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Abstract: Problems are frequently used in mathematics to introduce and convey new notions and skills. Hence, teachers transform and adjust those problems to their students' level. The present study focuses on this transformation process on the particular case of a geometric problem posed by two teacher educators in one French Institute for Teacher Training. The whole process is described as a trajectory of the problem through various institutions from training center to secondary school and back. Before presenting the notion of trajectory of the problem, some elements about a general theoretical frame which refers to didactics of mathematics are presented.

Keywords: Geometry, open problem, problem situation, problem solving, teacher training, technologies.

Introduction

The idea of grounding the teaching of mathematics on making students solve problems is not new, especially in primary education. From the 1970's on it has been very popular in many countries, undoubtedly as a reaction to the abstract teaching given during the so-called 'modern math' period. This pedagogical trend was variously structured according to the country, and the use of problems for learning maths depends to some extent on both cultural traditions and theoretical frames underlying teaching which are specific of each country. We became aware of these differences on the occasion of a joint research undertaken by a French team (from the LDAR, Paris-Diderot University) and a Mexican team (from Cinvestav, Mexico-city). The scope of this study, presented at the Cerme 7 Conference (Rzeszów, 2011) by Kuzniak, Parzysz, Santos and Vivier (2011), was the question of the initial training of teachers to the use of technologies for the teaching of maths. On the Mexican side, the implementation was based on the problem solving methodology, whereas on the French side the stress was put on the notion of open problem, in connection with Brousseau's Theory of Didactical Situations (TDS).

In this article we shall present in detail our approach for this research, within a training course for prospective mathematics secondary school teachers, with reference to some of the theoretical frames used by our team, and especially the notions of open problem and instrumental approach (sec. 1.1 and 1.2). Besides, the training course situation here studied belongs to what can be described as a training homology strategy (sec. 1.3). The problem at work is used to develop among pre-service teachers, not only their mathematical knowledge, but also their didactical knowledge.

After having exposed an a priori analysis of the problem (sec. 2), we describe in section 3 the work required from the students-teachers which is split into three steps. Then, we expose and analyse the various transformations of the problem chosen for the training.

Finally, in discussion section (4), we define a framework (sec. 4.1) intended to describe and analyse what we call the trajectory of the problem, that is its global evolution, from its use in the training course to its setting up in a regular classroom. We conclude the section (§4.2 *sq.*) with remarks on some important points related to teacher training.

1. Context and stake of the study

1.1 Problem solving in French context

As Artigue and Houdement (2007) underscore it, there does not exist a tradition of education research on problem solving in French didactic research even if Polya and Schoenfeld works are well known. This characteristic partly results form the influence of the Theory of Didactical Situations (TDS in the following) initiated by Brousseau (see Brousseau, 1997, for reference texts in English) and from the pedagogical approach developed by the IREM (Institut de Recherche sur l'Enseignement des Mathématiques). Both introduced two kinds of perspective on problem solving: problem situation and open problem.

The notion of "problem situation" appeared in France in the 1980's in Brousseau's TDS, which is based on a socio-constructivist conception of learning. A problem situation is a learning approach aiming at fostering the acquisition of a new knowledge by the students. Its setting up implies identifying previously their conceptions by analysing their errors. On this basis the teacher conceives of and sets up a situation presenting some specific features, namely:

- be relevant for the cognitive objective aimed at;
- have a meaning for the student;
- allow him/her to begin the search for a solution;
- be rich (in terms of mathematical and heuristic contents);
- be possibly formulated within several conceptual "settings" (Douady, 1986) or "semiotic registers" (Duval, 2006).

The notion of "open problem" was introduced at about the same time (Arsac *et al.*1988, Arsac & Mante, 2007). In comparison with the problem situation, the aim of an open problem is methodological rather more than cognitive. The students are induced to implement processes of a scientific type, *i.e.* experimenting, formulate conjectures, test them and validate them. The problem must belong to a conceptual domain in which students are somewhat familiar with, the wording (statement) has to be short and induce neither a solution nor a solving method. Here is an example taken from APMEP (1987): *What is the biggest product of two numbers which can be obtained by using once each of the digits 1, 2, 3,…,9 to write these numbers?*

In fact, open problem and problem situation refer to two complementary sides of a mathematician's work:

- in the case of an open problem the question is to find a genuine and personal solution, with one's own means, the general solution can be out of reach of the students (and possibly the teacher);
- in the case of a problem situation the question is, starting from a specific problem, to elaborate a more general knowledge (concept, process…) which is intended to be institutionalised, socially acknowledged and mastered by all the students.

The French official curricula for junior high school integrated recently − though without naming them − these two practices:

> *If solving problems allows the emergence of new elements of knowledge, it is also a privileged means to broaden its meaning and to foster its mastery. For that, more open situations, in which the students must autonomously appeal to their knowledge, play an important role. Their treatments require initiative and imagination and can be achieved by making use of different strategies, which must be made explicit and compared, without necessarily privileging one of them*. (BOEN 2008, page 10, our translation.)

The notion of research narrative (*narration de recherche*), which is explicitly linked with those of open problem and problem situation, appeared in France some twenty years ago, first at junior high school level, before being extended to senior high and primary school (Bonafé et *al*. 2002). It involves asking the student to write an account of the thought processes he/she has undertaken in order to solve a given problem, pointing out his/her ideas, successes, failures, etc. The features of the problem are the same as for an open problem, but its statement has often several questions and is such that the student must be able to start a research, test his/her results and validate them. And, if possible, different solutions can be considered.

1.2 Integration and influence of technologies

Pre-service teachers in maths are accustomed to solving mathematical problems with specific software, mainly of the symbolic calculation or dynamic geometry types but this does not mean that they are prepared to use them as future teachers. Research studies into teaching in technological contexts (see Laborde, 2001) show that the students (preservice teachers) do not have or have little knowledge of the teaching of mathematics, that is to say, they are unaware of the development of mathematical notions in teaching situations and they have difficulties in the use of software in a learning situation. This makes it necessary to integrate specific work in the form of understanding teaching using software into teacher training.

Specific studies on teacher training within a technological context (see Chacon and Kuzniak, 2011) are few. And they show the need to go more deeply into processes regarding proof and the structuring of different spaces of knowledge (teaching, mathematical, instrumental) which a teacher must structure when using dynamic software for geometric learning. Moreover future teachers have to be aware of secondary school students difficulties related to instrumental knowledge.

