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Abstract: The character of the mathematics education traditions on problem solving and proof are compared, and aspects of problem solving that occur in the processes of forming a proof, which are not well represented in the literature, are portrayed.

Keywords: heuristics; problem solving; proof

Introduction

Mathematics educators tend to compartmentalize the domains 'problem solving' and 'proof and proving'. This detachment seems somehow artificial as both deal with aspects of producing mathematical argumentation. However, problem solving tends to emphasize the thought processes in furthering on-going work; in contrast the proof tradition is concerned more in evaluating the soundness of the complete output. In this paper, we shall respect the distinction made between problem solving and proof, but at the same time we shall discuss issues that are common.

We use the words 'culture', 'tradition' and 'agenda' synonymously for general views broadly adopted by the research community on any given educational perspective. Both the problem-solving tradition and the proof tradition are diverse, so we restrict ourselves to particular stances, mostly attending to the upper school and university level. For problem solving, the subject is taken for it's own sake; hence the full weight of self-conscious decision-making becomes the scope of investigation. For proof, we distinguish the case where the practitioner possesses and implements the requisite mathematical tools to fully articulate the proof from the case where he/she does not. The various types of

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tools needed will be discussed, especially when the context lies in a mathematical theory currently been taught: then tools are adapted and appropriated from techniques that the theory avails. However, such tools have to be designed and then coordinated in the mind, so within the processes in obtaining a proof it is evident that substantial elements of problem solving must occur. The main focus of the paper is to give a preliminary account of these elements.

In the next section, we shall present a short, rather personal, description of problem solving. Largely supposing that the reader is familiar with the core principles laid out by Polya, it discusses more practical issues like the role of the teacher, implementation and assessment. The section that follows deals with the problem-solving component in proof making. Here no attempt has been made to give a coherent picture of the proof tradition; one reason is that proof and proving are, as an educational domain, particularly prone to contrasting standpoints. Rather we limit our attention to those facets of proof that differ from the problem-solving tradition but at the same time retain some problem-solving elements. The choice of papers referred to is made with this in mind. The discourse will not be unidirectional; some points made could be read as if the culture of proof is supporting problem-solving activity. The extended example given in the penultimate section illustrates this, as well as other matters. The epilogue, in part, raises the question how well the problem-solving tradition (as it stands presently) is equipped to cover the problem-solving elements in formulating proof.

On the problem-solving tradition and allied practical issues

The perspective of problem solving has a relatively compact core of ideas, mainly centered on heuristics, meta-cognition including executive control, accessing and applying the knowledge base, and identifying patterns of modes of thinking as students' work progresses, following the pioneering work of Polya (e.g., Polya, 1973) and later by Schoenfeld (e.g., Schoenfeld, 1985). However, problem solving, as a domain of mathematical activity is very general; it concerns the student's engagement on any mathematical task that is not judged procedural or the student does not have an initial overall idea how to proceed in solving the task. Many other perspectives taken by the educational literature would embrace this same arena; for these the term problem solving is likely to be invoked (after all it is a term that is quite natural to use generally), but it is not a term around which the main focus revolves.

On the practical level, to deserve autonomous attention, problem solving must have something to say about teaching and instruction. The function of problem solving has been broadly characterized in three categories: teaching for problem solving, teaching about problem solving, and teaching through problem solving (Schroeder and Lester, 1989). For the first category, tasks are chosen that force students to think more actively about whatever mathematical topic that is being studied, the third is about building up conceptualization via a program of deliberately sequenced tasks. For both, problem solving is given a utilitarian role. On the other hand, for the second category, problem solving is taken per-se as an integrated theme of discourse, and we will largely adopt this perspective in this paper. The teaching must be directed to the student's own awareness of the influences that form his/her processes to reach a solution. The teacher then has to teach not only mathematical content and method, but also general solving skills. Doing this necessarily needs elements of intervention on the side of the teacher; he/she has to induce habits of self-questioning and reflection that allow students to realize ideas critical in achieving a result. This means that there are aspects of teaching problem solving that can be regarded authoritative (but not authoritarian), see e.g. B. Larvor (2010).

