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Abstract: This study uses the perspective of schemes to analyze characteristics of arithmetic word problems that can influence the process of translation from the verbal statement to an arithmetical representation. One characteristic that we have detected in the two-step word problems is the presence of one or two connections (nodes) in schemes that represent them, and this paper explores whether the number of nodes affects the activation of the associated schemas. With students from the 5th and 6th grades of elementary school (11 and 12 years of age), we analyze the written productions and would stress that the number of connections influences the activation of the right schema. Results show that the double connection implicate a greater difficulty for obtaining a correct arithmetical representation. Likewise, the presence of a simple or double connection between the two relationships means that the students commit specific errors that we associate with this characteristic.

Keywords: Two-step word problems, arithmetic, schemes, double node, errors.

Introduction

Research on problem solving on mathematics education is a wide and varied field, and it is not limited to a single study focus; nor is it performed within a single theoretical framework (Castro, 2008; Santos, 2008). A good number of studies have centered on the use of arithmetic operations to solve word problems. Verschaffel, Greer, & Torbeyns, (2006) distinguish four focuses in the study of arithmetic problems: (a) conceptual structures (schemes) for representing and solving word problems; (b) word problems viewed from a problems-solving perspective; (c) a sociocultural analysis of performance on arithmetic word problems; and (d) the modeling approach.

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Since the 1990s, there have been a tendency in Mathematics Education to undervalue the educational value of word problems and stress situated and socially mediated approaches to solving authentic, complex problems. Despite this focus, Jonassen (2003) indicates that "story problems remain the most common form of problem solving in K-12 schools and universities" (p. 267). This paper treats arithmetic word problems whose statement contains two relationships between the data and that therefore require more than one operation to solve them (two-step arithmetic word problems). We perform our analysis from the perspective of schemes (Hershkovitz, & Nesher, 2003) and focus on characterizing the double node in two-step arithmetic word problems and the schemes to which they give rise, and on studying the influence of the double node on the activation of the schemes and the errors this causes. Enright, Morley, & Sheehan (2002) indicate that *problem features such as those described previously can be related theoretically to individual differences in cognition* (p. 51).

The scheme approach

From the semantic perspective on one-step arithmetic problems, *once the concepts and relationships involved are understood, the child has only to choose the correct operation and apply it* (Quintero, 1983, p. 102). In problems with several steps, it is also necessary to understand the concepts and relationships, but additional issues are involved as well. Quintero (1983) indicates that the child must plan and organize the order in which to apply the operations and identify the pairs of numbers to which to apply each operation. Shalin and Bee (1985) analysis of two-step problems leading to specific structures is based on the possible logical combinations of one-step problems. They represent the corresponding scheme of a simple arithmetic word problem by means of a diagram (Figure 1) of three connected components in terms of part-whole relationships.

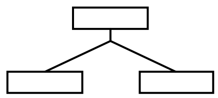


Figure 1. Notation of the triad of components present in the part-whole relationship

If the diagram in Figure 1 represents a mathematical object, we can construct more complex mathematical relationships from it using more than one diagram and connecting them to each other, forming networks. Following this idea, Shalin & Bee (1985) obtain the structure of a two-step problem by combining two triads based on local relationships. Each of the different ways of combining two triads like that in Figure 1 constitutes a different global problem structure. These combinations (Figure 2) define three structures of two-step problems: hierarchy, sharing the whole and sharing a part.

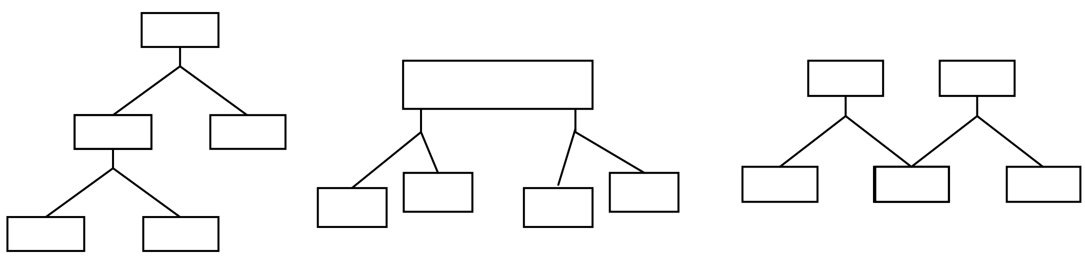


Figure 2. Hierarchical scheme, sharing the whole and sharing a part

Nesher & Herskovitz (1994, 2003) research the influence that the three schemes (Figura 2) have on the index of difficulty of composite problems. With a sample of students from third, fourth, fifth, and sixth grades in Israel, they find that the variable type of scheme has a significant effect on the index of difficulty of these problems. The "hierarchical" scheme is the easiest, followed by the "shared whole" and finally the "shared part" scheme. The study by Shalin & Bee (1985) also showed that children in the

3rd, 4th and 5th grades (elementary school) had a higher rate of success with the hierarchical scheme. In the following section, we will see that the results can be altered by other cognitive variables.