1.3 Teacher training

Till the end of 2010, IUFMs, Instituts Universitaires de Formation des Maîtres (French University Training Colleges), have been in charge of the formation of preservice teachers. The IUFMs were accepting, after a first selection, maths graduate students from any University (three years of study). During one year, students were preparing a competitive examination with academic maths knowledge. The successful candidates received a theoretical and practical education of one year (the "second year") in the Institute and were in charge of a class for six hours a week; they received a salary. Nowadays, students need to have a master and pass competitive examination to become teachers. Preservice secondary teachers could follow a master in teacher education (two years) at University, they are not in charge of a class and are not paid during the second year. Our experimentation was made in 2010 before the new system.

As it is well known, preservice teachers need a set of knowledge on maths and teaching, usually described with the notion of Pedagogical Content Knowledge (PCK) introduced by Shulman (1986) to complement subject content knowledge, and based on this idea, various refinements have been made to describe knowledge that is really needed to teach mathematics known as Mathematical Knowledge for Teaching (MKT). Teaching mathematics is obviously connected to Mathematical Content Knowledge but also to other ones that are not automatically owned by a specialist of mathematics and that are more or less close to mathematics like history, epistemology, didactics, psychology or pedagogy. This large set of knowledge is classified in two parts. The first one, that is made explicit and structured clearly within the frame of didactical theories, constitutes Didactical Content Knowledge. The second one, that is not explicitly written and theorised, but exists in the professional action of each teacher is what is called "third knowledge" (Houdement & Kuzniak, 2001). Within this framework, the question is how to introduce and combine the various types of knowledge. And how to give to students who are specialists of mathematics at university level, a level in school mathematics which are often far away from the first one.

The combination between various types of knowledge can take different forms: they can be suggested to or developed by the students; they can be juxtaposed or connected; the connection can be explained or not…So we have distinguished various strategies which differ concerning the explanation of knowledge, the combination between them, the position they give to the students. Strategies also depend on the knowledge considered as dominant and on the transposition made by the teacher trainer.

During our experimentation, we followed a strategy firstly based on homology and then on transposition. That means that we first use the lack of knowledge of content and teaching for the classroom of the preservice teachers as a pretext to build a learning situation close to a conception of teaching favoured by French curriculum. The preservice teachers, or student teachers, are considered similarly as maths students searching a problem and supposed to analyse the teaching session to pinpoint elements of didactical knowledge and the "third" knowledge. The strategies based on transposition favour didactical knowledge. Then, we tried to know more about the phenomena of transposition of knowledge that might be a bias in every teaching situation (Chevallard, 1985). Student teachers are considered as teachers examining their own teaching way. We detail this with the notion of problem trajectory for the training.

2. Presentation of the problem the *folded square* and a priori analyses

The problem we discussed in this paper is the core of a pre-service teachers' training course that conveys didactical knowledge about problem use in the class. For this reason, this problem was asked to fulfil several conditions:

- To be an « open » problem easily integrated in the teachers' training process.

- To allow the link between several semiotic registers (Duval, 2006) and the use of various mathematical settings (Douady, 1986) related to French curriculum.
- To be solved in different technological contexts especially those using dynamic software.
- To be open to a number of exploitation and transformation in class with pupils and training session with future teachers. This point relates to our idea of problems trajectory.

To these various constraints linked to a training context, we added one more related to the context of a comparative study. For that, we chose a problem or a kind of problems already given by other researchers using other theoretical approaches.

The problem posed to the students belongs to a kind of problems named "shop-sign problems" as used in Artigue, Cazes and Vandebrouck (2011). In such problems, with geometric support, two areas representing a shop-sign are determined by a point situated within a square or a circle or a rectangle... Both areas change in function of the position of the point in the square. These problems are introduced in a geometric setting but to solve them, a change to algebra or calculus settings is generally required. Changes of semiotic registers with algebraic or functional notations are also needed to get a solution. The functions used are quadratic polynomial functions which allow a mathematical treatment in synchronization with the secondary school curriculum.

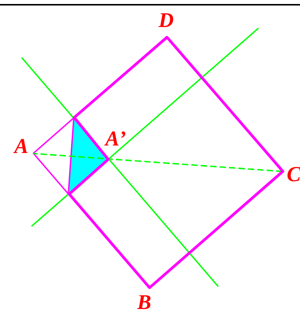
By using dynamical geometric software as *Geogebra*, it is also possible to solve such problems in a graphical setting by focusing on the covariation of areas without the use of a functional or algebraic writing. It is indeed possible of drawing a graphical representation of the phenomena studied without any algebraic writing of the function:

the curve is defined as a locus of points. The number of solutions that students can find and understand is increased by the use of technological tools introducing an experimental perspective in the implemented working space.

The problem was presented in a real context with material and not with a writing in mathematical form: A square, cut in a bi-color sheet, is given to the students. And they have to fold it along a diagonal and compare the areas of both visible parts of different colours. Students are entirely in charge of the problem representation according to the first step of the modelling circle in Blum and Leiss (2005) view. By doing that, we do not favour any mathematical approaches and frames but to control the task effectively made by student teachers and reach our training objectives on the use of technologies for teaching, student teachers have been encouraged to use some software as it will be detailed in sec 3.1. on problem trajectories.

The problem is not original and was used in French and Mexican contexts (Kuzniak et al., 2011) with the following form, Mexican Task, which will give the reader an easier access to the mathematical stake of the problem.

Mexican Task. A square piece of paper *ABCD*, the side of which is *l*, has a white front side and a blue back side. Corner *A* is folded over point *A′* on the diagonal line *AC*. Where should point *A′* be located on this diagonal (or: how far is *A′* from the folding line) in order to have the total visible area half blue and half white?



In this version, a figure is associated to the text and that orients and makes easier the mathematical work of students. It is no more necessary to fold the square and the problem for students is to find the mathematical expression of both areas: area A_1 of the blue triangle and area A_2 of the white hexagon. Moreover, the side of the square is given as a parameter l and the question is exactly on the place of point A' on the diagonal. Visual adjustments are invalidated by calculations for the area of the triangle seems larger than the other in the case of equality[1]. So, to solve the problem students need to reason on an elaborate and high level.

Two great types of reasoning are expected:

- In the first one, students need to determine an algebraic expression for each area and solve a quadratic equation; in France, this approach is only possible without help in grade 11.
- In the second one, it is possible to reason in figural register. Indeed, the drawing given in the text makes visible three "useful" areas, the two areas to compare and a new area A_3, equal to A_1: the area of the triangle of vertex A completing A_1 to make the square of diagonal AA'. This new area does not exist in the real folding since the triangle does not have a material existence in this case. With the use of this new area, it is possible to find, almost without any calculation, a solution of the problem. The drawing makes clear a decomposition of the square $ABCD$ which implies the equality $2A_1 + A_2 = l^2$ between the areas and in the case where $A_1=A_2$, we get $3A_1= l^2$.