If a problem-solving approach is adopted in teaching, there are associated issues about design and evaluation. What constitutes a 'good' problem? For this question, one could simply say that any problem for which there is not immediately an obvious line of attack is suitable. But do other factors come in? A sense of the attractiveness of the solution is one, a sense of achievement the solver experiences is another. A measure of a 'good' problem is how far a solution, or an attempt to achieve a solution, would inspire the solver to form related problems (Crespo & Sinclair, 2008). Another possible measure might be the plurality of different directions that the problem can be treated so that connections can be formed (Leikin & Levav-Waynberg, 2007), though problems that have a particular 'catch' in the solution can also be useful because of the better control afforded to the teacher/researcher. The evaluation of a student's complete output, then, is not straightforward; a model is given in Geiger & Galbraith (1998). Another factor is the gap of experience between the setter and the solver; here lies the danger that either the setter assumes that students have more experience than they really possess or the experience of the setter leads him / her to expect an over-involved solution blinding a more elementary path. Hence it is difficult to gauge how challenging a particular problem is. Further if you credit an argument by its plausibility rather than its logical security, you bring in a subjective factor. Such points of loss of control in terms of evaluation makes problem solving, taken as an overall guiding principle in teaching, open to criticism. For example, a recently published paper bears the rather provoking title: "Teaching General Problem-Solving Skills Is Not a Substitute for, or a Viable Addition to, Teaching Mathematics" (Sweller, Clark & Kirsher, 2010). The basis of the authors' contention is well represented by their claim: "in over a half century, no systematic body of evidence demonstrating the effectiveness of any general problem-solving strategies has emerged". The position taken by the paper might well seem to be extreme and partisan, but it does reflect the difficulty in designing comprehensive, large-scale studies assessing the success (or otherwise) of teaching about problem solving, not least on what exactly should be measured.

The principles behind problem solving can be significant to the working of a student of any age and of any ability, and there are numerous educational studies that advance the cause of problem solving convincingly to populations ranging from pre-primary school level to professional mathematicians. But making conscious decisions about which heuristics to use as well as how other metacognitive dimensions should be employed need mature deliberation. In danger of seeming elitist, we consider there are two groups of students that are most able to cope with and profit from problem-solving based instruction; the so-called mathematically 'gifted' student at school, and the undergraduate student studying mathematics. (A third group might be teacher- students, as they have to learn how to reflect on and attend to the difficulties of their future students.) We are not saying that other students cannot gain from problem-solving activities, but for them the gain could well be qualified. For instance, in Perrenet & Taconis (2009), it is stated "(university mathematics) students show aspects of the development of an individual problem-solving style. The students explain the shifts mainly by the specific nature of the mathematics problems encountered at university compared to secondary school mathematics problems". Other papers offer models on how traits of thinking are different for the gifted and the expert over the 'average' solver (e.g., Gorodetsky & Klavir, 2003).

What sources cater for these special groups? First, there is now a plethora of 'problem solving' texts on the market; these usually list many challenging problems with exposition of some 'model' solutions. However, most are composed in the spirit 'you learn as you practice', without much commentary on the educational level. Typically the organization has some chapters based on general aspects of problem solving and others on problem solving that is mathematically domain-specific. (It would be misleading to identify such books as textbooks because the aim is not to cover a fixed curriculum of mathematical content.) The style of presentation can be daunting, but some tomes are particularly attractive and reader friendly. One, written by P. Zeitz (Zeitz, 2007, p. xi), includes in its preface a list of principles guiding its writing that is surprisingly close to the tenets held by educational research on problem solving:

- Problem solving can be taught and can be learned.
- Success at solving problems is crucially dependent on psychological factors. Attributes like confidence, concentration and courage are vitally important.
- No-holds-bared investigation is at least as important as rigorous argument.
- The non-psychological aspects of problem solving are a mix of strategic principles, more focused tactical approaches, and narrowly defined technical tools.
- Knowledge of folklore (for example, the pigeonhole principle or Conway's Checker problem) is as important as mastery of technical tools.