Decrease-increase relationship

In the research performed by the Numerical Thinking Group of the University of Granada, with 4th, 5th, and 6th grade elementary school children from Granada (Spain), the results obtained by comparing the indices of difficulty of the different combinations of the relationships of increase or decrease show that the combinations of increase and decrease affect the difficulty of the two-step problems (Castro, Rico, Castro, & Gutiérrez, 1994; Castro, et al., 1996); Rico, Castro, González, & Castro, 1994; Rico, et al., 1997).

The four classes of problem are determined by whether the relationship is one of increase or decrease.

Type (I, I). Two relationships of *increase*. The *whole* of the first initial relationship is a *part* of the second relationship (*hierarchical* scheme).

Type (D, D). Two relationships of *decrease*. One *part* of the first relationship is the *whole* of the second relationship (*hierarchical* scheme).

Type (I, D). First relationship of increase and second of decrease. The two relationships share the whole (*sharing the whole* scheme).

Type (D, I) The first relationship is one of decrease and the second one of increase. The two relationships share a part (*share a part* scheme).

Presented in order of increasing difficulty, they are:

(I, I), (I, D), (D, I) and (D, D)

where the type (D, D) is the most difficult. These results contradict the argument that the hierarchical scheme is generally less difficult than the other two schemes. Shalin & Bee (1985) and Nesher & Hershkovitz (1994) find that the problems associated with the hierarchical scheme are less difficult than the others. However, in Castro, et al., (1996) study with additive problems, the problems corresponding to the two extreme combinations—the easiest, increase-increase (I, I) and the most difficult decrease-decrease (D,D)—correspond to the same scheme: the hierarchical scheme. The difficulty of the hierarchical scheme is consequently affected by the relationships of increase or decrease used to state the problem. Other cognitive variables also appear in two-step problems, however, such as the number of connections between the components of the basic structure, as we will see in the next section.

Problems with two nodes

One of the key issues in understanding the structure of two-step word problems is understanding the nature of the two elements that compose the basic triad of the part-whole scheme and the way of connecting these elements between two triads. To determine how this is done, Nesher & Hershkovitz (1994) perform a textual analysis of the problems, breaking them into components. They distinguish three components in a one-step problem. Two of these provide numerical information explicitly (complete components) and the other, the question, is missing numerical information (incomplete component).

In the composite schemes for two-step problems (Nesher & Hershkovitz, 1994), the connection between the two one-step problems is created by a new component, which they call the latent component of the problem (see Figure 3) and which is common to the two simple structures.

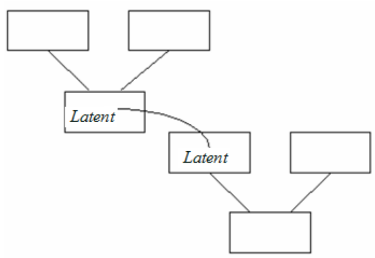


Figure 3. Latent component

From a representational point of view, we say in this situation that there is a *nexus* or *node* between the two simple structures that produce the corresponding composite scheme. Thus, the two simple structures share a component within a two-step problem. For example, in Problem 1:

Problem 1. I bought 5 books. Each book cost 8 euros. If I pay 50 euros, how much money will I get back?

In the first structure, the latent component is the question of the first problem:

- I bought 5 books
- Each book cost 8 euros
- How much do all of the books cost?

In the second structure, the latent component becomes a complete component:

- All of the books cost 40 euros.
- I pay 50 euros.
- How much money will I get back?

In this problem, the latent component (the price of all the books) is shared by the first and second arithmetic structures. This latent component, which is not stated explicitly in the wording of the problem, connects the two structures. The price of the books is obtained in the first structure, where it has the function of incomplete component. This price is then used in the second structure as a complete component. This function of connection between the two structures is what leads us to call it a node or nexus of union between the two.