If x denotes the side of the square made by the two rectangle and isosceles triangles, as $A_1= x^2/2$, then $3x^2/2=l^2$, hence $x^2=(2/3)l^2$.

It should be noted that if we take the unknown d on the diagonal, d is the height of one of the rectangle and isosceles triangles, then $x^2=2d^2$ and so $d^2=l^2/3$. This way gives a simple solution to the original problem posed by Carlson et Bloom (2005):

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[1] Let's note that these invalidations are operational since the grade 6 (it has been noted with the class of the student teacher STe).

> A square piece of paper is white on the frontside and black on the backside and has an area of 3 in². Corner *A* is folded over point *A′* which lies on the diagonal *AC* so that the total visible area is half white and half black. How far is *A′* from the folding line. (*op. cit.* p. 55)

In the case chosen by the authors, the area of the square is of three square inches and we get immediately $d^2=1$ and therefore $d=1$. This initial formulation of the problem is really more complex than those used in our study with a real folding and material that allow the student chose a more « natural » variable as the side or the diagonal or in Mexican Task approach where a drawing and the variable are provided. The form used by Carlson and Bloom is not geometric meaningful because it gives only the area of the square. This probably explains much of the difficulties[2] encountered by their students, though advanced in mathematics.

The requested use of a software in the task posed in our study changes again the nature of the task. The software – *Geogebra* – gives an area immediately to each of the surfaces and, as mentioned, it allows – and to some extent encourages – the use of graphics, without the need for an algebraic notation. One could represent graphically A_1 and A_2 in function of x (or d) and then solves the problem by considering the curve intersection (see figure 1 in sec 3.2.1). It is also possible to solve the problem by drawing the graph of point which coordinates are (A_1,A_2) – it is a straight line – and considering the intersection with the line $y=x$.

With this first analysis, it is already clear that the same initial problem can be transformed in different ways leading to very different tasks, depending on the support and tools provided to students or preservice teachers and obviously on curriculum

[2] In the adaptation of STb, described in section 3.2.1, the square has an of area 27 cm^2 but the square is given to students (within the *Geogebra* software).

content. These tasks may also depend strongly on institutional constraints integrated by teachers and their trainers. This is the subject of the study presented in the third section.

3. Transformations of the problem for teacher training

In this section we study the various transformations of a single problem P_0 inside the French educational system through two institutions: a training center for teachers and secondary school classes. More precisely, this study involves two groups of student teachers and two teacher educators, named TEa and TEb in the following. The aim of the research is to grasp the impact of an initial training of math secondary schoolteachers on their actual teaching in a classroom: *what remains of the training when these teachers are back with their students with real constraints?* Due to this aim, our study is not based on Brousseau's theory nor on problem solving but on a specific framework presented in section 4.1. We suppose that the changes of institutions motivate and make necessary some transformations, the study of which will enable to better understand some constraints lying on teachers, together with some usual practices of the profession.

3.1 The transformations of the problem

In the training course involved in the present study we shall distinguish three stages of transformations of problem P_0. In this section we describe these stages.

Stage 1. First transformation: from problem P_0 to problem P_1

Problem P_0 (section 2) required a first transformation in order to be given in the initial training of secondary schoolteachers. The students are prospective math teachers and the aim of the educators (TEa and TEb) is twofold: at the beginning it is a matter of insuring that their students have well understood the problem with its educational potential, the various ways for solving it and the possible difficulties of the solutions. In a second time they will be asked to transform this problem in order to use it in their own training classrooms.

Here is the form chosen for P_1 by TEb, together with the working instructions given to the student teachers (the form chosen by TEa was very close).

> You have at your disposal a square of paper, one side of which is white and the other is grey. A fold shows a diagonal of the square
> A type of folding bringing a vertex of the square on this diagonal, like the one performed on the enclosed square, is considered.
> One intends to compare the white and grey areas obtained in that kind of folding.

For both groups TEa and TEb, problem P_1 was based on this 'minimalist' presentation making use of a model: TEa showed the student teachers the folding with a material square and TEb decided upon sending the instructions with a material square by mail.

The student teachers are asked to work on the problem and show their entire solution process (Schoenfeld 1985). This solution is complemented by a research narrative (cf. section 1.1). It is during this research phase that the student teachers, here in a 'student' position, had to use at least one technological tool[3] to explore the problem favouring experimental approach according to the French curriculum.

The choice of a problem as 'bare' as possible from the mathematical point of view has also a didactical aim, conveyed by homology: encourage the prospective teachers to use, on one side problems with an open question, and on the other side technologies for solving them. By so doing the educators hoped that the student teachers would feel free to operate their own choices, both from a mathematical point of view (cf. a priori analysis in

[3] To be chosen among: spreadsheet, dynamic geometry software, calculator.

section 2) and as regards the actual modes of class implementation by integrating technological tools (cf. section 1.2).

Stage 2. Second transformation: from problem P_1 to problem P_2

Again in the training center, the student teachers were asked to write down the wording of a problem and to make explicit the modes of implementation for their students in their classrooms. Actually, the students involved are also math teachers in a secondary class (junior or senior high school). At this stage, the issue is not to pose the problem in a class but, in the training center, to think about the form that the problem could take if it were posed to a class. In that sense it may be considered as *a virtual* problem P_2 which marks the outcome of the work for TEa's training group. This stage could possibly have been carried on, but its existence and its control had not explicitly been anticipated in the course specific for this group of training students. A description of the work of TEa's group is developed in Kuzniak, et al. (2011).

Stage 3. Third transformation: from problem P_2 to problem P_3

In TEb's group, after a session of the 'seminar' type in which the students had to expose their work in stages 1 and 2, they were asked to write down a problem P_3, again with making explicit its modes of implementation and its aim, and above all to actually pose it to their own students. Then they had to present in the training center, again during a session of the seminar type, and a posteriori analysis of problem P_3 posed in their class, illustrating it with their students' writings. This shift from the training center (virtual problem P_2) to the classroom (real problem P_3) supposes a sharper adaptation of the problem to the trainee's class, in particular because of the real constraints.

3.2. Description of complete trajectories developed by student teachers

We call the set of stages transforming problem P_0 which has been exposed above a *trajectory* of this problem. Of course, every student teacher develops his/her own trajectory, which can even be trajectories because classroom is an important factor influencing the transformations of a problem.

Below are described the complete trajectories of P_0 elaborated by five student teachers of the TEb group, named STa, STb,... STe. In fact, the differences between these trajectories are essentially due to the mathematical aims linked with the teaching contents of each class and with standard activities of textbooks at the different teaching levels. The student teachers try to design and develop teaching activities which are as close as possible to what we call suitable mathematical working space (Kuzniak 2011).

