency. Data were collected only once, with findings presented from the single analysis. This type of static research of mathematical knowledge is useful in order to identify areas of concern for a subsequent, careful examination of PTs' conceptions and the development thereof.

Table 3

*Motion Studies of Learning Versus Snapshot Studies of Knowledge for Each Content Area*

	Motion Studies of Learning	Static Studies of Knowledge	Total
Whole Number	2 (8%)	24 (92%)	26
Fractions	1 (3%)	33 (97%)	34
Decimals	0	14	14
Geometry & Measurement	0	21	21
Algebra	5 (29%)	12 (71%)	17
TOTAL	8 (7%)	104 (93%)	112

We use the term *motion studies of learning* to describe a careful examination of learning. In such studies it is not enough to report pre data, describe the treatment, report post data, and indicate potential change; with motion studies, a clear description of the learning, the treatments implemented, any developmental change in learning, and an examination of possible correlations of the developmental change to implemented treatments are needed. Examples of such studies could be case studies conducted during an extended period of time assessing how any interventions were related to the learning or a systematic analysis of a sequence of learning segments determining the strength of correlations between treatment and learning. Richardson, Bereson, and Staley (2009) provide an example of a motion study of learning in their study with PTs focusing on algebraic reasoning. Their teaching experiment focused on the processes of teaching PTs how to generalize and justify rules, noting critical moments in the PTs' development of

algebraic reasoning during the experiment, finding positive associations between the tasks the instructors developed and the PTs' learning.

We note that Mewborn's (2001) review of research on PTs' mathematical knowledge also found a prevalence of "snapshot studies" (p. 33) and a dearth of "video-taped" (p. 33) studies in the literature up to that point. More than a decade later, we are still making this same observation. Very few studies of PTs' content knowledge have analyzed the processes by which that knowledge develops (see Table 3). While understanding PTs' knowledge is a critical component of our understanding of how PTs learn (since we want to build on the knowledge they bring with them to their preparation programs), we also need to focus on understanding how PTs learn and construct knowledge so we can help them build the mathematical understandings from which they will need to teach.

### **Descriptive Themes of the Summarized Research: Two Commonalities**

In the next sections we highlight two common themes that emerged in the findings across all of the content areas we examined. We highlight a few examples for each theme and refer the reader to the individual summary papers within this Special Issue for a more in-depth reading of the themes.

#### **Most Research Focuses on Deficit Descriptions**

One noted theme that was found within the static studies of knowledge across the content areas was the focus on identifying and describing deficits in PTs' content knowledge as opposed to (a) providing a useful characterization of the PTs' conceptions, and (b) identifying knowledge that can serve as a resource in learning. While establishing

what PTs know and do not know is essential, and thus deficit studies are useful, we need to move beyond those studies to understand what PTs do know and how learning happens.

For example, the work of Tirosh and Graeber (1989, 1990a, 1990b, 1991) highlighted PTs' misconceptions about multiplication and division. Findings of such "static" work can prompt subsequent research to focus on the how such misconceptions develop in PTs' learning. Yet, still what is needed is a characterization of PTs' conceptions, such as what is presented in Thanheiser (2009, 2010), that details how PTs think about number. Whitacre (2013) argued for the need to view PTs' prior knowledge as a resource in their learning. He offered several examples from his research of ways in which PTs' prior knowledge—including their knowledge of procedures—can be built upon productively. Another example that goes beyond identifying and describing deficits in PTs' content knowledge comes from geometry, where the work of Battista, Wheatley, and Talsma (1982, 1989) examined the importance of spatial visualization in learning geometry.

### **PTs' Focus on Procedures Rather Than Concepts**

Related to a focus on deficits in PTs' content knowledge was the theme related to PTs' procedural understanding. The studies reviewed for this Special Issue highlighted PTs' tendency to focus on procedures rather than concepts across all content areas; several examples are shared below.

When PTs were asked to reason about alternative algorithms or nonstandard strategies when working with whole numbers, Harkness and Thomas (2008) found that only 7 of 71 PTs were able to explain conceptually why the algorithm worked. Fifteen more PTs showed some understanding but gave incomplete explanations. The remaining PTs'

arguments relied on comparing the alternative algorithm to the standard multiplication algorithm.