Beyond problem-solving books, there is the collected 'wisdom' from the many dedicated teachers involved in 'training' students for mathematics contests and special examinations. There are now some regularly held conferences aimed not only to attract such teachers but also researchers in education, such as one titled "Creativity in

Mathematics and the Education of Gifted Students". The ensuing interaction between the two interested communities, the one more theoretically inclined, the other more practically minded, is valuable, and has enriched the educational literature published on problem solving especially over the last decade or so. The facet of 'training' in particular is interesting, for it does not at first seem to be quite consonant to the idea of flexible thinking as espoused in the problem-solving tradition held by educators.

Another source is problem-solving courses offered in the curriculum of some university mathematics departments. The content and 'style' of the delivery of a class usually is a mixture of: introductory statements made by the instructor, students' attempting to solve particular problems quite often conducted in small groups, students criticizing peers work, a class discussion about how solutions were instigated and how completed arguments functioned to realize the solution. The instructor perhaps in the end winds up the session by relating the class activity to terminology found in the problem-solving culture. For such a free ranging course, an accompanying textbook is out of place; rather a succession of class-plans by the individual teacher is followed ad lib. This raises the issue of the demands put on a teacher when teaching a problem-solving course, and whether they have to be trained to teach in a special way (see e.g. A. Karp, 2010). Another awkward point concerning courses oriented towards problem solving is that the level of interaction is high, so really are suitable only for classes of a relatively small size. The yearly intake of students to a mathematics department can be in the hundreds, with the result that a problem-solving course is usually selective and non-compulsory. Thus the course effectively becomes a special interest class on a par to 'standard' courses that present particular mathematical theory. Where then is the universal need for undergraduates to be instructed in problem solving? Indeed, it is reported in Yosof and Tall, 1999, that students who took a problem-solving course mostly enjoyed the experience but they found difficulties to apply what they learnt in other courses.

(At this point, we should clarify our position on the use of textbooks; as we asserted above, textbooks perhaps do not have a place in teaching *about* problem solving, but they certainly have their place in teaching *for* problem solving. The idea is that the whole structure of the book consists of carefully sequenced problems leading up to major theorems in a main field of mathematics. It replaces a 'standard' presentation of a topic in the curriculum. An example is found in a book by Polya & Szego for Analysis first published in 1924 (translated into English in 1978); more recently R. P. Burn has written several textbooks (e.g., R. P. Burn, 1992) in the same kind of spirit. A natural question arises: does a course based on such a textbook infuse general problem-solving sensibilities?)

Much that we have discussed so far is addressed to practical matters; the focus for the remaining part of the paper will be based on theoretical lines. There are many expositions extant that have elaborated on the core ideas, i.e. heuristics, metacognition and observing phase patterns during the solution process. Some have a local perspective, some attempt to present overall models to refine the character of the whole field. For the latter, (Carlson & Bloom, 2005) is a good example; the authors develop what they call a 'Multidimensional Problem-Solving Framework', which tabulates items along two axes: activity phases against resources, heuristics, affect and monitoring. Within the framework it is stressed that various aspects of cycling in types of activity occur in problem-solving behavior. The paper clearly is in the fold of the problem-solving tradition. But for many papers, this is not so clean-cut; in them there is substantial material that seems to be in accord to the tenets of problem solving but the ostentatious perspective lies elsewhere. In Mamona-Downs and Downs (2005), it was argued that if we wished to form an 'identity' of problem solving we must examine how other mathematics education topics impinge. One topic brought in was 'proof'. The 'terrain' of proof obviously encompasses many reasoning processes that are common to problem solving, so it is a natural candidate for comparison. In the next section we shall discuss the confluences (and to some degree the disparities) between the domains; references are made to papers that are ostensibly placed in the proof agenda but betray interesting problem-solving traits.