Hence, the aims of each problem are different according to the mathematical contents aimed at. On the other hand, a teacher will only give his students a problem on the condition that it fits well in the syllabus. For this reason it is necessary to supply the student teachers with problems having strong potentialities and open to varied adaptations. In the present case, problem P_1 (cf. section 2), elaborated after discussion by the teacher educators, is adequate and, as will be seen, might give rise to adaptations at all secondary education levels. Another common characteristic that we noticed is that the problem was always used to introduce a new knowledge and never an assessment of an old knowledge.

3.2.1 Two pre-service teachers' trajectories at grade 10

In this first case we consider two student teachers, STa and STb, teaching in *seconde* grade (grade 10), which in France is the first course of senior high school. In spite of different modes, essentially due to the real constraints of the two classes, the two trajectories presented here are very close to each other. Such closeness can be explained by the fact that the aims chosen, depending on the teaching program of the class, were practically identical, that is, a global study of polynomial functions. Indeed, problem P_1 is close to a standard type of exercises which can be found in many textbooks at this education level: a geometric statement followed by a modelling by a quadratic function enabling to solve the initial problem.

Stage 1. Solving problem P_1

STa and STb solved the problem in a similar manner and used the graphs of the two functions defined by modelling the two areas (triangle and hexagon) generated by the use of *Geogebra* software. The variable chosen, called *x*, is the length of the side of the small square. The intersection point of the two curves gives an approximate solution: the common measure of the areas is its ordinate while the measure of the side of the small square in the case of equality is its abscissa. However, the use of *Geogebra* by the two student teachers was very different:

- STa constructed, in a same file, the square simulating the folding and drew the two curves representing the areas as functions of the distance between the folded vertex and a free point on the folding diagonal (cf. figure 1).
- STb as well made a construction with Geogebra to simulate the folding (two constructions were proposed) but functions are used in another file. She first got the two algebraic functions then graphically represented them (cf. figure 2). In this case Geogebra was in fact used as a graphics software and not as a dynamic geometry software.

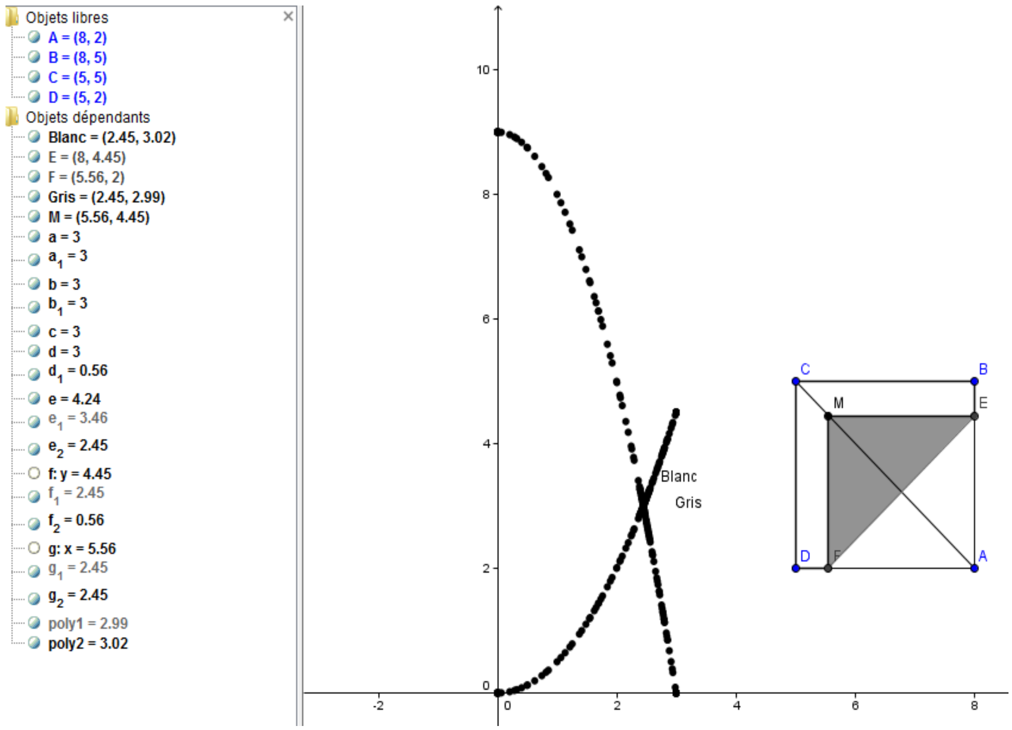

Figure 1. Use of Geogebra by STa for the solving of P_1

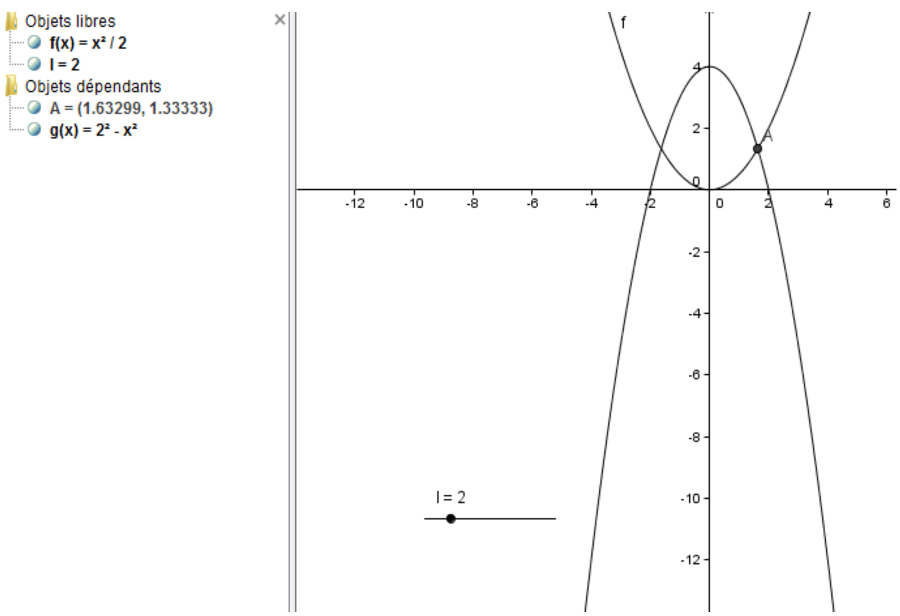

Figure 2. Use of Geogebra by STb for the solving of P_1

Another difference between STa and STb appears in how each of them considers the square length *l* with the software:

- STa fixed the value of the square length to $l = 3$ cm though he received a 5 cm length square by mail: he considered this value inadequate because it did not allow a good representation of the two curves on the computer window, the size of the objects being estimated too big. This last point shows that he has a quite poor knowledge of the software since he modified the situation instead of using the Geogebra potentiality to manage the mathematical situation.
- STb did not fix the square length since the parameter l is managed by the software through a cursor and the two functions introduced are defined using this parameter. So the abscissa of the intersection point of the two curves gives the searched value of x as a function of l.