Similar procedural thinking was exhibited in decimal place value understanding when PTs worked with converting  $12.34_{\text{five}}$  to base ten (Khoury & Zazkis, 1994; Zazkis & Khoury, 1993) by relating the fractional part of a number to the base in the number in non-standard ways. For example, in  $12.34_{\text{five}}$ , some PTs suggested that the 3 was in the 0.5 position and the 4 was in the 0.05 position, reasoning that is aligned with the consistent use of 1 in decimal notation for tenths (0.1) and hundredths (0.01). Other PTs ignored the fractional part of the number, noting that decimals exist only in base ten (Zazkis & Khoury, 1993). The digits after the decimal were unchanged, while the integer part of the number was converted using a conventional strategy.

PTs demonstrated the ability to use algorithms to multiply, divide, and compare fractions, but were unable to explain why these procedures worked (e.g., Ball, 1990; Borko et al., 1992) or to stray from them, even if using number sense would be more appropriate (e.g., Yang, Reys, & Reys, 2008).

Baturo and Nason (1996) found PTs relied on procedural understandings of area and that they could not explain why one must divide by 2 in the area formula for a triangle, as the rule had not been connected to any concrete experiences.

And finally, PTs generally had strong procedural skills in the context of linear functions such as calculating a slope (Nillas, 2010; You & Quinn, 2010). However, results of both studies suggest that many PTs struggled to (a) interpret the slopes of the graphs of linear functions in real-world contexts, and (b) flexibly translate between multiple representations of a function.

Both themes of deficit and procedural understanding arise from the wealth of static studies summarized across content areas, showing what we know in the moment, that PTs struggle when it comes to learning and understanding mathematics. Again, these common themes strongly suggest our need to examine how the PTs can be successful in learning mathematics so we can move beyond these noted limitations of understanding.

### **Conclusions**

Through the collective summary of research from 1978–2012 on the mathematical content knowledge of PTs, we found that:

1. The number of peer-reviewed research articles focusing on elementary PTs' content knowledge that have been published in journals has increased from roughly one study per year in the 1970s and 80s, to six or more per year since 2007, with the trend suggesting this count per year will continue to be maintained and possibly increase.
2. Within the five specific content areas examined (whole numbers and operations, decimals, fractions, geometry and measurement, and algebra), research on fraction content knowledge generally had the highest frequencies of published work (twice out of the three time periods).
3. Similar research on PTs' mathematical content knowledge was conducted in the United States and several other countries with similar findings.
4. Far more static studies of knowledge (104) were reported than motion studies of learning (8).
5. While many individual research findings were summarized, two themes across all five content areas emerged: (1) PTs' reliance on procedural understanding,

and (2) the tendency of the literature to focus on describing deficits in PTs' understandings.

We also realize there are limitations to these findings in that:

1. Reviewed articles are published in mathematics education research journals. There may be other relevant articles published in journals in other fields.
2. Almost all of the articles reviewed were in English. Given the known work in international venues, relevant studies published in other languages unfortunately were missed and not examined.
3. We did not attempt to systematically search for all related research published prior to 1998. There were limited available resources for an exhaustive review of research.

As stated several times in this Special Issue, the goal of our work was to provide a summary of the research (as of 2012) conducted on PTs' mathematical content knowledge needed for teaching and to inform the research community on (a) what we currently know, (b) what we do not know, and (c) what we yet need to know. If we use these three points to frame our summative findings, we see that *we currently know* and have identified many misconceptions that PTs hold across all content areas, and we have some more nuanced descriptions of PTs' conceptions (e.g., Thanheiser, 2009, 2010). Our summaries also suggest *we do not know* enough yet about how PTs learn, showing a lack of nuanced descriptions for PTs' conceptions across most content areas, as well as a dearth of studies on PTs' learning. What we do not know provides a sufficient context for what *we yet need to know*. Thus, we suggest that more research articulate characterizations of the PTs' conceptions and focus on PTs' learning, conducting more of what we have described as



*motion studies* in learning that examine the PTs' learning process and describe any associations between what is done in the classroom and developmental changes in learning. We see these types of studies as particularly fertile ground for future research that can help mathematics teacher educators understand how to support PTs' critical development of important content knowledge.

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## APPENDIX

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