In the schemes of two-step problems defined by Shalin & Bee (1985) and subsequently used by Hershkovitz & Nesher (1996) and Nesher & Hershkovitz (1994, 1996, 2003), the latent quantity is the only nexus of union between the two simple structures. However, the condition of a node does not imply being a latent quantity, nor does it mean that this is the only quantity with this condition. The node can also be a piece of information given explicitly in the statement and that is shared by more than one simple structure within a two-step problem. It is possible to find two-step problems that have two simple structures connected by two nodes, as occurs in the following problem:

> *Problem 2. John has 5 balls. His grandfather gives him triple the number he had. How many balls does John have now?*

This problem 2 combines two simple schemes: one multiplicative scheme and one additive. Both schemes have two quantities, “John’s 5 balls” and “the balls that John’s grandfather gives him,” which are shared. In Figure 4, we see the representation of the two simple schemes and how both contain the shared quantities. This kind of two-step problem has only two pieces of information or, from another perspective, three pieces, but one of these is repeated or has a double function. Therefore, two components are shared by the two simple structures, one of these the latent component (balls that the grandfather gives) and the other the repeated piece of information (John’s 5 balls) in the problem.

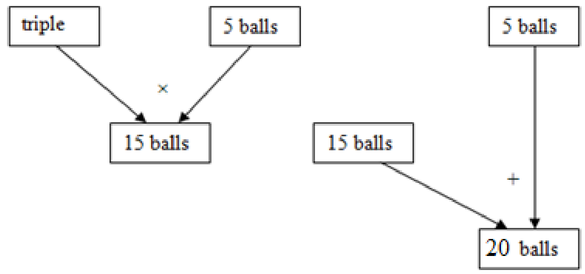


Figure 4. Simple schemes of a two-node problem

The quantities that are shared by various simple structures within a composite problem have, therefore, the condition of node, independently of whether these quantities are given pieces of information or intermediate unknowns (latent quantities) in the problem.

Types of two-node schemes

The problem we have used as an example of a double node is hierarchical in kind (see Figure 5), and the 15 balls constitute the latent variable, which is the intermediate unknown quantity.

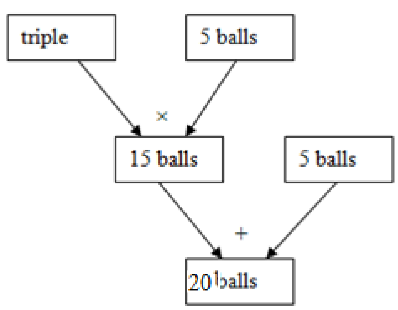


Figure 5. Hierarchical scheme

We can see that the quantity of 5 balls appears in the two simple structures. If we merge both boxes into a single one, as shown in Figure 5, we get a sub-scheme of the hierarchical scheme, but one in which two quantities are shared by the two simple schemes.

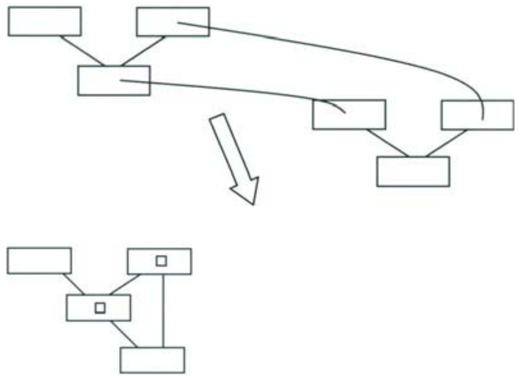


Figure 6. Composite scheme HP

When a part and the whole of one simple scheme matches with the part and part of the other simple scheme (P, W = P, P), we call this composite scheme HP, since it can be obtained from the hierarchical scheme H, since one part of each simple scheme coincides with the other (see Figure 6).

In the other two structures of two-step problems, sharing part and sharing whole, new substructures also emerge with this condition of considering the double node. In the case of the structure "sharing part" (SP), we can generate a substructure by making it agree with the other part of the two simple schemes (see Figure 7). We call this substructure SPP.

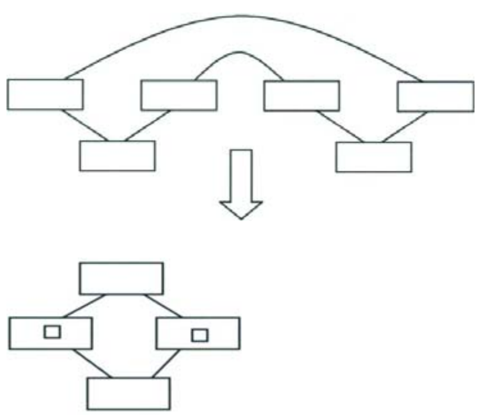


Figure 7. Composite scheme SPP

An example of a problem corresponding to the structure SPP is

> *Problem 3. John has 7 pieces of clothing, 3 shirts and the rest pants. How many ways can he combine shirts and pants to get dressed?*

A new subscheme emerges when one part and the whole of a simple scheme matches with one part and the whole of the other (P, W = P, W). In this case, we obtain the composite scheme that we labelled SWP (Sharing Whole and Part) (see Figure 8). This subscheme can be obtained both from the composite scheme "sharing whole" SW, if we make a part of each simple scheme coincide. It can also be obtained from the composite scheme "sharing part" SP, if we match the whole of each simple scheme.