Nevertheless, neither STa nor STb undertook a deeper exploration of the situation within the software. They only gave approximate values[4] of the solution:

- STa wondered whether the same reasoning is still valid when the value of l − that is the square size − is changed but it seems that he did not try answering this dilemma.
- STb did not try to search the link between the solution, which is the abscissa x_A of point A in figure 2, and the parameter l given by the cursor. Indeed, the graph of function $l \rightarrow x_A(l)$ could be easily obtained by considering the point of coordinates (l,x_A). Then, one can easily see that this graph is a straight line.

During the exploration of the possible solutions, the two student teachers did not use any other software. Their researches within a paper and pencil environment are also

[4] STa obtained the approximate value 2.46 for $l = 3$ cm; STb gave the approximate solution values with 5 decimals. STb noticed that these approximate solutions were also approximate values of 1/3 (for $l = 1$) or 4/3 (for $l = 2$, cf. figure 2). But this remark was without any consequence on splitting the square area into three thirds: STb stuck to her approximate determination of x.

very close. The configuration studied is general and both use the *l* parameter to name the side of the square and a variable (or unknown) *x* to name the side of the small square. After calculation of the two areas as functions of *x*, the problem was solved in the case of equality, with the answer $\sqrt{2/3}\,l$ accompanied by a justification for not considering the negative root of the equation. The comparison of the areas was made by using the extreme values x=0 and x=l, as well as an argument (implicit for STa) about continuous functions.

On the other hand, a notable difference between STa and STb appeared in the management of the geometric setting. Using properties of orthogonal symmetry STb developed a detailed proof on the nature of the triangles which seem to be isosceles and rectangle. STa apparently remained at a visual stage (of the GI type, see (Houdement & Kuzniak, 1999)) since he did not make any remarks on the geometric configuration, although he fully used it in his calculations.

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

For both STa and STb these problems were integrated in the chapter on polynomial functions of degree two.

For STa, the statement of the virtual problem P_2 is identical to P_1 (with the exception of the length of the side of the square which is fixed to 5 cm) with the use of *Geogebra* in half-classes. Though the precision "the length of the side is not given" can be noticed, the statements of the real problem P_3 and P_2 are almost identical (and so is the case for P_1). However, P_3's implementation modes are very different. It is finally given as homework, the choice being left to students to send a *Geogebra* file by Internet or to give back a paper-and-pencil work. Contrary to P_2, the use of the software is not required. Sending works by electronic mail had already been used in the year but none of SPa's students chose this option for this work, and finally all of them achieved a paper-and-pencil work (presumably using calculator).

In both problems, P_2 and P_3, STa encouraged his students to make the folding by themselves. However, in P_2 the square was given whereas in P_3 the square had to be constructed by the students themselves: therefore they had to choose the length of the side.

For STb problem P_2 is close to P_1 but, with the addition of specific questions, it became a closed problem. The l side was fixed to 6 cm and only the case of equality was asked; the actual folding was required (for this a bi-colored square on which a diagonal had been drawn was given to every student); a question asked to prove the existence of an isosceles rectangle triangle; notations for geometric points and the variable x were provided and use of *Geogebra* was considered – in half-classes – to represent the two curves and thus allow a graphical resolution of the problem (let's notice that this type of task has already been asked in this class).

Although if in P_3 there is no question about the nature of the triangle, STb mentioned that the nature of this triangle would be assumed. Finally l was fixed to $3\sqrt{3}$ (more or less like in problem P_0, although STb did not know of it) and the question was then more open, no procedure was imposed anymore, the students had the choice between *Geogebra* software and paper-and-pencil environment. Two questions were asked: one on the case of equality and the second on the comparison of areas. The students, by groups of three, had to cut out a square. Two different aids had been prepared by STb: for students who choose *Geogebra* (the square $3\sqrt{3}$ size was already constructed) a hint indicates some *Geogebra* tools, and for those who chose the paper-and-pencil environment several possibilities for choosing the unknown, or variable *x*, were given (this help was not immediately provided and was limited to cases of blockage).

3.2.2 A pre-service teacher's trajectory at grade 8

One student teacher, STc, was in charge of a grade 8 class. At this level, two mathematical contents, obviously in relation with the syllabus, were considered: mathematical proof in geometry (chosen by STc) and algebraic calculation.

Stage 1. Solving problem P_1

STc produced a long research, exploring various points of view on the problem, remaining mostly in a geometrical setting. He produced proofs using geometrical tools and notions: isometric triangles, intercept theorem (known in France as the *théorème de Thalès*), orthogonal symmetry, Pythagoras' theorem, perpendicular bisector, square, bisector, sum of angles of a triangle. He chose a variable *x* on the diagonal (he *instinctively* did not consider the side of the square) and calculated the areas but he could not solve the problem.

In his research on problem P_1, STc made a clear distinction between geometrical paradigms GI and GII (Houdement & Kuzniak, 1999) which constitutes one of the stakes of the teaching of geometry at junior secondary school. An attempt to cut out figures for determining areas (especially for the hexagon) was also noticed but STc concluded that it was impossible to find a solution without using the above mentioned geometry tools.

He also used the *Geogebra* software to simulate the folding and visualize the hexagonal area by a curve, using sizes measured by the software (length and area). Like for STa, the value of *l* leads to a curve that does not fit well in the graphical window. But instead of modifying the value of *l*, STc divided the ordinates of the points by 10. He stopped when seeing that he got a parabola as his calculations had shown him. He did not solve the problem, neither with the software (contrary to STa and STb), nor by using the notion of function (the curve shows only that there is a parabola).

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

STc did not produce a virtual problem P_2 (this comes probably from an omission or misunderstanding of the statement), and in his real problem for his class he put the stress on the teaching of proof. The problem P_3 he proposed was stated only in a paper-and-pencil environment, and there are multiple reasons for this:

- he points out constraints in the use of the computer room;
- he thinks that his students are not able to use a software for making a conjecture without being guided and he wants to keep the character open of the problem;
- he thinks his grade 8 class is a 'good' one.

