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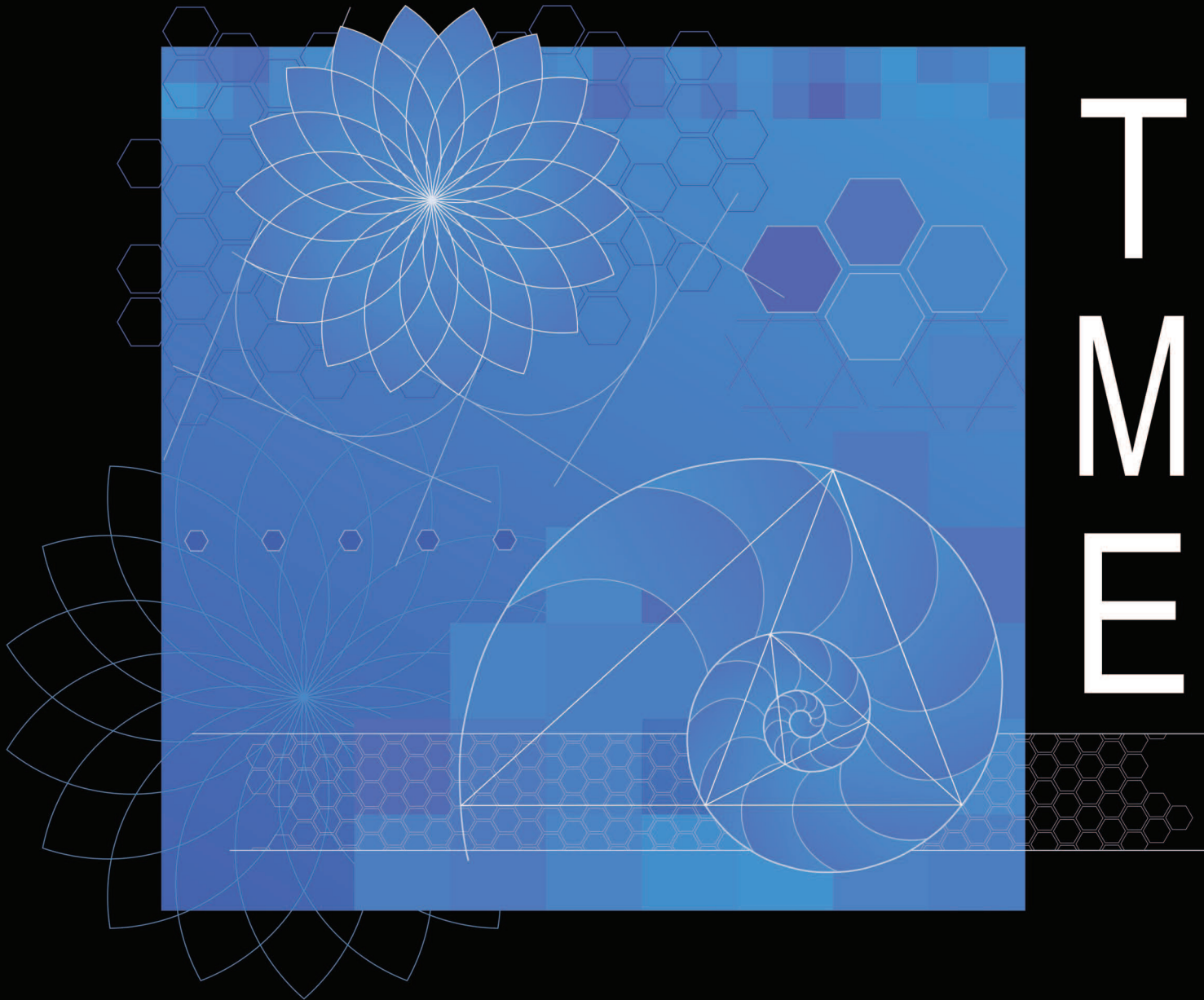
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THE MATHEMATICS ENTHUSIAST

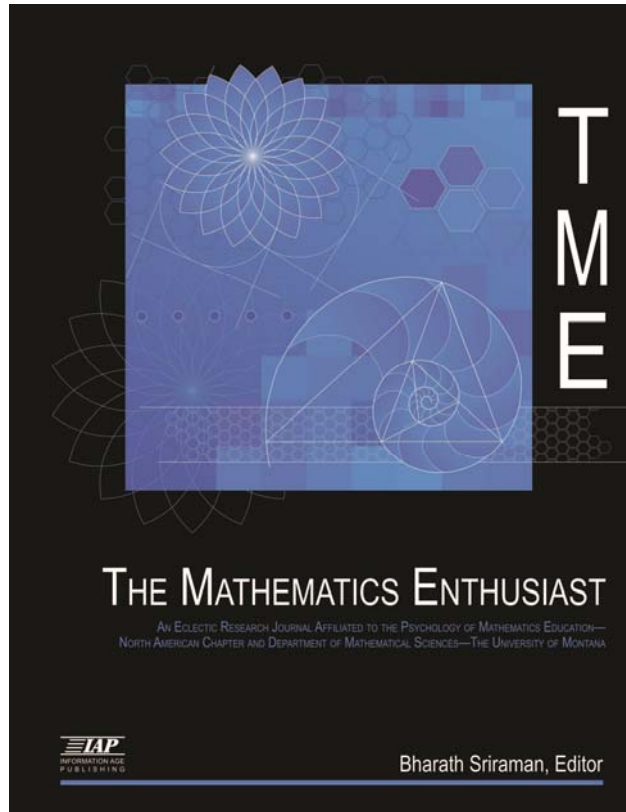
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The Mathematics Enthusiast (TME) is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy.

The journal is published on a print-on-demand basis by Information Age Publishing and the electronic version is hosted by the Department of Mathematical Sciences- The University of Montana. The journal supports the professional association PMENA [Psychology of Mathematics Education- North America] through special issues on various research topics. TME strives to promote equity internationally by adopting an open access policy, as well as allowing authors to retain full copyright of their scholarship contingent on the journals' publication ethics guidelines: http://www.math.umt.edu/TMME/TME_Publication_Ethics.pdf The journal is published tri-annually.

Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning. The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-12 months.

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Editorial: Why (yet) another issue on Problem Solving?

Bharath Sriraman

This is the 10th volume of *The Mathematics Enthusiast*, consisting of 500+ pages in 18 articles that give reflections, directions and the state of the art of mathematical problem solving as it relates to the field of mathematics education. This impressive collection compiled and guest edited by Manuel Santos-Trigo and Luis Moreno-Armella contains a treasure trove of scholarship from both the pioneers of this area of research (Alan Schoenfeld, Richard Lesh, Frank Lester, among others) as well as reports on new areas of study from Mexico, France and Spain. Two of the articles (Mamona-Downs & Downs, Selden& Selden) discuss the connections between problem solving and proof, and one piece (Flores & Braker) explores an interesting open-ended problem. There are many themes in this double issue- For instance those interested in advances in problem solving as a result of new technologies such as haptic devices will find articles (e.g., Hegedus) that report on cutting edge investigations. Others interested in cognition and learning trajectories as a result of problem solving practices will find articles that cater to this particular taste. The reflections by forerunners such as Alan Schoenfeld and Frank Lester are well worth reading for anyone that wants to catch up with developments in problem solving in the last 40 years.

Mathematics education (in the U.S) has been victimized as not having “really” progressed in terms of experimental research by the National Mathematics Advisory Panel (see Greer, 2008), which prescribed algebra as a panacea to cure our students mathematical ills. As noted in an earlier survey (English & Sriraman, 2010) and numerous articles in this double issue, there have been tremendous advances in the area of problem solving which unfortunately did not translate into curricular or “test-score” gains as measured by the testing industry. Problem solving as implemented in schools in the 90’s also became a fad caught in the pendulum swing of mathematics education reform. Polya style heuristics that capture the nuances of real mathematical thinking became didactically transposed and dogmatized by the textbook industry into prescriptive condition-action rules or flowcharts (Lesh & Sriraman, 2010). Several articles in this double issue revisit Polya style heuristics and capture its real essence. Some provide existence “proofs” of the mathematical thinking young children are capable of when presented with semi structured open-ended problems in conditions that foster novelty (see Lesh, English, Riggs & Sevis). This should offer the community hope that problem solving research and well documented empirically validated skill sets can be promoted and made relevant for the new generation of school children, particularly in an age where thinking across the disciplines is necessary in many professions. Hopefully this answers the question posed in the title of the editorial.

The journal imposed an 18 month embargo on submissions (**which will end on 04/2014**) to clear up the backlog of articles as well as make room for special issues in the works. In 2011, the journal received the honor of being selected by National Science Foundation's Math and Science Partnership (MSP) program committee to assemble and publish a set of papers over the next two years to expand avenues for more MSP projects to share what they are learning about

mathematics and science education through an internationally recognized peer-reviewed journal that is widely available. Over the next two years some special issues, starting with Vol10, no3 [July 2013] will feature articles reporting on MSP projects. These projects include large partnerships targeting science and/or mathematics teaching and learning in specific grade bands or disciplinary areas, institute partnerships focusing on developing teacher leadership, partnership incubator (or “Start”) projects focusing on learning about institutional partnership development.

Another important change to be noted is that TME now allows *authors to retain full copyright of their work* as opposed to transferring it to publishing entities that use our work to generate profit (Sriraman, 2012). Indeed the journal now exists as an independent entity, with open access, as well as supports professional associations like PMENA and other grass roots research groups by providing a peer reviewed outlet for ongoing research. Vol.11, no.3 [July 2014] will feature articles synthesizing 5+ years of research within the PMENA working group on *Pre-service Elementary Mathematics Teacher Content Knowledge*. This topic is particularly poignant to me since the first issue of this journal (vol1, no1, 04/2004) was the result of four idealistic elementary school teachers believing in the mission of this journal and writing about their attempts to reconcile the mathematics content they were learning in a mathematics for elementary school teachers course with existing mathematics education research found in practitioner’s journals as well as standards imposed by institutions framing policy.

I am thankful to the community for supporting the mission and the existence of this journal. Ten years ago, I dared to dream and imagine the possibilities *of and for* this journal. Time and dedicated work have allowed it to flourish in myriad uncharted directions and benefit many people. I wish the editors, authors and readers of The Mathematics Enthusiast a Happy 2013- Unlike the doomsday soothsayers predictions things continue to exist! To that end for T(i)ME to continue to exist (pun intended), I ask for your continued support...

Bangalore, India
Jan 4, 2013.

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Introduction to International Perspectives on Problem Solving Research in Mathematics Education

Luis Moreno Armella¹ & Manuel Santos-Trigo² (Mexico)

Any field of research and innovation must be exposed to revisions, criticisms and to an intense scrutiny not only to discuss the state of the art but, hopefully, to identify prospective changes and new areas of study and exploration as well.

Problem Solving has been such an area, with a prominent place in mathematics education and whose contributions continuously appear in conference proceedings, handbooks, journals, books and, more recently, in digital endeavors. Problem Solving involves an approach that fosters reflection and delving into mathematical ideas to explain individuals' cognitive behaviors within social media. Here, we argue that ideas do not live by themselves isolated from the semantic networks that sustain the life of cognition: meaning. These networks constitute a key ingredient for developing understanding and structural perspective of concepts through problems. In the long term, (and maybe not that long) these networks provide integration of knowledge that learners need to construct and integrate in order to gain a wide perspective.

Problem Solving drives developments through research programs, curriculum design, teachers' mathematical education, and mathematical instruction at the level of the classroom. Taking this and more into due account, and in order to identify current trends in mathematical problem solving and to foster a further exchange of ideas within our

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community, we discussed the present project with Bharath Sriraman, the editor of *The Mathematics Enthusiast*, who generously accepted to devote a special issue of the Journal to Mathematical Problem Solving. We invited colleagues, who have made significant contributions to this field, to contribute to the special issue. Previously, we had identified some lines of development to eventually frame their contributions. Of course this was not meant as a restriction to their freedom; rather to orient their possible directions. Some questions were posed and discussed to help us identify possible themes to consider in the volume. Thus, we tried to answer: *What are the current trends in problem solving research and what are the main results that influence teachers' practices and curricula design?* In addition, with the significant development of digital tools and environments, we are in need to understand to what extent the present research agenda is being or need to be transformed by their influence. This touches, for instance, deep epistemic issues concerning the nature of valid mathematical reasoning and results in mathematics in the classroom. We have to take into account that mediation tools are not neutral, neither from the cognitive nor from the epistemic viewpoint. That the knowledge students generate and/or appropriate, is intertwined with these tools. However, we cannot forget that a school culture always leaves significant marks on students' and teachers' values. Artigue (2005, p. 246) states that "these [previous] values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution." Thus, the school culture requires the gradual re-orientation of its practices to gain access to new habits of mind and to the new environments resulting from a serious presence of digital technologies.

We consider that how we understand the learning of mathematics through a problem-solving approach is deeply related, today, with the presence of the mediation tools that students will find and use in and outside the classroom.

We have shared with the invited authors a list of themes; we would like to mention some of them we consider particularly relevant.

Mathematical Problem Solving Foundations. Any domain of study needs to make explicit tenets and principles that support and justify its academic agenda. As we mentioned before, we are interested in documenting the extent to which theoretical and pragmatic frameworks are helpful to explain the problem solvers' development of new mathematical knowledge. Besides taking into account the role of mediational tools as the foundation of a research program, we need to consider as well the contrasts with a modeling approach to problem solving.

It is relevant to investigate how the presence of digital tools has transformed the agenda of problem solving approaches, which, in its early stages, has developed within a culture of paper and pencil mathematics. Of course, the lines of reasoning supported with and within, the new expressive media reflects what we have, before, termed the cognitive and epistemological consequences of the digital tools.

Mathematical Problem Solving and International Students' Mathematical Assessment. Currently, results from international assessments like PISA or TIMSS are used to compare or contrast students' mathematical competences among different

countries. In general, the media use the results to talk about the success or failure of national educational systems in science, mathematics, and language. Thus, it becomes important to discuss issues regarding the role of problem solving activities in the students' development of competences associated with those types of assessments. Some questions to discuss in this section involve:

- a) How are PISA and TIMSS goals and ways to assess students' mathematical achievement related to mathematical problem solving? Is the PISA framework consistent with frameworks used in mathematical problem solving?
- b) What makes a good task or problem foster and evaluate the students' mathematical thinking? The role of routine and non-routine problems in problem solving approaches.
- c) How can a problem be used for teaching and evaluating the students' comprehension of mathematical concepts?
- d) To what extent should we expect students to pose and solve their own problems?
- e) To what extent international assessments like PISA or TIMSS actually evaluate problem-solving competences, including those that demand the use of computational technology?

Mathematical Problem Solving and Curriculum Frameworks. A distinguishing feature of some current curriculum proposals is that they are structured to enhance problem-solving activities through all school levels. However, there is a need to discuss

what those proposals entail and should include in terms of contents and mathematical processes. Thus, relevant questions to discuss in this section involve the structure and organization of a curriculum centered in problem solving activities. Besides, we need to identify fundamental ideas and processes that are central to foster students' appropriation of mathematical knowledge and the ways digital media can be incorporated within the eventual proposals. Needless to say, the assessments conundrum will be lurking turning the corner.

As a consequence, the presence of digital technologies in education calls us to address this fundamental issue that curricular structures eventually will be inhabited by these technologies. It has already happened in the past: the technology of writing and the technology of positional notation of numbers are two of the milestones in the history of semiotic representations with a living impact on education.

Future developments of Mathematical Problem Solving. It is widely recognized that students should develop abilities, mathematical resources, and ways of thinking that help them formulate and solve not only school problems but also situations that they encounter outside the institutional settings. In this context, it becomes important for students to examine and explore phenomena in which they have the opportunity to examine information embedded in a variety of contexts in order to formulate, explore, and formulate meaningful mathematical questions. Thus, we will be in need to research the extent to which students can transfer problem-solving experiences learnt within the school context to new situations. This will of course, demands from the students the

abilities to move across the semantic field of a mathematical notion. This is far from being a trivial activity.

For instance how could students through problem solving phases that involve gathering data, modeling activities, find solutions and provide interpretations?

We received a positive response to our invitation letter from the authors and their contributions often address several issues discussed above. We hope that readers will find the contents of these two special issues useful to reflect on and extend their views about problem solving and we invite all to continue the discussion directly with the authors and other members of the problem solving community.

Reflections on Problem Solving Theory and Practice

Alan H. Schoenfeld¹

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Abstract: In this article, the author reflects on the current state of mathematical problem solving, both in theory and in instruction. The impact of the book *Mathematical Problem Solving* (Schoenfeld, 1985) is also discussed, along with implications of problem solving today with the advent of 21st century technologies.

Keywords: Mathematical problem solving; Mathematics teaching; Mathematical learning

Introduction

My book *Mathematical Problem Solving* (Schoenfeld, 1985), which I shall refer to as *MPS*) was published more than 25 years ago. *MPS*, which was fundamentally concerned with research and theory, had been developed in dialectic with a course in problem solving at the university level. The book provided a theoretical rationale for the course, and evidence that it worked; the course was an existence proof that, with the “right” kinds of instruction, students could become more effective problem solvers. The book-plus-course addressed a series of theoretical and pragmatic questions, some of which they answered, some of which they suggested answers to, and some of which they left unaddressed. Either directly or by logical extension the ideas in the book had the potential for significant curricular impact, if the “lessons” in them were taken seriously.

The question is, what has been the fate of the ideas that the book and the course embodied? Which ideas survived, which flourished? Which evolved in unpredictable

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ways, which withered with unfulfilled promise? I am grateful to the editors for the opportunity to reflect on the past and to think about future opportunities.

I begin by describing what, in my opinion, were the achievements, failures, and potential of that early work (which, of course, built upon and reflected the state of the field in 1985). This is followed by a characterization some of the main outcomes of the evolution of problem solving research and development. There is, of course, a huge literature on problem solving. It is impossible to do justice to that literature, and my comments will be selective. My most general comments are based, in part, on the volume *Problem solving around the world – Summing up the state of the art* (Törner, Schoenfeld, & Reiss, 2008). That volume provides a recent overview of theory and practice (and to some degree, curricular politics) in a wide variety of nations. This article will update my article in that volume (Schoenfeld, 2008), characterizing recent and potentially significant events in the U.S.

Problem Solving as of 1985 – a retrospective view

In theoretical terms, what *MPS* offered in 1985 was a *framework* for the analysis of the success of failure problem solving attempts, in mathematics and hypothetically in all problem solving domains. “Problem solving” at its most general was defined as trying to achieve some outcome, when there was no known method (for the individual trying to achieve that outcome) to achieve it. That is, complexity or difficulty alone did not make a task a problem; solving a system of 100 linear equations in 100 unknowns without the use of technology might be a real challenge for me, but it is not a *problem* in the sense that I know how to go about getting an answer, even if it might take me a very long time and I agonize over the computations.

The core theoretical argument in *MPS*, elaborated slightly in Schoenfeld (1992), was that the following four categories of problem solving activity are necessary and sufficient for the analysis of the success or failure of someone's problem solving attempt:

- a) The individual's knowledge;
- b) The individual's use of problem solving strategies, known as heuristic strategies;
- c) The individual's monitoring and self-regulation (an aspect of metacognition);
and
- d) The individual's belief systems (about him- or herself, about mathematics, about problem solving) and their origins in the students' mathematical experiences.

Regarding (a), little needs to be said; one's mathematical knowledge is clearly a major determiner of one's mathematical success or failure. Regarding (b): In 1985 I singled out heuristic strategies for special attention, because my major intuition when I began doing research on problem solving was that, with the right kinds of help, students could learn to employ the heuristic problem solving strategies described by Pólya (1945/57, 1954, 1962,65/81). Regarding (c): research over the course of the 1970s and early 1980s had revealed that how well problem solvers "managed" the resources at their disposal was a fundamental factor in their success or failure. When working complex problems, effective problem solvers monitored how well they were making progress, and persevered or changed direction accordingly. Unsuccessful problem solvers tended to choose a solution path quickly and then persevere at it, despite making little or no progress (see, e.g., Brown, 1987; Garofalo & Lester, 1985). Finally, regarding (d): by the

time that *MPS* was published, many counterproductive student beliefs, and their origins, had been documented. For example, students whose entire mathematical experience consisted of working exercises that could be solved in just a few minutes came to believe that “all problems can be solved in five minutes or less,” and ceased working on problems that they might have been able to solve had they persevered.

By these categories of behavior being “necessary and sufficient” for the analysis of problem solving success or failure, I meant that:

They were necessary in the sense that if an analysis of a problem solving failed to examine all four categories, it might miss the cause. It was easy to provide examples of problem solving attempts for which each of the four categories above was the primary cause of success or failure.

They were sufficient in that (I posited that) no additional categories of behavior were necessary – that the root cause of success or failure would be found in categories (a) through (d) above.

In *MPS* I claimed that the framework described above applied for all of mathematical problem solving; I had ample evidence and experience to suggest that that would be the case. I conjectured, on the base of accumulated evidence in other fields, that the framework applied to *all* problem solving domains, broadly construed. If you take problem solving in any of the sciences, there was a face value case for the framework. The relevant knowledge and strategies would be different in each domain – knowledge and heuristic strategies are different in physics or chemistry than in mathematics – but it was easy to see that the framework fit. But the potential application was broader. Consider writing, for example. Someone who sits down to write an essay, for example, is engaging

in a problem solving task – the task being to create a text that conveys certain information, or sways the opinion, of a particular audience. Various kinds of knowledge are relevant, both factual and in terms of text production. Writers use heuristic strategies for outlining, using topic sentences, etc. They can profit from monitoring and self-regulation; or they can lose track of their audience or argument, thus wasting time producing text that will ultimately be discarded. Finally, beliefs are critically important: the writer who believes that writing simply consists of writing down what you think will produce very different text from the writer who believes that crafting text is a challenging art requiring significant thought and multiple edits.

In sum, *MPS* offered a framework for analyzing the success or failure of problem solving, potentially in all problem solving domains. At the same time, the work reported in *MPS* had significant theoretical limitations. My analyses of problem solving all took place in the lab: one or two individuals sat down to work on problems that I had chosen. In various ways, this represented very significant constraints on their problem solving, and thus on my analyses. First, they were given the tasks. In most real-world problem solving, the tasks emerge in practice and have a history or context of some sort. Second, the goals were pre-determined (the students were to solve *my* problem) and the problems themselves were fixed. In problem solving “au naturel,” goals and the problems themselves often change or emerge in interaction. Third, the timescale was relatively short. Fourth, social interactions were minimal. Fifth and most important, *MPS* offered a framework, highlighting what was important to examine in order to explain success or failure. What *MPS* did not offer was a *theory* of problem solving – a characterization that allowed one to explain how and why people made the choices they did, while in the midst

of problem solving. All of these were limitations I wished to overcome. My ultimate theoretical goal has been to provide a theoretical explanation that characterizes, line by line, every decision made by a problem solver while working on a problem (trying to achieve one or more complex goals) in knowledge intensive, highly social, goal-oriented activities. In 1985 that goal was far beyond what the field could do.

Let me now turn to issues of practice. First and foremost, *MPS* was an existence proof, at multiple levels. At the macro level, the book provided evidence that my problem solving courses really worked – that my students became much more effective problem solvers, being able to solve more and more difficult *unfamiliar* problems after the course than before. At a finer level of grain size, examining students' work after the course showed that it was indeed possible for the students to master a range of problem solving heuristics; that they could become more effective at monitoring and self-regulation; and that on the basis of their experiences in the course, students were able to evolve much more productive beliefs about themselves and mathematics. At a yet finer level of grain size, *MPS* offered a methodological blueprint for developing problem solving instruction.

The challenge in 1975, when I began my problem solving work, was that heuristic strategies “resonated” – when mathematicians read Pólya's books his descriptions of problem solving strategies felt right – but, it had not yet been possible to teach students to use such strategies effectively. A major realization was that Pólya's descriptions of the strategies were too broad: “Try to solve an easier related problem” sounds like a sensible strategy, for example, but it turns out that, depending on the original problem, there are at least a dozen different ways to create easier related problems. Each of these is a strategy in itself; so that Pólya's name for any particular strategy was in fact a label that identified

a *family* of strategies. Once I understood this, I could “take apart” a family by identifying the main strategies that fell under its umbrella. I could teach each of those particular strategies (e.g., solving problems that had integer parameters by looking at what happened for $n = 1, 2, 3, 4 \dots$; looking at lower-dimensional versions of complex problems; etc.), and when the students had learned each of these, they had mastery of the family of strategies that Pólya had named. What that meant was that understanding and teaching Pólya’s strategies was no longer a theoretical challenge, but an empirical one. One could imagine a purely empirical, pragmatic program: take the main heuristic strategies identified by Pólya; consider each as a family of strategies and decompose them into their constituent parts; and work out a straightforward instructional program that enabled students to learn each of the constituent strategies. In this way, it should be possible to make problem solving accessible to all students. I hoped that some such work would take place.

A quarter-century later . . .

Issues of theory

Here there is good news, both in terms of what has been achieved and how the theoretical horizon has expanded. As noted above, the major challenge with regard to problem solving was to build a *theory* of problem solving, rather than a *framework* for examining it. More broadly, the challenge was to build a theory of goal-oriented decision making in complex, knowledge-intensive, highly social domains. Mathematical problem solving or problem solving in any content area, is an example. The goal is to solve the problem; knowledge (including knowledge of various strategies) is required; and, depending on the context, the problem solving activities may be more or less socially

engaged. Mathematics (or other) teaching is another, much more complex activity. The goals here are to help students learn mathematics. Achieving those goals calls for a huge amount of knowledge and strategy, and for deploying that knowledge amidst dynamically changing circumstances: when a student suddenly reveals a major misconception, for example, or it becomes apparent that the class does not have a good grip on something that the teacher thought they understood, the current “game plan” has to be revised on the spot and something else put in its place. In fact, if you can model decision making during teaching, it is straightforward to model decision making in other complex knowledge-intensive domains such as medical practice, electronic trouble-shooting, and more.

By “model” I mean the following. One needs to specify a theoretical architecture that says what matters, and say how decision making takes place within that architecture. Then, given any instance of such decision making (e.g., problem solving or teaching), one should be able to identify the things that matter in that instance, and show how the decision making took place in a principled way (that is, through a structured model consistent with the theoretical architecture), using only the constructs in the theoretical architecture to build and run the model. By way of crude analogy, think of Newton’s theory of gravity as providing a theoretical architecture (the inverse square law) for characterizing the motion of a set of objects. For each object (say the planets in our solar system, plus the sun) the some parameters need to be specified: mass, position, direction, velocity. The model of the solar system is given these data for time T , and the theory is used to specify these parameters for time $T+1$. The theory, then, is general; each model (whether of our solar system or some other galactic system) is a specific instantiation of the theory. The quality of any particular model is judged by how well the behavior of the

objects represented in the model corresponds to the behavior of the objects being represented. (A model of the solar system had better produce motion that looks like the motion of the planets in our solar system!) The quality of the theory is judged by its accuracy and its scope – what is the range of the situations for which it can generate accurate models? (A theory that only modeled two-body gravitational systems wouldn't be very exciting.)

Twenty-five years after *MPS* was published, my new book *How We Think* (Schoenfeld, 2010) builds on the earlier work and lays out the structure of a general theory of in-the-moment decision making. The architecture it describes is straightforward: what one needs for a theoretical account of someone's decisions while that person is engaged in a familiar goal-oriented activity such as problem solving, teaching, or medical practice is a thorough description of:

- a) The goals the individual is trying to achieve;
- b) The individual's knowledge (and more broadly, the resources at his or her disposal);
- c) The individual's beliefs and orientations (about himself and the domain in which he or she is working); and
- d) The individual's decision-making mechanism.

These categories represent the natural evolution of the categories in the 1985 framework. Regarding (a), depicting the goals is necessary in that the theory describes a much broader spectrum of behavior than problem solving. Depending on context, one's highest priority goal may be, for example: to solve a problem; to make sure that one's students understand a particular body of mathematics; or to diagnose a patient

appropriately and set him or her on a path toward recovery. Regarding (b), the role of knowledge is still central, of course: what one can achieve depends in fundamental ways on what one knows. In my current theoretical view I fold access to and implementation of heuristic strategies into the category of knowledge. I always viewed problem solving strategies as a form of knowledge, of course – but, in the problem solving work I was trying to validate their importance and utility, so they were separated out for special attention. In addition, I add “resources” into the category of “what the individual has to work with”: the approach one takes to a problem may vary substantially depending on, for example, whether one has access to computational tools on a computer. Regarding (c), beliefs still play the same central role in shaping what the individual perceives and prioritizes as in my earlier work. I have chosen to use the word orientations (including preferences, values, tastes, etc.) as a more encompassing term than beliefs because, for example, choices of what to purchase for dinner and how to cook it, while modelable in terms of the architecture I specify, aren’t necessarily a matter of beliefs.

Regarding (d), the decision making mechanism in the theory is implemented in two ways. If circumstances are familiar – that is, one is collecting homework or going over familiar content in class – people use various mechanisms described in the psychological literature (scripts, frames, schemata, etc.) that essentially say what to do next. If circumstances suddenly vary from the predictable – e.g., a student makes a comment indicating a serious misconception, an explanation obviously doesn’t work – then it is possible to model the individual’s decision making using a form of subjective expected utility. (The various options that might be used are evaluated in light of their perceived value to the person being modeled, and the higher a valuation an option

receives the more likely the option is to be chosen.) Monitoring and self-regulation, which were a separate category in *MPS*, still play a centrally important role – but here they are placed as a major component of decision making.

To my mind *How We Think* has roughly the same status today that *MPS* had in 1985. The book offers a number of very detailed case studies, showing how a wide range of mathematics teaching can be modeled, and an argument suggesting (by virtue of the breadth of “coverage” in the cases) that the model applies to all teaching. Then, there are suggestions that the theory should suffice to describe goal-oriented decision making in all knowledge-intensive fields. This is a heuristic argument similar to the argument I made in 1985, that the problem solving framework I explicated for mathematics should apply to all problem solving disciplines. Time will tell if the theory holds up.

While *How We think* brings to fruition one theoretical line of inquiry, it also opens up a number of others – lines of investigation that I think will be fruitful over the coming decades. These may or may not strike the reader as falling under the banner of “problem solving” – but, they should, if the question is, what do we need to know about thinking, teaching, and learning environments to help students become more effective mathematical thinkers and problem solvers? (I will revisit this question directly when we turn to practical issues.)

My work to date has examined problem solving through the lens of the individual, at any point in time. That is, the question has been, how and why does the individual go about making decisions in the service of some (problem solving) goals, given what he or she knows? These are serious limitations. First, the focus on what is happening in the moment ignores questions of learning and development. The person who has worked on,

and solved, a problem, is not the same person who began working on it. He or she approaches the next problem knowing more than before. So, one question is, how can issues of learning and development be incorporated into a theory of decision making? This is a deep theoretical question, which may not have immediate practical applications – but, if we can trace typical developmental trajectories with regard to students’ (properly supported) ability to engage in problem solving, this might help shape curriculum development. More generally, if our goal is to theorize cognition and problem solving, such issues need to be addressed.

Second, individuals do not work, or learn, in a vacuum. As will be seen below, characterizing productive learning environments – and the norms and interactions that typify them – is an essential endeavor, if we are to improve instruction. But learning environments are highly interactive, and the ideas that individuals construct are often built and refined in collaboration with others. At minimum, a theory of learning and cognition that explains how ideas grow and are shared in interaction is critical. There is much to be done on the theoretical front.

Issues of practice

Here, the question is whether one wishes to view the metaphorical glass as being half empty or half full. There is reason to be disheartened, and reason to be encouraged. And there is work to be done.

On the one hand, there are ways in which we could and should be much further along in curricular development (and the research that would undergird it) than we are. As explained above, there was an implicit blueprint for progress in *MPS*: the methods I described for decomposing heuristic strategies into families of more fine-grained

techniques, and finding out how much instruction was necessary for those techniques to become learnable, were well enough characterized for others to implement them. That is, 25 years ago it was theoretically possible to begin a straightforward program of development that would result in successful instruction on a wide range of problem solving strategies. “All” that was needed was a huge amount of work! That work did not get done. There are systemic reasons for this, which Hugh Burkhardt and I (Burkhardt & Schoenfeld, 2003) have explored. University reward systems work against this kind of work. There is no theoretical “glory” in working through such pragmatic issues, either for the individual or in terms of promotion decisions at research universities. Making significant progress at the curricular level calls for a team of people, and university reward structures are stacked against that as well – our system tends to reward individual achievements, and to give less credit for collaborative work. Perhaps for those reasons, perhaps because there are fads and fashions in educational research (as in all fields), an area that I considered to be fertile ground for practical development went unexplored. I think that’s a shame.

At the same time, some good things have happened in K-12 education. A global summary of developments can be found in Törner, Schoenfeld, & Reiss, 2008. Here I will summarize the optimistic view regarding the past 25 years in the U.S., and then point to the fact that we are at a turning point, where much hangs in the balance.

In contrast to some nations where a ministry of education or its equivalent makes curricular decisions that are implemented nationwide, the U.S. has had what is best described as a “loosely coupled” system. For almost all of its history, each of the states had its own educational system, which was responsible for setting statewide standards.

Historically, textbook decisions have been local – each of the roughly 15,000 school districts in the U.S. could choose its own textbooks. Until the past decade, few states had statewide assessments, so there was little pressure to “teach to the test.” There were homogenizing factors, of course. There were a small number of textbook publishers, so textbook choice, though theoretically unconstrained, was limited in practice; and, most school districts aimed at preparing their college-intending students for the (essentially universal) college calculus course, so the goal state was clearly established. By tradition, grades K-8 focused on arithmetic and then pre-algebra; algebra I was taken in 9th grade, plane geometry in 10th, algebra II and possibly trigonometry in 11th, and pre-calculus in 12th. Some students accelerated through calculus in high school; many students dropped out of the pipeline altogether. (The generally accepted figure in the 1980s was that each year, some 50% of the students at each grade level in secondary school failed to take the next year’s mathematics course.) There was huge variation in the courses students took, but “traditional” instruction focused mostly on conceptual understanding and mastery of skills and procedures. There was a negligible amount of “problem solving,” by any real definition.

In 1989 the National Council of Teachers of Mathematics produced the *Curriculum and Evaluation Standards for School Mathematics*. The volume was intended for teachers, and had few references; but the authors knew the problem solving research, and it showed. For the first time in a major policy document, there was a significant emphasis on the *processes* of doing mathematics: The first four standards at every grade level focused on problem solving, reasoning, communication using mathematics, and connections within and outside of mathematics. The U.S. National Science Foundation,

recognizing that commercial publishers would not build such textbook series on their own, issued a request for proposals for the creation of “Standards-based” texts. In each of these texts, the authors elaborated their own vision of what it meant to learn according to the *Standards*. This variety was a good thing: different visions of a richer mathematics, focused on problem solving and reasoning, began to emerge. It is hard to get precise figures, but some estimates are that 20-25% of the K-12 textbooks in use today are Standards-based. Given the vagaries of the “loosely coupled” educational system in the U.S., that’s a non-trivial impact for research ideas! (Of course it took 25 years, and the ideas don’t necessarily reflect, and may sometimes be contradictory to, the views of the original researchers. But that’s the way the system works.)

So, there has been curricular progress in K-12 mathematics in the U.S., if not as much as one would like. Recent political events mean that the progress will either be accelerated or blocked, in the near future. As part of an attempt to improve mathematics instruction called the “Race to the Top” initiative (see <http://www2.ed.gov/programs/racetothetop/index.html>), the federal government offered fiscal incentives to *collections* of states that produced high quality standards and plans for assuring that students reached them. Given the short time frame to apply for funding, the U.S. National (State) Governors Association and the Council of Chief State School Officers supported an effort to construct a set of standards for mathematics, known as the Common Core State Standards (see <http://www.corestandards.org/>, from which the Common Core State Standards for Mathematics (CCSSM) can be downloaded.)

As of this writing, 44 of the 50 states have committed to the Common Core initiative, meaning that they will replace their current state standards with CCSSM. By

federal statute, they will need to use assessments (tests) that are deemed consistent with the CCSSM in order to measure student progress toward the goals of CCSSM. Two national consortia have been funded to produce assessments consistent with the CCSSM: PARCC (the Partnership for Assessment of Readiness for College and Careers; see <http://www.achieve.org/PARCC>) and the SMARTER Balanced Assessment consortium (see <http://www.k12.wa.us/smarter/>). By the time this article appears, both consortia will have published their “specs” for assessments consistent with CCSSM. Simply put, those assessments (tests) will shape the mathematics experiences of the students in the states that have committed to the Common Core State Standards initiative. As we know, testing – especially high-stakes testing – determines the foci of classroom instruction. The CCSSM place significant emphasis on what they call mathematical *practices*, claiming that people who are mathematically proficient:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

If the tests produced by the consortia provide students with opportunities to demonstrate such mathematical habits of mind, the tests will serve as a lever for moving the K-12 system in productive directions. But, if they consist largely of short answer

questions aimed at determining students' mastery of facts and procedures, they will serve impede the kind of progress we have been making over the past 25 years.

In sum, progress in K-12 has been slow but steady; it may get a boost or a setback in the immediate future, depending on the high-stakes tests that the two assessment consortia adopt. But there has been significant progress. I wish I could say the same about collegiate mathematics over the same time period. For a while calculus reform flourished, but it seems to have stabilized and become "same old, same old." There have been glimmers of excitement surrounding innovations in linear algebra and differential equations (stimulated in some part by technology), but not so much that the general zeitgeist of collegiate mathematics instruction is noticeably different from what it was when I was a math major. And that was a long time ago!

Rethinking "problem solving"

I got my "start" in problem solving and I still think that, in some ways, it deserves to be called "the heart of mathematics" (Halmos, 1980). More broadly, there is a view of mathematics as the "science of patterns" (Steen, 1988). What I like about this framing is that one thinks of science as consisting of systematic explorations – and mathematics as we practice it certainly has that character. This broad a framing includes problem posing as well as problem solving, and a certain form of empiricism, which was made explicit in Pólya's (1954) title, "patterns of plausible inference." In doing mathematics we explore; we seek systematicity; we make conjectures; and, we use problem solving techniques in the service of making and verifying those conjectures. Yet more broadly, we engage in

“thinking mathematically” – a title used by John Mason, Leone Burton, and Kaye Stacey (1982, 2010) and which I wish I’d thought of. Here I want to push things one step further.

At heart, doing mathematics – whether pure or applied – is about sense-making. We observe an object, or a relationship, or a phenomenon, and we ask: What properties must it have? How do we know? Do all objects that look like this have the same property? Just what does it mean to “look like this”? Are there different ways to understand this? With that mindset, simple objects or observations become the starting points for explorations, some of which become unexpectedly rich and interesting. Third graders observe that every time they add two odd numbers the sum is even. Must it always be so? How would one know? We observe that some numbers can be factored, others can’t. How many of the unfactorable kind are there? How can I measure the height of a tree without climbing it? How many different crayons do I need to color a map, so that every pair of countries that share a border have different colors?

What I strive to do in my problem solving courses is to introduce my students to the idea that mathematics is about the systematic exploration and investigation of mathematical objects. Elsewhere (see, e.g., Schoenfeld, 1989; see also Arcavi, Kessel, Meira & Smith, 1998; English & Sriraman, 2010) I have described our first-week discussions of the magic square. We start with the 3×3 , which my students solve easily. But that is just a start. Did we have to get an answer by trial and error, or are there reasons that even numbers go in the corners, and that 5 goes in the center? Is the solution unique (modulo symmetry), or are there distinct solutions? Having finished with the original 3×3 , we ask: what if I had nine other integers? Say 2 through 10, or the odd numbers from 1 through 17, or any arithmetic sequence? Can we find a magic square for

which the sum of each row, column, and diagonal is 87? How about 88? We observe that if we multiply all the cells in a magic square by a constant, we get a magic square; if we add a constant to each cell, we get a magic square. Thus, we can generate infinitely many 3 x 3 magic squares. But can we generate all the 3 x 3 magic squares this way?

The reason for this discussion is that I want to introduce my students to what it means to *do* mathematics. I want them to understand that mathematics isn't just about mastering facts and procedures, but that it's also about asking questions (problem posing, if you will) and then pursuing the answers in reasoned ways. The problem solving strategies are tools for sorting things out, seeing what makes the mathematical objects and relationships "tick." So yes, we are solving problems, but as part of a larger sense-making enterprise. That, in part, is why attending to my students' beliefs is so important a part of the course. Having been "trained" by their prior experience to understand (believe) that doing mathematics means "mastering" content selected and organized by others; that all problems can be solved in short order, usually by the techniques the teacher has presented within the past week; that proofs have nothing to do with discovery; and so on², my students needed to be "untrained" or "retrained" by their experiences in my course. Thus I give them extended opportunities to make observations and conjectures, and provide them with the tools that enable them to experience the doing of mathematics as a sense-making activity.

My question is, can't we approach all mathematics teaching this way? I believe that all of K-12 mathematics, and a good deal of collegiate mathematics, can be seen as a set of sensible answers to a set of sensible questions. Given the pace at which K-12

² See Schoenfeld (1992) for a list of counterproductive beliefs that students typically develop.

mathematics proceeds, I am sure that this could be done without any formal loss of content. What would be gained is that students would experience mathematics as an exciting sense-making domain, which is the way we see it as mathematicians. If K-12 students truly experienced mathematics that way, I'm willing to bet that similarly oriented collegiate instruction could build on well-established habits of mind, and proceed much more effectively than it currently does. (Despite the fact that the students in my problem solving courses through the years could be labeled "the best and the brightest" – it's a non-trivial achievement to get into Berkeley – my feeling has always been that my problem solving courses have been remedial in a significant way. The vast majority of the students who entered those courses were unaware of basic mathematical problem solving strategies, and, as a function of their experience, did not view mathematics as a domain that they could make sense of.)

Rethinking Research on Classroom Environments

A comment made by one of the advisory board members of one of my projects ("Classroom Practices that Lead to Student Proficiency with Word Problems in Algebra", NSF grant DRL-0909815), struck me as particularly interesting. Megan Franke (2011) noted that, of the various classroom variables she had looked at, the one that seemed to have the strongest impact on student learning was the amount of time students spent explaining their ideas. This resonates, not only with the discussion of sense-making above, but with an emerging body of research focusing on the character of classroom environments that support the kinds of rich student engagement and thinking that one would like. Engle, for example (Engle, in press; Engle and Conant, 2002) has developed

a “productive disciplinary engagement framework.” Reviewing the best-known examples of rich learning environments in mathematics and science, Engle concludes that the most powerful learning environments all include aspects of:

- Problematizing – students participate in the act of framing meaningful questions, which the class explores.
- Agency and authority – students are empowered to seek information, distill it, craft arguments, and explain them.
- Disciplinary accountability – students learn what it is to make claims and arguments that are consistent with disciplinary norms.
- Resources – when tools or information is needed, the students have access to them.

For an elaboration of these ideas with examples drawn from my problem solving courses, see Schoenfeld (2012). Gresalfi, Martin, Hand, & Greeno (2009) offer a framework (see Fig. 1) that indicates ways on which classroom participation structures can lead to differential outcomes in terms of student agency, argumentation, and accountability.

<i>Negotiation of</i>	Participation structure	How an idea enters the common ground <i>Other people need to be able to understand</i>	<i>Leads to definitions of</i>	What students are accountable for <i>Making sure that their symbols make sense</i>
		Who is expected to author or critique ideas <i>Other students</i>		
		Norms of argumentation <i>students need to talk to and convince each other</i>		
	Task as realized	Requirements for sense-making <i>doing mathematics—coordinating different records of the trip</i>		Who students are accountable to <i>Other students and the teacher</i>
		Openness of the task <i>moderate—inventing symbols that others' could understand</i>		
		Requirements for successful completion <i>Students need to be able to understand others' representations</i>		What kinds of agency students can exercise. <i>Conceptual</i>

Fig. 1. A model of how competence gets constructed in the classroom.
From Gresalfi, Martin, Hand, & Greeno (2009), p. 54, with permission.

This kind of framework can be useful, as we seek to understand both how to craft classrooms more focused on sense making and to document their effects.

Rethinking Technology (a.k.a. entering the 21st century)

When I took my first statistics class, all of the examples were “cooked.” This was before the days of widespread access to calculators and computers, so everything I did had to be hand-computable. As a result, the variation of every distribution I worked with was a perfect square! The presence of computational technology should have radicalized the ways in which our students can engage with statistics, in that no data analysis is now an obstacle. Students should be able to ask their own questions and gather their own data. Yet, few students have this experience. In Schoenfeld (2012) I give the example of how, sitting at my desk in early June and watching it rain, I wondered whether this was

atypical. The San Francisco Bay Area is supposed to have a “dry” season, and it seemed that we had gotten more rain than we should have. Using Google I was quickly able to find data regarding annual monthly rainfall and recent rainfall, at which point I could do some simple statistical analyses to verify that this year’s June rainfall was anomalously high. (In fact, it went on to set a record.) From my perspective, I was clearly doing mathematics. My question is, where do today’s students learn to gather such information and to operate on it? Asking questions, seeking data, building models, and drawing inferences should be everyday experiences for our students.

Similarly, the presence of computational tools – whether symbolic calculators, graphers, or Wolfram Alpha (see <http://www.wolframalpha.com/>) – has the potential to radically reshape the knowledge to which students have access in mathematics classrooms, and the ways they can operate on it. Pure mathematics can become an empirical art for students in ways that it was not, even for mathematicians, until recently. Where are students learning to harness these skills – not for the sake of learning to be fluent with technology, but as means to mathematical ends? There have been some inroads along these lines, for example with dynamic geometry software, but for the most part these positive examples are the exceptions that probe the rule.

Concluding Comments

I thank to the editors for the opportunity to think about the current state of mathematical problem solving, both in theory and in instruction. I became a mathematics educator many years ago because of my love for mathematics and my wish to share it with students, who were typically deprived of the pleasures that I consistently experienced as a mathematician. Problem solving provided a way into the joys of doing

mathematics and the pleasures of discovery. I firmly believe that problem solving – or a broader conception of mathematics as sense making – still can do so, and I hope to see us make progress along those lines.

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Problem Solving in the Primary School (K-2)

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Lesh: “Do you really think your children can do this?”

Riggs: “So far, nobody has taught them yet about what they can’t do.”

Abstract: This article focuses on problem solving activities in a first grade classroom in a typical small community and school in Indiana. But, the teacher and the activities in this class were not at all typical of what goes on in most comparable classrooms; and, the issues that will be addressed are relevant and important for students from kindergarten through college. Can children really solve problems that involve concepts (or skills) that they have not yet been taught? Can children really create important mathematical concepts on their own – without a lot of guidance from teachers? What is the relationship between problem solving abilities and the mastery of skills that are widely regarded as being “prerequisites” to such tasks? Can primary school children (whose toolkits of skills are limited) engage productively in authentic simulations of “real life” problem solving situations? Can three-person teams of primary school children really work together collaboratively, and remain intensely engaged, on problem solving activities that require more than an hour to complete? Are the kinds of learning and problem solving experiences that are recommended (for example) in the USA’s Common Core State Curriculum Standards really representative of the kind that even young children encounter beyond school in the 21st century? ... This article offers an existence proof showing why our answers to these questions are: Yes. Yes. Yes. Yes. Yes. Yes. And: No. ... Even though the evidence we present is only intended to demonstrate what’s possible, not what’s likely to occur under any circumstances, there is no reason to expect that the things that our children accomplished could not be accomplished by average ability children in other schools and classrooms.

Keywords: Common core standards; elementary mathematics education; problem solving in elementary school;

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Can Children Solve Problems involving Concepts they have not been Taught?

Most people's ordinary experiences are sufficient to convince them about the truth of two important assumptions about learning and problem solving.

- First, the kinds of things that students can learn, and the kinds of problems that they can solve, tend to be strongly influenced by the things they already know and are able to do. So, the accompanying “common sense assumption” is that these prerequisites must be mastered before students are expected to learn relevant new ideas, or solve relevant new types of problems. And consequently, learning is viewed as a long step-by-step process in which prerequisites are checked off one at a time.
- Second, concepts and abilities do not go from unknown to mastered in a single step. They develop! And, so do associated abilities. In fact, especially for the most important “big ideas” in the K-12 curriculum, development typically occurs over time periods of several years, and along a variety of dimensions – such as concrete-abstract, intuition-formalization, situated-decontextualized, specific-general, or increasing representational fluency, or increasing connectedness to other important concepts or abilities. So, in situations which are meaningful and familiar to students, rapid developments often occur for clusters of related concepts and abilities. And, in these contexts, students' ways of thinking often integrate ideas and abilities associated with a variety of textbook topic areas – so that the resulting knowledge and abilities are organized around experiences as much as around abstractions.

For readers who are familiar with Vygotsky's zones of proximal development, the title of this section poses a question that is clearly naïve. Learning does not occur in this all-or-nothing manner. For example, in a series of projects known collectively as The Rational Number Project (RNP, 2011), it is well known that the "difficulty level" of a given task can be changed by years – simply by changing the context or the representational media in which problems are posed (e.g., written symbols, written language, diagrams or graphs, concrete models, or experience-based metaphors). Consequently, when students encounter a problem in which some type of mathematical thinking is needed, all of the relevant concepts and abilities can be expected to be at some intermediate stages of development – not completely unknown, yet not completely understood – regardless of whether these concepts or abilities have been formally taught.

In fact, for researchers who have investigated what it means to "understand" the most powerful and important ideas in the elementary school curriculum, it has become clear that most of the "big ideas" that underlie the K-12 curriculum begin to develop in early years– in topic areas ranging from rational numbers and proportional reasoning (RNP, 2011), to measurement and geometry (e.g., Krutetskii, 1976), to statistics and probability (e.g., Zieffler, Garfield, delMas, & Reading, 2008), to early ideas in algebra (English, in press; Thompson, 1996) or calculus. In fact, in each of these domains of mathematical thinking, many important understandings typically begin to develop even in the primary grades (K-2). Such observations are reminiscent of Bruner's claim, long ago, that: *Any child can be taught any concept at any time – if the concept is presented in a form that is developmentally appropriate* (Bruner, 1960). Of course, the "if clause" in this quote is very significant. That is, in order for remarkable developments to occur,

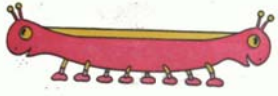


relevant concept and abilities need to be accessible in the forms that are developmentally appropriate.

For the problems that will be described in this chapter, the two primary tests of developmentally appropriateness are: (a) Do the children try to make sense of the problem using their own “real life” experiences – instead of simply trying to do what they believe that some authority (such as the teacher) considers to be correct (even if it doesn’t make sense to them)? (b) When the children are aware of several different ways of thinking about a given problem, are they themselves able to assess the strengths and weaknesses of these alternatives – without asking their teacher or some other authority? When these two criteria are satisfied, children are able to go from “first-draft of thinking” to “Nth-draft of thinking” without interventions from an outside authority.

When referring to “real life” sense-making abilities, it is important to emphasize that we are not assuming that a first grader’s “real life” interpretations of experiences are the same as an adult’s one. For example, for first graders, children’s stories often engage their sense making abilities more than situations that an adult might consider to be a “real life” situation. So, for the problems that we’ll describe in this article, the tasks were presented in the context of stories such as *Two Headed Stickbugs*, *The Proper Hop* (for *Beauregard the Frog*), *Fussy Rug Bugs*, *Isabelle Talks*, *The Royal Scepters*, or *Tubby the Train* (see Figure 1) – most of which appeared first in Scott Foresman’s longest running kindergarten book - written by Lesh & Nibbelink (1978).

For our purposes in this article, some other important of “real life” characteristics that we tried to build into our problems include the following. (a) The product that the children are challenged to produce often is not just a “short answer” to a pre-

mathematized question. It is a sharable and reusable tool or artifact that needs to be powerful (in the given situations) and also sharable (with other people) and reusable (in situations beyond the given situation). (b) The children need to know who needs their product - and why? Otherwise, they won't be able to assess the strengths and weaknesses of alternative products; and, they won't be able to judge: is a 3-second answer sufficient; or, is a 3-minute or 30-minute needed? (c) Product development is likely to require several cycles of testing and revising – similar to first, second, or third drafts of written descriptions of drawn pictures. (d) Products will often need to integrate ideas and procedures drawn from a variety of textbook topic areas. One reason for that this is true is because “real life” problem often involves partly conflicting constraints and trade-offs (e.g., low cost but high quality, simple but powerful).

<h3>Two-Headed Stickbugs</h3> <p>Two-headed stickbugs are cute little stinkers and having two heads makes them very good thinkers. Now if you should see one with two heads a-showing you might not be sure if it's coming or going.</p>  <p>All two-headed stickbugs are equally long and all little stickbugs are said to belong to a family of bugs who think it's a ball to measure and say just how long and how tall.</p>  <p>To answer each question, they stand head to head on the thing to be measured, and then it is said, they decide on how many of them it took to measure a stick, or the height of a book.</p>	<h3>The Proper Hop</h3> <p>As swamps go, none was finer for frogs than Sugar Swamp. During warm days, frogs hopped and flopped around whenever and wherever they liked. But mainly, they just sat around, listening to the ducks quack, or waiting for flies to come too close.</p> <p>Just before sunset, all the frogs would climb onto their lily pads. The last frog to climb onto his pad was always Beauregard, the biggest, grandest bullfrog in the swamp.</p> 
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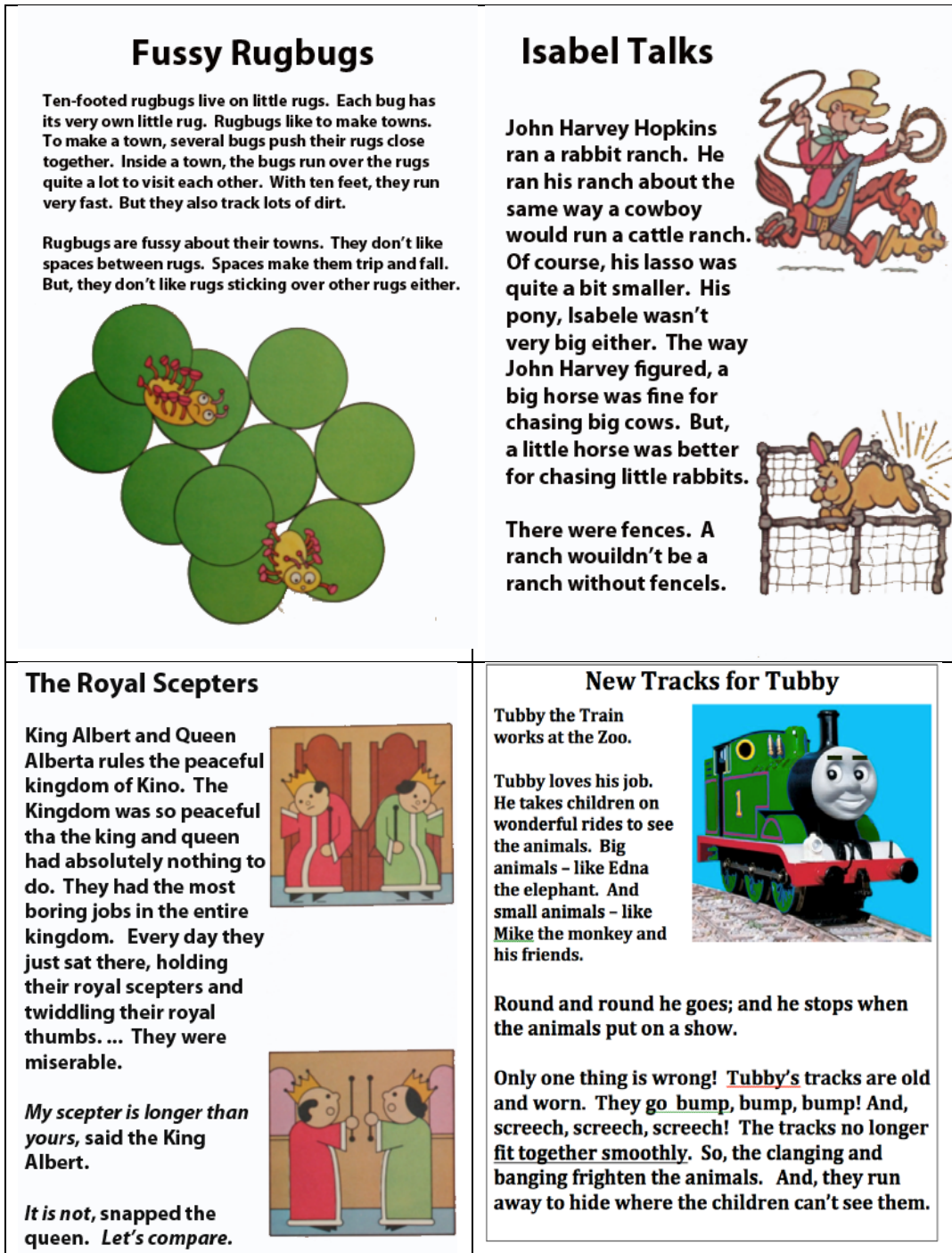

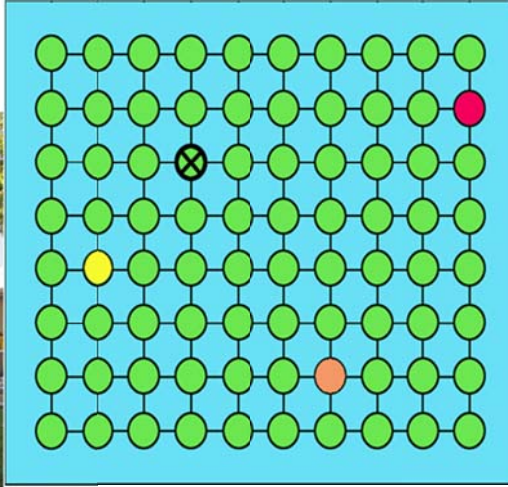


Figure 1. The First Pages from Six Stories

Figure 1 shows the six contexts that were used for the problems which will be described in this article. Then, Figure 2 briefly describes the tasks that accompany each of these stories. For each task, the children worked in groups of three; the work spaces

usually were at least the size of large poster boards; and, the products usually needed to include letters to “someone else” describing how they could do and what our students did. This letter-writing aspect of the tasks was emphasized partly because the school in which we worked was focusing on writing abilities. But, it also was used because, to be consistent with our published principles for designing *model-eliciting activities* (Lesh, et. al., 2000), it is important for the products to be more than single solutions to isolated problems. Solutions to our problems also needed to be sharable and reusable. So, one straightforward way to achieve this goal was to ask children to write letters describing tools or artifacts that can be used by others.

<p>Two-Headed Stickbugs</p> <p>Two different sized “stickbugs” were used – one made with popsicle sticks, and the other made with meter-long strips of wood. In a warm-up activity, the children work in groups of three, and use meter-long “stickbug” to measure as many distances as possible in the playground of their school. (note: On boring playgrounds, markers can be placed using highway markers or signs.) The teacher should record these distances by drawing arrows and points on a poster-sized photograph (or drawing) of the school yard. Then, the next day, children should again work in groups of three, and use the teacher’s notes and popsicle-sized stickbugs to create miniature scale-models of their playground.</p> 	<p>The Proper Hop</p> <p>The green dots shown below are lily pads. The lines between dots indicate “proper hops” (which must be horizontal or vertical hops to adjacent lily pads). The red, yellow, and orange dots are where three of Beauregard’s friends live. And, the children’s task is to find a place where Beauregard (x) should live so that the sum of the distances to his three friends’ houses is as small as possible.</p> 
<p>Fussy Rugbugs</p> <p>What is the largest number of rugbugs that can live totally inside the area shown below? (Give each group enough post-it stickies to cover the area; and, give approximately one-third of the stickies to each child. Either square or round stickies can be used. But, in either case, the stickies should (a) be completely inside the area indicated, (b) not overlap, and (c) fit together as closely as possible.</p>	<p>Isabel Talks</p> <p>The “trees” shown below should be unevenly distributed. They represent apple trees, with big juicy apples that Isabel loves to eat while standing in their shade. The goal is to build a closed fence which encloses the largest number of apple trees. The fence is a loop of soda straws or coffee stirrers – strung on a closed loop of string.</p>

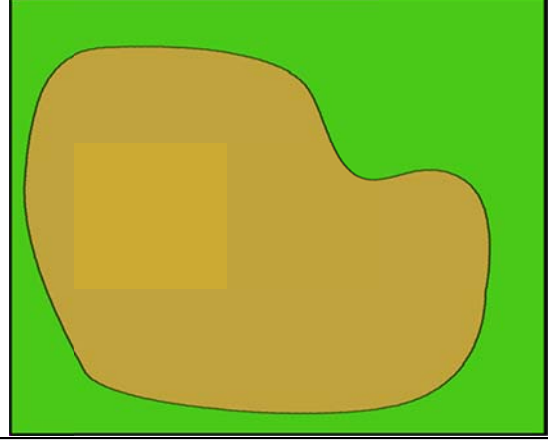
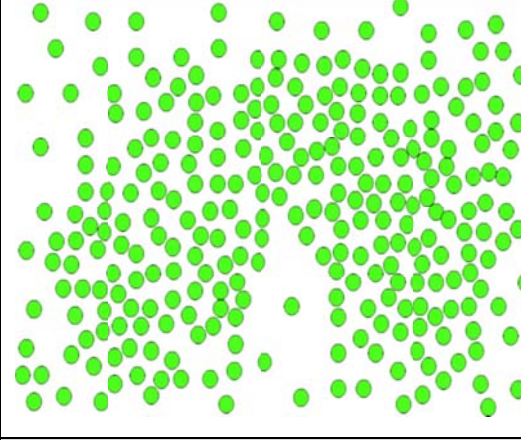


	
<p>The Royal Scepters</p>	<p>New Tracks for Tubby</p>
<p>Several different templates for gingerbread houses (like the one shown below) can be downloaded from the internet – and the sizes of the pieces can be marked using small “toothpick” rulers. Then, different groups of children should cut cardboard pieces to make parts for several houses – e.g., with one group specializing in roofs, and other groups specializing in side walls, or end walls. The goal is for the pieces to fit together for each house.</p>	<p>For the Tubby the Train Problem, the train track pieces look like the ones shown here. The goal is to make a track for Tubby so that:</p> <ol style="list-style-type: none"> 1. As many as possible of the pieces are used up – so that none will be wasted. 2. The track makes as many closed loops as possible – so that many different animal houses can be put inside of loops. 3. The pieces of track fit together smoothly – so there will be no bumps and screeches. 4. The track should not have any dead ends. <div style="display: flex; align-items: center; justify-content: center;">  <div style="margin-left: 10px;"> <p>4 Right Forks</p> <p>4 Left Forks</p> <p>2 Splitters</p> <p>27 Straits</p> <p>18 Curves</p> </div> </div> <p>Notice that all of the train track pieces are marked on hexagonal shapes so that it is as clear as possible when the tracks fit together well.</p>
	

Figure 2. Tasks for Six of the Stories

Can Children Create Important Mathematical Concepts on Their Own?

If we examine the title of this section, two opposing answers to this question seem to be equally obvious - depending on how the question is interpreted. If the question is intended to ask: Can children invent the most fundamental concepts in algebra, geometry,

or calculus?, then the answer clearly must be: No! It took years for some of history's most brilliant mathematicians to invent these concepts. So, average ability children cannot be expected to do such things during single class period? But, if the question is asking: Can children use numbers to describe mathematically interesting situations in the mathematical "objects" involve more than simple counts of discrete objects (i.e., cardinal numbers), then one of the main points of this paper is that the answer to this question is: Yes! For example, the six problems that we describe in this article involve using numbers to describe locations (coordinates, or ordinal numbers), lengths or distances (or other types of measurable quantities), signed quantities (negative numbers), directed quantities (vectors), actions (operators, transformations, functions), changing quantities (rates or intensive quantities), or accumulating quantities (calculus). In particular, for the six stories described here:

- Children's responses to the Stickbug Problem often use numbers to describe lengths, distances, and sometimes even coordinates – if the "map" is thought of as a simple kind of grid.
- Children's responses to the Proper Hop Problem often use numbers to describe locations, actions (hops), number patterns, or quantities that have both a magnitude and a direction.
- Children's responses to the Fussy Rugbugs Problem often use numbers to describe areas or dimensions (concerning how "rugs" are aligned within shapes).
- Children's responses to the Isabel Talks Problem often use numbers to describe relationships between areas and perimeters, and even negative

numbers (because when borders are rearranged to include some new “trees” and other “trees” tend to be lost).

- Children’s responses to the Royal Scepters Problem often use numbers to describe scaling-up, proportions, ratios, lengths, distances, shapes (e.g., rectangles, triangles), and sometimes angles or areas .
- Children’s responses to the Tubby the Train Problem often use numbers to describe lengths, angles, and negative quantities (which occur pieces of tracks are inserted or deleted in order to eliminate dead ends, or in order to enlarge or shrink enclosed areas).

Of course, from a child-eye view, the preceding situations are not about ordinal numbers, coordinates, signed numbers, vectors, operators – or areas, volumes, or densities. To the children, they are simply contexts in which numbers are used to describe things such as: hops, measuring sticks, sticky post-it notes, straws, or paths. Nonetheless, because the tasks require children to externalize their thinking in forms that are visible to the students themselves (as well as teachers and researchers), the seeds are apparent for many of the most important “big ideas” that span the entire K-12 mathematics curriculum.

In general, what research based on models & modeling perspectives (Lesh & Doerr, 2003) shows that, if children clearly recognize the need for a specific kind of mathematical description, diagram, artifact, or tool, and if the children themselves are able to assess strengths and weaknesses of alternative ways of thinking, then remarkably young children are often able to produce impressively powerful, reusable, and shareable tools and artifacts in which the mathematical “objects” being described involve far more

than simple counts. However, even though children are able to generate such descriptions without guidance from adults, this claim does not imply that there is no role for teachers. For example, even if children succeed in developing a powerful, sharable, and reusable artifact or tool in response to a problem, they usually lack powerful ways to visualize underlying constructs, and they are not often aware of strengths and weaknesses of alternative ways of thinking. Furthermore, because their results often integrate concepts and procedures drawn from a variety of textbook topic areas, they usually have not unpacked these ideas-or, expressed them using elegant language and notations.

Can Teams of Primary School Children Work Collaboratively, and Remain Intensely Engaged, on Problem Solving Activities that Require an Hour to Complete?

Lesh: *How long do you think primary school children are able to work on these kinds of tasks? And, what is it about such activities that stimulate sustained work from children?*

Riggs: *In general, the children worked on one modeling activity for two or three consecutive days for an hour or more each day. The fourth day was reserved for sharing explanations of their modeling to their classmates. Due to the cooperative nature of the activities, complemented by children's engagement in problem solving, the children were highly motivated and often requested additional time to devote to the task. Through sharing, children learned to appreciate diversity in problem solving. I believe that introducing concepts through interesting children's stories gives the children a purpose for their learning; this purpose is what stimulates them to complete the task no matter the amount of time required or how challenging it seemed. The children viewed learning as*

something they wanted to do instead of something they were required to do; modeling activities provide that motivation. The activities were designed to open and close within a week. One reason for this policy was because class time is precious. These stories served as “chunks” that children could use to organize ideas and skills related to a central “big idea”. If these “chunks” got too large, the children would lose sight of the “big idea”. Memorable stories also help children remember what they have learned. The children continued to think about the “big idea” after class - and after we moved to other topics. Weeks after they had finished activities directly associated with one of our stories, they often referred back – saying: This is like Stickbugs, or Beauregard, or Tubby the Train. Then, they would use concepts and abilities that they had developed during those tasks. ... So again, several smaller stories are better than one big story.

Lesh: How much and what kind of guidance did you need to provide in order for children to be successful for these tasks?

Riggs: When the children work in groups, they tend to persevere when they otherwise might have given up. But also, in every one of our activities, children worked together to build some concrete tools or artifacts – such as pathways, fences, villages, maps, or scaled-up houses. So, as long as they clearly understood what was needed and why, and as long as they were able to test their thinking without asking me “Am I done?” or “Is this right?”, they were able to move from first-draft thinking to second and third-draft thinking without much guidance from me.

Self-assessment is important because, in complex activities, if children need to wait for their teacher’s approval at each step, then things move too slowly, and young

children lose interest. Also, when children express their thinking in forms that are visible, it's easier for teachers to wisely pinpoint when and what kind of help is needed. Usually, the kind of help that is most effective involves reflecting more than guiding. For example: interventions that tend to be most helpful are questions like: Who needs this? Why do they need it? What do they need to do? Does everybody agree? ... They are not commands like: Think of a similar problem that you have solved. Or: Everybody listens to Alice. ... The goal is for children to iteratively express, test, and revise their own ways of thinking. The goal is not for them to superficially adopt the teacher's ways of thinking.

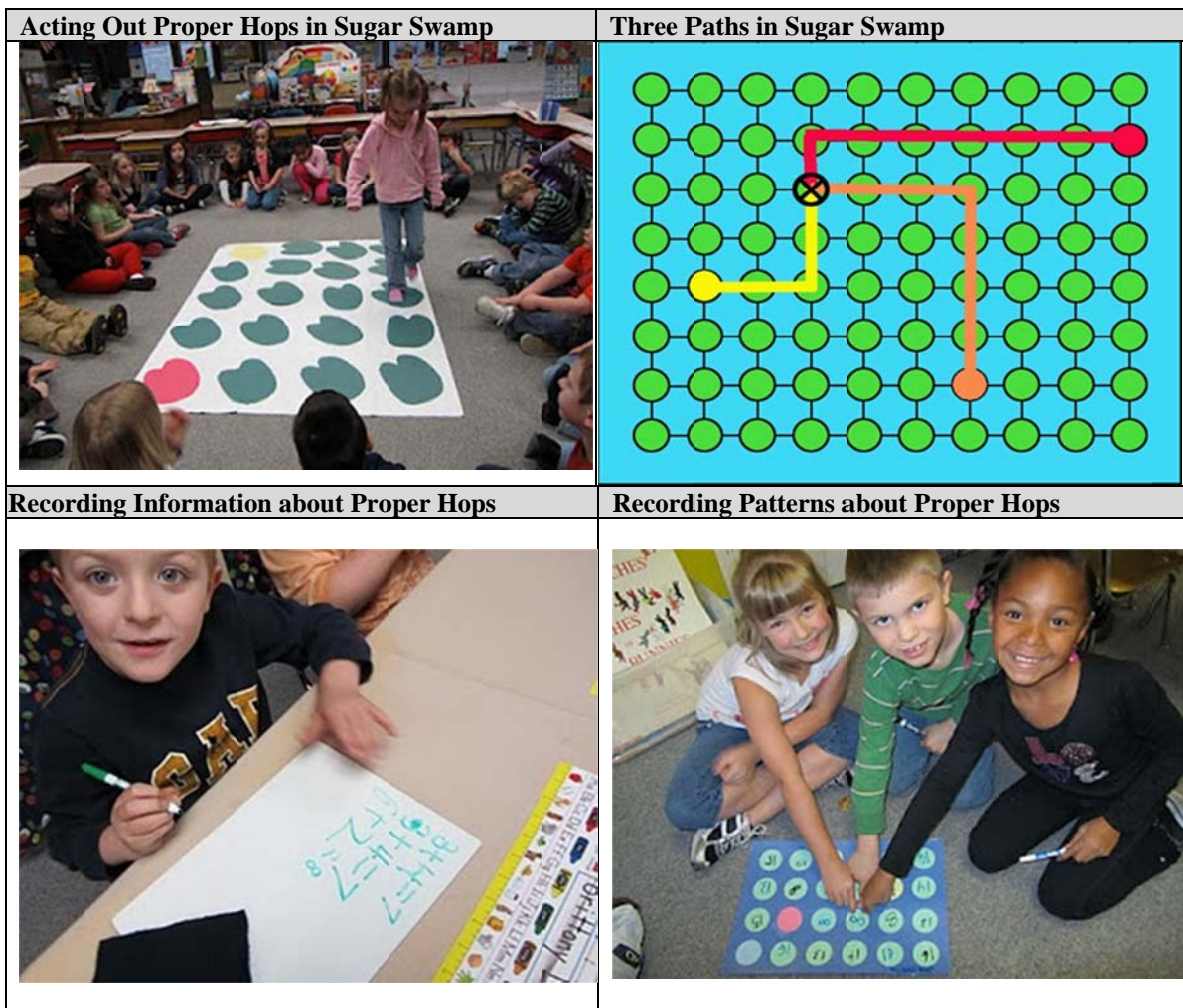


Figure 3. The Proper Hops Problem

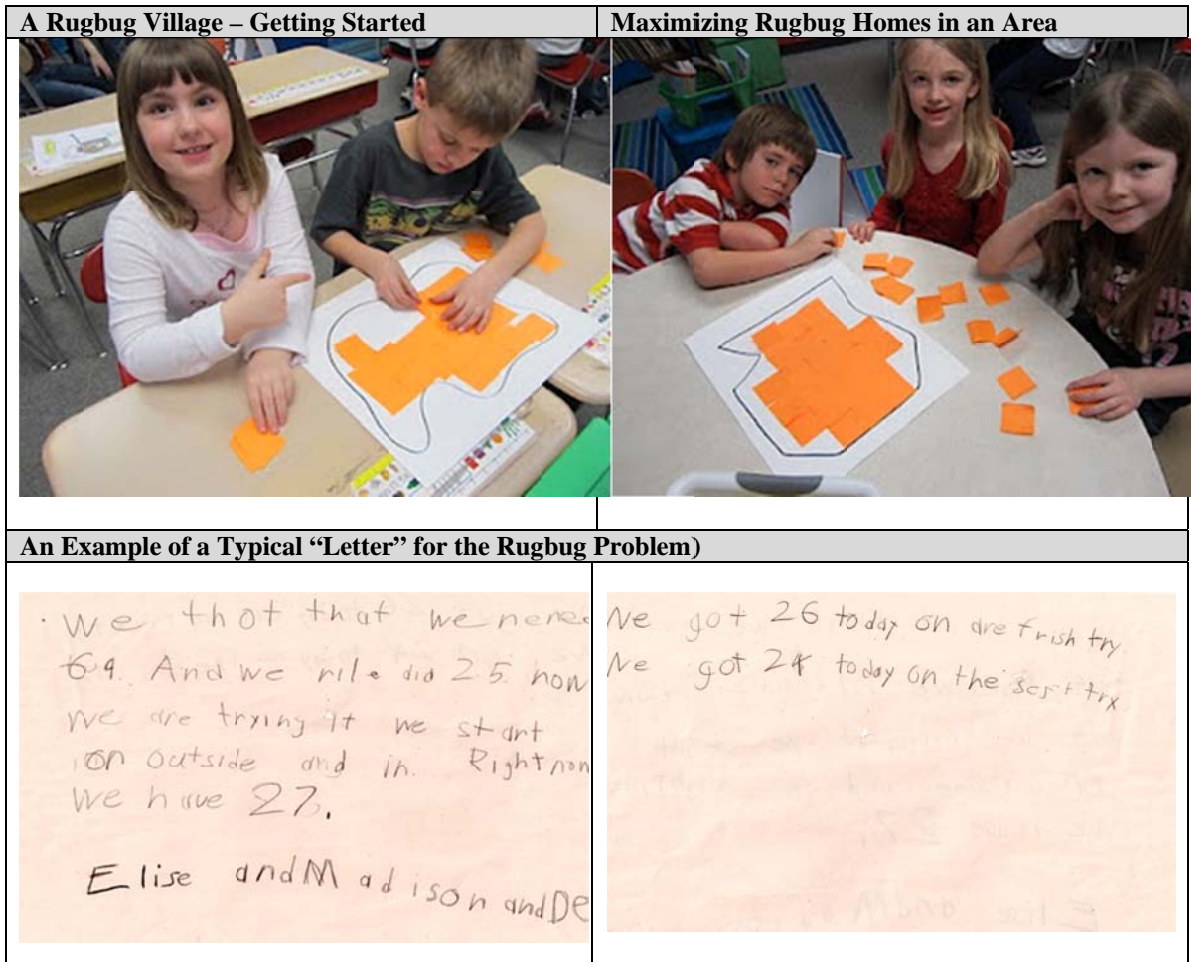


Figure 4. The Rugbug Problem

As the “letter” in the Figure 4 shows, 1st graders’ letters cannot be expected to communicate objectively to another person. But, in this particular study, the school where we were working had made a school-wide commitment to focus on writing. So, the opportunity was too good to miss. For our purposes, the main point of trying to craft letters was not to press the children for writing excellence. The main purpose was (a) for the children to understand that “someone else” wanted to use the information, tool, or artifact that they produced, and (b) to emphasize the tools and artifacts needed to be useful to other people to use. In other words, we wanted to make it as clear as possible to the children that their procedure (or tool) need to be sharable and reusable. ... On the one

hand, the solutions that children produce in such situations are highly situated forms of knowledge. On the other hand, sharable and reusable solutions also are transferrable.

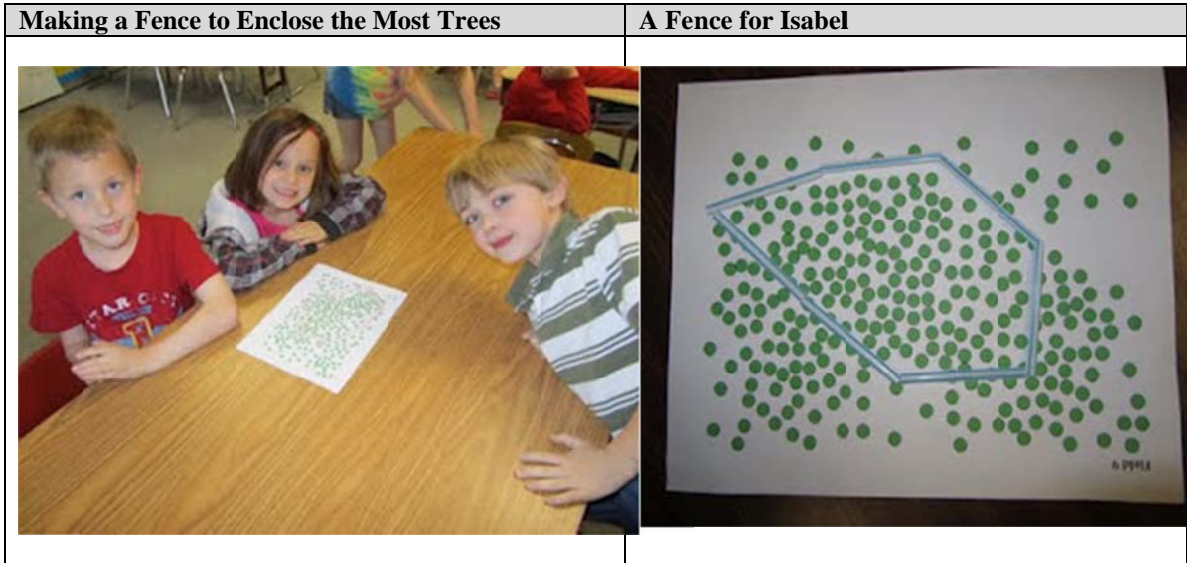


Figure 5. Apples for Isabelle Problem

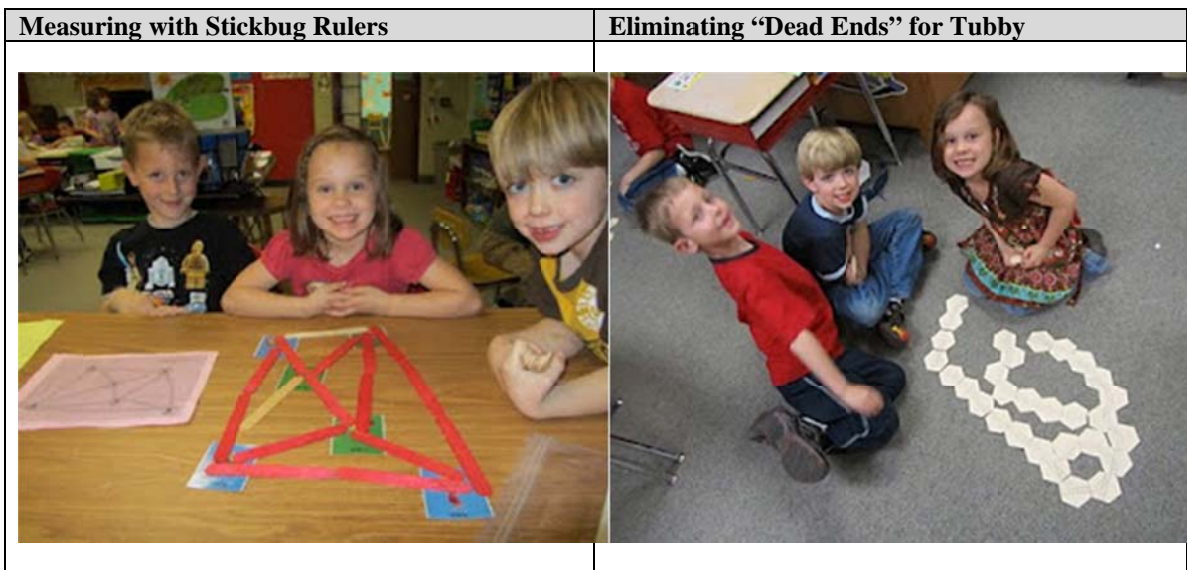


Figure 6. Measuring with Royal Scepters & Laying Tracks for Tubby the Train

What is the Relationship between Problem Solving & the Development of “Prerequisite” Concepts & Skills?

Lesh: This project was not an experiment that treated your children like guinea pigs in a laboratory. It was simply a joint effort that you and our research team decided to provide the best kind of learning experiences for your children. Yet you, like most teachers, administrators, and schools on these days, are being held accountable for learning gains which are measured by standardized tests which (I believe) don't measure much beyond low-level skills. So, even though we didn't have any experimental “control group”, how do you think your students will perform, compared to others, on standardized tests that are relevant to you and others in your school?

*Riggs: I believe that my students will perform as well, if not better, on standardized assessments after using the model eliciting activities. Given that the children learn to problem solve in ways that make sense to them, and they can see their results from the models created, the model eliciting activities provided a knowledge base where information can be retrieved and applied as needed. The students' ability to apply what they had learned became evident when they would remember the "big idea" weeks after we had finished the activity, and when they would apply it to situations in their own lives. One example: Three weeks after completing *The Proper Hop*, a student stated that living in an apartment complex is like living in Sugar Swamp - there are a lot of lily pads. After helping Beauregard to find the best lily pad closest to his friends, this student understood why her Mom didn't want her to walk all the way over to the other side of the complex to visit a friend. It was too far away; it was like Beauregard hopping 20 hops. She said that her Mom allowed her to go next door to visit a friend; for Beauregard, it would only*

be one or two hops. This student also wished she could pick the location of her apartment to be close to her friends - just like she helped Beauregard find his home in The Proper Hop.