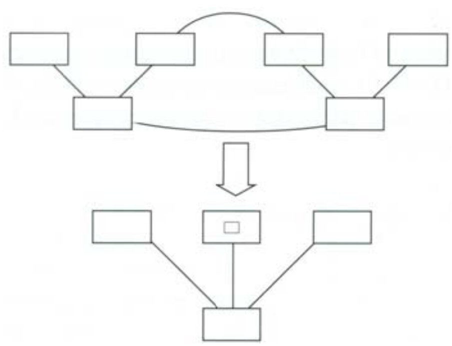


Figure 8. Composite scheme SWP

The following problem is an example of the scheme SWP:

> *Problem 4. Peter has 15 marbles. Peter has 10 marbles more than John. How many times more marbles does Peter have than John?*

In summary, if we consider two nodes to connect the two simple structures that form a two-step problem, we get the following composite subschemes (see Figure 9):

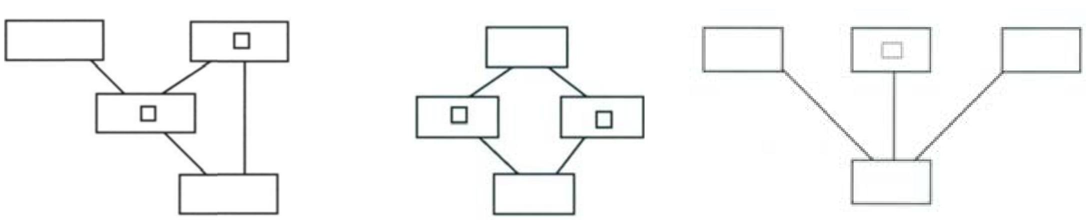


Figure 9. Composite subschemes with two nodes

The double node as a characteristic of some two-step problems can be related theoretically from the cognitive point of view to individual differences or different success rates (Frías, & Castro, 2007). This is due, for example, to the limited capacity of the work memory or, as Embretson (1983) suggests, to the fact that "the characteristics of the stimuli of the items in the tests determine the components that are involved in finding the solution" (p. 181). From the foregoing considerations, we find it important to study whether the two-node problems have different cognitive effects on the subjects.

For the specific case of two-step problems, the variable node takes more than one value. We have described two-step problems with one node and two-step problems with two nodes. We now ask whether the number of nodes in a scheme is a cognitive variable that can influence the problem-solving process for two-step problems in students who are finishing their elementary education.

Our conjecture is that the number of nodes in a composite two-step problem affects the way in which the advanced elementary school students represent two-step problems internally. This difference should become visible in issues such as the success rate and the emergence of errors specifically involving the number of nodes.

Method

Participants

We performed a study to compare the competence of students from the fifth and sixth grades of elementary education (ages ranging from ten to twelve) in two-step arithmetic problems and to determine whether the number of nodes in the problem influences the process of solving it. 172 students from public elementary schools in the city of Almería (Spain) participated in the study, 86 students from 5th grade and 86 from 6th.

Variables

Given the wide variety of two-step problems, we limited the study to using a carefully-defined set of problems. The first condition we imposed on the two-step problems used in the study was that the semantic category corresponding to the first simple structure of the problem be comparison (additive or multiplicative) and the semantic category corresponding to the second simple structure of the problem be combination, whether additive or a cartesian product. We imposed this restriction to control for the possible effect that the kind of semantic category in each of the simple schemes could have on the overall solving of the two-step problem.

Once we established this condition, the problems we used were chosen using factorial design with four factors or independent variables of the problems, which are:

First factor

The first factor, which we call A, includes whether one of the simple structures that make up the two-step problem has an additive or multiplicative character. We understand the additive structure here to include problems that are solved with one addition or subtraction. Likewise, we understand by multiplicative structure problems solved with one multiplication or division. The variable A refers to the double arithmetic relationship present in the two-step problem and in this study takes two values, corresponding to the possible combinations of a problem composed of two steps, a simple additive structure and another multiplicative structure:

- A_1 for an additive structure followed by a multiplicative structure (+, ×).
- A_2 for a multiplicative structure followed by an additive structure (×, +).