He then considers a paper-and-pencil work in small groups, planned for two sessions. The problem P_3 he poses asks to cut out a 6 cm sided square, with the students achieving actual folding, and includes only one question: "How to achieve this folding so that the grey area is equal to the white area?"

The aim is twofold, as it can be noticed in the planned institutionalization: proof of the fact that the hexagon is obtained by removing a small square and calculation of the position giving equal areas. Besides, after the first session a student proposed to cut out the square into three figures having the same area (hexagon and two isosceles rectangle triangles) but without being able to justify it. Then STc adjusted his plans and thought of proposing a solution based on the areas: the area of the small square must be equal to the two thirds of the total area, and therefore the side of the small square (which gives the solution) is $\sqrt{2/3} \times 6$ cm or $\sqrt{24}$ cm. However, in the class reality, the aspects linked to geometrical proof were hardly tackled during the session.

3.2.3 Two pre-service teachers' trajectories at grade 6

At grade 6 level, the mathematical notions that the students know do not allow the use of the previous mathematical supports (functions, algebraic calculation, geometrical proof). It seems that the calculation of areas of polygonal figures is the only possible mathematical support at this level. Thus, it is not surprising that this very content constitutes the choice of both student teachers, STd and STe, who are considered in this section.

Stage 1. Solving problem P_1

STe used *Geogebra* for modelling the folding. A visual adjustment with the measures of the two areas allowed him reducing the gap between them in order to solve the problem in an approximate way. Then, in order to make a conjecture, STe tried searching for a notable value, the approximate solution could be an approximation of it. His attempts were not successful in spite of two constructions depending on whether the mobile point is on the side of the square or on the diagonal – these lengths being, in each case, fixed to 10 cm for making the research of a conjecture easier.

Then STe shifted to paper-and-pencil environment. After fixing the length of the square to 1, he produced two calculations of the solution by taking two unknowns, respectively the side x of the small square, and $1–x$. For STe, it is explicit that equal areas corresponds to cutting out the square into three thirds, but the general comparison of areas is not taken in account.

In her research for a solution, STd started with working in a paper-and-pencil environment; she named x the length of the side of the small square and l the length of the side of the initial one, calculated the two areas and solved the problem of their equality. Let us remark that she wrote, without justification, that the comparison of areas is solved with the help of the equality case. The comparison of areas was made with respect to the value $l/\sqrt{1.5}$. Then STd carried out the folding with her square: "I measured l (7,3cm), I did the calculation, which gave x=5.96". STd found that, visually, there seemed to be a little difference between the areas and she thought that it is due to an optical illusion. She then gave a construction of the folding with the help of the *Geogebra* software. STd regrets that this only provides an approximate value of the solution, like the ones obtained with a square of paper: measures and area calculations.

Stages 2 and 3. Problems P_2 (virtual) and P_3 (real)

STe proposed a statement of the virtual problem P_2 identical to P_1's, but he fixed the side of the square to l=12 cm. The scenario he considered includes three steps:

- an initiation, during about 20 min, in a session that involved an actual folding of a particular square, a statement of the problem and first attempts of solution;
- a second stage, in the computer room, to determine an approximate solution with the help of Geogebra;
- a last stage, working in pairs, aiming to justify the solution found with the help of a cutting out of the square (this last step being not explicit).

He proposed a 'dressing' of the problem in order to make it more concrete for his students: a square field inherited by three brothers has to be divided between them. The eldest receives the total big square minus a small square (situated in 'a corner'), this remaining small square being shared between the two others. The question is: "do the three brothers have equitable parts?". This dressing, not taken up in problem P_1, changes significantly the problem because it turns it onto cutting the initial square into three polygons of equal areas. There is not folding anymore and nothing is said on how the small square is shared between the two younger brothers (nor even if it is equitable).

The real problem P_3 took up this idea of contextualisation, but remains closer to problem P_1: a firm wants to make a logo defined by the folding of a square of side 12 cm and the constraint of equality of the two areas. STe also took up the idea of three phases, only slightly modified:

1. a first activity, on paper, to understand the problem;
2. a second activity, with *Geogebra* (construction and research are very guided), to find out an approximate value, which is quite suitable for the realization of the logo;
3. here the justification was replaced by a actual construction of the logo on paper, using this approximate value (this third step was planned in the same session than point 2).

The student teacher STd proposed a problem P_2 taking up problem P_1 and modifying the question in the same way as STe: "*How has the black corner to be folded so that it has the same area than the white surface?*" The possibilities for using calculator as well as the *Geogebra* software were mentioned (under the condition of not asking to draw the diagram, judged too complex for this level). In particular, STe planned to have the students work in groups of four in a computer room and let them choose their environment.

The real problem P_3, differs notably from P_1 by the fact that one the interest is only in the equality (like P_2) and especially the fact that an approximate value is explicitly asked: "*Determine as precisely as possible a folding of this type, so that the white part and the colourful part have the same area*". The work was organised in groups of three students in the computer room, with a possibility to use *Geogebra* or only paper and pencil. Each group was provided a square of paper, the size of which was 3 cm, 4 cm, 5 cm or 6 cm (STd explicitly adjusted this choice of the didactical variable: multiple of 3 or not).

4. Discussion

4.1 About trajectories

In this section we propose an original frame to organize and analyse the emerging trajectories to deal with the problem, like those which have been set out in section 3. The aim of this frame is to take into account various dimensions of a problem (institution and persons involved, goal(s) aimed at) and study the nature and the dynamics of the changes which take place through the successive 'moves' of this problem from one institution to another.

At the start there is a problem, not necessarily mathematical, coming from an institution I, that may involve an everyday life or any domain of knowledge. Then there are several didactical institutions I_1, I_2,… in which successive alternative forms ('avatars') of the initial problem will show up. In each institution I_k ($k \geq 1$) one or several individuals T_k in a 'teacher' (or 'educator') position, as well as individuals S_k in a 'student' (or 'trainee') position, will be distinguished.

These institutions will be concatenated between them in the following way: the problem was introduced in I_k under the P_k avatar by T_k who poses it to the S_k with a given purpose. Then one of the S_ks, who in institution is in a 'teacher' position ($T_{k+1} = S_k$), poses the problem to his/her students S_{k+1}s under the avatar P_{k+1}, with a purpose which is generally different (*figure 3*).