Ever since the seminal work of William Brownell (1970), it has been known that, even if we only care about skill-level knowledge, “varied practice” is far more effective than “routine practice” (or drills that are repeated again and again). Brownell identified three kinds of varied practice. The first type involves mixed activities in which attention shifts among several skills – rather than emphasizing just one. This is effective partly because “understanding” involves more than knowing how to do something; it also involves knowing when to do it. The second type of varied practice involves practicing skills in a full range of situations in which they are intended to be useful. This is effective partly because useful skills need to be flexible, not rigid. And, the third type of varied practice involves using skills during complex activities – similar to the way excellent chefs not only know how to use each of the tools sold in chef’s catalogues, but they also know how to orchestrate the use of these tools during the development of complex meals.

Can Primary School Children Engage Productively in Authentic Simulations of “Real Life” Problem Solving Situations?

According to the models & modeling perspectives that underlie our work (Lesh & Doerr, 2003), we reject the notion that children learn, or learn to be effective problem solvers, by first learning concepts and skills, and then learning to use them in meaningful “real life situations.” By far the most important characteristic of the models & modeling

perspectives that distinguish our work from traditional research on problem solving is the recognition that – regardless of whether investigations focus on decision making by medical doctors, business managers, chess players, or others in real life decision-makers - in virtually every field where learning scientists have investigated differences between ordinary and exceptionally productive people, it has become clear that exceptionally productive people not only do things differently, but they also see (or interpret) things differently. Furthermore, when problem solvers interpret situations they don't simply engage models that are completely mathematical or logical in nature. Their interpretations also tend to include feelings, values, dispositions, and a variety of metacognitive functions. But, instead of mastering these other higher-order functions separately, and then attaching them to mathematical models, research on models and modeling shows that they develop as integral parts of the relevant interpretation systems (Lesh, Carmona & Moore, 2010).

- Traditionally, problem solving has been characterized as a process of (a) getting from givens to goals when the path is not obvious, and (b) putting together previously learned concepts, facts, and skills in some new (to the problem solver) way to solve problems at hand. But, when attention shifts toward models & modeling, problematic situations are goal directed activities in which adaptations need to be made in existing ways of thinking about givens, goals, and possible solution steps. So, modeling is treated as a way of creating mathematics (Lesh & Caylor, 2007); and, modeling and concept development are expected to be highly interdependent and mutually supportive activities – especially for young children.

- Traditionally, problem solving strategies and metacognitive functions have been specified as lists of condition-action rules – and have been thought of as providing answers to the question: What should I do when I’m stuck (i.e. when I am not aware of any productive ways of thinking about the problem at hand). But, when attention shifts toward models & modeling, the goal of metacognitive processes is to help problem solvers develop beyond their current ways of interpreting the situations, rather than helping them identify “next steps” within current ways of thinking.
- Traditionally, problem solving in mathematics education has focused on individual students working without tools on textbook word problems. But, because research on models and modeling tends to focus on simulations of “real life” situations, problem solvers often are diverse teams of students each of whom are likely to have access to a variety of specialized technical tools and resources. So, capabilities that become important include: modularization, communication, explanation, and documentation - as well as planning, monitoring, and assessment – all of which tend to be overlooked in the traditional mathematics education problem solving literature; and, all of which emphasize modern socio-cultural perspectives on learning.

Because model development activities are, above all, research sites for directly observing the development of interpretation systems that involve some of the most important aspects of what it means to “understand” many of the most important concepts and “big ideas” in mathematics education, research on models and modeling has led to new views about: (a) how the modeling cycles that students go through during one 60-

minutes model-eliciting activity often are remarkably similar to developmental sequences that Piagetian psychologists have identified during timespans of several years based on normal everyday experiences, (b) how average ability students often develop (locally) through several Piagetian stages during single 60-minute problem solving episodes, (c) how students' final-draft solutions often embody mathematical thinking that is far more sophisticated than traditional curriculum materials ever dared to suggest they could be taught, (d) how student solutions which are expressed in the form of sharable and reusable tools often enable students to exhibit extraordinary abilities to remember and transfer their tools to new situations, (e) how the processes that enable students to move from one model to another seldom look anything like currently touted "learning trajectories" which describe learning and problem solving using the metaphor of a point moving along a path, (f) how the tools and underlying models which students produce in "real life" model development often integrate concepts and abilities associated with a variety of textbook topic areas, (g) how students' early interpretations often involved collections of partial interpretations – which tend to be both poorly differentiated and poorly integrated, (h) how later interpretations tend to notice patterns of information, rather than the kind of pieces of information that tend to dominate earlier interpretations, (i) how model development tends to involve gradually sorting out and integrating several earlier interpretations, (j) how model development often occurs along a variety of interacting dimensions – such as concrete-abstract, intuition-formalization, specific-general, global-analytic, and so on, (k) how the origins for final interpretations often can be traced back to several conceptual grandparents, and (l) how final models tend to include not only systems of logical/mathematical "objects", relations, operations, and

patterns, but they also usually included dispositions, feelings, and a variety of relevant metacognitive functions.

Are the Learning & Problem Solving Experiences Recommended (for example) in the USA's Common Core State Curriculum Standards Representative of Those Children Encounter beyond School in the 21st Century?

For mathematics in the primary school (K-2), the main themes of the CCSC Standards are clear. One of its laudable overall goals is to focus on deeper “conceptual” treatments of fewer standards. Another is to emphasize research-based learning progressions about how students’ mathematical knowledge, skill, and understanding develop over time. And, another is to treat mathematical understanding and procedural skill as being equally important.

- What do the CCSC Standards mean by focusing on deep treatments of a small number of “big ideas”? They say: *Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to other topics.*
- What does mathematical understanding look like? They say: *One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from.*

- Modeling with mathematics is mentioned in only one small paragraph in these standards. And, what do the CCSC Standards mean by “modeling with mathematics”? They say: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*
- The goal of describing and comparing measurable attributes is mentioned in precisely one sentence in the CCSC Standards for the primary grades. But, this sentence is overwhelmed with statements and examples focusing on number operations, and on counts of discrete objects in sets.

The preceding prejudiced portrayal of a view of mathematics, learning, and modeling that is extremely different than the one described briefly in this article. The CCSC preoccupation with counts is not focused. It is narrow. And, it is not at all consistent with the kinds of situations that even young children encounter where numbers and arithmetic outside their school classrooms. Similarly, the CCSC’s notion of what it means to “understand” important concepts and processes completely overlooks the development of powerful sense-making systems - that is, models for describing (quantifying, dimensionalizing, coordinatizing, or in general: mathematizing) situations in forms so that the concepts and procedures that they profess to emphasize will be useful beyond mathematics classrooms (Lesh & Sriraman, 2010; Lesh, Sriraman & English, 2013).

Similarly, the notion of modeling in the CCSC as “applying mathematics that they know to solve problems arising in everyday life” is not at all what we have described in this paper – where 1st grade children learned to actively develop impressively

sophisticated descriptions of meaningful situations – similar to those that occur beyond school classrooms. And finally, the CCSC’s notion of “research progressions” completely ignores the large literature on situated cognition – where knowledge is recognized as being organized around mathematically rich experiences (like our stories) as much as around the kind of decontextualized abstractions that the CCSC Standards continues to emphasize in the examples and detailed descriptions of curriculum goals that are given. Why is this oversight so important? One reason is because most “learning progressions” of the type that the CCSC appears to have in mind envision long strings of prerequisites as being necessary to “master” before children can proceed to more important milestones. So, learning is thought of as a long and arduous process – which looks nothing like the rapid local developments that we describe in this article.

Certainly “real life” situations where number and arithmetic concepts are useful involve many kinds of mathematical “objects” including beyond counts. Examples include locations, actions, weights, likelihoods, and so on. But, unlike the word problems that fill K-12 textbooks, which can be characterized as situations described by a single rule (or function) going in one direction. “Real life” situations often involve several “actors” or several functions – so that feedback loops and 2nd-order effects are important, and where issues such as maximization, minimization, or stabilization occur regularly. For example, in the story-based problems that we have emphasized here, most of them involved several interacting arithmetic operations, as well as issues such as minimization or maximization.

Most of all, this article is intended to portray mathematical model development as an important aspect of mathematical “understanding” that is unabashedly optimistic about

the level of mathematical thinking that is accessible – even to primary school children, and to students of average-ability as measured on standardized achievement tests.

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Prospective Teachers' Interactive Visualization and Affect in Mathematical Problem-Solving

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Abstract: Research on technology-assisted teaching and learning has identified several families of factors that contribute to the effective integration of such tools. Focusing on one such family, affective factors, this article reports on a qualitative study of 30 prospective secondary school mathematics teachers designed to acquire insight into the affect associated with the visualization of geometric loci using GeoGebra. Affect as a representational system was the approach adopted to gain insight into how the use of dynamic geometry applications impacted students' affective pathways. The data suggests that affect is related to motivation through goals and self-concept. Basic instrumental knowledge and the application of modeling to generate interactive images, along with the use of analogical visualization, played a role in local affect and prospective teachers' use of visualization.

Keywords: problem-solving strategies, visual thinking, interactive learning, drawing, diagrams, teacher training, visual representations, reasoning, GeoGebra.

1. Experimental conditions and research questions addressed

At present, the predominant lines of research on problem-solving aim to identify underlying assumptions and critical issues, and raise questions about the acquisition of problem-solving strategies, metacognition, and beliefs and dispositions associated with problem-solvers' affect and development (Schoenfeld, 1992; Lester and Kehle, 2003). Problem-solving expertise is assumed to evolve multi-dimensionally (mathematically, metacognitively, affectively) and involve the holistic co-development of content, problem-solving strategies, higher-order thinking and affect, all to varying degrees (English & Sriraman, 2010). This expertise must, however, be set in a specific context.

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Future research should therefore address the question of how prospective teachers' expertise can be holistically developed.

The research described here was conducted with a group of 30 Spanish mathematics undergraduates. These future teachers took courses in advanced mathematics in differential and Riemannian geometry, but worked very little with the classical geometry they would later be teaching. They were accustomed to solving mathematical problems with specific software, mainly in areas such as symbolic calculation or dynamic geometry, but were not necessarily prepared to use these tools as future teachers. Research on teaching in technological contexts (Tapan, 2006) has shown that students are un- or ill-acquainted with mathematics teaching, i.e., they are unaware of how to convey mathematical notions in classroom environments and find it difficult to use software in learning situations. Hence the need to specifically include the classroom use of software in teacher training.

This paper addresses certain understudied areas in problem-solving such as visualization and affect, with a view to developing discipline awareness and integrating crucial elements for mathematics education in teacher training. As defined by Mason (1998), teachers' professional development is regarded here as development of attention and awareness. The teacher's role is to create conditions in which students' attention shifts to events and facts of which they were previously unaware. Viewed in those terms, teaching itself can be seen as a path for personal development.

The main aim of this essay is to explain that in a dynamic geometry environment, visualization is related to the viewer's affective state. The construction and use of imagery of any kind in mathematical problem-solving constitute a research challenge

because of the difficulty of identifying these processes in the individual. The visual imagery used in mathematics is often personal in nature, related not only to conceptual knowledge and belief systems, but laden with affect (Goldin, 2000; Gómez-Chacón, 2000b; Presmeg, 1997). These very personal aspects are what may enhance or constrain mathematical problem-solving (Aspinwall, Shaw, and Presmeg, 1997; Presmeg, 1997), however, and as such should be analyzed, especially in technological contexts.

Gianquinto (2007) and Rodd (2010) contend that visualization is “epistemic and emotional”. Gianquinto suggests that visual experience and imagining can trigger belief-forming dispositions leading to the acquisition of geometrical beliefs that constitute knowledge. According to Rodd (2010), the nature of belief-forming dispositions is not confined to perception, but incorporates the results of affect (or emotion-perception relationships). Hence, the belief-forming dispositions that underlie geometric visualization are affect-laden.

The present study on teaching geometric loci using GeoGebra forms part of a broader project involving the design, development and implementation of multimedia learning scenarios for mathematics students and teachers². The solution of geometric locus problems using GeoGebra was chosen as the object of study because a review of the literature revealed that very little research has been conducted on teaching that aspect of geometry. A recent paper (Botana, 2002) on computational geometry reviewed current approaches to the generation of geometric loci with dynamic geometry systems and compared computerized algebraic systems to dynamic symbolic objects. However, it did not address the educational add-ons needed by teachers. Several authors have compared

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the visual (and sometimes misleading) solutions generated by dynamic geometry systems to the exact solutions obtained using symbolic computational tools (Botana, Abánades and Escribano, 2011). The approximate solution problem affects all dynamic geometry systems, due to the numerical nature of the calculations performed. The GeoGebra team has been working on improving this feature as part of the GSoC³ project. In the meantime, however, external tools must be used to obtain accurate solutions⁴.

This article specifically explores the role of technological environments in the development of students' competence as geometers and future teachers. More precisely, it focuses on the relationship between technology and visual thinking in problem-solving, seeking to build an understanding about the affect (emotions, values and beliefs) associated with visualization processes in geometric loci using GeoGebra. The questions posed are: how does affect impact visual thinking through dynamic geometry software (GeoGebra)? and how does interactive visualization impact affect in learning mathematics? The difficulties encountered in training students to build strategic knowledge for the classroom use of technology, which weaken personal problem-solving, are also explained.

The rest of the paper is organized as follows. A description of the scientific theory underlying the research is followed by a presentation of the training and research methodology used. A subsequent section discusses the results of all the analyses, including tentative answers to the questions formulated above. A final section addresses the preliminary conclusions of the study and suggestions for future research.

2. Theoretical considerations

³ <http://www.geogebra.org/trac/wiki/Gsoc2010>

⁴ <http://nash.sip.ucm.es/LAD/LADucation4ggb/>

Different theoretical approaches to the analysis of visualization and representation have been adopted in mathematics education research. In this study the analysis of the psychological (cognitive and affective) processes involved in working with (internal and external) representations when reasoning and solving problems requires a holistic definition of the term visualization. Arcavi's proposal (Arcavi, 2003: 217) has consequently been adopted: *“the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings”*.

Analysis of those two complementary elements, image typology and use of visualization, was conducted as per Presmeg (2006) and Guzmán (2002). In Presmeg's approach, images are described both as functional distinctions between types of imagery and as products (concrete imagery (“picture in the mind”), kinesthetic imagery, dynamic imagery, memory images of formula, pattern imagery). In Guzman they are categorized from the standpoint of conceptualization, the use of visualization as a reference and its role in mathematization, and the heuristic function of images in problem-solving (isomorphic visualization, homeomorphic visualization, analogical visualization and diagrammatic visualization⁵). This final category was the basis adopted in this paper for addressing the handling of tools in problem-solving and research and the precise

⁵ *Isomorphic visualization*: the objects may correspond “exactly” to the representations. *Homeomorphic visualization*: inter-relationships among some of the elements afford an acceptable simulation of the relationships between abstract objects They serve as a guide for the imagination. *Analogical visualization*: the objects at hand are replaced by that are analogously inter-related. Modeling process. *Diagrammatic visualization*: mental objects and their inter-relationships in connection with aspects of interest are merely represented by diagrams that constitute a useful aid to thinking processes. (Guzmán, 2002).

distinction between the *iconic* and *heuristic function of images* (Duval, 1999; Souto and Gómez-Chacón, 2011) to analyze students' performance. The *heuristic function* was found to be related to *visual methods* (Presmeg, 1985) and cognitive aspects as part of visualization: the effect of basic knowledge, the processes involved in reasoning mediated by geometrical and spatial concepts and the role of imagery based on analogical visualization that connects two domains of experience and helps in the modeling process.

The reference framework used to study affective processes has been described by a number of authors (DeBellis and Goldin, 1997 & 2006; Goldin, 2000; Gómez-Chacón, 2000 and 2011), who suggest that local affect and meta-affect (affect about affect) are also intricately involved in mathematical thinking. Goldin (2000: 211) contends that affect has a representational function and that the affective pathway exchanges information with cognitive systems. According to Goldin, the potential for affective pathways are at least in part built into the individual. Both these claims were substantiated by the present data. For these reasons, while social and cultural conditions are discussed, the focus is on the individual and any local or global affect evinced in mathematical problem-solving in the classroom or by interviewees. This aspect of students' problem-solving was researched in terms of the model summarized in Figure 1 and used in prior studies (Goldin, 2000: 213; Gómez-Chacón, 2000b: 109-130; Presmeg and Banderas-Cañas, 2001: 292), but adapted to technological environments.

Affective pathway 1 (enabling problem-solving): curiosity →puzzlement→ bewilderment →encouragement→ pleasure →elation →satisfaction →global structures of affect (specific representational schemata, general self-concept structures, values and beliefs)

Affective pathway 2 (constraining or hindering problem-solving): curiosity → puzzlement → bewilderment → frustration → anxiety → fear/despair → global structures of affect (general self-concept structures, hate and rejection of mathematics and technology-aided mathematics)

Figure 1. Emotions and meta-affective aspects in problem-solving

This idealized model illustrates how local affect might influence the heuristic applied by a problem solver. This model was used in individual case studies because it provides insight into how visual processes, emotions and cognitive strategies interact. It also helps detect mental blocks and emotional instability where confusion and perceived threat are significant, generating high anxiety levels, and therefore conditioning visual thinking and attitudes. Here, emotions are not mere concomitants of cognition, but are intertwined with and inseparable from it. Most importantly, they are bound up with the individual's self-image and self-concept and the global affective dimension where purpose, beliefs and goals have a substantial impact.

3. Training and the research methodology used

The qualitative research methodology used consisted of observation during participation in student training and output analysis sessions as well as semi-structured interviews (video-recording). The procedure used in data collection was student problem-solving, along with two questionnaires: one on beliefs and emotions about visual reasoning and the other on emotions and technology (one was filled in at the beginning of the study and the other after each problem was solved). All screen and audio activity on the students' computers was recorded with CamStudio software, with which video-based information on problem-solving with GeoGebra could be generated. Consequently, at least four data sources were available for each student.

Six non-routine geometric locus problems were posed, to be solved using GeoGebra during the training session. Most of the problems were posed on an analytical register (Table 1: for a fuller description see Gómez-Chacón and Escribano, 2011).

Finding the solutions to the problems called for following a sequence of visualization, technical, deductive and analytical steps.

Table 1: Geometric locus problems

PROBLEM	
<p><i>Problem 1:</i> find the equation for the locus formed by the barycenter of a triangle ABC, where $A = (0, 4)$, $B = (4, 0)$ and C is a point on circle $x^2 + y^2 + 4x = 0$.</p>	<p><i>Level: basic</i> <i>Geometric locus:</i> the wording of the problem determines the steps to be followed.</p>
<p><i>Problem 2:</i> assume a variable line r that cuts through the origin O. Take point P to be the point where line r intersects with line $Y=3$. Draw line AP from point $A = (3,0)$, and the line perpendicular to AP, s. Find the locus of the intersection points Q between lines r and s, when r is shifted.</p>	<p><i>Level: medium</i> <i>Geometric locus:</i> in this problem, the difficulty is to correctly define a variable line. That done, the rest is fairly straightforward. The instructions for using GeoGebra are stated explicitly in the problem.</p>
<p><i>Problem 3:</i> assume a triangle ABC and a point P. Project P on the sides of the triangle: Q1, Q2, Q3. Are Q1, Q2 and Q3 on the same line? Define the locus for points P when Q1, Q2 and Q3 are aligned.</p>	<p><i>Level: medium – advanced</i> <i>Geometric locus:</i> the locus cannot be drawn with the “locus” tool in GeoGebra, because it is non-parametric. There is no mover point.</p>
<p><i>Problem 4:</i> the top of a 5-meter ladder rests against a vertical wall, and the bottom on the ground. Define the locus generated by midpoint M of the ladder when it slips and falls to the ground. Define the locus for any other point on the ladder.</p>	<p><i>Level: medium – advanced</i> <i>Geometric locus:</i> the problem does not give explicit instructions on the steps to follow. The situation is realistic and readily understood, but translation to GeoGebra is not obvious. An ancillary object is needed.</p>
<p><i>Problem 5:</i> find the locus of points such that the ratio of their distances to points $A = (2, -3)$ and $B = (3, -2)$ is $5/3$. Identify the geometric object formed.</p>	<p><i>Level: Advanced</i> <i>Geometric locus:</i> the problem is simple using paper and pencil. The difficulty lies in expressing “distance” in GeoGebra.</p>
<p><i>Problem 6:</i> find the equation for the locus of point P such that the sum of the distances to the axes equals the square of the distance to the origin. Identify the geometric object formed.</p>	<p><i>Level: Advanced</i> <i>Geometric locus:</i> the problem is simple using paper and pencil. The difficulty lies in expressing “distance” in GeoGebra.</p>

Geometric locus training was conducted in three two-hour sessions. Prior to the exercise, the students attended several sessions on how to use GeoGebra software, and were asked to solve problems involving geometric constructions.

In the two first sessions, the students were required to solve the problems individually in accordance with a proposed problem-solving procedure that included the steps involved, an explanation of the difficulties that might arise, and a comparison of paper and pencil and computer approaches to solving the problems. Students were also asked to describe and record their emotions, feelings and mental blocks when solving problems.

The third session was devoted to common approaches and the difficulties arising when endeavouring to solve the problems. A preliminary analysis of the results from the preceding sessions was available during this session.

The problem-solving results required a more thorough study of the subjects' cognitive and instrumental understanding of geometric loci. This was achieved with semi-structured interviews conducted with nine group volunteers. The interviews were divided into two parts: task-based questions about the problems, asking respondents to explain their methodologies and a series of questions designed to elicit emotions, visual and analytical reasoning, and visualization and instrumental difficulties.

A model questionnaire proposed by Di Martino and Zan (2003) was adapted for this study to identify subjects' belief systems regarding visualization and computers to study their global affect and determine whether the same belief can elicit different emotions from different individuals. In this study, students were asked to give their opinion of a belief and choose the emotion (like/ dislike) they associated with it, e.g.:

Table 2: Example of items of student questionnaire on beliefs and emotions

- Visual reasoning is central to mathematical problem solving.		
- Visual reasoning is not central to mathematical problem solving.		
Give reasons and examples. How do you feel about having to use problem representations or visual imagery?		
I like it.	I don't like it.	I'm indifferent.
.....Explain the reasons for your feelings.		

A second questionnaire, drawn up specifically for the present study, was completed at the end of each problem. The main questions were:

Table 3: Student questionnaire on the interaction between cognition and affect

Please answer the following questions after solving the problem:
1. Was this problem easy or difficult? Why?
2. What did you find most difficult?
3. Do you usually use drawings when you solve problems? When?
4. Were you able to visualize the problem without a drawing?
5. Describe your emotional reactions, your feelings and specify whether you got stuck when doing the problem with pencil and paper or with a computer.
6. If you had to describe the pathway of your emotional reactions to solving the problem, which of these routes describes you best? If you do not identify with either, please describe your own pathway.
<i>Affective pathway 1</i> (enabling problem-solving): curiosity →puzzlement→ bewilderment →encouragement→ pleasure →elation →satisfaction →global structures of affect (specific representational schemata, general self-concept structures, values and beliefs).
<i>Affective pathway 2</i> (constraining or hindering problem-solving): curiosity → puzzlement → bewilderment → frustration → anxiety → fear/despair → global structures of affect (general self-concept structures, hate or rejection of mathematics and technology-aided mathematics).
7. Now specify whether any of the aforementioned emotions were related to problem visualization or representation and the exact part of the problem concerned.

The protocols and interviewee data were analyzed for their relationship to affect as a representational system and the aspects described in section two.

4. Findings

The results shown here attempt to answer the concerns formulated in the introduction. The affective pathways reported for each problem consistently showed: a)

the effect of subjects' beliefs and goals on the preference and use of visual thought/knowledge in computerized environments; b) that students proved to have a poor command of the tools, especially the locus tool; c) that notwithstanding, beliefs on the potential of GeoGebra helped them maintain productive affective pathways. As a qualitative study, the aim here was to describe the findings in detail. Consequently, the cases that best exemplified the results that were consistent across the entire group (30 students) and the nine volunteers were chosen and characterized by: gender, mathematical achievement, visual style, beliefs about computer learning, computer emotion, beliefs about visual thinking, feelings about visualization processes and global affect.

4.1. Beliefs about visual reasoning and emotion typologies

The data showed that all students believed that visual thinking is essential to solving mathematical problems. However, different emotions were associated with this belief. Initially, these emotions toward the object were: like (77%), dislike (10%), indifference (13%). The reasons given to justify these emotions were: a) pleasure in knowing that expertise can be attained (30% of the students)⁶; b) pleasure when progress is made in the schematization process and a smooth conceptual form is constructed (35%); c) pleasure and enjoyment afforded by the generation of in-depth learning and the control over that process (40%); d) pleasure and enjoyment associated with the entertaining and intuitive aspects of mathematical knowledge (20%); e) indifference about visualization (13%); f) dislike or displeasure when visualization is more cognitively demanding (10%).

⁶ Some students cited several reasons.

A similar response was received when the beliefs explored related to the use of dynamic geometry software as an aid to understanding and visualizing the geometric locus idea. All the students claimed to find it useful and 80% expressed positive emotions based on its reliability, speedy execution and potential to develop their intuition and spatial vision. They added that the tool helped them surmount mental blocks and enhanced their confidence and motivation. As future teachers they stressed that GeoGebra could favour not only visual thinking, but help maintain a productive affective pathway. They indicated that working with the tool induced positive beliefs towards mathematics itself and their own capacity and willingness to engage in mathematics learning (self-concept as a mathematical learner).

4.2. Cognitive and instrumental difficulties: student's geometric constructions with GeoGebra

This section describes the solution typologies for the six problems.

Typology 1: static constructions (discrete treatment). In this typology, the students used GeoGebra as a glorified blackboard (Pea, 1985), but none of its dynamic features. They repeated the constructions for a number of points. To draw the geometric locus, they used the “5-point conic” tool. This underuse of potential appeared in problems 1 and 4.

Typology 2: incorrect definition of the construction. The students solved the problem (imprecisely), but with solutions that implied that the GeoGebra tools were unusable. The “locus” tool can only be used if the defining points are correctly determined (they may not be free points). Adopting this approach, at best the students

could build a partially valid construction, but since the GeoGebra tools couldn't be used, no algebraic answer was obtained.

This typology appeared in problems 2 and 4. In problem 2, the sheaf of lines had to be defined by a point on an ancillary object such as a line, and not as a free point. Otherwise, the approximate visual solution obtained was unusable with GeoGebra. The students concerned were absolutely convinced that their solution was right and wholly unaware of any flaw in the solution.

The difficulty in problem 4 was to define a point that was not the mid-point. The locus tool could not be used for a free point on the ladder.

Typology 3: incorrect use of elements. For example, in problems 1, 2, 4 and 6, some students used the “slider” tool to move the “mover point”. They realized that the “mover point” had to be controlled, which is what the slider is for. In GeoGebra, however, the slider is a scalar and can't be used with the locus tool.⁷

Problem 2 is a case in point. Some students defined the sheaf of lines as the lines passing through the origin on a point in the circle, and this point in the circle was moved with the slider. For example, student 9 said: “This problem is similar to the one before it. I built the construction while reading the problem. The hardest step was to construct the variable line. First, I thought I'd use a slider for the slope of the line passing through the origin, but that way I never got a vertical line, so I used the slider as in the preceding problem to build point C that revolves around the origin, and then to build the line connecting C and O. After that, I just followed the instructions in the problem, and I was very careful about the way I named the elements” (student 9, problem 2).

⁷ <http://www.geogebra.org/help/docues/topics/746.html>

Typology 4: failure to use the locus tool. Here, the construction was correct, but the student did not use the locus tool. To use it, the point that projects the locus (tracer) must be distinguished from the point that moves the construction (mover). The mover must be a point on an object. Some students were apparently unable to make that distinction, which prevented them from using the tool correctly.

This misunderstanding arose in problems 1, 2, 3 and 4. Student 8 exemplifies this type of reasoning: “The first thing I had to do was find the center and radius of the circle to draw, to complete the square in the equation: $(x + 2)^2 + y^2 = 4$. Therefore, point C is in a circle with a center at $(-2, 0)$ and a radius of 2. (I didn’t actually need this because in GeoGebra I could enter the equation directly and draw the circle). Now, to solve the problem I had to know what a barycenter was. I took point C on the circle (creating an angular slider so the point would run along the entire circumference of the circle) and drew the triangle ABC. I calculated the triangle barycenter (I drew the medians as dashed green lines to make it easier to see that G is the barycenter). Using animation to project point G gave me the locus. Since the locus was a circle, I was able to solve the equation by finding three points, G1, G2, G3, and activating the “circle through three points” tool. Then I entered the data in GeoGebra: $(x - 0.66)^2 + (y - 1.34)^2 = 0.44$ ” (student 8, problem 1).

4.3. Maintaining productive affective pathways

As noted in the preceding paragraph, the belief that visual thinking is essential to problem-solving and that dynamic geometry systems constitute a visualization aid, particularly in geometric locus studies, was widely extended across the study group. That belief enabled students to maintain a positive self-concept as mathematics learners in a

technological context and to follow positive affective pathways with respect to each problem, despite their negative feelings at certain stages along the way and their initial lack of interest in and motivation for computer-aided mathematics.

A comparison of the affective pathways reported by the students revealed: a) concurrence between the use of visualization typologies and associated emotion; b) that the availability of and subsequent decision to use GeoGebra was often instrumental in maintaining a productive affective pathway. This section addresses three examples, in two of which the affective pathway remained productive and one in which it did not. It discusses the determinants for positive global affect and positive self-concept as mathematical learners. The key characteristics of the case studies are given in Table 4.

Table 4: Three case studies: characteristics

Case	Gender	Mathematical achievement	Visual style	Beliefs about computer learning	Feelings about computers	Beliefs about visual thinking	Feelings about visualization	Global affect
Student 19	Male	High	Visualizing student	Positive	Likes	Positive	Likes	Positive self-concept
Student 20	Female	Average	Non-visualizing student	Positive	Dislikes	Positive	Dislikes	Positive self-concept
Student 6	Female	Low	Style not clear	Positive	Dislikes	Positive	Likes	Negative self-concept

Problem 4 (Table 1) was chosen for this analysis. The students' affective pathways for this problem are given in Table 5.

Student 19 is a visualizer. In the interview he said that the pleasure he derives from visualization is closely associated with the mathematics view. He regards visual reasoning as essential to problem-solving to monitor and generate in-depth learning, to contribute to the intuitive dimension of knowledge and to form mental images.

When he was asked whether his feelings were related to visualization and problem-solving and to specify the parts of the problem where they were, he replied: “curiosity predominated in visualization. Since the problem was interesting and seemed to be different from the usual conic problems, I was keen on finding the solution. I had a major mental block when it came to representing the problem and later, as I sought a strategy. I was unable to define a good strategy to find the answer. I was puzzled long enough to leave the problem unsolved and try again later. When I visualized the problem in a different way, I found a strategy: construct a circle with radius 5 to represent the ladder and another smaller circle to represent the point in question. When I reached that stage, I felt confident, happy and satisfied” (student 19).

Student 20 is a non-visualizing thinker with positive beliefs about the importance of visual reasoning. However, she claimed that her preference for visualization depends on the problem and that she normally found visualization difficult. It was easier for her to visualize “real life” than more theoretical problems (the difference between problems 4 and 5, for instance).

Her motivation and emotional reactions to the use of computers were not positive, although she claimed to have discovered the advantages of GeoGebra and found its environment friendly. She also found that working with GeoGebra afforded greater assurance than manual problem-solving because the solution is dynamically visible. Convincing trainees such as student 20 that mathematical learning is important to teaching their future high school students helps them keep a positive self-concept, even if they don’t always feel confident in problem-solving situations (Table 5).

Student 6's visual thinking style could not be clearly identified. She expressed a belief in the importance of positive visual reasoning (“because visual reasoning helps gain a better understanding of the problem and consequently the solution”). This confirmed a liking for visualization and representation because it made it easier to understand the problem and she found formalization helpful. She added, however, that she felt insecure applying technological software to mathematics, although she believed GeoGebra, specifically, to be useful. In her own words, “I don’t like it and never will. I feel a little nervous and insecure, not because of GeoGebra but because computers intimidate me because I don’t understand them completely. But when I managed to represent the problem with GeoGebra, I felt more satisfied with the result than when I solved it with paper and pencil”. Although student-6’s pathway was essentially negative in problem 4, she persisted until she found the solution. In some cases students were unaware of their mistakes and misunderstandings, however.

GeoGebra can be used to solve problem 4, although an average student cannot be expected to build the entire construction from scratch. The visual and instrumental challenge is to deploy the sliding segment, and that calls for an auxiliary circle (which may be concealed to simulate the effect of the ladder). The point in the ladder must be chosen with care to use the locus tool. Just any “point in segment” will not do; the “middle point” tool or a more sophisticated construction must be used.

While none of the three students applied the “locus” command, student 19 used the visual power of the technology to gain a better mathematical understanding of the problem. That inspired a change in context which facilitated notion and property applications. He used GeoGebra as a genuine mathematical modeling tool. He did not

solve the problem with the geometric locus command, however, even though he came up with the right answer by modeling. A comparison of this student's pathways in the six problems revealed that the interaction between visual reasoning and negative feelings arose around the identification of interactive representation strategies and the formulation of certain representations in which the identification of parametric variations plays a role. This student's command of the use of concrete, kinesthetic and analogical images was very helpful and contributed to his global affect and his positive overall self-concept when engaging in computer-aided mathematics.

An analysis of the relationship between these three students' affective pathways (Table 5) and their cognitive visualization shows that visualization - negative feelings interactions stem essentially from students' lack of familiarity with the tools and want of resources in their search for computer-transferable analogical images and their switch from a paper and pencil to a computer environment in their interpretation of the mathematical object.

Behavior such as exhibited by student 6 denotes a need to include construction with locus tools in teacher training. Although no general methodology is in place, any geometric problem that aims to determine locus must be carefully analyzed. This calls for identifying three categories of geometric elements in such problems: fixed (position, length, dimension); mobile (position, length, variable points); and constant (length, dimension).

The data also revealed the relationship between beliefs, goals and emotional pathways. The analysis of student 20's responses showed that while she had no inclination to use computers, the importance she attached to mathematics and IT in

specific objectives and the structuring of her overall objective kept her on a productive affective pathway (McCulloch, 2011). Student 20's solution to problem 5 (Table 1), for instance, constitutes a good example of a productive pathway: despite negative feelings, she maintained a positive mathematical self-concept, which she reported when she explained her global affect. (Her self-reported pathway in problem 5 was: curiosity → confusion /frustration → desperation → puzzlement → satisfaction → a negative mathematical self-concept in terms of technology for problem 5, but a positive global affect regarding computer use in solving the six problems). Questions designed to elicit the reasons for her positive mathematical self-concept in terms of technology showed that objectives, purposes and beliefs were clearly interrelated. Her own words were: "I think that computers, not only the GeoGebra program, are an excellent tool for anyone studying mathematics. Nowadays, the two are closely linked: everyone who studies mathematics needs a computer at some point... mathematics is linked to computers and specifically to software like GeoGebra (if you want to teach high school mathematics, for instance. I at least am trying to learn more to be a math teacher) (student 20)".

Table 5: Affective pathways and visual cognitive processes reported for this problem by three students

Problem 4	COGNITIVE-EMOTIONAL PROCESS	
Student 19 Own pathway	Curiosity	Reading and understanding problem
	Confusion	Drawing (patterns and lines/figure) Analytical
	Puzzlement. Mental block	(Search for mental image) (specific figure/illustration and dynamic image)
	Confidence	Search for mental image
	Perseverance-motivation	Search for mental image
	Excitement and hope	Physical manipulation - kinetics Kinesthetic learning Mental image Identification mathematical object
	Confidence	Technological manipulation with the computer Representing circle radius (specific illustrations)
	Confidence, joy	Interactive image generation, slider (analogical)
	Joy and happiness	Interactive image generation, slider (analogical)
	Perceived beauty	Specific illustration with interactivity (analogical)
	Satisfaction	Analytical-visual Memorized formulaic typology
	GLOBAL AFFECT	Positive self-concept
Student 20 Own pathway	Curiosity	Problem reading
	Frustration	Global visualization of problem Pictorial image
	Confusion	Search for mental image Inability to visualize the ladder as the radius of a circle
	Puzzlement	Search for mental image Dynamic and interactive image with GeoGebra
	Stimulus, motivation	Technological manipulation with the computer Pictorial representation with GeoGebra
	Satisfaction	Pictorial representation with “trace on” GeoGebra Full construction from scratch Come up with a final solution
	GLOBAL AFFECT	Positive self-concept
Student 6 Pathway-2	Curiosity	Problem reading
	Puzzlement	Global visualization of problem Pictorial image
	Bewilderment	Search for an instrumental image with GeoGebra
	Frustration	Computer handling skills
	Anxiety	Inability to visualize the ladder as the radius of a circle and using “trace on”
	Fear/despair	Needing help to find the solution
	GLOBAL AFFECT	Negative self-concept

Conclusion, limitations and further research

The results of this study suggest that various factors are present in conjunction with visual thinking. The first appears to be the study group's belief that visual thinking and their goal to become teachers would be furthered by working with technology (Cobb, 1986). The data shows that all the student teachers believed that visual thinking is essential to solving mathematical problems. That finding runs counter to other studies on visualization and mathematical ability, which reported a reluctance to visualize (e.g., Eisenberg, 1994). However, different emotions were associated with this belief. The belief about using computers and that software is a tool that contributes to overcoming negative feelings has an impact on motivated behavior and enhances a positive self-concept as a mathematical learner. Despite this advantage, however, student teachers may still misunderstand or misinterpret and therefore misuse computer information, unknowingly in some cases, and surrender all authority to the computer.

While prospective teachers resort to GeoGebra software to help maintain a productive affective pathway and foster visual thinking, student 20's experience with problem 5 is significant, for it shows that the tool by itself is not enough. If the software is unable to deliver the dynamic geometric capability that students want to use for the concepts at hand, it is useless and may even have an adverse impact on their affective pathway, possibly resulting in feelings of defeat such as reported by student 20. Her experience provides further evidence of the importance and complexity of mathematics teacher training, as documented by researchers studying the issue from an instrumental approach (e.g., Artigue, 2002). The mere provision of tools cannot be expected to necessarily raise the frequency of productive affective pathways. Rather, thought needs to

be given to how those tools are integrated into classrooms to support the development of visualization skills. Some students (as in item 4.2) think of graphs as a photographic image of a situation due to a primarily static understanding of functional dependence. That might be attributed to the fact that the pointwise view of mathematical objects tends to prevail in the classroom, where the dynamic view is underrepresented (institutional dimension of visualization).

The results of this study bring to mind the *progressive modelling in visual thinking* notion introduced by Rivera (Rivera, 2011: 270). Furthering visualization processes in teaching involves more than just drawing “pretty pictures”: it requires sequenced progression of the thought process. This in turn calls for awareness of the transition in dimensional modelling phases from the iconic to the symbolic and the change of mindset. For the problem proposed, “geometric locus”, each transition can be associated with mathematical explanations and symbol notation and the proficient use of the visual tool to reify the mathematical concept. Therefore, one question that would be open for research is the definition of the components of an overarching theory of visualization for problem-solving in technological environments where this progression is explicit. While this study was conducted in a classroom context, it focuses on the individual only, not on interaction among individuals. Future studies might profitably explore the role of external affect and others’ (i.e., teachers’, community’s, institution’s) external affective representations. Such interaction impacts meta-affect and may potentially either help maintain or interrupt productive affective pathways.

Finally, as explained in the introduction, the teacher training model pursues the development of students’ awareness and ability to apply their knowledge in complex

contexts, integrating knowledge with their own attitudes and values and therefore developing their personal and professional behavior. From this standpoint, teacher training programs should adopt a more holistic approach (cognitive, didactic, technical and affective). The present paper aims to provide a preliminary framework to help teacher educators or mathematical cognitive tool designers select and analyze interaction techniques. A secondary aim is to encourage the design of more innovative interactive mathematical tools.

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Young Children Investigating Advanced Mathematical Concepts With Haptic Technologies: Future Design Perspectives

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1. Introduction

In this chapter, we focus on how new technologies can be used with young children to investigate mathematical ideas and concepts that would normally be introduced at a later age. In particular, we focus on haptic technologies that allow learners to touch and feel objects through force feedback in addition to visual images on a screen. The main purpose of this paper is to describe how these technologies can be used to enable young learners to construct meaning about geometric shapes and surfaces as well as attributes of particular mathematical constructions in multiple dimensions (particularly 2D and 3D for purposes of this chapter). Such learning environments enable various forms of mediation both through the devices and software used as well as socially, as students work together to develop meaning and create models of complex ideas.

We begin by describing how and why young learners in particular should be working in such learning environments in order to provide a rationale for our work. In Section 2, we provide some background on how these technologies have evolved and their use in other disciplines and how we have built on prior research in the use of dynamic geometry in mathematics education. Section 3 presents how relevant these new learning environments can be with some specific examples from preliminary work at the Kaput Center. The section also contains some theoretical reflections on how we can

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begin to analyze and understand how students work and construct meaning in such environments. Section 4 then concludes by offering some design principles for future research and development.

How Should Young Children be Doing Mathematical Problem-Solving in the Future?

We believe that the answer to this question lies in three areas that focus on the early introduction of mathematical ideas, the use of technology, and engagement.

Early introduction. Several researchers have promoted the idea of introducing mathematical ideas earlier in the curriculum and even introducing the foundation of advanced mathematical thinking in the early grades (Kaput, Carraher & Blanton, 2008; Kaput, 1994). If not then, many children will never be exposed to important mathematics and engage in fruitful and relevant investigations. This can have detrimental effects throughout a child's educational career, reducing their desire to want to learn mathematics because of its lack of relevance or inaccessible representations.

Technology use. Technology is often not a major part of elementary school classroom teaching due to a lack of resources and perception of its role and use. The predominant form of technology use in most elementary school classrooms in the U.S. is PowerPoint presentations. Some researchers (Carraher & Schliemann, 2000) believe that the introduction of technology is not enough:

It is important to provide a social analysis in consonance with a cognitive one. Because technology does not act directly on learners, but only exerts an influence on the social activities and contexts in which it is employed, introducing technology into the mathematics classroom ultimately entails questions such as the following: What is the teacher's role; what are the students trying to achieve in the tasks (p. 174)

While we agree that these questions are important, new technologies can have a more participatory and collaborative role rather than be a prosthetic device to prop up existing pedagogical practices. New technologies can actually re-structure interaction in the classroom and allow the introduction of advanced mathematical ideas through radically new mathematical representation systems. The interactions of teachers, students and technologies within a learning environment can modify and transform activity structures (Jonassen, 2000).

Technological affordances can also be mathematical affordances providing a symbiotic link between how mathematical activity can occur. Mathematizing technological affordances is an important step and one we discuss in detail later.

Engagement. By integrating activity structures with the affordances of new technologies, the learning environment should be simple enough to establish engagement—to motivate curious young minds to explore, question, and be encouraged to want to continue to learn. It should allow them to construct meaning in open-ended tasks, which have been carefully designed to have mathematical purpose. It should allow them to share, collaborate, and feel free to use non-scholastic language as they conduct their mathematical investigation.

We take a very broad view of what is mathematical problem-solving viewing it as an enterprise of collaborative investigation where multiple approaches are valid. It is not just about solving a specific problem, which has a specific answer or application into the real world, but rather it is an investigation that might have multiple approaches and where students can make multiple observations. Also, most of our activities might best be described as “tasks” rather than “problems.”—that is, they are goal directed activities.

Students are seldom at a loss for ideas to pursue. They are not stuck; they are not frustrated; and, their progress often does not fit the metaphor of moving along a single path that is somehow temporarily blocked. Instead, our environments are carefully engineered so that students can make parallel progress along a variety of interacting paths. Our initial tasks involve exploring, categorizing attributes of geometric shapes or objects, making sense of a set of objects and constructing broad and specific meaning. These tasks, in a broad sense, could be described as modeling (Lesh, 2007). We will continue to use the phrase mathematical problem-solving throughout this chapter but in the spirit of the position described above.

We have referred to new technologies, but we focus on a particular type of learning environment that utilizes haptic or multi-modal devices. Multiple modalities are used in real-world applications. We make sense of problem conditions in the world by using sight, touch, and hearing to name a few. Hence, in our research and development, we have focused on new technologies that use multiple modes of input in early mathematics classrooms. First, let us describe the evolution of such technologies in contrast to the predominance of visualization software in mathematics education.

2. Background to New Technology

Haptic literally means “ability to touch” or “ability to lay hold of” (Revesz, 1950) and has evolved to be an interface for users to virtually touch, push, or manipulate objects created and/or displayed in a visual environment (McLaughlin, Hespanha, & Sukhatme, 2002). Recently, this has rapidly evolved to include multi-touch environments. In these environments, learners literally lay their hands on objects via a screen interface,

mathematical objects can be manipulated and resultant actions be investigated. Let us first examine the background of educational technology involved in dynamic visual mathematics before extending to haptic technologies which is the focus of this chapter.

Traditionally, dynamic, interactive, mathematical, visual environments—including Computer Algebra Systems such as Mathematica and Maple, as well as multi-dimensional Graphing software such as Avitzur's Graphing Calculator—are used to aid students to visualize complex surfaces in various coordinate systems and complete computationally intensive tasks. The Geometer's Sketchpad® is used in classrooms ranging from elementary grades through to undergraduate programs to allow users to construct, interact and explore geometric figures and shapes, and so engage in model-eliciting activities in various mathematical topics. But these environments are not responsive to users' physical interactions apart from mouse pointing.

The experience of visual mathematics, particularly three-dimensional mathematics, is often very brief for U.S. mathematics and science students. Following a school curriculum of Euclidean Geometry rarely expanding to non-Euclidean geometry or solid geometry, there is a rapid progression in most university curriculum from three-dimensional geometry, which is embedded in third or fourth semester Calculus courses, to the abstract intangibles of higher dimensional mathematics. In fact, given the very nature of multi-dimensional mathematics—that it can examine real life objects and phenomena all around us—it is interesting that such a small proportion of a student's formal mathematical life is spent examining the subject. Such mathematics provides a vocabulary for understanding fundamental modeling equations, for example, weather, heat, planetary motion, waves, and later, multi-dimensional mathematics, finance,

epidemiology, quantum mechanics, bioinformatics and many more. Yet, there is a growing emergence of technologies in the scientific workplace that apply, manipulate, and model three-dimensional representations

A wide range of technologies are used in the teaching and learning of multi-dimensional mathematics in various contexts, ranging from relatively expensive Computer Algebra Systems (CAS) such as Mathematica™ and Maple™ (Meel, 1998; Park & Travers, 1996), industrial design packages, (e.g., AutoCAD™), through to Java Applets freely downloadable from the WWW. During reform periods, Mathematics and Science departments have been encouraged to integrate CAS technology into their classes as it can help students with visual and conceptual problems (Zorn, 1987; 1992). As technology becomes more sophisticated, the opportunity cost of training time and money spent on learning how to use a particular software and how to successfully integrate it into school curriculum is sufficiently high to dissuade teachers from the investment.

Dynamic geometry environments offer point-and-click tools to construct geometric objects that can be selected and dragged by mouse movements. All user-defined mathematical relationships are preserved, thus providing environments for students to conjecture and generalize by clicking and dragging hotspots on the object. These hotspots dynamically re-draw and update information on the screen as the user drags the mouse, and in doing so, efficiently testing large iterations of the mathematical construction (Moreno & Sriraman, 2005; Moreno & Hegedus, 2009; Moreno, Hegedus & Kaput, 2008).

Such environments aim to develop spatial sense and geometric reasoning by allowing geometric conjectures to be tested, offering “intelligent” constructivist tools that

constrain users to select, construct or manipulate objects that obey mathematical rules (Mariotti, 2003)— that are largely used in secondary and not primary schools.

In summary, these dynamic mathematic environments are responsive to users' interaction but are still more structured in their feedback and lack the expressive capabilities of using physical interaction and force-feedback.

We believe that students naturally need more haptic, kinesthetic avenues through such the combination of dynamic visual environments and haptic technologies to explore the mathematics of change and variation in a more sensory environment to connect to the symbolic formalisms of the mathematical ideas (Nemirovsky & Borba, 2003). Change and variation occurs in multiple school subjects, in particular algebra, geometry and data analysis. In allowing students the combined affordances of multi-touch interaction, visual feedback and force feedback where possible, the technological environment can become a semiotic mediator of mathematical thinking and investigation. Young learners can have access to new forms of mathematical problem-solving or investigation through direct manipulation of mathematical objects linked to varying attributes (e.g. area).

To this aim, we have focused on integrating two types of haptic technologies: (1) Sensable's PHANTOM Omni—a force-feedback device and (2) iPad with a dynamic geometry application—a multi-touch/multi-input device.

Sensable's PHANTOM Omni® (<http://www.sensable.com/haptic-phantom-omni.htm>)—hereon referred to as Omni—is a desktop haptic device with six degrees of freedom for input (x, y, z, pitch, roll, yaw), and three degrees of output (x, y, z). The Omni's most typical operation is via a stylus-like attachment that includes two buttons (see Figure 1a). The Omni has a very robust community and SDK behind it. The SDK

(OpenHaptics-Academic Edition) allows for two levels of programmatic control: precise programmer created feedback—such as vibration—and pre-programmed feedback—such as springs and dynamic/static friction. The Omni provides up to 3 forces of feedback for x, y, and z. It is primarily used in research, with a significant presence in dentistry and medicine but growing in mathematics education (Hegedus & Moreno, 2011).

Figure 1b is an example of how the Omni can be used with a graphical environment. Users can click, push, or drag a “bug” across a surface using the haptic device. Here, the bug is being pushed around a saddle surface. A saddle surface has a maximum and a minimum at the same place (the place where you would sit on the saddle). A sample activity would be to ask the student to push the bug to that place on the surface and describe how the surface changes during its motion, both in terms of the visualization but more importantly in terms of its feel. Mathematically speaking this potentially enables an interpretation of differentiability of the surface. This approach can offer new access to profoundly important mathematical ideas for younger children.

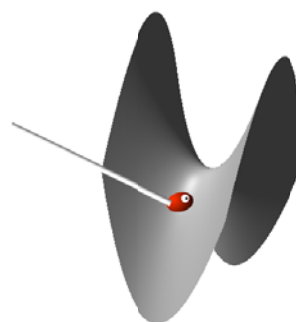


Figure 1a. Sensable’s Omni Haptic *Figure 1b.* Graphical interface prototype.

Device.

In addition to shape, feedback from the Omni can be linked to particular mathematical attributes, particularly varying quantities. Figure 2 highlights a triangle whose shape can be dynamically transformed by moving the vertices (A, B, or C) or by

moving the parallel lines apart (via D). Here the feedback can be linked to the area of the triangle and children can investigate what properties impact the area before being introduced to a formal equation.

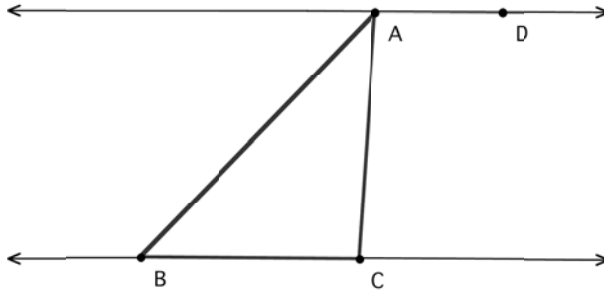


Figure 2. Area of a triangle.

It should also be noted that many mass-market whiteboards, tablet PCs and gesture-based PDAs (e.g., iPod Touch/iPhone) and larger forms such as the iPad now feature pressure-sensitive styli of some form; this pressure-input becomes a limited form of force-feedback in software that “responds” differentially to variable pressure. At this stage, feedback is visual, and does not include the additional sensory information of forces or kinesthetics that are directly linked to the properties of the objects being manipulated on a screen. But the growing spread of these devices and their use with educational applications creates an entry point for the use of peripheral computational devices that extend the use of mouse as a physical pointing action, to an environment where users can touch, push or move objects more directly.

In multi-touch/multi-input platforms such as the iPad, the learner can use multiple modes of input and outputs— their natural modes of seeing and feeling, to make sense of the task. The iPad offers a direct (almost zero-interface) mode to touch and directly

manipulate mathematical objects, and offer multiple inputs to one mathematical object hitherto impossible on a single-input computer (mouse as pointer and selector).

With both hardware, the technological affordances are tightly coupled with mathematical affordances in such that the technology offers mathematical meaningful tools or avenues to investigate. Hence a new form of mathematical problem-solving originates because of new mathematization routes. We will exemplify these affordances in the next section. In terms of software deployment, key representational features include high-resolution visualization of mathematical objects and constructions which can be made transparent to see their interaction with other objects, and direct manipulation of objects allowing users to rotate and navigate “around” objects and flexible notation systems to allow users to observe outputs (e.g., changing area of a shape) based upon their input.

We now describe how such environments are relevant to mathematics education and offer examples of how they can advance mathematical investigations and inquiry.

3. Relevance: Future Mathematical Problem-Solving

Situations inside and outside of formal learning environments involve visualization, multi-modal investigations, using and interpreting multiple representations, connecting mathematical attributions and concepts to real world phenomenon, e.g., form, shape of objects and models (visual surfaces), features and attributes. So what is modeling in a problem-solving context for early learners? And is it relevant or necessary for early learners to be introduced to such ideas? We think it is and it goes deep into what a mathematical problem-solving environment is for a life-long learner. In addition,

making sense of the environment in a mathematical way is not just physical or visible (i.e., tangible) but also occurs at the nano-level. Macro images, surfaces, and objects can simulate phenomena, which cannot be seen or felt, e.g., cell structures. Census datasets cannot be understood at a macro level without a deep understanding of the micro. What constitutes a dataset? In a similar vein, what constitutes variation at all across mathematical models? The heart of the research reported here is to establish conditions by which early learners can advance their mathematical inquiry at stages that are hitherto not required by standardized frameworks but are still complimentary. Our primary research question is: Can we establish learning environments by which advanced mathematical ideas can be more readily accessed, understood and used to solve problems? Such understanding is mediated through the affordances of technological devices and at the same time social interaction between peers. Young children can construct meaning through collaboration (one form of mediation) but supported and additionally mediated through the tools afforded to them through mathematically-enhanced technologies.

Holland et al. (2004) outlines the role of a mediating device:

A typical mediating device is constructed by the assigning of meaning to an object or a behavior. This symbolic object or behavior is then placed in the environment so as to affect mental actions. (p.36)

In our design of a learning environment integrating new technologies in mathematically relevant ways, we adhere to a socio-cultural perspective of learning and analyze the interaction of the students in terms of mathematically-relevant discourse as mediated by the various tools and supports available to them. The affordances of the technological environment are cultural devices. Children can modify the environment to make sense of the attributes of the geometric objects and configurations through

investigation and interaction with each other. Vygotsky (1980) explains how activity structures the social environment of interaction and the very behavioral routines of members of that environment. We adhere to that position in our design and observe the technological devices to not be the only mediating device in the learning environment but the interaction between the children as meaning-making becomes a collaborative enterprise. Both are forms of semiotic mediation and result from co-action (Moreno & Hegedus, 2009) between the various participants. The children guide the discussion by interacting with visuals on a screen, receiving visual and haptic feedback loops, which are iteratively discussed and compared within the group and as such the technology reciprocally guides the resulting investigation, decisions in how to further interact, and conjectures or refutations from the resulting actions. Such embodied actions of pointing, clicking, grabbing and dragging parts of the geometric construction also allows a semiotic mediation (Falcade, Laborde & Mariotti, 2007; Kozulin, 1990; Mariotti, 2000; Pea, 1993) between the object and the user who is trying to make sense of, or induce some particular attribute of the diagram or prove some theorem.

Based upon this theoretical perspective, we present two different technologies from our preliminary research and development at the Kaput Center. These have been field tested in informal and formal learning settings. This preliminary work was conducted with 4th graders in a high achieving elementary school in Massachusetts.

Omni Force-Feedback Device

In the Omni environment, we developed an exploration activity using solids and a plane to explore how these objects interact—in particular, what different types of planar intersections can be constructed. Our environment includes crisp visuals of these objects,

which can be navigated by dragging and moving the stylus on the Omni so that different views of the objects could be explored. Through iterative design, we found that certain colors and use of transparency helped the young learners focus their attention and interpretation on the interaction and their reference to certain attributes. In addition, we combined the haptic affordances of the Omni to add additional feedback to the investigation. We found that magnetism was an important design principle to further aid the learners to focus their attention and aid their discovery. In magnetizing the surfaces, the children could lock onto the intersection of the two shapes and consider what they felt in conjunction with what they saw. Two examples are shown in Figures 3a and 3b. The first shows the planar intersection of a cube, which can result in a set of intersections from a point (plane resting on a vertex), a line (plane on an edge), and 3-gon to 6-gons. The second illustrates the planar intersection of a square based pyramid, which can result in a similar set of intersections up to a 5-gon. Children, in groups of 4 with one device, mainly explored a variety of triangles, quadrilaterals and pentagons. Such an activity is challenging to undergraduates and the children had no prior experience with such an investigation, but we discovered that their engagement in discovering various types of intersection was immediate and endured for almost an hour. They did have prior experience with 2D geometric shapes such as 3-gons to 5-gons but had only a basic knowledge of the attributes of these shapes. For example, they did not classify 4-gons as quadrilaterals but squares and rectangles. They did know how many sides each shape should have which gave rise to interesting discussions as they explored what they saw and how it contrasted with what they felt. In one investigation, the children thought they saw a pentagon, but on tracing around the magnetized shapes they felt a 4-sided shape

(by counting edges) and concluded it was trapezoid through group discussion. This illustrates a classic issue of cross-modality where our vision and touch can be in conflict. The pseudo-3D representation on a flat screen is not sufficient, even with dynamic interaction tools such as rotating and navigating the objects—more feedback is necessary for young learners to make sense of certain specific mathematical attributes of the overall geometric configuration. More work is needed in establishing activity structures that help students make mathematical classifications of varying shapes. For example, can force feedback help develop a sense of angle measure (acute, right, obtuse) in classifying all types of triangles?

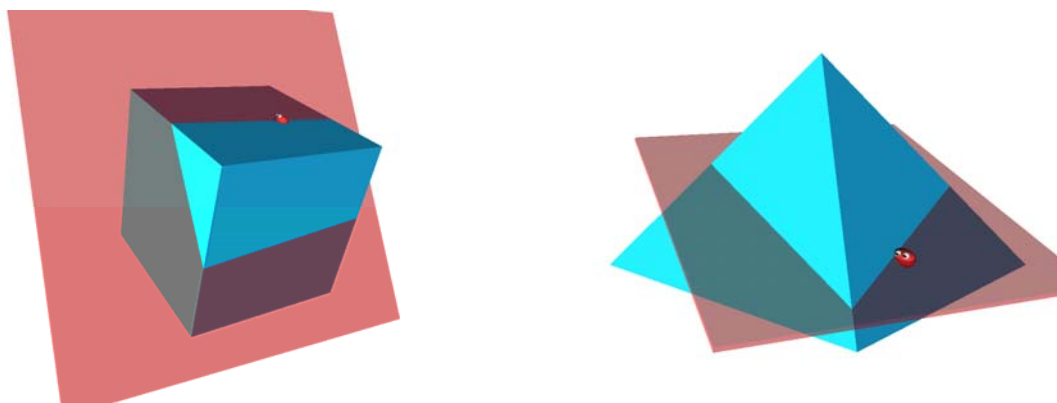


Figure 3a. Planar intersection of a cube. *Figure 3b.* Planar intersection of a square-based pyramid.

In collaboration with KCP Technologies, we developed a set of activities for use with SketchExplorer for the iPad, a viewer application of the widely popular Geometer's Sketchpad® software. This application is available in the Apple Store. Activities were constructed in Sketchpad and then transferred to the iPad through email or other forms of file exchange. All activities are pre-configured for the children to use—as no construction tools are presently available in this version for the iPad. Children directly interacted with objects in the pre-configured activity including geometric objects (e.g., points), iterative

counters through flicking, or buttons that had been configured to perform a set of operations (e.g., reflection of an image). Two examples are illustrated below. The first (Figure 4a) allows students to make successive attempts at translating a pre-image onto its pre-destined image (i.e., it has been fixed). They interact by moving the reflection line and pressing the reflect button. This activity calls for two reflections to make one translation. We found that all children in our preliminary field work in 4th grade classrooms eventually discovered how to complete this activity through a variety of methods, and develop an understanding of the relationship between reflections and translations.

The second activity (Figure 4b) maximizes the affordance of multi-touch in a mathematical way. Point 1 can be moved laterally and Point 2 vertically (they are constrained to move along two perpendicular lines that have been hidden). The output of these movements is a blob. This blob will simultaneously move in the directions of the two input Points 1 and 2. The size of the blob can be changed by moving Point H along a slider and the color can be changed by moving a point across the spectrum. In this activity, we asked students to make the blob trace a circle. This was a rich mathematical activity in that two inputs can make one output and many of the children in our preliminary field work discovered this idea. More formally, the construction of a circle is parameterized with two perpendicular actions. Again, this activity was extremely engaging, especially when we added the time to establish a competition of who can make the best circle in the least amount of time. Here, haptics is in the form of multi-touch and can be done by one child (multiple fingers) or single-touch by multiple children. We

found the latter to be more fruitful as children had to make sense of the effect of each other's actions and to collaborate to complete the task.

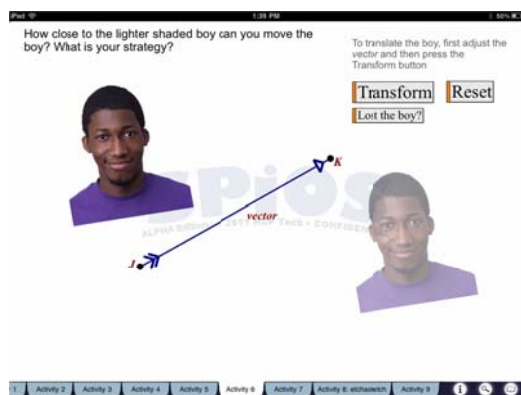


Figure 4a. Translation as a composition of two reflections

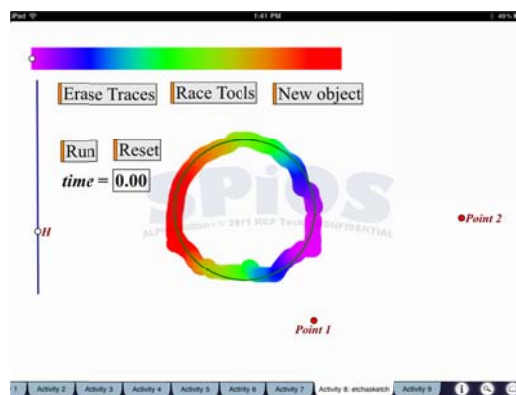


Figure 4b. Etchasketch

Our preliminary work with two integrated visual and haptic environments have yielded many positive results in terms of mathematical investigation in which young learners can be engaged. Engagement is of a mathematical kind. Indeed, many of our participating children were highly excited about the possibilities of using new technologies (particularly the iPad which over half of them had at home). But the engagement rapidly became one of a mathematical kind with many forms of scholastic and non-scholastic language being used. Interaction was heightened since not every child had one of either of the devices; collaborative group work was imperative. Such devices enable meaning-making to rise to a small group or whole class level. Discussions resulted from two forms of modality—the visual and the haptic. Sometimes these caused some interference but it was still fruitful in enabling the children to infer, construct and refute ideas in constructing meaning.

Finally, mathematical activities can be designed to create investigations that are difficult or impossible in other technological environments.

4. Future Design Principles

Technological affordances should become mathematical affordances and it is in the mathematization of technological affordances that meaningful integration of new multi-modal learning environments can be developed. We conclude with a set of design principles that have evolved from our preliminary work, introduced in this paper, that have the potential to profoundly affect teaching and student learning in the early grades.

Executable Representations

Mathematical objects and configurations should allow learners to dynamically manipulate and execute operations on the representations in the learning environment. Instead of dealing with static objects or computational outputs, representations that are flexible allow young learners to adapt the configuration and test out their conjectures in an iterative manner.

Co-action

The learner and learning environment should be collaborative. In dealing with flexible and executable representations, the actions of the learner can guide the environment (re-configure representations) and be guided by the resulting actions of the learning environment.

Navigation

The integration of dynamic visuals with meaningful haptic feedback forms should allow the learner to navigate the various attributes of the mathematical configuration and construct meaning.

Manipulation and Interaction

Objects in such learning environments should be manipulable, and deformed into a wide (if not infinite) set of similar objects, e.g., recall our triangle-area activity earlier, in such a setup all triangles can be configured through direct manipulation.

Variance/Invariance

Understanding how quantities vary or not under certain interactions allows a large wealth of mathematics to be explored. In addition to annotations such as measurement, linking variation to force feedback allows meaningful feedback to help guide the learner to make sense of important features, co-varying relationships or invariance.

Mathematically Meaningful Shape & Attributes

We naturally use touch to explore the composition of objects in nature as well as varying attributes. In addition to shape, form and texture, haptic feedback can be linked to attributes to aid the learner in their investigation.

Magnetism

A natural force is magnetism and this can be used to help learners focus on particular features or relationships between geometric shapes and surfaces. Some objects, or features of objects (where there is a particular mathematically-meaningful interest) can be magnetized and all other attributes de-magnetized.

Pulse/Vibration

Pulse in the form of vibro-tactile feedback or oscillating devices (such as the Omni) can similarly aid learners to focus their attention on certain parts of the activity, or offer some form of numerical feedback. For example, the frequency and amplitude of the pulse/vibration can be regulated to vary with some quantity.

Construction

Building on the affordances of dynamic geometry, allowing learners to use visual and haptic tools to construct mathematical configurations can help learners to make sense of what objects relate to each other (e.g., co-varying quantities) and communicate with others their understanding or production of a mathematical model.

Aggregation

Learning environments often have the affordance of wireless connectivity. Constructions, or evolving discoveries within the learning environment can be easily shared across networks as part of larger models to be aggregated on another computer, or to be contrasted with the work of other students working on the same project. Consider transferring a haptic force with a visual across a network where others can “feel” what you have felt.

We hope that these principles and our preliminary work provide ground-breaking insights into effective generative activity design by future researchers and developers in the future.

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Cognitive processes developed by students when solving mathematical problems within technological environments

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Abstract: In this paper we document and discuss how the use of digital technologies in problem solving activities can help students to develop mathematical competences; particularly, we analyze the characteristics of reasoning that students develop as a result of using Cabri Geometry software in problem solving. We argue that the dynamical nature of representations constructed with Cabri, and the availability of measure tools integrated to it are important elements that enhance students' ability to think mathematically and foster the implementation of several heuristic strategies in problem solving processes.

Keywords: Problem solving, digital technologies, mathematical thinking.

Introduction

Mathematical problem solving has been widely recognized as a framework to analyze learning mathematical processes in which it plays dual relevant roles. On one side, it guides performing research in mathematics education (Schoenfeld, 1985) and on the other hand, it supports the development of curricular proposals (NCTM, 2000). In learning approaches, based on problem solving, it is considered that students construct mathematical knowledge by solving problems (Harel, 1994) in a community that fosters development of an inquisitive attitude. Students' participation in a community of practice has been recognized as a fundamental element of what constitutes mathematical thinking (Schoenfeld, 1992; Santos-Trigo, 2010), since in this community they have opportunities to reflect on their own thought processes through listening and reflecting upon ideas of other members of it.

[In a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection;

inquiry is developed as one of the forms of practice within the community and individual identity develops through reflective inquiry. (Jaworski, 2006, p. 202)

Problem solving is an activity involving conceptualization of the discipline “as a set of dilemmas or problems that need to be explored and solved in terms of mathematical resources and strategies” (Santos-Trigo, 2007, p. 523) and that promotes students’ engagement in a variety of cognitive actions that can allow them to relate diverse mathematical concepts, facts, procedures and forms of reasoning to construct learning with understanding (Hiebert et al., 1997) through posing and pursuing relevant questions.

In problem-solving learning approaches, students need to conceptualize the construction of mathematical knowledge as an activity in which they have to actively participate in order to identify and communicate ideas that emerge when they are approaching mathematical situations (Moreno-Armella & Sriraman, 2005), as well as to pose questions around problematic tasks that lead them to recognize relevant information needed to give meaning to mathematical concepts. In this line of thinking, Santos-Trigo (2010, p. 301) has stated that: “An overarching principle that permeates the entire problem-solving process is that teachers and students should transform the problem statement into a set of meaningful questions to be examined”.

Some classical approaches to problem solving have identified necessary steps for solving problems (Polya, 1945), and central variables that influence students’ behaviors and ways of reasoning. For instance, Schoenfeld (1985) considers four categories of variables that are useful to characterize students’ mathematical performance: (i) resources, (ii) heuristics, (iii) control and (iv) belief systems. However, since these

theoretical categories were developed based on experiences carried out in paper and pencil environments, when using technological tools, those categories necessary have to be reviewed since the use of technological tools offers students new opportunities to discuss mathematical tasks from perspectives where visual and empirical approaches are widely enhanced and by doing this, students can gain a deeper understanding of mathematical concepts.

Technology based tools are now used on a daily basis in fields ranging from the sciences to the arts and the humanities, as well as in professions from agriculture to business and engineering [...] And, these new conceptual tools are more than simply new ways to carry out old procedures; they are radically expanding the kind of problem solving and decision-making situations that should be emphasized in instruction and assessment. (Lesh & Doerr, 2003, p. 15)

Technological tools allow students experiment, observe mathematical relations, formulate conjectures, construct proofs, and communicate results in ways that can enhance and complement paper and pencil approaches, supporting mathematical learning by offering opportunities to expand students' capabilities to visualize, experiment, obtain feedback, and consider the need to prove mathematical results (Arcavi & Hadas, 2000).

In order to examine the potential of using particular computational tools, in terms of characteristics of reasoning developed by students when solving problems, and the type of cognitive processes performed by learners as a result of the use of these tools, in the international research agenda in mathematical problem solving it has been identified some important questions that can shed light on our understanding about the effect of using these tools in learning mathematics through problem solving, such as: To what extent does the systematic use of technological tools help students to think mathematically? Which aspects of mathematical thinking can be enhanced through the

use of digital technologies in mathematical problem solving? What type of reasoning do students develop as a result of using diverse computational tools in problem solving? (Santos-Trigo, 2007). In this line of thinking, the aim of this paper is to identify and analyze how the use of computational tools could help teachers, enrolled in a master program in mathematics education, to propose problem solving strategies and give arguments to justify and validate conjectures that emerge in the course of solving optimization problems.

Digital technologies as cognitive reorganizers

According to Pea (1987), cognitive technologies are media that help us transcend some limitations of mind such as capacity for storing and processing information based only on biological memory. These cognitive technologies are characterized by externalizing the intermediate thinking products, allowing us to operate, analyze and reflect upon them. Furthermore, the representations that can be constructed with technological artifacts are dynamical and manipulable. This dynamical character of computational representations enable students to construct, for instance, families of configurations, and to establish links among diverse representations, so that when a representation is modified, the change is reflected immediately on the other representations, allowing students to interact, operate or modify the representation and its relations more directly than in a paper and pencil environment.

How does the systematic use of digital technologies impact cognitive structures? Digital technologies can be considered as amplifiers or reorganizers of human cognition. The term “amplify” means doing the same things that one could do without technology,

but performing it in a faster or a better way, without transforming qualitatively our actions; for example, a calculator is an amplifier if it is used only to perform arithmetic computations. On the other hand, “reorganize” means doing new things that one cannot do without technology, or those that were not practical to do. A technological tool can be considered as a reorganizer if it modifies cognitive processes and allows us to establish a dialectical relationship among our actions, forms of thinking and tool’s functionalities, which affect our modes of approaching the acquisition of knowledge.

The use of digital tools in learning activities, promotes that students pay attention on the structural aspects of problem solving, by facilitating the performance of routine procedures, opening the possibility of approaching problems which were difficult to discuss within paper and pencil settings, and modifying the cognitive processes that they develop to construct or to operate representations of mathematical objects. For instance, to sketch the graph of a function within a paper and pencil environment, students could proceed to explore and evaluate an algebraic expression, defining the function, for some values of the variable, then those values need to be plotted in a coordinate system, and finally students sketch the graph. However, graphing a function with a calculator or a computational tool only requires introducing in the system the algebraic expression that defines the function, and the software performs intermediate steps required to sketch the graph. That is, computational tools simulate cognitive processes that formerly were exclusive of human beings, attribute that Moreno-Armella and Sriraman (2005) have called *executability*.

Although the use of computational tools offers students advantages to learn mathematics, technological tools by themselves are not enough for constructing learning

with understanding. Mathematical learning with understanding also requires developing an appreciation to practice genuine mathematical inquiry, and disposition to construct connections among diverse mathematical concepts, ideas and procedures. The ability to construct connections is supported by the conceptual structure of the problem solver, a term that we use to indicate how the problem solver's resources are used to approach the examination and solution of a problem; that is, the extent in what these resources can be coordinated in order to articulate different concepts and results when students develop a mathematical activity.

The use of technology to approach learning activities involves considering its impact on the principles and concepts associated with the frame that guides research or instructional processes. As Santos-Trigo and Barrera-Mora (2007, p. 84) have stated "...any conceptual framework or perspective constantly needs to be examined, refined or adjusted in terms of the development of the use of tools (particularly computational tools) that influences directly the ways students learn the discipline". Thus, it is important to consider to what extent the systematic use of technology allows us to examine, test, refine and expand some elements of mathematical thinking considered in problem solving frameworks such as (i) students' access to basic resources or knowledge, (ii) implementation of problem solving strategies that involves ways to represent and analyze the problems, (iii) the use of metacognitive strategies and, (iv) the construction of justifications to validate conjectures and mathematical results.

[...] mathematical problem solving as a research and practice domain has evolved along the development and availability of computational tools and, as a result, research questions and instructional practices need to be examined deeply in order to characterize principles and tenets that support this domain. (Santos-Trigo, 2007, p. 524)

We argue that computational tools “incorporates a mathematical knowledge accessible to the learner by its use” (Mariotti, 2000, p. 37), and by doing this, several consequences arise. Among them, the use of technology allows that some resources inherent to the tool could be incorporated to students’ resources when they solve problems. For example, when students solve problems using computational tools, they need a lesser amount of explicit mathematical resources to approach a task since students can develop forms of reasoning based on visual and empirical approaches, enhanced by the tools, and therefore their mathematical conceptual structure can be extended incorporating to it some inherent tool’s characteristics.

Methodology

Six high school teachers (Jacob, Sophia, Daniel, Emily, Peter and Paul) participated in three hours-weekly problem-solving and problem-posing sessions during one semester. These teachers were enrolled in a master program in mathematics education. They had some experience in using computational tools such as Cabri-Geometry and a hand-held calculator (Voyage 200). All teachers had completed a Bachelor Science degree, majoring in mathematics or engineering, and they had teaching experience ranging from one to five years.

During the semester there were twenty work sessions. The first two sessions were employed to show teachers basic functionalities of Cabri-Geometry through the construction of some common geometrical figures, and to illustrate the form to implement several heuristic strategies such as: to consider that the problem has been solved, relaxing problem conditions, add auxiliary elements to geometric configurations

or to solve a simpler problem. The aim of these sessions was that teachers should comprehend that a valid construction in Cabri geometry must be based in the properties and relationships defining the geometrical figures, and that dynamic behavior of figures is based on the hierarchy of construction procedure.

The core of the dynamics of a DGE figure, as it is realized by the dragging function, consists of preserving its intrinsic logic, that is, the logic of its construction. The elements of a figure are situated in a hierarchy of properties; this hierarchy is defined by the construction procedure and corresponds to a relationship of logical conditionality. (Drijvers, Kieran & Mariotti, 2009, p. 119)

In the following three sessions, teachers discussed The Church View Task: A car is driven on a straight roadway. Aside, there is an old church and the driver wants to stop so that his friend (the passenger) can appreciate the facade of the church. At what position of the roadway should the driver stop the car, so that his friend can have the best view? In the process to solve this task, teachers used Cabri to construct a dynamic model of the situation, and developed numerical and graphical strategies to quantify and understand the relationship between the car's position on the roadway, and the view of the church's facade. Besides, through exploration of relationships among elements of the dynamic configuration, teachers transformed the original problem in an equivalent geometrical problem: draw a tangent circle to line l (roadway) that passes through points A and B (representing the church's facade). They conjectured that tangency point of the circle and line l is the place where the observer gets the best view of the church (Santos-Trigo & Reyes-Rodríguez, 2011).

The analyzed tasks in this paper were developed within the sixth to eighth sessions. During the sessions, the teachers were encouraged to use Cabri Geometry and a

hand held calculator to solve problems involving construction of dynamic configurations.

The teachers worked on solving problems that come from different contexts:

mathematical, hypothetical and real world (Barrera-Mora & Santos-Trigo, 2002). The researchers documented how the tools helped teachers to propose strategies to solve the problems and give arguments to justify and validate conjectures that emerged in the course of the solution process.

The didactical approach employed during the sessions involved teachers working in pairs and plenary discussions in which each pair of teachers communicated and discussed their approaches and strategies employed to solve the problems. Two researchers coordinated the sessions and participated as members of a community, encouraging the development of an inquisitive approach to perform the tasks, and promoting a collaborative work not only to solve the problems, but also to review and reflect on mathematical content and ideas that emerged during problem-solving processes.

The sessions were video recorded and recordings were transcribed. Each pair of teachers handed in a report that included the software files. The transcripts and teachers' reports constituted basic research data. The unit of analysis was the work shown during the sessions by pairs of teachers, however sometimes attention was focused on the work the entire community. The reduction of data was performed by identifying and selecting some chunks of the transcripts or reports, which offered information about strategies employed by teachers to solve the problems or forms of reasoning used to justify their conjectures.

The main tasks analyzed in this paper are three: (a) find the rectangle of maximum area among all rectangles of given perimeter, (b) find the rectangle of maximum perimeter among all rectangles of given area, (c) given a wire, split it into two parts; and with one of the parts construct a square and with the other, construct a circle. Where should you cut the wire so that the sum of the areas of the square and the circle will be minimal?

First task

Peter and Paul constructed a dynamic configuration in Cabri to solve the first problem. They drew a segment AB representing the given perimeter of the rectangle that they wanted to construct. Then, they obtained midpoint M of segment AB , traced segment MB and put a point C on segment MB . The teachers transferred measures MC and CB to the horizontal and vertical axes, respectively, to construct a rectangle. After that, they verified that the dynamic construction fulfilled the conditions of the problem (the perimeter of the rectangle should be equal to the length of the segment AB) measuring the length of the segments MC and CB , and comparing these lengths with the length of rectangle's sides (Figure 1). The aim of these actions was to verify that there were no mistakes during the construction process, and to provide evidences that the dynamic construction works properly.

Peter computed the rectangle's area using Cabri tools, and dragged point C until he obtained a numerical approximation of the maximum area, conjecturing that this one is not attained when the rectangle is a square. His conjecture was based on numerical results, since apparently the maximum area is reached when the measures of the

rectangle's sides are 3.42 cm and 3.29 cm. In this phase of teachers' activity, the tool acted as a cognitive reorganizer since it enabled them to formulate conjectures based on the relationship between visual and numerical representations mediated by dragging, as well as to construct justifications supported and expressed via the software's resources.

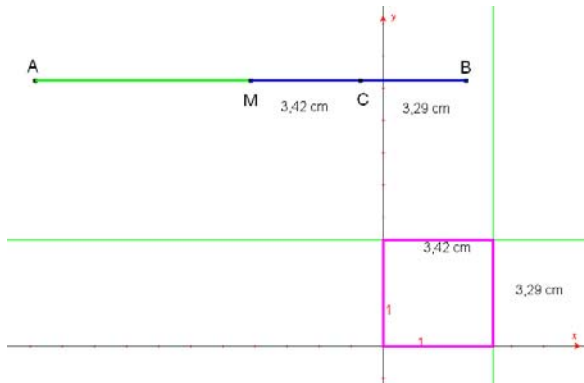


Figure 1. Dynamical model constructed in Cabri Geometry.

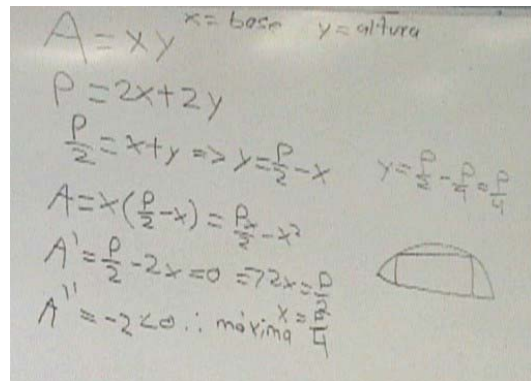


Figure 2. Algebraic procedure developed by Peter to obtain the problem solution.

Peter and Paul considered necessary to take an algebraic approach in order to obtain the “exact” solution of the problem using calculus techniques. Peter and Paul denoted by x and y the base and height of the rectangle, respectively. Then, they represented algebraically the area A as a function of x , and differentiated this function to obtain the critical points and the value that maximizes the area of the rectangle (Figure 2). Based on this algebraic procedure, Peter and Paul were convinced that the maximum area is attained when the rectangle is a square, and obtained evidence that their initial conjectures was wrong. This conjecture was based on both, visual perception obtained by manipulating the dynamic configuration, and prior problem solving experiences of Peter with other optimization problems whose solution do not correspond to a square. For instance, the following problem: find the rectangle of maximum area inscribed in a semicircle (see bottom right corner from figure 2).

Peter and Paul obtained additional certainty about the correctness of the solution associating the geometric problem with a similar algebraic problem: maximize the product of two numbers whose sum is given.

At first, I thought that the rectangle of maximum area would not be square, since there is a classical problem of finding the maximum area of a rectangle inscribed in a semicircle, and the square is not the figure with maximum area. After we obtained the solution $x = y$ algebraically, the result of the problem became logical to me, because if the area is equal to xy , you can prove that the product of two numbers, whose sum is given, is maximum if both numbers are equal (Extracted from Peter and Paul's report corresponding to the sixth session).