Second factor

Since the two-step problems that compose the instrument we have used all contain a simple scheme of comparison, we have limited the possible variants of these comparison problems to two kinds, consistently worded comparison problems and inconsistently worded comparison problems (Lewis & Mayer, 1987). Attending to these two kinds of wording for comparison problems, we consider the variable to be the kind of wording in the comparison, which we have called variable E and which takes two values:

- E_1 if the wording of the comparison is consistent.
- E_2 if the wording of the comparison is inconsistent.

E_1 Consistent wording	E_2 Inconsistent wording
John has 15 marbles Peter has 3 times more marbles than John How many marbles do they have altogether?	Peter has 15 marbles Peter has 3 times more marbles than John How many marbles do they have altogether?

Third factor

Each of the simple relationships involved in a two-step problem can be of either increase or decrease (Castro, et al., 1996; Castro, Rico, Castro, & Gutiérrez, 1994; Rico, Castro, González, & Castro, 1994). We call R the variable that combines the two possibilities in the double relationship. In this study, we will take into account two values:

- R_1 for the relationship increase-increase (I I).
- R_2 for the relationship increase-decrease (I D).

From the point of view of direct translation based on key words, this variable provides information most specifically about the arithmetic relationship that can be used. Increase will refer to addition or multiplication and decrease to subtraction or division.

Fourth factor

The fourth factor is the variable, our main focus of attention. It includes the number of nodes in the two-step problem. The number of nodes, which we call the variable nodes (N), has two values in this study:

- N_1 for two-node problems.
- N_2 for one-node problems.

N_1 two-node problems	N_2 one-node problems
Mary has 15 trading cards. George has 3 times more trading cards than Mary. How many trading cards do George and Mary have between the two of them?	Mary has 15 trading cards, and Paula has 90 cards. George has 30 more cards than Mary. How many cards do George and Paula have between the two of them?

Instrument and procedure

The instrument used in this experiment was a questionnaire with sixteen problems. The sixteen problems correspond to the possible combinations that emerge from crossing the four factors mentioned above in a factorial design. So as not to overwhelm the study subjects with too many problems, we divided this set of sixteen problems into two questionnaires of eight problems each, according to the following distribution:

		N_1		N_2	
		A_1	A_2	A_1	A_2
E_1	R_1	Q1	Q2	Q1	Q2
	R_2	Q2	Q1	Q2	Q1
E_2	R_1	Q1	Q2	Q1	Q2
	R_2	Q2	Q1	Q2	Q1

Q1 Questionnaire Nº 1 Q2 Questionnaire Nº 2

The problems in these questionnaires were solved by the children individually and silently in the classroom using pen and paper. Each child was given a questionnaire at random.

Results

The answers given by the subjects to the problems posed were evaluated as correct or incorrect, taking into account the choice and execution of the operations, as well as the expression of the result. We have classified a response as correct when the subject has chosen the right two operations between the corresponding data and has expressed the solution correctly, writing it in the space provided for the result the expression of the relationship that each problem required according to the instructions provided on the questionnaires. This circumstance occurred in different ways. The most common was to perform two operations, executing the corresponding algorithms, and to conclude with the full expression by answering the question posed in the problem. However, we have also considered correct those answers in which this was done implicitly. For example, given the problem:

Javier has 12 pairs of pants. Javier has 3 more shirts than pairs of pants.
How many ways can Javier combine pants and shirt?

Some subjects did one of the operations (12+3=15) mentally, so that the only explicit operation that appears is 12×15=180. In cases like this, we have evaluated the answer as correct, since we understand that student chose the two correct operations, performed one as a mental calculation and the other as a written algorithm, and provided the correct answer: "Javier can combine his shirts and pants in 180 different ways." We have also considered answers to be correct if the answer was expressed elliptically, for example, "They can be combined in 180 ways." In cases where students chose the operations to be performed correctly and used the correct data but committed a calculation error in the algorithm, we have considered the answer to be correct, even though the result shows a quantity different from the correct one. In this case, we believe that this kind of error does not affect the subject's understanding of the problem.

The answers were evaluated as incorrect when one of the two operations to be performed was not the correct one or the subject did not perform the operation with the proper data. No response on one of the operations was also qualified as incorrect, since it shows that the subject did not understand at least one of the two relationships in the problem. No answer at all was also evaluated as incorrect.

The success rates at which the children in the study were able to translate each of the questionnaire problems into its arithmetic representation are shown in Table 1 as percentages. They range from 20% for the most difficult problem to 90% for the least difficult. This result shows that some of the factors that define the problem influence their difficulty. To highlight which variables have a significant influence, we have applied a variance analysis to the four factors.