The interface between problem solving and proving

Proof and proof production is associated with deductive reasoning. From the perspective of problem solving, the notion of deductive reasoning can seem artificial; employing deductive reasoning on its own cannot help students to build up the ideas involved in the making of a strategy, it can only inform the student that any particular act is 'legal' or the whole argument a-posteriori is logically sound. On the other hand induction, i.e. obtaining evidence from considering cases that are not exhaustive, is useful for explorative work but is insufficient to establish the desired result. Over the years numerous models of reasoning have been put forward to fill the gap between inductive and deductive argumentation. To mention a few, there is representational reasoning (Simon, 1996), abductive reasoning due to Pierce (see Cifarelli, 1999, for a contemporary summary), and plausible reasoning derived from Polya himself (Polya, 1954). Such models differ in detail, but all deal in one way or another with a shift from a speculative mode of thinking to one that has an anticipatory character. Any type of mathematical reasoning suggests an a-posteriori summation of the lines of thought taken before; the question "what is your reasoning here" is a request to track back. Despite this, it still concerns on-going working; an advance in reasoning depends on the problem-solving decisions preceding it. In this respect some authors like to discriminate between reasoning and argumentation; for example, Lithner (2001) takes argumentation as the 'substantiation' that convinces you that the "reasoning is appropriate". Argumentation then suggests a completion as well as a check that your reasoning is, in loose terms, getting the job done; as such, argumentation should be closely related to proof, in cognitive terms at least. It has always been contentious what a proof is; perhaps the range of interpretation today is as wide as it has ever been. Here is not the place to give even a skeleton sketch of the current views taken about the role and character of proof; a comprehensive account from the mathematical education perspective is to be found in the book Reid & Knipping (2010). In the last recommendation for 'Directions for Research' in this book, it is stated, "the relationship between argumentation and proof is far from clear" (despite the numerous theoretical theses forwarded in this area). In explaining this relationship it might be instructive to explore what different problem-solving skills we would expect vis-à-vis argumentation and proof.

Is proof for every student and for every age? Leading up to answer this question we regard 'deductive' or 'formal' proof as an ideal rarely adhered to. What, then, do professional mathematicians tend to produce? We believe that this issue is nicely expressed in a public lecture given by Hyman Bass (2009) where the image of a proof providing certification is replaced by an image of a proof supporting a claim:

> *" Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists…the convinced listener feels empowered by the argument, given sufficient time, incentive, and resources, to actually construct a formal proof".* *p. 3*

Hence a typical proof provided by a mathematician is an argument for which there is a potentiality to convert it into formal proof (in principle at least). Bass then considers the notion of generic proof (see also, e.g., Leron & Zavlasky, 2009) as a type of proof that mathematicians often accept and adopt, and convincingly relates an incident where a primary school child was able to express a generic proof (in joint work with D. Ball). The child was able to explain the proposition that 'the sum of two odds are even' by mentally imagining two separate collections each with an odd number of objects, pairing off objects that lie in the same collection as far as possible and pairing the two objects 'left over' one from each collection. One might say the argumentation takes place on the perceptional level and so cannot be regarded as a proof. On the other hand, the reasoning is executed through properties that suggest both abstraction and generalization are involved; from this criterion, perhaps it should qualify as a proof after all. The obvious stance to take is to acknowledge different levels of proof. For example, in the opening document for ICMI Study 19 (2009) on proof, the terms 'developmental proof' and 'disciplinary proof' are introduced. A major factor in this distinction is that students at school do not usually possess the requisite tools to allow them to articulate disciplinary proof, whereas in the culture of advanced mathematical thinking (pertaining mostly to tertiary level study) pains are taken to explicitly define the tools needed to process proof in a particular field. For example, if the Intermediate Value Theorem of Real Analysis is mentioned at school it is usually taken as a truism, but at university either it has to be proved or explicitly recognized as a premise. Hence the learning of mathematics at the university level is 'privileged' in terms of proof making; in principle, the tools are available, or the tools are at hand to 'design' further finely-honed tools for your own specific purposes.