Daniel and Emily drew the segment AB and its midpoint (C). After this, they placed point E between points B and C , without considering that C should move on segment BC . For this reason, point E can be dragged over the entire segment AB and not only over BC . Teachers also drew point D , symmetric to point E respect to point C , but this point was not used. Teachers transferred lengths EB and CE to the horizontal and vertical axes, respectively, to draw a rectangle (Figure 3).

Daniel and Emily computed the rectangle's area and transferred this value to the vertical axis; then employed the "Locus" tool to construct a graphic representation of the area function (Figure 3). The teachers were astonished to observe the graph (Figure 3), since they expected that it was only a portion of a parabola. The graph behavior was due to the way that Daniel and Emily developed the geometrical construction, since this rectangle does not always meet the problem's conditions. For some positions of point E , rectangle's perimeter is greater than the length of segment AB (Figure 3, right). It is important to notice that Daniel and Emily, unlike Peter and Paul, did not verify that their

geometrical model satisfied the problem conditions, although the graph they visualized on the screen did not correspond to what the teachers had anticipated.

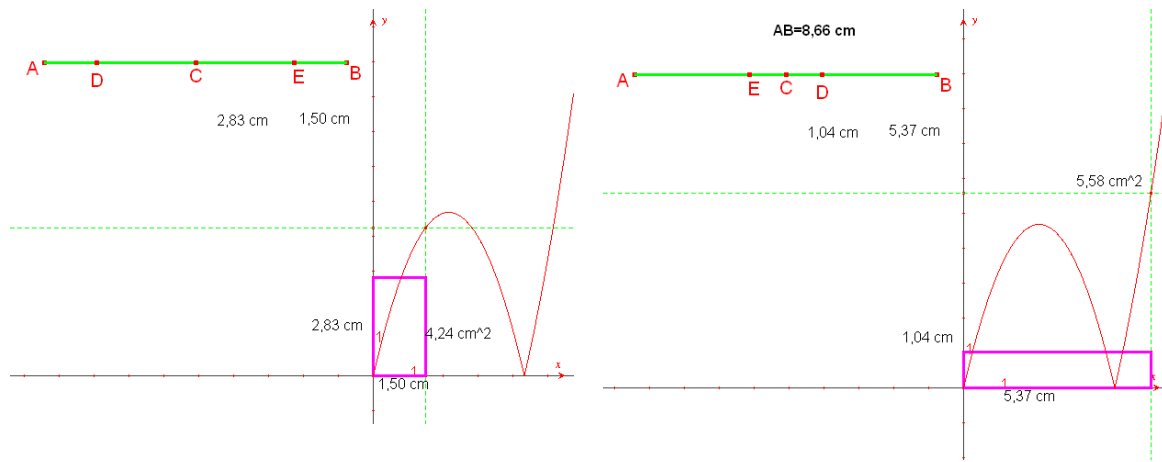


Figure 3. Dynamical model elaborated by Daniel and Emily.

The problem solving behavior shown for this pair of teachers to approach the first task is representative of the activity performed by them to approach all tasks. Daniel and Emily had some difficulties to construct dynamic configurations that met the conditions of problem statement. Additionally, in this task, they did not consider relevant to use the resources offered by the software, such as measure tools to check the accuracy of their dynamical construction. However, these teachers employed the graph of the function area to conjecture that the maximum area is attained when the rectangle is a square, so the use of the tool allowed teachers to formulate conjectures, which is an important element of what constitutes mathematical thinking.

Other important feature of Emily and Daniel's problem solving behavior was that they showed difficulties to implement algebraic procedures to exploring solution routes, although, plenary presentations allowed them consider the importance to develop this type of strategies to prove or refute conjectures posed using the resources offered by Cabri. In this context, the use of a dynamic software offered teachers opportunities to

approach tasks with less amount of algebraic resources in relation to the requirements of a paper and pencil setting.

Jacob and Sophia traced a segment AB , a point C on AB , and midpoint (M) of segment AC . They transferred lengths of segments MC and CB to the vertical and horizontal axes, respectively, and traced a rectangle based on these measures. Teachers verified that rectangle's sides had the same measures as segments CB and MC , however they did not realize that rectangle's perimeter is not the length of segment AB . The mistakes made in the constructions process led them to formulate a wrong conjecture: the base of the rectangle of maximum area must be twice its height (Figure 4).

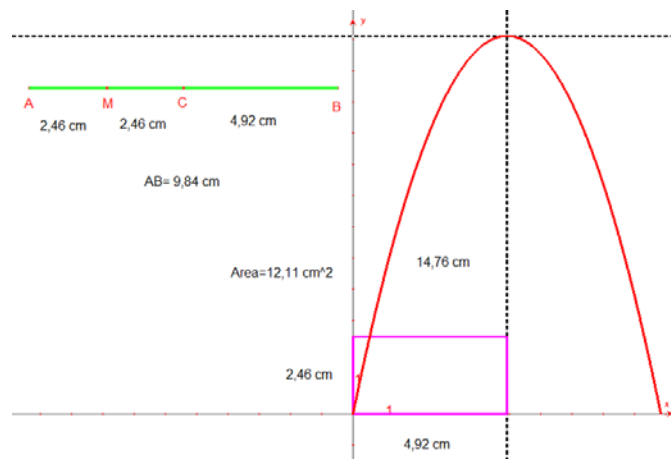


Figure 4. Dynamic model elaborated by Sophia and Jacob.

Comments

The results of this task show that, in general, Cabri acted as a reorganizer, since it allowed teachers to develop procedures to approach the task that could not be done in paper and pencil environments, such as formulating conjectures based in the observation of variation of numerical attributes of figures, as was the case of Peter and Paul's approach; or the visualization of a relationship between two quantities obtained without the previous formulation of an algebraic expression as in the approaches developed by

Daniel, Emily, Jacob and Sophia. That is, the teachers were able to access the resources incorporated in the tool, specifically numerical and graphical resources available in Cabri, to develop a particular form of thinking to approach the problem.

Concerning the justification process, Peter and Paul considered important to verify empirically that the construction satisfied the conditions stated in the problem and elaborated an algebraic proof of their conjecture. In this case, the use of measure tools was a mean to establish the validity of their construction; and the algebraic proof was employed to obtain an “exact” and not only an “approximated” solution. However, it can be observed that not all teachers verified that geometric configurations were constructed properly, neither all of them were aware of the importance to provide justifications using the means offered by the tool or external to it. These results differ from other research works that analyze the same problem. In those, it is concluded that the transition from a geometric conjecture to an algebraic proof, emerges from a discrepancy between a conjecture and the approximate results obtained with the tool, which suggested a different result (Olive, 2000).

The plenary discussion supported Daniel, Emily, Jacob and Sophia to identify pitfalls in their work and reflect about some important mathematical ideas such as the domain of a function and the importance to provide justifications. Besides this, the interaction among member of the learning community allowed Peter and Paul to incorporate a visual approach to their repertoire of problem solving strategies.

Second task

Peter and Paul selected a point A on the horizontal axis. The distance between point A and the origin O of the coordinate system represents the length of a side of the rectangle that teachers wanted to construct. Teachers used the “Numerical Edit” tool to define a quantity that represents the area of the rectangle, and to calculate the length of other side of it, they divided the length of segment AO by the area, and obtained the value c . Then, they transferred c to the vertical axis and obtained point B . Then, teachers drew rectangle $OABC$, calculated its area and dragged point A to verify that the area remain constant. Next, teachers obtained the perimeter of rectangle $OABC$ to construct a graph relating a side of the rectangle and the corresponding rectangle’s area (Figure 5). In this problem, Peter and Paul incorporated to their repertoire of strategies the graphical approach discussed in the plenary session corresponding to the first task.

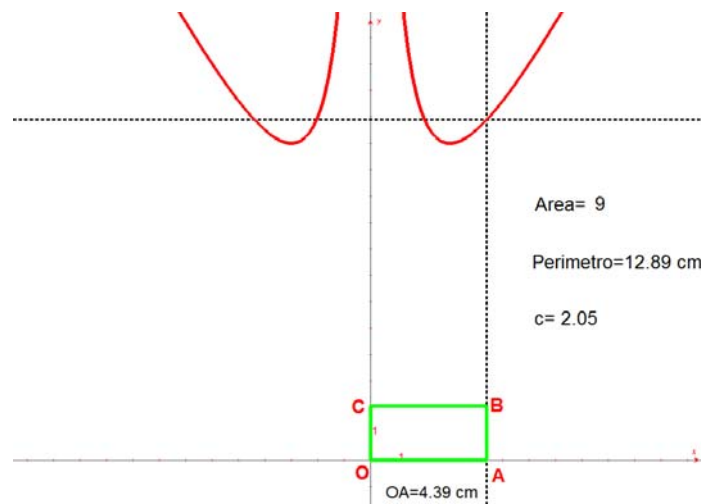


Figure 5. Perimeter of rectangle $OABC$, as a function of a length of side OA . (Graph, elaborated by Peter and Paul)

Teachers conjectured that the graph of the perimeter, as a function of side OA , consists of a branch of a hyperbola. Peter and Paul determined that although the locus was split in two branches, it was enough to consider one of them. Teachers tried to test their conjecture, first, by using the “Equation or Coordinates” tool, but the software did

not display the equation corresponding to the locus. Secondly, teachers selected five points on one of the branches for tracing a conic that, visually overlapped the graph, and by this mean they were convinced that the locus corresponded to a hyperbola. In the same way as in the first task, the software acted as a reorganizer, since it allowed Peter and Paul to develop graphical approaches to obtain evidence support their conjectures that are difficult to implement in paper and pencil settings.

Peter and Paul also conjectured that minimum perimeter is reached when the rectangle is a square. Teachers did not construct an algebraic proof of their conjectures; they were convinced of their results based on the visual and numerical evidence provided by the software. The problem solving behavior of these teachers differs from that shown by them to solve the previous problem, in which they considered important to formulate and solve the problem algebraically.

Sophia and Jacob approached this problem drawing a segment AB whose length represents the rectangle's area. Then, they put a point C on AB , and stated that the length AC would represent one of rectangle's sides. Sophia and Jacob transferred the measure of AC to the horizontal axis to obtain a point X . Then, they obtained the length of the other rectangle side computing the quotient AB/AC , and transferred this measure to the vertical axis to obtain point Y , finally they drew the rectangle $OXYZ$ (Figure 6). Later, teachers measured rectangle's area to verify that this measure coincided with the length of segment AB . In this action, it can be observed the effect of interaction in a learning community, since in the previous problem; this pair of teachers does not considered the use of measure tools to verify their construction was correct.

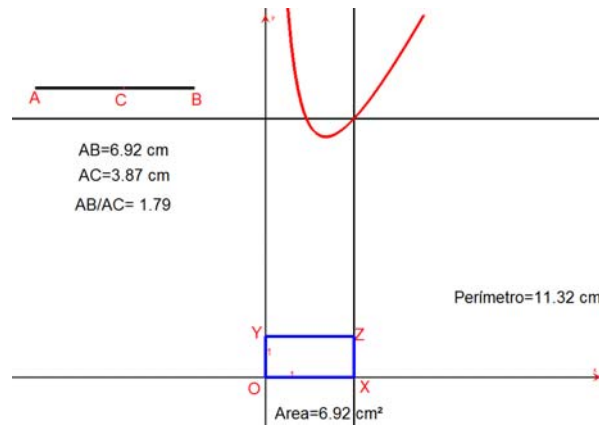


Figure 6. Graph of the perimeter of rectangle $OXZY$, as a function of side OX , elaborated by Sophia and Jacob.

Then, Sophia and Jacob constructed the graph that relates the perimeter of rectangle $OXZY$ and the length OX , and conjectured that the minimum perimeter is reached when the rectangle is a square, based on dragging point C and the visualization of perimeter function. They did not elaborate an algebraic justification of their conjecture.

Emily and Daniel had difficulties to build a rectangle of constant area in Cabri, and they tried to solve a simpler problem with the aim of using this to solve the original problem. They proposed constructing a triangle of constant area, and carried out the construction fixing the triangle's base, and putting the third triangle's vertex on a parallel line to the base of a triangle. Teachers verified, with the "Area" tool, that the triangle they constructed satisfied the condition of having constant area, and conjectured that the triangle of minimum perimeter is an isosceles triangle (Figure 7). In the process to approaching this task it can be observed that Daniel and Emily incorporated the use of measure tools to their repertoire of resources to verify accuracy of a dynamical construction, strategy which was discussed during plenary discussion of the first task.

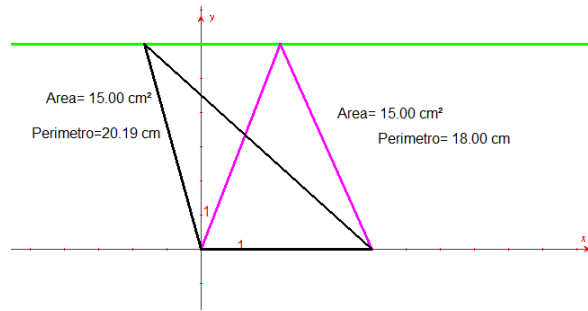


Figure 7. A family of triangles of constant area.

In the same line of thinking, Daniel and Emily considered relevant to provide justifications. For instance, the teachers were able to justify that triangles they constructed have constant area since the base is fixed and all triangles of the family have the same height. Daniel and Emily also tried to use algebraic procedures to verify that the triangle of minimum perimeter is an isosceles one, but they were unable to algebraically formulate the problem, as can be observed in the figure 8. The analysis of the activity developed by Daniel and Emily, allows us to obtain evidence that the use of Cabri increases the number of problems that students, with a low ability to manage algebraic procedures, can tackle.

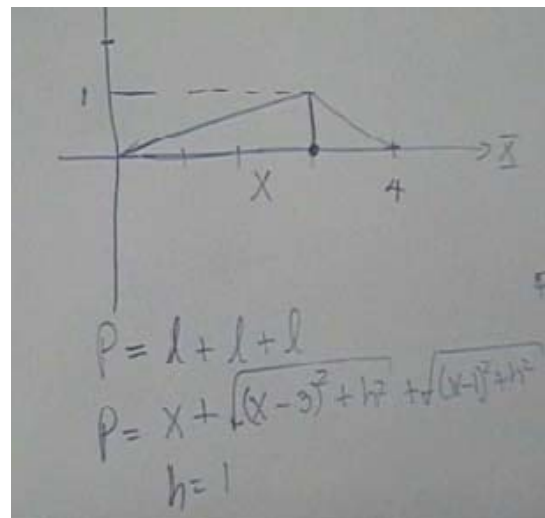


Figure 8. Algebraic formulation for the minimum perimeter triangle problem. (Elaborated by Daniel and Emily)

Sophia and Jacob were interested in solving the previous problem, and they tried to find, by algebraic means, the triangle of minimum perimeter given the conditions stated by Emily and Daniel. The teachers formulated algebraically the problem (Figure 9, left) and using calculus tools and a hand held calculator to perform algebraic operations, they obtained the point that maximizes the perimeter of the triangle, and concluded that the triangle of maximum perimeter is an isosceles triangle. (Figure 9, right).

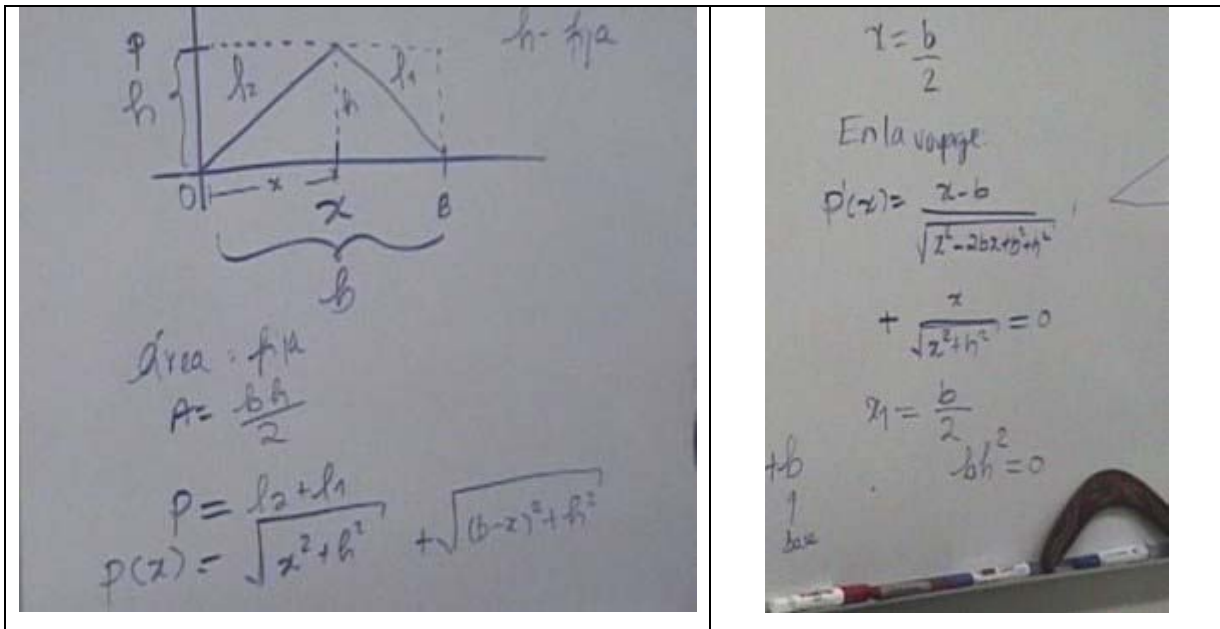


Figure 9. Algebraic solution proposed by Sophia and Jacob.

It was observed that discussion developed into the community, influenced the problem solving behavior of Sophia and Jacob, since these teachers incorporated the use of algebraic procedures to their repertory of justifications. On the other hand, when teachers solved this task, they used calculator Voyage 200 as an amplifier, since the tool was only employed to perform computations such as the derivative of the perimeter function and to solve the equation . However, the use of the calculator allowed teachers to reflect about the results not encountered in paper and pencil settings. When Sophia and Jacob solved the equation , they obtained as a result

. Sophia commented that the expression means that the minimum perimeter is also attained if the base or height of the triangle is equal to zero, but in this case the triangle disappears.

Comments

This task allowed us to observe that Cabri transformed teachers' forms of thinking and reasoning. For instance, approaching tasks within a paper and pencil environment leads to consider the meaning of a variable with restricted properties, basically based on representing it with a symbol, say x . Meanwhile, using a dynamic software to approach the task, allowed teachers to construct the idea of a variable, not only as a symbol, rather as an amount that changes, as it can be observed when teachers dragged the point representing the independent variable to approximate the value that produces the minimum perimeter. That is, the use of Cabri, particularly the executability of representations, gives rise to a different meaning of the concept of variable, since the tool helps to perceive the idea of variation as the work of Peter, Paul, Jacob and Sophia has shown. We argue that the exploration of ideas such as variation and co-variation, through the use of a dynamic software, favors a reorganization of students' cognitive processes, since it helps them to give meaning to ideas and concepts involved in the solution of optimization problems, such as the function concept. It is attained by means of visualization and perceiving how one quantity changes when the other does.

Third task

To approach this task Peter and Paul drew a segment AB that represents the length of the wire. Then, they located a point P on AB . The lengths AP and PB were used to construct the square and the circle, respectively (Figure 10). To construct the square Peter and Paul divided the segment AB in four parts, the length of each of these parts is the length of the square's side. To construct the circumference, the teachers obtained the radio using the calculator introducing the formula , where perimeter is the length of segment PB .

With the “Area” tool the teachers computed the area of each of the figures, added them up and plotted the graph of area as a function of length AP . Based on visual perception, teachers conjectured that the graph is a parabola and approximated visually the value of segment AP that minimizes de sum of areas dragging point D .

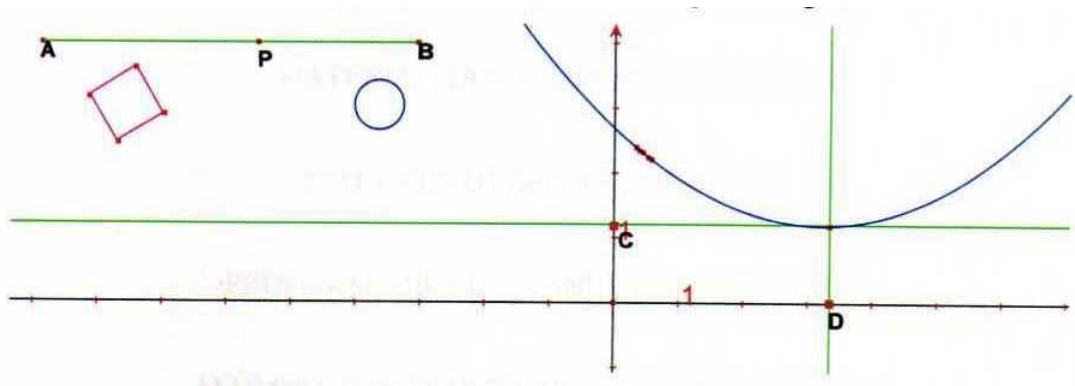


Figure 10. Graph of sum of areas of a square and a circle as a function of a length AP .

To obtain the algebraic solution of the problem Peter drew on the board a segment AB and a point P on the segment, in a similar way as he did in the software. He denoted the length of segment AP as x , then he said that the length of segment PB is equal to . Since AP is the perimeter of rectangle, then the area of this rectangle is equal to

. Moreover, the area of the circle can be computed as (figure 11, left).

Then, Peter expressed the sum of areas as a function of x , and used the calculator to obtain the derivative of the function and its critical points (Figure 11, right).

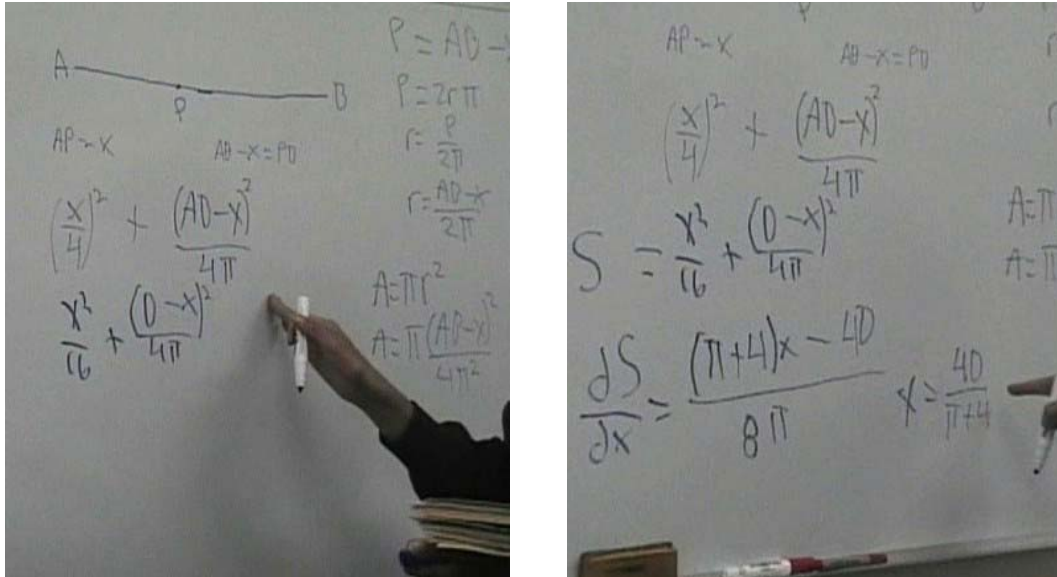


Figure 11. Algebraic formulation of the wire problem.

Peter expressed that in the dynamic configuration he approached the point which minimizes the sum of the areas and compared it with the result obtained by substituting the particular values into the algebraic solution.

The process employed by Daniel and Emily to solve the problem consisted in drawing a segment AB to represent the wire, and put a point C on AB which is the point where it is cut. Then, teachers constructed a square by considering as one of its sides the segment AC , they traced midpoint (D) of segment CB , and drew a circle with center D and radius DB . Daniel and Emily also computed the areas of the square and circle, and computed their sum S . Finally, teachers constructed a graph relating length of segment AC and the area S which is the sum of areas (Figure 12). The activity developed by the

teachers showed that they did not understand the problem statement, since the perimeter of the circle and square are not the lengths of segments AC and BC , respectively. Daniel and Emily had difficulties to understand the problem, even after Peter and Paul showed how they solved the problem; Daniel and Emily did not understand why the cable should be divided into four equal parts to construct the square.

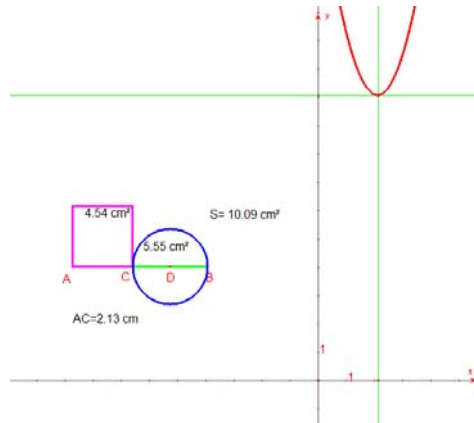


Figure 12. Dynamical configuration representing the wire problem, elaborated by Daniel and Emily.

Comments

Approaching this task, using technology, required by the problem solvers to think about the geometric objects in terms of actions, for instance, the actions to be considered to construct a square given a segment, are different from those when paper and pencil environment is used. The difference has to do with a “new quality” that the representation of the objects have when using Cabri, the executability property.

In analogy with the previous tasks, the use of Cabri software allowed the problem solvers to relate geometric and algebraic aspects of the problem as well as to coordinate them into a wider conceptual network. For instance, it allowed them to assign the variable, which represented the side of a square or the radius of a circle, a more concrete

meaning in terms of variation and not only its representations as a symbol. Besides, as in the previous tasks, the idea of dependence between variables acquired a more robust meaning in terms of the concept of a function. The plenary discussion allowed the participants to reflect about the properties of a function concerning points where it reaches maxima or minima as well as its domain.

Closing remarks

The use of digital tools allowed teachers not solely to remember facts or apply algorithms, but most importantly, it helped them to formulate conjectures, and develop visual schemas to provide justifications. Mainly, measuring attributes and dragging elements in geometric constructions allowed teachers to formulate conjectures (Arzarello, Olivero, Paola & Robutti, 2002) and observe relations among mathematical objects that can be a departure point to develop a deeper mathematical understanding.

It was observed that the use of technology helped teachers to develop ways of reasoning and forms of reflecting about the meaning and connections among mathematical objects. For example, the dynamic software enabled teacher to search for various forms of justifying a conjecture, in which the use and integration of visual, empirical and deductive arguments were useful.

Based on the activities developed, we noticed that teachers founded their forms of reasoning strongly on the visual representations, a result previously reported by Arcavi (2003). The dynamism of representations helped teachers to think about variation of particular instances and provided them with empirical basis to formulate conjectures. The software provided feedback to the teachers (Arcavi & Hadas, 2000), but not all teachers

were able to give meaning to this feedback, which was observed in the form that Sophia and Jacob, and Emily and Daniel have solved the first task.

The analysis of the tasks has shown a way in which the conceptual structure of the problem solver can be extended by incorporating the resources of the tool through the use of it in the process of solving problems. This was explicit when teachers used the tool to provide a visual representation of the information and by doing so, to approach a solution of the problem, which used algebraic setting as well as visual ones. That is, the capabilities of the tool as a cognitive reorganizer were based on the different possibilities that the tool offers to establish connections and to act as an extension of the cognitive structure of the teachers.

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Problem Solving and its elements in forming Proof

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Abstract: The character of the mathematics education traditions on problem solving and proof are compared, and aspects of problem solving that occur in the processes of forming a proof, which are not well represented in the literature, are portrayed.

Keywords: heuristics; problem solving; proof

Introduction

Mathematics educators tend to compartmentalize the domains ‘problem solving’ and ‘proof and proving’. This detachment seems somehow artificial as both deal with aspects of producing mathematical argumentation. However, problem solving tends to emphasize the thought processes in furthering on-going work; in contrast the proof tradition is concerned more in evaluating the soundness of the complete output. In this paper, we shall respect the distinction made between problem solving and proof, but at the same time we shall discuss issues that are common.

We use the words ‘culture’, ‘tradition’ and ‘agenda’ synonymously for general views broadly adopted by the research community on any given educational perspective. Both the problem-solving tradition and the proof tradition are diverse, so we restrict ourselves to particular stances, mostly attending to the upper school and university level. For problem solving, the subject is taken for its own sake; hence the full weight of self-conscious decision-making becomes the scope of investigation. For proof, we distinguish the case where the practitioner possesses and implements the requisite mathematical tools to fully articulate the proof from the case where he/she does not. The various types of

tools needed will be discussed, especially when the context lies in a mathematical theory currently been taught: then tools are adapted and appropriated from techniques that the theory avails. However, such tools have to be designed and then coordinated in the mind, so within the processes in obtaining a proof it is evident that substantial elements of problem solving must occur. The main focus of the paper is to give a preliminary account of these elements.

In the next section, we shall present a short, rather personal, description of problem solving. Largely supposing that the reader is familiar with the core principles laid out by Polya, it discusses more practical issues like the role of the teacher, implementation and assessment. The section that follows deals with the problem-solving component in proof making. Here no attempt has been made to give a coherent picture of the proof tradition; one reason is that proof and proving are, as an educational domain, particularly prone to contrasting standpoints. Rather we limit our attention to those facets of proof that differ from the problem-solving tradition but at the same time retain some problem-solving elements. The choice of papers referred to is made with this in mind. The discourse will not be unidirectional; some points made could be read as if the culture of proof is supporting problem-solving activity. The extended example given in the penultimate section illustrates this, as well as other matters. The epilogue, in part, raises the question how well the problem-solving tradition (as it stands presently) is equipped to cover the problem-solving elements in formulating proof.

On the problem-solving tradition and allied practical issues

The perspective of problem solving has a relatively compact core of ideas, mainly centered on heuristics, meta-cognition including executive control, accessing and

applying the knowledge base, and identifying patterns of modes of thinking as students' work progresses, following the pioneering work of Polya (e.g., Polya, 1973) and later by Schoenfeld (e.g., Schoenfeld, 1985). However, problem solving, as a domain of mathematical activity is very general; it concerns the student's engagement on any mathematical task that is not judged procedural or the student does not have an initial overall idea how to proceed in solving the task. Many other perspectives taken by the educational literature would embrace this same arena; for these the term problem solving is likely to be invoked (after all it is a term that is quite natural to use generally), but it is not a term around which the main focus revolves.

On the practical level, to deserve autonomous attention, problem solving must have something to say about teaching and instruction. The function of problem solving has been broadly characterized in three categories: teaching for problem solving, teaching about problem solving, and teaching through problem solving (Schroeder and Lester, 1989). For the first category, tasks are chosen that force students to think more actively about whatever mathematical topic that is being studied, the third is about building up conceptualization via a program of deliberately sequenced tasks. For both, problem solving is given a utilitarian role. On the other hand, for the second category, problem solving is taken per-se as an integrated theme of discourse, and we will largely adopt this perspective in this paper. The teaching must be directed to the student's own awareness of the influences that form his/her processes to reach a solution. The teacher then has to teach not only mathematical content and method, but also general solving skills. Doing this necessarily needs elements of intervention on the side of the teacher; he/she has to induce habits of self-questioning and reflection that allow students to realize ideas critical

in achieving a result. This means that there are aspects of teaching problem solving that can be regarded authoritative (but not authoritarian), see e.g. B. Larvor (2010).

If a problem-solving approach is adopted in teaching, there are associated issues about design and evaluation. What constitutes a ‘good’ problem? For this question, one could simply say that any problem for which there is not immediately an obvious line of attack is suitable. But do other factors come in? A sense of the attractiveness of the solution is one, a sense of achievement the solver experiences is another. A measure of a ‘good’ problem is how far a solution, or an attempt to achieve a solution, would inspire the solver to form related problems (Crespo & Sinclair, 2008). Another possible measure might be the plurality of different directions that the problem can be treated so that connections can be formed (Leikin & Levav-Waynberg, 2007), though problems that have a particular ‘catch’ in the solution can also be useful because of the better control afforded to the teacher/researcher. The evaluation of a student’s complete output, then, is not straightforward; a model is given in Geiger & Galbraith (1998). Another factor is the gap of experience between the setter and the solver; here lies the danger that either the setter assumes that students have more experience than they really possess or the experience of the setter leads him / her to expect an over-involved solution blinding a more elementary path. Hence it is difficult to gauge how challenging a particular problem is. Further if you credit an argument by its plausibility rather than its logical security, you bring in a subjective factor. Such points of loss of control in terms of evaluation makes problem solving, taken as an overall guiding principle in teaching, open to criticism. For example, a recently published paper bears the rather provoking title: “Teaching General Problem-Solving Skills Is Not a Substitute for, or a Viable Addition

to, Teaching Mathematics” (Sweller, Clark & Kirsher, 2010). The basis of the authors’ contention is well represented by their claim: “in over a half century, no systematic body of evidence demonstrating the effectiveness of any general problem-solving strategies has emerged”. The position taken by the paper might well seem to be extreme and partisan, but it does reflect the difficulty in designing comprehensive, large-scale studies assessing the success (or otherwise) of teaching about problem solving, not least on what exactly should be measured.

The principles behind problem solving can be significant to the working of a student of any age and of any ability, and there are numerous educational studies that advance the cause of problem solving convincingly to populations ranging from pre-primary school level to professional mathematicians. But making conscious decisions about which heuristics to use as well as how other metacognitive dimensions should be employed need mature deliberation. In danger of seeming elitist, we consider there are two groups of students that are most able to cope with and profit from problem-solving based instruction; the so-called mathematically ‘gifted’ student at school, and the undergraduate student studying mathematics. (A third group might be teacher- students, as they have to learn how to reflect on and attend to the difficulties of their future students.) We are not saying that other students cannot gain from problem-solving activities, but for them the gain could well be qualified. For instance, in Perrenet & Taconis (2009), it is stated “(university mathematics) students show aspects of the development of an individual problem-solving style. The students explain the shifts mainly by the specific nature of the mathematics problems encountered at university compared to secondary school mathematics problems”. Other papers offer models on

how traits of thinking are different for the gifted and the expert over the ‘average’ solver (e.g., Gorodetsky & Klavir, 2003).

What sources cater for these special groups? First, there is now a plethora of ‘problem solving’ texts on the market; these usually list many challenging problems with exposition of some ‘model’ solutions. However, most are composed in the spirit ‘you learn as you practice’, without much commentary on the educational level. Typically the organization has some chapters based on general aspects of problem solving and others on problem solving that is mathematically domain-specific. (It would be misleading to identify such books as textbooks because the aim is not to cover a fixed curriculum of mathematical content.) The style of presentation can be daunting, but some tomes are particularly attractive and reader friendly. One, written by P. Zeitz (Zeitz, 2007, p. xi), includes in its preface a list of principles guiding its writing that is surprisingly close to the tenets held by educational research on problem solving:

- Problem solving can be taught and can be learned.
- Success at solving problems is crucially dependent on psychological factors. Attributes like confidence, concentration and courage are vitally important.
- No-holds-bared investigation is at least as important as rigorous argument.
- The non-psychological aspects of problem solving are a mix of strategic principles, more focused tactical approaches, and narrowly defined technical tools.
- Knowledge of folklore (for example, the pigeonhole principle or Conway’s Checker problem) is as important as mastery of technical tools.

Beyond problem-solving books, there is the collected ‘wisdom’ from the many dedicated teachers involved in ‘training’ students for mathematics contests and special examinations. There are now some regularly held conferences aimed not only to attract such teachers but also researchers in education, such as one titled “Creativity in

Mathematics and the Education of Gifted Students". The ensuing interaction between the two interested communities, the one more theoretically inclined, the other more practically minded, is valuable, and has enriched the educational literature published on problem solving especially over the last decade or so. The facet of 'training' in particular is interesting, for it does not at first seem to be quite consonant to the idea of flexible thinking as espoused in the problem-solving tradition held by educators.

Another source is problem-solving courses offered in the curriculum of some university mathematics departments. The content and 'style' of the delivery of a class usually is a mixture of: introductory statements made by the instructor, students' attempting to solve particular problems quite often conducted in small groups, students criticizing peers work, a class discussion about how solutions were instigated and how completed arguments functioned to realize the solution. The instructor perhaps in the end winds up the session by relating the class activity to terminology found in the problem-solving culture. For such a free ranging course, an accompanying textbook is out of place; rather a succession of class-plans by the individual teacher is followed ad lib. This raises the issue of the demands put on a teacher when teaching a problem-solving course, and whether they have to be trained to teach in a special way (see e.g. A. Karp, 2010).

Another awkward point concerning courses oriented towards problem solving is that the level of interaction is high, so really are suitable only for classes of a relatively small size. The yearly intake of students to a mathematics department can be in the hundreds, with the result that a problem-solving course is usually selective and non-compulsory. Thus the course effectively becomes a special interest class on a par to 'standard' courses that present particular mathematical theory. Where then is the universal need for

undergraduates to be instructed in problem solving? Indeed, it is reported in Yosof and Tall, 1999, that students who took a problem-solving course mostly enjoyed the experience but they found difficulties to apply what they learnt in other courses.

(At this point, we should clarify our position on the use of textbooks; as we asserted above, textbooks perhaps do not have a place in teaching *about* problem solving, but they certainly have their place in teaching *for* problem solving. The idea is that the whole structure of the book consists of carefully sequenced problems leading up to major theorems in a main field of mathematics. It replaces a ‘standard’ presentation of a topic in the curriculum. An example is found in a book by Polya & Szego for Analysis first published in 1924 (translated into English in 1978); more recently R. P. Burn has written several textbooks (e.g., R. P. Burn, 1992) in the same kind of spirit. A natural question arises: does a course based on such a textbook infuse general problem-solving sensibilities?)

Much that we have discussed so far is addressed to practical matters; the focus for the remaining part of the paper will be based on theoretical lines. There are many expositions extant that have elaborated on the core ideas, i.e. heuristics, metacognition and observing phase patterns during the solution process. Some have a local perspective, some attempt to present overall models to refine the character of the whole field. For the latter, (Carlson & Bloom, 2005) is a good example; the authors develop what they call a ‘Multidimensional Problem-Solving Framework’, which tabulates items along two axes: activity phases against resources, heuristics, affect and monitoring. Within the framework it is stressed that various aspects of cycling in types of activity occur in problem-solving behavior. The paper clearly is in the fold of the problem-solving

tradition. But for many papers, this is not so clean-cut; in them there is substantial material that seems to be in accord to the tenets of problem solving but the ostentatious perspective lies elsewhere. In Mamona-Downs and Downs (2005), it was argued that if we wished to form an 'identity' of problem solving we must examine how other mathematics education topics impinge. One topic brought in was 'proof'. The 'terrain' of proof obviously encompasses many reasoning processes that are common to problem solving, so it is a natural candidate for comparison. In the next section we shall discuss the confluences (and to some degree the disparities) between the domains; references are made to papers that are ostensibly placed in the proof agenda but betray interesting problem-solving traits.

The interface between problem solving and proving

Proof and proof production is associated with deductive reasoning. From the perspective of problem solving, the notion of deductive reasoning can seem artificial; employing deductive reasoning on its own cannot help students to build up the ideas involved in the making of a strategy, it can only inform the student that any particular act is 'legal' or the whole argument a-posteriori is logically sound. On the other hand induction, i.e. obtaining evidence from considering cases that are not exhaustive, is useful for explorative work but is insufficient to establish the desired result. Over the years numerous models of reasoning have been put forward to fill the gap between inductive and deductive argumentation. To mention a few, there is representational reasoning (Simon, 1996), abductive reasoning due to Pierce (see Cifarelli, 1999, for a contemporary summary), and plausible reasoning derived from Polya himself (Polya, 1954). Such models differ in detail, but all deal in one way or another with a shift from a speculative

mode of thinking to one that has an anticipatory character. Any type of mathematical reasoning suggests an a-posteriori summation of the lines of thought taken before; the question “what is your reasoning here” is a request to track back. Despite this, it still concerns on-going working; an advance in reasoning depends on the problem-solving decisions preceding it. In this respect some authors like to discriminate between reasoning and argumentation; for example, Lithner (2001) takes argumentation as the ‘substantiation’ that convinces you that the “reasoning is appropriate”. Argumentation then suggests a completion as well as a check that your reasoning is, in loose terms, getting the job done; as such, argumentation should be closely related to proof, in cognitive terms at least. It has always been contentious what a proof is; perhaps the range of interpretation today is as wide as it has ever been. Here is not the place to give even a skeleton sketch of the current views taken about the role and character of proof; a comprehensive account from the mathematical education perspective is to be found in the book Reid & Knipping (2010). In the last recommendation for ‘Directions for Research’ in this book, it is stated, “the relationship between argumentation and proof is far from clear” (despite the numerous theoretical theses forwarded in this area). In explaining this relationship it might be instructive to explore what different problem-solving skills we would expect vis-à-vis argumentation and proof.

Is proof for every student and for every age? Leading up to answer this question we regard ‘deductive’ or ‘formal’ proof as an ideal rarely adhered to. What, then, do professional mathematicians tend to produce? We believe that this issue is nicely expressed in a public lecture given by Hyman Bass (2009) where the image of a proof providing certification is replaced by an image of a proof supporting a claim:

“ Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists...the convinced listener feels empowered by the argument, given sufficient time, incentive, and resources, to actually construct a formal proof”.
p. 3

Hence a typical proof provided by a mathematician is an argument for which there is a potentiality to convert it into formal proof (in principle at least). Bass then considers the notion of generic proof (see also, e.g., Leron & Zavlasky, 2009) as a type of proof that mathematicians often accept and adopt, and convincingly relates an incident where a primary school child was able to express a generic proof (in joint work with D. Ball). The child was able to explain the proposition that ‘the sum of two odds are even’ by mentally imagining two separate collections each with an odd number of objects, pairing off objects that lie in the same collection as far as possible and pairing the two objects ‘left over’ one from each collection. One might say the argumentation takes place on the perceptual level and so cannot be regarded as a proof. On the other hand, the reasoning is executed through properties that suggest both abstraction and generalization are involved; from this criterion, perhaps it should qualify as a proof after all. The obvious stance to take is to acknowledge different levels of proof. For example, in the opening document for ICMI Study 19 (2009) on proof, the terms ‘developmental proof’ and ‘disciplinary proof’ are introduced. A major factor in this distinction is that students at school do not usually possess the requisite tools to allow them to articulate disciplinary proof, whereas in the culture of advanced mathematical thinking (pertaining mostly to tertiary level study) pains are taken to explicitly define the tools needed to process proof in a particular field. For example, if the Intermediate Value Theorem of Real Analysis is mentioned at school it is usually taken as a truism, but at university either it has to be proved or explicitly recognized as a premise. Hence the learning of

mathematics at the university level is ‘privileged’ in terms of proof making; in principle, the tools are available, or the tools are at hand to ‘design’ further finely-honed tools for your own specific purposes.

How does the above concern problem solving? As regard to ‘developmental proof’, students have to rely on mental images and loosely grounded representations; the processes in initializing, collating and monitoring the argumentation as it evolves are very much in the field of problem solving. But because now mostly we are angling for an ‘informal’ justification of a general proposition, there is a tendency for properties to determine objects rather than vice-versa. Here, the notion of characterization comes to the fore; you ask which objects satisfy the conditions (rather than asking which properties a given object satisfies). Even though there is no real difficulty in designing tasks asking for a characterization in the problem-solving mold, this aspect is poorly represented as a theme in the literature. What quite often occurs is that the solver identifies a class of objects, C say, for which either all the objects that hold the given conditions are shown to be in C or all the objects of C are shown to satisfy the conditions. Several rounds can be made in restricting or expanding the class respectively, until analysis allows the removal or inclusion of any remaining isolated exceptions. Such a program probably is best illustrated in the literature by the framework of ‘example generation’, largely launched through some work of Mason and his colleagues (e.g. Watson & Mason, 2005). Also related is the Lakatosian notion of ‘heuristic refutation’, where counter examples are not taken to disprove but in order to reformulate the premises (e.g. De Villiers, 2000).

For disciplinary proof, in principle the tools to prove are at hand. But an informal discourse is usually kept to whilst engaged in the actual production of the proof, though

much technical terminology is retained. For the mathematician who holds well the topic concerned, the technical terminology is not a barrier to understanding, to the contrary it is empowering (Thurston, 1995). Whenever there is an interaction between an informal language and one that is more documentarily directed, problem solving has its place; strategy making is made in the informal language and ‘converted’ to the documentary style. The problem-solving aspects so evoked would have restrictions compared to general problem solving but they are nevertheless important. Below, particular angles of this issue are mentioned. In the context of building up a mathematical theory, a meta-cognitive examination of a proof is required to understand what is important to retain in the memory; the proof, the fact that the proof ascertains, both or none. For example, a method can be extracted from a proof, which has a potential to be applied elsewhere; in Hanna & Barbeau (2008), this phenomenon is discussed through various facets of problem solving. Structuring mathematical work into semi-independent units acts not only to produce a neat exposition but also represents essential ‘chunking’ of lines of thought without which the mind could well be overstrained. The deliberate design of modules in order to process the desideratum would seem to lie naturally in the realm of executive control. (For an illustration, see the worked example appearing in the next section.) For disciplinary proof the knowledge base typically is sophisticated and in flux; one possible consequence of this is that the solver could be tempted to apply knowledge that far exceeds the needs of the task/proof. An instance is given in Koichu (2010) from an example-generating activity. Also, the detection of applicability of theoretical knowledge is not often immediately apparent; in Mamona-Downs (2002) it is suggested that part of the teaching of a mathematical theory should include what the author terms

‘cues’, i.e., formats of knowledge that are not important theoretically but invite realization of certain types of application. (An instance is to be found in the worked example.) Changing track a little, we note that there is still a strand of informal discourse in disciplinary proof, so one might expect that some students would like to exploit it more than others. In this respect some research, mostly aimed at the tertiary level, has indeed identified students that have a strong inclination to work consistently either semantically or syntactically (e.g. Weber & Alcock, 2004). Such a marked preference must reflect the character of the problem-solving tools with which a particular student feels most at ease. Another but related issue is how to initiate a proof; Moore (1994) has noted that from fairly elementary but formally defined systems many students cannot deduce the simplest consequences, whereas Selden et al. (2000) talk about ‘tentative starts’. The first suggests that students cannot interiorize the abstraction that confronts them, the second suggests a more pro-active view that by ‘playing around’ with operations that they can do, even ‘blindly’, students can ‘click’ to openings in the underlying (or accommodating) structure yielded by the given situation. (Our worked task also features a tentative start.) The notion of structure, even though being somehow vague, seems to be a natural backdrop to combine the analytic tools given by formalization with problem-solving input, see Mamona-Downs & Downs (2008).

We wind up this section by raising a few points for which the difference between disciplinary and developmental proof is no longer relevant. First, whenever a shift of the mode of thinking occurs, the character of the supportive components of problem solving also changes. We have already discussed how an informal language reinforces (disciplinary) proof. The making of a conjecture also marks a change in the mode of

thinking; one could compare the style of argumentation made before the conjecture to that made after. If you adopt the more conventional propositional form of a proof perhaps it would be more appropriate to compare pre-strategy work with work done in effecting the strategy. Also, how does the reasoning allowing a first draft differ to the reasoning leading to the final presentation of a proof? In particular, whilst obtaining a proof sometimes recourse is made to diagrams and other kinds of representations that, quite often, are not ultimately referred to in the presentation. Another differentiation in mode of thought concerns what is processed in the mind against what is carried by 'authorized' symbolic usage. Even though these switches of thinking are documented in the literature, they have not been thoroughly examined in the problem-solving tradition.

Second, we resume our discourse regarding the essential difference between proof and problem solving. We center our problématique on discerning 'problem solving' tasks from 'proof' tasks. For a 'problem solving' task it is allowed to accept perceptual indications, as long as they seem self-evident. Verification itself does not feature strongly in the difference; rather a task requiring a proof differs in that the verification has to be articulated in officially accepted mathematical language; one might say we require an endorsement of the verification. Let's have an example. The task is to identify the different types of plane nets of a cube. It's a problem that can be tackled by a bright student of age eight, even though there are quite sophisticated things to do; first to interpret from the task environment what is meant by a 'type', and after to validate the answer by taking cases in an exhaustive manner. But the argument cannot be judged to be a proof because each net is just recognized as being one; there is no explicit mathematical warrant expressed that assures that when the squares are folded in the

appropriate fashion they do indeed ‘form’ a cube. Furthermore there is no expectation for the solver to undertake this ‘extra’ level of verification; thus what we have here is a problem-solving task.

An interesting offshoot of the above concerns mathematical modeling. Suppose a task is expressed in words that are not completely mathematical in form; it is modeled into a self-contained mathematical system in which it can be treated as a proof. However, the action of modeling certainly is subjective; so the task is better assigned as a ‘problem-solving’ task, but with a strong proof-making component. Also note that informal argumentation carried out within the task environment sometimes can be sustained right up to achieving a solution; if a more strict version is desired, modeling can be made not only of the task but also of the line of the context-held reasoning. This can be a valuable vehicle for students to realize the character of proof; proof in this regard is a channel to provide the tools to fully articulate the output of a problem-solving activity, see Mamona-Downs & Downs (2011).

An example

The example takes the form of an indicative solution path of a particular task; it exemplifies some points made in the previous section.

The task is relatively complicated to solve, though no complex mathematics is involved. A strategy is made without knowing which tools are needed to implement it. Designing these tools requires anticipatory reflection but the form of them rests on simple proofs, so we have a case where the proof culture contributes to problem solution processes. The style of discourse indeed is made very much in a problem-solving vein, but from it a presentation as a proof is readily extracted.

The task:

Does the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ have a partial sum equal to an integer apart from 1?

Preliminary observations; sizing up the situation

From previous knowledge, we know the harmonic series ‘converges’ to ∞ , i.e. for every $n \in \mathbf{N}$, there is a partial sum that exceeds n . Hence there are ‘infinitely’ many potential candidates for a partial sum to be an integer, so we cannot reduce the problem to a finite number of cases.

The first partial sum equals 1; can we make a preliminary guess whether it is likely for any other partial sum to be an integer? In response, take any $n \in \mathbf{N} \setminus \{1\}$ and consider the greatest $r \in \mathbf{N}$ for which $\sum_{i=1}^r \frac{1}{i} < n$. Then for n to be a partial sum necessarily:

$$n - \sum_{i=1}^r \frac{1}{i} \text{ must equal } \frac{1}{r+1}.$$

This condition seems stringent because of the appearance of r in both terms, so despite there being infinitely many candidates for n , it still would be a surprise to get a solution apart from 1. But this does not really help us to get a start for a strategy that would be conclusive.

Playing around; try out an action you can do

A partial sum is just a finite number of fractions added up; an action you can perform is to render this into a single fraction, i.e.

$$\sum_{i=1}^r \frac{1}{i} = \frac{\sum_{i=1}^r \frac{r!}{i}}{r!}. \quad (1)$$

This move is made as a tentative start; it is speculative in the sense that there is a hope (but not an expectation) that the new algebraic form might give us more handles to attack the task. Notice that the new form is one natural number divided by another; the question is reduced to analyzing whether the latter is a divisor of the prior.

Importing mathematical knowledge and its cue

The knowledge we import is the theorem that states any natural number greater than one can be expressed (uniquely) by a product of prime powers. The cue relevant to the task is if you think that n does not divide m , for $m, n \in \mathbf{N}$, then there will be a prime p such that the highest power of p dividing n is greater than the highest power of p dividing m .

The strategy

Suppose that the solver believes that there is not a solution apart from 1. In accordance he/she chooses the ‘simplest’ prime, i.e. 2, and tests whether the highest power of 2 dividing $r!$ is greater than the highest power of 2 dividing the nominator in (1). (If this test fails, one might choose another prime to check, or change the method from the experience gained by considering the case $p=2$).

Considerations how to implement the strategy

In order to analyze the highest power of 2 dividing

$$\sum_{i=1}^r \frac{r!}{i}$$

you have to search for more elementary and general properties about the highest power of 2 dividing integers. For convenience, we introduce the rather eccentric (but standard) symbolism: $2^a \parallel u$ where $a, u \in \mathbb{N}$ indicates that a is the highest power of 2 dividing u .

Designing the auxiliary tools

(i) Investigate: If $2^a \parallel u$, $2^b \parallel v$ and $2^c \parallel (u+v)$, what is c in terms of a and b ?

It is easy to demonstrate that

if $a \neq b$, then $c = \min(a, b)$

if $a = b$, then $c > a + b$

but nothing further can be said in general for the second implication. Hence in the case of $a \neq b$ we have more control than in the case of $a = b$.

(ii) In order to take advantage of the good control that occurs in the case of $a \neq b$ it is useful to have this fact established:

If $2^a \parallel u$ and $2^a \parallel v$ (where $u < v$), then there is a natural number w such that $w > u$, $w < v$ and $2^{a+1} \parallel w$. An immediate corollary is:

Suppose that $S := \{1, 2, 3, \dots, r\}$ and $l := \text{Max}\{l' : \exists k' \in S \text{ s.t. } 2^{l'} \parallel k'\}$. Then there is a *unique* element k of S for which $2^l \parallel k$. (2)

Implementation of the strategy

Here we show how the auxiliary tools facilitate the original situation:

Let $2^N \parallel r!$ and $2^{N_i} \parallel i$ whenever $i \leq r$. Now the highest power of 2 dividing $\frac{r!}{i}$ is equal to $N - N_i$.

For k and l specified in (2), the highest power of 2 dividing $\frac{r!}{k}$ is equal to $N - l$ so is less or equal to $N - N_i$; the equality holds only when $i=k$ (second auxiliary tool).

Applying the first auxiliary tool (recursively) we know that

$$2^{N-l} \parallel \sum_{i=1}^k \frac{r!}{i}.$$

Now N_k is simply an alternative symbol for l , so for $i > k$, $N - N_i > N - N_k$ and we can again apply the first auxiliary tool to obtain

$$2^{N-l} \parallel \sum_{i=1}^r \frac{r!}{i}.$$

Now if $r > 1$, l is a positive integer, so the highest power of 2 dividing the nominator in (1) is less than that for the denominator. We are done.

Comments

1. The preliminary observations are made not only to ‘understand’ the task, but include comments concerning a ‘feasibility test’ in order to make an informed guess on what the most likely outcome would be. Note that this guess is not made on experimental evidence.
2. The preliminary observations did not give a lead how to approach the task. A blind algebraic operation is made, resulting in another algebraic form and a new perspective. This action was not motivated, so there is an element of luck here.
3. The new form is a quotient of natural numbers; the issue now is to try to show that the denominator does not divide the nominator. (Guided by our feeling that the solution space is empty). This issue then could be resolved through an application of the fundamental theorem of arithmetic. Would students notice this link? Suppose that this type of application was taught, would they be more likely to catch the ‘cue’?
4. We now have a strategy; we take a particular prime p in the hope the greatest power of p dividing the nominator is less than that of the

denominator. Things are still tentative; we don't know as yet whether the strategy is intractable or not, and we choose $p=2$ solely on the grounds that 2 is probably the easiest prime to work with. Our decisions here have as much to do with hopeful wishing as control. However, somehow we judge the direction we have taken is promising.

5. We have a strategy, but now we need a strategy to implement the strategy. Modules of a more theoretical character are designed deliberately directed to the implementation. These modules can stand on their own merit outside of the context of the original task with autonomous proofs. Designing such analytic tools vis-à-vis envisaging how they would fit in the particular argumentation can be challenging.
6. In the process of resolving the task, there are several places where the solver is not completely controlling the effect of the decisions or actions that have been made. Were we just lucky? No: luck comes in, but from a certain stage there is an anticipation that things would work out as envisaged. But this raises the question, how can we quantify the grounds of this anticipation.

Epilogue

The main drive of this paper is to consider how elements of the problem-solving tradition are evident in the formulation of a proof, and (to a lesser degree) vice-versa. Also occasionally we have drawn out diverging tendencies between the two traditions. One difference not as yet explicitly mentioned is that tasks in the domain of problem solving are intended to be challenging, whereas for proof no such intention is imputed. In this regard, it is now quite often to have a transition course aimed towards proof practices in the curriculum of a mathematics department, especially in the United States; the 'content' of the examples presented tend to be relatively elementary in order to concentrate on the exposition of logically based argumentation. Such courses have a completely different 'feel' to problem-solving courses. We have stated that there is some doubt that taking a problem-solving course really will help the student when more theoretical courses are studied, but a 'proof' course could also be problematic in other

ways. For example, in Alcock (2010) it is reported that some mathematicians felt that those students who can pass a proof course would be the ones that did not really need to take the course anyway. Given this, a mix of the two kinds of courses might be the most profitable. It would bring up, for example, situations for which a student can achieve a result informally, and then can be challenged to articulate it with consummate reasoning and grounds; also it would bring up situations for which the student cannot proceed without building up constructions of a formal character. Proof should not be shown just as an imposition, but as a channel that enhances our range of mathematical thought and potentialities. The two kinds of situations mentioned above surely pertain to problem solving as much as to proof, but bring out a tempered outlook towards the current tradition of problem solving. To the mathematician, answering a typical problem-solving task often is an enjoying and rewarding pursuit, but can seem frivolous if aspects resting on perception are not tied down mathematically.

In the paper by Alcock referred to above, the author identifies four modes of thinking whilst forming a proof (drawing on the comments of a small population of mathematicians); instantiation, structural, creative and critical. These bear a striking semblance to the set of phases in problem solving famously put forward by Polya, i.e., getting acquainted, working for better understanding, hunting for the helpful idea, carrying out the plan, looking back (Polya, 1973, p.p. 33-36). The main discrepancy between the two taxonomies is that ‘structural thinking’ is characterized by introducing appropriate definitions and by working according to the rules of logic. This is consonant to the notion of definitional tautness introduced in Mamona-Downs and Downs (2011) that is not currently stressed in the problem-solving tradition. But the meta-cognitive

thought involved in forming definitions whilst partially envisaging how to control their consequences to particular ends has problem-solving elements that should not be ignored; then, perhaps, problem solving might be just as relevant in proving as it is in more relaxed forms of argumentation. In this paper we have examined elements of problem solving in the context of proof construction, admittedly in a rather eclectic way. We suggest that further research in this direction should be undertaken in the future, involving both researchers primarily 'affiliated' to proof and those primarily concerned with problem solving.

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Becoming Aware Of Mathematical Gaps In New Curricular Materials: A Resource-Based Analysis Of Teaching Practice

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Abstract: The study featured in this article, with its central focus on resources-in-use, draws upon salient aspects of the documentational approach of didactics. It includes an a priori analysis of the curricular resources being used by a teacher for the first time, followed by detailed in situ observations of the unfolding of her teaching practice involving these resources. The central mathematical problem of the lesson being analyzed deals with families of polynomial functions. The analysis highlights the teacher's growing awareness of the mathematical gaps in the resources she is using, which we conjecture to be a first step for her in the evolutionary transformation of resource to document, as well as an essential constituent of her ongoing professional development.

Keywords: documentational approach of didactics, documentational genesis, curricular resources in mathematics, families of polynomial functions, mathematical gaps in resources, ongoing professional development, resources-in-use, research on teaching practice with new curricular resources.

Introduction

Mathematical problems suitable for use in high school classrooms can be obtained from a variety of resources, including the internet, newspapers and books, colleagues, and of course textbooks. There is general consensus that most mathematics teachers rely on textbooks for their day-to-day fare of problem-solving items for students (Schmidt, 2011). Over time, these problems and the ways in which they are presented to students get tinkered with and gradually become refined (Gueudet & Trouche, 2010, 2011). However, we are only now beginning to learn a little about the ways in which teachers

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interact with the mathematical resources available to them (Gueudet, Pepin, & Trouche, 2011). Chevallard and Cirade (2010) have raised an additional issue, that of the lack of adequate mathematical resources for teachers when the school program is changed and new problems and problem-solving approaches are introduced. Moreover, as pointed out by Artigue and Houdemont (2007), many teachers who teach mathematics – even at the level of secondary school – are not mathematics specialists and “are quite often not proficient in mathematics, and that the mathematics and didactic formation they receive during their training does not compensate for these limitations” (p. 376). Although a focus on the mathematical resources available to teachers, their supportive role, and their adaptation and adoption is not one that, up to now, has been central to the research agenda of the problem-solving research community, its importance can be argued for, at the very least, on pragmatic grounds: The ways in which resources support (or do not support) teachers in their problem-solving efforts in class clearly impact upon the problem-solving experience of students.

According to Remillard (2005) who conducted a seminal review of teachers’ use of curricular materials, the process by which mathematics teachers appropriate and transform such resources, as well as the support that these resources offer, is rather unexplored terrain. In 2000, Adler similarly proposed that “mathematics teacher education needs to focus more attention on resources, on what they are and how they work as an extension of the teacher in school mathematics practice” (Adler, 2000, p. 205). In one such study of teachers using reform-based curricular materials, Manouchehri and Goodman (1998) reported what they viewed as shortcomings in the guidance for teachers provided by the curricula, saying that the curricula “did not provide the teachers

with detailed methods of how to address the content development” (p. 36). Teaching with new resources can thus lead to situations where teachers are not suitably prepared, but which can provide the impetus for new awarenesses of both a mathematical and didactical nature. In this regard, Gilbert (1994) has said: “reflection-in-action occurs when new situations arise in which a practitioner’s existing stock of knowledge – their knowledge-in-action – is not appropriate for the situation” (p. 516). This reflection-in-action, which involves critical examination and reformulation of one’s existing knowings, is intimately connected to, and synergistic with, one’s evolving appropriation and transformation of resources, according to the documentational approach of didactics (Gueudet & Trouche, 2009, 2011).

The Documentational Approach of Didactics

Gueudet and Trouche (2009, 2011) have developed a theoretical research framework based on the premise that documentation work is at the core of teachers’ professional activity and professional change. Documentation work includes selecting resources, combining them, using them, and revising them. Even outside a particular reform or professional development program context, such work is deemed central to teaching activity. Gueudet and Trouche employ the term “resource” to describe the variety of artifacts that they consider – such as a textbook, a piece of software, a student’s work sheet, a discussion with a colleague. Like Adler (2000), a key aspect of Gueudet and Trouche’s (2011) approach is resource-in-use (in-class and out-of-class).

One of the pivotal constructs of their theory is that ‘resources’ become transformed into ‘documents’ via a process of documentational genesis – a construct

inspired by and adapted from the parallel process in the instrumental approach whereby artifacts become transformed into instruments via instrumental genesis (Rabardel, 1995). The instrumental approach distinguishes between an artifact, available for a given user, and an instrument, which is developed by the user – starting from this artifact – in the course of his/her situated action. Similarly, a document is developed by a teacher, starting from a resource, in the course of his/her situated action. Gueudet and Trouche represent this process of documentational genesis with the following simplified equation, where the ‘scheme of utilisation’ refers to the various personal adaptations that are made with respect to using the resource in accordance with a teacher’s evolving knowledge and beliefs (Gueudet & Trouche, 2009, p. 209): “*Document = Resources + Scheme of utilization*”. Documentational genesis is therefore considered to be a dialectical process involving both the teacher’s shaping of the resource and her practice being shaped by it.

In their description of this theoretical approach and its accompanying methodological principles, Gueudet and Trouche (2011) emphasize the professional growth that is intertwined with documentational genesis. They argue that:

Teachers “learn” when choosing, transforming resources, implementing them, revising them etc. The documentational approach proposes a specific conceptualisation of this learning, in terms of *genesis*. A documentational genesis induces evolutions of the teacher’s schemes, which means both evolutions of the rules of action (belonging to her practice) and of her operational invariants (belonging to knowledge and beliefs). Documentation being present in all aspects of the teacher’s work, it yields a perspective on teachers’ professional growth as a complex set of documentational geneses. (Gueudet & Trouche, 2011, p. 26)

The study featured in this article, with its central focus on resources-in-use within actual teaching practice, draws upon salient aspects of the documentational approach of didactics. More specifically, our research question centered on uncovering key moments

of teacher awareness, particularly those of a mathematical nature, in the process of using new curricular resources in class. We begin with an *a priori* analysis of the curricular resources being used by a teacher for the first time, followed by detailed in situ observations of the unfolding of her teaching practice involving these resources. The analysis highlights the teacher's growing awareness of the mathematical gaps in the resources she is using – conjectured to be a first step for her in the evolutionary transformation of resource to document, as well as an essential constituent of her ongoing professional development.

Methodological Aspects of the Study

The present study is situated within a multi-phase program of research whose current phase is the study of teaching practice in mathematics classes involving the use of digital technology in the teaching of algebra, in particular, the use of Computer Algebra System (CAS) technology. Previous phases of the research integrated tasks that had been designed by members of the research team (see, e.g., Kieran, Tanguay, & Solares, 2011). This phase examines teaching practice in technology-supported classroom environments where commercially-developed curricular resources, such as textbooks, are in use.

Participants in this phase of the study included three teachers from three different public high schools. They responded positively to our request for volunteers who were using technology in their regular teaching of high school algebra and who would be willing to be observed and interviewed for our research study. We observed and videotaped each teacher's practice for five consecutive days in all of their regular mathematics classes. We intended to capture, as much as would be possible under the videotaping circumstances, their natural teaching practice involving whatever resources

they had chosen to make use of. We also interviewed each teacher twice – once at the beginning of the week and once at the end. The analysis presented in this article focuses on the practice of one of the three teachers, Mae (a pseudonym), during one of her lessons of the week.

Mae taught all three of the senior year (17-year-old students) mathematics classes in her school. She was one of the pioneers of her school on the use of technology in the teaching of mathematics. In her own classes, she regularly used a whiteboard hooked up to her computer and all students had CAS calculators available to them. She was technically very savvy and could respond easily to all students' questions regarding the use of technology. Her academic background included a doctorate in education with a thesis on the use of graphing calculator technology. Her mathematical knowledge seemed, however, less developed than her technological skill. She made a regular practice of asking students to read ahead in the text because – as she mentioned during an interview – they would soon be graduating and had to learn to be autonomous adults who were responsible for their own learning. However, this practice also led students to pose questions of a mathematical nature that went beyond what they had been able to extract from their textbook. Such questions were not, in general, handled with the same expertise and knowledge base with which Mae handled their technological questions.

The analysis of Mae's teaching practice that we present in this article does not focus on her integration of technology into the teaching of mathematics, but rather on the mathematical content at stake in her lesson within the framework of the documentational approach of didactics (Gueudet & Trouche, 2009), a key construct of which is the evolutionary nature of documentational genesis whereby resources gradually become

transformed into documents. The resources that Mae was using during the period in which our classroom observations occurred were new to her that year. The provincial curriculum guidelines had changed the year before and new textbooks were developed that would adhere more closely to the new guidelines. Mae tended to rely on both the student textbook and accompanying teacher guide to plan the mathematical content of her lessons. We were interested in following the process of her integration of these resources into her teaching practice, the way in which she was adapting and transforming them, and the way in which they might be co-transforming her practice and her knowledge – that is, in capturing the reciprocal nature of the documentational genesis that was occurring.

Although the analysis we present in this article is focused on a very small part of Mae's teaching practice, on one lesson in fact, the approach to our analysis is broader in scope. We begin with an analysis of the two text-based resources she used for her lesson on families of polynomial functions, tracing back in these resources to some of the earlier notions that served as foundation for the development of the lesson's mathematical content. Then we analyze the videotape of the unfolding of the classroom lesson. This latter analysis attempts to draw out the dynamics and forces that came into play as the prepared mathematical content was elaborated in the classroom setting, examining in particular those moments that seemed critical to the further development of her teaching practice and to the evolutionary process whereby a resource becomes a document. The videotapes of the interviews with the teacher also serve to illuminate some of the underlying aspects of her teaching practice.

Analysis of the Resources Used by the Teacher in Preparing her Lesson

Herein we present pertinent extracts from the resources used by the teacher, Mae, in preparing her lesson, as well as some of our own mathematical and didactical commentary related to these extracts. The two resources she used were the student textbook *Advanced Functions 12* (Erdman, Lenjosek, Meisel, & Speijer, 2008a) and the accompanying teacher guide *Advanced Functions 12, Teacher's Resource* (Erdman et al., 2008b). The lesson was on Families of Polynomial Functions (Section 2.4 of Erdman et al., 2008a). Our analysis of the resources used by the teacher focuses primarily on the issue of the mathematical links between factors written in the form $(x - b/a)$ versus the form $(ax - b)$ for given families of polynomial functions.

The Student Textbook

Background mathematical material from Section 2.2. In Section 2.2 of the textbook (Erdman et al., 2008a), students are presented with the Factor Theorem (see Figure 1). This section of the textbook provides some of the support for the theoretical affirmations regarding the desired form for factors of a polynomial that are made in the later section on Families of Polynomial Functions with respect to rational roots.

<p>Factor Theorem $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$. Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.</p>
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Figure 1. The Factor Theorem (drawn from p. 95 of Erdman et al., 2008a)

This theorem allows for determining whether a certain binomial is or is not a factor of a given polynomial on the basis of numerical evaluation. The textbook does not prove this theorem; it merely provides the following affirmation, which allows for some misinterpretation: “With the factor theorem, you can determine the factors of a

polynomial without having to divide” (p. 95). No explanation is provided as to why the numerical evaluation $P(b/a)$, when it yields zero, should in fact be sufficient for determining a factor of the polynomial. However, the central issue for our analysis is the following: if $P(b/a) = 0$, why write the factor in the form $(ax - b)$ and not in the form of $(x - b/a)$? It clearly makes for an easier long-division calculation when written in the form of $(ax - b)$. But what happens, mathematically speaking, when one expresses $(x - b/a)$ as $(ax - b)$? Are the two forms equivalent? What mathematics is hidden in expressing the former form as the latter? How does one convert one form to the other and maintain equivalence?

Subsequent pages of the student textbook expand on the Factor Theorem by means of two additional theorems, the Integral Zero Theorem (p. 97) and the Rational Zero Theorem (p. 100), illustrated in Figure 2. However, once again, no further explanation is provided for the case of the polynomial $P(x)$ having a rational zero a/b , either as to why a should be a factor of the leading coefficient of $P(x)$ or the issue regarding the form to be used for the factor of $P(x)$ corresponding to the rational zero.

<p>Integral Zero Theorem</p> <p>If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.</p>	<p>Rational Zero Theorem</p> <p>Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then,</p> <ul style="list-style-type: none"> • b is a factor of the constant term of $P(x)$ • a is a factor of the leading coefficient of $P(x)$ • $ax - b$ is a factor of $P(x)$
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Figure 2. The Integral Zero and Rational Zero Theorems
(drawn from Erdman et al., 2008a, pp. 97 & 100)

The textbook provides several examples that show the advantages of using these two latter theorems when the task is to find the factors of a polynomial. However, the relevance of writing the factor in the form $(ax - b)$ when $x = b/a$ is a root of the

polynomial $P(x)$ is never discussed. This can have repercussions, didactically speaking, at the moment when the teacher introduces the theory underlying families of polynomial functions, coming up in Section 2.4. The intervening section 2.3, on the solving of polynomial equations, adds no further theory related to the Factor Theorem.

Families of Polynomial Functions. Before giving a general definition of families of polynomial functions, the textbook offers several examples that illustrate that one obtains different members of the same family of polynomial functions for different values of the parameter k (see Figure 3 for one such example).

Example 1 Represent a Family of Functions Algebraically

The zeros of a family of quadratic functions are 2 and -3 .

- Determine an equation for this family of functions.
- Write equations for two functions that belong to this family.
- Determine an equation for the member of the family that passes through the point $(1, 4)$.

Solution

- The factor associated with 2 is $x - 2$ and the factor associated with -3 is $x + 3$.
An equation for this family is $y = k(x - 2)(x + 3)$, where $k \in \mathbf{R}$.
- Use any two values for k to write two members of the family.
For $k = 8$, $y = 8(x - 2)(x + 3)$.
For $k = -3$, $y = -3(x - 2)(x + 3)$.
- To find the member whose graph passes through $(1, 4)$, substitute $x = 1$ and $y = 4$ into the equation and solve for k .

$$4 = k(1 - 2)(1 + 3)$$

$$4 = k(-1)(4)$$

$$4 = -4k$$

$$k = -1$$
 The equation is $y = -(x - 2)(x + 3)$.

Figure 3. Algebraic representation of a family of polynomial functions (drawn from Erdman et al., 2008a, p. 115)

As is illustrated in Figure 3, the family of polynomial functions that has as zeros 2 and -3 can be represented algebraically as $y = k(x - 2)(x + 3)$. But, if we look at part (b) of the solution of this example, the information that is given suggests that different values of k yield different members of the same family of polynomial functions. This can lead those who are using this textbook as a resource to a false mathematical conception if they

do not distinguish the crucial role being played by the root of the polynomial in terms of whether it is a whole number or a rational. In other words, if the zeros of the polynomial are not whole numbers, but rather rational numbers, then the value of k can vary according to the form of the factor, without changing the member of the polynomial family. For example, if the zeros of a family of polynomial functions are 3 and $-1/2$, then the family has as its function $P(x) = k(x - 3)(x + 1/2)$. And so, a member of this family is: $P(x) = 2(x - 3)(x + 1/2)$, if $k = 2$. But if we write the factor $(x + 1/2)$ as $(2x + 1)$, the value of k changes from 2 to 1 for the same member of the polynomial family, that is, $P(x) = 1(x - 3)(2x + 1)$. The algebraic transformation involved in changing the form of the factor $(x + 1/2)$ to $(2x + 1)$ is as follows: $(x + 1/2) = 2/2 (x + 1/2) = 1/2(2x + 1)$. Thus, the conversion of $(x + 1/2)$ to $(2x + 1)$ involves also multiplying the rest of the expression by $1/2$, thereby yielding the new k -value of 1 (from multiplying the previous k -value of 2 by $1/2$). This example shows that, if we have a family of polynomial functions expressed algebraically as $P(x) = k(x - a_1)(x - a_2)...(x - a_n)$, $k \in \mathfrak{R}$, $k \neq 0$, we cannot say that different values of the parameter k necessarily imply different members of a given family of polynomial functions, unless of course all the zeros are whole numbers.

The examples provided in the textbook are then followed by the general definition of families of polynomial functions (see Figure 4).

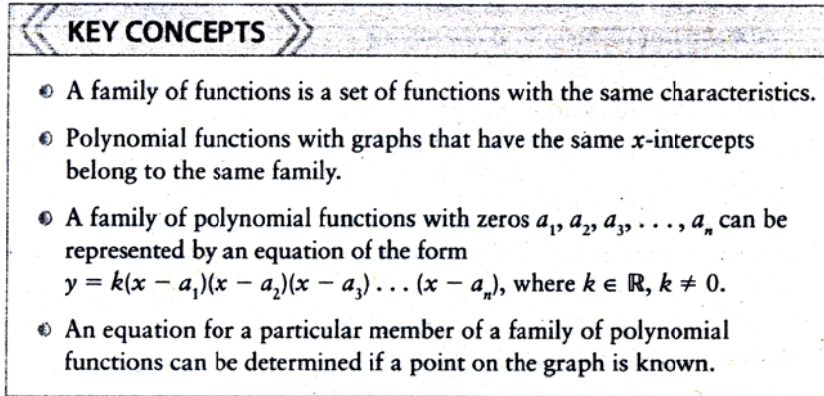


Figure 4. Definition of families of polynomial functions (drawn from Erdman et al., 2008a, p. 118)

With this general definition, where there are no conditions on the zeros of a polynomial function, the earlier suggestion to write the factor as $(ax - b)$ when the zero is rational receives no consistent theoretical support. In fact, the general definition would seem to suggest that the factored form for a given zero a , be it a rational or whole number, is of the form $(x - a)$.

The Teacher Guide

We now examine the nature of the support offered in the teacher guide with respect to preparing lessons on families of polynomial functions (Erdman et al., 2008b). This resource presents only a few teaching suggestions, most of which could be considered, at the very least, quite incomplete from a didactical point of view (see Figure 5).

- In **Examples 1 and 2**, point out that the equation for the family of functions must include the constant k to represent the leading coefficient. Encourage students to use fractions and not decimals where appropriate. Apply Chapter 1 skills to graph the polynomials, as shown in **Example 2 part d**).

Figure 5. A typical suggestion found in the teacher guide from the section dealing with families of polynomial functions (drawn from Erdman et al., 2008b, p. 52)

In fact, the guidance noted in Figure 5 where students are to be encouraged to use fractions and not decimals is contradicted in another suggestion within the same resource a few lines later (see Figure 6).

- The factors corresponding to the rational zeros in questions 10 and 11 should be expressed in the form $ax - b$. All equations should be expanded and simplified.

Figure 6. An explicit suggestion in the teacher guide (Erdman et al., 2008b, p. 52) to write all factors involving rational zeros in the form $(ax - b)$

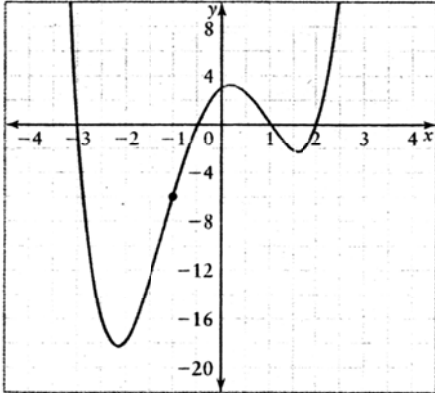
The advice displayed in Figure 6 is not accompanied by any justification for the use of the form $(ax - b)$, nor is there any discussion as to how a teacher might respond to potential students' questions regarding the issue as to why they are to use the form $(ax - b)$ and not $(x - b/a)$. In fact, the teacher is not even alerted to the possibility that such a question might arise. Additionally, no explanation is provided as to why "all equations should be expanded and simplified." Question 10, to which the suggestion given in Figure 6 refers, reads as follows: *Determine an equation for the family of quartic functions with zeros $-5/2$, -1 , $7/2$, and 3 .* In accordance with the directive given in Figure 6, the equation for the given family of quartic functions ought to be written as $P(x) = k(2x + 5)(x + 1)(2x - 7)(x - 3)$. But an obvious question is why one might not instead write the function in the following form: $P(x) = k(x + 5/2)(x + 1)(x - 7/2)(x - 3)$.

Analysis of the Unfolding of Mae's Lesson on Families of Polynomial Functions

The mathematical problem on which Mae had decided to focus in her lesson on families of polynomial functions was one that involved a rational root. It was a variation of an example that was worked out in the student textbook (see Figure 7).

Example 4 Determine an Equation for a Quartic Function From a Graph

Determine an equation for the quartic function represented by this graph.



Solution

From the graph, the x -intercepts are -3 , $-\frac{1}{2}$, 1 , and 2 .

The corresponding factors are $x + 3$, $2x + 1$, $x - 1$, and $x - 2$.

An equation for the family of polynomial functions with these zeros is

$$y = k(x + 3)(2x + 1)(x - 1)(x - 2).$$

Select a point that the graph passes through, such as $(-1, -6)$.

Substitute $x = -1$ and $y = -6$ into the equation to solve for k .

$$\begin{aligned} -6 &= k[(-1) + 3][2(-1) + 1][(-1) - 1][(-1) - 2] \\ -6 &= k(2)(-1)(-2)(-3) \\ -6 &= -12k \\ k &= 0.5 \end{aligned}$$

The equation is $y = 0.5(x + 3)(2x + 1)(x - 1)(x - 2)$.

Figure 7. The mathematical problem that was used as a basis for Mae's lesson on families of polynomial functions (drawn from Erdman et al., 2008a, pp. 117-118)

Mae had taught the same lesson in two other Grade 12 classes earlier the same day. We note that during each class, including the third, she had emphasized that all members of a family of polynomial functions share the same x -intercepts. The only features that differed for each member of a given family, she said, were the value of the leading coefficient k and the accompanying stretching or compressing of the graph. She had also drawn students' attention to the fact that there would be only one correct value of the parameter k for any given member of a family of polynomial functions. We now

recount the unfolding of the lesson as it occurred in the third class of the day. The reader will notice how closely Mae follows the textbook presentation of the problem she chose to use in her mathematics lesson.

The lesson on Families of Polynomial Functions began with Mae presenting the following three functions – all of them being members of one family of polynomial functions:

$$f_1(x) = 2x^3 - 4x^2 - 10x + 12$$

$$f_2(x) = -x^3 + 2x^2 + 5x - 6$$

$$f_3(x) = -2.5x^3 + 5x^2 + 12.5x - 15.$$

Mae displayed the definitions of the functions on the whiteboard at the back of the room [see Figure 8] and asked students to copy them down and then to graph them on their CAS calculators: “Open up a graphs page on your calculator. Given f_1 , f_2 , and f_3 , what do you notice about all three functions?”

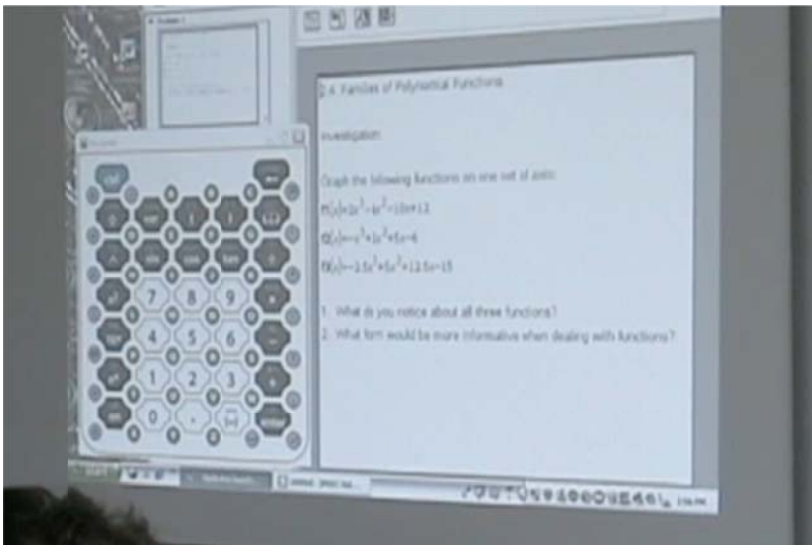


Figure 8. The opening of the lesson, with its accompanying CAS and whiteboard technology

After students had spent some time trying to find appropriate graphing windows, Mae asked them what common characteristics the functions shared. One student mentioned that they were all of degree three and another that they had the same x -intercepts. Following up on the latter idea, Mae asked if they were able to tell from looking at the given expanded forms that the three functions had the same x -intercepts. “So how could you make it more obvious?”, she asked. When one student suggested “factoring them”, Mae responded: “Yes, when you factor them, you have your function in a form where you can easily see that the x -intercepts are similar.” She then asked students to split their graphs page in two so that they could insert a calculator page for the factoring of the three functions. It is noted that a certain amount of time was devoted to taking care of technical aspects of the CAS, such as splitting a page in two and then copying the three functions to that page.