Table 1. Percentages of success in the questionnaire problems according to factors

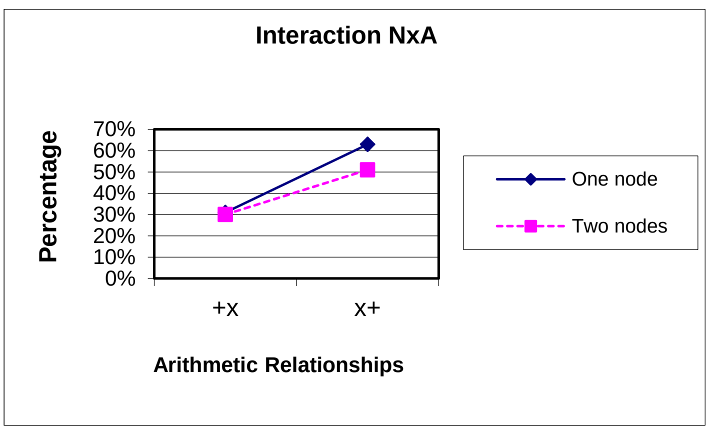
		N_1 -two nodes		N_2 -one node	
		A_1 +×	A_2 ×+	A_1+×	A_2 ×+
Consistent E_1	R_1 I I	37	80	36	90
	R_2 I D	34	57	33	58
Inconsistent E_2	R_1 I I	28	36	34	55
	R_2 I D	22	30	20	51

Using the success rate measured in percentages as a dependent variable, we have applied variance analysis to detect whether the four factors defined in the study had a significant effect on the success rate. The variance analysis applied to the data obtained shows a significant effect on the following cases:

- variable N number of nodes ($F = 6.677$, $p=0.010$). The percentage of success on problems with one or two nodes is: two nodes-N_1 (41%) and one nodes-N_2 (63%).
- variable R combinations of increase and decrease ($F=20.982$, $p=0.000$), with a percentage of success on the combinations of: increase-increase (49%) and decrease-increase (38%).
- variable E or kind of wording ($F=56.504$, $p=0.000$): Consistent (61%) and inconsistent (45%).
- variable A combination of the additive and multiplicative relationships ($F=116.760$, $p= 0.000$). The percentages of success on the combinations of additive and multiplicative relationships used were: A1(+×) combination (30%) and A2(×+) combination (57 %).

We interpret the marked difference in difficulty shown by combinations A_1 and A_2 according to the restriction imposed, that is, that the problems be comparison (additive or multiplicative) in the first step and either additive or Cartesian product combination in the second step. In problems of the type +×, we use the additive comparison in the first step and the Cartesian product in the second step. In problems ×+, we use the multiplicative comparison in the first step and the additive combination in the second. The presence of the Cartesian product in the simple scheme corresponding to the second step of the problems seems to cause the difference in difficulty.

The only significant interaction effect influenced by the variable of node is N×A (F=6.084, p=0.014). This interaction does not change the order of difficulty of the values of the variable node, however, as can be seen in graphic 1.



Graphic 1. Percentages of correct answers according to nodes and combinations of arithmetic relationships

In graphic 1, we can see that the problems with two nodes are more difficult to translate into a symbolic representation than the problems with one node for the two combinations of arithmetic operations. We can conclude from this analysis that the number of connections between the two relationships is a significant differentiating characteristic in two-step problems. The percentages of success on one-node problems (63%) and two-node problems (41%) show a significant difference in students' performance between these two kinds of problem. This difference does not depend on the other factors considered.

Error analysis

In written products, we found that in addition to typical errors already identified in one-step problems (such as the additive error or the inversion error), the sample subjects produced new errors in the two-step problems, errors that we identified as errors belonging to the double structure itself. Since our goal is to characterize the issues that differentiate the two-step problems, we will stick to the description of the errors specific to the double structure.

Type 1 error: performing only one operation

This error is characterized by using only one operation to solve a two-step problem. The operation may be either one of the two correct operations that should be performed or the wrong operation for another reason. In all of the cases, the subjects do not attempt to perform more operations but instead give as an answer the result of the only operation that they have performed with the two pieces of information from the problem. Most of the cases observed occurred in problems with two nodes (and only two pieces of information). In a few cases, this kind of error occurred with a problem of only one node (with three pieces of information). Table 2 shows examples of this kind of error.