How does the above concern problem solving? As regard to 'developmental proof', students have to rely on mental images and loosely grounded representations; the processes in initializing, collating and monitoring the argumentation as it evolves are very much in the field of problem solving. But because now mostly we are angling for an 'informal' justification of a general proposition, there is a tendency for properties to determine objects rather than vice-versa. Here, the notion of characterization comes to the fore; you ask which objects satisfy the conditions (rather than asking which properties a given object satisfies). Even though there is no real difficulty in designing tasks asking for a characterization in the problem-solving mold, this aspect is poorly represented as a theme in the literature. What quite often occurs is that the solver identifies a class of objects, C say, for which either all the objects that hold the given conditions are shown to be in C or all the objects of C are shown to satisfy the conditions. Several rounds can be made in restricting or expanding the class respectively, until analysis allows the removal or inclusion of any remaining isolated exceptions. Such a program probably is best illustrated in the literature by the framework of 'example generation', largely launched through some work of Mason and his colleagues (e.g. Watson & Mason, 2005). Also related is the Lakatosian notion of 'heuristic refutation', where counter examples are not taken to disprove but in order to reformulate the premises (e.g. De Villiers, 2000).

For disciplinary proof, in principle the tools to prove are at hand. But an informal discourse is usually kept to whilst engaged in the actual production of the proof, though much technical terminology is retained. For the mathematician who holds well the topic concerned, the technical terminology is not a barrier to understanding, to the contrary it is empowering (Thurston, 1995). Whenever there is an interaction between an informal language and one that is more documentarily directed, problem solving has its place; strategy making is made in the informal language and 'converted' to the documentary style. The problem-solving aspects so evoked would have restrictions compared to general problem solving but they are nevertheless important. Below, particular angles of this issue are mentioned. In the context of building up a mathematical theory, a meta-cognitive examination of a proof is required to understand what is important to retain in the memory; the proof, the fact that the proof ascertains, both or none. For example, a method can be extracted from a proof, which has a potential to be applied elsewhere; in Hanna & Barbeau (2008), this phenomenon is discussed through various facets of problem solving. Structuring mathematical work into semi-independent units acts not only to produce a neat exposition but also represents essential 'chunking' of lines of thought without which the mind could well be overstrained. The deliberate design of modules in order to process the desideratum would seem to lie naturally in the realm of executive control. (For an illustration, see the worked example appearing in the next section.) For disciplinary proof the knowledge base typically is sophisticated and in flux; one possible consequence of this is that the solver could be tempted to apply knowledge that far exceeds the needs of the task/proof. An instance is given in Koichu (2010) from an example-generating activity. Also, the detection of applicability of theoretical knowledge is not often immediately apparent; in Mamona-Downs (2002) it is suggested that part of the teaching of a mathematical theory should include what the author terms 'cues', i.e., formats of knowledge that are not important theoretically but invite realization of certain types of application. (An instance is to be found in the worked example.) Changing track a little, we note that there is still a strand of informal discourse in disciplinary proof, so one might expect that some students would like to exploit it more than others. In this respect some research, mostly aimed at the tertiary level, has indeed identified students that have a strong inclination to work consistently either semantically or syntactically (e.g. Weber & Alcock, 2004). Such a marked preference must reflect the character of the problem-solving tools with which a particular student feels most at ease. Another but related issue is how to initiate a proof; Moore (1994) has noted that from fairly elementary but formally defined systems many students cannot deduce the simplest consequences, whereas Selden et al. (2000) talk about 'tentative starts'. The first suggests that students cannot interiorize the abstraction that confronts them, the second suggests a more pro-active view that by 'playing around' with operations that they can do, even 'blindly', students can 'click' to openings in the underlying (or accommodating) structure yielded by the given situation. (Our worked task also features a tentative start.) The notion of structure, even though being somehow vague, seems to be a natural backdrop to combine the analytic tools given by formalization with problem-solving input, see Mamona-Downs & Downs (2008).