The factored form of the three functions was as follows:

$$f_1(x) = 2(x + 2)(x - 1)(x - 3)$$

$$f_2(x) = -(x + 2)(x - 1)(x - 3)$$

$$f_3(x) = -2.5(x + 2)(x - 1)(x - 3)$$

Mae then continued with her lesson, as illustrated by the following extract of classroom dialogue. It was soon to lead to the problem associated with a factor that corresponds to a given rational zero.

Teacher: So, in factored form, right away you can see that they all share -2 , 1 , and 3 as x -intercepts. So, if you are looking at all of these three graphs and they all share the x -intercepts, why do they look so different on your graphs page?

Student1: Coefficients and translations.

Student2: Leading coefficients.

Teacher: So you can express it in different ways: leading coefficients, stretches, compressions. OK, so if you look at the leading coefficients, there's a two in one of them, negative one in the other one, and negative two point 5. Alright.

So, this section (2.4) is titled, Families of Polynomial Functions. And by definition if you have polynomial functions, all with the same x -intercepts, they're within the same family. Is everyone OK with that?

So another way I can ask you questions would be something like this. So here [referring to the whiteboard where the general form for families of polynomial functions was displayed: $f(x) = k(x - a_1)(x - a_2)(x - a_3)\dots(x - a_n)$, where $k \in \mathfrak{R}$, $k \neq 0$] is the basic definition of the functions you were dealing with before, where if you have all the zeros, all the x -intercepts being the same, and the only thing that differs is your value – and here they label it k – in front, basically you can say that this family of polynomials, they share the same characteristics, they're in the same family.

Then I can ask you something like question #3 [which was then displayed on the whiteboard]:

A function has x -intercepts -3 , $-(1/2)$, 1 , and 2 , with point $(-1, -6)$ on the function. Determine the equation of the polynomial function.

What #3 is asking you to do, you're given specific x -intercepts, they want you to find the equation of the polynomial function. But along with the four x -intercepts, they also give you a point. What do you think the point is going to help you determine?

Student1: the k .

Teacher: Right, the k . Thank you very much. So try to give me the equation of the polynomial function. And remember there are two ways to present the equation of a

polynomial function, or two forms. It's up to you which form you want to present. But obviously in factored form, you can get the k easily.

After working on the problem, various students stated the values that they had obtained for k , not all of them arriving at the same value. So, Mae asked a few students with different answers to go to the board to show their work, but first insisted that they all use the basic form, which she wrote at the board as follows:

$f(x) = k(x - a_1)(x - a_2)(x - a_3)(x - a_4)$. She also asked that everyone show the factors they were using and how they were substituting-in the coordinates of the given point.

One student began writing at the board the following equation (see Figure 9):

$f(x) = k(x + 3)(2x + 1)(x - 1)(x - 2)$, clearly using the factored form $(ax - b)$ to represent the rational zero $-1/2$.

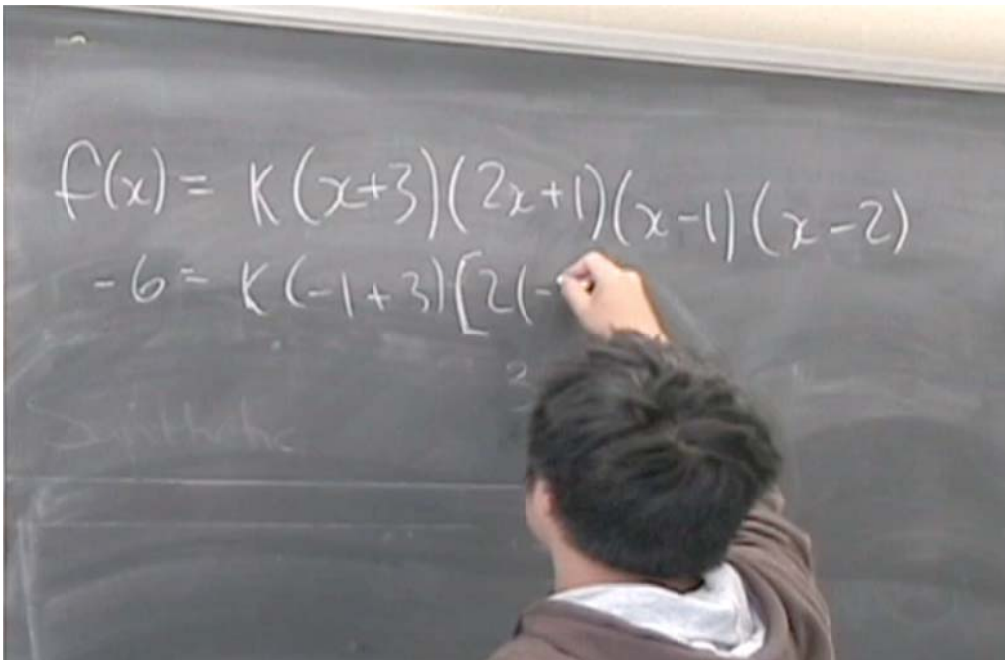


Figure 9. At the blackboard, one student writing his version of the requested equation

The other students who were working at the board used a similar form for the second factor. This was clearly a reflection of the work they had done earlier in the week on the Factor Theorem. Despite the fact that Mae had just a few minutes earlier mentioned that they all should employ the general form, whose factors were of the form, $x - a$, she did not remark on the students' use of the form $ax - b$. It conformed, after all, to the form suggested in the teacher's guide. The student, after substituting-in the coordinates of the point for the x 's and $f(x)$, obtained the result of $1/2$ for k . So too did all of the others who were showing their work at the board. The various erroneous values that they had earlier obtained for k were self-corrected.

Teacher: Well, so, we all got a half. So you all determined your polynomial function equation all in the same way. Did anyone happen to write their function differently?

Student1: Well, you could expand your function first and then plug it in.

Teacher: Actually, that's correct. So, it actually turns out to be the same thing. But did anyone write this part differently [pointing to the four factors of the expression]? [No one said anything]. So, everyone was able to write their factors as either x plus or minus b , or ax plus or minus b . Is everyone OK with that?

Student3: Why can't you use $(x+0.5)$ for the x -intercept of $-1/2$? Like for $(2x+1)$.

The teacher seemed unsure as to what Student3 was proposing. So, she asked him to come forward to write it at the board, which he did: [he wrote $x + 0.5$].

Teacher: Ooh! Very good question. So. Let's all try this. Instead of using $(2x + 1)$, use $(x + 1/2)$. Tell me what happens when you use $(x + 1/2)$ instead of $(2x + 1)$.

Student 4: You get 1.

Teacher: OK, you get 1. So you get something completely different. Right. So why do you get something completely different?

Student 5: Divide that part by 2 and then write in the rest of it [clearly referring to the $2x + 1$, but his technique was not clearly and completely stated].

Teacher: OK, good [without expanding on the student's partial suggestion], so your entire expression is actually completely different.

Here in lies the crux of the mathematical difficulty. The teacher appears to see the function with its different value of k as another member of the family of polynomial functions, and not as the same member: that is, that $f(x) = 1/2(x + 3)(2x + 1)(x - 1)(x - 2)$ and $f(x) = 1(x + 3)(x + 1/2)(x - 1)(x - 2)$ are two different members of the same family. We reiterate that neither of the resources she was using had led her to think otherwise. She attempted to explain this phenomenon to the class in the following manner, focusing on the fact that the zeros were the same, but the k 's were different:

Teacher: So your x -value here [in $2x + 1$] is -1 , so when you go 2 times -1 plus 1, you get -1 . But when you put -1 in here [in $x + 1/2$] plus $1/2$, you get $-1/2$. Right, so you get two totally different values, so your k will be different.

Student 1: Isn't that also right though?

Teacher: Is this [pointing to $2x + 1$] a different intercept from this [pointing to $x+1/2$]? We have $2x + 1$ and $x+1/2$ [she writes on the board $2x+1=0$ and $x+1/2=0$]. So, what does x equal in the two cases? So, they're the same answer, right [i.e., the same zero or x -intercept]. But we're getting different values [for each] because, in $2x+1=0$, you double something and then you add, and in this [$x+1/2$] you just add something. So, according to the order of operations, you get different values of k here. Right.

Student6: So how do you know it's not $(4x+2)$, because the x -intercept is still $-1/2$?

Teacher: That's very good, but you actually don't know that. You don't know if that would be $(4x+2)$. Although again what you're trying to do is figure out what kind of leading coefficient you have there. OK.

Mae's 'explanation' of the phenomenon at hand showed her to be oblivious at that moment to any consideration that the two algebraic forms might be equivalent. If she had realized that the factoring of $(2x+1)$ as $2(x+1/2)$, followed by the multiplication of the 2 with the k -value of $1/2$, would yield an equivalent second form of the given expression, the problem might have been resolved. Furthermore, Student6's question regarding the possibility of using $(4x+2)$ for the $(2x+1)$ factor (or any of an infinite number of other possibilities for the factor representing the x -intercept of $-1/2$) might have been discussed in terms of there being no difference whether one uses one form of the factor or another, because the resulting different value of k would maintain the equivalence. The following are all equivalent: $1(x+3)(x+1/2)(x-1)(x-2)$;

$1/2(x+3)(2x+1)(x-1)(x-2)$; $1/4(x+3)(4x+2)(x-1)(x-2)$; and so on. They are all the same member of a certain family of polynomial functions, despite their having different k 's. Mae's distinction between different members of the same family, based on the criterion of having different k 's, had failed to take into account the role played by different possible forms of a factor that represent the same x -intercept, or zero, when it is a rational number. The textbook resources she had just begun to use had not alerted her to this phenomenon.

As if to prove her point about the two functions being distinct members of the same family, Mae then suggested to the class that they expand the two – but was somewhat taken aback by the result. When the expanded results came out to be the same, the teacher wondered aloud if she had not mistakenly entered the same expression twice into her computer, which was hooked up to the whiteboard. The following classroom discussion ensued.

Student1: Even though the k is different, it is still the same thing. Whatever you do to the factor, you are also doing to k [not quite correct, but on the right track]

Teacher: I am not sure that you are all following this. For the second one, we got a different value of k . And what do you find when you do it [that is, expand the expression: $1(x+3)(x+1/2)(x-1)(x-2)$]?

Several students at once: The same thing!

Student1: Witchcraft!

Teacher: [recovering somewhat from her surprise, but still at a loss for words] Does it make a difference? [Looking around the class] Do you understand why that, even

though, because of how you are phrasing the question, or your factors, you are going to get your different values of k . Remember some people were saying that when you expand it, you should still get the same thing anyway [what had actually been suggested earlier by one of the students was related to expanding just one expression that was in factored form and not expanding two seemingly different expressions]. Well, when you expand it [the two seemingly different factored forms], you can see that the functions are still the same. Generally, we do use the $ax \pm b$ form, but obviously you can see that we are dealing with the same function. Right. So thank you very much for your question, Student3.

At this moment, the teacher quickly brought her lesson on families of polynomial functions to an end. The mathematical issues that had arisen clearly required further reflection on her part.

Discussion

The issues we wish to discuss here are threefold: the mathematical gaps of textbook resources, the process of becoming aware of and overcoming these gaps which constitutes a form of ongoing professional development for a teacher, and the evolutionary nature of documentational genesis whereby resources gradually become transformed into documents.

The new textbook and teacher guide that Mae had used as resources for her lesson had not provided the level of mathematical support that she needed. They had not alerted her to the issues surrounding the two forms of a factor representing a given rational zero of a function, and the accompanying impact on the value of the parameter k . The

resources had been silent about both the technique for converting from one factored form to another and the equivalent nature of the two. Chevallard and Cirade (2010) have discussed the question of missing mathematical resources and have identified this as a major praxeological problem for the profession.

It was in the act of teaching her three classes on a given day that Mae became aware of the mathematical deficiencies of the textual resources with which she had prepared her lesson on families of polynomial functions. She had not been equipped to handle the questions put to her by her students and had to react on the fly in an ad hoc and inadequate fashion. Nevertheless, she seemed to learn from the experience. Zaslavsky and Leikin (2004) have pointed out that, by listening to students and observing their work, and by reflecting on this work, teachers learn through their teaching. Mason (1998) has emphasized that it is one's developing awareness in actual teaching practice that constitutes change in one's knowledge of mathematics and mathematics teaching and learning.

By taking seriously her students' questions regarding the relationship between two seemingly different factored forms of a function, Mae became sensitized to mathematical aspects of the given area of study that she had not heretofore considered. Her knowledge of families of polynomial functions was in the process of being transformed by what transpired in her class, especially by the thought-provoking queries of her students. According to Zaslavsky and Leikin (2004), such in-practice activity can be an effective vehicle for teachers' own professional growth. Although Mae's primary preoccupation was the teaching of the material on families of functions, she was at the same time engaging in the problem that she was putting to the students. She, with the

collaboration of her students, was developing her knowledge of the mathematics of this area.

In their theoretical paper on documentation systems for mathematics teachers, Gueudet and Trouche (2009) introduce a general perspective for the study of teachers' professional evolution, where the researcher's attention is focused on the resources, their *appropriation* and *transformation* by the teacher or by a group of teachers working together. Their approach aligns with Adler's (2000), who claims that, "in mathematics teacher education, resources in practice need to become a focus of attention" (p. 221) and with Remillard's (2005) whereby the evolution of the curriculum material actually used and a teacher's professional development are viewed as two intertwining processes.

With respect to this intertwining process, Gueudet and Trouche (2009) point out that:

A teacher draws on resource sets for her documentation work. A process of genesis takes place, producing what we call *a document*. ... A given teacher gathers resources: textbooks, her own course, a previously given sheet of exercises... She chooses among these resources to constitute a list of exercises, which is given to a class. It can then be modified, according to what happens with the students, before using it with another class during the same year, or the next year, or even later. The document develops throughout this variety of contexts. (p. 205)

We suggest that the awarenesses acquired by Mae in her teaching of families of polynomial functions with new resources will be instrumental in enabling her to modify these resources, thereby leading to the gradual transformation of a resource into a document for her. However, Gueudet and Trouche (2009) emphasize that "documentational genesis must not be considered as a transformation with a set of resources as input, and a document as output. It is an ongoing process ... that *continues in usage*. We consider here accordingly that a document developed from a set of resources

provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of the *dialectical* relationship between resources and documents” (p. 206).

We close our discussion by turning to a relevant comment made by Adler (2000) that puts the focus not on producing more (or better) resources, but rather on better understanding how teachers use the resources they have, change them, and in the process engage in a form of ongoing, personal, professional development: “Our attention shifts away from unproblematised calls for more [resources] and onto the inter-relationship between teacher and resources and how, in diverse, complex contexts and practices, mathematics teachers use the resources they have, how this changes over time, and how and with what consequences new resources are integrated into school mathematics practice” (p. 221). In this article, we have attempted to illustrate the complex interrelationship within actual teaching practice between a teacher and a new set of resources, by describing the nature of the classroom experiences whereby a teacher becomes aware of the mathematical gaps of new resources and thus better positioned to make changes to them over time. Such an approach both situates resources and their adaptive use within a documentational framework and re-centers professional development within the actual practice of teaching.

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Mathematical Problem Solving in Training Elementary Teachers from a Semiotic Logical Approach

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Abstract: The aim of this article is to consider the professional knowledge and competences of mathematics teachers in compulsory education, and to propose basic tasks and activities in an initial training programme in the framework of a global proposal for “Immersion” in the curriculum of the educational phase which the trainee teacher would go on to work in. Problem-solving, in this context, is considered as being an inherent part of mathematics and this is described in terms of problem-solving, establishing connections between concepts, operations and implicit processes in the mathematical activity (conceptual field) and their relationships problem-solving; and it is assumed that the learning of problem-solving is an integrated part of learning in mathematics.

Keywords: Problem Solving, Teacher Training, Didactical Analysis, Semiotic Logical Approach (SLA).

Introduction

The analysis of the results obtained, in recent years, in different national (in Spain) and international assessments shows that the knowledge of mathematics (Problem Solving) of students in compulsory education (K-9 Grades) is insufficient in terms of the desired curriculum. What needs to be done to improve the learning and teaching of mathematics and, in particular problem solving in this educational stage? This question is addressed here by reflecting on the role played by teachers in primary and secondary education in the pursuit of an effective learning of mathematics and problem solving. At present, the initial training of teachers in primary and secondary education takes place

within the European Higher Education Area, where primary school teachers need four years training and secondary school teachers are required to have completed a mandatory Professional Masters degree. This initial teacher training has a great opportunity for improvement.

Problem solving in mathematics education

Problem solving has always been regarded as a basic component in the construction of mathematical culture. However, when mathematical knowledge is presented in its final state, what prevails is the conceptual organization of the objects of such knowledge in which problem solving appears again as a core of relevant mathematical knowledge. In the early eighties, in view of the primacy of the concepts and their properties as well as their algorithmic use, problem solving was vindicated as a key activity in the learning of mathematics, which has led to the development of an emerging theoretical and practical body of research in mathematical education, and a notable increase of its presence in the curriculum, either as a further block of contents or as cross content but specific to mathematics at the corresponding level (Santos-Trigo, 2007, Castro, 2008). The follow-up research on problem solving clearly shows that, despite all amount of effort, there are no significant data on the improvement in this on the part of the students and different questions arise ranging from the need to establish relationships and existing connections between the development of the understanding of mathematical contents and problem solving skills, to the need of having theoretical bases to guide problem solving (Lester and Kehle, 2003).

Some authors such as Lesh and Zawojewski (2007), suggest that the rise of research in problem solving was very important between 1980 and 1990, and that some trends are presently aimed at putting an emphasis on critical thinking, technology and mathematical problem solving, and analysis of how mathematics is used in other sciences and professions that does not match the way mathematics is taught in school, or the development of problem solving in other settings or contexts such as situated cognition, communities of practice or representational fluency. These directions and perspectives in solving mathematical problems are, at the present, promising lines of research.

The knowledge and professional skills of a mathematics teacher

The concern, from the point of view of mathematics education, regarding teacher's knowledge and professional skills has been and is, a constant research topic, and is based on the following conjecture: The knowledge and professional skills of the mathematics teacher must be acquired through different scientific domains: mathematics, mathematical didactics and educational sciences. The initial teacher training should enable the trainee teachers to increase their knowledge about mathematics and mathematical didactics as a specific field of professional competence (mathematics education) and a field of research, along with other issues arising from educational sciences.

Shulman pointed out in 1986, for the first time, the importance of the specific subject to teach in teacher training. This author identified three categories of professional knowledge of teachers: *Knowledge of the specific subject*, *pedagogical content knowledge* (PCK) or in the context here: the *didactical content knowledge* (DCK) and *curricular knowledge*. Subsequently, Bromme (1988, 1994) described the qualitative

characteristics of the major areas of professional knowledge: Knowledge of mathematics as a discipline, knowledge of mathematics as school subject, philosophy of mathematics schools, pedagogical knowledge and specific pedagogical knowledge of mathematics. The author proposed that the teachers' professional knowledge is not simply a conglomeration of these domains of knowledge, "but an integration of the same", which occurs during teaching practice or during professional teaching experience.

Semiotic Logical Approach (SLA). Assumptions

Semiotic Logical Approach (SLA) (Socas 2001a and b, 2007), when understood as a theoretical-practical proposal (formal-experimental), aims to provide tools for the analysis, description and management of problematic situations or phenomena of a mathematical didactical nature from a perspective based on semiotics, logic and competence models (semiosis), and takes one of the great problems of mathematics education, the study of difficulties and errors of students in learning mathematics as a reference (Freudenthal, 1981). Logical and Semiotic Aspects of SLA uses Peirce's Phenomenology (1987) as a reference. Peirce, starting from the logic conceived of as a science of language, describes the development of a science of signs and meanings called semiotics which can be used to analyse, within the semiotic constructs, different phenomena of logic, mathematics, physics and even psychology, which is why this phenomenology is used here. Semiotics is a theory of reality and knowledge that one can have of phenomena through signs which are the only means available. Semiotic inference emerges in sign analysis where what is analysed are the trademarks or observable and overt expressions of inference, which Pierce organized as a logical theory (semiotics) that has three references closely linked to one another. Therefore, if the aim is to study any

phenomenon (problem situation), which is the starting point in SLA, this will always be analysed from a given context and by means of three references organized as first, second and third, which is defined as primary or basic semiotic function which is determined by the sign, object and meaning references (Socas, 2001b). This can be used to determine the notion of representation as the semiosis determining such references. Therefore, the representation is a sign that:

- (1) has certain characteristics that are proper (context)
- (2) sets a dyadic relationship with the meaning
- (3) establishes a triadic relationship with the meaning via the object, this triadic relationship being such that it determines the sign of a dyadic relationship with the object and the object to a dyadic relationship with the meaning (Hernández, Noda, Palarea and Socas, 2004).

As far as the Educational System is concerned, SLA uses the Begle's diagram of school mathematics as a reference, which shows the mutual relationships between the different components in the training process and defends the need to set multiple perspectives and procedures in the field of the teaching / learning of school mathematics (cited in Romberg, 1992). To do this, two different parts must be distinguished: the "educational macro system", where both disciplinary knowledge and the institutions or persons involved intervene in the education system, and, the "educational micro system", which is made up of three references or basic elements: mathematical knowledge (mathematics), students and teachers, and their relationships in a context determined by the following components: social, cultural and institutional, which is shown in the figure below:

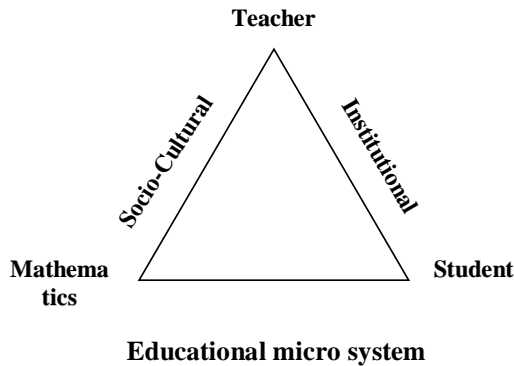


Figure 1: Elements of educational micro system

The three essential relationships are:

Relationship1: Between the mathematical knowledge and the student, which is called "school mathematics learning as a conceptual change".

Relationship 2: Between the mathematical knowledge and the teacher, called: "adapting the curricular mathematics content to be taught".

Relationship 3: Between the mathematical knowledge and the teacher via the student which is called: "interactions".

Thus, the three elements and the three essential relationships contextualized in the three components of the context determine the teaching / learning process in the regulated systems, thereby characterizing the six core contents that are a part of the mathematics teacher's professional knowledge, in addition to those derived from the three previous mentioned relationships: mathematical knowledge in a disciplinary sense, the curricular mathematics knowledge and the mathematics curriculum of an educational stage. It is in this framework that the difficulties, obstacles and errors that students have or make in the construction of mathematical knowledge are examined. SLA organizes three models of competence: Formal Mathematical Competence (FMC), Cognitive Competence (CC) and Teaching Competence (TC), which constitute the references that define the General

Semiosis which plans and manages research in the educational micro system (Socas, 2010a, 2001b, 2007 and 2010).

This General Semiosis can be used to plan and manage both the problems of teaching and learning in the educational micro system, and the didactical mathematical problems to be studied.

The Formal Mathematics Competence Model (FMC) can be used to describe the conceptual field of the mathematical object in the thematic level in which both their functions and phenomenology are being considered.

The Cognitive Competence Model (CC) is the second reference and takes into account the above mentioned Formal Mathematic Competence Model, it refers to the specific cognitive functions of students when they use the mathematical objects in question and structural aspects of learning.

The Teaching Competence Model (TC) is the third reference, and it also considers the above mentioned aspects (formal mathematical competence and cognitive competence) and describes the actions of the subjects involved, the communication processes, the mediators, the situations, the contexts, which occur in education.

Three basic assumptions of SLA are now proposed here: Mathematical Content Analysis, Didactical Analysis and the Curricular Organization.

The didactical analysis and the curricular organization are the concepts that SLA uses to characterize the knowledge of mathematical content from the professional point of view. The didactical analysis allows the comprehension of the professional problem, while the organization curricular plans his development.

Mathematical Content Analysis

The Formal Mathematics Competence Model (FMC) sets out the conception of mathematical literacy and the different relationships between the elements characterizing it. The FMC is organized by means of the semiosis that characterizes and relates the conceptual, phenomenological and functional aspects of mathematical content involved in the problematic situation to be addressed in the educational micro system, and would appear as below:

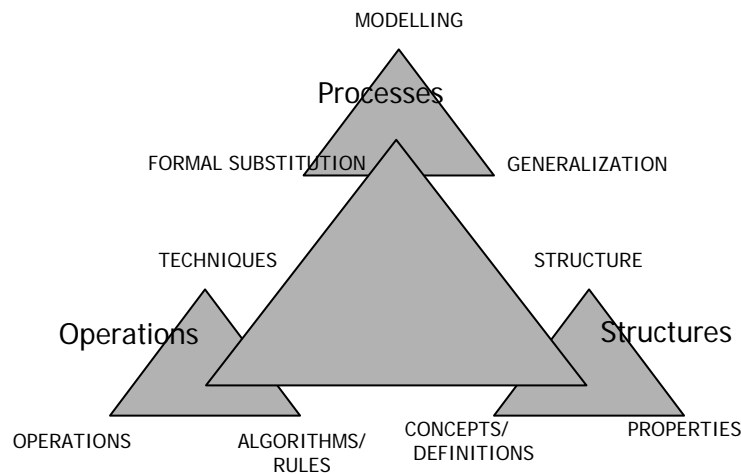


Figure 2: Domains of mathematical activity

The different domains of mathematical activity are expressed within this model in relation to the conceptual field from a formal perspective and its different relationships, i.e., it describes the duality of mathematical objects in relation to conceptual/procedural mathematical knowledge of the field in question. Any activity is described in relation to the three components: operations, structures and processes, and relationships, which we explain later. Each component, in turn, is determined by three others that describe a new semiosis: 1) The operations component for the semiosis: operations, algorithms (rules) and techniques, 2) the structures component for: concepts (definitions), properties and

structure, and 3) the processes component for: formal substitutions, generalization and modelling. This organization of the conceptual fields is contextualized in the problematic situations that are addressed in the language (representations) and in the arguments (reasoning) that are used in developing it.

The three context components are similarly determined by the respective semiosis. In the case of problematic situations: identification, approach and resolution; in representations (language): recognition, transformation (conversion) and elaboration (production), and in arguments: description, justification and reasoning. This organization of mathematics by the FMC can be used to consider problem solving as an inherent part of mathematics and to describe it in terms of problem solving. Hence, the following aspects characterize mathematical culture in SLA:

1. Mathematics is a multifaceted discipline
2. Mathematical culture emerges and develops as a human activity of problem solving
3. The problems have one common feature: the search for regularities (identification, approach and resolution). Modelling is the mathematical process par excellence
4. Mathematical culture creates a system of signs able to express regular behaviour
5. The set of regularities is organized into conceptual fields
6. The conceptual elements of these fields are mathematical objects

The Formal Mathematical Competence Model can also be used to establish the connections between concepts, operations and processes involved in mathematical

activity and their relationship to problem solving, which is, generally speaking, relevant for mathematics education, and particularly for problem solving.

Didactical Analysis in the Semiotic Logical Approach (SLA)

Semiosis can be used to identify and understand the didactical mathematical problem, whose reference framework is comprised of the curriculum organizers (Rico, 1997) and the initial notion that Freudenthal (1983) put forward for didactic analysis as follows: "the analysis of the curricular content of mathematics is performed to serve the organization of its teaching in educational systems".

Didactic analysis is organized according to the following triad: formally described curricular mathematics, semiotic representations and difficulties, obstacles and errors, it also facilitates the identification and understanding of the didactic problem to be addressed.

Didactic analysis implies, in relation to the curricular component, a review of the curricular contents from the formal perspective: operational, structural and processual (using processes), but also implies a necessary relationship with the students linked to their interests and motivation.

The semiotic representations component involves a review of the curricular content in relation to different forms of representation of the objects in question, as well as the presentation of information to students. The following states of the historical development of the mathematical object are considered in this section: semiotic, structural and autonomous, that also implies a necessary relationship with the students linked to the coordination between the forms of expression and representation and the interests and motivation of the students. The component difficulties, obstacles and errors,

require a review of the curricular content in relation to these three aspects, with a dual aim of prevention and remedy, making it possible, for example, from the perspective of prevention, to set the levels or cognitive skills required of students in relation to the mathematical object in question. The identification of the errors generated by the students needs analytical tools which can get into the complexity of learning difficulties in mathematics. One way to address this would be, as reported by Socas (1997), to take the three directions of analysis into consideration, like three coordinated axes which would more accurately identify the origins of the error and would enable the teachers to devise more effective procedures and remedies. These three axes would be determined by their origin: i) in an obstacle, ii) in the absence of meaning; iii) in affective and emotional attitudes.

Curricular Organization

Not only do mathematics teachers need knowledge about the discipline of mathematics and the curriculum, but they also require didactical mathematical knowledge (DMK) in order to organize the mathematical content for teaching.

This is professional knowledge that includes the appropriate elements of analysis to understand, plan and do a professional job. The teacher needs to expand and connect different perspectives on the curricular mathematics content, in such a way that its consideration is not only from the internal logic of the discipline, which may emerge as being too restrictive, formal and technical, but from the curricular dimension, a more open perspective and one which integrates the teaching of mathematical knowledge more, and this is not possible to put into practice from only the theoretical consideration of knowledge about the discipline of mathematics and the curriculum, to convert this into

the mathematical knowledge to be taught. This professional knowledge develops in the subject Didactic of the Mathematics for teachers, structured according to the didactical analysis and the curricular organization.

The curricular organization emerges from the organizers of the curriculum (Rico, 1997), and must be understood as those teaching skills that can be used to plan mathematical content for teaching, i.e., planning and evaluating mathematics classroom schedules, which is determined by the following triad: context, teaching/learning and assessment.

In so far as the context reference is determined by the semiosis described by the problem situation, which refers to the environments in which the activities take place, the contextualization, which is determined by the specific goals, specific skills and teaching content involved in the activity, and the levels, referring to the complexity of mathematical tasks: reproduction, connection and reflection, skills demanded by the same, taken from the PISA Project (Rico and Lupiáñez, 2008), or to stages of development: semiotics, structural and autonomous, taken from SLA (Socas, 1997).

Proposal for training mathematics teachers

The different areas of knowledge (mathematical and didactical mathematical) that can be used to support the training proposal have been described here in general terms. But before going on, it is necessary to make a few comments about the trainee teachers who this training proposal is aimed at. Several studies conducted at the University of La Laguna (Spain), in which students from several other Spanish universities have also participated, show that the students who start teacher training courses in primary

education teaching have huge gaps in basic mathematical knowledge. As regards problem solving, the situation is that 18 year old students, with more than 12 years of studying mathematics in the educational system and learning to solve problems, still tend to concentrate on the data of the problem as their general cognitive strategy, without demonstrating a clear understanding of the problem and without identifying operational, structural (conceptual) or processual relationships, given in the data, often providing solutions that cannot be valid for the conditions of the problem, which furthermore clearly shows a lack of cognitive strategies (heuristic methods) and a lack of critical thinking (Palarea, Hernández and Socas, 2001; Hernández, Noda, Palarea and Socas, 2002 and 2003). Subsequent studies show no improvement on the previous results, finding that students show a predominance of operational rather than structural and processual thinking, and it is this thinking that is mostly behind the solution to any mathematical task, which many times is unsuccessful, even when the applied operational knowledge is correct. This suggests that the emphasis that the teaching of mathematics puts on operational knowledge may be creating difficulties and obstacles for the student to apply, for example, heuristics and strategies to solving problems that are more associated with structural and even processual thinking, which creates difficulties in achieving mathematical competence (Socas et al., 2009).

As regards trainee teachers of mathematics in secondary education, the starting assumptions were that the design of the plan should take two essential aspects into account, on the one hand the mathematical training of future teachers (graduates in mathematics) and, on the other hand, the lack of a specific didactic training for professional work (teacher), except for that formed by existing knowledge, implicit

theories, values and beliefs that had come from their experiences as students of mathematics during their schooling, and which are, in many cases, an obstacle to properly channelling many aspects of professional thinking. The analysis of educational reforms leads one to believe that such reforms require the teacher to be able to take on the new curriculum changes which actually means confronting new tasks. The latter necessarily implies significant changes in training mathematics teachers which can be summarized in the following points:

- Scientific and educational training tailored to this new curricular change.
- Training to work with students who have a high degree of heterogeneity in basic skills, interests and needs.
- A change in attitudes among teachers so that they can develop the educational aspects of teaching, adopt flexible approaches and delve into a more interdisciplinary vision of culture.
- A conception of the curriculum as a research tool that can be used to develop concrete methods and strategies of consolidation and adaptation.
- Assessment and exercising of teamwork as well as the development of a strong professional autonomy (Camacho, Hernández and Socas, 1998).

Fundamentals of the Proposal

The analysis of the knowledge and skills that a maths teacher must have in compulsory education, shows that two essential questions need to be answered: What are the basic tasks and activities in an initial training plan for maths teachers in compulsory

education? And whether the theory and practice dichotomy is enough to provide a response to the tasks and basic activities of teacher training?

Llinares (2004, 2009 and 2011) proposes the articulation of three systems of activities or tasks to develop the knowledge and skills of a mathematics teacher: "Organise the mathematical content to teach it", "Analyze and interpret the production of the students" and "Manage the mathematical content in the classroom". A reflection and analysis of the two questions leads one to consider that the three aforementioned activities systems are at least necessary. These are the activities that also emerge as necessary and essential in all three relationships in the Semiotic Logical Approach (Socas, 2001a and 2007). As for the second question, one can see the need to make progress in the dichotomy between theory and practice with knowledge to develop the professional skills to design and manage teaching practice in mathematics. The general aspects of the basic proposal take the following as a reference: the analysis of mathematical content, the didactic analysis of curricular content and organization. This is a comprehensive proposal for the training of mathematics teachers, which aims to facilitate a reconciliation between disciplinary mathematical knowledge (DMK) to curricular mathematical knowledge (CMK), to pedagogical mathematical knowledge (PMK) and knowledge of educational practice (KEP). This can be achieved by means of a proposal that ranges from the general comprehensiveness of the curriculum and of the disciplinary mathematical knowledge, to the organized totality of curricular content as content to be taught. The situation is depicted in the graph below, which expresses the cyclical nature of the proposal.

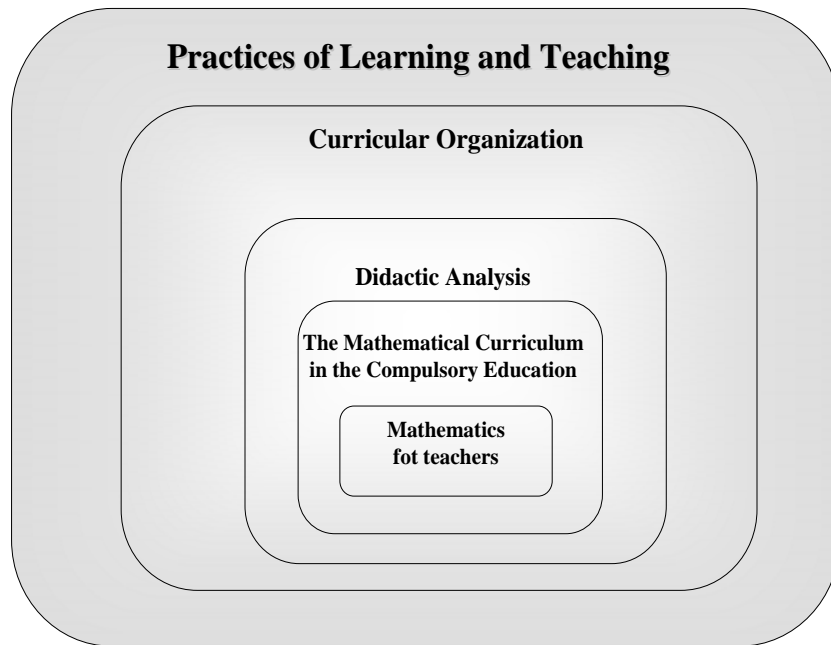


Figure 3: Proposal for the training of mathematics teachers

The analysis of the mathematical content plays a role, in this proposal, in the re-teachers' conceptualization of mathematics, and together with the didactic analysis of curricular content and organization, in the development of the school subjects of the didactics of mathematics and teaching practice of mathematics, where the three previously mentioned professional activities have a place.

Professional activity will be considered first, "organizing the mathematical content to teach it". This deal with solving a professional problem that requires analysis, understanding and planning, and can be represented by the following semiosis: curricular mathematical content, disciplinary mathematical content, and mathematical content for teaching.

First, the teacher needs to organize the curricular mathematical content (CMC), the desired mathematical content that is definable in the domain of the disciplinary mathematical content, although it is not organized under that logic. This CMC is

extracted via precise and precise mechanisms and organizations from the disciplinary content and is inserted in the curriculum. Once these actions have been performed by different elements of the educational system, curricular mathematical content knowledge is intrinsically different to the disciplinary knowledge, at least in its epistemological aspect, and supports interpretations from different perspectives, for example functional, as part of a common basic culture (Rico and Lupiáñez, 2008), the second derives from the discipline itself, scholarly mathematical knowledge, which we call disciplinary mathematical content (DMK) or formal mathematical knowledge (Socas, 2010a) and the third is the mathematical content for teaching (MCT), which includes both the taught and the mathematical content assessed (Hernández et al., 2010). The three components are interrelated in a process called transposition or adaptation of mathematical content, but have their own independent organization. The organization of curricular mathematical content comes from a pedagogical order implicit in the curriculum designers, and is associated with basic mathematical competence as part of a common culture. The organization of the mathematical content for teaching is compiled using the didactic order as a starting point, and is associated with the subjects' competence in didactical mathematical knowledge (DMK) and determines the sequence and level of the mathematical content in the teaching proposal with regard to basic mathematical skills and the other basic skills.

The professional task of organizing the mathematical content for teaching involves competence in the three areas of mathematical content. The question is now what happens to our students and how does one involve them in professional tasks that enable them to be competent professionals who can identify, analyze, understand and

plan for these three areas of mathematical content? As has been shown, students who begin teacher training for primary school have huge gaps in basic math skills which is why they need a revision of the discipline in terms of some "mathematics" to train them professionally, to improve not only their knowledge but their beliefs about the ends of this knowledge in compulsory education (Socas et al., 2009).

Mathematics for teachers in compulsory education

Teacher training programs have generally been designed to include in subjects, like mathematics, mathematical content as disciplinary knowledge, which is developed by explicating the different conceptual fields, and by considering mathematics as a fundamentally instructive tool that is organized primarily from the point of view of its internal logic, which means characterizing mathematical knowledge by using an organization based on its key concepts and on an introduction using a logical sequencing, i.e. the material is organized in the way a mathematicians would. On the other hand, the mathematical content of the curriculum that the teacher must impart has been determined by various agents of the educational macro system via a process that is generally unknown to the future teacher. The curriculum is organized by a list of contents that are related to the skills and competencies to be developed, the same happens with the evaluation process, and is immersed in a particular conception of understanding teaching and learning. Therefore, the curricular organization of the mathematical content, the object of education in a stage of education, needs to be seen as a systematic organization, which considers mathematical content as a fundamentally cultural and basic element, which is organized from an epistemological and phenomenological perspective capable of developing basic mathematical skills, and is introduced by means of an educational

organization as well as criteria for assessing the acquired knowledge and skills. The subject: mathematics for teachers in compulsory education would deal with revising different aspects of curricular mathematical content relevant to the stage of education in which the teachers have to exercise their profession from the disciplinary mathematical perspective, facilitating the teachers with a re-conceptualization of curricular mathematical content. This is a process of immersing the trainee teacher in curricular mathematical content which they will have to organize for teaching afterwards. This is, ultimately, a proposal for basic training in a closed curricular structure, which is approached from formal mathematical competence and basic mathematical competence, i.e. the analysis and understanding of curricular mathematical content in disciplinary terms with epistemological, phenomenological and applicability references, in which students complete their basic training related to such issues at the level of conceptual systems involved: operations, structures and processes in problem-solving situations, using the reasoning and the appropriate language for the thematic level in question by means of tasks and activities of differing natures but necessary for linking the school tasks and activities.

The didactic of mathematics for teachers in compulsory education

The next item to be considered is the second group of activities and tasks to be developed by the trainee teacher: "Analyzing and interpreting students' production" which refers to the knowledge and ability to mobilize different resources: analogical and digital mathematical representations, difficulties, obstacles and errors associated with the object of teaching mathematical content. Take, for example, the role of the difficulties, obstacles and errors of students in this analysis and interpretation. It is known that

learning mathematics creates many difficulties for the students and that these differ in nature. Some difficulties originate in the educational macro system, but generally speaking, they originate in the educational micro system: student, subject, teacher and educational institution. These difficulties are connected and reinforced in complex networks that, in practice, materialize in the form of obstacles and are manifested by the students in the form of errors. The error will have different roots, but will always be considered as the existence of an inadequate cognitive schema in the student, and not only because of a specific lack of knowledge or an oversight. The difficulties may be grouped into five major categories associated to 1) the complexity of the objects of mathematics, 2) mathematical thinking processes, 3) the teaching processes developed for the learning of mathematics, 4) cognitive development processes of students and 5) affective and emotional attitudes toward mathematics (Socas, 1997). In addition to the curricular and disciplinary mathematical knowledge, the trainee teacher of mathematics requires didactical mathematical knowledge (DMK) to be able to organize the mathematical content for teaching. This is specific professional knowledge that has to be provided by the subjects belonging to the didactics of mathematics, which includes the elements of analysis for adequately understanding, planning and conducting professional work. This knowledge is developed under the two constructs discussed above, didactic analysis and curricular organization.

Best Practices

The proposed teacher training should focus on the organization and development of best practices for the attainment of the skills required, these have to be developed within the framework of problem solving of a professional nature and associated with the

knowledge and resources that the teacher must mobilize to obtain the solution to the problem.

Thus, mathematical problems emerge from the situations developed in the curriculum and are addressed from the FMC in the subject of mathematics for the teachers in terms of the language and reasoning involved in the conceptual field in question. This immersion of the student continues because the problem solving must be organized for teaching, usually in the context of a classroom program, which must be considered from the training analysis. It involves incorporating the consideration of the difficulties, obstacles and errors of students in the different domains of mathematical activity. The trainee teachers of mathematics perform different activities and tasks of application, related to the various mathematical fields, and conclude in all situations with the elaboration of a map of the mathematical knowledge being dealt with, organized in terms of the six disciplinary mathematical content areas according to the FMC model, i.e., operations, structures, processes, representations, problems and reasoning. Certain tasks developed by the trainee teachers of mathematics in the course in a report format, all of which are from a questionnaire, are presented below:

Task 1: Report on numeration systems and decimal system

(For example, the first questionnaire has questions about the relations between the different numerical systems, the description of the numerical systems from the decimal representation and the representation in the number line of the different numbers).

- Analysis of the errors made and of the blank responses, as well as determination of their cause or origin.

- To characterize the D numeration system (Decimals) as is clear from the answers to the questions.
- Analysis of the representational procedures on the number line of the numbers proposed in the questionnaire.
- Decimal numbers in the curriculum of compulsory primary or secondary education.
- To elaborate a map of the numbers in the primary or secondary education.
- To elaborate a map of the procedures for representing numbers in primary or secondary education.

Task 2: Report on operational, structural and processual knowledge in mathematics

- Analysis and evaluation of the mathematical discipline according to SLA.
- Analysis of unanswered questions and the mistakes made in the questionnaire, determining the source of errors.
- Analysis of operational, structural and processual knowledge used in the questionnaire responses, both correct and incorrect.
- Self-evaluation of the type of knowledge used in the answers.
- Analysis of the mathematics curriculum in primary or secondary education. Choosing a course and a content block about numbers, algebra and functions, and analysing them from an operational, structural and processual perspective, identifying the systems they use for representing mathematical objects, the problems they give rise to and the reasoning they propose, with special emphasis on identifying the heuristic content.

- Analysis of a mathematics textbook in compulsory secondary education. Choosing two consecutive themes on numbers, algebra and functions, and then analyzing them from the aforementioned perspective.

Task 3: Report on mathematical problem solving

- Solving the problems correctly in various sessions.
- Analysis of the difficulties and errors made in the different sessions in solving the problems of the questionnaires.
 - a) To identify the following phases in each problem: acceptance, blockage and exploration
 - b) To determine the source of the difficulties and errors.
- To identify the different reasoning (and different heuristics) used in the given questionnaire responses.
- To analyze the map of the contents involved in solving the mathematical problems proposed in the questionnaire, paying special attention to the mathematical tools and reasoning (heuristics) used.
- To develop a new map of knowledge involved in the correct resolution of the proposed problems.

Final considerations

A proposal is suggested here, for training student teachers in primary and secondary mathematics to improve the learning and teaching of mathematics in these education stages because as Sowder said (2007), many of the difficulties that mathematics students have are to do with the teaching they receive, but what does preparing a trainee maths teacher competently really involve? This proposal opts to

develop three systems of basic activity that can determine the knowledge and skills of the teacher, presented as professional tasks from a global perspective in the context of problem solving in the case of their profession.

The three systems of professional activities categorize teachers according to different skills, for example, in the case of the activity: organizing the mathematical content to teach it, puts students in these skills areas: knowledge of the contents of mathematics from a global perspective in which the resolution of problems is an inherent part of the mathematical culture that should be taught and the ability to translate this into learning expectations, and the design and planning of learning sequences. In the case of the activity of analyzing and interpreting the students' mathematical production places students in the skills area regarding understanding and working based on the students' representations including their idiosyncrasies, and knowing and working with the difficulties, obstacles and errors of the students.

As regards the activity of knowing how to manage mathematical content in the classroom, this places students in the skills area of designing and controlling problematic situations appropriate to the different levels and possibilities of the students, and observing and assessing students in learning situations. The case of training maths teachers leads one to consider the basic situations of meaningful and effective work and how these should be dealt with by a professional comprehensive approach. The comprehensive approach is set in the context of trainee teachers, and it articulates and connects different subjects in a global proposal which seeks to ensure a comprehensive and inclusive vision of mathematics and of teaching and learning mathematics,

encouraging the active participation of students, which shows how we get closer to understanding reality through mathematical culture and how it is perceived by them.

Research has shown that when mathematics programs, from the disciplinary approach, are used with trainee teachers as a finished product, they are insufficient. Providing trainee teachers with an epistemological and phenomenological analysis of mathematical objects of teaching involves not only knowing the conceptual systems involved, their languages and problems, but also the usefulness of mathematical objects and their use, which could be successfully used to deal with the interpretation of the aims of the mathematics curriculum in this educational stage and confidently take on the didactical mathematical knowledge. The organization of mathematical knowledge using the phenomenology / epistemology pairing involves paying special attention to the use, management and function that this knowledge can have at a given time, without losing sight of its internal logic. Finally, it is important to emphasize that this global proposal for training mathematics teachers by "immersion" in the curriculum of the educational stage where they will work in the future, will allow them to develop, in this environment, the knowledge and skills needed in their professional work.

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Developing problem solving experiences in practical action projects

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Abstract: Problem solving doubtless is an essential element of mathematical learning, so that mathematics educators often are satisfied when finding situations that lead their students to such activity. But in many cases, the chosen situations and the ways to guide students' works are not sufficiently analyzed from a didactic point of view. Our goal in the present analysis is to underline the possible ways for managing the situations, and to exhibit the parameters that educators have at their disposition within their role as mediator between students and mathematical knowledge and know-how.

Keywords: Problem solving, problem for investigating, project based learning, a priori analysis, mediation.

1. Classical problem solving

Problem solving at its broad extent is clearly exposed by Alan Schoenfeld (Schoenfeld, 2006, p.41): “[Problem solving] *includes a child’s actions in interacting with its parents, a student working on a mathematics problem in class or in the laboratory, and a teacher’s decision-making while teaching a mathematics (or other) lesson.*” In this paper, we will only consider problem solving in mathematics instruction. Nevertheless, what we mostly found in mathematics educational literature and in learning material designed for students is a restricted use, with sometimes a reduction to drill for improving skills. In our opinion, this approach is too limited. Thus we first analyze some typical situations used in mathematics education with a problem solving purpose.

Our goal in the following analysis is to underline the possible ways for managing the situations, and to exhibit the parameters that educators have at their disposition within their role as mediator between students and mathematical knowledge and know-how.

1.1. Too easy... and how to do it better

The following *Math Problem Solving* example comes from a worksheet of Rhl School. We may consider it as a representative of certain trend in exploiting problem solving in mathematics education.

“Ryan’s Class

There are exactly twelve children in Ryan’s class. Only four of the children are boys. The following questions refer to a time when all the children are present in the class. There are no visitors in the class. There might be more than one correct answer to a question.

1. Which of the lettered statements **must** be true?
2. Which of the lettered statements **cannot** be true?
3. Which of the lettered statements could be true **or** not true?
 - a. There are **twice** as many girls as boys in Ryan’s class.
 - b. There are eight **more** girls **than** boys in Ryan’s class.
 - c. There are four **more** girls **than** boys in Ryan’s class.
 - d. If Ryan is sitting at a table with all the girls, there are exactly nine children at that table.
 - e. If **only** three of the boys are standing on their heads, one of the boys is not standing on his head.”

Math Problem Solving, Vol. 8, No. 1, May 5, 2003. Retrieved from

<http://www.rhlschool.com/math8n1.htm>

Comments about the task: Giving an answer to the lettered statements a, b, c and e only supposes to understand them. The only true *problem* here comes from the fact that we don’t know Ryan’s gender (while mostly used for boys, *Ryan* has been used for girls in the United States since the 1970s). And for the lettered statement d, if Ryan is a boy, the statement is true, and if Ryan is a girl, the statement is false; then the correct answer **for us** is the answer 3, but the correct answer would be different for somebody who

knows Ryan's gender. So we note that there is a subtle semantic distinction in that statement; nevertheless, in terms of mathematics knowledge, we will say that the proposed task is *too easy*. That means that the mathematical activity for answering is poor, even though understanding the statements may have a relatively high intellectual cost for the students. Compare with the situation that would result from the following questions, without changing the data (i. e. a class of 12 students, four of them being boys):

We try to form groups, all having the same number of students (for instance two groups of six students each), or all having the same numbers of girls and boys (for instance two groups, each one composed of four girls and two boys). What numbers of groups can we form in each case?

For such reasons, researchers later followed in France by official curricular commissions, introduced precisions about problem solving, considering *open problems* and *problem for investigating*. What follows is translated from a French text by Roland Charnay (Charnay, 1993).

“The team of the IREM of Lyon offers the following definition.
An open problem is a problem that has the following characteristics:

- The statement is short.
- The statement does not induce either method, no solution (no questions intermediate or questions like "show that"). This solution should never be reduced to the use or the immediate application of the latest results presented in class.
- The problem is in a conceptual domain to which students have enough familiarity. Thus, they can easily take "possession" of the situation and engage in testing, conjecture, draft resolution, counterexamples.

Example 1 (extracted from "Rencontres Pédagogiques", n°12, INRP)

I have 32 coins in my piggy bank.

I only have coins of 2 F and 5 F.

The total amount of my 32 coins is 97 F.

How many coins of each value are in my piggy bank?

Example 2 (extracted from "Situations problèmes", APMEP, Elem-math IX).

What is the biggest product of two natural numbers that we can obtain using each digit from 1 to 9 once for writing the numbers?"

The reader can find more precisions about Problem for Investigating in (Houdement, 2009), in particular a list of 24 representative statements, heuristic elements and perspective for research.

1.2. OK but limited except for students at an advanced level...

In this section, we present a situation well appropriate for sequences of investigation by 4 or 5 grade students. Thus it generates a good students' training in numerical treatments. But its institutionalization in Brousseau's meaning will be difficult until an advanced level, for instance undergraduate, because of the difficulty for enunciating and proving the general result. So in this case, the teacher can exploit the situation for younger students with an objective of arithmetical training or acquisition of methods, but not of learning new mathematics knowledge.

“A statement for students of all levels:

The number 23 can be written in many ways as a sum of natural numbers.

For example:

$23 = 11 + 5 + 7$. Among these sums, find the one whose product of the terms is the biggest (in our example, the product was $11 \times 5 \times 7 = 385$).

Other statement is: Among the additive decompositions of a natural number, find the one, whose product of its terms is the biggest.”

Retrieved from <http://educmath.inrp.fr/applet/exprime/plgrprge.pdf> and from <http://gilles.aldon.free.fr/ensemble/Ensemble/syplgrprt1.pdf>

In this situation, we can first discover the answer, either in particular cases (such as 23, that produces $4374 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 2$ as highest product) or in the general case in a descriptive way. At more advanced levels, a new possibility is to deepen the

problem, and express and prove the general solution in an algebraic way. So does Gilles Aldon when he writes what follows.

The Sloane Encyclopedia (the OEIS Foundation) gives some properties of this biggest product at <http://oeis.org/A000792>. It quotes a dozen of different definitions of the number $a(n)$ equal to the biggest product obtained with partitions of n in sum. If we take $a(0) = 1$, then we obtain the induction formula: $a(n) = \max\{(n-i)a(i), i < n\}$. This is the definition given in the Encyclopedia.

In such a situation, a teacher does not have possibility to adapt the situation with some changes in the statement. He/she has only to know that the situation is convenient for arithmetic training of young students, and for improving the use of induction by older students.

1.3. Too difficult because insufficiently explored by the interested teachers...

Analyzing a problem of a mathematical contest: the “Math Rally of Alsace”

We translate here the statement of one of the problems of a mathematical contest, the “Math Rally”, followed by the solution and comments given by the organizing team, retrieved from <http://irem.u-strasbg.fr/php/index.php?frame=.%2Fcompet%2Fcompet.php&m0=ral&categ=rallye>.

Exercise 3 (grade 11)

In the Valley of Bruche River, a clearing has a circular shape. A treasure is buried near the clearing. An old parchment shows the location of the treasure: “From the great fir located on the circle, go to the poplar in the clearing. Then turn right at a right angle and walk to the edge of the clearing. Still turn right at a right angle and walk as many steps as from fir to poplar. There is buried treasure.”

The clearing has a radius of 20 meters, and the only tree on the clearing is a poplar located at 4 meters from the centre. Unfortunately, on the edge of the clearing, firs disappeared long time ago. Can you find the distance between the centre of the clearing and the treasure?

Since $(a + c)^2 + (b + d)^2 = OT^2$, then $OT = 28$.

Thus the distance that we sought is 28 meters.

Comments by the organizing team

This is a geometrical exercise and it was the less successful for students. Yet its resolution only involves the Pythagorean Theorem, applied to several right triangles of course.

Candidates who worked on this exercise drew a figure that includes letters often undefined. Didn't they have the habit of labeling a figure in order to make it understandable? The resolution led them to the correct answer (28 meters), but most of them relied on a particular case – for example points S, O and P collinear – and they do not consider the general case. Some say that they were only considering a particular case, but many students seem not to realize it. (...)

Our analysis

Thus in its report, the organizing staff itself recognizes that this problem was unsuccessful. But we think that this poor result originates in a lack of mathematical analysis by the organizing team. The members of this team are very experienced teachers and competent mathematicians, but they were more preoccupied in this case by the design of an amazing story than by didactical considerations. As a matter of evidence, we refer to the solution given in the report, which is for us not satisfactory, because it is not convincing. It is like the rabbit that comes out of the hat of a magician: we measured lengths, and it appears that the final result does not depend on the variable elements. A

“good” solution of a problem produces a change in our mind with respect to the situation: from mysterious the result becomes evident for the reader. As an example, let us consider a classical result: The three altitudes of a triangle ABC intersect in a single point, called the orthocenter of the triangle. There are several proofs of this theorem, for instance by considering geocenter and circumcenter of the given triangle ABC , or studying angles and finding that the point of barycentric coordinates $(\tan \hat{A}, \tan \hat{B}, \tan \hat{C})$ is the searched orthocenter of the triangle. These proofs are of interest, but their cognitive routes are relatively complex. In contrast, the proof illustrated by Figure 2 easily could produce the change of mind that we want to emphasize: We trace respectively by A , B and C the lines parallel to the opposite side of the triangle, that create a new triangle $A'B'C'$; the altitudes of ABC are the perpendicular bisectors of the sides of $A'B'C'$. Thus it is obvious that they intersect at the point O equidistant of the three vertices A' , B' and C' .

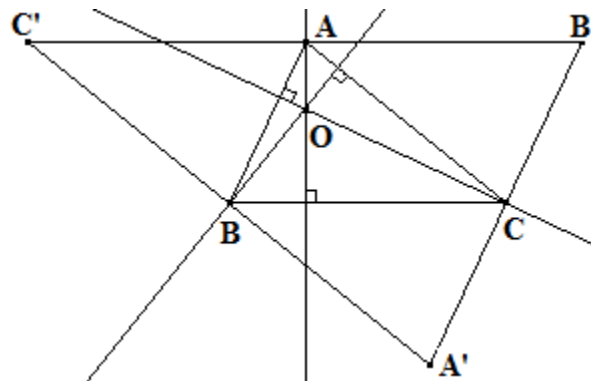


Figure 2: Altitudes of a triangle as perpendicular bisectors of a bigger triangle

For the Math Rally problem, we present below a solution approach that intends to promote the reader’s mathematical reflection and analysis. Then we will consider what changes could produce a better result in implementing this problem in the classroom.

To conclude that the distance between the center O of the circle and the point T does not depend on the location of S on the circle is the same as to affirm that the locus of

T when S moves along the circle is a new circle with O as its center. This assertion does not seem easier than the given statement of the problem (Figure 3 left), but...

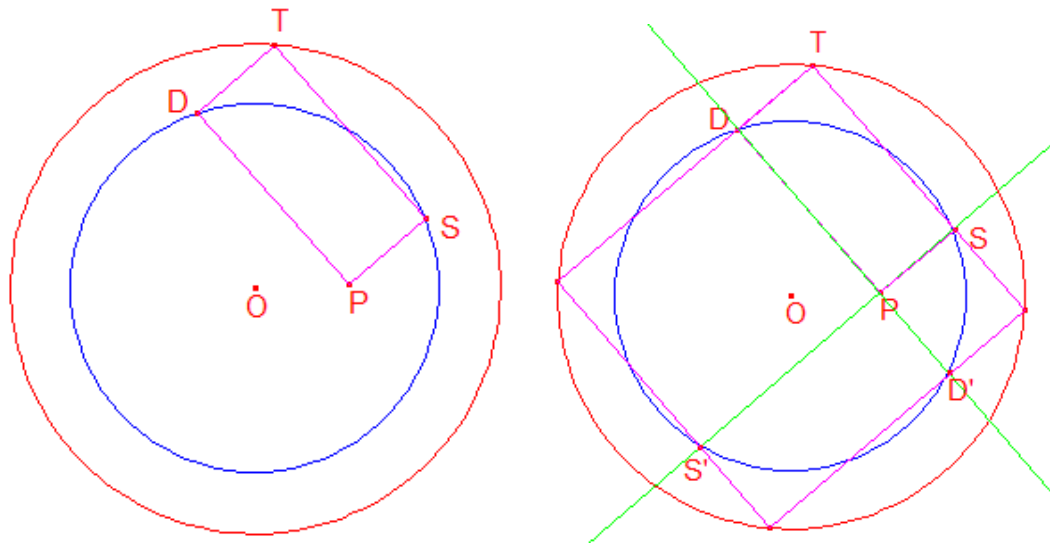


Figure 3. Left: incomplete figure, right: completed figure

If we observe that the situation is mathematically incomplete, our vision of the problem completely changes. And completing the figure is a natural idea when working with software such as Cabri or Geogebra. This involves drawing a complete line as a perpendicular to a given line or segment. After completing (figure above, right), a new statement is: *“Given a circle and a point P inside the disk, we consider two perpendicular lines passing through P and respectively cutting the circle at D and D’, and at S and S’.* Then the rectangle obtained by tracing by these points parallels to DD’ and SS’ has its center at O.” And this is a quite evident result, because the perpendicular bisectors of SS’ and DD’ are the axes of symmetry of the rectangle and pass through the center O of the circle.

The given statement of the problem did not lead the candidates, of the mathematical contest, a way to focus on a geometric construction. But we see that a teacher in his (her) class can present the same mathematical situation through another

statement and in a different environment that changes the didactical situation and facilitates the students' access to it. In this case, we pretend that a modified presentation could change the vision that students have of that geometric situation.

As a conclusion at this point, we will assert that the kind of students' work generated by dealing with a problem strongly depends on the way showed by the statement of the situation. But the form of working, particularly when using the technology, has an influence too. This enforces the role of mediator devoted to the teacher, not only when preparing the lesson but also during the class.

2. Project based learning (PBL or PjBL)

In the expression "Problem Solving" appears the word "Problem", which is also present in the pedagogical method called "Problem Based Learning" and often designated by the initials PBL. This method is in use in various disciplines such as biology or medicine or engineering, but problems for the associated topics do not refer to the same kind of mathematical problems that we studied in §1. For our part, in empiric experiments, we studied the potential impact of *projects of practical action* on the teaching-learning process. We were applying a teaching strategy, presented in Cuevas & Pluinage (2003), close to the method denominated Project Based Learning, which sometimes is also designated by the initials PBL. Here, like other authors, we use the letters PjBL in order to avoid confusions with Problem Based Learning.

Important features of our teaching strategy lie in the systematic use of registers of representation according to Duval's terminology (Duval, 1995): formation rules, treatments within a register, and conversions from one register to other. For this reason, a

first step in the study of a project of practical action is a descriptive phase, the objective of which being to introduce the formation in one register or various registers.

2.1. A paradigmatic example: concrete and virtual Tangram

Mexican educative institution presents Tangrams as a medium to organize learning activities at school. The Instituto Latinoamericano de la Comunicación Educativa (ILCE) has a web page on this subject at <http://redescolar.ilce.edu.mx/educontinua/mate/imagina/mate3z.htm>, with a link to instructions for cutting the puzzle. We retrieved the figure below from this web site. In English, for instance Wolfram's project gives a version at <http://demonstrations.wolfram.com/Tangram/>.

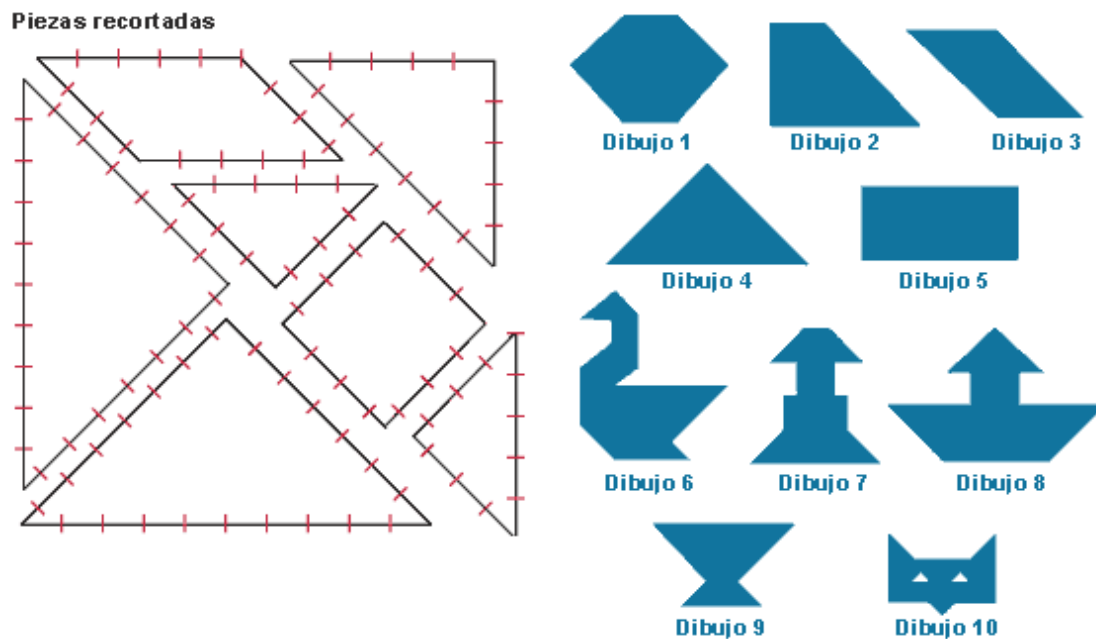


Figure 4: The seven pieces of a Tangram and shapes to be built

The web page of ILCE presents both possibilities for using Tangrams: concrete and virtual (on line). Wolfram's proposal is only a virtual one. We actually observed

some use of a concrete Tangram in class and related problems, but it remains to experiment the more extended use that we present in this section.

There are many problems that the students are able to pose by themselves. For example: Can we build a triangle with the seven pieces? With the seven pieces, can we construct a triangle different from the preceding? Can we build with the seven pieces a rhombus different from a square? How many distinct rectangles can we build with the seven pieces? How many distinct rectangles can we build with some of the pieces, but not necessary all pieces? Etc. In that situation, teacher's role can be to encourage the students to invent problems and not only to solve given problems.

An interesting activity in a class might be to classify these distinct problems. Organizing principles for such a classification arise from considerations of shape resulting from values of lengths, angles and areas. For example, the shape of each piece is a polygon with angles 45° , 90° or 135° , and this leads us to a strength restriction for the possible shape of a triangle made with the seven pieces: a triangle necessarily is a right isosceles triangle. With exact lengths, we enter in the symbolic world, because the lengths of the sides of a piece in two distinct directions are incommensurable magnitudes. In the table below, we present the exact lengths of the sides of pieces with the choice of the length of the shortest side as unit.

Table 1. Names of the Tangram pieces and exact lengths of their sides

Name	Side 1	Side 2	Side 3	Side 4
Square	1	1	1	1
Parallelogram	1	$\sqrt{2}$	1	$\sqrt{2}$
Triangle 1	1	1	$\sqrt{2}$	
Triangle 2	1	1	$\sqrt{2}$	
Triangle 3	$\sqrt{2}$	$\sqrt{2}$	2	
Triangle 4	2	2	$2\sqrt{2}$	
Triangle 5	2	2	$2\sqrt{2}$	

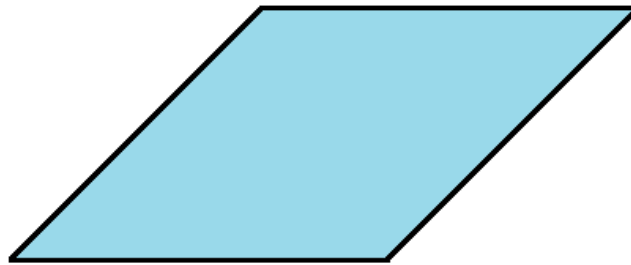


Figure 5: A shape impossible to construct with Tangram

Then, for instance when studying the situation of a rhombus different from a square (Figure 5), we can first observe that its angles might be 45° and 135° and then use

the algebraic register and employ inferences like $a + b\sqrt{2} = c + d\sqrt{2} \Rightarrow \begin{cases} a = c \\ b = d \end{cases}$.

Problem to be solved by the reader: Prove the impossibility of constructing a rhombus different from a square with the seven pieces of Tangram.

We suggest that all the problems we have seen can be included into a *project of practical action*, and in order to do this, we will replace the concrete material by a virtual one. Then the first step is to construct the material. We used Cabri geometer for this

purpose, but the design would be approximately the same if we were using Geogebra.

Thus it is necessary to solve a first problem in our project: *How to move a piece in the plane?*

There is a natural relationship between movements and geometric transformations. Nowadays, it is controversial if geometric transformations have a place in the curricula at elementary level. In our opinion, moreover with the reference to a genetic point of view, the geometric transformations are a topic of high interest in mathematics education. We suggest using the introduction of transformations in the learning of geometry at secondary level in order to facilitate the transition from Geometry I (natural geometry) to Geometry II (natural axiomatic geometry) (Houdement & Kuzniak, 1999).

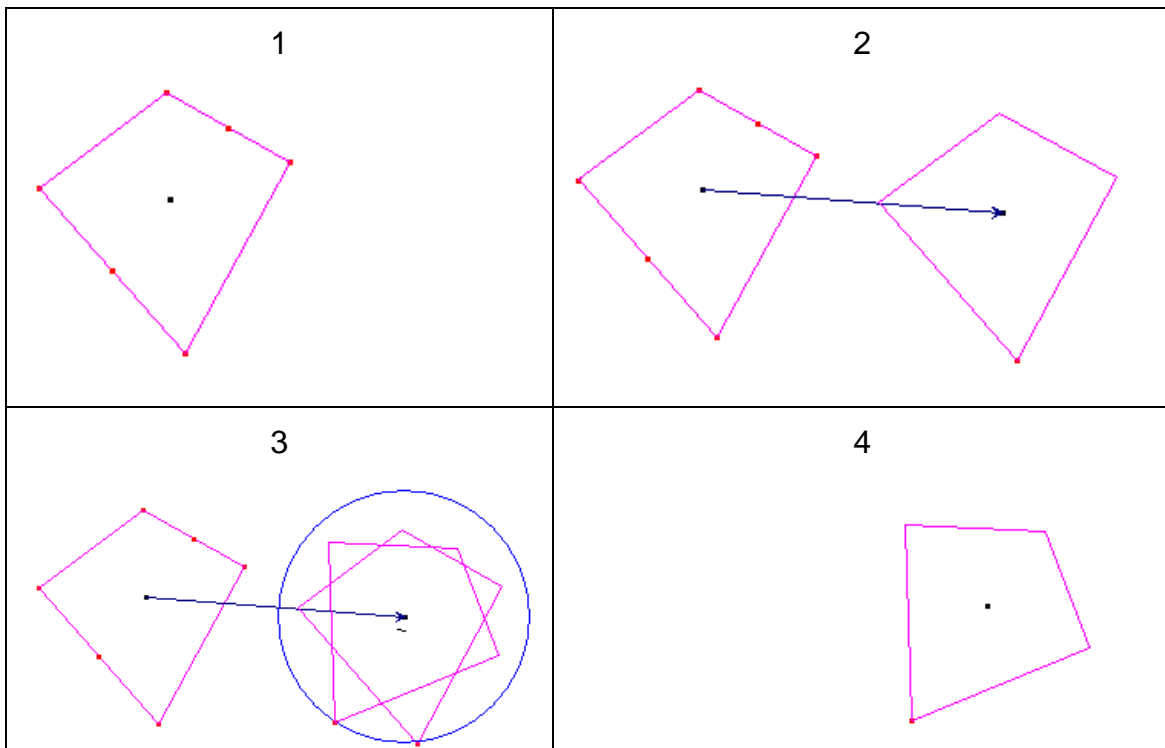


Figure 6. Modeling the displacement of a quadrilateral in the plane: 1- Choosing a center (here the midpoint of two midpoints), 2- Translating the piece, 3- Rotating the translated piece around its center, 4- Showing only the final figure with the two director points (black and red)

Figure 6 illustrates a solution obtained by a Cabri program. For each piece, we choose a center point that is represented by a black dot, and a vertex that is represented by a red dot. When moving the black dot, we translate the piece, and when moving the red dot, we rotate the piece around the black dot. In order to do that, first we build each piece and mark the black and red dots. Then we define the translation and the rotation and apply them to the piece. Finally we hide the initial piece and only show its last image.

The reader can imagine alternative possibilities, and also complete the universe of allowed transformations by adding the reflection of the parallelogram that we do not introduce in our program (with a concrete jigsaw puzzle, we can turn over the parallelogram). And one can find many interactive programs on line in Spanish and in English, but almost all such web pages are only game oriented. For example, we don't see there any place for verbal descriptions or for problems whose solutions are "impossible" to achieve. So, in this case, it seems to us preferable to promote students' work in classroom or at home. The teacher can construct a virtual Tangram and give his/her students a web page like that illustrated in the figure below, or ask them for constructing the virtual Tangram.

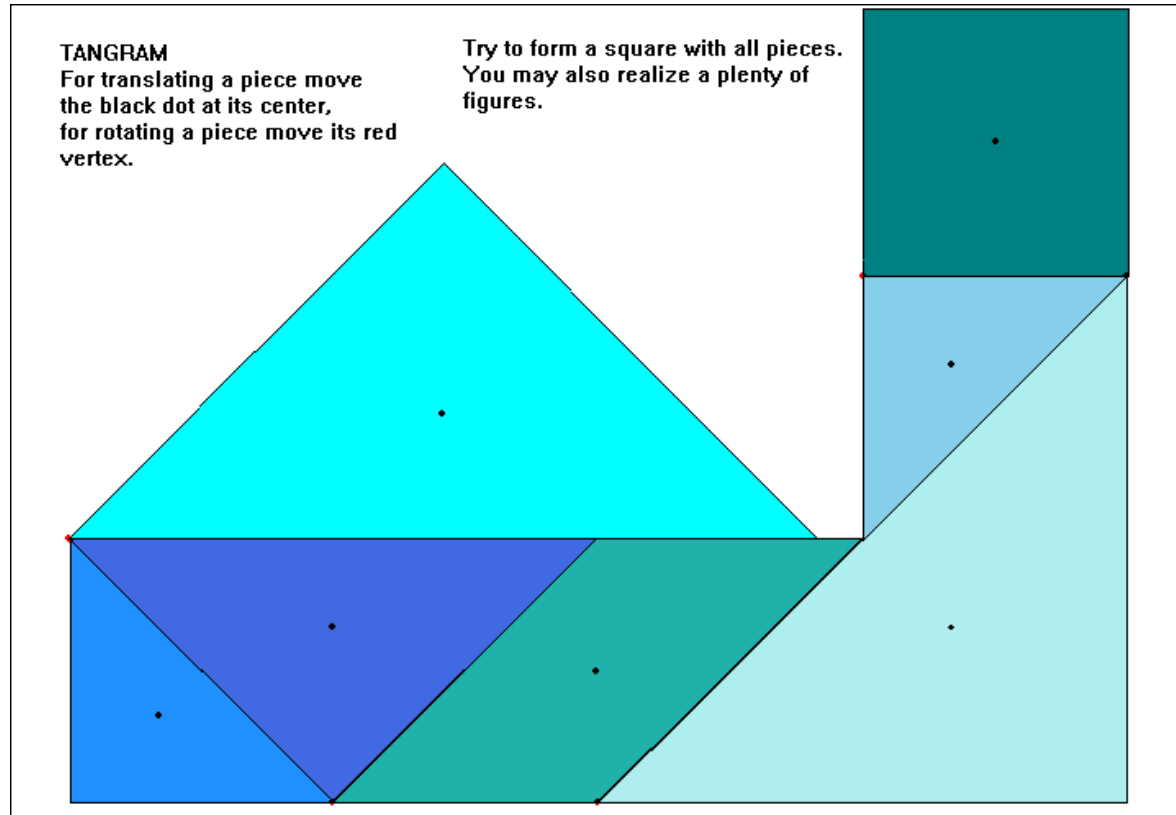


Figure 7: Virtual Tangram designed for students

Assigning the task of constructing a virtual Tangram to students by using geometric software is a good example of what Martin Wagenschein (1977) presents as *Exemplarisches Lehren*. The translation “exemplary teaching” does not exactly reflect the meaning of this expression, which refers to a specific kind of *case analysis*. The specificity of this method is that the studied cases are chosen as representative of the situations to be mastered by the student at the end of the learning process. Here, we pretend that the achievement of the construction of Tangram is significant for acquisitions of certain level of competency in mastering geometric transformations.

2.2. Some didactic observations resulting from empirical studies using projects

Most projects of practical action include spatial situations. So their modeling supposes to connect spatial geometry (3D-geometry) and 2D-geometry. This can be made

by perspective representations or plane sections and also by nets, a net being *an arrangement of edge-joined polygons in the plane which can be folded (along edges) to become the faces of the polyhedron* (see http://en.wikipedia.org/wiki/Net_%28polyhedron%29). A great family of projects of practical action is the construction of solids subject to certain constraints.

The study of nets associated to polyhedra is a particular case of these spatial projects, which can be applied at high school or more advanced level. When we were experimenting in undergraduate classes, we realized that the majority of students do not master the easiest spatial situations, because of the lack of both knowledge and experience. For instance it was a great surprise for them to know that all convex polyhedron has a net, and that the “tower” made by the superposition of two cubes, the edges of the higher being a third of the edges of the lower, is a polyhedron whose construction with thin cardboard requires two disconnect nets.

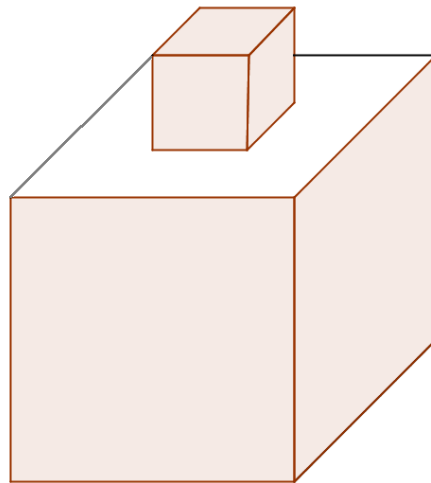


Figure 8. “Tower”: Non convex polyhedron that does not have a net

The preceding example illustrates one of the natural problems that arise from studying the nets associated to polyhedra. Other problems are those of uniqueness: Can we obtain two different polyhedra with the same plane net?

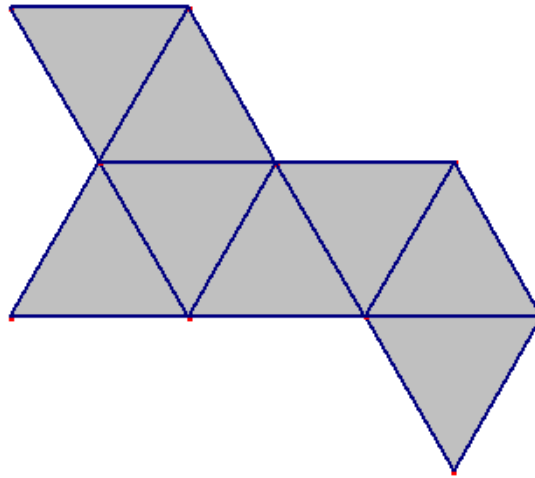


Figure 9: Same net for two different polyhedra?

Problem to be solved by the reader: The net above allows constructing a regular octahedron by folding paper. Try to find another (non convex) octahedron that the same net allows constructing.

Modeling spatial situations goes further than 3D-geometry. There is a lot of possible projects about containers (glasses, bottles, cans, recipients, etc.). A task for our students of the Master degree in mathematics education is to design a didactical project, and some of them choose this kind of subject. We present below the statement of a problem, which is a part of such a project that a student presented in a Web site, followed by its English translation and the representation of the solution that we made with Geogebra.

Optimización

Escrito por Paulo Angel Garcia Regalado

Domingo, 05 de Diciembre de 2010 01:32

- 1. Se pretende fabricar una lata de refresco de 335 mililitros de capacidad. ¿Cuáles deben ser sus dimensiones para que se utilice el mínimo posible de metal?*

Retrieved from <http://grupolycaon.com/matedu/>

Translation of the statement to English:

We want to make a soda can of 335 ml capacity. What should be its size in order to use the minimum amount of metal?

Can: Move the cross for changing the form of the can.

Area of metal in cm^2 for constructing this can: 267.02

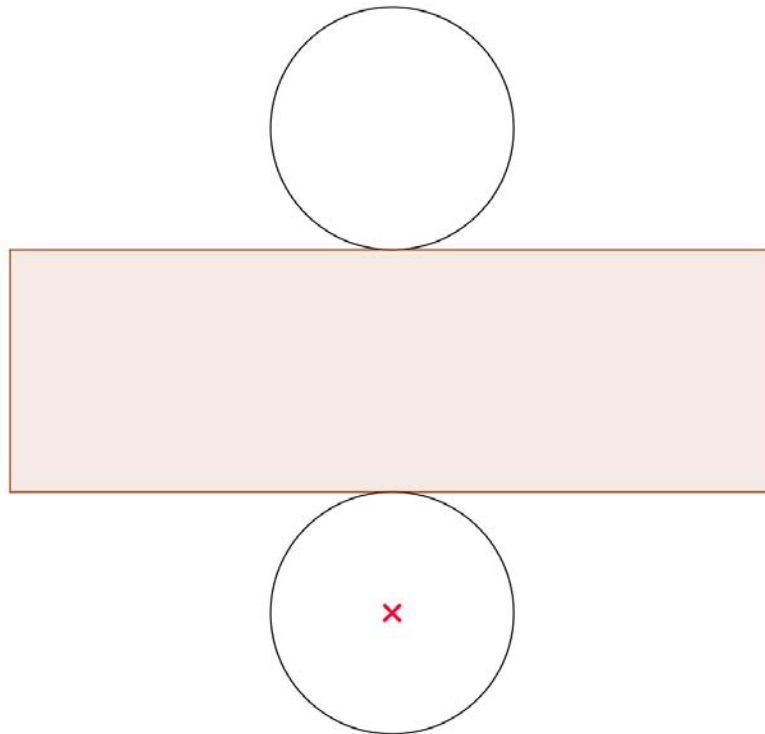


Figure 10. Problem to be solved by the reader: In the solution represented above for the problem of constructing a can with minimal quantity of metal, it seems that the height of the can is equal to the diameter of both disks at the bottom and the top. Is that rigorously exact?

In this last example, we saw calculus beside geometry. That is only a sample of the variety of mathematical domains and theories that the study of projects led us to encounter. Particularly, the project of modeling how recipients are filled by a regular flow of water has been a wealth field.

For instance 15-year students in Germany and France were asking by Stölting (2008) to represent in this situation the height of water in the recipient as a function of the time. We reproduce below the answers given to Stölting by a student. The last figure on the right is very interesting, because it shows a trend to the discretization of the

phenomenon. Previously we emphasized the importance of verbal descriptions of a phenomenon to be studied. Here we observe another important step in an investigation process for a complex phenomenon: a qualitative approach with the help of a representation. Both verbal description and figural representation were actually present in Stölting's research. Hence, students were interviewed to explain ways they drew the figures.