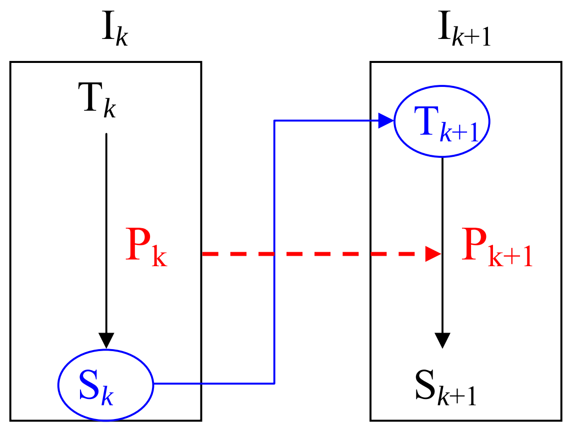

Figure 3. Concatenation of institutions

Of course this process can possibly be carried on from an institution to another (I_1, I_2,… , I_n), depending on the involved individuals. The succession of stages − and hence of avatars of the problem − constitutes the *trajectory* of the problem.

Example (*figure 4*).

Stage 1. In a training center for teachers (institution I_1) a math educator finds a problem written in everyday language in a magazine. He/she thinks that it could well give rise to a geometrical activity for his/her trainees. Then he/she transforms it into a geometrical wording and, within the training curriculum, asks the trainees to search ‘all possible solutions’ of the problem, regardless to the classroom level. (mathematical *a priori* analysis).

Stage 2. Again within the training curriculum (institution I_2=I_1), the teacher educator asks his/her trainees to transform the wording into a new one that could be posed as a research problem to a class of a given level (didactical *a priori* analysis).

Stage 3. Back to his/her school (institution I_3), each trainee undertakes posing the problem in his/her class. For that he/she transforms again the wording according to this particular class and poses it by asking his/her students to use their knowledge to find a solution to the problem.

Stage 4. Back to the training center (institution $I_4=I_1$), the educator asks the trainees, gathered in groups according to the level of their classes, to work out for that level a new formulation of the wording, in order to make it a research problem taking into account the implementation that they could observe in their own classes (*a posteriori* analysis).

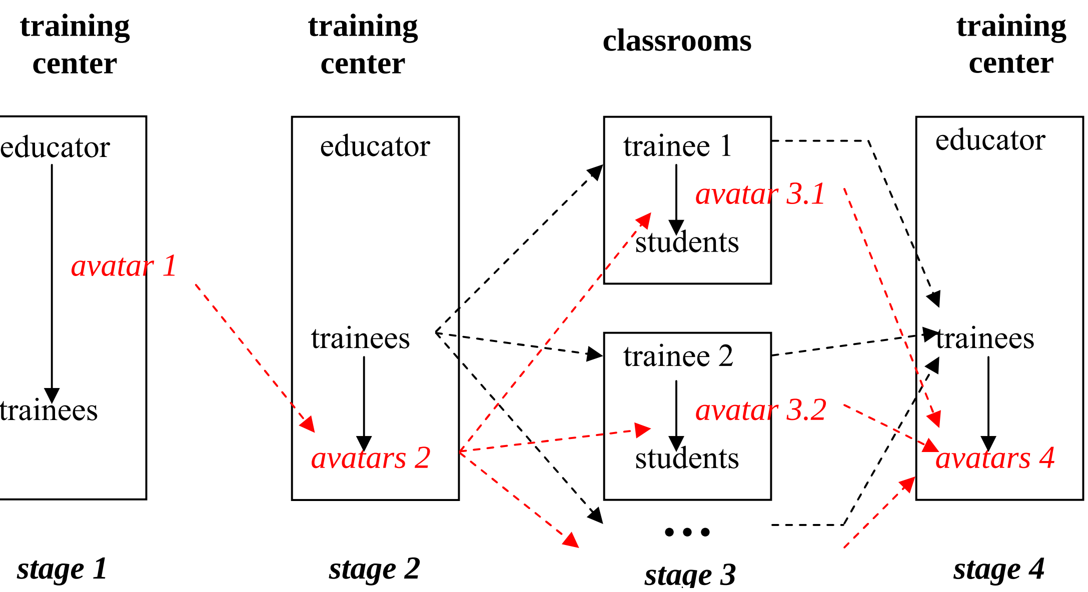

Figure 4 : Examples of trajectories of a problem

The first three stages correspond to the example of training constituting the study of section 3: $I_1=I_2$ is the training center and I_3 is one of the secondary school classes. Stage 4 could not be achieved during the training. It is nevertheless important, either being put into play in the training center or not, because it marks the start of a cycle of transformation of the problem taking into account the feedbacks from the students. This is a central component of the profession of teacher.

Moreover, one may quite consider conceiving trajectories in which other modes of transmission of problems intervene. For instance think of a continued training instead of an initial one or a debate between teachers of a same secondary school.

4.2 On training

During the first session dedicated to presentation of the problem P_1, the teacher educator TEb made an unsuccessful attempt to orientate the trajectories by encouraging the teacher students to think about the use of spreadsheet in the class. However, as we saw it in the class of STd, sixth grade students could generate values tables close to what they could get faster with spreadsheet. We could observe the teachers' difficulties to integrate spreadsheet in their actual practices despite an important focus during the training. It could suggest a training underperforming, but this opinion needs to be qualified because it seems that spreadsheet, according various studies, is a tool especially difficult to integrate into lessons by teachers. Indeed, Haspekian (2005) mentions some specific problems on spreadsheet instrumentation or teaching of particular notions related to spreadsheet (such as delicate and complex notion of cell) which do not exist or not under the same form in maths knowledge at this grade: that can interfere negativity with the teaching of algebra. Teachers can be aware of these difficulties and avoid the use of spreadsheet in class despite the official demand from educative institution. The interpretation is confirmed by the experiment of TEa. One group of teacher students had to prepare a session using spreadsheet. Convinced of the impossibility of using spreadsheet in their own class, they prepared a session dedicated to the teaching of algorithms without any actual adaptation to the level of their students. They argued that the use of a spreadsheet needs too much time and knowledge which is not of mathematical nature.

Open problems and problem-situations with a-didactic potential are largely favored by the training in teacher training institution I_1, especially to encourage student teachers to not only ask problems with closed questions to their students. As the problems P_3, posed in class, were generally open, we can conclude that prospective teachers were aware of this mathematics education complexity. This is perhaps due to the training based on homology that we gave to the students and which postulates that teachers students will reproduce the form of the teaching they received during their training in I_1 .

It should also be noted that the virtual problem P_2 does not provide a lot of information on the actual course in class, except to check that changes of the mathematical support could only be possible in the class in front of school students (see STc). Even when student teachers know they will have to manage the problem with their students, the real constraints of the class do not seem to be taken into account before they are involved in real teaching scenarios with their own students. This leads to significant differences between *laboratory* work in I_2 towards I_3 and the actual work in I_3 and could suggest that the training on problems prepared in I_2 is not representative and far away from the reality of class teaching - even if this work remains interesting for training. That too should lead teachers educators to complete the training by requiring prospective teachers to engage into an actual implementation in a class with an a posteriori analysis. This demand can also show them that, first, it is possible to implement in I_3 the requirement made in I_1 and, then, that the demand of the training institution is not opposite to the demand of school institution as some students think of it.