Table 2. Error in performing one operation

Problems	Errors	Comments
Example 1 *Anne has 12 pairs of pants* *Anne has 3 shirts fewer than pants. How many ways can he combine pants and shirts?*	*12 – 3 = 9* *Result: He can combine pants and shirts in 9 ways*	Omits the second operation
Example 2 *John has 24 balls* *John has 3 times fewer balls than Peter. How many balls do they have between the two of them?*	*24 + 3 = 27* *Result: Between the two of them, they have 27 balls*	Omits the first operation
Example 3 *Anne has 48 trading cards* *Mary has 4 times more trading cards than Anne. How many do they have between the two of them?*	*48 × 4 = 192* *Result: Between the two o them, they have 192 trading cards*	Omits the second operation

In the problems used in this study, the two relationships are ordered; the first one is always a comparison and the second a combination. For this kind of error, we can therefore distinguish the cases in which the subject forgot the first relationship from those in which the subject forgot the second:

1. Forgetting the first relationship

In this case, subjects take the two pieces of information in the problem and perform an operation without taking into account the first relationship in the context of the problem. They focus their attention on the second relationship, which is the one in which the problem's question appears. Examples 1 and 2 in Table 2 illustrate this case.

2. Forgetting the second relationship

In this case, they take the two pieces of information in the problem and work with them in the context of the first relationship stated in the problem, not taking into account the second relationship. In the result, they answer the question in the problem that corresponds to the second relationship although this value was obtained with the first. Example 3 in Table 2 fits this type of error.

Type 2 error. Ordered data

This error is characterized by choosing the data for performing the relationships in the same order in which they appear in the problem. In certain problems in our study, this leads to an error in the two relationships in the problem. The students take the first two pieces of information that appear in the word problem and perform the operation, then to perform another operation on the result and the third piece of information, and finally, with this result, to find the solution. An example of this error can be seen in the solution given to the problem in Table 3.

Table 3. Error in ordered data

Problem	Solution with Type 2 error
George *has 18 shirts and 6 belts.* George *has 3 shirts more than pairs of pants. How many ways can he combine pants and belt?*	*18 – 6 = 12; 12 × 3 = 36* *Result: He can combine pants and belt 36 ways*

This way of acting indicates recognizing the two relationships in the problem, even distinguishing between the two simple structures, one additive and the other multiplicative. But the subjects do not associate the data and the relationships in each structure correctly. This leads us to think that the choice of data is mechanical or algorithmic and that order of presentation takes precedence over any other characteristic of the problem. In many cases, we see that, if the correct order coincides with the order in which the data are presented, the subjects give the correct response, but when the correct order is different than the order in which the data are presented, students make mistakes.

These last two kinds of error, *one operation* and *ordered data,* occur in the same subjects; that is, that for the two-node problems they commit the error of *one operation* and for a one-node problem, that of *ordered data*.

Type 3 error. Repeating the unshared information

In the two-node problems, we saw an error that consisted of using twice the unshared piece of information in the two simple structures that compose the two-step problem, while using the shared piece of information only once. An example is shown in Table 4.

Table 4. Error of confusing repeated information

Problem	Solution with Type 3 error
Lucia has 15 shirts. Lucia has 3 fewer shirts than pairs of pants. How many ways can she combine shirts and pants?	*15 + 3 = 18; 18 × 3 = 36* *Result: She can combine shirts and pants in 36 different ways*

The previous solution that contains the Type 3 error shows that the subjects have recognized the two relationships and distinguished two structures, one additive and the other multiplicative. Further, the repetition of one piece of information from the problem in the calculations (in this case, the 3) seems to indicate that the subject recognizes that he or she must use this piece of information twice. The error occurs in choosing the right piece of information.

Type 4. Other errors

In this section, we include errors that do not fit any of those mentioned above, cases in which it is difficult to know what motivated the subjects' choice of operations. Most of these cases occur in problems with one node in which the student only recognizes as characteristic of the problem that there are always two or more operations but chooses the operation and/or the data related to it arbitrarily or by chance.

The distribution of the four kinds of error described according to levels of 5th and 6th grade are shown in Table 5. Here, we differentiate two subtypes two subtypes for the error *one operation* one type for the error *ordered data* and another for *repeat unshared datum,* whereas in classifying the others we include the unclassifiable wrong answers in the foregoing, as well as missing answers.

Table 5. Frequencies of each error at each level and total errors

Error type	Subtype	5th grade		6th grade		Total	
		Frequency	%.	Frequency	%.	Frequency	%
One operation	Forgetting the first relationship	16	16.16	10	11.24	26	14%
	Forgetting the second relationship	45	45.45	52	58.42	97	52%
Ordered data		22	22.22	14	15.73	36	19%
Repeat unshared datum		6	6.06	9	10.11	15	8%
Others		10	10.10	4	4.49	14	7%
Total		99		89		188	100%

As can be seen in Table 5, all kinds of error detected appear in the two levels (5th and 6th grades). Overall, the error in one operation has occurred with similar frequency at both levels, but this is due to the fact that the two subtypes compensate for each other. That is, students in 5th grade omit the first operation more frequently, whereas those in 6th omit the second more frequently. The next most frequent error is that of ordered data, which occurs with greater frequency in students in 5th grade than those in 6th.