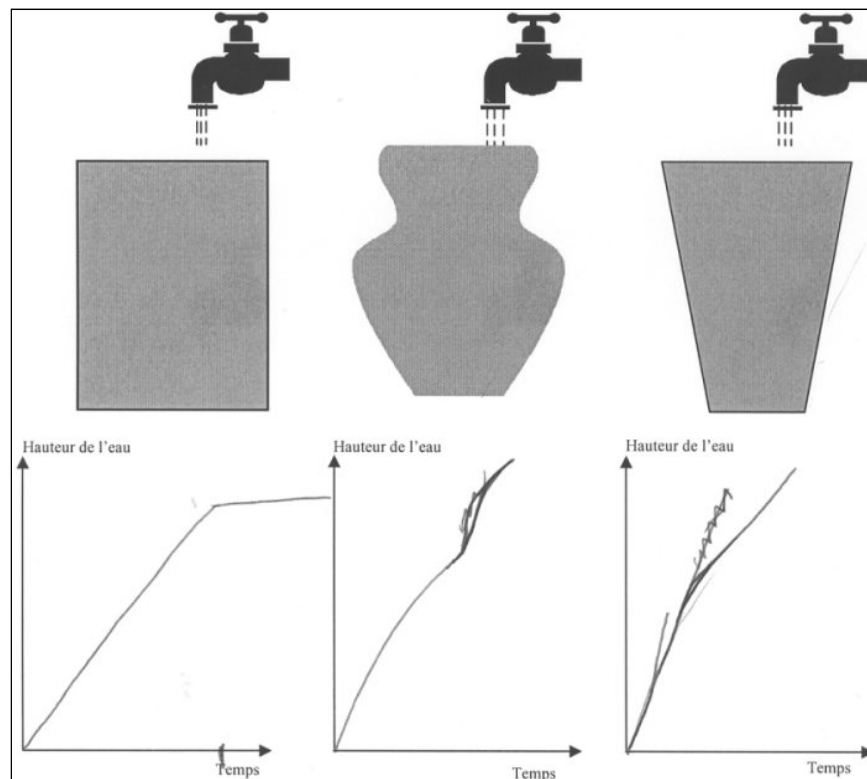


Figure 11: Answers of a (high achiever) French student

Interviews with high school students in order to deepen these observations with learning objectives are reported by Pluvinage & Marmolejo (2012) (see an illustration on next page), who describe the application in a class of the collaborative method ACODESA (Hitt, 2007), characterized by distinct phases of work: individual, group and collective, for exploiting the complete situation of filling recipients. We observed that a first step is important, namely to consider various cylindrical recipients. Indeed, many

students think that, in the case of a cylindrical recipient, diameter and height play the same role for the volume. So it is important to pay attention to this case studying the volumes of various cylindrical recipients, for instance as pouring water from one recipient to another that has double diameter and half height. Many students are surprised by the fact that the second recipient is not full. Moreover the same behavior as that related before was observed with our students. With access to the convenient mathematical tools, this would lead us to use an ordinary differential equation for solving the problem.

Nevertheless there is another easier way for modeling the phenomenon, changing the theoretical frame of reference. It is the consideration of reciprocal function. Indeed, the volume of a truncated cone of height h , radio of lower disk a , and radio of upper disk b , is given by $V_{total} = \frac{\pi h}{3}(a^2 + ab + b^2)$. Thus we can represent the volume as a function of the height, and then by symmetry we can conversely obtain the height as a function of the height. We used Geogebra in the illustration. But, and we do not know the precise reason, students do not spontaneously go on that way. This needs a help to the students from the teacher.

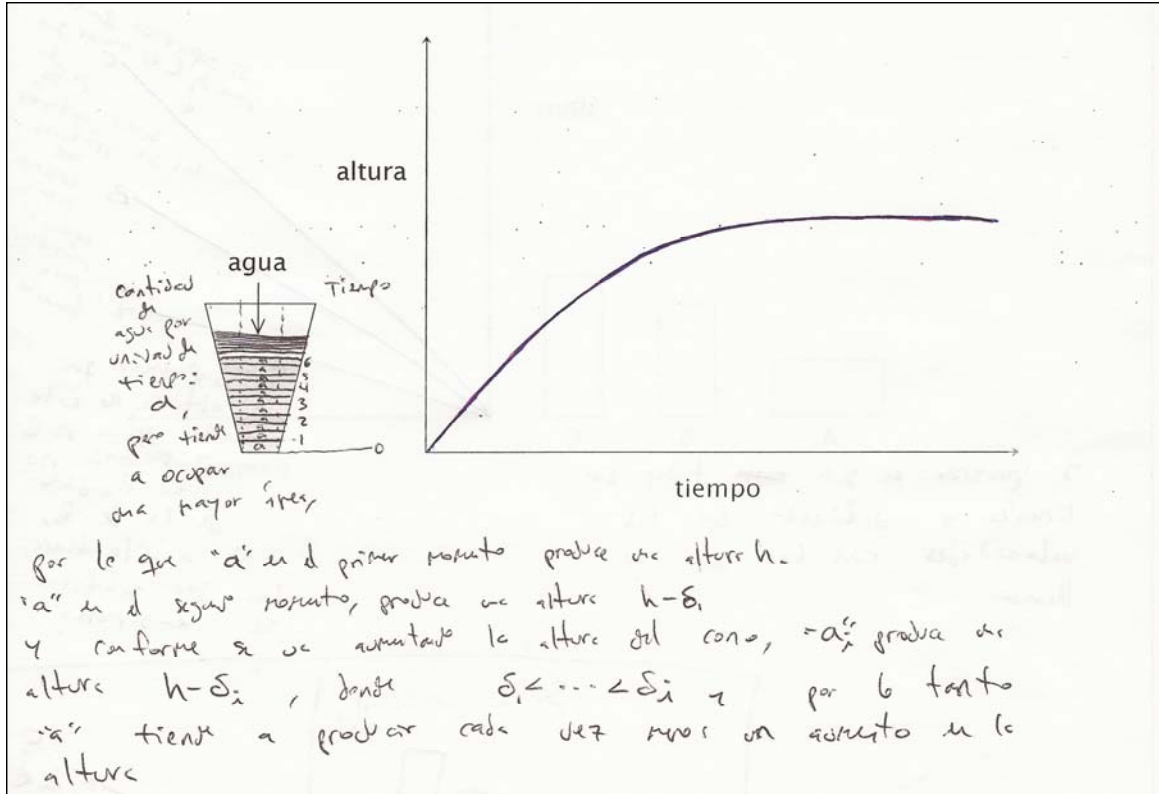


Figure 12. Answer given by an interviewed high school Mexican student: “The same amount of water produces a rise that always diminishes when the height increases”

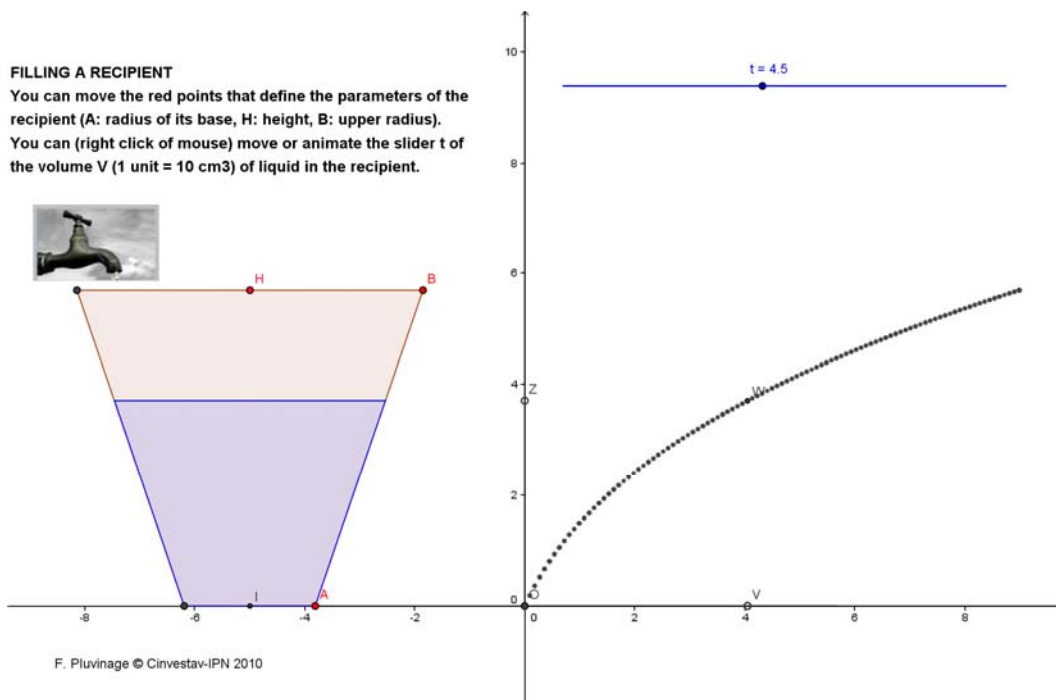


Figure 13: Solution resulting from consideration of the reciprocal function of volume as function of height

How could a teacher exploit these reported observations and considerations, for designing a possible instructional route? The first stages to do this are common to many practical action projects:

- (a) A diagnostic, here with a sheet of questions as we have seen about comparing volume of different cylindrical recipients and representing how various recipients are filled by a regular flow;
- (b) Some concrete experiment, for instance here to pour water or sand from a recipient to another and observe if the second is fulfilled or not. In our environment it is easy to get recipients that have the required characteristics: shape approximately cylindrical and respective diameters and heights approximately having simple ratios (for instance a box of coarse salt and a glass, or tin cans of varying size).

At the end of each stage, a discussion and a synthesis (e.g. volume of cylinder) will be useful. For the following stages, with the use of computation, the knowledge of a truncated cone and its volume formula are necessary. The teacher has the choice between giving the students these elements, and asking the students to obtain them by searching on Internet. Then the stages are:

- a) Figural representation of the transversal section of a truncated cone of varying height and diameters of upper disc and lower disc (see figure on the preceding page), and graphical representation of its volume V as a function of the height h of liquid in the recipient;
- b) In order to find now the inverse of the obtained function, i.e. the height h of liquid as a function of the volume V , first observe the difficulty to work only with a formula and, as a consequence of this difficulty, introduce the students into the work with geometric register, particularly with the use of reflection. Construction depends on software in use, for instance Cabri could reflect the graph of the function $V(h)$ and Geogebra not, so with Geogebra we had to locate a point on the graph and then to reflect this point and eventually obtain its locus.

The time devoted to this project is a good investment with the objectives of improvement of the functional thinking and acquisition of the difficult *procept* (amalgam

of process, mathematical object and symbol introduced by Eddie Gray and David Tall) of an inverse function.

3. Concluding

Teaching ways that we found rich of learning perspectives for the students consist in managing practical action projects at school, particularly in collaborative environment, for instance with the so called ACODESA method (Hitt, 2007). An important feature of these teaching ways is that the teacher is not the only one who is posing problems or ideas of problems. Nevertheless, his/her experience is essential for redacting in a correct mathematical language the statements suggested by students. And this step of writing mathematics also is extremely important for the students in terms of improvement of their mathematical experience.

The author is grateful to the reviewers, whose pertinent suggestions allowed to improve the first version of this paper.

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Thoughts About Research On Mathematical Problem- Solving Instruction

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Abstract: In this article, the author, who has written extensively about mathematical problem solving over the past 40 years, discusses some of his current thinking about the nature of problem-solving and its relation to other forms of mathematical activity. He also suggests several proficiencies teachers should acquire in order for them to be successful in helping students become better problem solvers and presents a framework for research on problem-solving instruction. He closes the article with a list of principles about problem-solving instruction that have emerged since the early 1970s.

Keywords: mathematical activity, problem solving, problem-solving instruction, proficiencies for teaching, craft knowledge, research design, teaching as a craft, teacher planning, metacognition..

Introduction

My interest in problem solving as an area of study within mathematics education began more than 40 years ago as I was beginning to think seriously about a topic for my doctoral dissertation. Since that time, my interest in and enthusiasm for problem solving, in particular problem-solving instruction, has not waned but some of my thinking about it has changed considerably. In this article I share some of my current thinking about a variety of ideas associated with this complex and elusive area of study, giving special attention to problem-solving instruction. To be sure, in this article I will not provide much elaboration on these ideas and careful readers may be put off by such a cursory discussion. My hope is that some readers will be stimulated by my ideas to think a bit differently about how mathematical problem solving, and in particular problem-solving instruction, might be studied.

Setting the stage

Most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction and how this goal is to be reached involves consideration by the teacher of a wide range of factors and decisions. For example, teachers must decide on the problems and problem-solving experiences to use, when to give problem solving particular attention, how much guidance to give students, and how to assess students' progress. Furthermore, there is the issue of whether problem solving is intended as the *end* result of instruction or the *means* through which mathematical concepts, processes, and procedures are learned. Or, to put it another way, should teachers adopt "teaching *for* problem solving,"—an *ends* approach—or "teaching *via* problem solving"—a *means* approach?¹ (I say more about means and ends later in this article.) In my view, the answer to this question is that both approaches have merit; problem solving should be both an end result of learning mathematics and the means through which mathematics is learned (DiMatteo & Lester, 2010; Stein, Boaler, & Silver, 2003). Whichever approach is adopted, or if some combination of approaches is used, research is needed that focuses on the factors that influence student learning.

Unfortunately, as far as I know, no prolonged, in-depth, programmatic research of this sort has been undertaken and, as a result, the accumulation of knowledge has been very slow. Moreover, the present intense interest in research on teachers' knowledge and

¹ It has become more common to refer to the "means" approach to teaching as teaching *through* problem solving. In Schroeder and Lester (1989) we discuss three approaches to problem-solving instruction: *teaching about*, *for*, and *via* problem solving. Teaching "via" problem solving is essentially the same as teaching *through* problem solving. Today, teaching *about* problem solving is not generally regarded as a legitimate instructional method, although I suspect that some (many?) teachers and curriculum writers subscribe to this approach.

proficiencies demands that future problem-solving research pay close attention to the mathematical and pedagogical knowledge and proficiencies a teacher should possess (cf., Ball, Thames, & Phelps, 2008; Hill, Sleep, Lewis, & Ball, 2007; Moreira & David, 2008; Zazkis & Leikin, 2010).

But before discussing problem-solving instruction, let me first say a few things about mathematical problem solving. This short discussion will highlight how my thinking has changed about the nature of problem solving and other forms of mathematical activity.

Some claims about Problem Solving

Among the many issues and questions associated with problem-solving instruction I have worried about during my career, several have endured over time. In this section I make five claims related to these enduring issues and offer brief discussions of my current thinking about them.

Claim 1. We need to rethink what we mean by “Problem” and “Problem Solving”

Although there have been at least four distinct problem-solving research traditions within (namely, Gestalt/Cognitive, Learning/S-R, Computer/Information Processing, and Psychometric/Component Analysis), they all agree that a problem is a task for which an individual does not know (immediately) what to do to get an answer (cf., Frensch & Funke, 1995; Holth, 2008). Some representative definitions illustrate this fundamental agreement:

A problem arises when a living creature has a goal but does not know how this goal is to be reached. (Duncker, 1945, p. 1)

A question for which there is at the moment no answer is a problem. (Skinner, 1966, p. 225)

A person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it. (Newell & Simon, 1972, p. 72)

Whenever you have a goal which is blocked for whatever reason . . . you have a problem. (Kahney, 1993, p. 15)

These definitions have two common ingredients: there is a goal and the individual (i.e., the problem solver) is not immediately able to attain the goal. Moreover, researchers irrespective of tradition, view problem solving simply as what one does to achieve the goal. Unfortunately, these definitions and descriptions, like most of those that have been given of *mathematical* problem solving – including those I and other mathematics educators have proposed – are unhelpful for thinking about how to teach students to solve problems or to identify the proficiencies needed to teach *for* or *via* problem solving. A useful description should acknowledge that problem solving is an activity requiring an individual (or group) to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine. Furthermore, these cognitive actions are influenced by a number of non-cognitive factors. And, although it is difficult to define problem solving, the following statement – which Paul Kehle and I devised a few years ago– comes much closer to capturing what it involves than most of those that have appeared in the literature.

Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an

effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. (Lester & Kehle, 2003, p. 510)

The advantage of this description of problem solving over the others lies in its identification of several key ingredients of success: *coordination of experience, knowledge, familiar representations, patterns of inference, and intuition*. So, to be a successful problem solver, an individual must have ample relevant experience in learning how to solve problems, strong content knowledge, proficiency in using a variety of representations² and a solid grasp of how to recognize and construct patterns of inference. Moreover, it recognizes the importance of intuition in successful problem solving³. With the possible exception of intuition, each of these ingredients should be attended to any program aimed at equipping prospective teachers with the proficiencies needed to teach mathematics either *for* or *via* problem solving. I say more about the implications of this description for the education of mathematics teachers later in this article. But first, let me continue with a few more observations about the nature of problem solving.

Claim 2. We know very little about how to improve students' metacognitive abilities.

So much has been written about metacognition and its place both in the teaching and learning of mathematics that a few comments about this elusive construct seem warranted. I remain convinced that metacognition is one of the driving forces behind

² The research perspective on the role of representation in doing mathematics provided by Goldin (2003) is particularly relevant to this discussion.

³ A reviewer pointed out to me that intuition is itself a very subtle notion and as such the definition we propose is unhelpful. To be sure, intuition is a subtle idea, but I think it essential to include it in any description of what problem solving entails because it serves to point out just how subtle the act of problem solving can be and, consequently, how difficult it has been to make progress in learning how to teach students to be better problem solvers.

successful problem solving (Garofalo & Lester, 1984), but we really know almost nothing about what teachers should do to develop students' metacognitive abilities. To be sure, it is essential that successful problem solvers be able to monitor and regulate their cognitive behaviors. But, almost no research has been done that demonstrates that students' can be taught these behaviors. Within the mathematics education community both Schoenfeld (1992) and I (Lester, Garofalo, & Kroll, 1989), among others, have conducted research aimed at enhancing students' metacognitive abilities, but neither of us has identified the proficiencies teachers need to do this. Instead, we have offered suggestions, with too little evidence to support them. So, any program designed to enhance mathematics teachers' proficiencies that pays heed to metacognition should do so only after acknowledging that there is no conclusive research evidence to support any particular method of metacognition instruction over another.

Claim 3. Mathematics teachers needn't be expert problem solvers; they must be serious students of problem solving.

It is natural to suggest that teachers must themselves be expert problem solvers before they are to be considered proficient mathematics teachers. But, I think this is asking too much of them! George Polya was an expert (and, hence, proficient) problem solver as well as an expert teacher of mathematics, but to expect all teachers to be experts is both unreasonable and unnecessary. After all, expert basketball coaches needn't have been expert basketball players and expert violin teachers needn't have been concertmasters. Of course, teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves, but care should be taken

not to confuse *proficiency* in teaching students to solve problems with *expertise* as problem solvers.

Claim 4. Problem solving isn't always a high-level cognitive activity.

A fourth observation is that the description of problem solving I have given above blurs the distinction between problem solving and other types of mathematical activity—I have more to say about this blurring later. The distinctions that led historically to the isolation of mathematical problem solving as a research focus from other areas of study and the subsequent distinctions that resulted from this isolation are due in part to strong traditions of disciplinary boundaries (Lester & Kehle, 2003).⁴ This isolation led to subsuming mathematical understanding under problem solving. But, the inverse makes more sense; that is, to subsume problem solving (and problem posing) under mathematical understanding and, hence, under mathematical activity. By so doing, emphasis is placed on several other constructs that are important in being able to do mathematics — e.g., model building, generation of representations, constructing patterns of inference — that too often are not considered when problem solving is isolated from other forms of mathematical activity.

Claim 5. Research tells us something about problem-solving instruction, but not nearly enough.

Although research on mathematical problem solving has provided some valuable information about problem-solving instruction, we haven't learned nearly enough (but see also the very last section of this article). In a paper I co-authored about 20 years ago,

⁴ Indeed, some years ago, one mathematics education researcher asked me why I (and most other problem-solving researchers) studied problem solving in isolation from learning specific mathematics concepts and processes. I had no good answer; she was correct and I couldn't provide a compelling reason why we did so. Today, I think the reason stems from our reliance on cognitive science for guidance in developing our research agendas and methods.

my co-author and I identified four reasons for this unfortunate state of affairs: (1) relatively little attention to the role of the teacher in instruction; (2) too little concern for what happens in real classrooms; (3) a focus on individuals rather than small groups or whole classes; and (4) the largely atheoretical in nature of problem-solving research (Lester & Charles, 1992). I have discussed the fourth reason elsewhere (Lester, 2005), so will not discuss it here. Instead, let me comment on the other three reasons. (Interested readers may wish to read the provocative analysis of the state of mathematical problem-solving research written by Lesh and Zawojewski (2007). In their analysis they take issue with the nature and direction of nearly all the research over the past 50 years.)

The role of the teacher. More than twenty-five years ago, Silver (1985) pointed out that the typical research report might have described in a general way the instructional method employed, but rarely was any mention made of the teacher's specific role. Some progress has been made since then (see, e.g., the edited volumes by Lester and Charles (2003) and Schoen and Charles (2003) and the review by Schoenfeld (1992)). But, as useful as these efforts have been, they fall short of what is needed. Instead of simply considering teachers as agents to effect certain student outcomes, their role should be viewed as one dimension of a dynamic interaction among several dimensions of a system involving: the role of the teacher, the nature of classroom tasks, the social culture of the classroom, the use of mathematical tools as learning supports, and issues of equity and accessibility. Changing any of the dimensions of this system requires parallel changes in each of the other dimensions (Hiebert et al., 1997).

Observations of real classrooms. Several years ago, my colleague, Randy Charles, and I conducted a large-scale study of the effectiveness of an approach to

problem-solving instruction based on ten specific teaching actions (Charles & Lester, 1984). The research involved several hundred fifth and seventh grade students in more than 40 classrooms over the period of one full school year. The results were encouraging: students receiving the instruction benefited tremendously with respect to several key components of the problem-solving process. However, despite the promise of our instructional approach, the conditions under which the study was conducted did not allow us to make extensive, systematic observations of classrooms. Ours is not an isolated instance. In particular, there has been a lack of descriptions of teachers' behaviors, teacher-student and student-student interactions, and the type of classroom atmosphere that exists. It is vital that such descriptions be compiled if there is to be any hope of deriving sound prescriptions for teaching problem solving. In the final section of this article I present a framework for research that, if used, might provide the sorts of rich, detailed descriptions I think we need.

Focus on individuals rather than groups or whole classes. Throughout most of the history research in mathematical problem solving (dating back about 50 years) the focus has been on the thinking processes used by individuals as they solve problems or as they reflect back on their work solving problems. When the goal of research is to characterize the thinking involved in a process like problem solving, a microanalysis of individual performance seems appropriate. However, when our concerns are with classroom instruction, we should give attention to groups and whole classes. To be sure, small groups can serve as an appropriate environment for research on teaching problem solving, but the research on problem-solving instruction cannot be limited to the study of small groups. In order for the field to move forward, research on teaching problem

solving needs to examine teaching and learning processes for individuals, small groups, and whole classes.

A Model of Complex Mathematical Activity⁵

In addition to the lack of attention to the role of the teacher in real classrooms and the focus on individuals rather than whole classes, the relative ineffectiveness of instruction to improve students' ability to solve problems can be attributed to the fact that problem solving has often been conceptualized in a simplistic way. This naive perspective has two levels or "worlds": the everyday world of things, problems, and applications of mathematics and the idealized, abstract world of mathematical symbols, concepts, and operations. In this naive perspective, the problem-solving process typically has three steps. Beginning with a problem posed in terms of physical reality, the problem solver first translates the problem into abstract mathematical terms, and then operates on this mathematical representation in order to come to a mathematical solution of the problem. This solution is then translated into the terms of the original problem. According to this view, mathematics may be, and often is, learned separately from its applications and (too often) with no attempt to connect new mathematics concepts to old ones. Teachers who adhere to this perspective are very concerned about developing skillfulness in translating (so-called) real-world problems into mathematical representations and vice versa. However, these teachers tend to deal with problems and applications of mathematics only after the mathematical concepts and skills have been introduced, developed, and practiced. Many of the "problems" found in textbooks often

⁵ The discussion in this section is excerpted with only minor revision from Lester and Kehle (2003).

can be solved exactly as this naive perspective indicates. But for more challenging, substantive problems, the problem solver cannot simply apply a previously learned procedure to solve the problem. In addition to translation and interpretation, these problems also demand more complex processes such as planning, selecting strategies, identifying sub-goals, choosing or creating appropriate representations, conjecturing, and verifying that a solution has been found. For non-routine tasks, a different type of perspective is required, one that emphasizes the making of new meanings through construction of new representations and inferential moves (refer back to the description of problem solving Kehle and I (2003) have proposed).

The new perspective, like the previous one, also contains two levels representing the everyday world of problems and the abstract world of mathematical concepts, symbols and operations. In this perspective, however, the mathematical processes in the upper level are "under construction" (i.e., being learned, as opposed to already learned; coming to be understood, as opposed to being understood) and the most important features are the relationships between steps in the mathematical process (in the mathematics world) and actions on particular elements in the problem (in the everyday world). It is in the forging of these relationships that results in the meaning making that is central to mathematical activity of all kinds. At times the problem solver is learning to make abstract written records of the actions that are understood in a concrete setting. This involves the processes of abstraction and generalization. And, at other times the problem solver attempts to connect a mathematical process to the real-world actions that the mathematical process represents. Also, a problem solver who had forgotten the details of a mathematical procedure would attempt to reconstruct that procedure by

imagining the corresponding concrete steps in the world in which the problem was posed. As a result, typically the problem solver moves back and forth between the two worlds—the everyday problem world and the mathematical world—as the need arises.

But, although this perspective is an improvement over the original, it too falls short of what is needed because it does not account for many of the most important actions (both cognitive and non-cognitive) involved during real problem solving. Even the modified perspective regards problem solving as somehow being different from other sorts of mathematical activity. In my view, what is needed is to subsume problem solving within a much broader category, "mathematical activity," and to give a prominent role to the metacognitive activity engaged in by the individual or group.⁶

Figure 1 below depicts mathematical behavior as a complex, involved, multiphase process that begins when an individual, working in a complex context (Box A), poses (or is given) a specific task to solve (the solid arrow between A and B). To start solving the task, the individual simplifies the complex setting by identifying those concepts and processes that seem to bear most directly on the problem. This simplifying and problem posing phase involves making decisions about what should be attended to and what can be ignored, developing a sense about how the essential concepts are connected, and results in a realistic representation of the original situation. This realistic representation is a model of the original context from which the problem was drawn because it is easier to examine, manipulate, and understand than the original situation.

Next comes the abstraction phase (solid arrow from B to C), which introduces mathematical concepts and notations (albeit perhaps idiosyncratic). This abstraction

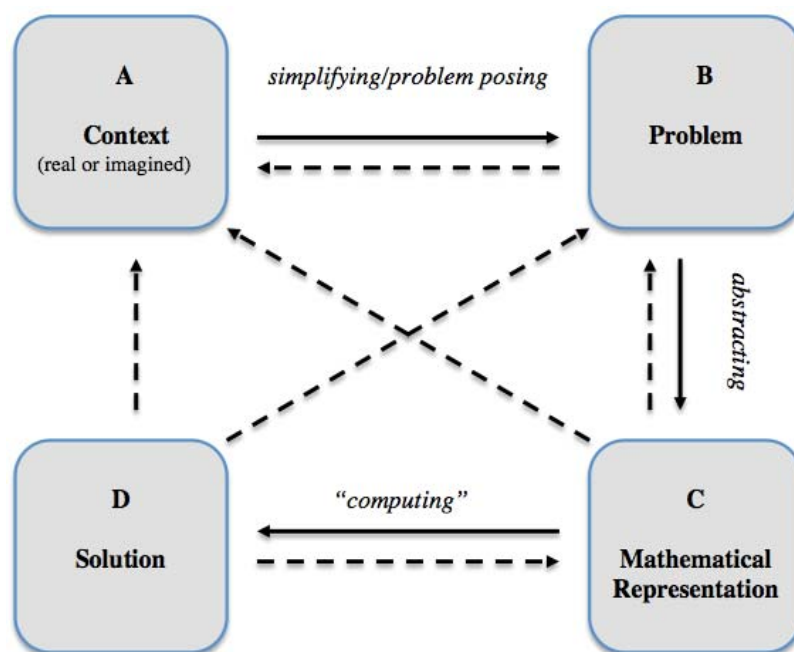
⁶ I should point out that this depiction is a representation of ideal, rather than typical, performance during an individual's work on some mathematical task. It is ideal in the sense that it denotes key actions in which the individual *should* engage in order to obtain acceptable results.

phase involves the selection of mathematical concepts to represent the essential features of the realistic model. Often the abstraction phase is guided by a sense of what a given representation makes possible in the subsequent computation phase. The explicit representation of the original setting and problem in mathematical symbolism constitutes a mathematical representation of both the setting and the task/problem.

Once a problem solver has generated a mathematical representation of the original situation, the realistic problem now becomes a specific mathematical problem related to the representation. This mathematical problem acquires a meaning all its own, becoming an isolated, well-defined mathematical problem (Box C).

The third phase of the process (from C to D) involves manipulating the mathematical representation and deducing some mathematical conclusions—depicted in the figure by the “computing” arrow. During this phase, the person draws upon her or his store of mathematical facts, skills, mathematical reasoning abilities, and so forth. For example, the problem might call for a solution of a system of equations and solving this system of equations does not depend on the original context of the initial problem. The final phase (from D to A, D to B, and D to C), then, should involve the individual in comparing the conclusions/results obtained with the original context and problem, as well as with the mathematical representation (refer to the dashed arrows between boxes). But, the act of comparing does not occur only after conclusions are drawn and a solution is obtained. Rather, it might take place at any time and at any point during the entire process. Indeed, this regular and continual monitoring—metacognitive activity—of one's work is a key feature of success on complex mathematical tasks. In general, the act of comparing the current state of one's work, thinking, and decisions denotes how complex

mathematical activity can be. The degree to which the individual chooses to compare her or his current state with earlier states can be considered a determinant of task complexity and, in fact, is the primary way to distinguish “routine” from “non-routine” tasks (i.e., problems). For example, performing routine calculations using whole numbers typically requires little comparing, whereas work on more complex tasks might necessitate quite a lot of comparing throughout ones work on it. In brief, then, the degree to which a task can be considered problematic can be determined by the amount of “comparing” involved.



Note. Dashed arrows denote “comparing”

Figure 1. A model of complex mathematical activity

To sum up, what Kehle and I have proposed is a blurring of the distinction between problem solving and other mathematical activity emerging from research on mathematical problem solving and constructivist thinking about learning. Furthermore, we have proposed a blurring of task, person, mathematical activity, non-mathematical

activity, learning, applying what has been learned, and other features of mathematical problem solving. A consequence of this blurring is that it necessitates some rethinking about the proficiencies mathematics teachers need. In the next section I discuss these proficiencies in light of the preceding discussions.

Proficiencies for Teaching Mathematics

The debate over the merits of direct (explicit) instruction versus constructivist instruction has been raging for at least 50 years and any consideration of the mathematical proficiencies needed for teaching mathematics must be made in view of this debate. More specifically, the identification of such proficiencies must take into account the assumptions that are being made about the nature of mathematics learning and instruction, as well as about instructional goals. For example, a proponent of direct instruction (e.g., Kirschner, Sweller & Clark, 2006) might view learning as simply a matter of making a change in students' long-term memories. But for a constructivist teacher, in addition to making a change in students' long-term memories, learning involves much more. Constructivist teachers are concerned with (among other things) how to help students select and use good procedures for solving problems (Gresalfi & Lester, 2009). Clearly, these quite different perspectives on what mathematics learning involves will have a tremendous influence on what teachers must be able to do in their classrooms (i.e., what proficiencies they need). Furthermore, there is the matter of the teacher's goals. If problem solving is intended as the *end* result of instruction, one set of proficiencies for teaching is needed, but if problem solving is the *means* through which mathematical concepts, processes, and procedures are learned, then a different set of

proficiencies may be called for. For example, the teacher for whom problem solving is a *means*, would likely need to be very proficient at listening to and observing students as they work on mathematical tasks (Davis, 1997; Yackel, 2003). And, quite naturally, listening to students would play a much less important role for a teacher who mostly lectures. Put more directly, consideration of how to include problem solving in a mathematical-proficiencies-for-teaching framework should be done in view of the assumptions the teacher makes about the nature of mathematics learning and the goals of instruction.

But, what of the proficiencies needed to help students learn how to solve problems? One consequence of subsuming problem solving under the broader heading “mathematical activity,” is that it becomes more difficult to specify a precise set of proficiencies teachers need. To illustrate, consider the task “Which is more $\frac{2}{3}$ or $\frac{2}{5}$?” Does this task involve any problem solving on the part of the student? Maybe, maybe not! Of course, one can “cross multiply” to determine that $\frac{2}{3}$ is more (or use some other previously learned procedure), or one may have had sufficient experience with fractions to simply “know” that $\frac{2}{3}$ is more. In these instances, one could argue that no problem solving is going on. But suppose you are a 3rd grader⁷ who does not know of any procedures to decide which is more. Without a prescribed method of attack, this task might be used to help you better understand the meanings of numerator and denominator and also help you see how useful one half can be as a fraction benchmark (Van de Walle, 2003). This is at the heart of what it means to teach *via* problem solving. But, what

⁷ One needn't be a 3rd grader to find this task problematic. Over the past 4 years I have been tutoring unemployed adults who hope one day to pass the US high school equivalency exam (GED). Almost to a person, they do not know how to solve this task when they begin to study with me.

proficiencies must teachers have who subscribe to a teaching *via* problem solving approach? Of course, they must be adept at selecting good problems, at listening and observing, at asking the right questions, at knowing when to prod and when to withhold comment, as well as a host of other actions⁸. These actions make up what Moore (1995) has called the “craft of teaching.” Moore’s “image of a [proficient] teacher is that of a skilled craftworker, a master machinist say, who knows exactly what she must do, brings the tools she needs, does the work with straightforward competence, and takes pleasure in a job well done. She does her work right every day, and every day's work fits the larger plan of her project” (p. 5). For Moore a craft is a “collection of learned skills accompanied by experienced judgment” (p. 5). So, the question is “How does one become a craftsman?”

Thirty years ago, Randy Charles and I wrote a book in which we laid out an instructional plan for teachers to follow in order to be effective in teaching students how to solve mathematics problems (Charles & Lester, 1982). The plan focused on three phases of instruction—Before, During, and After—and was organized around 10 “teaching actions.” Since then, the three phases have appeared in different guises in various American elementary and middle school textbook series (e.g., the middle grades *Connected Mathematics* series organizes activities around Launch, Explore, and Summarize (Pearson Education Inc. 2011)). The features of our plan that most clearly distinguish it from more “traditional” instructional plans have to do with the teacher’s role and the nature of the classroom environment. However, this is far from sufficient; knowing about the teaching actions is simply not enough! In addition to knowing *what* to

⁸ I, and various collaborators over the years, have used the word (teaching) “actions” to refer to what the teacher does during the act of teaching.

do, the teacher must also know *when* to do it and what the implications might be of the action taken. In particular, teachers must be adept at: (1) designing and selecting tasks and activities, (2) listening to and observing students as they engage with problem-solving activities, (3) making sure that instructional activities remain problematic for students, (4) focusing on the methods students use to solve problems and being familiar with problem-solving methods (e.g., heuristics, strategies) that are accessible to students, and (5) being able to tell the right thing at the right time (cf., Cai, 2010; DiMatteo & Lester, 2010; Hiebert, 2003). Moreover, teachers and students share responsibility for creating and maintaining a classroom atmosphere that is conducive to exploring and sharing ideas, cooperating with each other, and risk taking (Stephan & Whitenack, 2003). Thus, for me, in addition to myriad other knowledge and skills, a proficient mathematics teacher must be skillful at —

- Designing and selecting appropriate tasks for instruction
- Making sense of and taking appropriate actions after listening to and observing students as they work on a task
- Keeping tasks appropriately problematic for students
- Paying attention to and being familiar with the methods students use to solve problems
- Being able to take the appropriate action (or say the right thing) at the right time
- Creating a classroom atmosphere that is conducive to exploring and sharing.

To be sure, teachers who have command of these and related teaching actions and who also have considerable mathematics content knowledge appropriate for the level at

which they are teaching might be considered craftsmen. But, I think what separates a craftsman from others has to do with the amount of planning and reflection that he or she has done prior to and after instruction. Unfortunately, even though it seems clear to me that the type and amount of planning a teacher does have tremendous impact on what happens during instruction, teacher planning has been largely ignored as a factor of importance in research on problem-solving instruction. Indeed, in most studies teacher planning has not even been considered because the teachers in these studies have simply implemented a plan that had been predetermined by the researchers, not the teachers. Furthermore, it is no longer warranted to assume that the planning decisions teachers make are driven totally by the content and organization of the textbooks used and, therefore, need not be considered as an object of research. The challenge, then, is to determine ways to provide these teachers with opportunities to acquire the proficiencies needed to become craftsmen; opportunities that in my view are best provided through apprenticeship experiences in their real-world context and situation (Collins, Brown, & Newmann, 1990). To date, too little attention has been paid to studying the design and implementation of apprenticeship programs for teacher education. This lack of attention is unfortunate because I think apprenticeship training is the approach most likely to result in highly proficient teachers—that is, teacher craftsmen skilled at teaching mathematics *via* problem solving.

A Framework for Research on Problem-solving Instruction

Twenty years ago, Randy Charles and I developed a framework for research on problem –solving instruction that was a synthesis of previous conceptualizations of

teaching in general and mathematics teaching in particular (Lester & Charles, 1992).

Unfortunately, to my knowledge, other researchers have not adopted this framework. I still think it could serve us well in designing research on problem-solving instruction and I bring this article to a close by offering a slightly-modified sketch of what it consists of.

The framework is comprised of four broad categories of factors that we consider essential in the conceptualization and design of research studies: (1) Non-classroom factors, (2) Teacher planning, (3) Classroom processes, and (3) Instructional outcomes. Of course, the categories overlap and the factors within each interact both within and across categories.

Category 1: Non-classroom Factors

What goes on in a classroom is influenced by many things that exist or take place apart from actual classroom instruction. For example, the teacher's and students' knowledge, beliefs, attitudes, emotions, and dispositions all play a part in determining what happens during instruction. Furthermore, the nature of the tasks used as well as the contextual conditions present outside the classroom also affect instruction (e.g., course schedules, school structures). There are six types of factors: teacher presage characteristics, student presage characteristics, teacher knowledge and affects, student knowledge and affects, tasks features, and contextual (situational) conditions.

Teacher and student presage characteristics. These are characteristics of the teacher and students that are not amenable to change but which may be examined for their effects on classroom instruction. In addition, presage characteristics serve to describe the individuals involved. Typically, in experimental research these characteristics have potential for control by the researcher. But, awareness of these

characteristics can be useful in non-experimental research as well by helping researchers make sense of what they are observing. Among the more prominent presage characteristics are age, sex, and previous experience (e.g., teaching experience, previous experience with the topic of instruction). Factors such as previous experience may indeed be of great importance as we learn more about the ways knowledge teachers glean from experience influences practice.

Teacher and student knowledge and affects. The teacher's and students' knowledge (both cognitive and metacognitive) and affects (including beliefs) can strongly influence both the nature and effectiveness of instruction. As a category, these teacher and student traits are similar to, but quite different from, presage characteristics. The similarity lies in the potential for providing clear descriptions of the teacher and students. The difference between the two is that affects and knowledge may change, in particular as a result of instruction, whereas presage characteristics cannot.

Task features. Task features are the characteristics of the tasks used for instructional or assessment purposes. Historically, at least five types of features serve to describe tasks: syntax, content, context, structure, and process (see Goldin & McClintock, 1984). *Syntax features* refer to the arrangement of and relationships among words and symbols in a task. *Content features* deal with the mathematical meanings in the problem. Two important categories of content features are the mathematical content area (e.g., geometry, probability) and linguistic content features (e.g., terms having special mathematical meanings such as "less than," "function," "squared"). *Context features* are the non-mathematical meanings in the task statement. Furthermore, context features describe the problem embodiment (representation), verbal setting, and the format of the

information given in the problem statement. *Structure features* can be described as the logical-mathematical properties of a task. Structure features are determined by the particular representation that is chosen for a problem. For example, one student may choose to represent a task in terms of a system of equations, while another student may represent the same problem in terms of some sort of guessing process. Finally, *process features* represent something of an interaction between task and student. That is, although problem-solving processes (e.g., heuristic reasoning) typically are considered characteristics of the student, it is reasonable to suggest that a problem may lend itself to solution via particular processes. A consideration of task process features can be very informative to the researcher in selecting tasks for both instruction and assessment.

Contextual conditions. These factors concern the conditions external to the teacher and students that may affect the nature of instruction. For example, class size is a condition that may directly influence the instructional process and with which both teacher and students must contend. Other obvious contextual conditions include textbooks used, community ethnicity, type of administrative support, economic and political forces, and assessment programs. Also, since instructional method provides a context within which teacher and student behaviors and interactions take place, it too can at times be considered a factor within this category. I should add that that these six areas of consideration do not necessarily cover all possible influences; it is likely that there are other influences that may be at least as important as the ones I have mentioned. Rather, my intent is to point out the importance of paying attention to the wide range of factors that can have an impact on what takes place during instruction.

Category 2: Teacher Planning

Teacher planning is not clearly distinct from the other categories, in fact, it overlaps each of them in various ways. Of particular interest for research are the various decisions made before, during and as a result of instruction about student presage characteristics, instructional materials, teaching methods, classroom management procedures, evaluation of student performance, and amount of time to devote to particular activities and topics. Unfortunately, teacher planning has been given too little attention as a factor of importance in problem-solving instruction research. Indeed, in most studies teacher planning has not even been considered because the teachers in these studies have simply implemented a plan that had been predetermined by the researchers, not the teachers. Furthermore, it is no longer warranted to assume that the planning decisions teachers make are driven totally by the content and organization of the textbooks used and, therefore, need not be considered as an object of research. A teacher's behavior while teaching either *for* or *via* problem solving is certainly influenced by the teacher's knowledge and affects. However, some of this behavior is likely to be determined by the kinds of decisions the teacher makes prior to entering the classroom. For example, a teacher may have planned to follow a specific sequence of teaching actions for delivering a particular problem-solving lesson knowing that the exact ways in which these teaching actions are implemented evolve situationally during the lesson. Or, if the knowledge teachers use to plan instruction is knowledge gleaned from previous instructional episodes, then we would search for those cases that significantly shape the craft knowledge teachers use as a basis for planning and action. Future research should consider how teachers go about planning for problem-solving instruction and how the decisions made during planning influence actions during instruction.

Category 3: Classroom Processes

Classroom processes include the host of teacher and student actions and interactions that take place during instruction. Four dimensions of classroom processes are apparent: teacher knowledge and affects; teacher behaviors; student knowledge and affects; and student behaviors.

Both the teacher's and the students' thinking processes and behaviors during instruction are almost always directed toward achieving a number of different goals, sometimes simultaneously. For example, during a lesson the teacher may be assessing the appropriateness of the small-group arrangement that was established prior to the lesson, while at the same time trying to guide the students' thinking toward the solution to a problem. Similarly, a student may be thinking about what her classmates will think if she never contributes to discussions and at the same time be trying to understand what the task confronting her is all about. In our framework, we have restricted consideration to what the teacher thinks about and does to facilitate the students' thinking and what the student thinks about and does to solve a problem. We have not attempted to include a complete menu of objects or goals a teacher might think about during instruction.

Teacher knowledge and affects. These processes include those attitudes, beliefs, emotions, cognitions and metacognitions that influence, and are influenced by, the multitude of teacher and student behaviors that occur in the classroom during instruction. In particular, this dimension is concerned with the teacher's thinking and affects while facilitating students' attempts to understand a task, develop a plan for solving it, carry out the plan to obtain an answer, and look back over the solution effort.

Teacher behaviors. A teacher's knowledge and affects that operate during instruction give rise to the teacher's behaviors, the overt actions taken by the teacher during problem-solving instruction. Specific teacher behaviors can be studied with regard to use (or non-use) as well as quality. The quality of a teacher behavior can include, among other things, the correctness of the behavior (e.g., correct mathematically or correct given the conditions of the problem), the clarity of the action (e.g., a clear question or hint), and the manner in which the behavior was delivered (e.g., the verbal and nonverbal communication style of the teacher).

Student knowledge and affects. Similar to the teacher, this subcategory refers to the knowledge and affects that interact with teacher and student behaviors. The concern here is with how students interpret the behavior of the teacher and how the students' thinking about a problem, their affects, and their work on the problem affects their own behavior. Also of concern here is how instructional influences such as task features or contextual conditions directly affect a student's knowledge, affects, and behaviors.

Student behaviors. These behaviors include the overt actions of the student during a classroom problem-solving episode. By restricting our attention to the problem-solving phases mentioned earlier, we can identify several behaviors students might exhibit as they work on a task.

Category 4: Instructional Outcomes

The fourth category of factors consists of three types of outcomes of instruction: *student outcomes*, *teacher outcomes*, and *incidental outcomes*. Most instruction-related research has been concerned with short-term effects only. Furthermore, transfer effects,

effects on attitudes, beliefs, and emotions, and changes in teacher behavior have been considered only rarely.

Student outcomes. Both immediate and long-term effects on student learning are included in this category, as are transfer effects (both near and far transfer). Illustrative of a student outcome, either immediate or long-term, is a change in a student's skill in implementing a particular problem-solving strategy (e.g., guess and check, working backwards). An example of a transfer effect is a change in students' performance in solving non-mathematics problems as a result of solving only mathematics problems. Also, of special importance is the consideration of changes in students' beliefs and attitudes about problem solving or about themselves as problem solvers and the effect of problem-solving instruction on mathematical skill and concept learning; for example, how is computational skill affected by increased emphasis on the thinking processes involved in solving problems?

Teacher outcomes. Teachers, of course, also change as a result of their instructional efforts. In particular, their attitudes and beliefs, the nature and extent of their planning, as well as their classroom behavior during subsequent instruction are all subject to change. Each problem-solving episode a teacher participates in changes her or his craft knowledge. Thus, it is reasonable to expect that experience affects the teacher's planning, thinking, affects, and actions in future situations.

Incidental outcomes. Increased performance in science (or some other subject area) and heightened parental interest in their children's school work are two examples of possible incidental outcomes. Although it is not possible to predetermine the relevant

incidental effects of instruction, it is important to be mindful of the potential for unexpected “side effects.”

Research on teaching in general points to the important role a teacher's knowledge and affects play in instruction. Questions such as the following need to be investigated : What knowledge (in particular, content, pedagogical, and curriculum knowledge) do teachers need to be effective as teachers of problem solving? How is that knowledge best structured to be useful to teachers? How do teachers' beliefs about themselves, their students, teaching mathematics, and problem solving influence the decisions they make prior to and during instruction?

The forgoing analysis of factors to be considered for research on problem-solving instruction is intended as a general framework for designing investigations of what actually happens in the classroom during instruction. As I mentioned earlier, there may be other important factors to be included in this framework and that certain of the factors may prove to be relatively unimportant. Notwithstanding these possible shortcomings, this framework could serve as a step in the direction of making research in the area more fruitful and relevant.

A Final, More Positive Note

I do not intend for my remarks to give the impression that I think mathematical problem solving research has not amounted to much during the past 40 years or that current research efforts are misguided. Indeed, quite the opposite is the case! Several important principles have slowly emerged from the research since the early 1970s. I end this article by listing these principles without comment: each principle could serve as the

basis for an article or monograph. My hope is that this list, like much of the rest of my article, will stimulate discussion among those who are interested in pursuing a research agenda that includes problem solving at its core.

1. *The prolonged engagement principle.* In order for students to improve their ability to solve mathematics problems, they must engage in work on problematic tasks on a regular basis, over a prolonged period of time.
2. *The task variety principle.* Students will improve as problem solvers only if they are given opportunities to solve a variety of types of problematic tasks (in my view, principles 1 and 2 are the most important of the seven).
3. *The complexity principle.* There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, heuristics, skills, control processes, and awareness of one's own thinking develop concurrently with the development of an understanding of mathematical concepts. (This principle tells us that problem-solving ability is best developed when it takes place in the context of learning important mathematics concepts.)
4. *The systematic organization principle.* Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.
5. *The multiple roles for the teacher principle.* Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a

facilitator of students' metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver.

6. *The group interaction principle.* The standard arrangement for classroom instructional activities is for students to work in small groups (usually groups of three or four). Small group work is especially appropriate for activities involving new content (e.g., new mathematics topics, new problem-solving strategies) or when the focus of the activity is on the process of solving problems (e.g., planning, decision making, assessing progress) or exploring mathematical ideas.
7. *The assessment principle.* The teacher's instructional plan should include attention to how students' performance is to be assessed. In order for students to become convinced of the importance of the sort of behaviors that a good problem-solving program promotes, it is necessary to use assessment techniques that reward such behaviors.

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Framing the use of computational technology in problem solving approaches

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Abstract: Mathematical tasks are key ingredient to foster teachers and students' development and construction of mathematical thinking. The use of distinct computational tools offers teachers a variety of ways to represent and explore mathematical tasks which often extends problem solving approaches based on the use of paper and pencil. We sketch a framework to characterize ways of reasoning that emerge as result of using computational technology to solve a task that involves dealing with variation phenomena.

Keywords: problem solving, framework, the use of computational tools.

Introduction

It is widely recognized that the use of computational technology offers teachers and students different ways to represent and explore mathematical problems or concepts. There is also evidence that different tools might offer learners different opportunities to think of problems in order to represent, explore, and solve those problems. What tools and how should teachers integrate them in their teaching environments? What instructional goals should teachers aim with the use of technology? In accordance to Hegedus & Moreno-Armella (2009) "technology is here to transform thinking, and not to serve as some prosthetic device to prop up old styles of pedagogy or curriculum standards" (p. 398). Thus, it becomes important for teachers to discuss approaches to use technology in order to guide their students to develop ways of thinking that favour their comprehension of mathematical concepts and problem solving experiences. In particular, teachers should discuss the extent to which the use of the tools helps them represent and

explore mathematical tasks in ways that enhance and complement problem solving processes that rely on the use of paper and pencil environment. The use of computational tools in learning scenarios implies that teachers need to pay attention to and reflect upon aspects that involve:

- (a) The process shown by the subject to transform the artefact (material object) into an instrument to represent, to comprehend mathematical ideas, and to solve problems;
- (b) The type of tasks used to foster students' mathematical thinking;
- (c) The ways of reasoning exhibited by the subjects during problem solving activities;
- (d) The role of teachers during problem solving sessions; and in general,
- (e) The structure and dynamics of scenarios that promote the use of different tools to learn mathematics and solve problems.

We introduce a pragmatic framework for teachers to organize learning activities that promote the systematic use of technology. The framework provides teachers with the opportunity to discuss aspects related to the presentation and exploration of mathematical tasks through the use of a dynamic software in problem solving environments. The aim is to identify and reflect on possible routes that teachers or researchers can follow to structure and organize problem-solving activities that enhance the use of technology with the purpose of furthering mathematics learning. We highlight a set of questions that teachers can think of as a way to delve into the problem through the use of technology.

To this end, we chose a generic¹ task that involves a variation phenomenon to illustrate how the use of the tool fosters an inquiring approach to make sense of the posed statement and to promote different ways of reasoning to explore and solve the task (NCTM, 2009). Thus, focusing on ways to represent a variation phenomenon through the tool demands that teachers identify, express, and explore mathematical relationships in terms of visual, numeric, graphic, and algebraic approaches. “Conceptualization of invariant structures amidst changing phenomena is often regarded as a key sign of knowledge acquisition” (Leung, 2008, p. 137). Thus, teachers need to work on tasks where the use of the tools provides them a set of affordances to identify and perceive what parameters vary and what are maintained invariant within the problem structure.

Background and Rationale

Lester (2010) quotes the online *Encarta World English Dictionary* to define a framework: “a set of ideas, principles, agreements, or rules that provides the basis or the outline for something that is more fully developed at a later stage” (p. 60). Our notion of framework includes initial arguments that describe patterns associated with the use of a dynamic software in mathematical problem solving. “A framework tells you what to look at and what its impact might be” (Schoenfeld, 2011, p. 4). It is a pragmatic framework that consists of episodes that could help practitioners re-examine and contrast those frameworks that explain learners competences exhibited in paper and pencil environments. It becomes a scaffolding tool to reflect on issues related to the use of tools in learning scenarios.

¹ Generic in the sense that the task represents a family of tasks where it is possible to explore or examine optimization behaviours of the parameters involved in the task.

Schoenfeld (1985) proposed a framework to explain students' problem solving behaviours in terms of what he calls basic resources, cognitive and metacognitive strategies, and students' beliefs. Schoenfeld's framework came from analyzing and categorizing experts and students' problem solving approaches that involve mainly the use of paper and pencil tools. What happens when subjects use systematically computational tools to make sense of problem statement, represent, explore and solve problems? We argue that the use of technology introduces new information to characterize the problem solver's proficiency. For instance, one of the tasks used by Schoenfeld involves asking the students to draw with straightedge and compass a circle that is tangent to two intersecting lines where one point of tangency is a given P on one line. Schoenfeld reports that students formulated several conjectures about the position of the centre of such a tangent circle: (a) The centre of the tangent circle C is the midpoint of the line segment between P and the point Q , where P and Q are equidistant from the point of intersection V (Figure 1a); (b) The centre of the circle is the midpoint of segment of the circular arc from P to Q that has centre V and radius $|PV|$ (Figure 1b), etc. (Schoenfeld, 2011, p. 31).

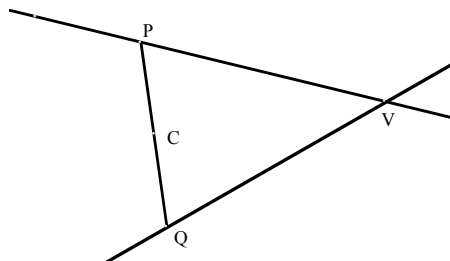


Figure 1a: A student conjecture

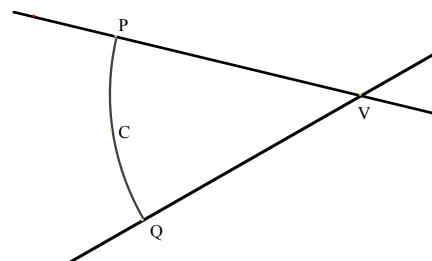


Figure 1b: Another student conjecture

Schoenfeld stated that the students picked up the straightedge and compass, tried out their conjecture, and either accepted or rejected it on the basis of how good their

drawing looked. With the use of a dynamic software “good drawing” doesn’t depend on subject’s skills to manage the straightedge and compass; rather, the tool provides the affordances (precision of drawings, parameter movement, quantification of parameters, loci, etc.) to deal or explore conjectures. That is, the use of a dynamic software provides teachers ways to initially visualize and test empirically conjectures and, they often access or develop relevant knowledge needed to verify and prove those conjectures (Moreno-Armella & Sriraman, 2005; Santos-Trigo, 2010). For example, in Figures 2a and 2b, the dotted circle drawn with the software provides elements to reject the corresponding conjectures. Thus, the use of the tool offers relevant information to characterize and foster the students’ problem solving competences. For example, students can explore visually that the centre of the tangent circle lies on the perpendicular line to line PV at P (Figure 2c) and use that information to construct a formal approach based on properties embedded in that visual approach.

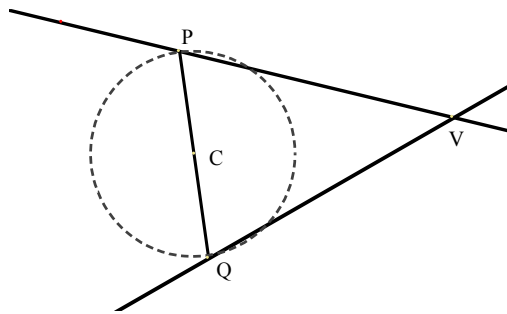


Figure 2a: A student conjecture

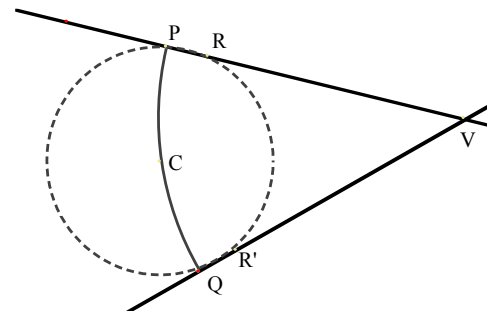


Figure 2b: Another student conjecture

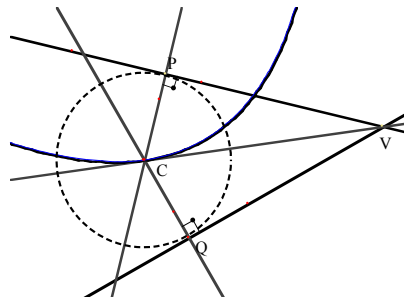


Figure 2c: The centre of the tangent circle lies on the perpendicular to PV that passes through P.

We argue that practitioners interested in using computational tools in their learning activities can find in the problem solving episodes described in the next section a quick reference to the type of mathematical discussions that might emerge during the problem solving sessions. In addition, the episodes might provide directions to structure a lesson plan where empirical, visual, graphic, and formal approaches can be considered to organize a didactic route. We contend that the episodes can provide relevant information that relates to what Jackiw and Sinclair (2009) call first and second order effects of the use of the software (referring to *The Geometer's Sketchpad*) in learning. “First-order effects are a direct consequences of the affordance of the environment; second-order effects are then a consequence of these consequences, and usually relate to changes in the way learners think, instead of changes in what learners *do*” (p. 414). That is, teachers could use the affordances associated with the software to encourage their students to think of novel ways to represent dynamically problem situations. Software’ affordances (dragging, finding loci, quantifying parameters, etc.) provide ways to observe changes or invariance of involved parameters. As a consequence, the use of the tool allows the problem solver to develop ways of reasoning to examine parameters behaviours that emerge as a result of moving mathematical objects within the task representation or configuration. Heid & Blume (2008) stated “[t]he nature of a mathematical activity depends not only on the mathematical demands of the task but also on the process of the task as constructed by the doer” (p. 425). Thus, teachers with the

use of the tool might guide their students to think about the problem in different ways and to discuss concepts and processes that appear during the exploration of the task.

A problem-solving episodes to deal with phenomena of variation

An example is used to illustrate, in terms of episodes, a route to think of the use of technology to represent and explore the area variation of an inscribed parallelogram. The first episode emphasizes the relevance for the problem solvers to comprehend the statement in order to construct a dynamic representation that can help them visualize parameter behaviours.

The task

Given any triangle ABC , inscribe a parallelogram by selecting a point P on one of the sides of the given triangle. Then from point P draw a parallel line to one of the sides of the triangle. This line intersects one side of the given triangle at point Q . From Q draw a parallel line to side AB of the triangle. This line intersects side AC at R . Draw the parallelogram $PQRA$ (Figure 3). How does the area of inscribed parallelogram $APQR$ behave when point P is moved along side AB ? Is there a position for point P where the area of $APQR$ reaches a maximum value? (Justify).

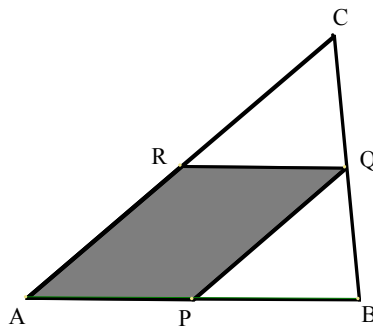


Figure 3: Drawing a parallelogram inscribed into a given triangle.

Comprehension Episode

Polya (1945) identifies the process of understanding the statement of a problem as a crucial step to think of possible ways for solving it. Understanding means being able to make sense of the given information, to identify relevant concepts, and to think of possible representations to explore the problem mathematically. The use of technology could help teachers focus on the construction of a dynamic model as a means to pose and explore questions that lead them to comprehend and make sense of tasks.

The comprehension stage involves questioning the statement and thinking of the use of the tool to make sense and represent the task. For instance, what does “for any given triangle” mean and how this can be expressed through the software?, what information does one need to draw *any* triangle?, are there different ways to inscribe a parallelogram into a given triangle?, and how can one draw a dynamic model of the problem? are examples of questions where the problem solver could rely on the tool to explore and discuss the problem. Thus, a route to answer these questions might involve using *Cabri-Geometry* or *The Geometer’s Sketchpad* to draw triangle ABC (Figure 4) and from P on AB draw a parallel line to CB (instead of AC). This line intersects side AC and from that point of intersection, one can draw a parallel line to AB that intersects BC, thus, the two intersection points and point P and B form an inscribed parallelogram, the problem solver can ask: how is the former parallelogram related to the one that appears in Figure 3? Do they have the same area for the same position of P? How can we recognize that for different positions of point P the area of the parallelogram changes? This problem

comprehension phase is important not only to think of the task in terms of using the software commands, but also to identify and later examine possible variations of the task. For example, how does the area of a family of inscribed parallelograms, generated when P is moved along AB, change (Figure 4)?

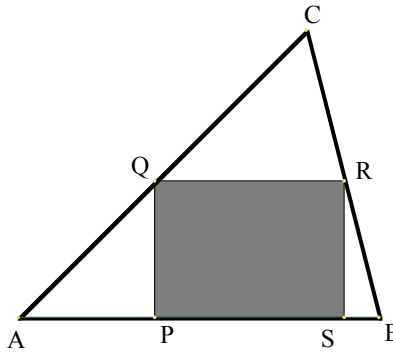


Figure 4: Another way to inscribe a parallelogram in a given triangle.

Comment

Making sense of the problem statement is a crucial step in any problem solving approach. The use of a dynamic software plays an important role in initially conceptualizing the statement as an opportunity to pose and explore a set of questions. That is, the use of the tool demands that the problem solver thinks of the statement in terms of mathematical properties to use the proper software commands to represent and explore the problem (Santos-Trigo & Espinosa-Pérez, 2010). In this case, teachers can work on the task in order to identify task's sketches that can help their students focus their attention to particular concepts or explorations. Of course, the posed questions don't include all possible routes to examine the statement; rather they illustrate an inquiry method to guide the problem solver's reflection.

A Problem Exploration Episode

Teachers can use the software to draw a triangle by selecting three non-collinear points and discuss conditions needed to draw it when for example three segments (instead of three points) are given (the triangle inequality). The use of the software allows moving any vertex to generate a family of triangles. This process broadens the cases for which the problem can be analyzed. Then, they can select a point P on side AB to draw the corresponding parallels to inscribe the parallelogram. With the help of the software it is possible to calculate the area of the parallelogram and observe area values change when point P is moved along side AB . Thus, it makes sense to ask whether there is a position of P in which the area of the inscribed parallelogram reaches either its maximum or minimum value. By setting a Cartesian system (an important heuristic) as a reference and without using algebra, it is possible to construct a function that associates the length of segment AP with the area value of the corresponding parallelogram. Figure 5 shows the graphic representation of that function. The domain of the function is the set of values that represents the lengths of AP when point P is moved along side AB . The range of that function is the corresponding area values of the parallelogram associated with the length AP . This graphic representation can be obtained through the software by asking: What is the locus of point S (the coordinates of point S are length AP and area of $APQR$) when point P moves along the side AB ? It is important to observe that the graphic representation is obtained without defining explicitly the algebraic model of the area change of the parallelogram.

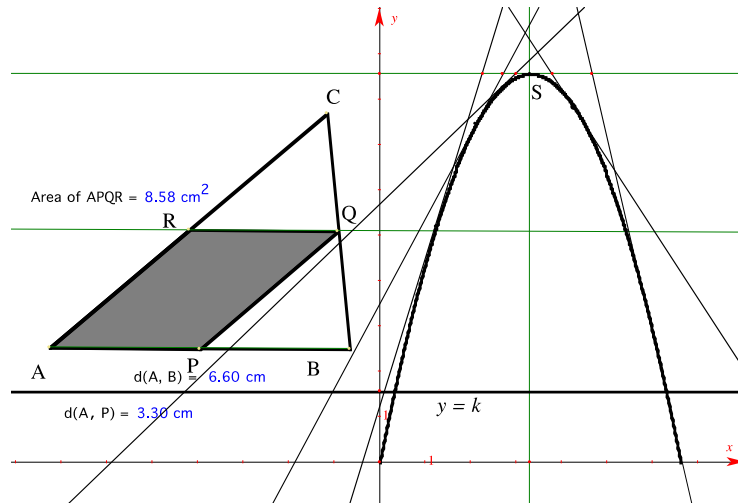


Figure 5: Representation and visual exploration of the problem.

This graphic approach to solve the problem provides an empirical solution. Both visually and numerically it is possible to observe that in the given triangle the maximum area of the inscribed parallelogram is obtained when P is situated at 2.30 cm from point A. At this point, the area value of the parallelogram is 8.56 cm^2 . Based on this information a conjecture emerges: *When P is the midpoint of segment AB, then the corresponding inscribed parallelogram will reach the maximum area value.* Graphically the behaviour of tangent line to the curve behaves at different points can be observed (Figure 5). It can be seen that when the slope of the tangent line to the area graph is positive the function increases, but when the slope is negative the function area decreases.

Are there other ways to inscribe a parallelogram in triangle ABC? Figure 6 shows three ways to draw an inscribed parallelogram and all of them have the same area for different positions of point P. Also, Figure 7 shows that when point P is the midpoint of side AB then triangle ABC can be divided into four triangles with the same areas.

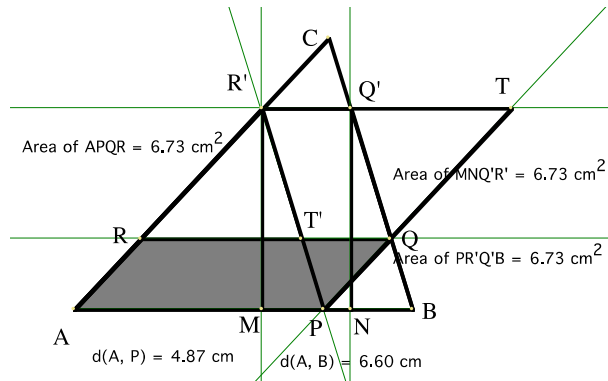


Figure 6: Inscribing three parallelograms in triangle ABC.

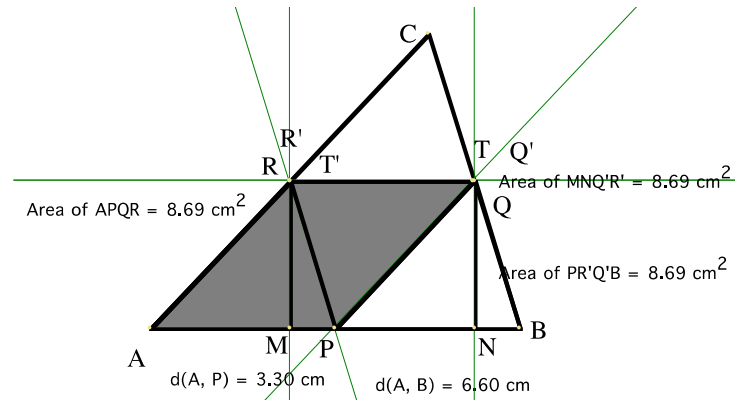


Figure 7: When point P is situated at the midpoint of segment AB, then triangles APR, PQR, PBQ, and RQC have the same area.

From Figures 6 and 7 two conjectures emerge: (i) the three inscribed parallelograms always have the same area for different positions of point P, and (ii) when point P is the midpoint of segment AB, the four triangles always have the same area and the maximum area of the inscribed parallelogram is half the area of the original or given triangle. Thus, the use of the tool provides an opportunity for the problem solver to simultaneously examine properties of figures that within the configuration. These conjectures are proved further down.

Comment

The dynamic representation becomes a source that generates mathematical conjectures as a result of moving objects within the configuration. Exploring different ways to inscribe the parallelogram leads to formulate two related conjectures. In addition, the use of the tool allows graphing the area's variation without defining explicitly an algebraic model. Thus, it is possible to think of a functional approach, without defining the function algebraically, that associates the position of point P (for example, the distance between AB, BP or AC) with the corresponding area value. Figure 5 provides a visual and numerical approach to describe the parallelogram's area behaviour.

The Searching for Multiple Approaches Episode

We argue that if students are to develop a conceptual understanding of mathematical ideas and problem solving proficiency, they need to think of different ways to solve a problem or to examine a mathematical concept. In this context, the visual and empirical approaches used previously to explore the problem provide a basis to introduce other approaches. We argue that each approach to the problem demands that the problem solver not only think of the problem in different ways; but also to use different concepts and resources to solve it.

Analytical approach

In this approach, the use of the Cartesian system becomes important to represent the objects algebraically. The problem can be thought in general terms as shown below.

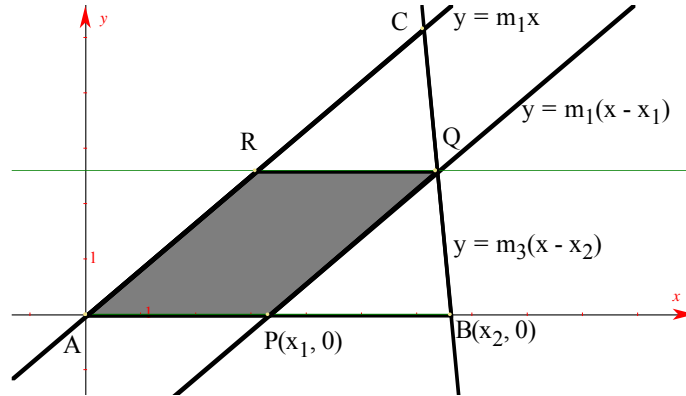


Figure 8: Using a Cartesian system to construct an algebraic model of the problem

General case

Without losing generality, we can always situate the Cartesian system in such a way that one side of the given triangle can be on the x -axis and the other side on line $y = m_1x$ (Figure 8). Point P will be located on side AB and its coordinates will be $P(x_1, 0)$. Point $B(x_2, 0)$ is vertex B of the given triangle (Figure 8). The general goal is to represent the area of parallelogram $APQR$ in terms of known parameters. This process leads to represent the area in terms of one variable ($AP = x_1$) as:

$$A(x_1) = \frac{m_1 m_3 (x_1^2 - x_2 x_1)}{m_1 - m_3}. \text{ The roots of } A(x_1) \text{ (a quadratic function) are } 0 \text{ and } x_2.$$

Also, this function has a maximum value if and only if $\frac{m_1 m_3}{m_1 - m_3} < 0$. We are

assuming that $m_1 > 0$. The assumption on the triangle location guarantees that

m_3 and $m_1 - m_3$ have opposite signs. By a symmetric argument, $A(x_1)$ reaches its

maximum at the midpoint of the interval $[0, x_2]$, that is, at $x_1 = \frac{x_2}{2}$.

Another way to determine the maximum value of this expression is by using calculus

concepts: $A'(x_1) = \frac{m_1 m_3 (2x_1 - x_2)}{m_1 - m_3}$, the critical points are obtained when $A'(x_1) = 0$,

we have that $x_1 = \frac{x_2}{2}$ which is the solution of the equation, then the function $A(x_1)$

will reach its maximum value at $x_1 = \frac{x_2}{2}$. This is because $A''(x_1) = \frac{m_1 m_3}{m_1 - m_3} < 0$. Thus,

this result supports the conjecture formulated previously in the graphic approach.

General case

It is possible to use a hand-held calculator to find the maximum area for the case shown in Figure 9. In this case, we have that $m_1 = 72/85$; $m_3 = -10.33$; and $x_2 = 6.6\text{cm}$.

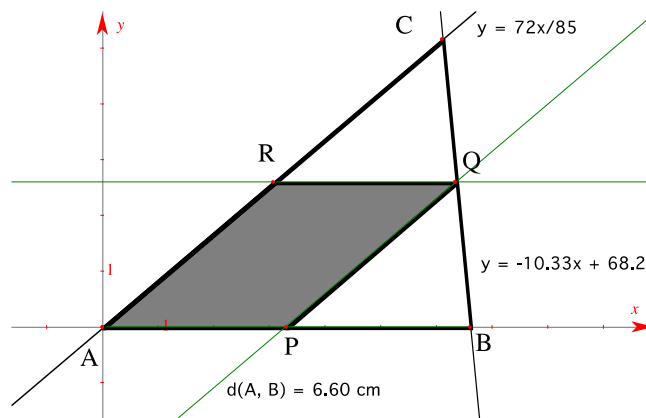


Figure 9: Finding the equations of lines with the use of the tool.

Figure 10 shows the algebraic operation carried out to get the point where the function reaches its maximum value and Figure 11 shows its graphic representation.

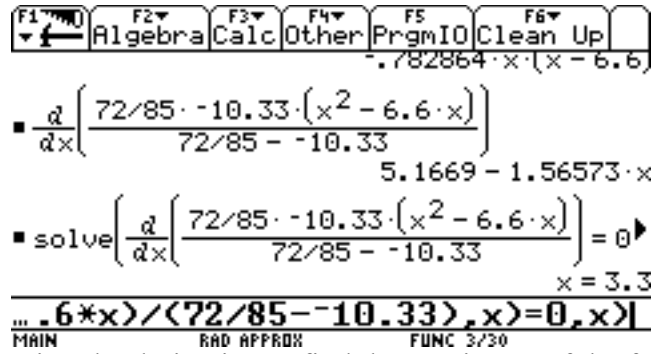


Figure 10: Using the derivative to find the maximum of the function area.

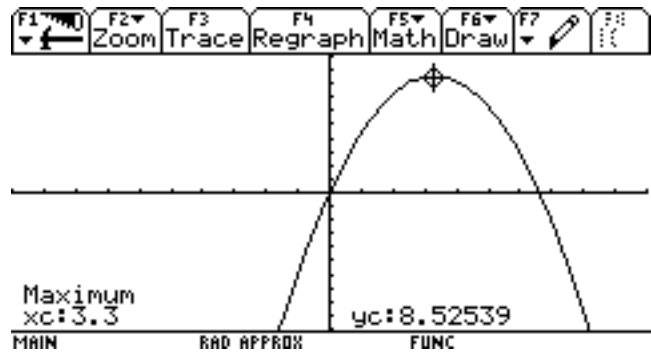


Figure 11: Graphic representation of the function area.

A Geometric approach

The goal is to use geometric properties embedded in the problem's representation to construct an algebraic model. In Figure 12, it can be seen that:

Triangle $\triangle ABC$ is similar to triangle $\triangle PBQ$, this is because angle $\angle PQB$ is congruent to angle $\angle ACB$ (they are corresponding angles) and angle $\angle ABC$ is the same as angle $\angle PBQ$. Based on this information,

$h_1 = \frac{h(a-x)}{a}$ and the area of $\triangle APQR$ can then be expressed as $A = xh_1$, that is,

$A(x) = x \left(\frac{h(a-x)}{a} \right)$. This latter expression can be written as

$A(x) = xh - \frac{hx^2}{a}$. This expression represents a parabola. $A'(x) = h - \frac{2hx}{a}$, now if

$A'(x) = h - \frac{2hx}{a} = 0$, then $x = a/2$. Now, we observe that $A'' < 0$ for any point on the domain defined for $A(x)$, therefore, there is a maximum relative for that value.

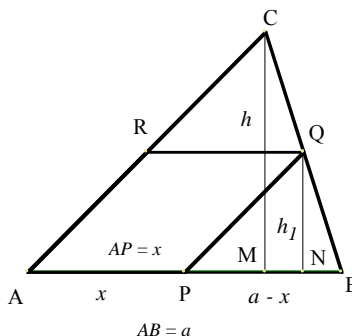


Figure 12: Relying on geometric properties to construct an algebraic model.

During the comprehension and exploration episodes two conjectures emerged, the first one (area of parallelogram APQR is the same as area of parallelogram PBQ'R') can be proved by considering parallelogram APTR' (Figure 7). It is observed that triangles APR' and TR'P are congruent and triangles RR'T' and TQQ' are also congruent (SSS). Then, we have that quadrilaterals APT'R and T'QQ'R' have equal areas, also, the area of triangle PQT' is the same as the area of triangle PQB. Based on this information, we have that the area of APQR is the same as area of PBQ'R'.

The second conjecture that involves showing that the four triangles have the same area can be proved by observing that the triangles are part of three parallelograms (APQR, PBQR and PQCR) that overlap each other (Figure 7). Then the overlapping triangle PQR has the same area as the others because they share a diagonal as a side of each corresponding parallelogram. Therefore, the maximum value of the inscribed parallelogram is half the area of the given triangle ($\triangle ABC$).

Comment

An important feature of the frame is that teachers should always look for different ways to solve and examine the tasks. The common goal in the task is to represent and explore the area model, however the approaches used to achieve this goal offer teacher the opportunity to focus on diverse concepts and resources as a way to construct the model. For example, the algebraic model relies on representing and operating mathematical objects analytically while the geometric approach is based on using triangles' properties to define the area model. It is also observed that the general model can be tested by assigning particular coordinates to the original triangle vertices. Thus, problem solvers have the opportunity to test their initial conjectures obtained visually and empirically by using now the general result (Figure 10 and 11). The use of a hand-held calculator, in general, makes easy to operate the algebraic expressions and as a consequence learners could focus their attention to discuss the meaning of the results. Each approach relies on using different concepts and ways to deal with the involved relations. As a consequence, the problem solver can contrast strengths and limitations associated to each approach.

An extension

In figure 13, we draw a line passing through points PR (vertices of parallelogram APQR). With the use of the software, we ask for the locus of line PR (envelope) when point P is moved along side AB. Visually, the locus (tangent points) seems to be a conic section, the goal is to show that it holds properties that define that figure.

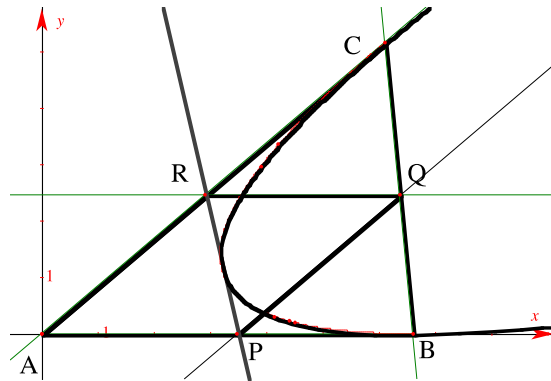


Figure 13: What is the locus of line PR when point P is moved along side AB?

Again, with the use of the tool it is shown that the locus is a parabola whose focus and directrix are identified in Figure 14. It is also shown that when point M is moved along the locus the distance from that point to the directrix (L) and to point F (focus of the parabola) is the same (this property defines a parabola).

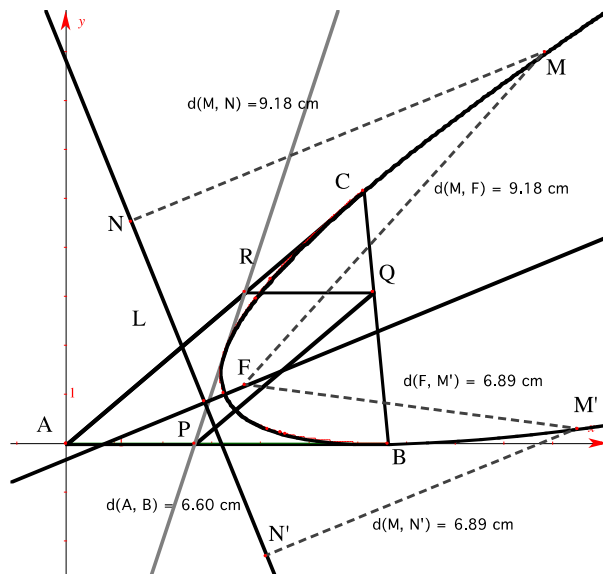


Figure 14: The locus of line PR when point P is moved along segment AB is a parabola.

Comment

Some serendipitous results or relations might appear as a result of introducing other objects within the configuration. In this case, adding a line PR to the configuration led to identify a conic section. Thus, the use of the tool offers a means to think of

mathematical connections that are not easy to identify with the only use of paper and pencil approaches.

The Integration Episode and Reflections

It is important and convenient to reflect on the processes involved in the distinct phases that characterize an approach to solve mathematical problems that fosters the use of computational technology. Initially, the comprehension of the problem's statements or concepts involves the use of an inquiry approach to make sense of relevant information embedded in those concepts or statements. This enquiry process provides the basis to relate the use of the tools and ways to represent dynamically the problem or situation. A dynamic model becomes a source from which to explore visually and numerically the behaviour of parameters, as a result of displacing some elements within the problem representation. In particular, it might be possible to construct a functional relationship between a variable, for example the variation of the side AP of the parallelogram and its corresponding area.

Two distinct ways to construct an algebraic model of the area variation were pursued; one involves the use of the Cartesian system to identify the equations associated with some elements of the model. The second way relies on identifying similar triangles in the inscribed parallelogram whose properties led to the construction of the area model. Both approaches, the analytic and geometric, converge in the search for the algebraic model. The algebraic model represents the general case and it can be "validated" by considering the information of the triangle used to generate the visual model. In addition, it can be used to explore some of the relations that were detected

during the visual approach. For example, to identify the intersection points of line $y = k$

and the area model $A(x_1) = \frac{m_1 m_3 (x_1^2 - x_2 x_1)}{m_1 - m_3}$ (Figure 5) we solve the equation

$k = \frac{m_1 m_3 (x_1^2 - x_2 x_1)}{m_1 - m_3}$ for x_1 . Thus, the discriminant of this quadratic equation

$\Delta = (m_1 m_3 x_2)^2 + 4m_1 m_3 (km_3 - km_1)$ provides useful information to interpret the

relationship between line $y = k$ and the graph of the area model

$A(x_1) = \frac{m_1 m_3 (x_1^2 - x_2 x_1)}{m_1 - m_3}$. When the discriminant is zero the line intersects the graph at

the maximum point, when it is greater than zero, there are two intersection points and

when the discriminant is less than zero, then the line does not intersect the area's graph.

Concluding, the systematic use of computational tools in problem solving approaches led us to identify a pragmatic framework to structure and guide learning activities in such a way that teachers can help the students develop mathematical thinking. A distinguishing feature of the problem solving episodes is that constructing a dynamic model of the phenomena provides interesting ways to deal with them from visual and empirical approaches. Later, analytical and formal methods are used to support conjectures and particular cases that appear in those initial approaches. The NCTM (2009) recognizes that reasoning and sense making activities require for students to gradually develop levels of understanding to progress from less formal reasoning to more formal approaches.