4.3 On the choice of the specific technological context

All prospective teachers have chosen to use *Geogebra* software to approach their problem research in response to the demand of using a technological context. This sole choice of *Geogebra* could be explained by some factors. First, training in I_2 favors this software which is widely used in French secondary school system. On other hand, *Geogebra* which is a multi-purpose software is well adapted to the problem: P_1 is generally seen as a geometrical problem and therefore the use of a dynamic geometry software is somehow natural, and for grade 10 the problem is also connected to functions as modeling tools and the use of *Geogebra* to make graphics is well suited.

For the problem posed in class, three different environments are employed for solving it: a paper and pencil environment or *Geogebra* (STa, STd), only *Geogebra* (STb, STe) and no use of software (STc). Moreover, there are few mentions of the use of a calculator (STd is the unique teacher who speaks explicitly about it) while school students use it widely. Perhaps, this lack of allusion to calculator is due to the fact that teachers do not perceive it as a technological environment (despite the instructions see sec 3.1) and they think of a computer. It is also possible that its use is now considered transparent and routine for prospective teachers and they feel no need to mention it.

4.4 On the folding

All prospective teachers keep the idea of the folding to present the problem to their students. Probably this anchoring to the real world supports the devolution of the problem as the attitude of STe suggests it: he left aside the idea of folding in the virtual problem P_2, but it takes again this idea when he poses the real problem P_3 to his students in class.

However, the folding is not easy to define as we can see it in I_1 where the teacher educators had been obliged to mention other geometric terms than the area like square and diagonal and vertices. The diagonal could be also drawn (an even marked by a fold as STe did it). Other ways are possible: STc pointed out the vertex to fold on the diagonal by coloring the good corner to use; STd has not defined the folding with enough precision and students did not well understand the instructions so that STd added some comments during the session in class; STa and STb made an unequivocal coding of the figure (like in the Mexican task in sec. 2).

4.5 On the problem

It is indeed a problem with a high potential that can be addressed at all levels of secondary education. The student teachers have all agreed without hesitation to pose it with their own transformations to their classes and students and, according to their comments, the students were interested in solving the problem P_3.

Many adjustments were made especially concerning modalities of implementation. But despite the diversity of educational levels where the problem was given, the core of the mathematical problem stays stable with few changes. Among the changes, we can note essentially: the value of l (except for STa) and the research of the equality (except for student teachers teaching in grade 10, STa and STb). The biggest adjustment was made by STd, who introduced the concept of precision of the solution. By and large, the problem P_1 *did the job*.

We can conclude that the transformations of the problem P_1 to give it in class are simultaneous oriented by the researches of the mathematical solution and by the official syllabus of the grades involved in the teaching. It would be interesting to know what will be the use of the problem by the teachers some years later and how the trajectory of the problem continues evolving. We intend to make an interview with the prospective teachers involved in this study in the future. Another point of interest is the impact of such problems on school students and some material need to be used to precise this crucial point.

References

Arsac, G., Germain, G., Mante, M. (1988). Problème ouvert et situation-problème. Ed. IREM de Lyon.

Arsac, G. & Mante, M. (2007). Les pratiques du problème ouvert. Ed. IREM/CRDP Lyon.

Artigue, M., Cazes, C. & Vandebrouck, F. (2011). The shop sign family : a project for supporting teachers' use of ICT. Poster at Cerme7. University of Rzeszow.

Artigue, M. & Houdement, C. (2007). Problem solving in France: didactic and curricular perspectives. ZDM Mathematics Education 39:365–382

BOEN (Bulletin officiel de L'Éducation nationale) (2008). Programme de mathématiques du collège. Bulletin spécial 6. Ed. Ministère de l'Éducation nationale.

Bonafé, F., Chevalier, A., Combes, M.-C., Deville, A., Dray, L., Robert, J.-P. & Sauter M. (2002), Les narrations de recherche de l'école primaire au lycée. Ed. IREM/APMEP 151.

Blum, Werner & Leiss, Dominik (2005). "Filling Up" - the problem of independence-preserving teacher interventions in lessons with demanding modelling tasks. Proceedings for the CERME4, WG 13 Modelling and Applications.

Brousseau, G. (1997). Theory of Didactical Situations in Mathematics (1970–1990). Dordrecht: Kluwer.

Carlson, M. & Bloom, I. (2005). The cyclic nature of problem solving:an emergent multidimensional problem solving framework. Educational Studies in Mathematics, 58, 45-75.

Gomez-Chacon, I.M. & Kuzniak, A. (2011). Les Espaces de Travail Géométrique de futurs professeurs en contexte de connaissances technologiques et professionnelles. Annales de didactique et de sciences cognitives, 16, 187- 216.

Chevallard, Y. (1985). La transposition didactique. Grenoble : La Pensée Sauvage.

Douady, R. (1986). Jeux de cadres et dialectique outil-objet. Recherches en Didactique des mathématiques 7/2, 7-31.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics, Educational Studies in Mathematics, 61, 103-131.

Haspekian, M. (2005). An ‘‘instrumental approach’’ to study the integration of a computer tool into mathematics teaching: the case of spreadsheets, International Journal of Computers for Mathematical Learning, 10, pp. 109–141.

Houdement, C. & Kuzniak, A. (1999). Sur un cadre conceptuel inspiré de Gonseth et destiné à étudier l'enseignement de la géométrie en formation des maîtres, Educational Studies in Mathematics, 40/3, 283-312.

Houdement, C., & Kuzniak, A. (2001). Pretty (good) didactical provocation as a tool for teachers’ training 795 in Geometry. Proceedings of CERME2 (pp. 292–304). Praha: Charles University.

Kuzniak, A. (2011). L'espace de Travail Mathématique et ses genèses, Annales de didactique et de sciences cognitives, 16, 9-24.

Kuzniak, A., Parzysz, B., Santos, M. & Vivier, L. (2011). Problem solving and open problem in teachers’ training in the french and mexican modes, Proceedings of Cerme 7, 9-13/02/2011 Rzeszów, Poland (to appear).

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabrigeometry. International Journal of Computers for Mathematical Learning 6(3), 283–317.

Schoenfeld, A. (1985). Mathematical problem solving. New York: Academic Press.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. Educational Researcher, 15 (2), 4–14. 827