Since we chose and identified the problems based on the four factors (N, E, R, A), it is reasonable to attempt to relate the types of error defined to these factors. We classified the association between the errors as belonging to two-step problems. The four factors are shown in Table 6, which includes the distribution of frequencies for each of the problems, according to the combination of factors.

Table 6. Frequency of errors in the combination of four factors

Factors				Type of error				
N	E	R	A	One operation		Ordered data	Repeating the unshared information	Others
				Forgetting the first relationship	Forgetting the second relationship			
N_1	E_1	R_1	A_1	7	15	0	3	0
			A_2	0	13	0	0	0
		R_2	A_1	3	18	0	1	0
			A_2	1	10	0	0	0
	E_2	R_1	A_1	4	11	0	1	0
			A_2	0	10	0	1	0
		R_2	A_1	5	13	0	3	0
			A_2	5	4	0	0	0
N_2	E_1	R_1	A_1	2	0	9	0	2
			A_2	0	0	1	1	0
		R_2	A_1	0	1	6	0	5
			A_2	0	0	4	0	0
	E_2	R_1	A_1	0	0	11	1	2
			A_2	0	0	2	0	1
		R_2	A_1	1	0	3	3	4
			A_2	0	0	0	1	0

Conclusions

In this study, we have demonstrated a new characteristic associated with two-step word problems: the number of connections between the two simple structures that compose the problem, which we have called “node.” We have established a specific class of two-step arithmetic word problems that contain only two known quantities in their wording. We have shown that these problems have a common characteristic: they are formed of additive and/or multiplicative structures connected by two nexus or nodes. Our starting hypothesis is that the number of nodes affects the difficulty of translating the wording of the problem into a mathematical representation. With a sample of students in the last two grades of elementary school in Spain, we have confirmed this hypothesis, in the sense that the two-step arithmetic word problems with two nodes are more difficult to translate into arithmetic expressions than similar problems with one node. Further, we have significant evidence that the result is not influenced by other variables that also influence the difficulty of translating arithmetic expressions, such as whether the relationship of comparison is expressed in consistent or inconsistent language or whether the additive and multiplicative relationships are of increase or decrease. The result is also independent of the combinations of additive and multiplicative structures that compose the scheme of the two-step problem. Although there is significant interaction between the factor node and the factor that represents the combinations of additive and multiplicative structures, the analysis of this interaction shows that the order of difficulty in the two-step problems remains the same.

Likewise, from an analysis of the errors committed by the children, we have found that in addition to the errors already identified in one-step problems and reviewed in the literature, there are patterns of error associated with two-step problems; that is, errors that do not occur in one-step problems. We stress the presence of three of these: performing only one operation, working with the data in the order in which they appear in the statement of the problem, and using one piece of information twice, in the two operations, when in reality it should be used only once in one operation. The error of performing one operation occurred with greater frequency in the two-node problems, whereas the error of working with the data in the order in which they appear occurred more often in one-node problems. Therefore, the number of nodes is an issue that enables us to differentiate between types of problems and to explain part of the difficulty that two-step arithmetic word problems pose to children. When the students have to solve word problems, the number of nodes in a two-step problem is shown to be a cognitive variable that influences the problem-solving process.

The limitations of the study performed are related to the kind of problem, the students' level, and the research focus adopted. Within the different semantic categories of the problems identified in the additive and multiplicative structure, our study imposed the restriction that the first relationship stated in the problem corresponds to the semantic category of additive or multiplicative comparison. Likewise, the second relationship always corresponds to an additive combination or a multiplicative combination. These conditions can mediate the results obtained in terms of difficulty, kind of error, and frequency of error. The results obtained must also be restricted to the students' level. In our case, these are students at the end of their elementary education. The results cannot therefore be extrapolated to lower levels, although similar results could emerge in the first year of the next educational level, the first year of secondary education. Although the methodology employed is valid for achieving the goal we proposed and the evidence shows the representations that the students produce in response to the two-step word problems, they are sensitive to the presence of one or two connections between the relationships. This is already a significant result from the point of view of the development of the school curriculum. This study could be continued by tackling from a qualitative point of view the psychological reasons for the different student errors in problems with one and two nodes.

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