The use of computational tools provides a basis not only to introduce and connect empirical and formal approaches, but also to use powerful heuristics as dragging objects

and finding loci of particular objects within the dynamic problem representation. As Jackiw & Sinclair (2009) pointed out “Dynamic Geometry is revealed as a technological capability to produce seemingly limitless series of continuously-related examples, and in so doing, to represent visually the entire phase-space or configuration potential of an underlying mathematical construction” (p. 414). Throughout the problem solving episodes we show that it is important for teachers to conceive of a task or problem as an opportunity for their students to represent, explore and examine the task from diverse perspectives in order to formulate conjectures and to look for ways to support them. The diversity of approaches allows them to contrast and relate different concepts and ways to reason about their meaning and applications. In this context, the use of the tools opens up new windows to frame and encourage teachers and students’ mathematical discussions

Remarks

Is there any way to characterize forms or ways of mathematical reasoning that emerge as a result of using computational tools in problem solving approaches? In which ways does this reasoning complement problem solving approaches that rely on the use of paper and pencil? Thinking of the task in terms of the affordances provided by the tools demands that problem solvers focus their attention on ways to take advantage of the opportunities offered by the tool to represent and explore the problem. For example, the use of the tool to construct a dynamic model of a task not only becomes relevant to identify and formulate series of conjectures or mathematical relations but also to reason about the task in terms of graphic and visual approaches without relying, at this stage, on an analytic model. In addition, with the use of the software becomes natural and easy to

extend the analysis of a case to a family of cases. For example, by moving any vertex of triangle ABC, it is possible to verify that all the relations found during the analysis of the task are also true for a family of triangles that result when moving one the vertices. With the use of the tool it is often possible to generate loci of points or lines within the model or to identify parameter behaviours without defining the corresponding algebraic model. In addition, the empirical and visual approaches often provide important information to present formal arguments to support conjectures. In this context, it is clear that the software approach could play an important role to complement and construct formal or analytic approaches.

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Proof and Problem Solving at University Level

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Abstract: This paper will be concerned with undergraduate and graduate students' problem solving as they encounter it in attempting to prove theorems, mainly to satisfy their professors in their courses, but also as they conduct original research for theses and dissertations. We take Schoenfeld's (1985) view of problem, namely, a mathematical task is a problem for an individual if that person does not already know a method of solution for that task. Thus, a given task may be a problem for one individual, who does not already know a solution method for that task, or it may be an exercise for an individual who already knows a procedure or an algorithm for solving that task.

Keywords: Hungary, mathematics education, mathematics competition, Olympiads, international comparative mathematics education, problem solving, creativity, mathematically talented students.

A Continuum of Tasks from Very Routine to Very Non-routine

While what is a problem depends on what a solver knows, it is possible for most mathematics teachers to judge what is difficult for most students in a given class. Thus, we see mathematical tasks for a given class, such as a calculus class, on a continuum from those that are very routine to those that are genuinely difficult problems (Selden, Selden, Hauk, & Mason, 2000). At one end, there are *very routine* problems which mimic sample worked problems found in textbooks or lectures, except for minor changes in wording, notation, coefficients, constants, or functions that are incidental to the way the problems are solved. Such problems are often referred to as exercises (and might not be considered to be problems at all in the problem-solving literature).

The vast majority of exercises in calculus textbooks are of this nature. Lithner (2004) distinguished three possible solution strategies for typical calculus textbook exercises: identification of similarities (IS), local plausible reasoning (LPR), and global plausible reasoning (GPR). In IS, one identifies surface features of the exercise and looks for a similar textbook situation -- an example, a rule, a definition, a theorem. Without consideration of intrinsic mathematical properties, one simply copies the procedure of that situation. In LPR, one identifies a slightly similar textbook situation, but one in which a few local parts may differ. The solution strategy is to copy as much as possible from that similar situation, modifying local steps as needed. In GPR, the strategy is mainly based on analyzing and considering intrinsic mathematical properties of the exercise, and using these, a solution is constructed and supported by plausible reasoning. Lithner selected a textbook used in Sweden [Adams' *Calculus: A Complete Course* (5th ed.), Addison-Wesley], and worked through and classified solution strategies for 598 single-variable calculus exercises. He found 85% IS, 8% LPR, and 7% GPR. Furthermore, he concluded that "it is possible in about 70% of the exercises to base the solution not only on searching for similar situations, but on searching only the solved examples."

Moving toward the middle of the continuum, there are *moderately routine* problems which, although not exactly like sample worked problems, can be solved by well-practiced methods, for example, ordinary related rates or change of variable integration problems in a calculus course.¹ Moving further along the continuum, there are *moderately non-routine* problems, which are not very similar to problems that students

¹ Sandra Marshall (1995) has studied how students can develop schema (well-practiced routines) to reliably guide the solution of arithmetic word problems.

have seen before and require known facts or skills to be combined in a slightly novel way, but are "straightforward" in not requiring, for example, the consideration of multiple sub-problems or novel insights. This is the type of problem we used on the non-routine test in our three studies of undergraduate students' calculus problem solving. One of those problems was: *Find values of a and b so that the line $2x+3y=a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x=3$.* (Selden, Selden, Mason, & Hauk, 2000, p. 133).

Finally, at the opposite end of the continuum from routine problems, there are *very non-routine* problems which, while dependent on resources in one's knowledge base, may involve considerable insight, the consideration of several sub-problems or constructions, and use of Schoenfeld's (1985) behavioral problem-solving characteristics (heuristics, control, beliefs). For such problems a large supply of *tentative solution starts* (Selden, Selden, Mason, & Hauk, 2000, p. 145), built up from experience, might not be adequate to bring to mind the resources needed for a solution, while for moderately novel problems it probably would. Often the Putnam Examinations include such very non-routine problems.²

²The following problem was on the 59th Annual William Lowell Putnam Mathematical Competition given on December 5, 1998: Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b) , one on the x -axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

This appears to be a calculus problem, but it only requires clever use of geometry. An elegant solution (posted by Iliya Bluskov to the sci.math newsgroup) involves extending the construction "outward" by reflecting across both the lines $y = x$ and the x -axis and noticing that the perimeter of the triangle equals the distance along the path from (b, a) to $(a, -b)$. Probably only very experienced geometry problem solvers could have previously constructed images of problem situations containing a tentative solution start that would easily bring this method to mind.

Most U.S. university mathematics teachers would probably like undergraduate students who pass their lower-level courses, such as calculus, to be able to work a wide selection of routine, or even moderately routine, problems. In addition, we believe that many such teachers would also expect their better students to be able to work moderately non-routine problems, and would think of the ability to do so as functionally equivalent to having a good conceptual grasp of the course. In other words, we conjecture that the ability to work moderately non-routine problems based on the material in a university mathematics course, such as calculus, is often considered part of the implicit curriculum and taken as equivalent to good conceptual grasp. However, no research has yet been done to substantiate this conjecture.

Tentative Solution Starts

An individual who has reflected on a number of problems is likely to have seen (perhaps tacitly) similarities between some of them. He or she might recognize (not necessarily explicitly or consciously) several overlapping problem situations, each arising from problems with similar features. For example, after much exposure, many lower-level university students would probably recognize a problem as one involving, for example, factoring, several linear equations, or integration by parts.³ Such problem situations can act much like concepts (perhaps without signs or labels). While they may lack names, for a given individual they are likely to be associated with mental images,

³Although the kinds of features noticed by students in mathematical problem situations do not seem to have been well studied, the features focused on in physics problem situations have been observed to correspond to an individual's degree of expertise. Novices tend to favor surface characteristics (e.g., pulleys), whereas experts tended to focus on underlying principles of physics (e.g., conservation of energy) (Chi, Feltovich, & Glaser, 1981).

that is, strategies, examples, non-examples, theorems, judgments of difficulty, and the like. Following Tall and Vinner's (1981) idea of concept image, we have called this kind of mental structure a *problem situation image* and have suggested that some such images may, and others may not, contain what we have called *tentative solution starts* (Selden, Selden, Hauk, & Mason, 2000, p. 145). These are tentative general ideas for beginning the process of finding a solution. The linking of problem situations with one or more tentative solution starts is a kind of (perhaps tacit) knowledge. For instance, the image of a problem situation asking for the solution to an equation might include "try getting a zero on one side and then factoring the other." It might also include "try writing the equation as $f(x) = 0$ and looking for where the graph of $f(x)$ crosses the x -axis," or even "perhaps the maximum of f is negative so $f(x) = 0$ has no solution." We suggest that an individual's problem-solving processes are likely to include the recognition of a problem as belonging to one or more problem situations, and hence, bring to mind one or more tentative solution starts contained in that individual's problem situation image. This, in turn, may mentally prime the recall of resources from that individual's knowledge base. Thus, a tentative solution start may link recognition of a problem situation with the recall of appropriate resources. We have suggested that problem situations, their images, and the associated tentative solution starts all vary from individual to individual and that the process of mentally linking recognition (of a problem situation) to recall (of requisite resources) through problem situation images might occur several times in solving a single problem, especially when an impasse occurs (Selden, Selden, Mason, & Hauk, 2000, pp. 145-147).

The Genre of Proof

We consider proofs, those that occur in advanced university mathematics textbooks and research journals, as being written in a special genre. It is clear that not every mathematical argument can be considered a proof, and much has been written in the mathematics education research literature about the distinction between argumentation and proof. (See, for example, Duval, as reported in Dreyfus, 1999, and Douek, 1999.) In this paper, we are considering proofs of the sort that advanced undergraduate students and beginning graduate mathematics students are expected to produce for their professors. We are aware that many upper-level U.S. mathematics majors just beginning their study of proof-based courses such as abstract algebra and real analysis often have great difficulty producing such proofs, despite the fact that many of them have previously taken a transition-to-proof course (Moore, 1994), usually in their second year of university. Students in such transition-to-proof courses often have trouble knowing what to write, especially when asked to prove simple set theory theorems, perhaps because the results are “too obvious” or are verifiable using examples or Venn diagrams. Thus, learning the genre of proof is important. Indeed, to help students learn the genre of proof, we have considered two aspects (or parts) of a final written proof: the formal-rhetorical part and the problem-centered part. The *formal-rhetorical* part of a proof (also sometimes referred to as a *proof framework*) is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call the remaining part of a

proof the *problem-centered* part. It is the part that *does* depend on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts involved. (See Selden & Selden, 2009).

A sample proof framework is given below for a proof of the following theorem: If f and g are real valued functions of a real variable continuous at a , then $f + g$ is continuous at a .

Proof. Let f and g be functions and suppose they are continuous at a . Suppose ε is a number > 0 . Because f is continuous, there is a $\delta_f > 0$ so that for all x , if $|x - a| < \delta_f$, then $|f(x) - f(a)| < \frac{\varepsilon}{2}$. Also because g is continuous, there is a $\delta_g > 0$ so that for all x , if $|x - a| < \delta_g$, then

$$|g(x) - g(a)| < \frac{\varepsilon}{2}. \text{ Let } \delta = \min\{\delta_f, \delta_g\}. \text{ Note that } \delta > 0. \text{ Let } x \text{ be a number.}$$

Suppose that $|x - a| < \delta$. Then $|f(x) + g(x) - (f(a) + g(a))| = \dots < \varepsilon$. Thus, $|f(x) + g(x) - (f(a) + g(a))| < \varepsilon$. Therefore $f + g$ is continuous at a .

The problem-centered part of the proof consists of cleverly filling in the blanks using, for example $\frac{\varepsilon}{2}$, minimum, and the triangle inequality. This is not to say that filling in the blanks is easy. Indeed it can be very difficult for an individual with little or no experience with such real analysis proofs.

Being able to write a proof framework can be very helpful for students because it not only improves their proof writing, bringing it in line with accepted community norms, but also because it can reveal the nature of the problem(s) to be solved. Having once learned to write a number of proof frameworks, students can then concentrate their creative energies on solving the actual mathematical problems involved. In addition, for students, writing the formal-rhetorical part of a proof, and whatever else they can

regarding the actual problem(s) to be solved, can enable their university mathematics teachers to give more helpful and targeted criticism and advice.

The Close Relationship Between Problem Solving and Proving

A number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Mamona-Downs & Downs, 2009; Moore, 1994), and our division of proofs into their formal-rhetorical and problem-centered parts (described above) can make this explicit for students. However, having good ideas for how to solve the problem-centered part of a proof is not equivalent to having a proof. Mamona-Downs and Downs (2009) have given university students informal arguments suggesting a way to solve tasks and asked them to convert those arguments into acceptable mathematical form. They concluded that “proof production [from an intuitively developed argument] can involve significant problem solving aspects. ... A particularly frustrating circumstance for a student is when he/she can ‘see’ a reason why a mathematical proposition is true, but lacks the means to express it as an explicit [mathematical] argument.” Thus, there are actually two distinct kinds of problem solving that can occur during proof construction, namely, solving the actual mathematical problem(s) that enable one to get from the given hypotheses to the given conclusion, and converting one’s (informal) solution into acceptable mathematical form. Neither of these problem-solving tasks is easy and students may require instruction and practice with each. How informal arguments are converted into acceptable mathematical form has been very little researched.

Sometimes one's informal argumentation (developed during one's initial problem solving) can be converted rather seamlessly into a written proof that is acceptable to the mathematical community. Boero, Douek, Morselli, and Pedemonte (2010, p. 183) suggested that "in some cases this [problem-solving] argumentation can be exploited by the student in the construction of a proof by organizing some of the previously produced arguments into a logical chain," and in such situations, writing proofs is easier for students. They refer to this situation as one of structural *cognitive unity*.

We have experienced such structural cognitive unity with beginning graduate mathematics students. They find the following theorem comparatively easy to prove: *If g is a function continuous at a and f is a function continuous at $g(a)$, then $f \circ g$ is continuous at a .* We conjecture that, if they think of an "appropriate representation," (see Figure 1), they can sketch an epsilon neighborhood of $f(g(a))$, and invoking the continuity of f , "pull it back" to a delta neighborhood of $g(a)$. Then they can use that delta, δ_g , as a new epsilon in applying the continuity of g to arrive at the needed final delta, δ_f .

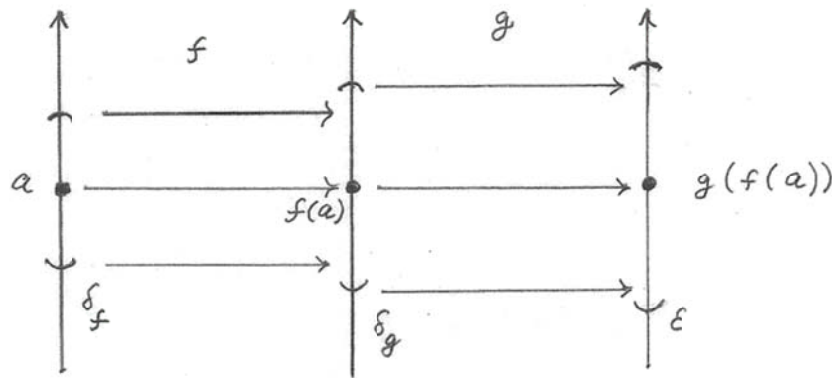


Figure 1. A “picture” of $g \circ f$ with the epsilon and delta neighborhoods indicated.

However, the theorem whose proof framework was illustrated above, namely, *If f and g are real valued functions of a real variable continuous at a , then $f + g$ is continuous at a* , and whose proof involves using minimum and the triangle inequality cannot be easily obtained from informal intuitive argumentation about adding together the ordinates of the Cartesian graphs of f and g .

The Importance of Problem Reformulation and Selection of Appropriate

Representations

A number of researchers (e.g., Boero, 2001; Gholamazad, Liljedahl & Zazkis, 2003; Zazkis & Liljedahl, 2004) have noted that reformulating a problem by making an appropriate choice of representation is often useful, sometimes even necessary, to make progress. Furinghetti and Morselli (2009) reported the unsuccessful problem-solving behavior of two fourth-year Italian university mathematics education students during attempts to prove that *The sum of two numbers that are prime to one another is prime to each of the addends*. One student, with the pseudonym Flower, after some initial panic and working with examples, succeeded in producing a potentially helpful representation (using the Prime Factorization Theorem), but could not exploit it. The other student, with the pseudonym Booh, first chose the representation of Least Common Multiple that “synthesizes [captures the essence of] the property, but doesn’t allow algebraic manipulation ... being non-transparent” and realized that it was “without future.” So he considered another representation (also using the Prime Factorization Theorem), but also could not exploit it. In a separate earlier paper, Furinghetti & Moriselli (2007) reported

the unconventional, metaphorical thinking of another student who chose to think of, and draw, a frog jumping from stop to stop (i.e., from integer to integer on the number line) and successfully proved the same theorem. They noted that “The choice of the representation ... may foster or hinder transformational and anticipatory thinking, which are two key issues in the proving process” (Furinghetti & Morselli, 2009, p. 74).

Concepts can have several (easily manipulated) symbolic representations or none at all. For example, prime numbers have no such representation; they are sometimes defined as those positive integers having exactly two factors or being divisible only by 1 and themselves. It has been argued that the lack of an (easily manipulated) symbolic representation makes understanding prime numbers especially difficult, in particular, for preservice teachers (Zazkis & Liljedahl, 2004).

Symbolic representations can make certain features transparent and others opaque.⁴ For example, if one wants to prove a multiplicative property of complex numbers, it is often better to use the representation $re^{i\theta}$, rather than $x+iy$, and if one wants to prove certain results in linear algebra, it may be better to use linear transformations, T , rather than matrices. Students often lack the experience to know when a given representation is likely to be useful. More research is needed on the effect of one’s choice of representation(s) on successful problem-solving behavior.

How Mathematicians Solve Problems

⁴ For example, representing 784 as 28^2 makes the property of being a perfect square transparent and the property of being divisible by 98 opaque. For more details, see Zazkis and Liljedahl (2004, pp. 165-166).

It would be very informative to have research on how advanced university mathematics students or mathematicians actually construct proofs in real time, but such a study has not yet been conducted. However, there is research on how mathematicians solve mathematical problems of various kinds: Carlson and Bloom (2005) investigated how mathematicians manage their well-connected conceptual knowledge and make decisions during problem solving; DeFranco (1996) replicated Schoenfeld's work on the use of resources, heuristics, control, and beliefs with mathematicians; and Stylianou (2002) investigated how mathematicians use diagrams in problem solving.

However, the problems given to the mathematicians in these studies were not the sort encountered by advanced undergraduate or graduate mathematics students when constructing proofs for their courses or by professors when conducting research. For example, one problem whose solution Carlson and Bloom (2005, p. 55) discussed at length was: *A square piece of paper ABDC is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is $\frac{1}{2}$ white and $\frac{1}{2}$ black. How far is A' from the fold line?* One problem used by DeFranco (p. 212) was: *In how many ways can you change one-half dollar?*

Still some of the results are interesting, so we briefly recall them here. Based on interviews with 12 mathematicians, Carlson and Bloom (2005) developed a "problem solving framework" that has four phases (orientation, planning, executing, and checking). As part of the planning phase, there was a *conjecture-imagine-evaluate* sub-cycle, in which the mathematicians typically imagined a hypothetical solution approach, followed by "playing out" and evaluating whether that approach was viable. If it was not viable,

the *conjecture-imagine-evaluate* sub-cycle was repeated until a viable solution path was identified. Carlson and Bloom (2005, p. 45) stated, “The effectiveness of the mathematicians in making intelligent decisions that led down productive paths appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their ability to manage their emotional responses.”

DeFranco (1996) studied the problem-solving behaviors of eight research mathematicians who had achieved national or international recognition in the mathematics community (e.g., had altogether 12 honorary degrees and had been awarded prizes such as the National Medal of Science) and eight who had not achieved such recognition, but had published from three to 52 articles. He concluded that the former were problem-solving experts, as well as content experts, and had superior metacognitive skills, whereas the latter were content experts with only modest problem-solving skills.

Stylianou (2002) was interested in the interplay between visualization and analytical thinking and asked mathematicians the following problem: *Given a right circular cylinder cut at an angle* (shown in her accompanying diagram), *describe the resulting truncated cylinder’s net, that is, the “unrolled” truncated cylinder.* She observed that the “mathematicians consistently attempted to infer additional consequences from their visual action. Each time a mathematician either constructed a new diagram or modified a previously constructed one, he took a few seconds to ‘extract’ any additional information . . . and to understand any possible implications.”

In addition, there have been studies (e.g., Burton, 1999) that have included the reflections of mathematicians upon their own ways of working; however, these are often too general to be useful for an in-depth understanding of problem solving or for obtaining

suggestions for teaching. For example, Burton (1999) found some of the mathematicians likened problem solving and research to working on jigsaw puzzles or to climbing mountains.

The Role of Affect in Proving and Problem Solving

While strong affect can play both a positive and a negative role during proving and problem solving, more research is needed on the role of various kinds of affect from beliefs and attitudes to emotions and feelings. Furinghetti and Morselli (2009, p. 82) considered how negative affective factors influenced the problem-solving behavior of their two unsuccessful students. They noted that Flower panicked immediately after reading the statement of the theorem writing, “Help! I’m not familiar with prime numbers!” Later, after constructing some examples, Flower wrote, “Help! I cannot do it, I do not see anything. Deepest darkness.” Then when she came up with the prime factorizations, Flower apparently expected to “conclude the proving process in an almost automatic way ... [without] the possibility of dead ends and failures,” illustrating that beliefs and expectations are also important factors influencing problem-solving outcomes.

In their study of mathematicians’ problem solving, Carlson and Bloom (2005) concluded, “The effectiveness of the mathematicians ... appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their *ability to manage their emotional responses* [italics ours].” In a study of non-routine problem solving, McLeod, Metzger, and Craviotto (1989) found that both experts (research mathematicians) and novices (undergraduates enrolled in college-level

mathematics courses), when given different experience appropriate problems, reported having similar intense emotional reactions such as frustration, aggravation, and disappointment, but the experts were better able to control them.

DeBellis and Goldin (1997, 2006) have considered affect (i.e., values, beliefs, attitudes, and emotions) as an internal representational system that is not merely auxiliary to cognition, but as “a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition.” They have introduced the construct of *meta-affect*, by which they mean not only affect about affect, but also cognition that acts to monitor and direct one’s emotional feelings. They also suggested that one might characterize individuals’ *affective competencies*, such as the ability to act on curiosity or to see frustrations as a signal to modify strategy, but did not suggest how to do so.

In addition, we see nonemotional cognitive feelings of appropriateness and of rightness or wrongness as giving direction to one’s problem-solving efforts. As Mangan (2001, Section 6, Paragraph 3) said, “In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand.” Below we give the example of Mary, a returning graduate student, who did not get a feeling of appropriateness with regard to using fixed, but arbitrary elements in her real analysis proofs for at least half a semester.

Also there have been working groups on affect and mathematical thinking at several recent CERME conferences,⁵ but the discussions there seem to have been mainly concerned with methodological issues and such topics as changing teachers’ and students’ motivation and attitudes towards mathematics. Still we feel that the interplay

⁵ CERME is the Congress of the European Society for Research in Mathematics Education, many of whose Proceedings are available online.

between cognition and affect during problem solving and proving needs further investigation.

How University Mathematics Students Prove Theorems

Much of the research on university students' proving has been concerned with difficulties they encounter or competencies needed. These include the use of logic, especially quantifiers (Dubinsky & Yiparaki, 2000; Epp, 2009; Selden & Selden, 1995); the necessity to employ formal definitions (Edwards & Ward, 2004); the need for a repertoire of examples, counterexample, and nonexamples (Dahlberg & Housman, 1997); the requirement for a deep understanding of the concepts and theorems involved (Weber, 2001); the need for strategic knowledge of which theorems are important (Weber, 2001), the selection of appropriate representations (Kaput, 1991); and the importance of being able to validate (i.e., read and check) one's own and others' proofs for correctness (Selden & Selden, 2003).

Teaching Proving to University Mathematics Students

For several years, we have been developing methods for teaching proof construction to advanced undergraduate and beginning graduate mathematics students. We have developed an inquiry-based Modified Moore Method course (Mahavier, 1999; Coppin, Mahavier, May, & Parker, 2009) for advanced undergraduate and beginning graduate mathematics students who need help with proving (hereafter referred to as the "proofs course" and described in Selden, McKee, & Selden, 2010) and a voluntary

proving supplement for undergraduate real analysis (hereafter referred to as the “supplement” and described in McKee, Savic, Selden, & Selden, 2010).

In the proofs course, the students are given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students construct their proofs at home and present them in class. The proofs are then critiqued, sometimes extensively, and additionally suggestions for improvements in the notation used and the style of writing are given. There are no formal lectures, and all comments and conversations are based solely on students’ work. The specific topics covered are of less importance than giving students opportunities to experience as many different kinds of proofs as possible so we select theorems from sets, functions, real analysis, semigroups, and topology.

We have developed some theoretical underpinnings for the two courses. One such theoretical underpinning involves having students develop a proof framework first in order to reveal the mathematical problem(s) to be solved. (See the above description in “The Genre of Proof” section.) While students with little experience in proof writing, at first can find constructing a proof framework to be a problem of moderate difficulty, eventually through practice, writing a proof framework can become routine or very routine.

In addition, our proofs course notes are constructed to give students opportunities to prove theorems that are successively more non-routine. But non-routineness is not unidimensional; it is not simply a matter of whether the students have seen the concepts before or have the necessary factual knowledge but cannot bring it to mind (as was the case for the students in our calculus studies). In our proofs course notes we have “built

in” non-routine theorems, which we refer to as theorems of Types 1, 2, and 3. *Type 1 theorems* have proofs that can depend on a previous result in the notes. These theorems are included to encourage students to look for helpful previous results, as we have found that students often attempt to prove theorems directly from the definitions without recourse to previous results. *Type 2 theorems* require formulating and proving a lemma not in the notes, but one that is relatively easy to notice, formulate, and prove, whereas *Type 3 theorems* require formulating and proving a lemma not in the notes, but one that is hard to notice, formulate, and prove. An example of a Type 3 theorem is: *A commutative semigroup S with no proper ideals is a group*, given after a brief introduction to the ideas of semigroup and ideals thereof. What is needed for a proof of this theorem is the observation that aS is an ideal and hence $aS=S$. (This is the first lemma needed.) This is followed by the nontrivial observation that $aS=S$ implies that equations of the form $ax=b$ are solvable for any b in S . Using some clever instantiations of this equation, one can obtain an identity and inverses, and hence, conclude S is a group. To date only two students have been able to produce a proof without help or hints, and several mathematics faculty (whose speciality is not semigroups) have found that proving this theorem takes time and a certain amount of reformulation. This convinces us that this theorem can be considered at least moderately non-routine. More research is needed on what makes a problem non-routine (for an individual or a class), that is, what are the various dimensions or characteristics contributing to non-routineness.

The Co-construction of Proofs in the Supplement

We have implemented this method three times to date in small (at most 10 students) supplementary voluntary proving classes for real analysis. The supplement is

intended for students who feel unsure of how to proceed in constructing real analysis proofs. At the beginning of a supplement class period, the statement of a theorem entirely new to them, but similar to a theorem assigned for homework, but not a template theorem, is written on the board. The students themselves, or one of us if need be, offer suggestions about what to do, beginning with the construction of a proof framework. For each suggested action, such as writing up the hypotheses or an appropriate definition, drawing a sketch, or introducing cases, one student is asked to carry out the action at the blackboard. The intention is that all students reflect on the actions and later perform similar actions autonomously on their assigned homework (McKee, Savic, Selden, & Selden, 2010).

For example, if the students are to prove a sequence $\{a_n\}_{n=1}^{\infty}$ converges to A , they would typically begin by writing the hypotheses, leave a space, and write the conclusion. After unpacking the conclusion, they would write “Let $\varepsilon > 0$ ” immediately after the hypotheses, leave a space for the determination of N , write “Let $n \geq N$ ”, leave another space, and finally write “Then $|a_n - A| < \varepsilon$ ” prior to the conclusion at the bottom of their nascent proof. This would conclude the construction of a proof framework and bring them to the problem-centered part of the proof (Selden & Selden, 2009), where some “exploration” or “brainstorming” on the side board would ensue. The entire co-construction process, and accompanying discussions, is a slow one – so slow that only one theorem can be proved and discussed in detail in a 75-minute class period. More research is needed on how to foster such mathematical “exploration” and “brainstorming.”

Theoretical Underpinnings: Actions and Behavioral Schemas

Actions in the Proving Process

We see proving as an activity, that is, as a sequence of actions, that are either physical (such as writing or drawing) or mental (such as attempting to recall a definition or theorem). Each action is paired with, and is a response to, a situation in a partly completed proof. By a situation we mean a reasoner's inner, or interpreted, situation as opposed to an outer situation that may be visible to an observer. Although we are referring to a person's inner situation, we have found in teaching that we can often gauge approximately what the inner situation is from the outer, observable, situation and the ensuing action. For example, below we will interpret Sofia's staring blankly at the blackboard during tutoring sessions as the situation of not knowing what to do next.

If a person engages in proving several theorems, then he or she is likely to experience a number of similar situations yielding similar actions. The first such situation-action pair is likely to have a conscious warrant based on, say, heuristics, logic, strategy, or known mathematics. However with time and (sometimes considerable) repetition, the need for a conscious warrant may disappear. The situation may then become linked, in an automated way, to a tendency to carry out the corresponding action; and the individual will not be conscious of anything happening between the situation and the action. We see such automated situation-action pairs as persistent mental structures and have called the smallest of them *behavioral schemas* (Selden, McKee, & Selden, 2010; Selden & Selden, 2011). By a small situation-action pair, we mean one that is not equivalent to any sequence of smaller such pairs. While the word "schema" has been used in several ways in the literature, we only mean such a persistent mental structure.

Behavioral schemas

The formation of behavioral schemas, whether beneficial or detrimental, requires the development of a way of recognizing particular kinds of situations, and in response, enacting particular kinds of actions. It is possible that neither the kind of situation nor the kind of action for a potential behavioral schema exists as a concept in the surrounding culture. In that case, constructing a behavioral schema entails noticing, either explicitly or implicitly, similarities among situations and among the corresponding actions, and eventually reifying these into what amounts to conceptions (usually without any need for formal designations).

Properties of Behavioral Schemas

- (1) Within very broad contextual considerations, behavioral schemas are immediately available. They do *not* normally have to be recalled, that is, searched for and brought to mind.
- (2) Behavioral schemas operate outside of consciousness. A person is not aware of doing anything immediately prior to the resulting action – he/she just does it. Furthermore, the enactment of a behavioral schema that leads to an error is not under conscious control, and one should not expect that merely understanding the origin of the error would prevent its future occurrences.
- (3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.
- (4) A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral

schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot “chain together” behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

(5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person’s actions, but such influence is normally not precise enough for doing mathematics. Also, non-conscious stimuli that lead to action usually originate outside of the mind, not within it (as often happens in proof construction).

(6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing detrimental behavioral schemas, many of which have been tacitly acquired, requires similar, perhaps longer, practice (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

Implicitly acquired detrimental behavioral schemas can be enacted automatically in problem-solving situations. For example, some experienced teachers may have noticed that giving a counterexample to a student who consistently makes an errorful calculation, such as $(3a + b)/3c = (a + b)/c$ or $\sqrt{a^2 + b^2} = a + b$, is often not very effective. This can be so even when the student seems to understand the counterexample. Our view of

behavioral schemas suggests an explanation. If an incorrect algebraic simplification is caused by the enactment of a behavioral schema, then the resulting action (the incorrect simplification) would follow directly from the situation, that is, would not be under conscious control. To change the student's behavior, one might try to change the detrimental behavioral schema not only by providing a counterexample, but also by suggesting a number of relevant problems and some monitoring.

Using our Theoretical Underpinnings to Teach Proving

Having students write a proof framework first, enables them to “get started” on writing a proof and reveals the mathematical problem(s) to be solved. What happens next depends on a student's ability to solve various mathematical problems. Informally, one of our graduate students has reported that writing a proof framework helped her organize her thoughts on her high stakes mathematics PhD comprehensive examinations. Also, looking for students' detrimental behavioral schemas and trying to help them replace them with beneficial schemas has enabled us to help students with proof construction. Sometimes acquiring a beneficial schema can take a long time.

Mary's Reaction to Considering Fixed, but Arbitrary Elements

There are theorems, particularly in real analysis, that involve several quantifiers. For example, proving a function f is *continuous* at a involves proving that for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all x , if $|x-a| < \delta$ then $|f(x)-f(a)| < \varepsilon$. For such proofs, one needs to consider a fixed, but arbitrary ε . Students are often reluctant to do this. We conjecture this is because they do not feel it right or appropriate to do so.

Mary, an advanced mathematics graduate student, was interviewed about events that took place two years earlier when she was taking both a pilot version of our proofs course and Dr. K's graduate real analysis course. In the homework for Dr. K's course, Mary needed to prove many statements that included phrases like 'For all real numbers $\varepsilon > 0$,' where ε represented a variable (the situation). In her proofs, Mary needed to write something like 'Let $\varepsilon > 0$,' where ε represented an arbitrary, but fixed number (the action).

When Mary was interviewed about this situation-action pair she said the following:

Mary: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to "do it", so I'm going to do it because I want to get full credit. And so I didn't have a sense of why it worked.

Interviewer: Did you have any feeling ... if it was positive or negative, or extra ...

Mary: Well, I guess I had a feeling of discomfort ...

Interviewer: Did this particular feature [having to fix ε] keep coming up in proofs?

Mary: ... it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense ... By the end of the semester this was very comfortable for me.

Mary told us that, after completing each such proof, she attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof.

However, only after repeatedly executing this situation-action pair, and convincing

herself that her individual proofs were correct, did she develop a feeling of appropriateness.

Willy's Focusing Too Soon on the Hypotheses

We have observed that after writing little more than the hypotheses, some students turn immediately to focusing on using the hypotheses, rather than unpacking the conclusion to see what is to be proved, after which they often cannot complete a proof. For example, late in our proofs course, Willy was asked to prove the theorem: *Let X and Y be topological spaces and $f : X \rightarrow Y$ be a homeomorphism of X onto Y . If X is a Hausdorff space, then so is Y .* Because only ten minutes of class time remained and Willy had indicated that he had not yet proved the theorem, we asked him to “do the set-up”, that is, construct a proof framework (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

On the left side of the blackboard, Willy wrote:

Proof. Let X and Y be topological spaces.

Let $f : X \rightarrow Y$ be a homeomorphism of X onto Y .

Suppose X is a Hausdorff space.

...

Then Y is a Hausdorff space.

Then, on the right side of the board which was for scratch work, he listed one after the other: “homeomorphism, one-to-one, onto, continuous (f is open mapping)”. He then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, “And, I was just trying to just think,

homeomorphism means one-to-one, onto, ...” After some discussion about the meaning of homeomorphism, the first author said, “There is no harm in analysing what stuff you might want to use, but there is more to do before you can use any of that stuff”, meaning that the conclusion should be examined and unpacked first.

We inferred that Willy was enacting a behavioral schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining the conclusion. We conjectured that Willy and other students, who are reluctant to look at, and unpack, the conclusion feel uncomfortable about this, or perhaps feel it more appropriate to begin with the hypotheses and work forward.

Sofia’s Reaction to Not Having an Idea

Sofia was a diligent first-year graduate student; however, as our proofs course progressed, an unfortunate pattern in her proving attempts emerged. When she did not have an idea for how to proceed, she often produced what one might call an “unreflective guess” only loosely related to the context at hand, after which she could not make further progress. Although we could sometimes speculate on the origins of Sofia’s guesses, we could not see how they could reasonably have been helpful in making a proof, nor did she seem to reflect on, or evaluate, them herself. We inferred that Sofia was enacting a behavioral schema: she was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing any approach that usually was only loosely related to the problem at hand without much reflection on its usefulness.

Using our idea of behavioral schemas, we devised an intervention that was used in tutoring sessions with Sofia. We attempted to deflect implementation of her “unreflective guess” schema, by suggesting that she write the first and last lines of a proof, unpack the conclusion, and then do something else, such as draw a diagram, review her class notes, or reflect on everything done so far. These suggestions and guidance helped Sofia construct a beneficial behavioral schema. As the course ended, this intervention of directing Sofia to do something else was beginning to show promise. For example, on the in-class final examination Sofia proved that if f , g , and h are functions from a set to itself, f is one-to-one, and $f \circ g = f \circ h$, then $g = h$. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-centered parts of at least a few proofs by the end of the course, and suggests her “unreflective guess” behavioral schema was weakened (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

Future Research on Proof and Problem Solving

The above discussion has not only synthesized some of the literature on proof and problem solving, it has highlighted several areas that could use more research. These are: how informal arguments are converted into acceptable mathematical form; how representation choice influences an individual’s problem-solving and proving behaviour and success; how students’ and mathematicians’ prove theorems in real time (especially when working alone); how various kinds of affect, including beliefs, attitudes, emotions, and feelings, are interwoven with cognition during problem solving; which characteristics

make a problem non-routine (for an individual or a class), that is, what are the various dimensions contributing to non-routineness; and how one might foster mathematical “exploration” and “brainstorming” as an aid to problem solving.

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Cognition and Affect in Mathematics Problem Solving with Prospective Teachers¹

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Abstract: Recent studies relating the affective domain with the teaching and learning of mathematics, and more specifically with mathematics problem solving, have focused on teacher education. The authors of these studies have been ever more insistently pointing to the need to design educational programs that take an integrated cognitive and affective approach to mathematics education. Given this context, we have designed and implemented a program of intervention on mathematics problem solving for prospective primary teachers. We here describe some results of that program.

Keywords: Mathematics Teaching, Problem Solving, and the Affective Domain

Problem solving (PS) has always been regarded as a focal point of mathematics, and in the last 30 years its presence in curricula has increased notably (Castro, 2008; Santos, 2007). It is regarded as the methodological backbone to approach mathematics content since it both requires and helps develop skills in analysis, comprehension, reasoning, and

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application. At the same time, it is now being proposed as an item of curricular content in its own right as a core competence that students need to acquire. Castro (2008) and Santos (2008) recognize that attempts to teach students general PS strategies have been unsuccessful. Also, it seems important to emphasize the lack of attention in textbooks to learning heuristic problem solving strategies (Schoenfeld, 2007; Pino & Blanco, 2008).

The results of the Programme for International students Assessment (PISA) of 2003, 2006, and 2009 have highlighted the importance of mathematics problem solving (MPS) in compulsory education. One of the aspects tacitly accepted in the curricula at this educational level is the influence of affect on the teaching and learning of mathematics in general, and of MPS in particular. Already in the 1980s, Charles & Lester (1982) were observing that: "*The problem solver must have sufficient motivation and lack of stress and/or anxiety to allow progress towards a solution*" (p.10). In their work, they recognized that factors involving cognition, experience, and affect influence the MPS process. Among the affective factors that they explicitly noted were interest, motivation, pressure, anxiety, stress, perseverance, and resistance to premature closure. It is currently accepted that the cognitive processes involved in PS are susceptible to the influence of the affective domain in its three fundamental areas: beliefs, attitudes, and emotions (Sriraman, 2003).

Initial Primary Teacher Education, the Affective Domain (Beliefs, Attitudes, and Emotions), and Problem Solving

Research on the affective domain has also expanded to the field of initial teacher education and the professional development of in-service teachers. It is considered that, in their actions in the classroom, teachers cannot dissociate affect from content when faced with a specific activity for pupils at a specific level.

Influence of beliefs

"When prospective primary teachers enter an Initial Education Centre they bring with them the educational baggage of many years in school. They thus naturally have conceptions and beliefs concerning Mathematics and the teaching/learning of Mathematics that derive from their own learning experience" (Blanco, 2004, p.40). Furthermore: "Few apparent changes in their beliefs were affected as a result of traditional mathematics method courses" (Chapman, 2000, p.188).

It is important to distinguish the beliefs of prospective primary teachers (PPTs) about mathematics as an object – about its teaching and learning, beliefs which depend on affect – and their beliefs about themselves as learners – beliefs related to their self-concept, self-confidence, expectations of control, etc.

Beliefs about mathematics and problem solving

According to Llinares and Sánchez (1996), prospective teachers acquire a technical school culture that conditions their approach to mathematics tasks and learning as future teachers. For them mathematics teaching is the transmission of specific information and mathematics learning is done through repetition. The teacher's role consists of presenting the content in a way that is clear and concise, and the learner's role consists of listening and repeating.

According to Szydlik, Szydlik & Benson (2003), research has shown that prospective teachers tend to "*see mathematics as an authoritarian discipline, and that they believe that doing mathematics means applying memorized formulas and procedures to do textbook exercises*" (Szydlik, Szydlik & Benson, 2003, p.254).

Prospective primary teachers have a very traditional idea of mathematics problems that are quite different from the suggestions of current curricular proposals (Blanco, 2004; Johnson, 2008). This leads to "*a contradiction between their personal experience, which they judge as having been negative and monotonous, and their conception of mathematics as linked to reasoning and rigour*" (Blanco, 2004, p.42). Furthermore, these beliefs constitute a kind of lens or filter through which they interpret their own personal learning processes and orient their teaching experiences and behaviours (Chapman, 2000), thus limiting their possibilities for action and understanding (Barrantes & Blanco, 2006). Moreover, "*these beliefs are [internally] consistent*" (Blanco, 2004, p.41).

For Schoenfeld (1992), beliefs form a particular view of the world of mathematics, setting the perspective from which each person approaches that world, and they can determine how a problem will be tackled, the procedures that will be used or avoided, and the time and intensity of the effort that will be put into the task. Consequently, these beliefs need to be taken into account in teacher education, which, if necessary, will have to try to promote their change and the generation of new beliefs (Blanco, 2004).

Prospective primary teachers regard MPS as a rote mechanical process, have few resources with which to represent and analyse problems, never look for alternative

strategies or methods for their solution, and make no use of the different guidelines and hints they might be given to help them towards a solution (Blanco, 2004; Córcoles & Valls, 2006; NCTM, 2003), thereby generating a vision of themselves as incompetent problem solvers (NCTM, 2003).

The beliefs that most influence motivation and achievement in mathematics are *students' perceptions about themselves in relation to mathematics* (Kloosterman, 2002; Skaalvik & Skaalvil, 2011). Hernández, Palarea & Socas (2001) and Blanco et al. (2010) note that PPTs generally do not see themselves as capable or skilled as problem solvers, with most of them experiencing feelings of uncertainty, despair, and anxiety which block their approach to the task – in sum, most of them consider themselves to be incompetent at PS. A major difference between successful and unsuccessful problem solvers lies in their beliefs about MPS, about themselves as solvers, and about how to approach the task (NCTM, 2003).

Influence of attitudes

What students think about mathematics influences the feelings that surface towards the subject and their predisposition to act in consequence. That is, if students have negative beliefs about mathematics or its teaching, they will tend to show adverse feelings towards related tasks, in particular presenting avoidance behaviour or simple rejection of those tasks. This predisposition towards certain personal intentions and behaviours is what one calls attitudes.

One can distinguish between mathematics attitudes themselves and attitudes towards mathematics as a subject. Mathematics attitudes have a marked cognitive component, and relate to general cognitive skills that are important in mathematics tasks.

Studies in Spain have shown that PPTs have few mathematics attitudes in this sense on aspects related to PS (Blanco, 2004; Corcoles & Valls, 2006).

In attitudes towards mathematics, the affective component predominates. It is manifest in interest, satisfaction, and curiosity, or, on the contrary, in rejection, denial, frustration, and avoidance of mathematics tasks. Positive interest and attitudes towards mathematics seem to decline with age, especially during secondary education (Hidalgo, Maroto, Ortega & Palacios, 2008).

Influence of emotions

The emotions aroused in students by a mathematics task are affective responses characterized by high intensity and physiological arousal reflecting the charge of positive or negative meaning that a task has for them. Studies of emotion have focused on the role of anxiety and frustration and their impact on achievement in mathematics, noting that one of the difficulties of mathematics education is seeing its teaching as essentially cognitive, and detached from the field of emotions (De Bellis & Goldin, 2006).

Emotions appear in response to an internal or external event which has a charge of positive or negative meaning for the person. Thus, in facing a mathematical task a pupil may encounter difficulties which lead to the frustration of their personal expectations, causing the appearance of essentially negative valuations of the subject. Various authors agree that anxiety interacts negatively with cognitive and motivational processes, and therefore with the pupil's overall performance (De Bellis & Golding, 2006; Zakaria & Nordim, 2008). In this regard, Hidalgo, Maroto, Ortega & Palacios

(2008) found a strong negative correlation between pupils' levels of anxiety towards mathematics and their final marks at the end of the course. This correlation is also present when comparing the levels of anxiety and positive attitudes towards mathematics. The relationship between anxiety and mathematics education has also been transferred to the case of prospective teachers, for which there is already a substantial literature (Peker, 2009).

Recent work has established relationships between anxiety and self-confidence. Thus, pupils with more anxiety towards mathematics have less confidence in their mathematical abilities and as learners of mathematics (Gil, Blanco & Guerrero, 2006; Isiksal, Curran, Koc & Askun 2009). *"Many of the negative emotional attitudes towards mathematics are associated with anxiety and fear. Anxiety to be able to complete a task, fear of failure, of making mistakes, etc., generate emotional blockages of affective origin that have a repercussion on the students' mathematics activity"* (Socas, 1997, p.135). Zevenbergen, (2004) notes that PPTs show *"low levels of mathematics knowledge as well as considerable anxiety towards the subject"* (Zevenbergen, 2004, p.5).

With respect to mathematics teaching and learning, there are various moments at which the relationship between emotions and cognitive processes becomes visible: when, following the proposal of a mathematics task, the structure of the activity is understood or relevant information is retrieved; when problem-solving strategies are being designed, including the recall of formulas or mechanical procedures; and when the PPTs are involved in the process of the control and regulation of their own learning coupled with a clear methodological approach to teaching the mathematics which they had come to reject.

It therefore seems appropriate to consider studying the beliefs, attitudes, and emotions of prospective teachers when they are dealing with PS. The lack of reflection on these issues is one reason for the persistence of PPTs' inappropriate conceptions and attitudes. In their passage through initial teacher education, they have not been led to re-conceptualize their role as primary teachers. Authors such as Mellado, Blanco & Ruiz (1998), Chapman (2000), Uusimaki & Nason (2004), and Malinsky, Ross, Pannells & McJunkin (2006) suggest that the origin of the negative beliefs of prospective teachers in their initial teacher education could be attributed to the influence of their own experiences as learners of mathematics, i.e., to their experiences when they themselves were being taught mathematics in school and to their teachers at that time, and to the mathematics courses in their teacher education programs.

A Research Project with Prospective Primary Teachers on Cognition and Affect in Problem Solving

The above references clearly show that the cognitive and the affective are closely related, that beliefs, attitudes, and emotions influence knowledge, and that knowledge in turn affects those same three aspects.

The study of this relationship between affect and cognition has also been explored with teachers. Teachers' concepts and values determine the image of mathematics in the classroom, and condition the type of teacher-pupil relationship. Conceptions influence attitudes, and both of them influence the teacher's behaviour and the pupils' learning (Ernest, 2000). In order to foster change in our prospective teachers' views of teaching,

we shall have to incorporate conceptions and attitudes as part of a process of discussion and reflection in our initial teacher education programs (Mellado, Blanco & Ruiz, 1998; Stacey, Brownlee, Thorpe & Reeves, 2005; Johnson, 2008). There thus seems to be a clear interest in studying the affective and emotional factors involved in the mathematics education of PPTs since, as future teachers, their beliefs and emotions towards mathematics will influence both the level of achievement and the beliefs and attitudes towards the subject of their pupils.

De Bellis & Goldin (2006) and Furinghetti & Morselli (2009) note that studies of students' performance and problem solving have traditionally concentrated primarily on cognition, less on affect, and still less on cognitive-affective interactions. However, a growing number of studies recognize the importance of integrating the affective and cognitive dimensions into the teaching and learning of mathematics (Amato, 2004; Zan, Bronw, Evans & Hannula, 2006; Furinghetti & Morselli, 2009; Blanco et al., 2010).

Some authors, such as Furinghetti & Morselli (2009), specifically note the need to simultaneously develop cognitive and affective factors in teacher education programs. In this regard: "*The role of teacher education is to develop beginning teachers into confident and competent consumers and users of mathematics in order that they are better able to teach mathematics*" (Zevenbergen, 2004, p.4).

In this context, we considered that there was a need to undertake a research project on MPS in initial primary teacher education with the consideration of its cognitive and affective aspects. Initial teacher education is conceived of as being just one part of a continuous and permanent process in a teacher's professional life in which emotional education is an indispensable complement to cognitive development. Indeed, cognitive

and affective aspects are essential elements in the development of teaching as a profession.

We believe that gaining the capacity to solve mathematics problems should be an achievable goal in an educational environment in which students are allowed to generate their own PS strategies and compare them with other alternatives. In particular, we believe that the way in which PS is approached is highly personal. Each student will have to be helped to discover their own particular style – their own capabilities and limitations. We must avoid conveying to our students only heuristic rules or methods, but instead be sure to help them develop positive attitudes and emotions towards MPS based on their own past and present experiences.

Objective of the research project

In our research study, we set ourselves the following general objective: to design, develop, and evaluate an intervention program to enhance the performance of PPTs in MPS, and to lay the foundations for them to learn to teach MPS at the primary school

level, integrating in a single model cognitive aspects of PS and emotional education (Annex 1).

Additionally, we set different specific objectives relating to the study population, two of which were:

- To describe the participating prospective teachers' conceptions about MPS.
- To describe and analyse their attitudes, beliefs, and emotions related to MPS, and in particular their expectations of control.

In addition, two specific objectives relating to the teacher education program were pursued:

- To evaluate the development of the program with respect to the PPTs' levels of anxiety.
- To describe the aspects of the program which they found to be most significant.

During the 2007-08 academic year, we conducted a pilot study that served to fine-tune the program. We performed the actual field-work during 2008-09.

Data collection and analysis

The nature of the research problem and the data collection led us to use a combination of qualitative and quantitative methods to relate, compare, and contrast the different types of evidence. The implementation of the program followed an action research approach since the ultimate goal is to help the participants develop their thinking, modify their attitudes, and seek ways to overcome their difficulties in MPS.

Annex I presents the plan of the 13 sessions comprising the program, specifying the objectives of each session, the instruments used to obtain information (open and closed questionnaires, diaries, and forums), the nature of the data, and the corresponding type of analysis. The participants in the program were a core group of 55 PPTs in the Education Faculty of the University of Extremadura (Spain) in the third year of their course.

All the program sessions lasted two hours, and were audio and video recorded, accompanied by field notes. The Moodle Virtual Platform was used as support for the program's documents, information, and forums, as also is indicated in Annex I.

Apart from the open and closed questionnaires specifically indicated in Annex I, the following research tools were employed:

- *Observation of the behaviour in the classroom* of both teacher and students, video recorded with two cameras, with subsequent transcription and analysis.
- The *Moodle platform* (Universidad de Extremadura) is a useful tool for the presentation of information and communication. It allows information to be stored for later analysis (both qualitative and quantitative), with the date and time and the subject contributing the information being reliably logged. It allows one to evaluate the participation, and to see whether the students have attained specific learning objectives, providing feedback as well as motivation to the students. A reference to the use of this platform in the present research can be found in Caballero, Blanco & Guerrero (2010).
- *Diaries* (Nichols, Tippins & Wieseman, 1997), kept on the Moodle virtual platform. These allow the collection of observations, sensations, reactions, interpretations, anecdotes, introspective remarks about feelings, attitudes, motives, conclusions, etc.
- *A forum*, also via the Moodle virtual platform, on some of the specific content or situations arising in class.

For the data analysis, we used the program packages SPSS 15.00 for the quantitative analysis of the questionnaires that we are given in Annex I. For the qualitative analysis, we followed the recommendations of Goetz & LeCompte (1984) and Wittrock (1986), establishing a process similar to that described in Barrantes & Blanco (2006) based on units of analysis (Goetz & LeCompte, 1984) and the categories noted in the instruments.

Analysis of Results and Discussion

The breadth of the research study and the characteristics of this present communication only allow us to present some partial results. In particular, we shall refer to some of the results on the PPTs' conceptions about PS, on certain aspects related to the affective domain, particularly those concerning the students' expectations of control, and on some general aspects of the program's evaluation.

What do the prospective teachers understand by a mathematics problem?

Our analysis of the questionnaires showed the prospective teachers to hold very traditional conceptions about mathematics problems. Thus, they referred to them as closed statements which explicitly or implicitly indicate the procedure to follow for their solution. The responses to the items of the questionnaire on "*What do I understand by a mathematics problem?*" (Annex II) reflect the classifications noted by some authors in the literature. In this sense, their formulation of a problem is in the form of a text which gives all the information to be resolved, which Borasi (1986, p.135) calls a "*word problem*"; and the method of solution explicitly or implicitly suggested in the text involves a translation of the words of the problem to a mathematical expression, which Charles & Lester (1982, p.6) call a "*simple or complex translation problem*"; and the solution of this expression involves using a known algorithm, which Butts (1980, p.24) refers to as an "*application problem*".

The basic referents of their problems are arithmetic operations, algebraic algorithms, or, to a lesser extent, calculations of areas. It was interesting to note that the contexts they describe are those that have been traditional in mathematics textbook problems since the late nineteenth century. Thus, in both years of the study, there are

references to problems of taps, the number of heads and legs of farm animals, trains and distances, and the comparison of ages. It stood out that in no case was there any reference to specific situations of their or their potential pupils' immediate environment, or to such everyday resources as mobile phones or personal hobbies. This result, which we did not find in the literature we reviewed, seems especially important because it is necessary to link problem solving with the pupils' interests and relate the problems to their immediate environment.

Of a total of 178 problems, 126 (70.8%) were arithmetic with a structure involving addition or multiplication, representing elementary shopping or business situations². Another 31 (17.4%) were questions of arithmetic proportionality³. There were 7 problems (3.9%) involving equations in which the situations were related to ages, taps, speeds of trains or cars, and farm animals⁴. Geometry problems accounted for 5.7%, and were very basic, referring to the calculation of areas⁵.

In analysing this PS situation in class with the students (4-XI-2008), we thought that it was convenient to focus on the following question:

- *Do you think there are other types of problems? If so, write down two examples.*

² There are 47 apples in an apple tree. Mary has picked 37 apples. How many apples are left in the tree?
In a fruit shop, 1 kg of apples costs 1.75 euros. If Laura buys two kilos, how much money has she spent altogether?

³ We know that Juan has eaten $\frac{2}{3}$ of a cake, and his brother Sergio $\frac{5}{6}$ of the rest. How much of the cake is left?
Three friends have 40 euros to spend. The first spent $\frac{2}{5}$ of the total, and the second $\frac{2}{3}$ of what the first spent. How much did the third spend?

⁴ On a farm there are horses and chickens. In total there are 74 feet and 12 beaks. How many horses and how many chickens are there on the farm?

⁵ Calculate the area of a square whose side is 2 cm.

Observation of the recordings and the analysis of this last question brought out the difficulty they were having in establishing mathematics activities that were different from those they had proposed, and which had been analysed previously in that same program session.

Thus, 14 participants answered directly that there are no different types of problem. Two examples of these responses are the following:

- *"I think not, because throughout my school life I always had problems of the same type."*
- *"The truth is that I have no idea. The maths problems that I know are those of always."*

Another 34.5% again insisted on the same kinds of problem noted above, but involving situations concerned with other mathematics content such as statistics or probability that had not specifically appeared previously.

The question prompted some students to guess that there really must exist other types of mathematics problems, but they found themselves unable to give any examples:

- *"After what we have seen today, there must be other types, but right now I can't think of any."*
- *"Clearly there must be, but I am unable to find any examples."*

This conception of MPS contrasts with what is imagined in today's curricula which consider a much broader view of problems, different perspectives (in terms of content, application, and methods), and in the usual classifications such as those we

presented in the program which show a variety of different possibilities. Consequently, initial teacher education programs should intensify the attention given to these issues.

The PPTs' expectations as problem solvers themselves

For the 5th session, we adapted the Battery of Scales of Generalized Expectations of Control, BEEGC-20 (Palenzuela et al., 1997), to the context of PS. This adaptation consisted of a closed questionnaire, with responses on a scale of 1 to 10, targeted at determining the students' expectations of control when faced with MPS. We wanted to examine whether they believed their success or failure in PS would be a true reflection of their actions, or rather be simply at the mercy of luck or chance. We also wanted to determine their expectations of self-efficacy, i.e., to what extent they felt themselves capable of solving mathematics problems. This was the second specific objective that we indicated above in Sec. 3, and whose partial results we shall consider in the following paragraphs.

The results showed the participating students had a high expectation of contingency on their actions (perseverance, effort, commitment, ability), and a low external locus-of-control reflecting luck or chance.

Thus, their responses to Item 1 (*"My success in solving mathematics problems will have much to do with the effort I put into it"*), with a mean score of 6.69, showed that they see effort as being crucial for success in MPS. The result was similar for Item 15 (*"If I try hard and work, I will be able to solve successfully the mathematics problems that I am set"*) which was directly related to the dependence of success in problem solving on effort and application. Additionally, 54.9% indicated on Item 11 (*"In general, success or*

failure in solving a mathematics problem will depend on my actions") that success would depend on their own actions.

Rinaudo, Chiecher & Donolo (2003) and Martínez (2009) also refer to high levels of control, and the subjects studied by Orozco-Moret & Diaz (2009) and Yara (2009) attributed success in MPS to ability and effort. However, many prospective teachers become blocked when faced with these mathematics tasks, and in many cases end up by abandoning the effort, as was shown in the observations of their behaviours in the fourth and eighth sessions. This reflected a certain contradiction between what they expressly stated and their actions in class in dealing with these mathematics tasks. These observations also revealed their lack of knowledge of procedures and heuristics with which to tackle mathematics problems.

With respect to their expectations of self-efficacy, these prospective teachers showed little confidence in themselves and their abilities when solving quite normal problems of mathematics. Thus, 70.58% said they had *"thoughts of insecurity when doing MPS"* (Item 14) and 64.7% *"had doubts about their ability to solve mathematics problems"* (Item 2). In this regard, Caballero, Guerrero & Blanco (2008) and Hernández, Palarea & Socas (2001) also note that PPTs in initial teacher education do not see themselves as capable or skilled in mathematics.

That the PPTs mainly attributed their success or failure in solving mathematics problems to their own actions and not to helplessness or luck means that they are largely attributing success to internal, unstable, and controllable factors. This is beneficial for their future learning situations. On the contrary, their low expectations of self-efficacy,

i.e., their lack of confidence in their capacity to solve the mathematics problems they will be set, would seem to be prejudicial for the future learning.

Their high expectations of contingency together with low expectations of self-efficacy foster the development of negative attitudes towards solving mathematics problems, leading the PPTs to consider that failure in this activity is due to a lack of ability rather than to any lack of effort. As suggested by Martínez (2009), the result is to severely lower their expectations of success, and to encourage them to abandon any persistence in trying to learn how to solve mathematics problems. Similarly, their low expectations of self-efficacy and their not very high expectations of achievement are suggestive of an algorithmic approach to learning.

Some results of the program of MPS and emotional education

To evaluate the program, one of the instruments we used was the State/Trait Anxiety Index (STAI) self-assessment questionnaire adapted from Spielberger (1982). We presented this questionnaire on three occasions – at the beginning of the program, on its completion, and four months after its completion. In the present communication, we shall compare the results of four of its items:

- *When I am solving mathematics problems I feel calm.* (calm)
- *When I am solving mathematics problems I feel secure.* (security)
- *I feel comfortable when I am solving a mathematics problem.* (comfort)
- *I feel nervous when I am faced with a mathematics problem.* (nervousness)

The results indicated a positive trend in the period covered by the program, with a decrease in anxiety about MPS continuing four months after the program. Even though

there was a slight regression relative to the actual moment of completion of the program, the data were better than those obtained at the beginning. This reflects a major advance with respect to the control of anxiety following participation in the intervention program.

The participants also declared a change in attitude: "*The program has changed our attitude to MPS, even though the content we have acquired throughout our lives is impossible to change in just 13 sessions*" (10 FS 1). Other evidence shows their desire to integrate cognitive with affective aspects: "*This program has also been useful in that, when we are teaching, we will know to take into account not only what our pupils know, but also their attitudes and emotions, which, by my own experience, I know have a great influence both positive and negative*" (7 DP 4).

The debate that took place in the evaluation session and in the forums showed important reflections and concerns which we interpreted as an attempt to approach PS in a systematic and orderly fashion, as a result of the procedures and heuristics worked on in the general model during the program. Thus, in the evaluation session (Session 13), one student notes that: "*The execution of the steps [in a problem solving strategy] helps me to concentrate, to analyse the problem, and to understand it better*" (18 MV 2). Another student says: "*In my case, it helped me to be more orderly in presenting problems*" (10 FS 2), and expresses a desire to apply it in her future work as a teacher: "*We dealt with aspects that we shall subsequently transmit to our own pupils, and with methods that we will use such as the steps to follow for problem solving and relaxation methods*" (18 MV 3).

Conclusions

The present work has confirmed the importance of considering in an integrated form the cognitive and affective aspects of mathematics teaching and learning at different levels, especially in initial teacher education. This can help foster the change in our prospective teachers' beliefs and attitudes along the line laid out in current curricular proposals.

As one of the students stated, it is difficult in just 13 class sessions for our PPTs to learn both MPS for themselves and how to teach it to their future primary pupils. But it is gratifying to note that a change in attitude was initiated, and that they themselves valued positively their first-person experience in the program, and that the content of the program fell within their expectations as future teachers.

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Annex I. Intervention Program

The program ran from October to December 2008 except for the evaluation session which took place in April 2009, four months after completion of the program.

Session 1. (27 and 28 October) Presentation of the program to students. 55 PPTs.

27-X. The students were provided with extensive information about the workshop, working methods, and objectives.

27-X. Initial questionnaire - Commitment to the workshop. Objective: To determine the participants' self-perception as problem solvers, and their degree of commitment to the workshop. (Open questionnaire; qualitative analysis.)

28-X. Conceptions and knowledge of MPS. What do I understand by a problem? Objective: To analyse students' conceptions and knowledge about mathematics problems. (Open questionnaire; qualitative analysis.)

28-X. Affective domain in MPS. Objective: To examine the students' affective factors (beliefs, emotions, and attitudes) that influence their development in MPS. Adaptation to MPS of the questionnaires of Gil, Blanco & Guerrero (2006), and Caballero, Guerrero & Blanco (2008) on the affective domain in mathematics. (Closed questionnaire; quantitative analysis.)

Session 2. (4 November) Conceptions and affective aspects of MPS. 53 PPTs.

Presentation and discussion of the results of the previous questionnaires. We analyse the PPTs' conceptions of MPS, comparing them with the perspectives outlined in the curriculum (as specific content and as method) and with those described by different authors. Likewise, the results of the questionnaire on affect are discussed, expanding them with other previous results (Gil, Blanco & Guerrero, 2006; Caballero, Guerrero & Blanco, 2008).

Session 3. (7 November) Problems vs exercises. 55 PPTs.

Differentiation of exercises and problems, and presentation of other types of problems based on different classifications (Borassi, 1986; Butts, 1980; Charles & Lester, 1982; etc.).

Forum on the Moodle platform concerning the content of Sessions 2 and 3.

Session 4. (11 November) How do I approach MPS? Before, during, and after. 55 PPTs.

Pre-test.

Objective: To evaluate the participants' own impressions that arise at different moments of MPS. Two problems proposed for solution which will allow us to observe their knowledge and affects at different stages of the PS process at this early stage of the program. They will be asked to respond to the same open questionnaire on three occasions – before seeing the problem, while they are solving it, and after having had to deal with the activity. (Open questionnaire; qualitative analysis.)

Adaptation of the STAI (state / anxiety) to MPS – pretest.
Objective: To determine the level of anxiety that MPS provokes in the students.
Adaptation of the STAI (Spielberger, 1982) to MPS.
(Closed questionnaire; quantitative analysis.)

Session 5. (14 November) Personal involvement. Causal attributions, and behaviour and stress. 54 PPTs.

BEEGC-20 adapted to MPS.
Objective: To examine the causal attributions relating to MPS (expectations of success and of the locus-of-control, of helplessness, and of self-efficacy).
Adaptation of the BEEGC-20 Questionnaire of Palenzuela et al. (1997) to MPS.
(Closed questionnaire; quantitative analysis.)

Initiation of a discussion in class on the content of the questionnaire, which will be followed by a specific forum on the Moodle virtual platform.

Session 6. (18 November) Emotional coping: relaxation, breathing, and self-instruction. 55 PPTs.

Presentation of results of the previous questionnaire, and analysis of the interventions in the forum.

Information and explanation of different aspects of emotional education and its relationship with PS. Mainly the topics covered in the questionnaire.

Session 7. (21 November) Overview of the Integrated Model⁶ of MPS I. 55 PPTs

Sessions 8–10. (25 November, 2 December, 5 December) Application of the Integrated Model, with specific problems.

Session 11. (9 December) Specific activities of the Integrated Model for primary pupils. 55 PPTs.

In this session, we present specific activities adapted to the primary level that allow the PPTs to work with pupils aged 6 to 12 at different stages of the general model – basically, the comprehension and analysis of problems, and the design of strategies.

Session 12. (12 December) General model of MPS. 55 PPTs.

The PPTs work specifically on problems in a complete and continuous application of the Integrated Model.

STAI adapted to MPS (state / anxiety). Post-test I.
Objective: To determine the level of anxiety that MPS provokes in the students

⁶ Blanco et al. (2010) proposed a theoretical model based on general models of PS (Bransford & Stein, 1987; Polya, 1957; Santos, 2007), on the cognitive-behavioural models of Zurilla & Goldfried (1971) and on the systemic model of De Shazer and the Milwaukee group (De Shazer, 1985). It consists of a process of experimentation and reflection based on the general model, and structured into five steps: (i) accommodation / analysis / understanding / familiarization with the situation; (ii) search for and design of one or more PS strategies; (iii) execution of the strategy or strategies; (iv) analysis of the process and the solution; and (v) How do I feel? What have I learnt?

after the workshop. (One problem.) Adaptation of the STAI of Spielberger (1982) to MPS. (Closed questionnaire; quantitative analysis.)

Session 13. (16 December) Evaluation of the PPTs and the workshop. 55 PPTs.

Evaluation: Proposal of a problem for the PPTs to solve by following the Integrated Model, in order to evaluate the knowledge they have acquired about the General Model.

Workshop evaluation. How have I managed my resources? Objective: To determine the strengths and weaknesses of the workshop, and the progress the participants have made in MPS. (Open questionnaire; qualitative analysis.)

Classroom discussion about how the workshop has functioned in relation to its proposed objectives, and about the participants' individual goals and commitments. Audio and video recordings and field notes, and opening a forum on the Moodle virtual platform.

Session to evaluate the research. (April 2009) 34 PPTs.

STAI (state / anxiety). Post-test II. Objective: To determine the level of anxiety that MPS provokes in the students four months after completion of the program.

Annex II

"What do I understand by a mathematics problem?"

a. Full name _____

1. Formulate the statement of three mathematics problems.
2. Indicate why mathematics problem solving is important in compulsory education.
3. Write down some personal reflections about your own experience in mathematics problem solving in primary and secondary school.
4. What consideration does mathematics problem solving merit on your part?
5. Add something that you find significant, and have not written.

Developing the art of seeing the easy when solving problems

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Introduction

For Leonardo da Vinci “*saper vedere*”, that is, knowing how to see, or having the art to see, was the key to unlocking the secrets of the visible world. *Saper vedere* included a precise sensory intuitive faculty as well as artistic imagination (Heydenreich, 1954) which were at the root of Leonardo’s inventiveness and creativity. According to Leonardo, to understand, you only have *to see things properly* (Bramly 1994, p. 264). Knowing how to see is also important in mathematics. The Italian mathematician Bruno de Finetti (1967) stresses this importance in his book on “*Saper vedere*” in mathematics. He highlights several aspects of knowing how to see in mathematics, such as knowing how to see the easy, how to see the concrete things, and how to see the economical aspects. He also discusses in what ways knowing how to see also helps us to better recognize the meaning of general and systematic methods of mathematics represented in formulas. His book starts by highlighting the importance of reflection for learning the art of seeing.

Reflection also plays a central role in Polya’s *Looking back* stage in problem solving. Polya’s heuristics also provide a language to help problem solvers think back about their problem solving experiences. As Lesh and Zawojewski (2007) point out, “*by describing their own processes*, students can use their reflections to develop flexible prototypes of experiences that can be drawn on in future problem solving” (p. 770).

Reading Polya's heuristics and looking at the examples he gives, we can concur with Lesh and Zawojewski that Polya's heuristics are intended to help students go *beyond* current ways of thinking about a problem, rather than being intended only as strategies to help students function better *within* their current way of thinking.

Lesh and Zawojewski (2007, p. 769) point out that when solving problems in complex problematic situations the abilities related to "seeing" are as important as abilities related to "doing". Schoenfeld (1985) found that individuals select solution methods to problems based on what they "see" in problem statements. Schoenfeld's data indicate that mathematical experts decide what problems are related to each other based upon the *deep structure* of the problems, whereas novices tend to classify problems by their *surface structure* (p. 243). Krutetskii (1976) found in his research that one trait of mathematically able students was to strive for a clear, simple, short, and thus "elegant" solution to a problem (p. 283). He also mentions that "a striving for simplicity and elegance of methods characterizes the mathematical thought of all prominent mathematicians" (p. 283-284). Krutetskii also describes how all the capable students, after finding the solution to a problem, continued to search for a better variant, even though they were not required to do so (p. 285). In contrast, average students paid no particular attention in his experiments to the quality of their solutions if there were no special instructions from the experimenter in that respect. Krutetskii observed that capable students "were usually not satisfied with the first solution they found. They did not stop working on a problem, but ascertained whether it was possible to improve the solution or to do the problem more simply" (p. 285-286).

In this article we will focus on learning the art of seeing the easy, by using an example of a problem posed to future secondary mathematics teachers. De Finetti indicates that it is often difficult to see the easy things, that is, to be able to distinguish, in the complexity of circumstances present in a problem, those that are enough to formulate the problem or that allow one to do the formulation as several successive steps that can be carried out easily.

The problem presented below was posed as part of a modeling course. Lesh and Doerr (2003) point out that from a modeling perspective, traditional problem solving is viewed as a special case of model-eliciting activities. Lesh and Doerr emphasize that “for model-eliciting activities that involve a series of modeling cycles, the heuristics and strategies that are most useful tend to be aimed at helping students find productive ways to adapt, modify, and refine ideas that they do have.” (p. 22). According to Lesh and Doerr, we need to put “students in situations where they are able to reveal, test, and revise/refine/reject alternative ways of thinking” (p. 26).

We will first present the strategy used by a group of future teachers, and then an approach gained by looking back at the problem and trying to see it at a glance. We finish with a brief discussion of why it would be worthwhile for prospective teachers to look back at the this and other problems..

The problem

During a course for prospective high school teachers, one of the assignments was to present a problem to their fellow students that could be modeled or solved with high school mathematics. The second author posed the following problem to her classmates.

You are attempting to bathe a cat in your kitchen. Unfortunately, the cat is not as open to the bath as you were hoping, and as a result you spill 3 gallons of water in your kitchen. Which brand of paper towel should you use to clean up the spill?

Brand A	Brand B
Paper towel is $\frac{1}{32}$ inches thick	Paper towel is $\frac{1}{64}$ inches thick
Total diameter of roll is 5 inches	Total diameter of roll is 6 inches
Diameter of hollow inside is 2 inches	Diameter of hollow inside is 2 inches
One sheet absorbs 1.5 fluid ounce	One sheet absorbs 1 fluid ounce
Each sheet is 10 inches long	Each sheet is 10 inches long

The assumption is that the price for the roll is the same for both brands. Notice that it is not necessary to know the width of the sheets, because we know how much each sheet absorbs for each brand. Remember that 1 gallon = 128 fluid ounces.

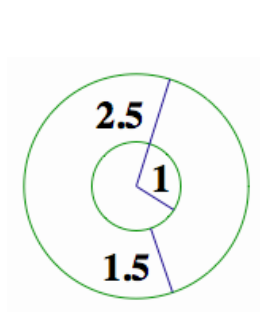


Figure 1a. Cross section of Brand A roll

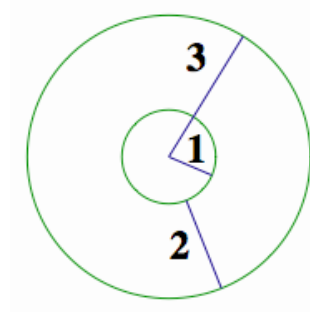


Figure 1b. Cross section of Brand B roll

The future teachers used an approximation by modeling the spiral cross section of the roll of paper as a series of concentric circles. Each successive layer was a little longer because the thickness of each sheet increased the diameter. The approach used by all the future secondary teachers to solve the problem was to find how many rolls of each brand

were needed to clean up the spill. To find this number they decided to compute how much water can be absorbed by one roll of each brand, finding first how many sheets are in each roll.

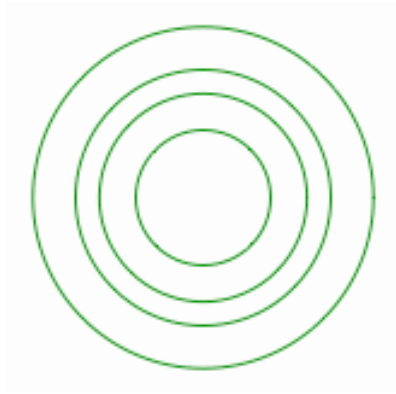


Figure 2. Concentric layers

Thus for Brand A the first layer has a length of $C_1 = 2\pi \times 1$ inches.

For the second and third layers the length in inches is

$$C_2 = 2\pi \times \left(1 + \frac{1}{32}\right)$$

$$C_3 = 2\pi \times \left(1 + \frac{2}{32}\right)$$

and in general, the length in inches of the k -th layer is

$$C_k = 2\pi \times \left(1 + \frac{k-1}{32}\right)$$

The number of layers, n , is given by dividing the thickness of the roll, 1.5 inches,

$$n = \frac{1.5}{\frac{1}{32}} = 48$$

by the thickness of each sheet, 1/32 inches, so

$$\begin{aligned}
 \text{The total length is thus } C_1 + C_2 + \dots + C_{48} &= 2\pi + 2\pi \times \left(1 + \frac{1}{32}\right) + \dots + \\
 2\pi \times \left(1 + \frac{47}{32}\right) & \\
 = 2\pi \left(48 + \frac{1}{32}(1 + 2 + \dots + 47)\right) & \\
 = 2\pi \left(48 + \frac{1}{32} \times \frac{47 \times 48}{2}\right) & \\
 = 96\pi + \frac{141}{2}\pi \approx 523 \text{ inches.} &
 \end{aligned}$$

Thus the total length of a roll of brand A is 523 inches. The length of each sheet is 10 inches, so there are about 52 sheets per roll. These sheets together can absorb $52 \times 1.5 = 78$ ounces of water. Thus each roll absorbs 78 fl. oz. of water. To clean 3

gallons = 3×128 fl. oz. = 384 fl. oz. we need $\frac{384 \text{ fl. oz}}{78 \frac{\text{fl. oz}}{\text{roll}}} = 4.9$ rolls. That is, we need almost 5 rolls of Brand A to clean the spilled water.

For Brand B the length of each layer is $C_k = 2\pi \times \left(1 + \frac{k-1}{64}\right)$ inches and the number of layers is $n = \frac{2}{\frac{1}{64}} = 128$. The total length is $C_1 + C_2 + \dots + C_{128}$

$$\begin{aligned}
 &= 2\pi + 2\pi \times \left(1 + \frac{1}{64}\right) + \dots + 2\pi \times \left(1 + \frac{127}{64}\right) \\
 &= 2\pi \left(128 + \frac{1}{64} \times \frac{127 \times 128}{2}\right) \approx 1602 \text{ inches.}
 \end{aligned}$$

The total length is thus about 1602 inches.

Because each sheet is 10 inches long, that is about 160 sheets. Each sheet absorbs one

fluid ounce, so one roll absorbs 160 fl. oz. To clean 3 gallons we need $\frac{384 \text{ fl. oz}}{160 \frac{\text{fl. oz}}{\text{roll}}} = 2.4$

rolls. Brand B is clearly the better choice for this problem.

Looking back

Polya points out that when we have obtained a long and involved solution we naturally want to see whether there is a more direct and clear way to solve the problem. He advises one to ask the questions: *Can you derive the result differently? Can you see it at a glance?* (Polya 1973, p. 61). He also points out that even when we have found a satisfactory solution we may still benefit from finding a different solution, which may give us further understanding or allow us to look at the problem from a different perspective. Polya encourages us to study a result and try to understand it better, to see a new aspect of it (p. 64). In the same way that we might get a better perception of an object by using two senses, we might get a better understanding of a problem by finding a solution in two ways. Future teachers need to learn to guide their students on how to find in a result itself indications of a simpler solution.

The approach used by the future teachers described above has several advantages. One is that it highlights the use of an arithmetic sequence and how the average of its terms can be used to obtain their sum. Because $1 + 2 + \dots + n$ is an arithmetic series, the average of all the terms will be the average of the first and last terms, $\frac{n+1}{2}$. One way to read the formula for $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ is that to obtain it we multiply the average

of the terms, $\frac{n+1}{2}$, by the number of terms n . Another advantage is that we can actually find how many rolls of paper we need.

In terms of the original problem posed we may want to look back and ask ourselves what are the essential differences between the two types of paper rolls in this problem. In the situation described, we really want to compare the efficiency to absorb water of the rolls relative to each other in order to determine which brand to use. Once we determine what brand to use then we can compute how many rolls of that brand we need.

Use of proportional reasoning to compare paper towels

A key insight for solving this problem in a different way is to realize that when comparing the rolls, we need to compare their ratios with respect to the different factors that affect their number of sheets and absorption capacity.

In the solution above, the average of the lengths of the layers played an important role. Here we will see how we can use the average in a different way. The number of sheets in a roll will be proportional to the area of the circular ring cross section. The area of this ring can be obtained by multiplying the circumference of the average circle by the width of the ring (Figure 3). If r_1 is the radius of the hollow circular center, r_2 the radius of the paper roll, and d_1 and d_2 their respective diameters, then the area of the cross section is given by

$$A = \frac{1}{2}(d_1 + d_2)\pi(r_2 - r_1) \quad (1)$$

Exercise 1. Derive formula (1) for the area of the ring using the difference of areas of concentric circles.

Exercise 2. Discuss in what ways formula (1) is analogous to the formula for the area of a trapezoid.

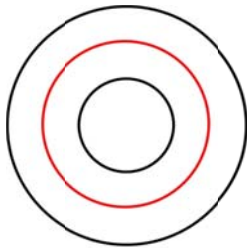


Figure 3. The average circle

Thus, a good way to compute how many sheets are around the roll is by using the circumference of the average circle, in this case the mid circle between the hollow core

and the outer layer. For Brand A this average circle has a diameter of $\frac{2+5}{2} = 3.5$ inches,

for Brand B a diameter of $\frac{2+6}{2} = 4$ inches. The number of sheets in each roll is

proportional to the diameters of its mid circles, and proportional to the useful cross-sectional width of the rolls. The corresponding ratios comparing Brand B to Brand A will

be thus $\frac{4}{3.5}$ for the diameters of the average circles, and $\frac{2}{1.5}$ for the useful cross-

sectional widths of the rolls. The number of sheets will be inversely proportional to the

thickness of each sheet, so the ratio between Brand B and Brand A is $\frac{1/32}{1/64} = \frac{64}{32} = \frac{2}{1}$. The

two brands have the same lengths of sheets, so to get the ratio of the number of sheets of

Brand B to the number of sheets of Brand A, we just need to multiply these three ratios,

which yields $\frac{4}{3.5} \times \frac{2}{1} \times \frac{2}{1.5}$. Because the ratio of the absorption efficacy per sheet of

Brand B to Brand A is $\frac{1}{1.5}$, the ratio of number of ounces of water absorbed by a roll of

Brand B to the number of ounces of water absorbed by Brand A is

$\frac{4}{3.5} \times \frac{2}{1} \times \frac{2}{1.5} \times \frac{1}{1.5} = \frac{128}{63}$. So Brand B is about twice as good as Brand A for this task.

This agrees with our previous result that the ratio of rolls needed is $\frac{4.9}{2.4}$. With this

alternative approach of multiplying ratios it would be easy to make adjustments in case

the length of the sheets or the price was not the same for both brands. All we would have

to do is to multiply the previous product of ratios by the ratios of the prices, and by the

ratio of the length of the sheets. In these cases, as with the thickness of the sheets, we

would be dealing with inverse ratios.

To find how many rolls of Brand B we would actually need, we can find the

number of sheets in a roll, using the average circumference (4π), multiplying it by the

number of layers that fit in the usable cross-sectional width ($2 \div \frac{1}{64}$), and dividing by the

length of the sheets (10). So the number of sheets is $4\pi \times 128 \div 10 \approx 161$. (Notice that this result is very close to the result obtained with the other method.) Because each sheet of Brand B absorbs one ounce of water, this is also the number of fluid ounces that each roll can absorb. The total number of rolls required is $\frac{384}{161} \approx 2.4$.

Exercise 3. Derive formula (1) as the limit of polygonal rings formed by trapezoids (see Figure 4).

Exercise 4. Discuss in what ways is formula (1) analogous to the formula for the volume of a torus obtained by rotating a circle around an axis outside the circle. The volume of the torus is equal to the product of the area of the circle times the circumference traced by its center.

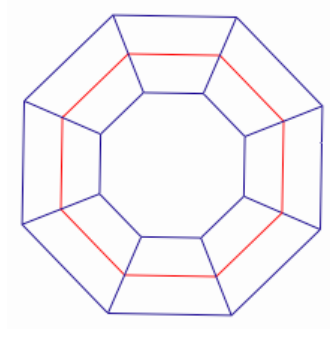


Figure 4. A ring formed by trapezoids

Concluding remarks

When teachers pose a mathematical problem to their students, they often do so because the problem can be solved with a mathematical approach that the teachers want to illustrate. Problem solving can be used as a powerful means to learn mathematical concepts and procedures (Lester & Charles, 2003; Schoen & Charles, 2003). In the above problem, the intent of the preservice teacher who posed it was that students have an

opportunity to use an arithmetic progression and the formula for its sum. Problems can be excellent ways to foster the development and understanding of particular mathematical concepts and procedures. However, students might use an alternative solution process that does not require the concept or process that the teacher wanted to emphasize. Teachers thus need to be aware that students might find alternative solutions that do not involve those concepts or procedures. In that case, the teachers needs to decide at what point, and to what extent, they should discuss those alternative approaches. It is important that teachers look at problems they pose from multiple perspectives, and try to foresee alternative solutions. That way teachers can better plan how and when to use those alternatives so that it becomes an enriching experience for all the students, rather than becoming a situation where some students have the opportunity to develop their thinking with respect to specific mathematical concepts and methods and others do not. Of course, sometimes students may surprise us and find an approach we did not foresee.