We wind up this section by raising a few points for which the difference between disciplinary and developmental proof is no longer relevant. First, whenever a shift of the mode of thinking occurs, the character of the supportive components of problem solving also changes. We have already discussed how an informal language reinforces (disciplinary) proof. The making of a conjecture also marks a change in the mode of thinking; one could compare the style of argumentation made before the conjecture to that made after. If you adopt the more conventional propositional form of a proof perhaps it would be more appropriate to compare pre-strategy work with work done in effecting the strategy. Also, how does the reasoning allowing a first draft differ to the reasoning leading to the final presentation of a proof? In particular, whilst obtaining a proof sometimes recourse is made to diagrams and other kinds of representations that, quite often, are not ultimately referred to in the presentation. Another differentiation in mode of thought concerns what is processed in the mind against what is carried by 'authorized' symbolic usage. Even though these switches of thinking are documented in the literature, they have not been thoroughly examined in the problem-solving tradition.

Second, we resume our discourse regarding the essential difference between proof and problem solving. We center our problématique on discerning 'problem solving' tasks from 'proof' tasks. For a 'problem solving' task it is allowed to accept perceptual indications, as long as they seem self-evident. Verification itself does not feature strongly in the difference; rather a task requiring a proof differs in that the verification has to be articulated in officially accepted mathematical language; one might say we require an endorsement of the verification. Let's have an example. The task is to identify the different types of plane nets of a cube. It's a problem that can be tackled by a bright student of age eight, even though there are quite sophisticated things to do; first to interpret from the task environment what is meant by a 'type', and after to validate the answer by taking cases in an exhaustive manner. But the argument cannot be judged to be a proof because each net is just recognized as being one; there is no explicit mathematical warrant expressed that assures that when the squares are folded in the appropriate fashion they do indeed ‘form’ a cube. Furthermore there is no expectation for the solver to undertake this ‘extra’ level of verification; thus what we have here is a problem-solving task.

An interesting offshoot of the above concerns mathematical modeling. Suppose a task is expressed in words that are not completely mathematical in form; it is modeled into a self-contained mathematical system in which it can be treated as a proof. However, the action of modeling certainly is subjective; so the task is better assigned as a ‘problem-solving’ task, but with a strong proof-making component. Also note that informal argumentation carried out within the task environment sometimes can be sustained right up to achieving a solution; if a more strict version is desired, modeling can be made not only of the task but also of the line of the context-held reasoning. This can be a valuable vehicle for students to realize the character of proof; proof in this regard is a channel to provide the tools to fully articulate the output of a problem-solving activity, see Mamona-Downs & Downs (2011).

An example

The example takes the form of an indicative solution path of a particular task; it exemplifies some points made in the previous section.

The task is relatively complicated to solve, though no complex mathematics is involved. A strategy is made without knowing which tools are needed to implement it. Designing these tools requires anticipatory reflection but the form of them rests on simple proofs, so we have a case where the proof culture contributes to problem solution processes. The style of discourse indeed is made very much in a problem-solving vein, but from it a presentation as a proof is readily extracted.

The task:

Does the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ have a partial sum equal to an integer apart from 1?

<u>Preliminary observations; sizing up the situation</u>

From previous knowledge, we know the harmonic series 'converges' to ∞, i.e. for every $n \in \mathbf{N}$, there is a partial sum that exceeds n. Hence there are 'infinitely' many potential candidates for a partial sum to be an integer, so we cannot reduce the problem to a finite number of cases.

The first partial sum equals 1; can we make a preliminary guess whether it is likely for any other partial sum to be an integer? In response, take any $n \in \mathbf{N} \setminus \{1\}$ and consider the greatest $r \in \mathbf{N}$ for which $\sum_{i=1}^{r} \frac{1}{i} < n$. Then for n to be a partial sum necessarily:

$$n - \sum_{i=1}^{r} \frac{1}{i} \text{ must equal } \frac{1}{r+1}.$$

This condition seems stringent because of the appearance of r in both terms, so despite there being infinitely many candidates for n, it still would be a surprise to get a solution apart from 1. But this does not really help us to get a start for a strategy that would be conclusive.