Learning to see the easy is one of the possible benefits of looking back at a problem and reflecting on its solution. Finding a simpler solution does not mean that our original approach was less valuable. The first method that occurred to us very likely gave us some insight into mathematical relations of a certain kind in the given situation, and perhaps used mathematical ideas that were freshest in our minds. Furthermore, often we find a simpler path only after we are able to solve the problem in another way. By taking time to consider alternatives once they have found a solution, students may find an easier solution. Students may realize it is not always necessary to apply the most complicated mathematical concepts that they know in order to solve even what appear to be difficult problems.

However, even when we find a simple solution first, it is worthwhile to take a second look at a problem and look for a different solution. The second solution may give us a different kind of insight. As Polya points out, there are also other benefits of looking back, such as establishing connections. A few connections were hinted at above, but a full treatment would go beyond the main focus of this paper.

There are other authors who emphasize the importance of reflection when solving problems. Shulman states that “the more complex and higher-order the learning, the more it depends on reflection—looking back—and collaboration—working with others.” (Shulman 2004, p. 319). The importance of reflection is not restricted to mathematics learning. Shulman also describes how studies of expertise in the solving of physics problems indicate that the most able problem solvers do not learn by just doing, that they do not learn from simply practicing the solving of physics problems. Rather they learn from looking back at the problems they have solved and learn by reflecting on what they have done to solve them. Able problem solvers learn, not just by doing, but also by thinking about what they have done. (Shulman, 2004, p. 319).

Good teachers understand and convey to their students the benefits of looking back at a problem. Learning to see the easy is one of the benefits.

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Two-step arithmetic word problems

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Abstract: This study uses the perspective of schemes to analyze characteristics of arithmetic word problems that can influence the process of translation from the verbal statement to an arithmetical representation. One characteristic that we have detected in the two-step word problems is the presence of one or two connections (nodes) in schemes that represent them, and this paper explores whether the number of nodes affects the activation of the associated schemas. With students from the 5th and 6th grades of elementary school (11 and 12 years of age), we analyze the written productions and would stress that the number of connections influences the activation of the right schema. Results show that the double connection implicate a greater difficulty for obtaining a correct arithmetical representation. Likewise, the presence of a simple or double connection between the two relationships means that the students commit specific errors that we associate with this characteristic.

Keywords: Two-step word problems, arithmetic, schemes, double node, errors.

Introduction

Research on problem solving on mathematics education is a wide and varied field, and it is not limited to a single study focus; nor is it performed within a single theoretical framework (Castro, 2008; Santos, 2008). A good number of studies have centered on the use of arithmetic operations to solve word problems. Verschaffel, Greer, & Torbeyns, (2006) distinguish four focuses in the study of arithmetic problems: (a) conceptual structures (schemes) for representing and solving word problems; (b) word problems viewed from a problems-solving perspective; (c) a sociocultural analysis of performance on arithmetic word problems; and (d) the modeling approach.

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Since the 1990s, there have been a tendency in Mathematics Education to undervalue the educational value of word problems and stress situated and socially mediated approaches to solving authentic, complex problems. Despite this focus, Jonassen (2003) indicates that “story problems remain the most common form of problem solving in K-12 schools and universities” (p. 267). This paper treats arithmetic word problems whose statement contains two relationships between the data and that therefore require more than one operation to solve them (two-step arithmetic word problems). We perform our analysis from the perspective of schemes (Hershkovitz, & Nesher, 2003) and focus on characterizing the double node in two-step arithmetic word problems and the schemes to which they give rise, and on studying the influence of the double node on the activation of the schemes and the errors this causes. Enright, Morley, & Sheehan (2002) indicate that *problem features such as those described previously can be related theoretically to individual differences in cognition* (p. 51).

The scheme approach

From the semantic perspective on one-step arithmetic problems, *once the concepts and relationships involved are understood, the child has only to choose the correct operation and apply it* (Quintero, 1983, p. 102). In problems with several steps, it is also necessary to understand the concepts and relationships, but additional issues are involved as well. Quintero (1983) indicates that the child must plan and organize the order in which to apply the operations and identify the pairs of numbers to which to apply each operation. Shalin and Bee (1985) analysis of two-step problems leading to specific structures is based on the possible logical combinations of one-step problems. They

represent the corresponding scheme of a simple arithmetic word problem by means of a diagram (Figure 1) of three connected components in terms of part-whole relationships.

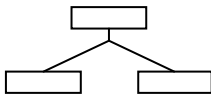


Figure 1. Notation of the triad of components present in the part-whole relationship

If the diagram in Figure 1 represents a mathematical object, we can construct more complex mathematical relationships from it using more than one diagram and connecting them to each other, forming networks. Following this idea, Shalin & Bee (1985) obtain the structure of a two-step problem by combining two triads based on local relationships. Each of the different ways of combining two triads like that in Figure 1 constitutes a different global problem structure. These combinations (Figure 2) define three structures of two-step problems: hierarchy, sharing the whole and sharing a part.

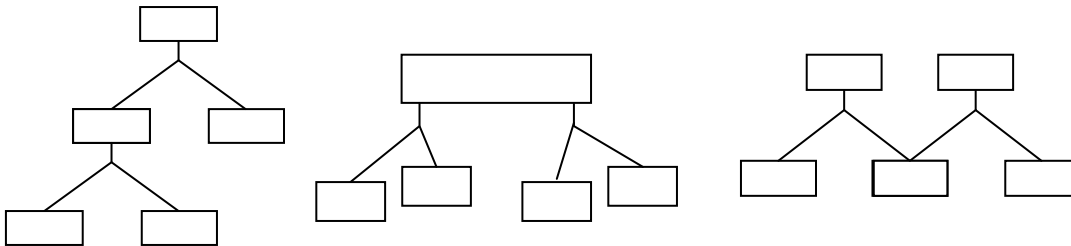


Figure 2. Hierarchical scheme, sharing the whole and sharing a part

Nesher & Herskovitz (1994, 2003) research the influence that the three schemes (Figure 2) have on the index of difficulty of composite problems. With a sample of students from third, fourth, fifth, and sixth grades in Israel, they find that the variable type of scheme has a significant effect on the index of difficulty of these problems. The “hierarchical” scheme is the easiest, followed by the “shared whole” and finally the “shared part” scheme. The study by Shalin & Bee (1985) also showed that children in the

3rd, 4th and 5th grades (elementary school) had a higher rate of success with the hierarchical scheme. In the following section, we will see that the results can be altered by other cognitive variables.

Decrease-increase relationship

In the research performed by the Numerical Thinking Group of the University of Granada, with 4th, 5th, and 6th grade elementary school children from Granada (Spain), the results obtained by comparing the indices of difficulty of the different combinations of the relationships of increase or decrease show that the combinations of increase and decrease affect the difficulty of the two-step problems (Castro, Rico, Castro, & Gutiérrez, 1994; Castro, et al., 1996); Rico, Castro, González, & Castro, 1994; Rico, et al., 1997).

The four classes of problem are determined by whether the relationship is one of increase or decrease.

Type (I, I). Two relationships of *increase*. The *whole* of the first initial relationship is a *part* of the second relationship (*hierarchical* scheme).

Type (D, D). Two relationships of *decrease*. One *part* of the first relationship is the *whole* of the second relationship (*hierarchical* scheme).

Type (I, D). First relationship of increase and second of decrease. The two relationships share the whole (*sharing the whole* scheme).

Type (D, I) The first relationship is one of decrease and the second one of increase. The two relationships share a part (*share a part* scheme).

Presented in order of increasing difficulty, they are:

(I, I), (I, D), (D, I) and (D, D)

where the type (D, D) is the most difficult. These results contradict the argument that the hierarchical scheme is generally less difficult than the other two schemes. Shalin & Bee (1985) and Nesher & Hershkovitz (1994) find that the problems associated with the hierarchical scheme are less difficult than the others. However, in Castro, et al., (1996) study with additive problems, the problems corresponding to the two extreme combinations—the easiest, increase-increase (I, I) and the most difficult decrease-decrease (D,D)—correspond to the same scheme: the hierarchical scheme. The difficulty of the hierarchical scheme is consequently affected by the relationships of increase or decrease used to state the problem. Other cognitive variables also appear in two-step problems, however, such as the number of connections between the components of the basic structure, as we will see in the next section.

Problems with two nodes

One of the key issues in understanding the structure of two-step word problems is understanding the nature of the two elements that compose the basic triad of the part-whole scheme and the way of connecting these elements between two triads. To determine how this is done, Nesher & Hershkovitz (1994) perform a textual analysis of the problems, breaking them into components. They distinguish three components in a one-step problem. Two of these provide numerical information explicitly (complete components) and the other, the question, is missing numerical information (incomplete component).

In the composite schemes for two-step problems (Nesher & Hershkovitz, 1994), the connection between the two one-step problems is created by a new component, which

they call the latent component of the problem (see Figure 3) and which is common to the two simple structures.

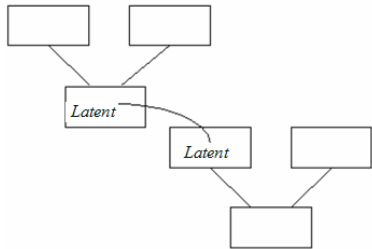


Figure 3. Latent component

From a representational point of view, we say in this situation that there is a *nexus* or *node* between the two simple structures that produce the corresponding composite scheme. Thus, the two simple structures share a component within a two-step problem.

For example, in Problem 1:

Problem 1. I bought 5 books. Each book cost 8 euros. If I pay 50 euros, how much money will I get back?

In the first structure, the latent component is the question of the first problem:

- I bought 5 books
- Each book cost 8 euros
- How much do all of the books cost?

In the second structure, the latent component becomes a complete component:

- All of the books cost 40 euros.
- I pay 50 euros.
- How much money will I get back?

In this problem, the latent component (the price of all the books) is shared by the first and second arithmetic structures. This latent component, which is not stated

explicitly in the wording of the problem, connects the two structures. The price of the books is obtained in the first structure, where it has the function of incomplete component. This price is then used in the second structure as a complete component. This function of connection between the two structures is what leads us to call it a node or nexus of union between the two.

In the schemes of two-step problems defined by Shalin & Bee (1985) and subsequently used by HersHKovitz & Nesher (1996) and Nesher & HersHKovitz (1994, 1996, 2003), the latent quantity is the only nexus of union between the two simple structures. However, the condition of a node does not imply being a latent quantity, nor does it mean that this is the only quantity with this condition. The node can also be a piece of information given explicitly in the statement and that is shared by more than one simple structure within a two-step problem. It is possible to find two-step problems that have two simple structures connected by two nodes, as occurs in the following problem:

Problem 2. John has 5 balls. His grandfather gives him triple the number he had. How many balls does John have now?

This problem 2 combines two simple schemes: one multiplicative scheme and one additive. Both schemes have two quantities, “John’s 5 balls” and “the balls that John’s grandfather gives him,” which are shared. In Figure 4, we see the representation of the two simple schemes and how both contain the shared quantities. This kind of two-step problem has only two pieces of information or, from another perspective, three pieces, but one of these is repeated or has a double function. Therefore, two components are shared by the two simple structures, one of these the latent component (balls that the grandfather gives) and the other the repeated piece of information (John’s 5 balls) in the problem.

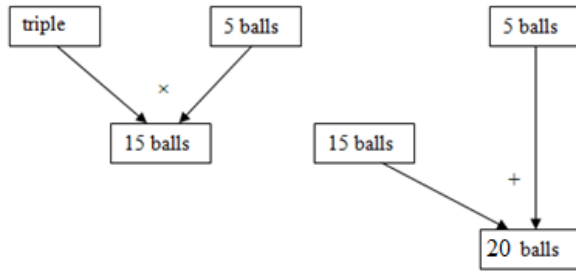


Figure 4. Simple schemes of a two-node problem

The quantities that are shared by various simple structures within a composite problem have, therefore, the condition of node, independently of whether these quantities are given pieces of information or intermediate unknowns (latent quantities) in the problem.

Types of two-node schemes

The problem we have used as an example of a double node is hierarchical in kind (see Figure 5), and the 15 balls constitute the latent variable, which is the intermediate unknown quantity.

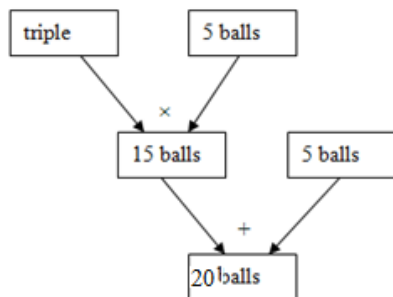


Figure 5. Hierarchical scheme

We can see that the quantity of 5 balls appears in the two simple structures. If we merge both boxes into a single one, as shown in Figure 5, we get a sub-scheme of the hierarchical scheme, but one in which two quantities are shared by the two simple schemes.

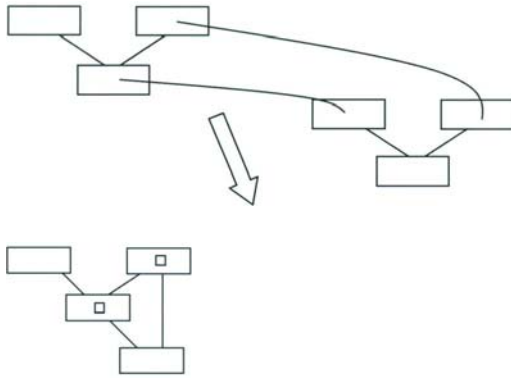


Figure 6. Composite scheme HP

When a part and the whole of one simple scheme matches with the part and part of the other simple scheme ($P, W = P, P$), we call this composite scheme HP, since it can be obtained from the hierarchical scheme H, since one part of each simple scheme coincides with the other (see Figure 6).

In the other two structures of two-step problems, sharing part and sharing whole, new substructures also emerge with this condition of considering the double node. In the case of the structure “sharing part” (SP), we can generate a substructure by making it agree with the other part of the two simple schemes (see Figure 7). We call this substructure SPP.

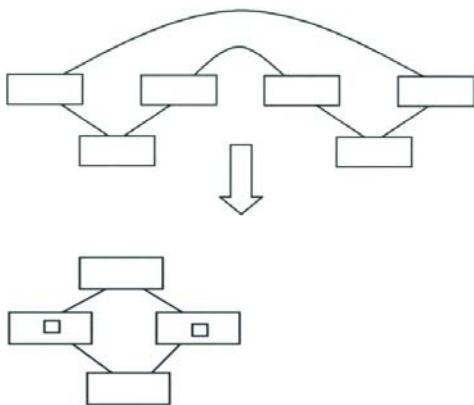


Figure 7. Composite scheme SPP

An example of a problem corresponding to the structure SPP is

Problem 3. John has 7 pieces of clothing, 3 shirts and the rest pants. How many ways can he combine shirts and pants to get dressed?

A new subscheme emerges when one part and the whole of a simple scheme matches with one part and the whole of the other (P, W = P, W). In this case, we obtain the composite scheme that we labelled SWP (Sharing Whole and Part) (see Figure 8). This subscheme can be obtained both from the composite scheme “sharing whole” SW, if we make a part of each simple scheme coincide. It can also be obtained from the composite scheme “sharing part” SP, if we match the whole of each simple scheme.

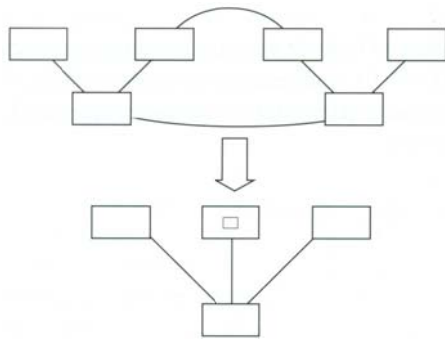


Figure 8. Composite scheme SWP

The following problem is an example of the scheme SWP:

Problem 4. Peter has 15 marbles. Peter has 10 marbles more than John. How many times more marbles does Peter have than John?

In summary, if we consider two nodes to connect the two simple structures that form a two-step problem, we get the following composite subschemes (see Figure 9):

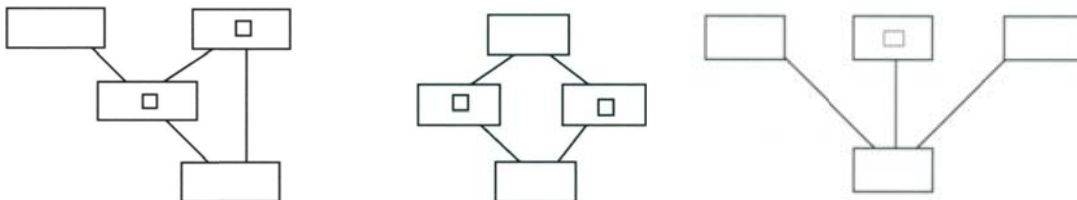


Figure 9. Composite subschemes with two nodes

The double node as a characteristic of some two-step problems can be related theoretically from the cognitive point of view to individual differences or different success rates (Frías, & Castro, 2007). This is due, for example, to the limited capacity of the work memory or, as Embretson (1983) suggests, to the fact that “the characteristics of the stimuli of the items in the tests determine the components that are involved in finding the solution” (p. 181). From the foregoing considerations, we find it important to study whether the two-node problems have different cognitive effects on the subjects.

For the specific case of two-step problems, the variable node takes more than one value. We have described two-step problems with one node and two-step problems with two nodes. We now ask whether the number of nodes in a scheme is a cognitive variable that can influence the problem-solving process for two-step problems in students who are finishing their elementary education.

Our conjecture is that the number of nodes in a composite two-step problem affects the way in which the advanced elementary school students represent two-step problems internally. This difference should become visible in issues such as the success rate and the emergence of errors specifically involving the number of nodes.

Method

Participants

We performed a study to compare the competence of students from the fifth and sixth grades of elementary education (ages ranging from ten to twelve) in two-step arithmetic problems and to determine whether the number of nodes in the problem influences the process of solving it. 172 students from public elementary schools in the

city of Almería (Spain) participated in the study, 86 students from 5th grade and 86 from 6th.

Variables

Given the wide variety of two-step problems, we limited the study to using a carefully-defined set of problems. The first condition we imposed on the two-step problems used in the study was that the semantic category corresponding to the first simple structure of the problem be comparison (additive or multiplicative) and the semantic category corresponding to the second simple structure of the problem be combination, whether additive or a cartesian product. We imposed this restriction to control for the possible effect that the kind of semantic category in each of the simple schemes could have on the overall solving of the two-step problem.

Once we established this condition, the problems we used were chosen using factorial design with four factors or independent variables of the problems, which are:

First factor

The first factor, which we call A, includes whether one of the simple structures that make up the two-step problem has an additive or multiplicative character. We understand the additive structure here to include problems that are solved with one addition or subtraction. Likewise, we understand by multiplicative structure problems solved with one multiplication or division. The variable A refers to the double arithmetic relationship present in the two-step problem and in this study takes two values, corresponding to the possible combinations of a problem composed of two steps, a simple additive structure and another multiplicative structure:

- A_1 for an additive structure followed by a multiplicative structure
(+, ×).
- A_2 for a multiplicative structure followed by an additive structure (×, +).

Second factor

Since the two-step problems that compose the instrument we have used all contain a simple scheme of comparison, we have limited the possible variants of these comparison problems to two kinds, consistently worded comparison problems and inconsistently worded comparison problems (Lewis & Mayer, 1987). Attending to these two kinds of wording for comparison problems, we consider the variable to be the kind of wording in the comparison, which we have called variable E and which takes two values:

- E_1 if the wording of the comparison is consistent.
- E_2 if the wording of the comparison is inconsistent.

E_1 Consistent wording	E_2 Inconsistent wording
John has 15 marbles Peter has 3 times more marbles than John How many marbles do they have altogether?	Peter has 15 marbles Peter has 3 times more marbles than John How many marbles do they have altogether?

Third factor

Each of the simple relationships involved in a two-step problem can be of either increase or decrease (Castro, et al., 1996; Castro, Rico, Castro, & Gutiérrez, 1994; Rico, Castro, González, & Castro, 1994). We call R the variable that combines the two possibilities in the double relationship. In this study, we will take into account two values:

- R_1 for the relationship increase-increase (I I).
- R_2 for the relationship increase-decrease (I D).

From the point of view of direct translation based on key words, this variable provides information most specifically about the arithmetic relationship that can be used. Increase will refer to addition or multiplication and decrease to subtraction or division.

Fourth factor

The fourth factor is the variable, our main focus of attention. It includes the number of nodes in the two-step problem. The number of nodes, which we call the variable nodes (N), has two values in this study:

- N_1 for two-node problems.
- N_2 for one-node problems.

N_1 two-node problems	N_2 one-node problems
Mary has 15 trading cards. George has 3 times more trading cards than Mary. How many trading cards do George and Mary have between the two of them?	Mary has 15 trading cards, and Paula has 90 cards. George has 30 more cards than Mary. How many cards do George and Paula have between the two of them?

Instrument and procedure

The instrument used in this experiment was a questionnaire with sixteen problems. The sixteen problems correspond to the possible combinations that emerge from crossing the four factors mentioned above in a factorial design. So as not to overwhelm the study subjects with too many problems, we divided this set of sixteen problems into two questionnaires of eight problems each, according to the following distribution:

		N ₁		N ₂	
		A ₁	A ₂	A ₁	A ₂
E ₁	R ₁	Q1	Q2	Q1	Q2
	R ₂	Q2	Q1	Q2	Q1
E ₂	R ₁	Q1	Q2	Q1	Q2
	R ₂	Q2	Q1	Q2	Q1

Q1 Questionnaire N° 1 Q2 Questionnaire N° 2

The problems in these questionnaires were solved by the children individually and silently in the classroom using pen and paper. Each child was given a questionnaire at random.

Results

The answers given by the subjects to the problems posed were evaluated as correct or incorrect, taking into account the choice and execution of the operations, as well as the expression of the result. We have classified a response as correct when the subject has chosen the right two operations between the corresponding data and has expressed the solution correctly, writing it in the space provided for the result the expression of the relationship that each problem required according to the instructions provided on the questionnaires. This circumstance occurred in different ways. The most common was to perform two operations, executing the corresponding algorithms, and to conclude with the full expression by answering the question posed in the problem. However, we have also considered correct those answers in which this was done implicitly. For example, given the problem:

*Javier has 12 pairs of pants. Javier has 3 more shirts than pairs of pants.
How many ways can Javier combine pants and shirt?*

Some subjects did one of the operations ($12+3=15$) mentally, so that the only explicit operation that appears is $12\times 15=180$. In cases like this, we have evaluated the answer as correct, since we understand that student chose the two correct operations, performed one as a mental calculation and the other as a written algorithm, and provided the correct answer: “Javier can combine his shirts and pants in 180 different ways.” We have also considered answers to be correct if the answer was expressed elliptically, for example, “They can be combined in 180 ways.” In cases where students chose the operations to be performed correctly and used the correct data but committed a calculation error in the algorithm, we have considered the answer to be correct, even though the result shows a quantity different from the correct one. In this case, we believe that this kind of error does not affect the subject’s understanding of the problem.

The answers were evaluated as incorrect when one of the two operations to be performed was not the correct one or the subject did not perform the operation with the proper data. No response on one of the operations was also qualified as incorrect, since it shows that the subject did not understand at least one of the two relationships in the problem. No answer at all was also evaluated as incorrect.

The success rates at which the children in the study were able to translate each of the questionnaire problems into its arithmetic representation are shown in Table 1 as percentages. They range from 20% for the most difficult problem to 90% for the least difficult. This result shows that some of the factors that define the problem influence their difficulty. To highlight which variables have a significant influence, we have applied a variance analysis to the four factors.

Table 1. Percentages of success in the questionnaire problems according to factors

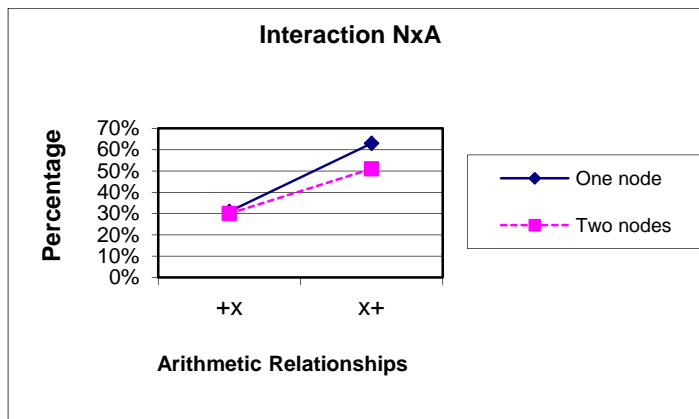
		N ₁ -two nodes		N ₂ -one node	
		A ₁ +×	A ₂ ×+	A ₁ +×	A ₂ ×+
Consistent E ₁	R ₁ I I	37	80	36	90
	R ₂ I D	34	57	33	58
Inconsistent E ₂	R ₁ I I	28	36	34	55
	R ₂ I D	22	30	20	51

Using the success rate measured in percentages as a dependent variable, we have applied variance analysis to detect whether the four factors defined in the study had a significant effect on the success rate. The variance analysis applied to the data obtained shows a significant effect on the following cases:

- variable N number of nodes ($F = 6.677$, $p=0.010$). The percentage of success on problems with one or two nodes is: two nodes-N₁ (41%) and one nodes-N₂ (63%).
- variable R combinations of increase and decrease ($F=20.982$, $p=0.000$), with a percentage of success on the combinations of: increase-increase (49%) and decrease-increase (38%).
- variable E or kind of wording ($F=56.504$, $p=0.000$): Consistent (61%) and inconsistent (45%).
- variable A combination of the additive and multiplicative relationships ($F=116.760$, $p= 0.000$). The percentages of success on the combinations of additive and multiplicative relationships used were: A1(+×) combination (30%) and A2(×+) combination (57 %).

We interpret the marked difference in difficulty shown by combinations A_1 and A_2 according to the restriction imposed, that is, that the problems be comparison (additive or multiplicative) in the first step and either additive or Cartesian product combination in the second step. In problems of the type $+×$, we use the additive comparison in the first step and the Cartesian product in the second step. In problems $×+$, we use the multiplicative comparison in the first step and the additive combination in the second. The presence of the Cartesian product in the simple scheme corresponding to the second step of the problems seems to cause the difference in difficulty.

The only significant interaction effect influenced by the variable of node is $N×A$ ($F=6.084$, $p=0.014$). This interaction does not change the order of difficulty of the values of the variable node, however, as can be seen in graphic 1.



Graphic 1. Percentages of correct answers according to nodes and combinations of arithmetic relationships

In graphic 1, we can see that the problems with two nodes are more difficult to translate into a symbolic representation than the problems with one node for the two combinations of arithmetic operations. We can conclude from this analysis that the

number of connections between the two relationships is a significant differentiating characteristic in two-step problems. The percentages of success on one-node problems (63%) and two-node problems (41%) show a significant difference in students' performance between these two kinds of problem. This difference does not depend on the other factors considered.

Error analysis

In written products, we found that in addition to typical errors already identified in one-step problems (such as the additive error or the inversion error), the sample subjects produced new errors in the two-step problems, errors that we identified as errors belonging to the double structure itself. Since our goal is to characterize the issues that differentiate the two-step problems, we will stick to the description of the errors specific to the double structure.

Type 1 error: performing only one operation

This error is characterized by using only one operation to solve a two-step problem. The operation may be either one of the two correct operations that should be performed or the wrong operation for another reason. In all of the cases, the subjects do not attempt to perform more operations but instead give as an answer the result of the only operation that they have performed with the two pieces of information from the problem. Most of the cases observed occurred in problems with two nodes (and only two pieces of information). In a few cases, this kind of error occurred with a problem of only one node (with three pieces of information). Table 2 shows examples of this kind of error.

Table 2. Error in performing one operation

Problems	Errors	Comments
<p>Example 1</p> <p><i>Anne has 12 pairs of pants</i></p> <p><i>Anne has 3 shirts fewer than pants.</i></p> <p><i>How many ways can he combine pants and shirts?</i></p>	<p>$12 - 3 = 9$</p> <p><i>Result: He can combine pants and shirts in 9 ways</i></p>	<p>Omits the second operation</p>
<p>Example 2</p> <p><i>John has 24 balls</i></p> <p><i>John has 3 times fewer balls than Peter. How many balls do they have between the two of them?</i></p>	<p>$24 + 3 = 27$</p> <p><i>Result: Between the two of them, they have 27 balls</i></p>	<p>Omits the first operation</p>
<p>Example 3</p> <p><i>Anne has 48 trading cards</i></p> <p><i>Mary has 4 times more trading cards than Anne. How many do they have between the two of them?</i></p>	<p>$48 \times 4 = 192$</p> <p><i>Result: Between the two of them, they have 192 trading cards</i></p>	<p>Omits the second operation</p>

In the problems used in this study, the two relationships are ordered; the first one is always a comparison and the second a combination. For this kind of error, we can therefore distinguish the cases in which the subject forgot the first relationship from those in which the subject forgot the second:

1. Forgetting the first relationship

In this case, subjects take the two pieces of information in the problem and perform an operation without taking into account the first relationship in the context of the problem. They focus their attention on the second relationship, which is the one in which the problem’s question appears. Examples 1 and 2 in Table 2 illustrate this case.

2. Forgetting the second relationship

In this case, they take the two pieces of information in the problem and work with them in the context of the first relationship stated in the problem, not taking into account

the second relationship. In the result, they answer the question in the problem that corresponds to the second relationship although this value was obtained with the first. Example 3 in Table 2 fits this type of error.

Type 2 error. Ordered data

This error is characterized by choosing the data for performing the relationships in the same order in which they appear in the problem. In certain problems in our study, this leads to an error in the two relationships in the problem. The students take the first two pieces of information that appear in the word problem and perform the operation, then to perform another operation on the result and the third piece of information, and finally, with this result, to find the solution. An example of this error can be seen in the solution given to the problem in Table 3.

Table 3. Error in ordered data

Problem	Solution with Type 2 error
<i>George has 18 shirts and 6 belts. George has 3 shirts more than pairs of pants. How many ways can he combine pants and belt?</i>	$18 - 6 = 12$; $12 \times 3 = 36$ <i>Result: He can combine pants and belt 36 ways</i>

This way of acting indicates recognizing the two relationships in the problem, even distinguishing between the two simple structures, one additive and the other multiplicative. But the subjects do not associate the data and the relationships in each structure correctly. This leads us to think that the choice of data is mechanical or algorithmic and that order of presentation takes precedence over any other characteristic of the problem. In many cases, we see that, if the correct order coincides with the order in which the data are presented, the subjects give the correct response, but when the correct order is different than the order in which the data are presented, students make mistakes.

These last two kinds of error, *one operation* and *ordered data*, occur in the same subjects; that is, that for the two-node problems they commit the error of *one operation* and for a one-node problem, that of *ordered data*.

Type 3 error. Repeating the unshared information

In the two-node problems, we saw an error that consisted of using twice the unshared piece of information in the two simple structures that compose the two-step problem, while using the shared piece of information only once. An example is shown in Table 4.

Table 4. Error of confusing repeated information

Problem	Solution with Type 3 error
<i>Lucia has 15 shirts. Lucia has 3 fewer shirts than pairs of pants. How many ways can she combine shirts and pants?</i>	$15 + 3 = 18$; $18 \times 3 = 36$ <i>Result: She can combine shirts and pants in 36 different ways</i>

The previous solution that contains the Type 3 error shows that the subjects have recognized the two relationships and distinguished two structures, one additive and the other multiplicative. Further, the repetition of one piece of information from the problem in the calculations (in this case, the 3) seems to indicate that the subject recognizes that he or she must use this piece of information twice. The error occurs in choosing the right piece of information.

Type 4. Other errors

In this section, we include errors that do not fit any of those mentioned above, cases in which it is difficult to know what motivated the subjects' choice of operations. Most of these cases occur in problems with one node in which the student only

recognizes as characteristic of the problem that there are always two or more operations but chooses the operation and/or the data related to it arbitrarily or by chance.

The distribution of the four kinds of error described according to levels of 5th and 6th grade are shown in Table 5. Here, we differentiate two subtypes two subtypes for the error *one operation* one type for the error *ordered data* and another for *repeat unshared datum*, whereas in classifying the others we include the unclassifiable wrong answers in the foregoing, as well as missing answers.

Table 5. Frequencies of each error at each level and total errors

Error type	Subtype	5 th grade		6 th grade		Total	
		Frequency	%.	Frequency	%.	Frequency	%
One operation	Forgetting the first relationship	16	16.16	10	11.24	26	14%
	Forgetting the second relationship	45	45.45	52	58.42	97	52%
Ordered data		22	22.22	14	15.73	36	19%
Repeat unshared datum		6	6.06	9	10.11	15	8%
Others		10	10.10	4	4.49	14	7%
Total		99		89		188	100%

As can be seen in Table 5, all kinds of error detected appear in the two levels (5th and 6th grades). Overall, the error in one operation has occurred with similar frequency at both levels, but this is due to the fact that the two subtypes compensate for each other. That is, students in 5th grade omit the first operation more frequently, whereas those in 6th omit the second more frequently. The next most frequent error is that of ordered data, which occurs with greater frequency in students in 5th grade than those in 6th.

Since we chose and identified the problems based on the four factors (N, E, R, A), it is reasonable to attempt to relate the types of error defined to these factors. We classified the association between the errors as belonging to two-step problems. The four factors are shown in Table 6, which includes the distribution of frequencies for each of the problems, according to the combination of factors.

Table 6. Frequency of errors in the combination of four factors

Factors				Type of error				
N	E	R	A	One operation		Ordered data	Repeating the unshared information	Others
				Forgetting the first relationship	Forgetting the second relationship			
N ₁	E ₁	R ₁	A ₁	7	15	0	3	0
			A ₂	0	13	0	0	0
		R ₂	A ₁	3	18	0	1	0
			A ₂	1	10	0	0	0
	E ₂	R ₁	A ₁	4	11	0	1	0
			A ₂	0	10	0	1	0
		R ₂	A ₁	5	13	0	3	0
			A ₂	5	4	0	0	0
N ₂	E ₁	R ₁	A ₁	2	0	9	0	2
			A ₂	0	0	1	1	0
		R ₂	A ₁	0	1	6	0	5
			A ₂	0	0	4	0	0
	E ₂	R ₁	A ₁	0	0	11	1	2
			A ₂	0	0	2	0	1
		R ₂	A ₁	1	0	3	3	4
			A ₂	0	0	0	1	0

Conclusions

In this study, we have demonstrated a new characteristic associated with two-step word problems: the number of connections between the two simple structures that compose the problem, which we have called “node.” We have established a specific class

of two-step arithmetic word problems that contain only two known quantities in their wording. We have shown that these problems have a common characteristic: they are formed of additive and/or multiplicative structures connected by two nexus or nodes. Our starting hypothesis is that the number of nodes affects the difficulty of translating the wording of the problem into a mathematical representation. With a sample of students in the last two grades of elementary school in Spain, we have confirmed this hypothesis, in the sense that the two-step arithmetic word problems with two nodes are more difficult to translate into arithmetic expressions than similar problems with one node. Further, we have significant evidence that the result is not influenced by other variables that also influence the difficulty of translating arithmetic expressions, such as whether the relationship of comparison is expressed in consistent or inconsistent language or whether the additive and multiplicative relationships are of increase or decrease. The result is also independent of the combinations of additive and multiplicative structures that compose the scheme of the two-step problem. Although there is significant interaction between the factor node and the factor that represents the combinations of additive and multiplicative structures, the analysis of this interaction shows that the order of difficulty in the two-step problems remains the same.

Likewise, from an analysis of the errors committed by the children, we have found that in addition to the errors already identified in one-step problems and reviewed in the literature, there are patterns of error associated with two-step problems; that is, errors that do not occur in one-step problems. We stress the presence of three of these: performing only one operation, working with the data in the order in which they appear in the statement of the problem, and using one piece of information twice, in the two

operations, when in reality it should be used only once in one operation. The error of performing one operation occurred with greater frequency in the two-node problems, whereas the error of working with the data in the order in which they appear occurred more often in one-node problems. Therefore, the number of nodes is an issue that enables us to differentiate between types of problems and to explain part of the difficulty that two-step arithmetic word problems pose to children. When the students have to solve word problems, the number of nodes in a two-step problem is shown to be a cognitive variable that influences the problem-solving process.

The limitations of the study performed are related to the kind of problem, the students' level, and the research focus adopted. Within the different semantic categories of the problems identified in the additive and multiplicative structure, our study imposed the restriction that the first relationship stated in the problem corresponds to the semantic category of additive or multiplicative comparison. Likewise, the second relationship always corresponds to an additive combination or a multiplicative combination. These conditions can mediate the results obtained in terms of difficulty, kind of error, and frequency of error. The results obtained must also be restricted to the students' level. In our case, these are students at the end of their elementary education. The results cannot therefore be extrapolated to lower levels, although similar results could emerge in the first year of the next educational level, the first year of secondary education. Although the methodology employed is valid for achieving the goal we proposed and the evidence shows the representations that the students produce in response to the two-step word problems, they are sensitive to the presence of one or two connections between the relationships. This is already a significant result from the point of view of the

development of the school curriculum. This study could be continued by tackling from a qualitative point of view the psychological reasons for the different student errors in problems with one and two nodes.

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Trajectory of a problem: a study in Teacher Training

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Abstract: Problems are frequently used in mathematics to introduce and convey new notions and skills. Hence, teachers transform and adjust those problems to their students' level. The present study focuses on this transformation process on the particular case of a geometric problem posed by two teacher educators in one French Institute for Teacher Training. The whole process is described as a trajectory of the problem through various institutions from training center to secondary school and back. Before presenting the notion of trajectory of the problem, some elements about a general theoretical frame which refers to didactics of mathematics are presented.

Keywords: Geometry, open problem, problem situation, problem solving, teacher training, technologies.

Introduction

The idea of grounding the teaching of mathematics on making students solve problems is not new, especially in primary education. From the 1970's on it has been very popular in many countries, undoubtedly as a reaction to the abstract teaching given during the so-called 'modern math' period. This pedagogical trend was variously structured according to the country, and the use of problems for learning maths depends to some extent on both cultural traditions and theoretical frames underlying teaching which are specific of each country. We became aware of these differences on the occasion of a joint research undertaken by a French team (from the LDAR, Paris-Diderot University) and a Mexican team (from Cinvestav, Mexico-city). The scope of this study, presented at the Cerme 7 Conference (Rzeszów, 2011) by Kuzniak, Parzysz, Santos and Vivier (2011), was the question of the initial training of teachers to the use of technologies for the teaching of maths. On the Mexican side, the implementation was

based on the problem solving methodology, whereas on the French side the stress was put on the notion of open problem, in connection with Brousseau's Theory of Didactical Situations (TDS).

In this article we shall present in detail our approach for this research, within a training course for prospective mathematics secondary school teachers, with reference to some of the theoretical frames used by our team, and especially the notions of open problem and instrumental approach (sec. 1.1 and 1.2). Besides, the training course situation here studied belongs to what can be described as a training homology strategy (sec. 1.3). The problem at work is used to develop among pre-service teachers, not only their mathematical knowledge, but also their didactical knowledge.

After having exposed an a priori analysis of the problem (sec. 2), we describe in section 3 the work required from the students-teachers which is split into three steps. Then, we expose and analyse the various transformations of the problem chosen for the training.

Finally, in discussion section (4), we define a framework (sec. 4.1) intended to describe and analyse what we call the trajectory of the problem, that is its global evolution, from its use in the training course to its setting up in a regular classroom. We conclude the section (§4.2 *sq.*) with remarks on some important points related to teacher training.

1. Context and stake of the study

1.1 Problem solving in French context

As Artigue and Houdement (2007) underscore it, there does not exist a tradition of education research on problem solving in French didactic research even if Polya and Schoenfeld works are well known. This characteristic partly results from the influence of

the Theory of Didactical Situations (TDS in the following) initiated by Brousseau (see Brousseau, 1997, for reference texts in English) and from the pedagogical approach developed by the IREM (Institut de Recherche sur l'Enseignement des Mathématiques). Both introduced two kinds of perspective on problem solving: problem situation and open problem.

The notion of “problem situation” appeared in France in the 1980’s in Brousseau’s TDS, which is based on a socio-constructivist conception of learning. A problem situation is a learning approach aiming at fostering the acquisition of a new knowledge by the students. Its setting up implies identifying previously their conceptions by analysing their errors. On this basis the teacher conceives of and sets up a situation presenting some specific features, namely:

- be relevant for the cognitive objective aimed at;
- have a meaning for the student;
- allow him/her to begin the search for a solution;
- be rich (in terms of mathematical and heuristic contents);
- be possibly formulated within several conceptual “settings” (Douady, 1986) or “semiotic registers” (Duval, 2006).

The notion of “open problem” was introduced at about the same time (Arsac *et al.*1988, Arsac & Mante, 2007). In comparison with the problem situation, the aim of an open problem is methodological rather more than cognitive. The students are induced to implement processes of a scientific type, *i.e.* experimenting, formulate conjectures, test them and validate them. The problem must belong to a conceptual domain in which students are somewhat familiar with, the wording (statement) has to be short and induce

neither a solution nor a solving method. Here is an example taken from APMEP (1987):

What is the biggest product of two numbers which can be obtained by using once each of the digits 1, 2, 3, ..., 9 to write these numbers?

In fact, open problem and problem situation refer to two complementary sides of a mathematician's work:

- in the case of an open problem the question is to find a genuine and personal solution, with one's own means, the general solution can be out of reach of the students (and possibly the teacher);
- in the case of a problem situation the question is, starting from a specific problem, to elaborate a more general knowledge (concept, process...) which is intended to be institutionalised, socially acknowledged and mastered by all the students.

The French official curricula for junior high school integrated recently – though without naming them – these two practices:

If solving problems allows the emergence of new elements of knowledge, it is also a privileged means to broaden its meaning and to foster its mastery. For that, more open situations, in which the students must autonomously appeal to their knowledge, play an important role. Their treatments require initiative and imagination and can be achieved by making use of different strategies, which must be made explicit and compared, without necessarily privileging one of them. (BOEN 2008, page 10, our translation.)

The notion of research narrative (*narration de recherche*), which is explicitly linked with those of open problem and problem situation, appeared in France some twenty years ago, first at junior high school level, before being extended to senior high and primary school (Bonafé et al. 2002). It involves asking the student to write an account of the thought processes he/she has undertaken in order to solve a given problem,

pointing out his/her ideas, successes, failures, etc. The features of the problem are the same as for an open problem, but its statement has often several questions and is such that the student must be able to start a research, test his/her results and validate them. And, if possible, different solutions can be considered.

1.2 Integration and influence of technologies

Pre-service teachers in maths are accustomed to solving mathematical problems with specific software, mainly of the symbolic calculation or dynamic geometry types but this does not mean that they are prepared to use them as future teachers. Research studies into teaching in technological contexts (see Laborde, 2001) show that the students (preservice teachers) do not have or have little knowledge of the teaching of mathematics, that is to say, they are unaware of the development of mathematical notions in teaching situations and they have difficulties in the use of software in a learning situation. This makes it necessary to integrate specific work in the form of understanding teaching using software into teacher training.

Specific studies on teacher training within a technological context (see Chacon and Kuzniak, 2011) are few. And they show the need to go more deeply into processes regarding proof and the structuring of different spaces of knowledge (teaching, mathematical, instrumental) which a teacher must structure when using dynamic software for geometric learning. Moreover future teachers have to be aware of secondary school students difficulties related to instrumental knowledge.

1.3 Teacher training

Till the end of 2010, IUFMs, Instituts Universitaires de Formation des Maîtres (French University Training Colleges), have been in charge of the formation of preservice teachers. The IUFMs were accepting, after a first selection, maths graduate

students from any University (three years of study). During one year, students were preparing a competitive examination with academic maths knowledge. The successful candidates received a theoretical and practical education of one year (the “second year”) in the Institute and were in charge of a class for six hours a week; they received a salary. Nowadays, students need to have a master and pass competitive examination to become teachers. Preservice secondary teachers could follow a master in teacher education (two years) at University, they are not in charge of a class and are not paid during the second year. Our experimentation was made in 2010 before the new system.

As it is well known, preservice teachers need a set of knowledge on maths and teaching, usually described with the notion of Pedagogical Content Knowledge (PCK) introduced by Shulman (1986) to complement subject content knowledge, and based on this idea, various refinements have been made to describe knowledge that is really needed to teach mathematics known as Mathematical Knowledge for Teaching (MKT). Teaching mathematics is obviously connected to Mathematical Content Knowledge but also to other ones that are not automatically owned by a specialist of mathematics and that are more or less close to mathematics like history, epistemology, didactics, psychology or pedagogy. This large set of knowledge is classified in two parts. The first one, that is made explicit and structured clearly within the frame of didactical theories, constitutes Didactical Content Knowledge. The second one, that is not explicitly written and theorised, but exists in the professional action of each teacher is what is called “third knowledge” (Houdement & Kuzniak, 2001). Within this framework, the question is how to introduce and combine the various types of knowledge. And how to give to students

who are specialists of mathematics at university level, a level in school mathematics which are often far away from the first one.

The combination between various types of knowledge can take different forms: they can be suggested to or developed by the students; they can be juxtaposed or connected; the connection can be explained or not...So we have distinguished various strategies which differ concerning the explanation of knowledge, the combination between them, the position they give to the students. Strategies also depend on the knowledge considered as dominant and on the transposition made by the teacher trainer.

During our experimentation, we followed a strategy firstly based on homology and then on transposition. That means that we first use the lack of knowledge of content and teaching for the classroom of the preservice teachers as a pretext to build a learning situation close to a conception of teaching favoured by French curriculum. The preservice teachers, or student teachers, are considered similarly as maths students searching a problem and supposed to analyse the teaching session to pinpoint elements of didactical knowledge and the “third” knowledge. The strategies based on transposition favour didactical knowledge. Then, we tried to know more about the phenomena of transposition of knowledge that might be a bias in every teaching situation (Chevallard, 1985). Student teachers are considered as teachers examining their own teaching way. We detail this with the notion of problem trajectory for the training.

2. Presentation of the problem the *folded square* and a priori analyses

The problem we discussed in this paper is the core of a pre-service teachers’ training course that conveys didactical knowledge about problem use in the class. For this reason, this problem was asked to fulfil several conditions:

- To be an « open » problem easily integrated in the teachers' training process.

- To allow the link between several semiotic registers (Duval, 2006) and the use of various mathematical settings (Douady, 1986) related to French curriculum.
- To be solved in different technological contexts especially those using dynamic software.
- To be open to a number of exploitation and transformation in class with pupils and training session with future teachers. This point relates to our idea of problems trajectory.

To these various constraints linked to a training context, we added one more related to the context of a comparative study. For that, we chose a problem or a kind of problems already given by other researchers using other theoretical approaches.

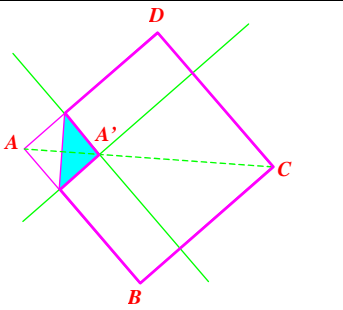
The problem posed to the students belongs to a kind of problems named “shop-sign problems” as used in Artigue, Cazes and Vandebrouck (2011). In such problems, with geometric support, two areas representing a shop-sign are determined by a point situated within a square or a circle or a rectangle... Both areas change in function of the position of the point in the square. These problems are introduced in a geometric setting but to solve them, a change to algebra or calculus settings is generally required. Changes of semiotic registers with algebraic or functional notations are also needed to get a solution. The functions used are quadratic polynomial functions which allow a mathematical treatment in synchronization with the secondary school curriculum.

By using dynamical geometric software as *Geogebra*, it is also possible to solve such problems in a graphical setting by focusing on the covariation of areas without the use of a functional or algebraic writing. It is indeed possible of drawing a graphical representation of the phenomena studied without any algebraic writing of the function:

the curve is defined as a locus of points. The number of solutions that students can find and understand is increased by the use of technological tools introducing an experimental perspective in the implemented working space.

The problem was presented in a real context with material and not with a writing in mathematical form: A square, cut in a bi-color sheet, is given to the students. And they have to fold it along a diagonal and compare the areas of both visible parts of different colours. Students are entirely in charge of the problem representation according to the first step of the modelling circle in Blum and Leiss (2005) view. By doing that, we do not favour any mathematical approaches and frames but to control the task effectively made by student teachers and reach our training objectives on the use of technologies for teaching, student teachers have been encouraged to use some software as it will be detailed in sec 3.1. on problem trajectories.

The problem is not original and was used in French and Mexican contexts (Kuzniak et al., 2011) with the following form, Mexican Task, which will give the reader an easier access to the mathematical stake of the problem.

<p>Mexican Task. A square piece of paper $ABCD$, the side of which is l, has a white front side and a blue back side. Corner A is folded over point A' on the diagonal line AC. Where should point A' be located on this diagonal (or: how far is A' from the folding line) in order to have the total visible area half blue and half white?</p>	
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In this version, a figure is associated to the text and that orients and makes easier the mathematical work of students. It is no more necessary to fold the square and the problem for students is to find the mathematical expression of both areas: area A_1 of the blue triangle and area A_2 of the white hexagon. Moreover, the side of the square is given

as a parameter l and the question is exactly on the place of point A' on the diagonal. Visual adjustments are invalidated by calculations for the area of the triangle seems larger than the other in the case of equality¹. So, to solve the problem students need to reason on an elaborate and high level.

Two great types of reasoning are expected:

- In the first one, students need to determine an algebraic expression for each area and solve a quadratic equation; in France, this approach is only possible without help in grade 11.
- In the second one, it is possible to reason in figural register. Indeed, the drawing given in the text makes visible three "useful" areas, the two areas to compare and a new area A_3 , equal to A_1 : the area of the triangle of vertex A completing A_1 to make the square of diagonal AA' . This new area does not exist in the real folding since the triangle does not have a material existence in this case. With the use of this new area, it is possible to find, almost without any calculation, a solution of the problem. The drawing makes clear a decomposition of the square $ABCD$ which implies the equality $2A_1 + A_2 = l^2$ between the areas and in the case where $A_1=A_2$, we get $3A_1 = l^2$.

If x denotes the side of the square made by the two rectangle and isosceles triangles, as $A_1 = x^2/2$, then $3x^2/2 = l^2$, hence $x^2 = (2/3)l^2$.

It should be noted that if we take the unknown d on the diagonal, d is the height of one of the rectangle and isosceles triangles, then $x^2 = 2d^2$ and so $d^2 = l^2/3$. This way gives a simple solution to the original problem posed by Carlson et Bloom (2005):

¹ Let's note that these invalidations are operational since the grade 6 (it has been noted with the class of the student teacher STe).

A square piece of paper is white on the frontside and black on the backside and has an area of 3 in^2 . Corner A is folded over point A' which lies on the diagonal AC so that the total visible area is half white and half black. How far is A' from the folding line. (*op. cit.* p. 55)

In the case chosen by the authors, the area of the square is of three square inches and we get immediately $d^2=1$ and therefore $d=1$. This initial formulation of the problem is really more complex than those used in our study with a real folding and material that allow the student chose a more « natural » variable as the side or the diagonal or in Mexican Task approach where a drawing and the variable are provided. The form used by Carlson and Bloom is not geometric meaningful because it gives only the area of the square. This probably explains much of the difficulties² encountered by their students, though advanced in mathematics.

The requested use of a software in the task posed in our study changes again the nature of the task. The software – *Geogebra* – gives an area immediately to each of the surfaces and, as mentioned, it allows – and to some extent encourages – the use of graphics, without the need for an algebraic notation. One could represent graphically A_1 and A_2 in function of x (or d) and then solves the problem by considering the curve intersection (see figure 1 in sec 3.2.1). It is also possible to solve the problem by drawing the graph of point which coordinates are (A_1, A_2) – it is a straight line – and considering the intersection with the line $y=x$.

With this first analysis, it is already clear that the same initial problem can be transformed in different ways leading to very different tasks, depending on the support and tools provided to students or preservice teachers and obviously on curriculum

² In the adaptation of STb, described in section 3.2.1, the square has an of area 27 cm^2 but the square is given to students (within the *Geogebra* software).

content. These tasks may also depend strongly on institutional constraints integrated by teachers and their trainers. This is the subject of the study presented in the third section.

3. Transformations of the problem for teacher training

In this section we study the various transformations of a single problem P_0 inside the French educational system through two institutions: a training center for teachers and secondary school classes. More precisely, this study involves two groups of student teachers and two teacher educators, named TEa and TEb in the following. The aim of the research is to grasp the impact of an initial training of math secondary schoolteachers on their actual teaching in a classroom: *what remains of the training when these teachers are back with their students with real constraints?* Due to this aim, our study is not based on Brousseau's theory nor on problem solving but on a specific framework presented in section 4.1. We suppose that the changes of institutions motivate and make necessary some transformations, the study of which will enable to better understand some constraints lying on teachers, together with some usual practices of the profession.

3.1 The transformations of the problem

In the training course involved in the present study we shall distinguish three stages of transformations of problem P_0 . In this section we describe these stages.

Stage 1. First transformation: from problem P_0 to problem P_1

Problem P_0 (section 2) required a first transformation in order to be given in the initial training of secondary schoolteachers. The students are prospective math teachers and the aim of the educators (TEa and TEb) is twofold: at the beginning it is a matter of insuring that their students have well understood the problem with its educational potential, the various ways for solving it and the possible difficulties of the solutions. In a

second time they will be asked to transform this problem in order to use it in their own training classrooms.

Here is the form chosen for P_1 by TEb, together with the working instructions given to the student teachers (the form chosen by TEa was very close).

You have at your disposal a square of paper, one side of which is white and the other is grey. A fold shows a diagonal of the square
A type of folding bringing a vertex of the square on this diagonal, like the one performed on the enclosed square, is considered.
One intends to compare the white and grey areas obtained in that kind of folding.

For both groups TEa and TEb, problem P_1 was based on this ‘minimalist’ presentation making use of a model: TEa showed the student teachers the folding with a material square and TEb decided upon sending the instructions with a material square by mail.

The student teachers are asked to work on the problem and show their entire solution process (Schoenfeld 1985). This solution is complemented by a research narrative (cf. section 1.1). It is during this research phase that the student teachers, here in a ‘student’ position, had to use at least one technological tool³ to explore the problem favouring experimental approach according to the French curriculum.

The choice of a problem as ‘bare’ as possible from the mathematical point of view has also a didactical aim, conveyed by homology: encourage the prospective teachers to use, on one side problems with an open question, and on the other side technologies for solving them. By so doing the educators hoped that the student teachers would feel free to operate their own choices, both from a mathematical point of view (cf. a priori analysis in

³ To be chosen among: spreadsheet, dynamic geometry software, calculator.

section 2) and as regards the actual modes of class implementation by integrating technological tools (cf. section 1.2).

Stage 2. Second transformation: from problem P_1 to problem P_2

Again in the training center, the student teachers were asked to write down the wording of a problem and to make explicit the modes of implementation for their students in their classrooms. Actually, the students involved are also math teachers in a secondary class (junior or senior high school). At this stage, the issue is not to pose the problem in a class but, in the training center, to think about the form that the problem could take if it were posed to a class. In that sense it may be considered as *a virtual* problem P_2 which marks the outcome of the work for TEa's training group. This stage could possibly have been carried on, but its existence and its control had not explicitly been anticipated in the course specific for this group of training students. A description of the work of TEa's group is developed in Kuzniak, et al. (2011).

Stage 3. Third transformation: from problem P_2 to problem P_3

In TEb's group, after a session of the 'seminar' type in which the students had to expose their work in stages 1 and 2, they were asked to write down a problem P_3 , again with making explicit its modes of implementation and its aim, and above all to actually pose it to their own students. Then they had to present in the training center, again during a session of the seminar type, and a posteriori analysis of problem P_3 posed in their class, illustrating it with their students' writings. This shift from the training center (virtual problem P_2) to the classroom (real problem P_3) supposes a sharper adaptation of the problem to the trainee's class, in particular because of the real constraints.

3.2. Description of complete trajectories developed by student teachers

We call the set of stages transforming problem P_0 which has been exposed above *a trajectory* of this problem. Of course, every student teacher develops his/her own

trajectory, which can even be trajectories because classroom is an important factor influencing the transformations of a problem.

Below are described the complete trajectories of P_0 elaborated by five student teachers of the TEb group, named STa, STb,... STe. In fact, the differences between these trajectories are essentially due to the mathematical aims linked with the teaching contents of each class and with standard activities of textbooks at the different teaching levels. The student teachers try to design and develop teaching activities which are as close as possible to what we call suitable mathematical working space (Kuzniak 2011).

Hence, the aims of each problem are different according to the mathematical contents aimed at. On the other hand, a teacher will only give his students a problem on the condition that it fits well in the syllabus. For this reason it is necessary to supply the student teachers with problems having strong potentialities and open to varied adaptations. In the present case, problem P_1 (cf. section 2), elaborated after discussion by the teacher educators, is adequate and, as will be seen, might give rise to adaptations at all secondary education levels. Another common characteristic that we noticed is that the problem was always used to introduce a new knowledge and never an assessment of an old knowledge.

3.2.1 Two pre-service teachers' trajectories at grade 10

In this first case we consider two student teachers, STa and STb, teaching in *seconde* grade (grade 10), which in France is the first course of senior high school. In spite of different modes, essentially due to the real constraints of the two classes, the two trajectories presented here are very close to each other. Such closeness can be explained by the fact that the aims chosen, depending on the teaching program of the class, were practically identical, that is, a global study of polynomial functions. Indeed, problem P_1 is

close to a standard type of exercises which can be found in many textbooks at this education level: a geometric statement followed by a modelling by a quadratic function enabling to solve the initial problem.

Stage 1. Solving problem P₁

STa and STb solved the problem in a similar manner and used the graphs of the two functions defined by modelling the two areas (triangle and hexagon) generated by the use of *Geogebra* software. The variable chosen, called x , is the length of the side of the small square. The intersection point of the two curves gives an approximate solution: the common measure of the areas is its ordinate while the measure of the side of the small square in the case of equality is its abscissa. However, the use of *Geogebra* by the two student teachers was very different:

- STa constructed, in a same file, the square simulating the folding and drew the two curves representing the areas as functions of the distance between the folded vertex and a free point on the folding diagonal (cf. figure 1).
- STb as well made a construction with *Geogebra* to simulate the folding (two constructions were proposed) but functions are used in another file. She first got the two algebraic functions then graphically represented them (cf. figure 2). In this case *Geogebra* was in fact used as a graphics software and not as a dynamic geometry software.

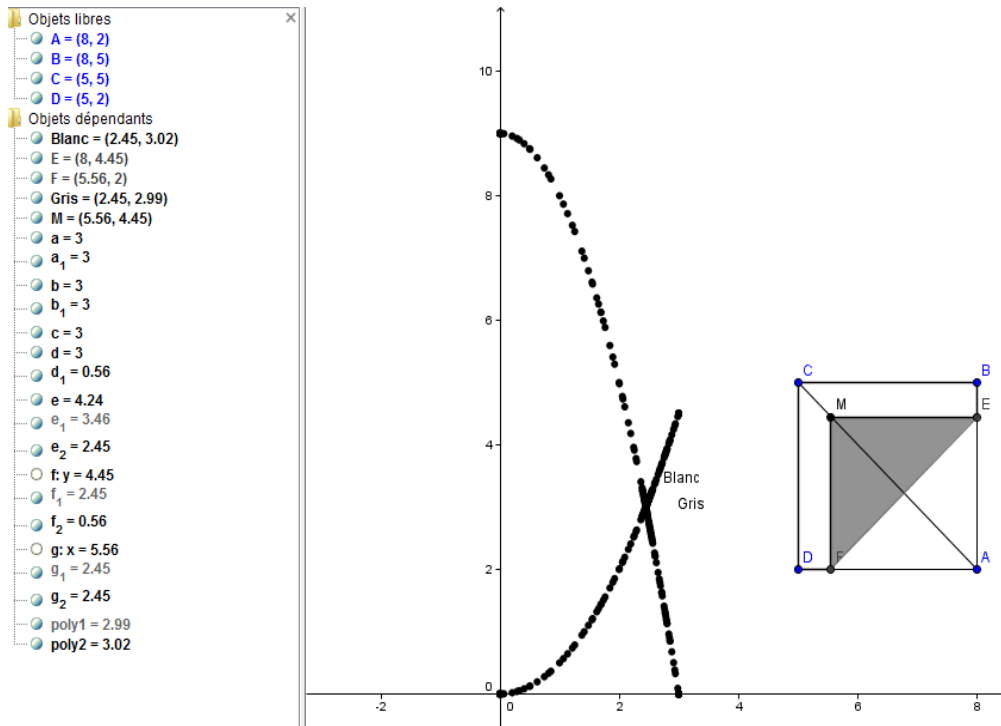


Figure 1. Use of Geogebra by STa for the solving of P₁

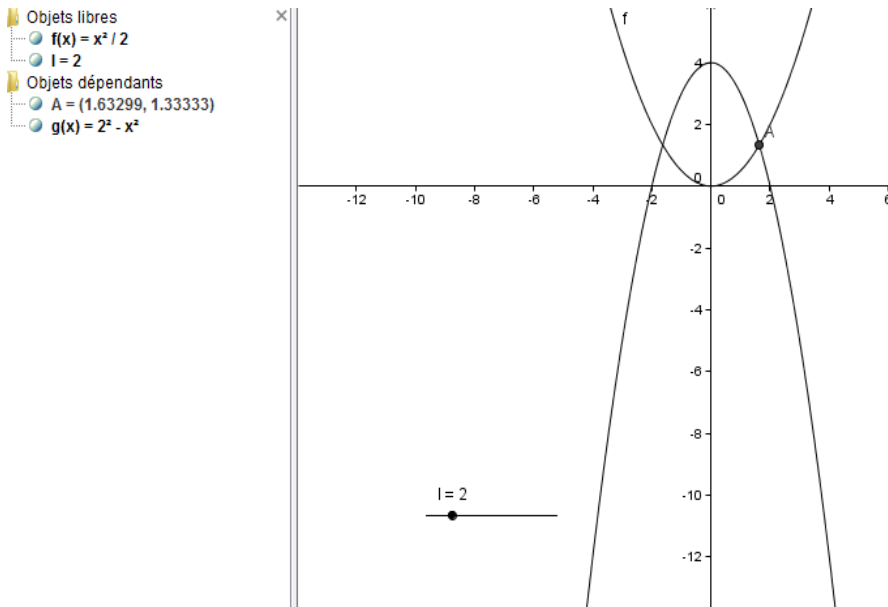


Figure 2. Use of Geogebra by STb for the solving of P₁

Another difference between STa and STb appears in how each of them considers the square length l with the software:

- STa fixed the value of the square length to $l = 3$ cm though he received a 5 cm length square by mail: he considered this value inadequate because it did not allow a good representation of the two curves on the computer window, the size of the objects being estimated too big. This last point shows that he has a quite poor knowledge of the software since he modified the situation instead of using the Geogebra potentiality to manage the mathematical situation.
- STb did not fix the square length since the parameter l is managed by the software through a cursor and the two functions introduced are defined using this parameter. So the abscissa of the intersection point of the two curves gives the searched value of x as a function of l .

Nevertheless, neither STa nor STb undertook a deeper exploration of the situation within the software. They only gave approximate values⁴ of the solution:

- STa wondered whether the same reasoning is still valid when the value of l – that is the square size – is changed but it seems that he did not try answering this dilemma.
- STb did not try to search the link between the solution, which is the abscissa x_A of point A in figure 2, and the parameter l given by the cursor. Indeed, the graph of function $l \rightarrow x_A(l)$ could be easily obtained by considering the point of coordinates (l, x_A) . Then, one can easily see that this graph is a straight line.

During the exploration of the possible solutions, the two student teachers did not use any other software. Their researches within a paper and pencil environment are also

⁴ STa obtained the approximate value 2.46 for $l = 3$ cm; STb gave the approximate solution values with 5 decimals. STb noticed that these approximate solutions were also approximate values of $1/3$ (for $l=1$) or $4/3$ (for $l=2$, cf. figure 2). But this remark was without any consequence on splitting the square area into three thirds: STb stuck to her approximate determination of x .

very close. The configuration studied is general and both use the l parameter to name the side of the square and a variable (or unknown) x to name the side of the small square. After calculation of the two areas as functions of x , the problem was solved in the case of equality, with the answer $\sqrt{2/3} l$ accompanied by a justification for not considering the negative root of the equation. The comparison of the areas was made by using the extreme values $x=0$ and $x=l$, as well as an argument (implicit for STa) about continuous functions.

On the other hand, a notable difference between STa and STb appeared in the management of the geometric setting. Using properties of orthogonal symmetry STb developed a detailed proof on the nature of the triangles which seem to be isosceles and rectangle. STa apparently remained at a visual stage (of the GI type, see (Houdement & Kuzniak, 1999)) since he did not make any remarks on the geometric configuration, although he fully used it in his calculations.

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

For both STa and STb these problems were integrated in the chapter on polynomial functions of degree two.

For STa, the statement of the virtual problem P_2 is identical to P_1 (with the exception of the length of the side of the square which is fixed to 5 cm) with the use of *Geogebra* in half-classes. Though the precision "the length of the side is not given" can be noticed, the statements of the real problem P_3 and P_2 are almost identical (and so is the case for P_1). However, P_3 's implementation modes are very different. It is finally given as homework, the choice being left to students to send a *Geogebra* file by Internet or to give back a paper-and-pencil work. Contrary to P_2 , the use of the software is not required. Sending works by electronic mail had already been used in the year but none of SPA's

students chose this option for this work, and finally all of them achieved a paper-and-pencil work (presumably using calculator).

In both problems, P_2 and P_3 , STa encouraged his students to make the folding by themselves. However, in P_2 the square was given whereas in P_3 the square had to be constructed by the students themselves: therefore they had to choose the length of the side.

For STb problem P_2 is close to P_1 but, with the addition of specific questions, it became a closed problem. The l side was fixed to 6 cm and only the case of equality was asked; the actual folding was required (for this a bi-colored square on which a diagonal had been drawn was given to every student); a question asked to prove the existence of an isosceles rectangle triangle; notations for geometric points and the variable x were provided and use of *Geogebra* was considered – in half-classes – to represent the two curves and thus allow a graphical resolution of the problem (let's notice that this type of task has already been asked in this class).

Although if in P_3 there is no question about the nature of the triangle, STb mentioned that the nature of this triangle would be assumed. Finally l was fixed to $3\sqrt{3}$ (more or less like in problem P_0 , although STb did not know of it) and the question was then more open, no procedure was imposed anymore, the students had the choice between *Geogebra* software and paper-and-pencil environment. Two questions were asked: one on the case of equality and the second on the comparison of areas. The students, by groups of three, had to cut out a square. Two different aids had been prepared by STb: for students who choose *Geogebra* (the square $3\sqrt{3}$ size was already constructed) a hint indicates some *Geogebra* tools, and for those who chose the paper-

and-pencil environment several possibilities for choosing the unknown, or variable x , were given (this help was not immediately provided and was limited to cases of blockage).

3.2.2 A pre-service teacher's trajectory at grade 8

One student teacher, STc, was in charge of a grade 8 class. At this level, two mathematical contents, obviously in relation with the syllabus, were considered: mathematical proof in geometry (chosen by STc) and algebraic calculation.

Stage 1. Solving problem P₁

STc produced a long research, exploring various points of view on the problem, remaining mostly in a geometrical setting. He produced proofs using geometrical tools and notions: isometric triangles, intercept theorem (known in France as the *théorème de Thalès*), orthogonal symmetry, Pythagoras' theorem, perpendicular bisector, square, bisector, sum of angles of a triangle. He chose a variable x on the diagonal (he *instinctively* did not consider the side of the square) and calculated the areas but he could not solve the problem.

In his research on problem P₁, STc made a clear distinction between geometrical paradigms GI and GII (Houdement & Kuzniak, 1999) which constitutes one of the stakes of the teaching of geometry at junior secondary school. An attempt to cut out figures for determining areas (especially for the hexagon) was also noticed but STc concluded that it was impossible to find a solution without using the above mentioned geometry tools.

He also used the *Geogebra* software to simulate the folding and visualize the hexagonal area by a curve, using sizes measured by the software (length and area). Like for STa, the value of l leads to a curve that does not fit well in the graphical window. But instead of modifying the value of l , STc divided the ordinates of the points by 10. He

stopped when seeing that he got a parabola as his calculations had shown him. He did not solve the problem, neither with the software (contrary to STa and STb), nor by using the notion of function (the curve shows only that there is a parabola).

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

STc did not produce a virtual problem P_2 (this comes probably from an omission or misunderstanding of the statement), and in his real problem for his class he put the stress on the teaching of proof. The problem P_3 he proposed was stated only in a paper-and-pencil environment, and there are multiple reasons for this:

- he points out constraints in the use of the computer room;
- he thinks that his students are not able to use a software for making a conjecture without being guided and he wants to keep the character open of the problem;
- he thinks his grade 8 class is a 'good' one.

He then considers a paper-and-pencil work in small groups, planned for two sessions. The problem P_3 he poses asks to cut out a 6 cm sided square, with the students achieving actual folding, and includes only one question: "How to achieve this folding so that the grey area is equal to the white area?"

The aim is twofold, as it can be noticed in the planned institutionalization: proof of the fact that the hexagon is obtained by removing a small square and calculation of the position giving equal areas. Besides, after the first session a student proposed to cut out the square into three figures having the same area (hexagon and two isosceles rectangle triangles) but without being able to justify it. Then STc adjusted his plans and thought of proposing a solution based on the areas: the area of the small square must be equal to the two thirds of the total area, and therefore the side of the small square (which gives the

solution) is $\sqrt{2/3} \times 6$ cm or $\sqrt{24}$ cm. However, in the class reality, the aspects linked to geometrical proof were hardly tackled during the session.

3.2.3 Two pre-service teachers' trajectories at grade 6

At grade 6 level, the mathematical notions that the students know do not allow the use of the previous mathematical supports (functions, algebraic calculation, geometrical proof). It seems that the calculation of areas of polygonal figures is the only possible mathematical support at this level. Thus, it is not surprising that this very content constitutes the choice of both student teachers, STd and STe, who are considered in this section.

Stage 1. Solving problem P₁

STe used *Geogebra* for modelling the folding. A visual adjustment with the measures of the two areas allowed him reducing the gap between them in order to solve the problem in an approximate way. Then, in order to make a conjecture, STe tried searching for a notable value, the approximate solution could be an approximation of it. His attempts were not successful in spite of two constructions depending on whether the mobile point is on the side of the square or on the diagonal – these lengths being, in each case, fixed to 10 cm for making the research of a conjecture easier.

Then STe shifted to paper-and-pencil environment. After fixing the length of the square to 1, he produced two calculations of the solution by taking two unknowns, respectively the side x of the small square, and $1-x$. For STe, it is explicit that equal areas corresponds to cutting out the square into three thirds, but the general comparison of areas is not taken in account.

In her research for a solution, STd started with working in a paper-and-pencil environment; she named x the length of the side of the small square and l the length of the

side of the initial one, calculated the two areas and solved the problem of their equality. Let us remark that she wrote, without justification, that the comparison of areas is solved with the help of the equality case. The comparison of areas was made with respect to the value $l/\sqrt{1.5}$. Then STd carried out the folding with her square: "I measured l (7,3cm), I did the calculation, which gave $x=5.96$ ". STd found that, visually, there seemed to be a little difference between the areas and she thought that it is due to an optical illusion. She then gave a construction of the folding with the help of the *Geogebra* software. STd regrets that this only provides an approximate value of the solution, like the ones obtained with a square of paper: measures and area calculations.

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

STe proposed a statement of the virtual problem P_2 identical to P_1 's, but he fixed the side of the square to $l=12$ cm. The scenario he considered includes three steps:

- an initiation, during about 20 min, in a session that involved an actual folding of a particular square, a statement of the problem and first attempts of solution;
- a second stage, in the computer room, to determine an approximate solution with the help of Geogebra;
- a last stage, working in pairs, aiming to justify the solution found with the help of a cutting out of the square (this last step being not explicit).

He proposed a 'dressing' of the problem in order to make it more concrete for his students: a square field inherited by three brothers has to be divided between them. The eldest receives the total big square minus a small square (situated in 'a corner'), this remaining small square being shared between the two others. The question is: "do the three brothers have equitable parts?". This dressing, not taken up in problem P_1 , changes

significantly the problem because it turns it onto cutting the initial square into three polygons of equal areas. There is not folding anymore and nothing is said on how the small square is shared between the two younger brothers (nor even if it is equitable).

The real problem P_3 took up this idea of contextualisation, but remains closer to problem P_1 : a firm wants to make a logo defined by the folding of a square of side 12 cm and the constraint of equality of the two areas. STe also took up the idea of three phases, only slightly modified:

1. a first activity, on paper, to understand the problem;
2. a second activity, with *Geogebra* (construction and research are very guided), to find out an approximate value, which is quite suitable for the realization of the logo;
3. here the justification was replaced by a actual construction of the logo on paper, using this approximate value (this third step was planned in the same session than point 2).

The student teacher STd proposed a problem P_2 taking up problem P_1 and modifying the question in the same way as STe: "*How has the black corner to be folded so that it has the same area than the white surface?*" The possibilities for using calculator as well as the *Geogebra* software were mentioned (under the condition of not asking to draw the diagram, judged too complex for this level). In particular, STe planned to have the students work in groups of four in a computer room and let them choose their environment.

The real problem P_3 , differs notably from P_1 by the fact that one the interest is only in the equality (like P_2) and especially the fact that an approximate value is

explicitly asked: "*Determine as precisely as possible a folding of this type, so that the white part and the colourful part have the same area*". The work was organised in groups of three students in the computer room, with a possibility to use *Geogebra* or only paper and pencil. Each group was provided a square of paper, the size of which was 3 cm, 4 cm, 5 cm or 6 cm (STd explicitly adjusted this choice of the didactical variable: multiple of 3 or not).

4. Discussion

4.1 About trajectories

In this section we propose an original frame to organize and analyse the emerging trajectories to deal with the problem, like those which have been set out in section 3. The aim of this frame is to take into account various dimensions of a problem (institution and persons involved, goal(s) aimed at) and study the nature and the dynamics of the changes which take place through the successive ‘moves’ of this problem from one institution to another.

At the start there is a problem, not necessarily mathematical, coming from an institution I, that may involve an everyday life or any domain of knowledge. Then there are several didactical institutions I_1, I_2, \dots in which successive alternative forms (‘avatars’) of the initial problem will show up. In each institution I_k ($k \geq 1$) one or several individuals T_k in a ‘teacher’ (or ‘educator’) position, as well as individuals S_k in a ‘student’ (or ‘trainee’) position, will be distinguished.

These institutions will be concatenated between them in the following way: the problem was introduced in I_k under the P_k avatar by T_k who poses it to the S_k with a given purpose. Then one of the S_k s, who in institution is in a ‘teacher’ position ($T_{k+1} = S_k$),

poses the problem to his/her students S_{k+1} s under the avatar P_{k+1} , with a purpose which is generally different (figure 3).

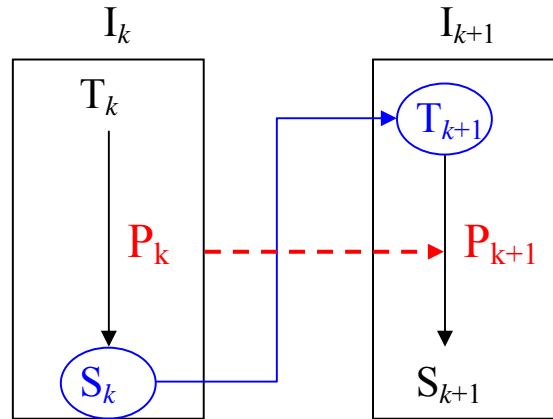


Figure 3. Concatenation of institutions

Of course this process can possibly be carried on from an institution to another (I_1, I_2, \dots, I_n), depending on the involved individuals. The succession of stages – and hence of avatars of the problem – constitutes the *trajectory* of the problem.

Example (figure 4).

Stage 1. In a training center for teachers (institution I_1) a math educator finds a problem written in everyday language in a magazine. He/she thinks that it could well give rise to a geometrical activity for his/her trainees. Then he/she transforms it into a geometrical wording and, within the training curriculum, asks the trainees to search ‘all possible solutions’ of the problem, regardless to the classroom level. (mathematical *a priori* analysis).

Stage 2. Again within the training curriculum (institution $I_2=I_1$), the teacher educator asks his/her trainees to transform the wording into a new one that could be posed as a research problem to a class of a given level (didactical *a priori* analysis).

Stage 3. Back to his/her school (institution I_3), each trainee undertakes posing the problem in his/her class. For that he/she transforms again the wording according to this

particular class and poses it by asking his/her students to use their knowledge to find a solution to the problem.

Stage 4. Back to the training center (institution $I_4=I_1$), the educator asks the trainees, gathered in groups according to the level of their classes, to work out for that level a new formulation of the wording, in order to make it a research problem taking into account the implementation that they could observe in their own classes (*a posteriori* analysis).

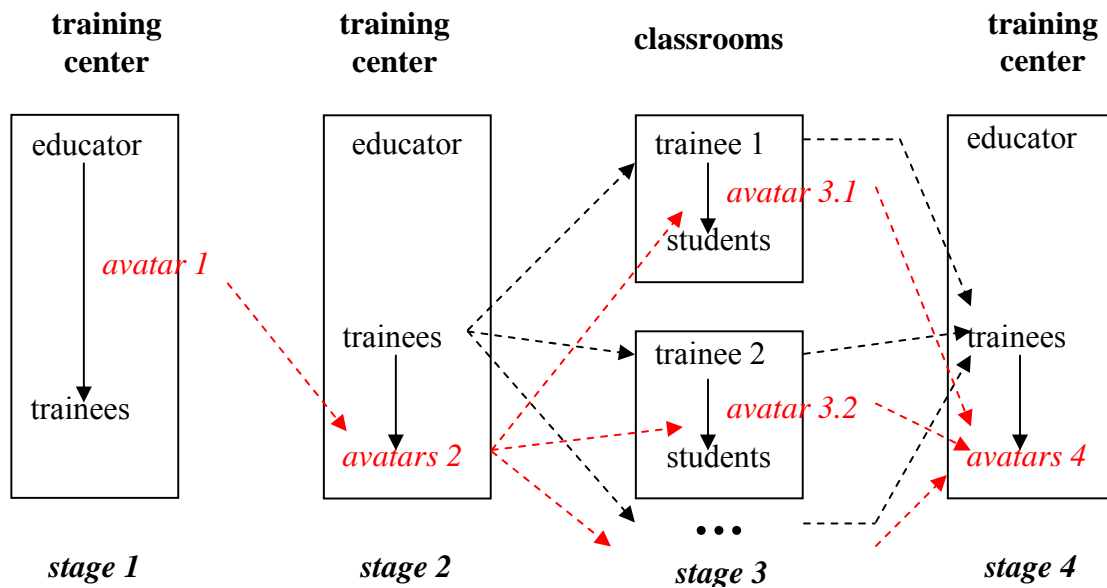


Figure 4 : Examples of trajectories of a problem

The first three stages correspond to the example of training constituting the study of section 3: $I_1=I_2$ is the training center and I_3 is one of the secondary school classes. Stage 4 could not be achieved during the training. It is nevertheless important, either being put into play in the training center or not, because it marks the start of a cycle of transformation of the problem taking into account the feedbacks from the students. This is a central component of the profession of teacher.

Moreover, one may quite consider conceiving trajectories in which other modes of transmission of problems intervene. For instance think of a continued training instead of an initial one or a debate between teachers of a same secondary school.

4.2 On training

During the first session dedicated to presentation of the problem P_1 , the teacher educator TEb made an unsuccessful attempt to orientate the trajectories by encouraging the teacher students to think about the use of spreadsheet in the class. However, as we saw it in the class of STd, sixth grade students could generate values tables close to what they could get faster with spreadsheet. We could observe the teachers' difficulties to integrate spreadsheet in their actual practices despite an important focus during the training. It could suggest a training underperforming, but this opinion needs to be qualified because it seems that spreadsheet, according various studies, is a tool especially difficult to integrate into lessons by teachers. Indeed, Haspekian (2005) mentions some specific problems on spreadsheet instrumentation or teaching of particular notions related to spreadsheet (such as delicate and complex notion of cell) which do not exist or not under the same form in maths knowledge at this grade: that can interfere negatively with the teaching of algebra. Teachers can be aware of these difficulties and avoid the use of spreadsheet in class despite the official demand from educative institution. The interpretation is confirmed by the experiment of TEa. One group of teacher students had to prepare a session using spreadsheet. Convinced of the impossibility of using spreadsheet in their own class, they prepared a session dedicated to the teaching of algorithms without any actual adaptation to the level of their students. They argued that the use of a spreadsheet needs too much time and knowledge which is not of mathematical nature.

Open problems and problem-situations with a-didactic potential are largely favored by the training in teacher training institution I_1 , especially to encourage student teachers to not only ask problems with closed questions to their students. As the problems P_3 , posed in class, were generally open, we can conclude that prospective teachers were aware of this mathematics education complexity. This is perhaps due to the training based on homology that we gave to the students and which postulates that teachers students will reproduce the form of the teaching they received during their training in I_1 .

It should also be noted that the virtual problem P_2 does not provide a lot of information on the actual course in class, except to check that changes of the mathematical support could only be possible in the class in front of school students (see STc). Even when student teachers know they will have to manage the problem with their students, the real constraints of the class do not seem to be taken into account before they are involved in real teaching scenarios with their own students. This leads to significant differences between *laboratory* work in I_2 towards I_3 and the actual work in I_3 and could suggest that the training on problems prepared in I_2 is not representative and far away from the reality of class teaching - even if this work remains interesting for training. That too should lead teachers educators to complete the training by requiring prospective teachers to engage into an actual implementation in a class with an a posteriori analysis. This demand can also show them that, first, it is possible to implement in I_3 the requirement made in I_1 and, then, that the demand of the training institution is not opposite to the demand of school institution as some students think of it.

4.3 On the choice of the specific technological context

All prospective teachers have chosen to use *Geogebra* software to approach their problem research in response to the demand of using a technological context. This sole

choice of *Geogebra* could be explained by some factors. First, training in I_2 favors this software which is widely used in French secondary school system. On other hand, *Geogebra* which is a multi-purpose software is well adapted to the problem: P_1 is generally seen as a geometrical problem and therefore the use of a dynamic geometry software is somehow natural, and for grade 10 the problem is also connected to functions as modeling tools and the use of *Geogebra* to make graphics is well suited.

For the problem posed in class, three different environments are employed for solving it: a paper and pencil environment or *Geogebra* (STa, STd), only *Geogebra* (STb, STe) and no use of software (STc). Moreover, there are few mentions of the use of a calculator (STd is the unique teacher who speaks explicitly about it) while school students use it widely. Perhaps, this lack of allusion to calculator is due to the fact that teachers do not perceive it as a technological environment (despite the instructions see sec 3.1) and they think of a computer. It is also possible that its use is now considered transparent and routine for prospective teachers and they feel no need to mention it.

4.4 On the folding

All prospective teachers keep the idea of the folding to present the problem to their students. Probably this anchoring to the real world supports the devolution of the problem as the attitude of STe suggests it: he left aside the idea of folding in the virtual problem P_2 , but it takes again this idea when he poses the real problem P_3 to his students in class.

However, the folding is not easy to define as we can see it in I_1 where the teacher educators had been obliged to mention other geometric terms than the area like square and diagonal and vertices. The diagonal could be also drawn (an even marked by a fold as STe did it). Other ways are possible: STc pointed out the vertex to fold on the diagonal

by coloring the good corner to use; STd has not defined the folding with enough precision and students did not well understand the instructions so that STd added some comments during the session in class; STa and STb made an unequivocal coding of the figure (like in the Mexican task in sec. 2).

4.5 On the problem

It is indeed a problem with a high potential that can be addressed at all levels of secondary education. The student teachers have all agreed without hesitation to pose it with their own transformations to their classes and students and, according to their comments, the students were interested in solving the problem P₃.

Many adjustments were made especially concerning modalities of implementation. But despite the diversity of educational levels where the problem was given, the core of the mathematical problem stays stable with few changes. Among the changes, we can note essentially: the value of l (except for STa) and the research of the equality (except for student teachers teaching in grade 10, STa and STb). The biggest adjustment was made by STd, who introduced the concept of precision of the solution. By and large, the problem P₁ *did the job*.

We can conclude that the transformations of the problem P₁ to give it in class are simultaneous oriented by the researches of the mathematical solution and by the official syllabus of the grades involved in the teaching. It would be interesting to know what will be the use of the problem by the teachers some years later and how the trajectory of the problem continues evolving. We intend to make an interview with the prospective teachers involved in this study in the future. Another point of interest is the impact of such problems on school students and some material need to be used to precise this crucial point.

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Primary school teacher's noticing of students' mathematical thinking in problem solving

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Abstract: Professional noticing of students' mathematical thinking in problem solving involves the identification of noteworthy mathematical ideas of students' mathematical thinking and its interpretation to make decisions in the teaching of mathematics. The goal of this study is to begin to characterize pre-service primary school teachers' noticing of students' mathematical thinking when students solve tasks that involve proportional and non-proportional reasoning. From the analysis of how pre-service primary school teachers notice students' mathematical thinking, we have identified an initial framework with four levels of development. This framework indicates a possible trajectory in the development of primary teachers' professional noticing.

Keywords: Professional noticing, proportional reasoning, pre-service primary teacher's learning, classroom artifacts.

INTRODUCTION

Teachers and problem solving: the role of understanding the students' mathematical thinking

Solving problem is a relevant task in mathematics teaching. However, teachers need to understand the students' thinking in order to manage problem solving situations in classroom. Teachers' abilities to identify the mathematical key aspects in the students' thinking during problem solving are important to performance teaching for

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understanding. The development of these abilities to interpret students' thinking may allow teachers to make appropriate instructional decisions, for instance, the selection and design of mathematical tasks in problem solving activities (Chamberlin, 2005).

Although the analysis of students' thinking is highlighted as one of the central tasks of mathematics teaching, identifying the mathematical ideas inherent in the strategies that a student used during the mathematical problem solving could be difficult for the teacher. However, teachers need to know how students understand the mathematical concepts in order to help them to improve their mathematical understanding (Schifter, 2001; Steinberg, Empson, & Carpenter, 2004). This approach is based on listening to and learning from students (Crespo, 2000) since, in this case, the teacher has to make decisions in which students' thinking is central.

Identifying the possible strategies used by students in problem solving allows teachers to interpret why a particular problem could be difficult and also to pose problems considering the characteristics of students' thinking. On the other hand, if teachers understand the mathematical ideas associated with problems in each particular mathematical domain, they may be able to interpret the mathematical understanding of students appropriately. This knowledge could help teachers to know which characteristics make problems difficult for students and why (Franke & Kazemi, 2001).

Considering these previous reflections about the relevant role of students' thinking in mathematics teaching, an important goal in some mathematics teachers programs is the development of teachers' ability to interpret students' mathematical thinking (Eisenhart, Fisher, Schack, Tassel, & Thomas, 2010). Some mathematics teacher education programs have reported findings that support this approach but have also

reported that the development of this expertise is a challenge (Llinares & Krainer, 2006). The findings in these studies have pointed out that the more or less success of programs depends on how pre-service teachers understand the mathematical ideas in the mathematical problems and the students' mathematical thinking activated in the problem solving activities (Norton, McCloskey, & Hudson, 2011; van Es & Sherin, 2002; Wallach & Even, 2005).

Teachers' professional noticing of students' mathematical thinking

Research on mathematics teacher development underlines the importance of the development of pre-service teachers' professional noticing in the teaching of mathematics (Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es & Sherin, 2002). Researchers and mathematics teacher educators consider the noticing construct as a way to understand how teachers make sense of complex situations in classrooms (Sherin, Jacobs, & Philipp, 2010). Particularly, Mason (2002) introduces the idea of awareness to characterize the ability of noticing as a consequence of structuring the teacher's attention about relevant teaching events. A particular focus implies the identification of key aspects of students' mathematical thinking and its interpretation to make decisions in the teaching of mathematics (Jacobs et al., 2010). Previous researches have indicated the relevance of pre-service teachers' interpretations of students' mathematical thinking to determine the quality of the teaching of mathematics (Callejo, Valls, & Llinares, 2010; Chamberlin, 2005; Crespo, 2000; Sherin, 2001). Therefore, the necessity that pre-service teachers base their decisions on students' understandings underlines the importance to characterize and understand the development of this skill (Hiebert, Morris, Berk, & Jansen, 2007). This fact justifies the necessity to focus our attention on how pre-service teachers identify and

interpret students' mathematical thinking in different mathematical domains (Hines & McMahon, 2005; Lobato, Hawley, Druken, & Jacobson, 2011).

Previous research on how students solve problems in specific mathematical domains has provided useful knowledge about the development of student's mathematical thinking in these domains that could be used in the study of the development of the noticing skill. One of these mathematical domains is the transition from students' additive to multiplicative thinking in the context of the proportional reasoning. Multiplicative structures in the domain of natural numbers that come from the expressions $a \times b = c$, have some aspects in common with additive structures, such as the multiplication as a repeated addition, but also have their own specificity that is not reducible to additive aspects (Clark & Kamii, 1996; Lamon, 2007; Fernández & Llinares, 2012-b). For example, tasks that involve the meaning of ratio such as: John has traveled by car 45 km in 38 minutes, how many km will he travel in 27 minutes? However, a characteristic of this transition is the difficulty that students of different ages (primary and secondary school students) encounter to differentiate multiplicative from additive situations. This difficulty is manifested in students who over-use incorrect additive methods on multiplicative situations (Hart, 1988; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985), and who over-use incorrect multiplicative methods on additive situations (Fernández & Llinares, 2011; Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2011-a, 2011-b; Van Dooren, De Bock, Janssens & Verschaffel, 2008). These previous researches have provided results that underline key ideas in the transition from additive to multiplicative structures. These ideas have allowed us the

opportunity to design instruments to analyze pre-service teachers' professional noticing of students' mathematical thinking.

The aim of this study is to characterize pre-service teachers' noticing of students' mathematical thinking in the domain of the transition from additive to multiplicative thinking, particularly, in the context of the proportional reasoning. Therefore, we are going to characterize how pre-service primary school teachers interpret students' mathematical thinking when they are analyzing the student's written work in mathematical tasks. Research questions are:

- Which aspects of students' mathematical thinking do pre-service teachers identify in multiplicative and additive situations?
- How do pre-service teachers interpret the aspects of involved students' mathematical thinking?

When we tried to answer these two questions, an additional result emerged: a framework with different levels that describes pre-service primary school teachers' noticing of students' mathematical thinking in the domain of students' transition from additive to multiplicative thinking in the context of proportional reasoning. In this sense, pre-service teachers' interpretations of the student's written work in the mathematical tasks help us to identify how they interpret the information about the way in which students have solved the problems. So, in this case, we hypothesized that students' solutions to the problems could help pre-service teachers interpret how students are thinking about the given situations.

METHODS AND PROCEDURES

Participants

The participants in this study were 39 pre-service primary school teachers that were enrolled in the last semester of their training program. The three years of teacher education program offers a combination of university-based courses and school-based practice. Pre-service teachers take foundational courses in education and method courses in different areas such as mathematics, language and social science, and a 12-week school teaching practicum. These pre-service teachers had still not made teaching practices at schools, but they had finished a mathematics method course in their first year of the training program (90 hours). This mathematics method course is focused on numerical sense, operations and modes of representation and, particularly, it has approximately 9 hours focused on the idea of ratio as an interpretation of rational numbers. We considered that characterizing pre-service teachers' noticing of students' mathematical thinking in problem solving could provide information about the development of pre-service teachers' learning during the teaching practices.

Instrument

Pre-service teachers had to examine six students answers to four problems (Figure 1), two proportional problems (modelled by the function $f(x) = ax, a \neq 0$) (problems 2 and 4) and two non-proportional problems with an additive structure (modelled by the function $f(x) = x+b, b \neq 0$) (problems 1 and 3). Additive and proportional situations differ on the type of relationship between quantities. For example, in Peter and Tom's problem (problem 1) the relationship between Peter's and Tom's number of boxes can be expressed through an addition: Tom's laps = Peter's laps + 60 (the difference between

quantities remains constant). On the other hand, in Rachel and John's problem (problem 2), the relationship between the number of flowers that Rachel and John have planted can be expressed through a multiplication: John plants 3 times more flowers than Rachel ($60 = 20 \times 3$). The first problem is an additive situation while the second situation is a proportional one. These differences among proportional and additive situations are considered in the problems with the sentences "they started together" or "Peter started later/David started earlier" and "John plants faster/Laura plants slower" or "they go equally fast".

The students' answers show different correct strategies used in proportional situations (the use of internal ratios, the use of external ratios, the building-up strategy, the unit rate and the rule of three as correct strategies) but they were used incorrectly in the additive problems. On the other hand, the additive strategy was used as correct strategy in additive problems but as incorrect strategy in proportional ones.

Pre-service teachers had to examine a total of 24 students' answers (four problems \times six students) and respond to the next three issues related to the relevant aspects of the professional noticing of students' mathematical thinking skill (Jacobs et al., 2010):

- "Please, describe in detail what you think each student did in response to each problem" (related to pre-service teachers' expertise in attending to students' strategies).
- "Please, indicate what you learn about students' understandings related to the comprehension of the different mathematics concepts implicated" (related to pre-service teachers' expertise in interpreting students' understanding).

- “If you were a teacher of these students, what would you do next?” (that is, documenting pre-service teachers’ expertise in deciding how to respond on the basis of students’ understandings).

The six students’ answers to the four problems were selected taking into account previous research on proportional reasoning. We focus our attention on the research findings that describe different profiles of primary and secondary school students when they solve proportional and non-proportional problems (Fernández & Llinares, 2012-a; Van Dooren, De Bock & Verschaffel, 2010). These students’ profiles are:

- students who solve proportional and additive problems proportionality,
- students who solve proportional and additive problems additively,
- students who solve both type of problems correctly, and
- students who solve problems with integer ratios using proportionality (regardless the type of problem) and solve problems with non-integer ratios using additive strategies.

So, four out of six students’ answers corresponded with one of these profiles and the other two students’ answers used methods without sense. These last two students’ answers were included as *buffer answers*. Furthermore, to avoid those results were affected by other specific variables of the test, problems and students’ answers order was varied. So, 20 different versions of the test were designed.

<p>Problem 1 Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded?</p>		
<p>Student 1</p> $\begin{array}{r} 60 \\ -40 \\ \hline 20 \end{array} \quad \begin{array}{r} 100 \\ +20 \\ \hline 120 \end{array}$ <p>Tom has loaded <u>120</u> boxes</p>	<p>Student 2</p> $\begin{array}{r} 60 \\ \times 40 \\ \hline 100 \end{array}$ <p>Tom has loaded <u>100</u> boxes</p>	<p>Student 3</p> $\begin{array}{r} 40 \mid 60 \\ 100 \mid 120 \end{array} \quad 60+60$ <p>Tom has loaded <u>120</u> boxes</p>
<p>Student 4</p> $\begin{array}{r} 60 \\ -40 \\ \hline 20 \end{array} \quad \begin{array}{r} 100 \\ +20 \\ \hline 120 \end{array}$ <p>Tom has loaded <u>120</u> boxes</p>	<p>Student 5</p> <p>Pedro $40+20=60$ Tomás $100+50=150$</p> <p>Tom has loaded <u>150</u> boxes</p>	<p>Student 6</p> $60+40=100 \quad 100+20=120$ <p>Tom has loaded <u>20</u> boxes</p>
<p>Problem 2 Rachel and John ara planting flowers. They started together but John plants faster. When Rachel has planted 4 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?</p>		
<p>Student 1</p> $\begin{array}{r} 4 \\ \times 5 \\ \hline 20 \end{array} \quad \begin{array}{r} 12 \\ \times 5 \\ \hline 60 \end{array}$ <p>John has planted <u>60</u> flowers</p>	<p>Student 2</p> $\begin{array}{r} 12 \\ +12 \\ \hline 24 \end{array}$ <p>John has planted <u>24</u> flowers</p>	<p>Student 3</p> $\begin{array}{r} 4 \mid 20 \\ 12 \mid 28 \end{array} \quad 12-4=8+20=28$ <p>John has planted <u>28</u> flowers</p>
<p>Student 4</p> $\begin{array}{r} 12 \\ -4 \\ \hline 8 \end{array} \quad \begin{array}{r} 20 \\ +8 \\ \hline 28 \end{array}$ <p>John has planted <u>28</u> flowers</p>	<p>Student 5</p> <p>Rachel $4 \rightarrow 20$ John $12 \rightarrow 60$</p> <p>John has planted <u>60</u> flowers</p>	<p>Student 6</p> $20 \div 4 = 5$ $5 \times 12 = 60$ <p>John has planted <u>60</u> flowers</p>
<p>Problem 3 Ann and David are manufacturing dolls. They work equally fast but David started earlier. When Ann has made 12 dolls, David has made 24 dolls. If Ann has made 48 dolls, how many dolls has David made?</p>		
<p>Student 1</p> $\begin{array}{r} 24 \\ \times 2 \\ \hline 48 \end{array}$ <p>David has made <u>48</u> dolls</p>	<p>Student 2</p> $\begin{array}{r} 24 \\ +24 \\ \hline 48 \end{array}$ <p>David has made <u>48</u> dolls</p>	<p>Student 3</p> $\begin{array}{r} 24 \mid 48 \\ 12 \mid 24 \end{array} \quad x=24 \div 12=2$ $\begin{array}{r} 48 \\ \times 2 \\ \hline 96 \end{array}$ <p>David has made <u>96</u> dolls</p>
<p>Student 4</p> $\begin{array}{r} 24 \\ -12 \\ \hline 12 \end{array} \quad \begin{array}{r} 48 \\ +12 \\ \hline 60 \end{array}$ <p>David has made <u>60</u> dolls</p>	<p>Student 5</p> <p>Ana $12 \rightarrow 48$ David $24 \rightarrow 96$</p> <p>David has made <u>96</u> dolls</p>	<p>Student 6</p> $24-12=12 \quad 12 \times 48=576$ <p>David has made <u>576</u> dolls</p>
<p>Problem 4 Laura and Peter are pasting stamps on postcards. They started together but Laura pastes slower. When Laura has pasted 80 stamps, Peter has pasted 280 stamps. If Laura has pasted 120 stamps, how many stamps has Peter pasted?</p>		
<p>Student 1</p> $\begin{array}{r} 120 \\ -80 \\ \hline 40 \end{array} \quad \begin{array}{r} 280 \\ +40 \\ \hline 320 \end{array}$ <p>Peter has pasted <u>320</u> stamps</p>	<p>Student 2</p> $\begin{array}{r} 280 \\ +280 \\ \hline 560 \end{array}$ <p>Peter has pasted <u>560</u> stamps</p>	<p>Student 3</p> <p>Laura $80 \rightarrow 280$ Peter $120 \rightarrow 420$</p> <p>Peter has pasted <u>420</u> stamps</p>
<p>Student 4</p> $\begin{array}{r} 280 \\ -80 \\ \hline 200 \end{array} \quad \begin{array}{r} 120 \\ +200 \\ \hline 320 \end{array}$ <p>Peter has pasted <u>320</u> stamps</p>	<p>Student 5</p> <p>Laura $80 \rightarrow 280$ Peter $120 \rightarrow 420$</p> <p>Peter has pasted <u>420</u> stamps</p>	<p>Student 6</p> $120 \div 80 = 1.5$ $1.5 \times 280 = 420$ <p>Peter has pasted <u>420</u> stamps</p>

Figure 1. Problems and students' answers used in the test

Analysis

Pre-service teachers' answers were analyzed by three researchers. From a preliminary analysis of a sample of pre-service teachers' answers, we generated an initial

set of rubrics to make visible aspects to characterize the professional noticing of students' mathematical thinking in the context of proportional reasoning. These initial rubrics were refined as the analysis was progressing. Finally, we generated four-level descriptors which were applied to all pre-service teachers' answers:

- Level 1. Proportional from additive problems are not discriminated
- Level 2. Discriminate proportional from additive problems without identifying the mathematical elements.
- Level 3. Discriminate proportional from additive problems identifying the mathematical elements but without identifying students' profiles.
- Level 4. Discriminate proportional from additive problems identifying the mathematical elements and the students' profiles.

Therefore, firstly, we classified pre-service teachers in two groups: pre-service teachers who discriminated proportional and additive situations, and pre-service teachers who did not discriminate both situations.

Secondly, focused on pre-service teachers who discriminated both situations, we analyzed if they discriminated the situations identifying the mathematical elements that characterize proportional and additive situations and if they were able to identify students' profiles. This second stage of the analysis tried to identify the quality of pre-service teachers' interpretations considering whether they have used specific mathematics elements to justify their interpretations. To do this, we took into account the mathematical elements of proportional and additive situations (Table 1) and the strategies used by students (Table 2). So, we analyzed if pre-service teachers identified the

strategies and integrated the mathematical elements in their written text produced (relating the characteristics of the problem and the strategy) when they answered the task.

We analyzed all pre-service teachers' answers but three out of thirty-nine pre-service teachers were not classified in one of these levels because their answers were incomplete.

Table 1. Mathematical elements of the situations

Proportional situation $f(x) = ax, a \neq 0$	Additive situation $f(x) = x + b, b \neq 0$
The function passes through origin " <i>they started together</i> "	The function does not pass through origin " <i>they started later or earlier</i> "
The value of the slope changes " <i>someone goes faster or slower</i> "	The value of the slope remains constant " <i>They go equally fast</i> "
External ratios are constant ($f(x)/x = a$) and internal ratios are invariant ($a/b = f(a)/f(b)$)	The difference between related quantities remains constant $f(x) - x = b$

Table 2. Students' strategies used to solve the problems

Proportional situations	Additive situations
The use of external ratios	Additive strategy
The use of internal ratios	
Unit-rate	
Building-up strategies	
Rule of three algorithm	

Results

In this section, we present the characterization of the different levels in the development of pre-service teachers' noticing of students' mathematical thinking skill in the mathematical domain of proportionality.

Level 1. Proportional from additive problems are not discriminated (25 out of 39 pre-service teachers).

In this level we classified pre-service teachers who did not discriminate proportional from additive situations. These pre-service teachers considered

- that all the problems were proportional (so proportional methods were the correct strategies to solve all these problems), or
- that all the problems were additive (so additive methods were the correct strategies to solve all these problems).

For example, a pre-service teacher gave the next argument in the answer of student 5 to problem 2 (proportional situation) (Figure 1): *“This answer is correct. The student has found out by how much Rachel goes from 4 to 20 and repeated the process with John”*. This pre-service teacher identified the multiplicative relationship between quantities used by the student 5 to solve the problem, but this pre-service teacher said in the answer of student 5 to problem 3 (additive situation): *“This answer is correct. The student has found out the multiplicative relationship between 12 and 48 and then has multiplied 24 by this number”*. In this case, the preservice teachers did not recognize the additive character of the situation.

Another pre-service teacher gave the next argument to the answer of student 4 to problem 3 (additive situation): *“The answer is correct. The student has obtained the*

difference between the dolls manufactured by David and Ann. Afterwards, the student has added 48 that are the dolls manufactured by Ann later". However, when this pre-service teacher interpreted the answer of student 5 to problem 2 (proportional situation), he did it erroneously "This student has used a correct method and has obtained a correct result. Firstly, the student has computed the difference between the flowers planted by John and Rachel and has obtained 8 flowers. After, taking into account this difference, the student has added this number (8 flowers) to the 20 flowers planted by Rachel obtaining how many flowers has John planted".

So, both pre-service teachers did not discriminate proportional from additive situations. Pre-service teachers in this level focus their attention on superficial features of the situations and show a lack of mathematical knowledge. As a consequence, their interpretations of students' answers mainly rely on the description of the operations carried out and not on the meanings.

Level 2. Discriminate proportional from additive problems without identifying the mathematical elements (2 out of 39 pre-service teachers).

We classified in this level pre-service teachers who discriminated proportional from additive situations but did not justify the difference between problems taking into account the mathematical elements of the situations. Therefore, these pre-service teachers identified the correctness of the strategies used by students in each type of problem (relating the situation with the strategy used by the student) but without justifying why the strategy is correct or incorrect taking into account the characteristics of the situations.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): *“The answer is correct. The student has determined how many boxes has Peter loaded, is that, the difference between the boxes loaded by Peter at the end (60 boxes) and the boxes loaded by Peter initially (40 boxes). So, this difference (20 boxes) is also the number of boxes loaded by Tom. So, $100 + 20 = 120$ ”*.

Pre-service teachers in this level only describe the operations carried out by students, but, in this case, the descriptions are related to the correctness of the strategy in each type of problem (subject matter knowledge).

Level 3. Discriminate proportional from additive problems identifying the mathematical elements but without identifying students' profiles (6 out of 39 pre-service teachers).

In this level we classified pre-service teachers who discriminated proportional from additive situations justifying the difference between situations taking into account some mathematical elements of the situations. However, these pre-service teachers were not able to identify students' profiles.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): *“The answer is correct. This student has computed the difference between the boxes loaded by Peter initially and later (20 boxes). As the problem said that the two people loaded equally fast but Peter started earlier, 20 are also the boxes loaded by Tom. So the student has added 20 boxes to the boxes loaded for Tom”*. This pre-service teacher justified the difference between situations with the mathematical elements

of the situations, in that case, mentioning two characteristics of the additive situations: “They loaded equally fast but someone started earlier”.

However, these pre-service teachers did not identify students’ profiles because they did not relate globally the behavior of each student to the four problems. For example, the pre-service teacher mentioned above identified the behavior of student 3 (student who solve both type of problems correctly): “*this student has solved all the problems correctly*” but this pre-service teacher were not able to identify the behavior of student 4 (student who solve all the problems additively) since he said “*this student has solved problems 1 and 3 (the same type) incorrectly and problems 2 and 4 (other type of problem) correctly*” neither the behavior of student 5 (student who solve all the problems using proportionality) “*this student has solved problems 1 and 3 (the same type) incorrectly and problems 2 and 4 (other type of problem) correctly*” because he/she did not identify that the student used the same strategy regardless the type of problem.

Level 4. Pre-service teachers who discriminate proportional from additive problems identifying the mathematical elements of the situations and the students’ profiles (3 out of 39 pre-service teachers).

In this level we classified pre-service teachers who discriminated proportional from additive problems justifying the difference between problems taking into account the mathematical elements of the situations and identifying the students’ profiles.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): “*The student has obtained the difference between the two Peter’s quantities and used it to obtain the number of boxes loaded by Tom. The answer is*

correct because the two people loaded equally fast and the difference has to be the same”.

This pre-service teacher was able to identify students’ profiles. In that way, this pre-service teacher indicated in relation to the answers of student 3 (student who solve all problems correctly) *“this student know the correct methods and apply them in the both type of problems”*, in relation to the answers of student 4 (student who solve all problems additively) *“This student only do correctly the problems where the speed is the same but someone starts earlier or later. This student apply the same method to the both type of problems”* and in relation to the answers of student 5 (student who solve all problems using proportionality) *“this student only do correctly problems where the speed is not the same. This student always applies the same method to all the problems”*. Pre-service teachers in this level are able to relate strategies within and across problems in order to see students’ overall performance to a certain type of problem focusing on a relation of relations.

DISCUSSION

Initially, the goal of this research was to characterize what pre-service teachers know about students’ mathematical thinking in the context of proportional and non-proportional problem solving before their teaching practices. However, the design of the test allows us to characterize a trajectory of the development of teachers professional noticing of students’ mathematical thinking. In the identified trajectory, pre-service teachers moved from the non-recognition of the characteristics of the situations towards the identification of the characteristics of the situations and the strategies used by students and the recognition of students’ profiles when solving problems. This last level

shows pre-service teachers' willingness and ability to analyze students' mathematical thinking in relation to the additive and multiplicative situations.

The development of a framework to characterize pre-service teachers professional noticing of students' mathematical thinking

Results show the difficulty of pre-service teachers to identify the relevant aspects of students' mathematical thinking in relation to the students' transition from additive to multiplicative thinking. This difficulty is manifested by pre-service teachers' difficulty in differentiate proportional from non-proportional situations (25 out of 39). This finding indicates a weakness in their own subject-matter knowledge about multiplicative and additive situations. Identifying the mathematical elements of additive and multiplicative situations is the first step to interpret properly students' mathematical thinking during the problem solving.

On the other hand, although some pre-service teachers could recognize the difference between both situations, they had difficulties in justifying why students' answers were or were not correct taking into account the mathematical elements of the situations. Furthermore, they had difficulties in interpreting globally all students' answers. This result shows the complex knowledge that pre-service teachers have to use to identify and interpret the way in which students solve the problems.

Another relevant result is the characterization of pre-service teachers' development of professional noticing of students' mathematical thinking. A framework consisted of four levels characterizing the development of this skill has been built. The transition from level 1 to 2 is determined when pre-service teachers are capable of analyzing the characteristics of situations to discriminate both types of problems. In level

2, pre-service teachers focus on the correctness of students' answers and tend to accept students' correct answers as evidence of understanding without making specific inferences about what or how students were or were not understand. The transition from level 2 to 3 is determined when pre-service teachers are capable to relate students' strategies with the characteristics of the problems justifying through the mathematical elements if the strategy is correct or incorrect. That is to say, pre-service teachers look beyond the surface of the student's answer. Finally, the transition from level 3 to 4 is determined when pre-service teachers are able to see student's overall performance to a certain type of problem. That is to say, pre-service teachers are able to relate strategies within and across problems in order to see how those strategies are related to other groups of problems. In this case, pre-service teachers display a greater attention towards the meaning of students' mathematical thinking rather than towards some surface features. Finally, the fact that some pre-service teacher focus on individual answers rather than characterizing the students' profiles could be related with the design of the task. For further researches, it is necessary to formulate more specific questions that address pre-service teachers to examine all the answers provided by each student to the four problems as a whole.

The different levels identified and the transition between them show how pre-service teachers professional noticing of students' mathematical thinking is developed and therefore, it allows us to begin to understand pre-service teachers learning (Figure 2). The key elements in this framework are how pre-service teacher use the evidence (students actions/operations) to describe what or how the student is thinking, and how they generate an explanation of what the student knows or thinks providing or not

evidence to support the explanation. The characteristics of this framework are similar to rubrics in the description of how pre-service teacher build a model of student thinking in a context of prediction assessments (Norton et al., 2011).

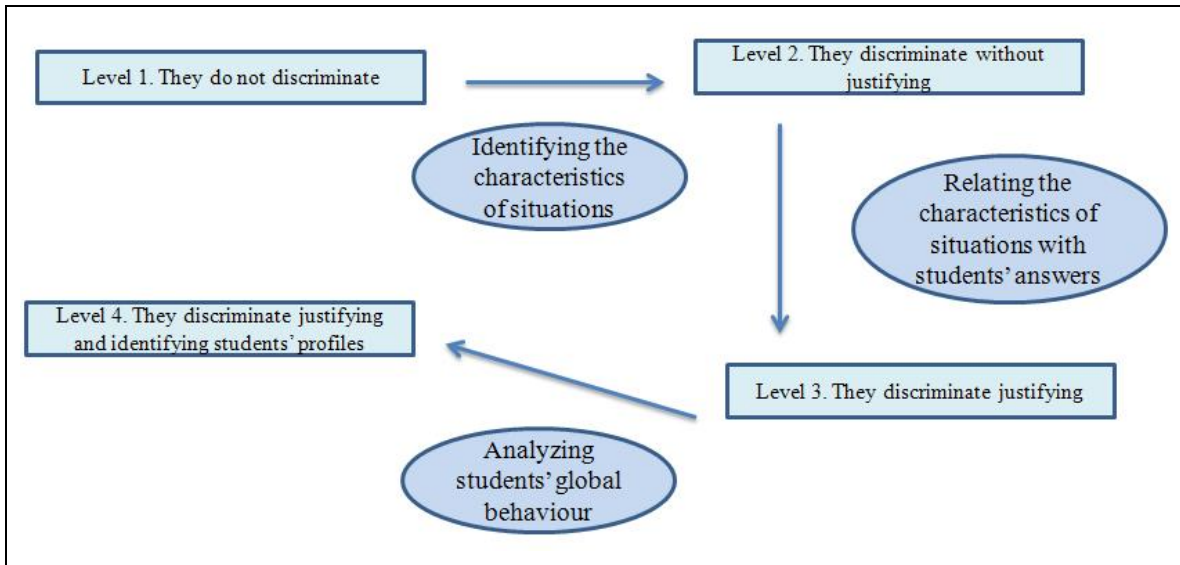


Figure 2. A framework to characterize pre-service teachers’ professional noticing of students’ mathematical thinking in the context of proportionality

The different levels in the framework support the idea that the subject-matter knowledge is necessary for teaching, but it is not a sufficient condition because teachers need to interpret the students’ behavior in problem solving situations using their understanding of mathematical knowledge (Crespo, 2000). Constructing a model for learning to notice students’ thinking, such as the framework presented, implies to focus on the organized knowledge about problems and on the range of strategies used by students to solve the problems (Franke & Kazemi, 2001).

In a previous research, van Es (2010) also provided a framework for learning to notice the student thinking articulating two central features of noticing: what teachers notice and how teachers notice. Van Es generated this framework using meetings with

seven elementary school teachers in which each teacher shared clips from his or her own classroom and discussed aspects of the lesson. Although van Es study and our research use different evidences and come from different contexts, it is possible to identify some features that provide insights about the noticing construct and its development. One of the relevant aspects showed in the two researches is how teachers or pre-service teachers go from a baseline to extent the noticing skill indicating how teachers/pre-service teachers go from noticing superficial aspects to consider the connections between different relevant aspects and meanings. However, there are also differences between the two frameworks: the role played by the mathematical content knowledge in the noticing skill and how it is integrated (as we have shown in the translation from one level to the next).

This framework should be considered as an initial approach to the characterization of the development of noticing. However, it points out two additional aspects that we should be considered. Firstly, the emergence of this framework is linked to a specific type of problems. Therefore, it is necessary more researches using different types of problems to extend and to validate this framework and this approach. Secondly, in the context of mathematics teacher education programs we could complement the written test (the questionnaire) with students' interviews.

Teacher education, problem solving and the development of the teacher's noticing skill

A goal in mathematics teacher education is the development of pre-service teachers' ability to model the student's thinking and to use evidences from the students' behavior when solving problems to construct this model (Norton et al., 2011). However, if pre-service teachers have a lack of content knowledge in solving the mathematical

tasks, they could have difficulties in building an appropriate model of the students' mathematical thinking. This is the case of pre-service teachers who did not differentiate the additive and multiplicative relations in the situations proposed in our study. As a consequence, the first level in the development of teachers' professional noticing of students' mathematical thinking is defined by the understanding of the mathematical knowledge. So, an aspect of pre-service teacher's content knowledge for teaching in the context of multiplicative and additive situations is related to the discrimination between proportional and non-proportional relationships. It is possible that the lack of knowledge that pre-service teachers have about proportionality may be due to the way in which proportionality is often taught at schools in which there is an over-use of missing-value problems and an overemphasis on routine solving processes (De Bock, Van Dooren, Janssens, & Verschaffel, 2007).

Since the proportionality is more than a four-term relation, in order to pre-service teachers could develop a professional noticing, it is necessary that they extend their understanding and consider other features of proportionality such as straight line graphs passing through the origin and the constant slope of such graphs identified with the coefficient of proportionality when it is adopting a functional approach. The differences between proportional and non-proportional situations should be another feature. Whether a good problem solver in a given domain is one who knows the connections between the different mathematical parts, a teacher who wants to interpret the students' mathematical thinking during a problem solving situation in the classroom also needs to know the mathematical structure of the domain. In this case a lack of pre-service teacher's content knowledge could limit his/her ability to model the student thinking. In this way, this

study examines pre-service teachers' capacities needed to make sense students' thinking about proportionality.

If teacher education programs require pre-service teachers to notice students' mathematical thinking in problem solving contexts then we should make an effort to document what is what prospective teachers notice in different mathematical domains and how the development of this skill could be characterized. Previous studies in initial mathematics teacher programs have reported improvements in noticing, going from a descriptive and evaluative noticing towards a more analytic and interpretative one (Crespo, 2000; Norton et al., 2011; van Es & Sherin, 2002). Furthermore, some studies underlined the benefits of teachers' discussions about students' written work. In a previous experience, seven prospective secondary school mathematics teachers solved the task proposed in this study and discussed it in an on-line debate (Fernández, Llinares, & Valls, 2012). Although, initially, prospective teachers had difficulties attending and interpreting the students' mathematical thinking in the domain of multiplicative and additive structures, when prospective teachers with a lower level of noticing interacted with other with a higher level of noticing in an on-line discussion, they changed their interpretations to reach mutual understanding. This process led prospective teachers with a lower level of noticing to develop a new understanding of students' mathematical thinking. From these preliminaries findings, we hypothesized that teachers could develop ways to elicit and listen to students' mathematical thinking when they focus their discussion on the students' written work. In this sense, focusing on students' written work remains an instrument for relating mathematics knowledge and students' mathematical thinking.

Our findings also provide additional information for the design of materials in teacher training programs that take into account the characteristics of pre-service teachers' learning and their understanding of proportionality (Ben-Chaim, Keret, & Ilany, 2007). In this sense, the instrument used in this research could be adapted as teaching material to create opportunities for the learning of pre-service teachers. These opportunities of learning should be focused on the development of pre-service teachers' skills to identify and interpret student's written work. In fact, a characteristic of our research instrument is that it is based on the details of students thinking and it is elaborated from the research based on students' understanding of additive and multiplicative structures (Fernández & Llinares, 2012-a; Van Dooren et al., 2010). So, firstly, pre-service teachers could solve the different problems and discuss on the possible different answers. Secondly, they could share the interpretations of students' solutions to the problem discussing on the mathematical understanding of each strategy and how particular strategies were elicited.

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2. An earlier version of this paper was presented as a research report at XV SEIEM- Ciudad Real, Spain. September 2011.

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A Proposal for a Problem-Driven Mathematics Curriculum Framework

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Abstract: A framework for a problem-driven mathematics curriculum is proposed, grounded in the assumption that students learn mathematics while engaged in complex problem-solving activity. The framework is envisioned as a dynamic technologically-driven multi-dimensional representation that can highlight the nature of the curriculum (e.g., revealing the relationship among modeling, conceptual, and procedural knowledge), can be used for programmatic, classroom and individual assessment, and can be easily revised to reflect ongoing changes in disciplinary knowledge development and important applications of mathematics. The discussion prompts ideas and questions for future development of the envisioned software needed to enact such a framework.

Keywords: Problem-based Mathematics, Curriculum frameworks, Mathematical Modeling, Model-Eliciting Activities.

Introduction

Curriculum frameworks are commonly organized around categories of mathematical topics (e.g., number, geometry), such as in the new Common Core School Mathematics Standards (NGA & CCSSO, 2011) and the National Council of Teachers of Mathematics (NCTM) standards documents (1989, 2000) for the United States (U.S.). Oftentimes, to convey the nature of mathematics teaching and learning, the content topics are cross-referenced with other types of mathematical behaviors, such as the “process standards” (e.g., problem solving, reasoning and proof) of the NCTM documents, and the “practices” (e.g., model with mathematics, attend to precision) of the CCSSM document.

Another approach is to formulate mathematics curriculum frameworks based on assumptions about learning mathematics, such as the Dutch curriculum framework described by van den Heuvel-Panhuizen (2003) (e.g., informal to formal, situated to generalized, individual to social). The developers of mathematics curriculum frameworks choose their organization and structure in order to communicate a mathematics curriculum to broad audiences (e.g., teachers, administrators, parents, students). The choices for content and the representation of curricula made by the framework developers, in turn, convey a distinctive perspective on mathematics curriculum, accompanied by inevitable (some intended, some unintended) consequences when users of the framework transform the represented curriculum into prescriptions for classroom experiences and assessment. A proposal for framing and representing a problem-driven mathematics curriculum is described in this article. The proposal envisions a framework that grows out of Lesh and colleague's work on models-and-modeling, which has focused on using modeling problems as sites for revealing and assessing students' thinking (e.g., Lesh, Cramer, Doerr, Post & Zawojewski, 2003), and more recently by Richard Lesh to teach data modeling (personal conversation, Dec. 21, 2012). The proposal also envisions a representational system that builds on a one originally posed by Lesh, Lamon, Gong and Post (1992), and is particularly poignant today because technology is now available that could carry out the proposal.

Why an Alternative Framework?

Assumptions about Curriculum Frameworks

Curriculum frameworks convey a view of mathematics learning to stake holders in education, influencing the full range of mathematics education activity—from implementation to assessment. For example, the two foundational NCTM curriculum documents (1989, 2000) contributed to a huge shift in views of mathematics curriculum in the U. S. Prior to the publication of these documents, schools, districts and state curriculum guides predominantly listed expected mathematical competencies by grade level, commonly referred to as *scope and sequence* documents. The NCTM standards documents introduced a process dimension (problem solving, reasoning, connections, communication) in addition to the common practice of describing mathematics competencies and performance expectations. Further, discussions about the mathematical processes and expected mathematical performances were embedded in the context of illustrative problems, teaching and learning scenarios, and ways of thinking about mathematics. These standards documents impacted not only state curriculum standards, but also resulted in the development of the now-famous NSF curricula (described in Hirsch, 2007a). Research on the standard-based curricula suggests that students using these curricula demonstrate enhanced learning of mathematical reasoning and problem solving (Hirsch, 2007b).

The new Common Core State Standards in Mathematics (NGA & CCSSO, 2011), adopted by 45 of the United States and 3 territories, lists mathematical learning objectives, or standards, organized by grade level, and is accompanied by a completely separate discussion of eight mathematical practices. There is no discussion in the document to help the practitioner envision what the implementation of the intended curriculum will look like—leaving the accomplished curriculum more dependent on

professional development and local school culture to fill in the picture. One advantage to the separation of mathematics competencies from the mathematical practices may be to avoid representing the mathematics curriculum as an array, which can inadvertently convey a view of mathematics curriculum as disaggregated into bits and pieces represented by each cell.

Consider, for example, the Surveys of Enacted Curriculum (SEC) (Porter, 2002), which are intended to drive assessment of student performance. The SEC is organized in a two-dimensional framework of cognitive demand (memorize, perform procedures, demonstrate understanding, conjecture/generalize/prove, and solve non-routine problems) vs. disciplinary topics (e.g., functions, data analysis, rational expressions). It divides the (K-12) mathematics topics dimension into 19 general categories, each of which is then divided into 4 to 19 smaller mathematical topics. “Thus, for mathematics, there are 1,085 distinct types of content contained in categories represented by the cells” (Porter, McMaken, Hwant, & Yang, 2011, p. 104). Porter’s fine-grained representation of curriculum is intended to ensure coverage of mathematical topics and types of cognitive demand while minimizing gaps and overlaps. However, such a representation may lead to an enacted curriculum prescribed by the “pieces” (i.e., the cells), and if educators are prompted to “teach to the test” an unintended emphasis on disconnected mathematics education may result. Further, once a framework like this is codified by formal external assessments, the mathematics content becomes more difficult to revise in response to the needs of evolving fields of science, engineering and technology.

An alternative may be found in the Dutch mathematics curriculum, rooted in Realistic Mathematics Education (RME) learning theory, initially developed by the well-

respected Dutch mathematics educator, Freudenthal (1991), and continued at the Freudenthal institute today. The work in RME portrays a vision of mathematics as a human activity that combines learning and problem solving as a simultaneous activity. Smith & Smith (2007) describe the three dimensions around which the RME-based mathematics curriculum framework is organized: informal to formal; situated to generalized; and individual to social. In practice, RME emphasizes curriculum designed to encourage students' development via progressive mathematization. van den Heuvel-Panhuizen (2003) describes progressive mathematization as the growth of an individual's mathematical knowledge from informal and connected to the local context, to an increasing understanding of solutions designed to reach some level of schematization (making shortcuts, discovering connections between concepts and strategies, making use of these new findings in a new way), and finally to an increasing understanding of formal mathematical systems.

The work on such progressive mathematization is growing (e.g., hypothetical learning trajectories as described by Clements & Sarama, 2004a; 2004b). But, questions have been raised by Lesh and Doerr (in press): Do all students optimally learn along a particular normalized path (learning line, learning trajectory)? Do all students learn the "end product" in the same way? Likely not. Rather than describing a particular learning objective or standard as a goal for learning, they use Vygotsky's (1978) "zone of proximal development" to describe particular goals for students' learning as regions around those goals that are individualistic and dependent on a variety of interacting factors. Such might include the scaffolding provided by the teacher, the language that the student has and the teacher uses, and the technology or manipulatives that may or may

not be available during the learning episode. Further, Lesh and Doerr, using Piaget's (1928, 1950) notion of *decalage*, describe how apparent learning of an objective may mask the partial development of an idea when "operational thinking" for one concept may occur years earlier or later than comparable levels of "operational thinking" for another closely related concept (Lesh & Sriraman, 2005). Lesh and Doerr emphasize that individuals learn in different ways and develop their understandings along different paths. They argue that intended "final products" (i.e., identified as standards or learning objectives) are likely to be in intermediate stages of development in most students, and open to revision and modification as they encounter new situations for which they need to form a mathematical interpretation.

Assumptions about Mathematics Learning

Lesh and Zawojewski (2007) refer to the work of various theorists and researchers (e.g., Lester & Charles, 2003; Lester & Kehle, 2003; Schoen & Charles, 2003; Silver, 1985; Stein, Boaler, & Silver, 2003) to establish a close relationship between the development of mathematical understandings and mathematical problem-solving. Their perspective on learning "treats problem solving as important to developing an understanding of any given mathematical concept or process [and]. . . the study of problem solving needs to happen in the context of learning mathematics . . ." (p. 765). In particular, Lesh and Harel (2003), and Lesh and Zawojewski's (1992) description of "local concept development" highlight the simultaneous increase in an understanding of a specific problem situation and the development of one's mathematization of the problem. "[S]tudents begin these type of learning/problem-solving experiences by developing [local] conceptual systems (i.e., models) for making sense of real-life situations where it

is necessary to create, revise or adapt a mathematical way of thinking” (Lesh & Zawojewski, 2007, p 783).

What is meant by local concept development and learning? Consider the Grant Elementary School Reading Certificate activity described in Figure 1, in which students are asked to create a set of “rules for awarding certificates” (i.e., a decision model). As described in Figure 1, the students generate a variety of models as an answer to this problem, and their answers provide windows to their mathematical thinking and learning—their local concept development.

<p><u>Grant Elementary School Reading Certificates Problem¹</u></p> <p>In this activity, third grade students are asked to create and apply a set of decision rules for awarding certificates to readers who read a lot and who read challenging books. The students are given sample sets of individual reader’s accomplishments, each presented in a table including the title of each book read, the number of pages for each book, and the difficulty level of the book (labeled as easy, medium, hard). The tension between the two criteria for earning a certificate (reading a lot of books and reading challenging material) was intentional, in order to enhance the potential for various reasonable models to be developed.</p> <p>Summary of Group #1 Response:</p> <ul style="list-style-type: none"> • Students should read either 10 books, or more than 1000 pages. • At least 2 of the books read should be <i>hard</i> books. <p>This group clearly communicates the decision rules (i.e., model) and takes into account both required conditions: reading many books, and reading challenging books. Readers can readily apply the rules to the given data sets. For example, in one data set, the reader had read 5 books (two of which were hard), and a total of 722 pages. Given the clarity of the decision rules, a reader can figure what he or she needs to do to earn a certificate. In this illustrative case, one way for the reader to earn a certificate is to read 5 more books (even if they are all easy). Another way is to pick one long book that has at least 279 pages.</p> <p>Summary of Group #2 Response:</p> <ul style="list-style-type: none"> • A student gets 1 point per page for easy-to-read books. • A student gets 2 points per page for hard-to-read books. • A student has to earn 1000 points to get a reading certificate. <p>This set of decision rules is clearly communicated, and a reader could easily apply the decision rules and self-assess. However, a reader could earn a reading certificate award by reading <i>only easy books</i>, not meeting the criteria that readers must read both hard and easy books. Therefore, the set of rules does not meet the requirements for a “good” set of rules.</p>

Figure 1. Two Illustrations of Local Concept Development

¹ This activity, in full, can be found in Yeatts, C. L., Battista, M. T., Mayberry, S., Thompson, D., & Zawojewski, J. S. (2004). Navigating through problem solving and reasoning in grade 3. Reston, VA: National Council of Teachers of Mathematics.

The mathematical goals of the activity are three-fold. First, each group of problem solvers is expected to generate a mathematical model, meaning they must develop a procedure or algorithm that meets the criteria given—that those earning a certificate must read a lot of books *and* read challenging books. In the generation of a model, many students engage in other types of mathematical knowledge development, such as quantifying qualitative information and differentially weighing and/or rank ordering factors. Each of the two responses described in Figure 1 represents different locally developed concepts, which are represented in the groups' model (i.e., a set of rules). Note that the first response meets the criteria, whereas the second does not. Note, also, how in each case, the model developed is situated in the context of the problem, and is also dependent on the knowledge that individuals bring to the group—about mathematics, about reading programs, about meaning of “challenging books” and meaning of “reading a lot.” A second goal is for students to practice basic skills, such as recognizing the need for and carrying out calculations, and comparing and ordering numbers. These take place as the students test their proposed models, and in the full activity, students are given further sets of data to conduct additional tests of the model they have generated. A third area of learning is generalization, which is driven by the design of the problem. In particular, a good response to this problem is one in which the model produced is reusable (reliably produces the same results for a given set of data), share-able (the decision rules are clearly and precisely communicated to all of the students, the teachers, and the parents, resulting in reliable application of the model across users), and modifiable (rationales and assumptions on which the model is built are articulated so others can make intelligent adjustments for new situations). Without assumptions or rationales

explained, intelligent modification of models can be quite difficult, if not impossible. Notice that neither of the two sample responses in Figure 1 meets the modifiability criteria for generalization, but they have addressed the re-usability and the share-ability criteria for generalization.

Over the years, Lesh and colleagues have reported on the local concepts developed by small groups of students as they engage in various problems, such as the one described in Figure 1. They indicate that individual students often pose initially primitive solutions, and as a result of social interactions, challenges, testing and revision, their initial solutions typically move toward a consensus model that is more stable. The learning of mathematics is described as an iterative process of expressing, testing and revising one's conceptual model. In particular, by using mathematical modeling as a way to think about mathematics learning, Lesh and Doerr (2003) describe a move away from behaviorist views on mathematics learning based on industrial age hardware metaphors in which the whole is viewed simply as a sum of the parts and involving simple causal relationships. Their perspective on mathematics learning also moves beyond software-based information processing metaphors, which involve layers of recursive interactions leading at times to emergent phenomena at higher levels that are not directly derived from the characteristics of lower levels. Instead, they align their models-and-modeling perspective on mathematics learning with a biology-based "wetware" metaphor, in which "neurochemical interactions . . . involv[e] logics that are 'fuzzy,' partially redundant, and partly inconsistent and unstable" (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008, p. 4). Assumed is that students arrive to school with dynamic mathematical conceptual systems already in place, that these conceptual systems are active and evolving before,

during and after problem solving and learning episodes, and that students must be motivated to engage in experiences by intellectual need (Harel, 2007) in order to learn. Thus, even when two students in a group may appear to have the same end product knowledge on one task, changing the task slightly, but keeping it mathematically isomorphic with the original, often reveals that the two students are thinking about the intended mathematical ideas in significantly different ways (Lesh, Behr, & Post, 1987; Lesh, Landau, & Hamilton, 1983).

What is the role of the small group in learning? Social aspects of acquiring knowledge from communities have been characterized in society over the decades (e.g., Mead, 1962, 1977, Thayer, 1982), and more recent work describes learning in communities of practice in various trades and occupations (Greeno, 2003; Boaler, 2000; Wenger & Snyder, 2000; Lave & Wenger, 1991; Wenger, 1998). These situations of social learning are characterized by the presence of a teacher, tutor, or mentor who models, teaches and collaborates with novices while engaged in the specific context of practice, rather than in a classroom. Other social aspects of learning have also been documented in situations where there is no teacher/tutor/mentor available. For example, researchers have documented successful collaborations among groups of diverse experts, where any needed leadership emerges flexibly from within the group in response to emerging challenges and opportunities (Cook & Yanow, 1993; Wenger, 2000; Wenger & Snyder, 2000; Yanow, 2000). Both perspectives on social aspects of learning are based on the assumption that all members of a group bring *some* understanding to the table, that the knowledge each brings is idiosyncratic, that the knowledge elicited by the problem is specific to the context, and that local concept development takes place among the group

members while simultaneously each individual in the group is adapting and modifying one's own understanding.

Social aspects of problem solving and learning are also related to the development of representational fluency, because interactions among collaborators require representations be used to communicate. When presenting initial solution ideas to peers, a problem solver typically describes one's own model using spoken words, written narratives, diagrams, graphs, dynamic action (e.g., gestures or using geometric software), tables, and other representations. The interpreting peer, who works to make sense of these representations, may request clarification, an additional explanation, or may point out inconsistencies, misrepresentations or other flaws. The peers, thus, iteratively negotiate a consensus meaning. Lesh and Zawojewski (2007) describe various social mechanisms that can elicit the use of representations, leading to the development of representational fluency, including: problem solvers making explanations to each other; groups or individuals keeping track of ideas they have tried; problem solvers making quick reference notes for new ideas to try as they continue in a current line of thinking; and problem solvers documenting their current line of reasoning when they must temporarily disrupt the work. These types of mechanisms, based largely on communication with others and oneself, provide the need to generate and use representations, and develop representational fluency.

Toward an Alternative Framework

Given the assumptions about learning grounded in problem solving, a number of challenges face the development of a framework for a problem-driven mathematics curriculum. How can a curriculum framework feature problem-solving activity as the

center of learning, while national and state standards documents highlight specific mathematical content as the central feature? How can a curriculum framework accommodate both the multi-topic nature of realistic mathematics problems and the pure mathematical nature of other mathematics investigations? How can a framework be represented to convey the complexities implied by the previous questions, yet be practical in meeting practical classroom needs? How can a framework be represented to inform the static nature of various standards documents, while also being responsive to changing societal needs and demands?

Envisioning a Curriculum Framework and It's Representation

What is Meant by a Problem-Driven Framework?

The development of problem-driven mathematics text series gained momentum in the U.S. in response to the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics*. In general, the NSF-funded texts (described in Hirsch, 2007a) are comprised of units of study organized around applied problems or mathematical themes. In many cases, these curricula use mathematical problems to launch and motivate learning sequences that progress toward development of understanding and proficiency for specified mathematical goals. For example, two of the design principles for developing the *Mathematics in Context* text series, which is based on the Dutch RME, are that the starting point of any instructional sequences “should involve situations that are experientially real to students” and “should . . . be justifiable by the potential end point of a learning sequence” (Web & Meyer, 2007, p. 82). The commitment to an experiential basis reflects the commitment to problem-solving as a means to learning, while the well-

defined mathematical end points correspond to a commitment to a curriculum framework organized around specific mathematical standards or learning objectives. In contrast, the problem-driven curriculum, *Mathematics: Modeling Our World*, described by Garfunkel (2007), is characterized by using mathematical models as end points. The dilemma for the *Mathematics: Modeling Our World* development team was coordinating the mathematics content naturally emerging from their model-based problem-driven curriculum with a standard mathematics topic driven curriculum framework. Garfunkel describes how the team grappled with the need to “cover” the scope and sequence of the required curriculum:

“[W]e believed (and still believe) that if we could not find, for a particular mathematics topic, a real problem to be modeled, that that topic would not be included in our curriculum. . . . Instead of ‘strands’ as they are usually defined we chose to organize curriculum around modeling themes such as Risk, Fairness, Optimization. We made an explicit decision . . . not to create a grid with boxes for mathematical and application topics. Instead, within the themes we chose areas and problems that we believed would carry a good deal of the secondary school curriculum. . . . For example, it was decided that one of the major mathematical themes of Course 1 was to be Linearity, so that each of the units in the course had to carry material leading to a deepening understanding not only of linear functions and equations, but also of the underlying concept of linearity.” (pp.161-162).

Garfunkel’s dilemma illuminates a fundamental mismatch between a curriculum framework that identifies a list of specific mathematical learning objectives or standards as outcomes, and the development of a curriculum framework driven by problem solving, and in particular, modeling. The “coverage” issue seems to force the enacted problem-driven curriculum to be a mix of problem-driven units accompanied by a collection of gap fillers to address missed content objectives. Thus, while *Mathematics: Modeling Our World* began the journey toward a problem-driven curriculum, it was challenged by the

coverage constraint, speaking to the question about what content *should* be included in a mathematics curriculum framework.

As a result, questions are raised about envisioning a problem-driven curriculum framework. Should a problem-driven curriculum framework have as final goals students' deep understanding of mathematical ideas that support certain types of problems, models or themes, or to demonstrate abilities about certain big mathematical ideas that were initiated in problem-solving settings? If the goal of a problem-driven curriculum is to cover certain mathematical models or themes, should the designers of a curriculum cover only those areas that naturally emerge in modeling or problem-solving work? If, on the other hand, the goal of a problem-driven curriculum framework is to accomplish certain big mathematical ideas, is the power of learning those ideas through problem solving to some extent defeated?

What is the Nature of Mathematics Content in a Problem-Driven Framework?

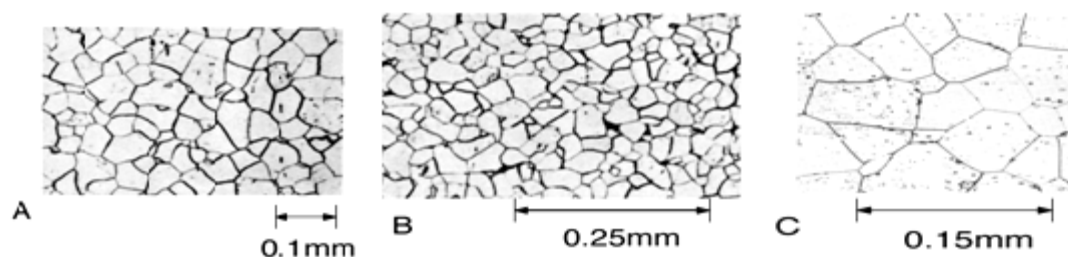
This larger question raises at least three issues about what mathematics to include in a problem-driven framework: What type of problems will the curriculum framework accommodate? What are the boundaries on the mathematics content to be covered? And how does the curricular framework adapt to evolving societal, scientific and technological needs concerning what mathematics is important?

A problem-driven curriculum framework would need to incorporate pure mathematical investigations, real-world applications, and modeling problems, among others. Whereas some problems nicely map onto a single mathematical big idea, others, especially applied and modeling problems emphasize multiple mathematical big ideas—

adding to the complexity of developing such a framework. Consider, for example, the Aluminum Crystal Size² MEA, included in Figure 2, as an illustration.

Aluminum Crystal Size Problem Description

The activity is situated in the context of the manufacture of softball bats that would resist denting, but also won't break. In materials science, one learns that the larger the typical size of crystals in a metal, the more prone to bending, and the smaller the typical size the more brittle the metal. A problem was posed that had two purposes. The first was to motivate the problem solver to quantify crystal size. The second was to establish a context where a client needs a procedure to quantifying crystal size as part of their quality control. The client in the problem "hires" the problem solver to create a way to measure, or quantify, aluminum crystal size using two-dimensional images, such as the ones here:



The images are given in different scales, making visual comparison of crystals in the three samples difficult. Therefore, the mathematical procedure would need to take scale into account.

A number of different approaches typically emerge, including:

- Draw a rectangular region to designate a sample within each image. Calculate the area of the rectangle in which the crystals are enclosed. Count the number of crystals in a rectangle drawn. Compute the average area per crystal. Compare samples.
- Select a sample of crystals within each image and estimate the area of each crystal (e.g., by measuring the distance across the widest part of a crystal, and the length of the distance perpendicular to that widest part, and then finding the product of those lengths). Compute the average area per crystal. Compare samples.

Figure 2. Aluminum Crystal Size Problem

In the Aluminum Crystal Size Problem, multiple big ideas in mathematics are relevant to producing a good solution. Spatial reasoning is important as the problem solver needs to figure out ways to quantify regions that are not consistently shaped nor consistently sized, yet must be considered collectively as a “class” tending toward a

² For the full activity, see: Diefes-Dux, H. A., Bowman, K. J., Zawojewski, J. & Hjalmarson, M. (2006). Quantifying aluminum crystal size part 1: The model-eliciting activity. *Journal of STEM Education and Research* (7) 1&2, 51-63.
Hjalmarson, M., Diefes-Dux, H. A., Bowman, K. J. & Zawojewski, J. S. (2006). Quantifying aluminum crystal size part 2: The Model-Development Sequence. *Journal of STEM Education and Research* h (7) 1&2, 64-73.

certain size. The problem solver must also consider what parts of the regions to use in the quantification. Measurement is another big idea addressed, since a definition of crystal *size* needs to be generated and mathematized. Proportional reasoning is needed because the micrographs are all shown to different scales, which needs to be accounted for in the development of the mathematical model for crystal size. Sampling is important when deciding what regions of the micrograph to use to determine the size of the crystals in the full image; a good solution will incorporate a method for selecting samples to include in the mathematical model. Measures of centrality are likely to emerge because quantifiable characteristics of the various crystals need to be summarized in some way to come up with a single measurement of crystal size. Finally, mathematical modeling is the centerpiece of the activity. If the Aluminum Crystal Size activity is used as the centerpiece of a unit of study, the problem context drives what mathematical topics are encountered. A framework, then, is needed to help make decisions about which topics to investigate more deeply, whether to stay within the problem context in those investigations, and whether or when to incorporate other more conventional lessons or purely mathematical investigations on the conventional topics.

The second consideration concerns the boundaries of mathematics curricular topics. For example, an economics problem may require designing a mathematical model that optimizes costs while producing the highest quality possible. An engineering problem may have ethical ramifications, where the “best” possible mathematical solution to attain cost-effectiveness may not meet equity considerations. A problem may lend itself to an elegant mathematical solution that uses cutting-edge technology, but the solution may not work with the *commonly* available technology. In the real world, when

clients want quantitative-based solutions that are cost-effective yet most powerful, thoroughly but quickly produced, and usable by a wide audience yet secure from abusers, the mathematical and non-mathematical considerations are inseparable. A collaboration of engineering educators have grappled with such an issue in the context of engineering education, where the goal has been to teach foundational engineering principles through mathematical modeling problems that carry competing constraints when considering ethical components (e.g., Yildirim, Shuman, Besterfield-Sacre, 2010).

The content of mathematics curriculum needs to be an entity that can evolve, and can be flexible and nimble as problems faced in the workplace and society evolve—the third consideration. To illustrate, two hundred years ago the computational algorithms needed for bookkeeper’s math were appropriately the main focus of school mathematics content. Now-a-days, research on current professional use of mathematics in fields such as engineering (e.g., Ginsburg, 2003, 2006), health sciences (Hoyles, Noss, & Pozi, 2001; Noss, Holyes & Pozi, 2002) and finance (Noss & Hoyles, 1996) reports an increasing need for students to develop or adapt mathematical models to solve novel problems and to flexibly interpret and generate representations. Zawojewski, Hjalmarson, Bowman, & Lesh (2008) indicate that “the real world uses of mathematics are described [in the studies referenced above] as often requiring that mathematical knowledge be created or reconstituted for the local [problem] situation and that content knowledge be integrated across various mathematics topics and across disciplines” (p. 3).

A Proposal for an Alternative Problem-Driven Curriculum Framework

Major dilemmas of constructing an over-arching curriculum framework were illuminated using the two problem driven curriculum frameworks described above. But, even when considered together, the RME and *Mathematics: Modeling Our World* do not necessarily accommodate all aspects of important mathematics to be learned. In particular, the RME framework is driven by problem-solving launches followed by a sequence of activities and instruction that lead to an increased understanding of formal mathematical systems. Garfunkel's *Mathematics: Modeling Our World* is organized around themes such as risk, fairness, optimization and linearity, each representing important areas of mathematics associated with formal mathematical modeling. Both generally aim toward formal mathematical goals, but do not have as end goals mathematics deeply embedded within broad contextual situations and areas such as ethics or equity. The alternative proposed here is based on a notion of model-development sequences that broadens the one described by Lesh, Cramer, Doerr, Post, & Zawojewski (2003). Like RME and Garfunkel's curricula's development, the underlying assumption is that powerful learning of mathematics emerges from students' mathematization of problematic situations. Going beyond RME and Garfunkel, a problem-driven mathematics curriculum framework built around model-development sequences has the potential to incorporate both formal mathematical big ideas/models and real world messy models that are intertwined with non-mathematical constraints.

Lesh, Cramer et al. (2003) describe model-development sequences as beginning with model-eliciting activities (MEAs), which are instantiated in the two problems presented so far (Figures 1 & 2). The main characteristic of a MEA is that the problem

requires students to create a mathematical model in response to the task posed, which could be extended to the production of smaller parts of formal mathematical systems. MEAs have traditionally been designed using six specific design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000) to devise “authentic” contexts, involving a client with a specified need for a mathematical model that facilitates making a decision, making a prediction, or explaining a reoccurring type of event in a system. Following the initial MEA are planned model-exploration activities (MXAs), which vary from comparing and contrasting trial models posed by peers in a class, to more conventional meaning-based instruction on various mathematical aspects of the model. For example, the Aluminum Crystal Size problem may be followed up with a lesson on the role and power of random sampling for making inferences, or an opportunity for students to compare and contrast their procedures for determining typical crystal size in micrograph samples. Similarly, one of the authors interviewed a teacher who enacted an MXA activity with her third grade students who had completed the Grant Avenue Reading Certificate Problem (Figure 1). After the teacher asked the third grade students to present their rules to each other, she asked students to identify similarities and differences among the sets of rules, and probed students perceptions of the pros and cons. By asking questions about what aspects of the situation each set of rules attended to and ignored, and how the choice of variables influenced the impacts the outcomes, she was teaching foundational ideas of modeling. For example, Group 2’s response (Figure 1) ignores the number of books in the data—depending only on the number of pages to represent “reading a lot.” Group 1 (Figure1), on the other hand, used all three types of data (number of pages, number of books and the rating of easy/medium/hard). Even though Group 2’s response did not fully

meet the criteria articulated in the problem, the use of page numbers only, and not the number of books, is defensible as an indicator of amount of reading. Helping the students articulate rationales for their decisions supports the development of an initial understanding that models are systems that represent larger systems, and inevitably capture some features of the original system, while ignoring other aspects.

A model-development sequence closes with a model-adaptation or model-application activity (MAA). To illustrate the power of a MAA, consider the full sequence of activities that has been used in the first-year engineering course (with students fresh out of high school) at Purdue University³. The opening Nano Roughness⁴ MEA, (see Figure 3) is “set in the context of manufacturing hip-joint replacements where the roughness of the surface determines how well the joint replacement moves and wears within the hip socket” (Hjalmarson, et al., p. 41). Given digital images of the molecular surface of different samples of metal, students were asked to create a procedure for quantifying the roughness of each sample, which resulted in a variety of models. The subsequent MXA introduced students to a conventional engineering model for quantifying roughness, the *average maximum profile (AMP)* method, and then asked them to compare their model for quantifying roughness to the conventional engineering model. The goal for this MXA was to enable students to identify and understand trade-

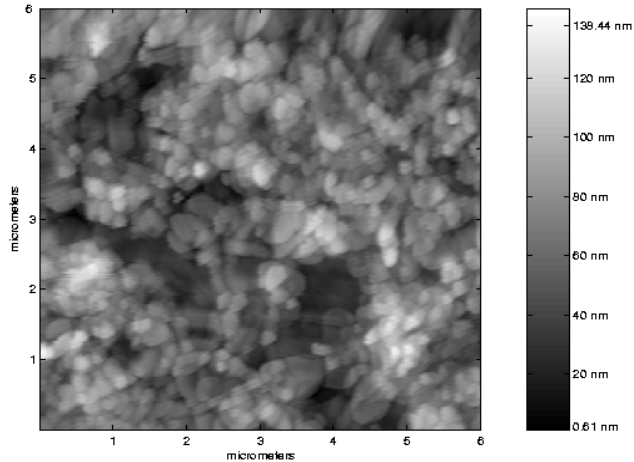
³ Purdue’s first-year engineering program has been using MEAs and model-development sequences for the past 9 years with approximately 1500 student per year in West Lafayette, Indiana, USA (Hjalmarson, Diefes-Dux & Moore, 2008).

⁴ The full activity can be found in J. S. Zawojewski, H. A. Diefes-Dux, & K. J. Bowman (Eds.), (2008). *Models and modeling in engineering education: Designing experiences for all students* (pp. 317-322). Sense: Rotterdam, The Netherlands. The lead author of the activity is Tamara J. Moore.

offs between models, and to identify and understand rationales and assumptions underlying different models.

Nano Roughness MEA

This goal of this activity is to produce a procedure to quantify roughness of metal surfaces at a microscopic level. Students are given atomic force microscope (AFM) images, similar to the one below, of three different samples of metal surfaces. At the atomic level, the lighter parts of the image represent higher surface, and the darker parts of the image represent lower surface. The gray scale indicator, to the side, provides information about the height of the surface. To motivate the problem situation, the students learn that the company, who is their client, specializes in biomedical applications of nanotechnology. They are planning to produce synthetic diamond coatings for use in orthopedic and biomedical implants, and need to have a way to quantify roughness of the coating surfaces. Given three top-view images of gold samples (illustrated in the one sample below), the modelers are asked to develop a procedure for quantifying the roughness of the material so the procedure could be applied to measure roughness in other types of metal samples.



Sample of an AFM image of gold surface (AFM data courtesy of Purdue University Nanoscale Physics Lab)

Figure 3. Nano Roughness MEA Description

The model-development sequence closes with a Model-Adaptation Activity (MAA) that requires students to adapt either *their* model for measuring roughness, or the conventional model, to a new situation. To do the work in the Purdue example, students were given a raw data set that had been used to produce a sample digital image. These raw data had been gathered by using an atomic force microscope (AFM), which uses a nano-scale probe dragged along the surface of the metal sample in lines at regular

intervals, measuring the relative heights along the bumps of the molecules. The students were asked to generate, using MATLAB, a cross sectional view of any line segment drawn on an image of the gold surface. In particular, they produced graph-like products that portrayed the relative heights of the bumps and valleys for any line segment drawn on an image. The mathematical learning goals for this MAA were to conduct 2-dimensional array manipulations of the data and to incorporate statistical reliability considerations into the process. Broader learning goals for the Nano Roughness problem include programming and fundamental engineering principles—illustrating how mathematics learned may be embedded and intertwined to what traditionally has been considered non-mathematical topics.

While MEAs, and their accompanying model-development sequences have traditionally been tied to authentic realistic modeling contexts, the basic concept of eliciting a mathematical model can be broadened to incorporate the more traditional modeling work, such as described by Garfunkel (2007). The model-development sequence framework can also be envisioned to include the elicitation of aspects of formal mathematical systems, such as what is the aim in RME. In other words, model-development sequences have a great deal of potential to serve as an umbrella framework that provides a way to unify problem-driven curricula frameworks, especially when considering the flexibility of Learning Progress Maps (LPMs), which is proposed as a possible way to represent problem-driven curriculum frameworks.

Envisioning a Representational System for Problem-Driven Curriculum

Using a metaphor of topographical maps, Learning Progress Maps (LPMs) can be thought of as a dynamic representation of mathematics curriculum *and* students' learning (Lesh, Lamon, et al., 1992; Lesh, unpublished manuscript). Lesh's goal in developing this concept has been to help teachers readily answer practical classroom questions such as: What concepts do my students still need to address in this unit I am teaching? Which topics would be strategic to address next? What are concepts or topic areas where my students appear to require more experience? Which students are having difficulty with specific concepts, and which have demonstrated learning in those areas? Single score results from large-scale measures do not provide useful information for these questions, whereas item-by-item information for every student might be overwhelming to use as an everyday tool to make decisions about classroom instruction. Portfolio assessment is difficult to define and standardize, let alone use for day-to-day classroom decisions about instruction. On the other hand, good teachers *do* develop their own personal methods to keep track of individual students' progress in a variety of ways, although their systems are idiosyncratic to the teacher, often very detailed, and usually perceived by others as too time consuming to maintain.

How Might a Learning Progress Map (LPM) Represent a Problem-driven Curriculum?

Consider a hypothetical topographical map representing a curriculum organized around mathematical big ideas, important mathematical models, or formal mathematical systems, presented in Figure 4. Lesh, Lamon, et al. (1992) describe the mountains of the landscape as corresponding to the "big mathematical ideas" of a given course (6 to 10 big ideas in this case), and the surrounding terrain of foothills and valleys as depicting facts and skills related to the big ideas. Using the topographical maps metaphor, one can think

about the height of the mountain as representing the relative importance of big mathematical ideas in the course, while relationships among the big ideas can be expressed by the proximity of the mountains to each other. The tops of the mountains would represent deep understanding of the big idea, abilities supporting the big ideas can be represented on the sides of the mountains, and associated tool skills (e.g., manipulations, skills, facts) can be represented by the regions of the surrounding foothills and valleys.

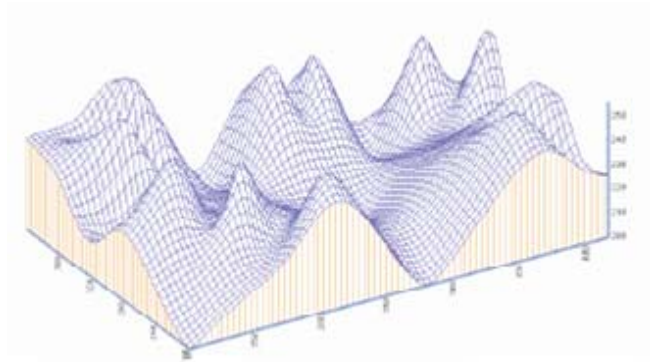


Figure 4. Representation of Big Ideas, Supporting Abilities, and Tools in a Course

A top-down view of the topographical curriculum map (illustrated in Figure 5), might delineate the interplay of the big mathematical ideas, supporting abilities and tool skills to be “covered” in the given course, in a way analogous to a traditional scope-and-sequence document.



Figure 5. Top Down View of Curriculum Scope and Sequence

On a LPM, problem-solving, modeling, deep insights into a designated mathematical big idea, and higher-order mathematical thinking about the idea would be designated in regions on the tops of the mountains. Thus, problems that involve multiple big mathematical ideas, such as the Nano Roughness MEA, could be represented by multiple mountains (e.g., 3-d geometry, proportions, sampling, measurement, mathematical models). The height of the mountains, and the arrangement of the regions around them, would represent the relative importance of the major mathematical areas with respect to the MEA. Supporting concepts and procedures would correspond to the sides of the mountain, and needed skills and facts would correspond to the adjoining valleys around each mountain. For example, in the Nano Roughness MEA, the fluent interpretation and manipulation of the scales would be an important component of proportional reasoning, and thus represented on the sides of a proportional reasoning mountain. The valleys nearby each mountain would represent the automatic skills and concepts that might be thought of as the tools of the trade for that big idea, such as masterful and precise computation or algebraic manipulation. Another illustration might be the linearity theme of *Mathematics: Modeling out World*, as described by Garfunkel (2007). Linearity might be the name of the mountain, and the idea of representing linear expressions in various forms (as narratives of situations, as tables, as graphs) may each correspond to regions along the side of that mountain, and fluent manipulation of linear equations might be represented in the valley nearby the mountain.

The potential flexibility of the proposed representational system is greatly enabled by the power of technology. For example, given that the concept of linearity is a major theme in Course 1 of *Mathematics: Modeling out World*, linearity may be important to a

number of units, and emerge in a variety of contexts. In the mountain representational system for each unit, the theme of linearity may be represented as an overlay of a particular colors or textures (e.g., striping, dotting) on all terrains. Further, theoretical perspectives on learning may also be represented using different intensities of colors to illustrate the three dimensions in RME, or an activity's classification in modeling sequences i.e., MEAs, MXAs and MAAs. The envisioned representation of a curriculum framework could provide teachers with the opportunity to manipulate the map, providing varied views of the curriculum. For example, a teacher may want to see how linearity emerges across chapters within a course by viewing any and all mountains that represent linearity across chapters. While one can imagine many useful scenarios of manipulation, the greatest potential for LPMs, however, is probably representing students' progress through the curriculum.

How Might a Learning Progress Map (LPM) Serve Assessment?

In a problem-driven curriculum framework, assessment of big ideas and models would be supported by LPMs which are envisioned as providing manipulable representations of students' attained curriculum. Specific assessment data can be used to "fill in" appropriate regions of a LPM for a particular student in a course. Since in a problem-driven curriculum, the students' mathematical experiences begin in problem-solving environments, and supporting skills may be learned or mastered later and at various levels, record-keeping is potentially very challenging. Planning assessment points to correspond with particular regions of the map would be a strategy for input points that would in turn help keep track of accomplishments by individuals, while also potentially providing a visual picture that organizes the assessment data for the individual students.

Assessment data points can be drawn from students' responses to problem-solving or modeling activities and used to guide subsequent instructional activity. To illustrate, consider the work of Diefes-Dux and colleagues, who have been very active in documenting students' modeling performance on iterations of revised solutions to MEAs. They have developed systematic ways to evaluate the development of mathematical models that students generate (e.g., Carnes, Diefes-Dux, & Cardella, 2011; Diefes-Dux, Zawojewski, & Hjalmarson, 2010). Their assessment rubric (Diefes-Dux, Zawojewski, Hjalmarson, & Cardella, 2012) that addresses four general characteristics of the models is made into task-specific versions for each MEA. Their recent work has focused on the challenge of identifying and implementing feedback to students with a goal of prompting students to rethink and revise their solution model to be more powerful and efficient (personal conversation with Diefes-Dux, January 17, 2012). One can imagine that this line of research would be enhanced with the proposed framework and representational system. For example, Diefes-Dux and colleagues' evaluate the generalizability of students' models based on three criteria. Assessment of a model's "re-usability" documents the stability of the model over its independent applications; that is, whenever the model is re-applied to a given data set the model will produce the same results each time. Assessment of model's "share-ability" documents whether the model is communicated well enough so that other users can apply the model independently and reliably. Finally, the assessment of the model's "adaptability" focuses on the articulation of critical rationales and assumptions on which the model is constructed, so that an external user would be able to intelligently modify the model for new, somewhat different, circumstances. These three dimensions could be easily represented and

manipulated in the envisioned framework to look for patterns and trends in students' series of revised models.

In a problem-driven curriculum framework, assessment of students' performance on concepts, skills, and procedures that support big ideas and models can be facilitated by LPMs and guided by available mathematics education research. For example, in the Ongoing Assessment Project (OGAP), Petit and colleagues (e.g., Petit, Laird, & Marsden, 2010) examined all available mathematics education research in selected domains, identified important benchmarks and "trouble spots" of understanding, and targeted those specific concepts and skills for the development of assessment items and activities. They have completed the work on fractions, multiplication and proportions. Such assessment items can be used as data points in the side regions and valleys of mountains corresponding to the big mathematical ideas. Further, in conjunction with the growing body of research on learning progressions (e.g., Clements, 2004), assessment points that have been embedded in the learning trajectories can become benchmarks that are carefully placed to track general progress as students eventually abstract from their variety of situations to generalized mathematical ideas.

How Might a Learning Progress Map (LPM) be Used to Inform Practice and Programs?

The envisioned dynamic LPMs would provide a means for teachers to quickly and easily identify information relevant to day-to-day questions for teaching and students' learning. For example: *What concepts do my students still need to address? Which topics would be strategic to address next? Which students are (or are not) having difficulty with specific concepts?* Using a keystroke, summarized students' assessment data could be displayed on the LPM, providing opportunities for nimble decision-making about

classroom practice. By illuminating the whole class's attainment on the LPM curriculum, teachers would be able to see what yet needs to be addressed in the course, and what may need some reteaching. Profiles of individual students' attainment could help teachers plan to group students for differentiated instructional experiences. LPMs could, for example, help teachers to form problem-solving groups by identifying students with a variety of expertise relevant to the problem. Individual profiles, when displayed side-by-side, could also inform teachers' decisions about students access to limited resources (e.g., volunteer tutors, particular technological assistance, advanced placement coursework).

Self-assessment could become a major component of classroom experience. Students could use their own individual profiles to self-assess their own progress, and perhaps even select problems through which they can address their own areas of need. In an advanced version of LPM, where the curriculum topics are linked to appropriate problems, perhaps students could select a context they like to think about (e.g., sports, health care), and be assigned an appropriate problem from the targeted area of need. By integrating an assessment system with the curricular map, LPMs could be used as a tool to guide students' selection of problems that have the potential to move them forward mathematically.

Professional development and program evaluation can also be enhanced through LPMs. Lesh, Lamon, et al., (1992) describe a variety of program level assessments that could be accomplished by dynamic LPMs. For example, a summary class attainment map that looks like the one in Figure 5, suggests instruction that is highly skill-based, and thus provides an opportunity to for a teacher to confront one's own (perhaps unconscious) assumption that problem solving and deep conceptual understanding can only be

addressed after all of the “basics skills” have been accomplished. On the other hand, a summary class attainment map that looks like the one illustrated in Figure 6 might suggest that a teacher is effectively implementing a problem-driven curriculum, given that the attained map illustrates splashes and spreads from multiple points near the tops of the mountains, and oozing downward to the sides of mountains and surrounding valleys.

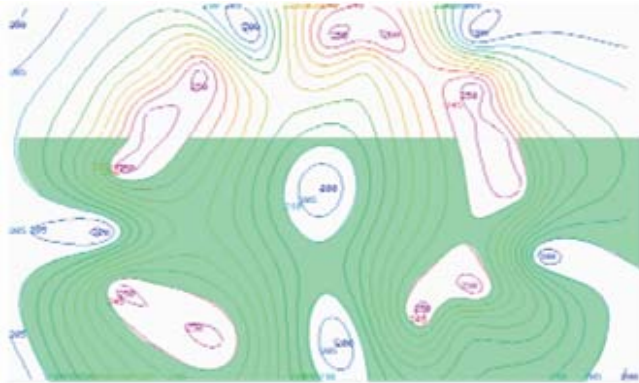


Fig 5: LPM (green) in Skill-based Attainment by Students



Fig 6: A LPM (green) in Multi-level Attainment by Students

Reflections

The envisioned problem-driven mathematics curriculum framework supported by a dynamic representational system, LPM, seems feasible. Given the potential of today’s technologies, design research (Kelly, Lesh, Baek, 2008) methodologies could be used to

simultaneously build, study and revise theoretical, pedagogical, and practical considerations of a problem-driven curriculum framework and its representational system. The LPM could be manipulated and revised quickly and easily in response to various changing conditions, such as changes in what constitutes important mathematics, changes in important problem context, changes in new content-driven state standards, and changes in interdisciplinary and social considerations. While the representational system has yet to be actualized, many aspects of problem-driven curricular frameworks are already under research and development. Imagining future work that links technology-driven LPMs and problem-driven curriculum frameworks brings a variety research questions and potential issues for investigation.

Given that problem-driven mathematics curriculum frameworks are grounded in the assumption that students learn mathematics while engaged in complex problem-solving activity, a question arises about how LPMs could be used to represent such curricula. What would a LPM look like for a course, or a unit of study? What will be identified as the “big ideas” or mountains around which the mathematical terrain is developed? What variables need to be represented in the LPM, beyond content topics? What needs to be fixed and what needs to be flexible in the software? These are only a few of the questions that need to be answered in interdisciplinary teams of mathematics educators, curriculum developers, assessment experts, and software developers in a design process.

How can LPMs be used to identify when, and the extent to which, problem-based instruction supports the given problem-based curriculum? Collaborative research and development would be needed to design software to display an image, such as the one in

Figure 5 that represents successful implementation of problem-based instruction. The design of the software would require the identification of variables and development of models to show the splashes and spreads from multiple points near the tops of the mountains, oozing downward, and eventually filling in the valleys. The needed data include the curriculum specifications, student assessment data, and teacher input about experiences implemented. The goal would be to provide real-time information to teachers and their support personnel concerning what students are learning, and to use that information to adjust instructional strategies to align with those appropriate for problem-based learning.

How might LPMs assist classroom teachers in their enactment of a problem-driven curriculum, yet help to keep an eye on “content coverage” as potentially required by other stake holders? To support implementation of problem-driven curricula in environments that are driven by standards and emphasize content coverage, teachers’ need to have tools that help them traverse the challenges of real world implementation. The envisioned LPMs must have embedded in them the ability to manipulate the representations so that teachers can easily check on “content coverage” while teaching a problem-based curriculum. Further, they need to be able to easily check on individual student progress in order to plan for reasonable differentiation. Challenges in implementing a problem-based curriculum must be addressed by well-designed LPMs that are easily used by teachers to inform *their* questions and issues.

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