<u>Playing around</u>; <u>try out an action you can do</u>

A partial sum is just a finite number of fractions added up; an action you can perform is to render this into a single fraction, i.e.

$$\sum_{i=1}^{r} \frac{1}{i} = \frac{\sum_{i=1}^{r} \frac{r!}{i}}{r!}. \qquad (1)$$

This move is made as a tentative start; it is speculative in the sense that there is a hope (but not an expectation) that the new algebraic form might give us more handles to attack the task. Notice that the new form is one natural number divided by another; the question is reduced to analyzing whether the latter is a divisor of the prior.

Importing mathematical knowledge and its cue

The knowledge we import is the theorem that states any natural number greater than one can be expressed (uniquely) by a product of prime powers. The cue relevant to the task is if you think that n does not divide m, for m, n $\in \mathbf{N}$, then there will be a prime p such that the highest power of p dividing n is greater than the highest power of p dividing m.

The strategy

Suppose that the solver believes that there is not a solution apart from 1. In accordance he/she chooses the 'simplest' prime, i.e. 2, and tests whether the highest power of 2 dividing r! is greater than the highest power of 2 dividing the nominator in (1). (If this test fails, one might choose another prime to check, or change the method from the experience gained by considering the case p=2).

Considerations how to implement the strategy

In order to analyze the highest power of 2 dividing

$$\sum_{i=1}^{r} \frac{r!}{i}$$

you have to search for more elementary and general properties about the highest power of 2 dividing integers. For convenience, we introduce the rather eccentric (but standard) symbolism: $2^a \parallel u$ where a, u $\in\mathbf{N}$ indicates that a is the highest power of 2 dividing u.

Designing the auxiliary tools

(i) Investigate: If $2^a \parallel u$, $2^b \parallel v$ and $2^c \parallel (u+v)$, what is c in terms of a and b?

It is easy to demonstrate that

if $a\neq b$, then $c=\min(a, b)$

if $a=b$, then $c>a+b$

but nothing further can be said in general for the second implication. Hence in the case of $a\neq b$ we have more control than in the case of $a=b$.

(ii) In order to take advantage of the good control that occurs in the case of $a\neq b$ it is useful to have this fact established:

If $2^a \parallel u$ and $2^a \parallel v$ (where $u<v$), then there is a natural number w such that $w>u$, $w<v$ and $2^{a+1} \parallel w$. An immediate corollary is:

Suppose that $S:=\{1, 2, 3,...,r\}$ and $l:=\mathrm{Max}\{l': \exists\, k'\in S \text{ s.t. } 2^{l'} \parallel k'\}$. Then there is a *unique* element k of S for which $2^l \parallel k$. (2)

Implementation of the strategy

Here we show how the auxiliary tools facilitate the original situation:

Let $2^N \| r!$ and $2^{N_i} \| i$ whenever $i \leq r$. Now the highest power of 2 dividing $\frac{r!}{i}$ is equal to $N - N_i$.

For k and l specified in (2), the highest power of 2 dividing $\frac{r!}{k}$ is equal to N - l so is less or equal to N - N_i; the equality holds only when i=k (second auxiliary tool). Applying the first auxiliary tool (recursively) we know that

$$2^{N-l} \| \sum_{i=1}^{k} \frac{r!}{i} .$$

Now N_k is simply an alternative symbol for l, so for i > k, $N-N_i > N-N_k$ and we can again apply the first auxiliary tool to obtain

$$2^{N-l} \| \sum_{i=1}^{r} \frac{r!}{i} .$$

Now if r>1, l is a positive integer, so the highest power of 2 dividing the nominator in (1) is less than that for the denominator. We are done.

Comments

1. The preliminary observations are made not only to 'understand' the task, but include comments concerning a 'feasibility test' in order to make an informed guess on what the most likely outcome would be. Note that this guess is not made on experimental evidence.
2. The preliminary observations did not give a lead how to approach the task. A blind algebraic operation is made, resulting in another algebraic form and a new perspective. This action was not motivated, so there is an element of luck here.
3. The new form is a quotient of natural numbers; the issue now is to try to show that the denominator does not divide the nominator. (Guided by our feeling that the solution space is empty). This issue then could be resolved through an application of the fundamental theorem of arithmetic. Would students notice this link? Suppose that this type of application was taught, would they be more likely to catch the 'cue'?
4. We now have a strategy; we take a particular prime p in the hope the greatest power of p dividing the nominator is less than that of the

denominator. Things are still tentative; we don't know as yet whether the strategy is intractable or not, and we choose p=2 solely on the grounds that 2 is probably the easiest prime to work with. Our decisions here have as much to do with hopeful wishing as control. However, somehow we judge the direction we have taken is promising.

5. We have a strategy, but now we need a strategy to implement the strategy. Modules of a more theoretical character are designed deliberately directed to the implementation. These modules can stand on their own merit outside of the context of the original task with autonomous proofs. Designing such analytic tools vis-à-vis envisaging how they would fit in the particular argumentation can be challenging.
6. In the process of resolving the task, there are several places where the solver is not completely controlling the effect of the decisions or actions that have been made. Were we just lucky? No: luck comes in, but from a certain stage there is an anticipation that things would work out as envisaged. But this raises the question, how can we quantify the grounds of this anticipation.

Epilogue

The main drive of this paper is to consider how elements of the problem-solving tradition are evident in the formulation of a proof, and (to a lesser degree) vice-versa. Also occasionally we have drawn out diverging tendencies between the two traditions. One difference not as yet explicitly mentioned is that tasks in the domain of problem solving are intended to be challenging, whereas for proof no such intention is imputed. In this regard, it is now quite often to have a transition course aimed towards proof practices in the curriculum of a mathematics department, especially in the United States; the 'content' of the examples presented tend to be relatively elementary in order to concentrate on the exposition of logically based argumentation. Such courses have a completely different 'feel' to problem-solving courses. We have stated that there is some doubt that taking a problem-solving course really will help the student when more theoretical courses are studied, but a 'proof' course could also be problematic in other ways. For example, in Alcock (2010) it is reported that some mathematicians felt that those students who can pass a proof course would be the ones that did not really need to take the course anyway. Given this, a mix of the two kinds of courses might be the most profitable. It would bring up, for example, situations for which a student can achieve a result informally, and then can be challenged to articulate it with consummate reasoning and grounds; also it would bring up situations for which the student cannot proceed without building up constructions of a formal character. Proof should not be shown just as an imposition, but as a channel that enhances our range of mathematical thought and potentialities. The two kinds of situations mentioned above surely pertain to problem solving as much as to proof, but bring out a tempered outlook towards the current tradition of problem solving. To the mathematician, answering a typical problem-solving task often is an enjoying and rewarding pursuit, but can seem frivolous if aspects resting on perception are not tied down mathematically.

In the paper by Alcock referred to above, the author identifies four modes of thinking whilst forming a proof (drawing on the comments of a small population of mathematicians); instantiation, structural, creative and critical. These bear a striking semblance to the set of phases in problem solving famously put forward by Polya, i.e., getting acquainted, working for better understanding, hunting for the helpful idea, carrying out the plan, looking back (Polya, 1973, p.p. 33-36). The main discrepancy between the two taxonomies is that 'structural thinking' is characterized by introducing appropriate definitions and by working according to the rules of logic. This is consonant to the notion of definitional tautness introduced in Mamona-Downs and Downs (2011) that is not currently stressed in the problem-solving tradition. But the meta-cognitive thought involved in forming definitions whilst partially envisaging how to control their consequences to particular ends has problem-solving elements that should not be ignored; then, perhaps, problem solving might be just as relevant in proving as it is in more relaxed forms of argumentation. In this paper we have examined elements of problem solving in the context of proof construction, admittedly in a rather eclectic way. We suggest that further research in this direction should be undertaken in the future, involving both researchers primarily 'affiliated' to proof and those primarily concerned with problem solving.

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